"Über die Formulierung der Naturgesetze mit fünf homogenen Koordinaten. Teil I: Klassische Theorie," Ann. d. Physik **18** (1933), 305-336.

# On the formulation of the laws of Nature with five homogeneous coordinates

## Part I: classical theory

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§ 1. Introduction. – § 2. Definition of tensors in homogeneous coordinates. – § 3. Metric. Connection with inhomogeneous coordinates. Decomposition of homogeneous tensors into affine parts. – § 4. Covariant differentiation by means of the three-index symbols $\Gamma^{\nu'}_{\lambda\mu'}$. – § 5. Parallel displacement of vectors. Geodetic lines. – § 6. Curvature. – § 7. The form of the laws of Nature. Variational principle. – *Appendix:* Generalization of the Ansatz for the $\Gamma^{\nu'}_{\lambda\mu'}$.

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### § 1. **Introduction**

It is well known that Kaluza and Klein have succeeded in formulating field laws of gravitation and electricity (at least, in the absence of charges and ponderable masses) in a unified way, and, in addition, presenting the law of motion for a charged mass point as a geodetic line in the five-dimensional continuum. This achievement of putting the foundations of physics on a geometrical footing stands in stark contrast with the following shortcomings, which keep many theoreticians from accepting the idea of a fifth dimension:

1. It must be formulated in such a way that the component $g_{\mu\nu}$ of the five-dimensional metric tensors shall not depend upon the fifth coordinate $x^5$, i.e., they shall be functions of only the first four coordinates. By means of this additional condition - the so-called *cylinder condition* – disturbs the general covariance of the theory, and the fifth coordinates appears to be an artificial appendage that must ultimately disappear. The first ten components of $g_{\mu\nu}$, namely, $g_{ik}$ with $i, k = 1, \ldots, 4$, can be identified with the ordinary metric tensor, and the four extra $g_{i5}$ ($i = 1, \ldots, 4$) components can be identified with the electromagnetic potentials (up to a numerical factor).

2. Since a physical interpretation of the component $g_{00}$ can be attained, one must arbitrarily set $g_{00} = 1$, which represents a new non-invariant condition. This reduces the number of equations for the $g_{\mu\nu}$ from 15, which is expressed by the vanishing of the contracted curvature tensor, to the correct number, 14, namely, the 10 equations for the $g_{ik}$ ($i, k = 1, \ldots 4$) and the 4 Maxwell equations. (There then exist 5 differential identities between them.)

An essential advance was then made by Einstein and Mayer. [1]) In this theory, the introduction of a fifth coordinate was completely absent, along with any cylinder

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† Translated by D.H. Delphenich.

[1]) A. Einstein and W. Mayer, Berl. Ber. (1931, pp. 541.

condition; rather, every point of the four-dimensional continuum was associated with a five-dimensional vector space. A particular relationship between these vectors is then posed axiomatically, in the form of their parallel displacement and how it relates to the ordinary Riemannian parallel displacement of the ordinary vectors of the parallel displacement. The electromagnetic potentials are completely absent from this theory, but only the field strengths.

This theory also has certain formal imperfections that were not present in the Kaluza-Klein formulation, but appeared for the first time in it, namely:

1. The field equations can not be derived from a variational principle. From this, one infers that such field equations that satisfy important differential identities (which a variational principle automatically guarantees) seem a bit artificial, which is also expressed in the form of the field equations. [1])

2. The first system of Maxwell equations (which is equivalent to the possibility of deriving the field strengths from potentials) does not follow from the assumed structure of space in this theory, but must be postulated explicitly (in which it can generally be connected with the curvature).

These imperfections are not very momentous in themselves, but one would not like to renounce the advantages that were already present in the Klein-Kaluza theory.

Here, we now introduce another way of presenting the Klein-Kaluza theory, namely, the *projective* way. In this presentation, the continuum is regarded as four-dimensional, but, as in projective geometry, five *homogeneous* coordinates (which will be denoted by $X^\mu$, $\mu = 1, \ldots, 5$) will be introduced; i.e., all coordinates that differ by the same factor belong to the same point of the continuum. All tensors (or, as one prefers to say in this case: "projectors") must then be homogeneous functions (of various differing degrees) of the coordinates. This way of regarding things was first applied to physics by Veblen and Hoffmann [2]), and indeed, to the interpretation of the Klein-Kaluza theory. However, these authors choose a formulation that, as a consequence of an unnecessary specialization of the coordinate system, distinguishes the fifth coordinate in a manner that is completely similar to the usual one that Klein-Kaluza introduced by the cylinder condition, and that therefore did not enter into the latter theory in an essential way.

The advance made by van Dantzig [3]) was to thoroughly examine projectors with homogeneous coordinates, along with their covariant differentiation by means of the three-index symbols $\Gamma^\nu_{\lambda\mu}$, geodetic lines, and the invariant form:

$$g_{\mu\nu} X^\mu X^\nu$$

of the metric that was introduced with the fundamental tensor $g_{\mu\nu} = g_{\nu\mu}$. Finally, Schouten and van Dantzig [4]) gave a formulation to field theory with the help of this general projective differential geometry (here, we will next discuss only the classical part

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[1]) Cf., the transition from $P_{tp}$ to $U_{tp}$ in loc. cit, § 5, eq. (41).

[2]) O. Veblen and B. Hoffmann, Phys. Rev. **36**, pp. 810, 1931. Summary by O. Veblen. Relativitätstheorie. Berlin, 1933. Further literature is contain therein.

[3]) D. van Dantzig, Math. Ann. **106**. pp. 400. 1932. – Confer the geodetic lines, in particular; Amst. Proc. **35**. pp. 524 and 534. 1932.

[4]) J.A. Schouten and D. van Dantzig, Zeitschr. f. Phys. **78**. pp. 639. 1932; furthermore, Ann. Math. (3) **34**. pp. 271. 1933. Citations to the earlier works can be found in these works.

that corresponds to the absence of material particles) that combines all of the advantages of Klein-Kaluza and Einstein-Mayer, while avoiding all of the disadvantages. The circumstance that all projectors depend homogeneously on the coordinates renders any particular cylinder condition superfluous. Furthermore, the condition $g_{55} = 1$ of the Klein-Kaluza theory appears here in the invariant form:

$$g_{\mu\nu} X^\mu X^\nu = \pm 1 ,$$

and can be stated as a normalization of the $g_{\mu\nu}$. The establishment of this condition by the variation:

$$\delta g_{\mu\nu} X^\mu X^\nu = 0$$

further reduces the number of field equations in a natural way to their correct number. The scalar curvature appears in the action integral as a Lagrangian function and is then varies with the $g_{\mu\nu}$. Finally, one does not need to introduce the electromagnetic potentials themselves into the theory, but they can be replaced (up to a numerical factor) by:

$$X_\mu = g_{\mu\nu} X^\nu.$$

The validity of the first system of Maxwell equations is then self-evident.

*We shall preserve these essential results of Schouten and van Dantzig here.* Therefore, as far as the three-index symbols $\Gamma^\nu_{\lambda\mu}$ are concerned, we take the position that the *symmetry requirement:*

$$\Gamma^\nu_{\lambda\mu} = \Gamma^\nu_{\mu\lambda}$$

(which is satisfied for Klein-Kaluza and Veblen-Hoffmann) is the most natural one for these quantities. Together with the requirement that the covariant derivatives of the $g_{\mu\nu}$, it then leads to the usual expression:

$$\Gamma^\nu_{\lambda\mu} = \frac{1}{2} g^{\nu\rho} \left( \frac{\partial g_{\rho\lambda}}{\partial X^\mu} + \frac{\partial g_{\mu\rho}}{\partial X^\nu} - \frac{\partial g_{\lambda\mu}}{\partial X^\rho} \right).$$

More generally, Schouten and van Dantzig introduced asymmetric $\Gamma^\nu_{\lambda\mu}$. Since this increases the number of possible mathematical invariants without any benefit to the physical interpretation (the particular grounds for *another* specialization of the general $\Gamma^\nu_{\lambda\mu}$ that Schouten and van Dantzig maintained do not seem valid to us), we shall forego this gratuitous generalization here and mention it only incidentally (cf., the Appendix).

The connection between projectors and ordinary tensors in inhomogeneous coordinates has a far-reaching formal analogy with the connection between tensors on five-spaces and those on the four-spaces of Einstein-Mayer. In the form presented in § 4 and § 6, this analogy is particular obvious.

We must apologize for the fact that the classical part of the theory will be represented so comprehensively, although it deviates from the work of Schouten and van

Dantzig in several essential points. On the one hand, this is done in order to have a certain foundation following section, and, on the other hand, in order to make understandable those specialized facts that are familiar from general relativity, but not the extended one that is discussed here.

## § 2. **Definition of tensors with homogeneous coordinates**

The description of an $n$-dimensional continuum by $n + 1$ homogeneous coordinates $X^1, \ldots, X^{n+1}$ can naturally permit only such transformations that make the new coordinates $X'^{\mu}$ homogeneous functions of the first degree of the old coordinates $X^{\mu}$. [1]) That is:

$$X'^{\mu} = f^{\nu}(X^1, \ldots, X^{n+1})$$

must satisfy:

$$f^{\nu}(\rho X^1, \ldots, \rho X^{n+1}) = \rho f^{\nu}(X^1, \ldots, X^{n+1}),$$

in which $\rho$ is an arbitrary function of the $X^{\mu}$. According to the Euler homogeneity condition, this requirement is equivalent to:

(1) $$X^{\mu}\frac{\partial X'^{\nu}}{\partial X^{\mu}} = X'^{\nu} .$$

In order for us to now proceed to the definition of vectors, we next define a contravariant vector $a^{\nu}$ (a covariant vector $b_{\nu}$, resp.) by the transformation laws:

(2) $$a'^{\nu} = \frac{\partial X'^{\nu}}{\partial X^{\mu}} a^{\mu}$$

(3) $$b'_{\nu} = \frac{\partial X^{\mu}}{\partial X'^{\nu}} b_{\mu},$$

such that:

(4) $$a^{\nu} b_{\nu} = a'^{\mu} b'_{\mu} = c$$

is a scalar.

For this reason, vector fields and scalar fields that are defined by homogeneous coordinates will be subjected to a further requirement that specifies, not their behavior under coordinate transformations, but their dependence on the coordinates in a fixed system of reference. *Namely, we will allow only those fields whose components are homogeneous functions of the coordinates.* This requirement is invariant under the homogeneous coordinate transformations (1) that are considered here, and, from this, the degree of the homogeneous functions that represent the field components remains invariant, since the coordinate transformations are homogeneous of first degree.

For the sake of later applications, *the components of a contravariant vector are always homogeneous of degree* +1, *those of a covariant vector are of degree* –1, *and*

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[1]) Here, and in the sequel, Greek indices shall always range from 1 to $n + 1$, and Latin ones, from 1 to $n$. Later, corresponding to the four-dimensional spacetime continuum, we shall set $n = 4$, $n + 1 = 5$.

*scalars are of degree* 0. From (4), this convention is usually in harmony with the construction of scalars by contraction. The following are then valid:

(2a) $$X^{\mu} \frac{\partial a^{\nu}}{\partial X^{\mu}} = a^{\nu}$$

(3a) $$X^{\mu} \frac{\partial b_{\nu}}{\partial X^{\mu}} = - b_{\nu}$$

(4a) $$X^{\mu} \frac{\partial c}{\partial X^{\mu}} = 0 \, .$$

It is a characteristic situation in the theory of homogeneous coordinates that *these coordinates* $X^{\nu}$ *themselves* (not their differentials) define a (contravariant) *vector field*. Then, according to (1), they satisfy, in fact, the necessary conditions (2) for a contravariant vector field [whereas relations (2a) are obviously satisfied identically]. By contrast, the differentials $dX^{\nu}$ may not be used as a vector field, since they are not homogeneous functions of first degree in the coordinates.

We can now easily generalize the definition of vector fields and scalar fields to arbitrary tensor fields. A general mixed tensor:

$$T^{\mu\nu\cdot}_{\cdot\cdot\cdot\rho\sigma\cdot}$$

satisfies the transformation law:

(5) $$T'^{\bar{\mu}\bar{\nu}\cdot}_{\cdot\cdot\cdot\bar{\rho}\bar{\sigma}\cdot} = \frac{\partial X'^{\bar{\mu}}}{\partial X^{\mu}} \cdot \frac{\partial X'^{\bar{\nu}}}{\partial X^{\nu}} \cdots \frac{\partial X^{\rho}}{\partial X'^{\bar{\rho}}} \cdot \frac{\partial X^{\sigma}}{\partial X'^{\bar{\sigma}}} \cdots T^{\mu\nu\cdot}_{\cdot\cdot\cdot\rho\sigma\cdot} \, .$$

Furthermore, it will be assumed that the *components of the tensor field* $T^{\mu\nu\cdot}_{\cdot\cdot\cdot\rho\sigma\cdot}$ *are homogeneous functions of the* $X^{\nu}$*, and also of degree r, which is equal to the difference between the number of upper and lower indices of* $T^{\mu\nu\cdot}_{\cdot\cdot\cdot\rho\sigma\cdot}$. One then has:

(5a) $$X^{a} \frac{\partial T^{\mu\nu\cdot}_{\cdot\cdot\cdot\rho\sigma\cdot}}{\partial X^{a}} = r T^{\mu\nu\cdot}_{\cdot\cdot\cdot\rho\sigma\cdot} \, .$$

It should be remarked that the volume element:

$$dX^1 \, dX^2 \ldots dX^{n+1}$$

(as opposed to the differentials $dX^{\nu}$ themselves) can possibly vanish. If it is multiplied by the transformation:

(6) $$X'^{\mu} = \rho X^{\mu} \, ,$$

in which $r$ is an arbitrary homogeneous function of null degree of the $X^{\nu}$, such that:

(6a) $$X^{\nu}\frac{\partial\rho}{\partial X^{\nu}} = 0\ ,$$

and similarly for $\rho^{n+1}$.

*Proof:* The assertion is equivalent to the statement that under the transformation (6) the determinant:

$$D(\rho) = \left|\frac{\partial X'^{\mu}}{\partial X^{\nu}}\right| = \left|\rho\delta^{\mu}_{\cdot\nu} + X^{\mu}\frac{\partial\rho}{\partial X^{\nu}}\right|$$

has the value $\rho^{n+1}$, or the statement that:

$$f(\rho) = \frac{1}{\rho^{n+1}}D(\rho) = \left|\delta^{\mu}_{\cdot\nu} + X^{\mu}\frac{1}{\rho}\frac{\partial\rho}{\partial X^{\nu}}\right|$$

has the value 1 .

In order to prove this, we consider:

$$f(t) \equiv f(\rho^{t}) = \left|\delta^{\mu}_{\cdot\nu} + X^{\mu}\frac{1}{\rho}\frac{\partial\rho}{\partial X^{\nu}}\right|$$

to be a function of $t$. This function has the following properties:

a) $f(0) = 1$ , $\qquad f(1) = f(\rho)$

b) $f'(0) = 0$ .

One then has:

$$f'(0) = \lim_{t\to 0}\frac{1}{t}\left[\left|\delta^{\mu}_{\cdot\nu} + tX^{\mu}\frac{1}{\rho}\frac{\partial\rho}{\partial X^{\nu}}\right| - 1\right] = X^{\nu}\frac{1}{\rho}\frac{\partial\rho}{\partial X^{\nu}}\ ,$$

which vanishes, from (6a);

c) $f(t_1 + t_2) = f(t_1)\,f(t_2)$,

which follows directly from the definition of $f(t)$ and the multiplication laws for determinants. From this, it further follows that:

$$f'(t) = f(t)\,f'(0)\ ,$$

hence, from (b):

$$f'(t) = 0\ , \qquad f(t) = \text{const.} = f(0) = 1\ .$$

Thus:

$$D(\rho) = \rho^{n+1} \qquad f(1) = \rho^{n+1}$$ Q.E.D.

The possibility of defining a meaningful volume element from homogeneous coordinates is important for the physical applications, since it allows one to define variational principles.

## § 3. **Metric. Connection with inhomogeneous coordinates. Decomposition of homogeneous tensors into affine parts.**

We introduce a metric tensor $g_{\mu\nu}$ in our space that satisfies the usual symmetry condition $g_{\mu\nu} = g_{\nu\mu}$ and distinguishes the quadratic form:

$$g_{\mu\nu}X^{\mu}X^{\nu} = F\ .$$

Corresponding to our convention, $g_{\mu\nu}$ is homogeneous of degree –2 and the scalar $F$ is homogeneous of null degree. Physics gives us a specialization of this Ansatz, in which the scalar $F$ is set equal to a *constant*. This constant can then be normalized to +1 or –1, such that we can set:

(7) $$g_{\mu\nu} X^\mu X^\nu = \varepsilon \quad (\varepsilon = +1 \text{ or } -1).$$

One can also express this in such a way that only the quotients $g_{\mu\nu}/F$, in which $F = g_{\mu\nu}X^\mu X^\nu$, shall enter into the equations thus defined. This specialization is analogous to the convention that $g_{55} = 1$ in the Klein-Kaluza theory. This specialization may seem arbitrary from a geometrical standpoint, but it is, in any event, an advantage of the use of homogeneous coordinates that the specialization can be formulated in an invariant way. The metric tensor can therefore be used to raise or lower tensor indices in an invariant manner.

For physical applications, it is essential for us to possess a method for deriving the ordinary tensors in $n$ inhomogeneous coordinates $x^k$ from the homogeneous tensors in the $n + 1$ homogeneous coordinates $X^\nu$, such that the $x^k$ are arbitrary homogeneous functions of null degree of the $X^\nu$:

(8) $$x^k = f^k(X^1, \ldots, X^{n+1}) .$$

The derivatives:

$$\gamma_\nu^{\cdot k} = \frac{\partial x^k}{\partial X^\nu}$$

satisfy the homogeneity condition:

(9) $$\gamma_\nu^{\cdot k} X^\nu = \frac{\partial x^k}{\partial X^\nu} X^\nu = 0,$$

and transform under arbitrary point transformations of the $x^k$ like a contravariant vector, and like a covariant vector under homogeneous transformation of the $X^\nu$, in which (9) remains invariant throughout.

Thus, the $\gamma_\nu^{\cdot k}$ can be used to, first, associate every covariant vector $a_k$ with a covariant vector $\bar{a}_\nu$ according to:

(10) $$\bar{a}_\nu = \gamma_\nu^{\cdot k} a_k ,$$

in which, from (9):

(10a) $$\bar{a}_\nu X^\nu = 0$$

(which is expressed by the overbar on $a$), and, second, to associate every contravariant vector $a^\nu$ with a contravariant vector $a^k$ according to:

(11) $$a^k = \gamma_\nu^{\cdot k} a^\nu ,$$

in which all vectors of the form $\rho X^\nu$ will be associated with the null vector. We can arrange that the quotient $X^\nu / X^{\nu+1}$ can be expressed in terms of the $x^k$ uniquely. The $n$

relations (9) are then the only dependencies that exist between the $\gamma_\nu^{\cdot k}$, and we have that the only solution of the equation:

$$\gamma_\nu^{\cdot k} v^\nu = 0$$

of the specified form is:

$$v^\nu = \rho X^\nu.$$

Any special $\bar{a}_\nu$ covariant vector that satisfies the relation (10a), namely:

$$\bar{a}_\nu X^\nu = 0$$

can then also be uniquely associated with a vector $a_k$ that satisfies (10). Namely, if one sets:

$$a_l = \gamma_\nu^{\cdot k} \bar{a}_\nu ,$$

then one must have:

(12) $$\gamma_\nu^{\cdot k} \gamma_{\cdot l}^\nu = \delta_{\cdot l}^k .$$

From the statements above, the new coefficients $\gamma_{\cdot l}^\nu$ are uniquely determined by these equations precisely, up to an additional term $X^\nu \rho_l$, according to:

$$\bar{\gamma}_{\cdot l}^\nu = \gamma_{\cdot l}^\nu + X^\nu \rho_l ,$$

which, from (10a), actually annuls the vector $\bar{a}_\nu$. Furthermore, according to (12), the vector $a^k$ is, by way of:

$$a^\nu = \gamma_{\cdot k}^\nu a^k$$

associated with a set of vectors of the form:

$$a^\nu + X^\nu (a^l \rho_l)$$

that all satisfy equation (11).

Up till now, the matrix $g_{\mu\nu}$ was not used in the coefficients $\gamma_{\cdot l}^\nu$ and $\gamma_\nu^{\cdot k}$ at all. Now, we can make the definition of the $\gamma_{\cdot l}^\nu$ unique by associating the same $\bar{a}_\nu$, for which one has:

(13a) $$\bar{a}_\nu X^\nu \equiv \bar{a}_\nu X^\mu g_{\mu\nu} = 0 ,$$

with all vectors of the form $a^\nu + \rho X^\nu$, with $a^\nu$ fixed and $\rho$ arbitrary. If we correspondingly establish the $\gamma_{\cdot l}^\nu$ uniquely by the condition that enters into (12):

(12a) $$\gamma_{\cdot l}^\nu X_\nu \equiv \gamma_{\cdot l}^\nu X^\mu g_{\mu\nu} = 0 ,$$

then we obtain the unique association of $\bar{a}^\nu$ with $a^k$ according to:

$$\bar{a}^\nu = \gamma^\nu_{\cdot k}\, a^k \qquad (\bar{a}^\nu X_\nu = 0) \tag{13}$$

and the unique association of $a_l$ with $a_\nu$ according to:

$$a_l = \gamma^\nu_{\cdot k}\; a_\nu , \tag{14}$$

in which $X_\nu$ is associated with the null vector.

From (11), (14), and the fact that $a_\mu = g_{\mu\nu}\, a^\nu$, and consideration of (12a), it follows that:

$$a_l = (\gamma^\mu_{\cdot l}\;\gamma^\nu_{\cdot k}\, g_{\mu\nu})\, a^k .$$

If one then demands that when $a_\mu = g_{\mu\nu}\, a^\nu$ the associated vectors $a_\mu$, $a_l$ must satisfy the metric relation:

$$a_l = g_{lk}\, a^k ,$$

then it follows that the metric tensor in inhomogeneous coordinates is:

$$g_{lk} = \gamma^\mu_{\cdot l}\;\gamma^\nu_{\cdot k}\, g_{\mu\nu} . \tag{15}$$

We now compute the sums:

$$d^\mu_{\cdot\nu} = \gamma^\mu_{\cdot k}\;\gamma^{\cdot k}_\nu ,$$

which defines a mixed tensor of rank two in the space of $X^\nu$. Next, from (12), we have:

$$\gamma^\nu_{\cdot k}\; d^\mu_{\cdot\nu} = \gamma^\mu_{\cdot k} , \qquad\qquad \gamma^{\cdot k}_\mu\; d^\mu_{\cdot\nu} = \gamma^{\cdot k}_\nu ,$$

hence:

$$\gamma^\nu_{\cdot k}\,(d^\mu_{\cdot\nu} - \delta^\mu_{\cdot\nu}) = 0 , \qquad \gamma^{\cdot k}_\mu\,(d^\mu_{\cdot\nu} - \delta^\mu_{\cdot\nu}) = 0 ,$$

hence:

$$d^\mu_{\cdot\nu} = \delta^\mu_{\cdot\nu} + X^\mu\,\rho_\nu , \qquad \rho_\nu\;\gamma^\nu_{\cdot k} = 0 , \qquad \rho_\nu = \rho\, X^\nu ,$$

$$d^\mu_{\cdot\nu} = \delta^\mu_{\cdot\nu} + \rho\, X^\mu\, X_\nu ,$$

$$d^\mu_{\cdot\nu}\, X^\nu = 0 , \qquad X^\mu + \rho\, X^\mu\,\varepsilon = 0 , \qquad \rho = -\varepsilon ,$$

according (7). Thus, finally, one has:

$$d^\mu_{\cdot\nu} = \gamma^\mu_{\cdot k}\;\gamma^{\cdot k}_\nu = \delta^\mu_{\cdot\nu} - \varepsilon\, X^\mu\, X_\nu . \tag{16}$$

The vector:

$$\bar{a}^\mu = d^\mu_{\cdot\nu}\, a^\nu$$

satisfies the equations:

$$\bar{a}^\mu X_\mu = 0 , \qquad \bar{a}^\mu\;\gamma^{\cdot k}_\mu = a^\mu\;\gamma^{\cdot k}_\mu = a^k ,$$

hence, we have:

(17a) $$\bar{a}^{\mu} = d^{\mu}_{\cdot\nu}\, a^{\nu} = \gamma^{\mu}_{\cdot k}\, a^{k} ,$$

and likewise:

(17b) $$\bar{a}_{\mu} = d^{\mu}_{\cdot\nu}\, a_{\nu} = \gamma^{\nu}_{\cdot k}\, a^{k} .$$

Finally, from (15), and the fact that:

$$g_{ik}\,\gamma_{\mu}^{\cdot i}\;\gamma_{\nu}^{\cdot k} = \bar{g}_{\mu\nu} ,$$

it follows that:

$$\bar{g}_{\mu\nu} = d^{\rho}_{\cdot\mu}\; d^{\sigma}_{\cdot\nu}\, g_{\rho\sigma} = (\delta^{\rho}_{\cdot\mu} - \varepsilon X^{\rho} X_{\mu})\,(\delta^{\sigma}_{\cdot\nu} - \varepsilon X^{\sigma} X_{\nu})\, g_{\rho\sigma} ,$$

or:

$$\bar{g}_{\mu\nu} = g_{\mu\nu} - \varepsilon X_{\mu} X_{\nu} ,$$

(18) $$g_{\mu\nu} = \bar{g}_{\mu\nu} + \varepsilon X_{\mu} X_{\nu} = g_{ik} + \varepsilon X_{\mu} X_{\nu} .$$

One further confirms by way of:

(18a) $$g^{\mu\nu} = \bar{g}_{\mu\nu} + \varepsilon X^{\mu} X^{\nu} = g^{ik}\gamma_{\mu}^{\cdot i}\;\gamma_{\nu}^{\cdot k} + \varepsilon X_{\mu} X_{\nu} .$$

the relation:

$$g^{\mu\rho} g_{\mu\sigma} = d^{\rho}_{\cdot\sigma} + \varepsilon X^{\rho} X_{\sigma} = \delta^{\rho}_{\cdot\sigma} .$$

We now have the means to characterize each tensor uniquely by affine parts. Thus, if $a^{\nu}$ is uniquely determined by means of the $\gamma^{\nu}_{\cdot k}$ and $\gamma_{\nu}^{\cdot k}$ through $a^k = a^{\mu} X_{\nu}$ and $a = a^{\nu} X_{\nu}$ then we have:

(19a) $$a^{\nu} = \gamma^{\nu}_{\cdot k}\, a^{k} + \varepsilon\, aX^{\nu} = \bar{a}^{\nu} + \varepsilon\, aX^{\nu} = d^{\nu}_{\cdot\rho}\, a^{\rho} + \varepsilon\, aX^{\nu} ,$$

just as $a_{\nu}$ is determined through $a_k = \gamma^{\nu}_{\cdot k}\, a_{\nu}$ and $a = a_{\nu} X^{\nu}$, according to:

(19b) $$a_{\nu} = \gamma_{\nu}^{\cdot k}\, a_{k} + \varepsilon\, aX_{\nu} = \bar{a}_{\nu} + \varepsilon\, aX_{\nu} = d_{\nu}^{\cdot\rho}\, a_{\rho} + \varepsilon\, aX_{\nu} .$$

Likewise, a tensor $T_{\mu\nu}$ of the second rank is characterized by the four affine quantities:

$$\gamma^{\mu}_{\cdot i}\;\gamma^{\nu}_{\cdot k}\, T_{\mu\nu} = T_{ik} , \qquad \gamma^{\mu}_{\cdot i}\, X^{\nu}\, T_{\mu\nu} = T_{i(0)} , \qquad X^{\mu}\, T_{\mu\nu} = T_{(0)i} ,$$

and:

$$X^{\mu} X^{\nu}\, T_{\mu\nu} = T_{(0)(0)} .$$

For a symmetric tensor $T_{\mu\nu}$, $T_{ik}$ is symmetric, and the vectors $T_{i(0)}$ and $T_{(0)i}$ are identical; it is thus characterized by the a symmetric tensor, a vector and a scalar in inhomogeneous coordinates. For the fundamental tensor $g_{\mu\nu}$, the vector $g_{i(0)}$ vanishes, in particular, according to (12a). If $T_{\mu\nu}$ is skew-symmetric then the scalar $T_{(0)(0)}$ is equal to null.

In general, a tensor with $N$ indices gives rise to $2^N$ affine tensors, between which certain linear dependencies can exist when the original tensor possesses a symmetry property.

We add a remark about the possibility of defining a vector that satisfies one of these requirements from the differentials $dX^\mu$, which do themselves satisfy the homogeneity condition, according to:

$$\bar{d}X^\mu = d^\mu_{\cdot\nu}\, dX^\nu = dX^\nu - \varepsilon X^\mu (X_\nu \; dX^\nu) \tag{20}$$

In fact, by replacing $X^\mu$ with $\rho X^\mu$, one has:

$$\bar{d}X'^\mu = d^\mu_{\cdot\nu}\, d(\rho X^\nu) = d^\mu_{\cdot\nu}\,(\rho\, dX^\nu + X^\nu\, d\rho) = \rho\; d^\mu_{\cdot\nu}\, dX^\nu = \rho\; \bar{d}X^\mu ,$$

since $X^\nu$ will be annulled by $d^\mu_{\cdot\nu}$. Moreover, one has:

$$\bar{d}X^\nu\; \gamma_\nu^{\cdot k} = \bar{d}X^\nu \frac{\partial x^k}{\partial X^\nu} = dX^\nu \frac{\partial x^k}{\partial X^\nu} = dx^k , \tag{20a}$$

hence:

$$\bar{d}X^\mu = \gamma^\mu_{\cdot k}\, dx^k . \tag{20b}$$

A curve $x^k = f^k(\mathrm{r})$ will be represented in homogeneous coordinates in the form:

$$X^\nu = \rho\, F^\nu(\mathrm{r}) = \rho\; \gamma^\nu_{\cdot k}\; f^k(\mathrm{r}) , \tag{21}$$

in which $\rho$ is an arbitrary homogeneous function of null degree in the $X^\nu$. Then, one has:

$$\frac{\bar{d}X^\nu}{d\mathrm{r}} = \rho\, \frac{dF^\nu}{d\mathrm{r}} = \rho\; \gamma^\nu_{\cdot k}\, \frac{df^k}{d\mathrm{r}} , \tag{21a}$$

whereas $dX^\nu/d\mathrm{r}$ has no simple meaning. [1])

## § 4. **Covariant differentiation by means of the three-index symbol $\Gamma^\nu_{\lambda\mu}$.**

As usual, the covariant differential calculus will be introduced by means of a $\Gamma^\nu_{\lambda\mu}$-field, for which one demands that:

$$a^\nu_{;\mu} = \frac{\partial a^\nu}{\partial X^\mu} + \Gamma^\sigma_{\lambda\mu} a^\lambda \tag{22a}$$

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[1]) The developments that were given here are largely analogous to the work of Einstein and Mayer that was cited in § 2. The quantity that was denoted by $\Sigma^\mu_{\cdot\nu}$ in that work is denoted by $d^\mu_{\cdot\nu}$ in ours, and furthermore, our coordinates $X^\mu$ play the same role as the vector $A^\mu$ of Einstein-Mayer.

(22b) $$b_{\nu\,;\,\mu} = \frac{\partial b_\nu}{\partial X^\mu} - \Gamma^\lambda_{\mu\nu} b_\lambda \,.$$

The second equation follows from the first one by the requirement that:

$$(a^\nu\, b_\nu)_{;\,\mu} \equiv a^\nu{}_{;\,\mu}\;\, b_\nu + a^\nu\, b_{\nu\,;\,\mu} \;=\; \frac{\partial (a^\nu b_\nu)}{\partial X^\mu}.$$

The differential calculus is extended in a well-known way for arbitrary mixed tensors by the demand that for two tensors $A^{\cdot\cdot}$ and $B^{\cdot}$ the product rule of ordinary differentiation is still valid:

$$(A^{\cdot\cdot}\, B^{\cdot}\,)_{;\,\rho} = A^{\cdot\cdot}{}_{;\,\rho}\;\, B^{\cdot}\;\, + B^{\cdot}{}_{;\,\rho}\;\, A^{\cdot\cdot}\,.$$

This then yields:

$$A^{\cdot\cdot}{}_{;\,\rho} \;= \frac{\partial A^{\cdot\cdot}}{\partial X^\rho} + \sum (\cdots)\,,$$

in which the sum has exactly as many summands as the tensor has indices, and indeed:

an upper index $A^{\cdot\mu}$ corresponds to a summand $+\Gamma^\nu_{\lambda\rho} A^{\cdot\mu}$,

a lower index $A^{\cdot\cdot}{}_\mu$ corresponds to a summand $-\Gamma^\lambda_{\nu\rho} A^{\cdot\cdot}{}_\lambda$.

Since $a^\nu{}_{;\,\mu}$ is a tensor when $a_\nu$ is a vector (from which, the tensor character of $b_{\nu;\mu}$ and general $A^{\cdot\cdot}{}_{\cdot;\rho}$ follows), the $\Gamma^\nu_{\lambda\mu}$ must obey the transformation laws:

(23) $$\Gamma'^{\bar\nu}_{\bar\lambda\bar\mu} = \frac{\partial X^\lambda}{\partial X'^{\bar\lambda}} \frac{\partial X^\mu}{\partial X'^{\bar\mu}} \frac{\partial X'^{\bar\nu}}{\partial X^\nu} \Gamma^\nu_{\lambda\mu} + \frac{\partial X'^{\bar\nu}}{\partial X^\rho} \frac{\partial^2 X^\rho}{\partial X'^{\bar\lambda} \partial X'^{\bar\mu}}\,.$$

The last term disturbs the tensor character of the $\Gamma^\nu_{\lambda\mu}$. Since the homogeneity requirement for the $A^{\cdot\cdot}{}_{\cdot;\rho}$ follows from the one for the $A^{\cdot\cdot}{}_{\cdot}$, the $\Gamma^\nu_{\lambda\mu}$ must be homogeneous of degree –1, as the general demands. From (23), it follows that part of $\Gamma^\nu_{\lambda\mu}$ that is skew-symmetric in the indices $\lambda$, $\mu$, which is therefore determined by:

(24) $$S_{\lambda\mu}^{\cdot\;\cdot\;\nu} = \frac{1}{2}(\Gamma^\nu_{\lambda\mu} - \Gamma^\nu_{\mu\lambda})\,,$$

is a tensor.

Further tensors (projectors, resp.) $\Pi_\mu^{\cdot\nu}$ and $Q_\mu^{\cdot\nu}$ are defined by:

(25a) $$X^\mu\, a^\nu{}_{;\,\mu} = \Pi_\mu^{\cdot\nu}\, a^\mu\,,$$

(25b) $$X^\nu\, b_{\mu\,;\,\nu} = -\Pi_\mu^{\cdot\nu}\, b_\nu\,,$$

(the latter relation follows from the first one due to the fact that:

$$X^\nu (a^\mu \; b_\mu)_{;\,\nu} = 0\ )$$

and by:

(26) $$X^\nu{}_{;\,\mu} = Q_\mu^{\cdot\,\nu} .$$

From the definition (22a) ((22b), resp.) of $a^\nu{}_{;\,\mu}$ ($b_{\nu\,;\,\mu}$, resp.), it follows that:

(27) $$\Pi_\mu^{\cdot\,\nu} = \delta_\mu^{\cdot\,\nu} + \Gamma^\nu_{\mu\lambda} X^\lambda ,$$

(28) $$Q_\mu^{\cdot\,\nu} = \delta_\mu^{\cdot\,\nu} + \Gamma^\nu_{\lambda\mu} X^\lambda ,$$

hence:

(29) $$\frac{1}{2} (\Pi_\mu^{\cdot\,\nu} - Q_\mu^{\cdot\,\nu}) = S_{\lambda\mu}^{\cdot\,\cdot\,\nu} X^\lambda .$$

A link between the $\Gamma^\nu_{\lambda\mu}$ and the metric will be given here by the condition:

(I) $$g_{\mu\nu\,;\,\rho} = \frac{\partial g_{\mu\nu}}{\partial X^\rho} - \Gamma^\lambda_{\mu\rho} g_{\lambda\nu} - \Gamma^\lambda_{\nu\rho} g_{\mu\lambda} = 0 .$$

From this, and the fact that:

(30) $$\Gamma^\lambda_{\mu\nu} g_{\lambda\rho} = \Gamma_{\mu\nu,\,\rho} , \qquad \Gamma^\lambda_{\mu\nu} = g^{\lambda\rho}\, \Gamma_{\mu\nu,\,\rho} ,$$

(30a) $$S_{\mu\nu}^{\cdot\,\cdot\,\lambda} g_{\lambda\rho} = S_{\mu\nu,\,\rho} , \qquad S_{\mu\nu}^{\cdot\,\cdot\,\lambda} = g^{\lambda\rho}\, S_{\mu\nu,\,\rho} ,$$

(1) implies the following relation:

(31) $$\Gamma_{\mu\nu,\rho} = \frac{1}{2}\left( \frac{\partial g_{\mu\rho}}{\partial X^\nu} + \frac{\partial g_{\nu\rho}}{\partial X^\mu} - \frac{\partial g_{\mu\nu}}{\partial X^\rho} \right) + (S_{\mu\nu,\,\rho} + S_{\mu\rho,\,\nu} + S_{\nu\rho,\,\mu}) .$$

A link between the metric and the $\Gamma^\lambda_{\mu\nu}$ and the $\gamma_\nu^{\cdot\,l}$ (and the $\gamma^\nu_{\cdot\,l}$) is then produced by the following requirement:

When:

$$\bar{a}^\nu X_\nu = 0, \qquad \text{hence} \quad \bar{a}^\nu = \gamma^\nu_{\cdot\,k}\, a^k ,$$

one must have:

(II) $$\gamma^\rho_{\cdot\,l}\, \gamma_\nu^{\cdot\,k} \bar{a}^\nu{}_{;\rho} = a^k{}_{;\,l} ,$$

in which $a^k{}_{;\,l}$ is defined in the normal way in terms of the Christoffel symbols $\begin{Bmatrix} l \\ nm \end{Bmatrix}$, according to:

$$a^k{}_{;\,l} = \frac{\partial a^k}{\partial x^l} - \begin{Bmatrix} k \\ lm \end{Bmatrix} a^m$$

$$\begin{Bmatrix} k \\ lm \end{Bmatrix} = \begin{Bmatrix} k \\ ml \end{Bmatrix} = \frac{1}{2} g^{kn} \left\{ \frac{\partial g_{nm}}{\partial x^l} + \frac{\partial g_{nl}}{\partial x^m} - \frac{\partial g_{lm}}{\partial x^n} \right\}.$$

We can formulate the requirement (II) in a different way when we define a covariant derivative $\gamma^{\nu}_{\cdot k;\rho}$ according:

(32)
$$\gamma^{\nu}_{\cdot k;\rho} \equiv \frac{\partial \gamma^{\nu}_{\cdot k}}{\partial X^{\rho}} + \Gamma^{\nu}_{\lambda\rho} \gamma^{\lambda}_{\cdot k} - \begin{Bmatrix} l \\ km \end{Bmatrix} \gamma_{\rho}^{\cdot m} \gamma^{\nu}_{\cdot l}\,.$$

This implies the product rule:

$$(\gamma^{\nu}_{\cdot k} a^k)_{;\rho} = \gamma^{\nu}_{\cdot k} a^k_{;l}\, \gamma_{\nu}^{\cdot l} + \gamma^{\nu}_{\cdot k;\rho} a^k$$
$$b_{k;\rho} = (\gamma^{\nu}_{\cdot k} b_{\nu})_{;\rho} = \gamma^{\nu}_{\cdot k;\rho} b_{\nu} + \gamma^{\nu}_{\cdot k} b_{\nu\,;\,\rho}\,.$$

Likewise, it follows that:

(32a)
$$\begin{aligned} \gamma_{\nu\,;\rho}^{\cdot k} &= \frac{\partial \gamma_{\nu}^{\cdot k}}{\partial X^{\rho}} + \Gamma^{\lambda}_{\nu\rho} \gamma_{\lambda}^{\cdot k} - \begin{Bmatrix} l \\ km \end{Bmatrix} \gamma_{\rho}^{\cdot m} \gamma_{\nu}^{\cdot l}\,, \\ a^k_{;\rho} &= (\gamma_{\nu}^{\cdot k} a^{\nu})_{;\rho} = \gamma_{\nu\,;\rho}^{\cdot k} a^{\nu} + \gamma_{\nu}^{\cdot k} a^{\nu}_{;\rho}\,, \\ (\gamma_{\nu}^{\cdot k} b_k)_{;\rho} &= \gamma_{\nu}^{\cdot k} b_{k;l}\, \gamma_{\rho}^{\cdot l} + \gamma_{\nu\,;\rho}^{\cdot k} b_k\,. \end{aligned}$$

Thus, (II) is equivalent to:

(II′)
$$\gamma^{\rho}_{\cdot l}\, \gamma_{\nu}^{\cdot k}\, \gamma^{\nu}_{\cdot m;\rho} = 0$$

or:

(II″)
$$\gamma^{\rho}_{\cdot l}\, \gamma^{\nu}_{\cdot k}\, \gamma_{\nu\,;\rho}^{\cdot k} = 0\,.$$

By antisymmetrization in the $\nu$ and $\lambda$, it then follows from the latter equation that:

(33)
$$\gamma^{\rho}_{\cdot l}\, \gamma^{\nu}_{\cdot k}\, \gamma_{\lambda}^{\cdot m}\, S_{\nu\rho}^{\cdot\cdot\lambda} = 0\,,$$

i.e., the affine part of $S_{\nu\rho}^{\cdot\cdot\lambda}$ must vanish. However, $S_{\nu\rho}^{\cdot\cdot\lambda}$ is then already determined by $X^{\nu} S_{\nu\rho}^{\cdot\cdot\lambda}$ and $X_{\lambda} S_{\nu\rho}^{\cdot\cdot\lambda}$, which can be expressed by way of $\Pi_{\mu}^{\cdot\nu}$ and $Q_{\mu}^{\cdot\nu}$ for a given $g_{\mu\nu}$ and its derivatives. [1])

From (25), (26), it now follows that:

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[1]) The $\Pi_{\mu\nu}$ and $Q_{\mu\nu}$ (with the indices lowered by means of the $g_{\mu\nu}$) yield only the conditions:

$$\Pi_{\mu\nu} = -\Pi_{\nu\mu}\,,$$
$$X^{\nu} Q_{\mu\nu} = X_{\nu} Q_{\mu}^{\cdot\nu} = 0\,,$$

[the latter follows from (7)] and:

$$\Pi_{\mu}^{\cdot\nu} X^{\mu} = Q_{\mu}^{\cdot\nu} X^{\mu} \;\; (= X^{\mu} X^{\nu}_{;\mu})\,.$$

Otherwise, $\Pi_{\mu\nu}$ and $Q_{\mu\nu}$ are arbitrary.

$$(34)\qquad \begin{cases} a^{\nu}{}_{;\rho} = \bar{a}^{\nu}_{;\rho} + \varepsilon(aX^{\nu})_{;\rho} \quad (a = X_{\nu}a^{\nu}, \quad \bar{a}^{\nu} = d^{\nu}{}_{,\mu}a^{\mu}) \\ a^{\nu}{}_{;\rho} = \gamma^{\nu}_{\cdot k}\gamma^{\rho}_{\cdot l}a^{k}_{\cdot l} + \varepsilon X_{\rho}\Pi_{l}^{\cdot k}a^{l} + \varepsilon X^{\nu}\left(\dfrac{\partial a}{\partial X^{\rho}} - \gamma_{\rho}^{\cdot l}Q_{lk}a^{k}\right) + \varepsilon a Q_{\rho}^{\cdot\nu}, \end{cases}$$

and likewise:

$$(34a)\qquad b_{\nu;\rho} = \gamma_{\nu}^{\cdot k}\gamma^{\rho}_{\cdot l}b_{k;l} - \varepsilon X^{\rho}\gamma_{\nu}^{\cdot k}\Pi_{k}^{\cdot l}b_{l} + \varepsilon X_{\nu}\left(\frac{\partial b}{\partial X^{\rho}} - \gamma_{\rho}^{\cdot l}Q_{l}^{\cdot k}b_{k}\right) + \varepsilon b Q_{\rho\nu}.$$

In addition, the $g_{\mu\nu}$ determine the $\Gamma^{\nu}_{\lambda\mu}$ uniquely by way of the fields $\Pi_{\mu\nu}$ and $Q_{\mu\nu}$.

Obviously, our requirements are still much too general to be useful to physics. The simplest case that comes into consideration is the one for which:

$$(\text{III})\qquad S_{\mu\nu,\rho} = 0,$$

i.e., in which the $\Gamma^{\lambda}_{\mu\nu}$ are assumed to be symmetric in $\mu$ and $\nu$, as one does in Riemannian geometry. This assumption seems to be the most natural to us. We would like discuss the more general case that Schouten examined later on as an appendix (§ 7).

If we accept the severe restriction (III) then one has, in particular:

$$(31')\qquad \Gamma_{\mu\nu,\rho} = \Gamma_{\nu\mu,\rho} = \frac{1}{2}\left\{\frac{\partial g_{\mu\rho}}{\partial X^{\nu}} + \frac{\partial g_{\nu\rho}}{\partial X^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial X^{\rho}}\right\}.$$

One then has:

$$\Gamma_{\lambda\mu,\nu}X^{\lambda} = \frac{1}{2}\left\{\frac{\partial g_{\mu\nu}}{\partial X^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial X^{\mu}} - \frac{\partial g_{\lambda\mu}}{\partial X^{\nu}}\right\}X^{\lambda}.$$

Due to the homogeneity condition, one further has:

$$X^{\lambda}\frac{\partial g_{\mu\nu}}{\partial X^{\lambda}} = -2g_{\mu\nu},$$

hence, due to the fact that:

$$(28')\qquad \Pi_{\mu\nu} = g_{\mu\nu} + \Gamma_{\mu\lambda,\nu}X^{\lambda},$$

$$(29')\qquad Q_{\mu\nu} = g_{\mu\nu} + \Gamma_{\lambda\mu,\nu}X^{\lambda},$$

one has, in our case:

$$(35)\qquad Q_{\mu\nu} = \Pi_{\mu\nu} = \frac{1}{2}\left(\frac{\partial g_{\lambda\nu}}{\partial X^{\mu}} - \frac{\partial g_{\lambda\mu}}{\partial X^{\nu}}\right)X^{\lambda}.$$

If, for the sake of what follows, we introduce the fundamental skew-symmetric tensor:

$$(36)\qquad X_{\mu\nu} = \frac{\partial X_{\nu}}{\partial X^{\mu}} - \frac{\partial X_{\mu}}{\partial X^{\nu}} = \left(\frac{\partial g_{\lambda\nu}}{\partial X^{\mu}} - \frac{\partial g_{\lambda\mu}}{\partial X^{\nu}}\right)X^{\lambda},$$

then we have:

(35a) $$Q_{\mu\nu} = \Pi_{\mu\nu} = \frac{1}{2} X_{\mu\nu} = -\frac{1}{2} X_{\nu\mu} .$$

Furthermore, we have:

$$X_{\mu\nu} X^{\nu} = - X^{\lambda} X^{\nu} \frac{\partial g_{\mu\lambda}}{\partial X^{\nu}} + X^{\lambda} X^{\nu} \frac{\partial g_{\nu\lambda}}{\partial X^{\mu}} ,$$
$$= - 2 X^{\lambda} g_{\mu\lambda} + \frac{\partial}{\partial X^{\mu}} (X^{\lambda} X^{\nu} g_{\lambda\nu}) + 2 X^{\lambda} g_{\mu\lambda} ,$$

hence, due to (7):

(37) $$X_{\mu\nu} X^{\nu} = 0 .$$

If we then define the skew-symmetric tensor:

(38) $$X_{ik} = \gamma^{\mu}_{\cdot i} \gamma^{\nu}_{\cdot k} X_{\mu\nu} = - X_{ki} ,$$

then one has, conversely:

(38a) $$X_{\mu\nu} = \gamma_{\mu}^{\cdot i} \gamma_{\nu}^{\cdot k} X_{ik} .$$

According to (36), $X_{\mu\nu}$ also satisfies the relations:

(36a) $$\frac{\partial X_{\mu\nu}}{\partial X^{\rho}} + \frac{\partial X_{\rho\mu}}{\partial X^{\nu}} + \frac{\partial X_{\nu\rho}}{\partial X^{\mu}} = 0 .$$

From (18), in which we have used (37), we can infer the following:

(39) $$\frac{\partial X_{ik}}{\partial X^{l}} + \frac{\partial X_{il}}{\partial X^{k}} + \frac{\partial X_{kl}}{\partial X^{i}} = 0 .$$

In order to simply the notation, we would like assert that if three indices are enclosed within brackets in an expression then what that means is that these indices will be cyclically permuted and added together in the expression. One then writes, e.g., (36a) and (39) as:

$$\frac{\partial X_{\mu\nu}}{\partial X^{\rho}}_{[\mu,\nu,\rho]} = 0, \qquad \frac{\partial X_{ik}}{\partial X^{l}}_{[ik,l]} = 0, \quad \text{resp.}$$

Now, it follows from (38a) and (36a) that:

$$0 = \frac{\partial(\gamma_{\mu}^{\cdot i} \gamma_{\nu}^{\cdot k} X_{ik})}{\partial X^{\rho}}_{[\mu,\nu,\rho]} = \frac{1}{2} \frac{\partial(\gamma_{\mu}^{\cdot i} \gamma_{\nu}^{\cdot k} - \gamma_{\mu}^{\cdot k} \gamma_{\nu}^{\cdot i})}{\partial X^{\rho}}_{[\mu,\nu,\rho]} \cdot X_{ik} + \gamma_{\mu}^{\cdot i} \gamma_{\nu}^{\cdot k} \gamma_{\rho}^{\cdot l} \frac{\partial X_{ik}}{\partial x^{l}}_{[\mu,\nu,\rho]} .$$

We have substituted $\gamma_{\rho}^{\cdot l} = \dfrac{\partial x^{l}}{\partial X^{\rho}}$ in the latter expression. Now, however, one has:

$$\gamma_\mu^{\cdot i}\gamma_\nu^{\cdot k} - \gamma_\mu^{\cdot k}\gamma_\nu^{\cdot i} = \frac{\partial x^i}{\partial X^\mu}\frac{\partial x^k}{\partial X^\nu} - \frac{\partial x^k}{\partial X^\mu}\frac{\partial x^i}{\partial X^\nu} = \frac{\partial}{\partial X^\mu}\left(x^i\frac{\partial x^k}{\partial X^\nu}\right) - \frac{\partial}{\partial X^\nu}\left(\frac{\partial x^k}{\partial X^\mu}x^i\right),$$

and it follows that:

$$\frac{\partial}{\partial X^\rho}(\gamma_\mu^{\cdot i}\gamma_\nu^{\cdot k} - \gamma_\mu^{\cdot k}\gamma_\nu^{\cdot i})_{[\mu,\nu,\rho]} = 0 .$$

Thus, one has:

$$\gamma_\mu^{\cdot i}\gamma_\nu^{\cdot k}\gamma_\rho^{\cdot l}\left.\frac{\partial X_{ik}}{\partial x^l}\right._{[\mu,\nu,\rho]} = \gamma_\mu^{\cdot i}\gamma_\nu^{\cdot k}\gamma_\rho^{\cdot l}\left.\frac{\partial X_{ik}}{\partial x^l}\right._{[i,k,l]} = 0 ,$$

from which, one immediately infers that:

$$\left.\frac{\partial X_{ik}}{\partial x^l}\right._{[i,k,l]} = 0 ,$$

i.e., one infers the validity of (39).

There thus exists a covariant vector $f_i$ such that:

$$X_{ik} = \frac{\partial f_k}{\partial x^l} - \frac{\partial f_i}{\partial x^k} . \tag{40}$$

From (36), (38a), and (40), it follows that:

$$\frac{\partial X_\nu}{\partial X^\mu} - \gamma_\mu^{\cdot i}\gamma_\nu^{\cdot k}\frac{\partial f_k}{\partial x^i} - \left(\frac{\partial X_\mu}{\partial X^\nu} - \gamma_\mu^{\cdot i}\gamma_\nu^{\cdot k}\frac{\partial f_i}{\partial x^k}\right) = 0 ,$$

or:

$$\frac{\partial X_\nu}{\partial X^\mu} - \gamma_\mu^{\cdot i}\gamma_\nu^{\cdot k}\frac{\partial f_k}{\partial x^i} - \left(\frac{\partial X_\mu}{\partial X^\nu} - \gamma_\mu^{\cdot i}\gamma_\nu^{\cdot k}\frac{\partial f_i}{\partial x^k}\right) = 0 ,$$

and, since:

$$\frac{\partial \gamma_\nu^{\cdot k}}{\partial X^\mu} = \frac{\partial \gamma_\mu^{\cdot k}}{\partial x^\nu} = \frac{\partial^2 x^k}{\partial X^\mu \partial X^\nu}$$

one has:

$$\frac{\partial}{\partial X^\mu}(X_\nu - f_\nu) - \frac{\partial}{\partial X^\nu}(X_\mu - f_\mu) = 0 ,$$

in which we have set:

$$f_\nu = \gamma_\nu^{\cdot k} f_k , \qquad (f_\nu X^\nu = 0) . \tag{41}$$

From this, it ultimately follows that:

$$X_\mu = f_\mu + \varepsilon\,\frac{1}{F}\frac{\partial F}{\partial X^\mu} = f_\mu + \varepsilon\,\frac{\partial \log F}{\partial X^\mu} , \tag{42}$$

in which the sign $\varepsilon$ is included, from which:

(42a) $$X^{\mu}\,\frac{\partial F}{\partial X^{\mu}} = F\,,$$

hence, $F$ is homogeneous of first degree.

Our metric is then exactly as general that it is (for a given $\gamma_{\nu}^{\cdot k}$) characterized by $g_{ik}$ and the skew-symmetric tensor $X_{ik}$ that satisfies (39). One is tempted to identify the latter, up to a proportionality factor, with the tensor $F_{ik} = - F_{ki}$ for the electromagnetic field strength. If $\kappa$ is the Einstein gravitation constant then:

(43) $$f_{ik} = \frac{\sqrt{\kappa}}{c} F_{ik}$$

has the dimensions of length, such that we can set:

(44) $$X_{ik} = r\,f_{ik} = r\,\frac{\sqrt{\kappa}}{c} F_{ik}\,,$$

in which $r$ is a real dimensionless numerical factor. With:

$$F_{\mu\nu}\,X^{\nu} = 0\,,$$

(44) may be extended to:

(44a) $$X_{\mu\nu} = r\,f_{\mu\nu} = r\,\frac{\sqrt{\kappa}}{c} F_{\mu\nu}\,.$$

It is satisfying that with this geometrical meaning for the electromagnetic field strength, the first system of Maxwell equation is automatically satisfied.

As for the potential $\varphi_k$, which is defined by:

(45) $$F_{ik} = \frac{\partial \Phi_k}{\partial x^i} - \frac{\partial \Phi_i}{\partial x^k}\,, \qquad f_{ik} = \frac{\partial \varphi_k}{\partial x^i} - \frac{\partial \varphi_i}{\partial x^k}\,, \qquad \varphi_k = \frac{\sqrt{\kappa}}{c}\,\Phi_k\,,$$

from (40), it can be identified with $f_i$, up to the factor $1/r$:

(46) $$f_i = r\,\varphi_i = r\frac{\sqrt{\kappa}}{c}\,\Phi_i\,,$$

hence, with:

(47) $$\bar{\varphi}_{\nu} = \gamma_{\nu}^{\cdot i}\,\varphi_i\,, \qquad \bar{\varphi}_{\nu} X^{\nu} = 0$$

(45a) $$\bar{f}_{\mu\nu} = \frac{\partial \bar{\varphi}_{\nu}}{\partial X^{\mu}} - \frac{\partial \bar{\varphi}_{\mu}}{\partial X^{\nu}}$$

(46a) $$\bar{f}_\nu = r\,\bar{\varphi}_\nu = r\frac{\sqrt{\kappa}}{c}\bar{\Phi}_\nu\,.$$

$\bar{f}_\nu$, like $\bar{\varphi}_\nu$, is defined only up to an additive gradient. As Schouten has remarked, the theory can, however, be formulated in such a way that $\bar{f}_\nu$ and $\bar{\varphi}_\nu$ do not appear explicitly, but the well-defined vector $X_\nu$ always appears in place of the ill-defined vector $f_\nu$.

## § 5. **Parallel displacement of vectors. Geodetic lines.**

One can define the parallel displacement of a vector $a^\nu$ ($b_\nu$, resp.) along a curve with the help of the modified coordinate differentials that were defined by (20), (20b):

$$\bar{d}X^\mu = d^{\mu}_{\cdot\nu}\,dX^\nu = \gamma^{\mu}_{\cdot\nu}\,dx^k\,,$$

namely:

(48) $$\left\{\begin{aligned}\delta a^\nu &= \bar{d}X^\mu a^\nu{}_{;\mu} = dx^k\left(\gamma^{\mu}_{\cdot k}a^\nu{}_{;\mu}\right)\\ &= dx^k\left(\gamma^{\mu}{}_{;k}\frac{\partial a^\nu}{\partial X^\mu} + \gamma^{\mu}{}_{;k}\Gamma^{\nu}_{\lambda\mu}a^\lambda\right),\end{aligned}\right.$$

or:

(48a) $$\left\{\begin{aligned}\delta b_\nu &= \bar{d}X^\mu b_{\nu;\mu} = dx^k\left(\gamma^{\mu}_{\cdot k}b_{\nu;\mu}\right)\\ &= dx^k\left(\gamma^{\mu}{}_{;k}\frac{\partial b_\nu}{\partial X^\mu} - \gamma^{\mu}{}_{;k}\Gamma^{\lambda}_{\nu\mu}b_\lambda\right),\end{aligned}\right.$$

resp. From (34), one can also write this as:

(49) $$\delta a^\nu = \left(\gamma^{\mu}_{\cdot k}a^l{}_{;k} + \varepsilon a\gamma^{\mu}_{\cdot k}Q_{\mu}^{\cdot\nu}\right)dx^k + \varepsilon\,da\,X^\nu\,,$$

(49a) $$\delta b_\nu = \left(\gamma_{\nu}^{\cdot l}b_{l;k} + \varepsilon b\gamma^{\mu}_{\cdot k}Q_{\mu\nu}\right)dx^k + \varepsilon\,db\,X_\nu\,,$$

resp., upon introducing the Riemannian parallel displacement:

$$\overset{R}{\delta} a^l = a^l{}_{;k}\,dx^k\,, \qquad \overset{R}{\delta} b_l = b_{l\,;\,k}\,dx^k\,,$$

and substitution of $X_\nu\,Q_{\mu}^{\cdot\nu} = X^\nu Q_{\mu\nu} = 0$:

(50) $$\delta a^\nu = \gamma^{\nu}_{\cdot l}\left(\overset{R}{\delta} a^l + \varepsilon aQ_k^{\cdot l}dx^k\right)dx^k + \varepsilon\,da\,X^\nu\,,$$

(50a) $$\delta b_\nu = \gamma_{\nu}^{\cdot l}\left(\overset{R}{\delta} b_l + \varepsilon bQ_{kl}dx^k\right)dx^k + \varepsilon\,db\,X_\nu\,.$$

We now define *geodetic lines* by saying that the vector $a^\nu$ is parallel displaced along them in the direction associated with:

(51) $$\frac{dx^k}{d\tau} = \gamma_\nu^{\cdot k} a^\nu = a^k ,$$

and postulate that the vector $a^\nu$ does not change under this motion. We shall then have:

(52) $$\frac{\delta a^\nu}{d\tau} = \frac{\bar{d} X^\mu}{d\tau} a^\nu{}_{;\mu} = 0 ,$$

when (51) is true. The latter is equivalent with:

(51a) $$a^\nu = \gamma_\nu^{\cdot k} \frac{dx^k}{d\tau} = \varepsilon\, a\, X^\nu ,$$

such that from (50) the requirement (52) decomposes into:

$$\frac{\overset{R}{\delta}}{d\tau} \frac{dx^k}{d\tau} + \varepsilon\, a\; Q_k^{\cdot l} \frac{dx^k}{d\tau} = 0$$

or:

(53) $$\frac{d^2 x^l}{d\tau^2} + \begin{Bmatrix} l \\ mn \end{Bmatrix} \frac{dx^m}{d\tau} \frac{dx^n}{d\tau} = -\,\varepsilon\, a\; Q_k^{\cdot l} \frac{dx^k}{d\tau}$$

and:

(54) $$\frac{da}{d\tau} = 0 , \qquad a = \text{const.}$$

If we now introduce the heretofore unused relation (35) ((35a), resp.) and take account of (44) then we have:

(53a) $$\frac{d^2 x^l}{d\tau^2} + \begin{Bmatrix} l \\ mn \end{Bmatrix} \frac{dx^m}{d\tau} \frac{dx^n}{d\tau} = -\,\varepsilon\, a\; Q_m^{\cdot l} \frac{dx^m}{d\tau} = \frac{1}{2} \varepsilon a r \frac{\sqrt{\kappa}}{c} F^l_{\cdot m} \frac{dx^m}{d\tau}$$

From the skew-symmetry of $F_{ik}$, it follows from this that:

$$g_{ik} \frac{dx^i}{d\tau} \frac{dx^k}{d\tau} = \text{const.},$$

such that we can normalize to:

(55) $$g_{ik} \frac{dx^i}{d\tau} \frac{dx^k}{d\tau} = -1 .$$

(The negative sign corresponds to the timelike character of the curve.) Comparing this with *the world lines of charged mass points* shows that they corresponds quite neatly with the generalized geodetic lines that we consider here, when we set the arbitrary integration constant $a$ to:

(56) $$\varepsilon\, a = \frac{c}{\sqrt{\kappa}}\frac{2}{r}\frac{e}{mc}$$

in which $e$ and $m$ mean the charge and mass of the mass point. If we introduce the impulse vector:

(57) $$p_\mu = m\, g_{\mu\nu}\, a^\nu = a_\mu = \gamma_{\mu}^{\cdot k}\, m\, g_{kl}\, \frac{dx^l}{d\tau} + \frac{c}{\sqrt{\kappa}}\frac{2}{r}\frac{e}{mc} X_\mu$$

then one has:

(58) $$\frac{\delta p_\mu}{d\tau} = 0\,.$$

We further remark that in many works the geodetic lines that satisfy (51) are defined by:

$$a^\mu\, a^\nu{}_{;\,\mu} = \left(\gamma^{\mu}_{\cdot k}\frac{dx^k}{d\tau} + \varepsilon a X^\mu\right) a^\nu{}_{;\,\mu} = \frac{\delta a^\nu}{d\tau} + \varepsilon\, a\, X^\mu\, a^\nu{}_{;\,\mu} = 0$$

instead of by (52), since that seems less natural to us.

Furthermore, from (42) and (46a), the vector $p_\mu$ that is given by (57) differs by a gradient from:

$$\gamma_{\mu}^{\cdot k}\, m\, g_{kl}\, \frac{dx^l}{d\tau} + 2\frac{e}{c}\bar{\Phi}_\mu$$

and not, as one might perhaps expect, from:

$$\gamma_{\mu}^{\cdot k}\, m\, g_{kl}\, \frac{dx^l}{d\tau} + \frac{e}{c}\bar{\Phi}_\mu\,.$$

We shall come back to this later (§ 7).

## § 6. **Curvature**

When $\Gamma^{\nu}_{\lambda\mu}$ is symmetric in $\lambda$, $\mu$, as we would like to assume here, the curvature tensor $P^{\mu}_{\cdot\nu\rho\sigma}$ is defined by:

(59) $$a^\mu{}_{;\rho;\sigma} - a^\mu{}_{;\sigma;\rho} = P^{\mu}_{\cdot\nu\rho\sigma}\, a^\nu\,.$$

Since:

$$(a^\mu\, b_\mu)_{;\,\rho\,;\,\sigma} - (a^\mu\, b_\mu)_{;\,\sigma\,;\,\rho} = (a^\mu{}_{;\,\rho\,;\,\sigma} - a^\mu{}_{;\,\rho\,;\,\sigma})\, b_\mu - a^\mu\,(b_{\mu\,;\,\rho\,;\,\sigma} - b_{\mu\,;\,\rho\,;\,\rho}) = 0\,,$$

it follows that:

(59a) $$b_{\mu\,;\,\rho\,;\,\sigma} - b_{\mu\,;\,\sigma\,;\,\rho} = -\,P^{\nu}_{\cdot\,\mu\rho\sigma}\,b_\nu\,.$$

The definition of $P^{\mu}_{\cdot\,\nu\rho\sigma}$ is skew-symmetric in $\rho$, $\sigma$, and carrying out the calculations yields:

(60) $$P^{\mu}_{\cdot\,\nu\rho\sigma} = \frac{\partial\Gamma^{\mu}_{\nu\rho}}{\partial X^{\sigma}} - \frac{\partial\Gamma^{\mu}_{\nu}}{\partial X^{\rho}} + \Gamma^{\tau}_{\nu\rho}\Gamma^{\mu}_{\tau\rho} - \Gamma^{\mu}_{\tau\rho}\Gamma^{\tau}_{\nu\sigma}\,.$$

We would now like to express the $P^{\mu}_{\cdot\,\nu\rho\sigma}$ in terms of the corresponding Riemann tensor $R^{i}_{\cdot\,klm}$. In order to do this, it is more convenient to compute the expression $b_{k\,;\,\rho\,;\,\sigma}$ − $b_{k\,;\,\rho\,;\,\sigma}$ by using the fact that:

(59′) $$\overset{R}{b}_{k;l;m} - \overset{R}{b}_{k;m;l} = -\,b_i\,R^{i}_{\cdot\,klm}\,.$$

Since:

$$b_{k\,;\,\rho} \equiv \gamma_{\rho}^{\cdot\,l}\,\overset{R}{b}_{k;l}\,,$$

it follows:

$$b_{k\,;\,\rho\,;\,\sigma} = \gamma_{\rho}^{\cdot\,l}\gamma_{\sigma}^{\cdot\,m}\,\overset{R}{b}_{k;l;m} + \gamma_{\rho\,;\sigma}^{\cdot\,l}\,\overset{R}{b}_{k;l}\,,$$

and since, from (32a), the symmetry of $\gamma_{\rho\,;\sigma}^{\cdot\,l}$ in $\rho$ and $\sigma$ follows from that of the $\Gamma^{\nu}_{\lambda\mu}$, we obtain, from $(59')$:

(61) $$b_{\mu\,;\,\rho\,;\,\sigma} - b_{\mu\,;\,\sigma\,;\,\rho} = -\gamma_{\rho}^{\cdot\,l}\gamma_{\sigma}^{\cdot\,m}R^{i}_{\cdot\,klm}\,b_i\,.$$

On the other hand, from the fact that:

$$b_k = \gamma^{\mu}_{\cdot\,k}\,b_\mu\,,$$

$$b_{k;\,\rho} = \gamma^{\mu}_{\cdot\,k;\rho}\,b_\mu + \gamma^{\mu}_{\cdot\,k}\,b_{\mu\,;\,\rho}\,,$$

one has:

$$b_{\mu\,;\,\rho\,;\,\sigma} - b_{\mu\,;\,\sigma\,;\,\rho} = (\gamma^{\mu}_{\cdot\,k;\rho;\sigma} - \gamma^{\mu}_{\cdot\,k;\sigma;\rho})\,b_\mu + \gamma^{\mu}_{\cdot\,k}\,(b_{\mu\,;\,\rho\,;\,\sigma} - b_{\mu\,;\,\sigma\,;\,\rho})\,.$$

Taking into account (61) and (59a), it follows:

(62) $$\gamma^{\mu}_{\cdot\,k}\,P^{\nu}_{\cdot\,\mu\rho\sigma} - \gamma_{\rho}^{\cdot\,l}\gamma_{\sigma}^{\cdot\,m}\gamma^{\mu}_{\cdot\,l}R^{i}_{\cdot\,klm} = \gamma^{\mu}_{\cdot\,k;\rho;\sigma} - \gamma^{\mu}_{\cdot\,k;\sigma;\rho}\,.$$

On the other hand, from (59), one has:

$$X^{\mu}_{;\,\rho\,;\,\sigma} - X^{\mu}_{;\,\sigma\,;\,\rho} = P^{\mu}_{\cdot\,\nu\rho\sigma}X^{\nu}\,,$$

hence, from (36) and (35a):

(63) $$P^{\mu}_{\cdot\,\nu\rho\sigma}X^{\nu} = \frac{1}{2}(X^{\cdot\,\mu}_{\rho;\sigma} - X^{\cdot\,\mu}_{\sigma;\rho})\,.$$

We can now compute the right-hand side of (62), if we know $\gamma^{\mu}_{\cdot k;\rho}$. From (II′), however, $\gamma^{\mu}_{\cdot k;\rho}$ is determined if we know:

$$X_\mu \, \gamma^{\mu}_{\cdot k;\rho}$$

and:

$$X^\rho \, \gamma^{\mu}_{\cdot k;\rho} .$$

However, one has:

$$X_\mu \, \gamma^{\mu}_{\cdot k;\rho} = -\gamma^{\mu}_{\cdot k} X_{\mu\,;\,\rho} = -\frac{1}{2} \gamma^{\mu}_{\cdot k} X_{\rho\mu}$$

and:

$$X^\rho \, \gamma^{\mu}_{\cdot k;\rho} = P_\rho^{\cdot\,\mu} \gamma^{\rho}_{\cdot k} = \frac{1}{2} \gamma^{\rho}_{\cdot k} X_\rho^{\cdot\,\mu} ,$$

so one has:

(64) $$\gamma^{\mu}_{\cdot k;\rho} = \frac{\varepsilon}{2} (X_\rho X_\nu^{\cdot\,\mu} - X^\mu X_{\rho\nu}) \gamma^{\nu}_{\cdot k} ,$$

and it likewise follows that:

(64a) $$\gamma_{\mu\;;\rho}^{\cdot\,k} = \frac{1}{2} (-X_\rho X_\mu^{\cdot\,\nu} + X_\mu X^{\nu}_{\cdot\,\rho}) \gamma_\nu^{\cdot\,k} .$$

From this, one further finds that:

(65) $$\begin{cases} \gamma^{\mu}_{\cdot k;\rho;\sigma} - \gamma^{\mu}_{\cdot k;\sigma;\rho} \\ \qquad = \gamma^{\nu}_{\cdot k} \left\{ \tfrac{1}{4} \varepsilon (-2 X_{\rho\sigma} X_\nu^{\cdot\,\mu} + X_\rho^{\cdot\,\mu} X_{\sigma\nu} - X_\sigma^{\cdot\,\mu} X_{\rho\nu}) \right. \\ \qquad\quad + \tfrac{\varepsilon}{2} (X_\rho X_{\nu\;;\sigma}^{\cdot\,\mu} - X_\sigma X_{\nu\;;\rho}^{\cdot\,\mu}) - \tfrac{\varepsilon}{2} X^\mu (X_{\rho\nu;\sigma} - X_{\sigma\nu;\rho}) \\ \qquad\quad \left. + \tfrac{1}{4} X^\mu X_\nu^{\cdot\,\tau} (X_\rho X_{\sigma\tau} - X_\sigma X_{\rho\tau}) \right\} . \end{cases}$$

From (62), (63), and (65), a brief calculation finally gives the ultimate formula for the curvature tensor:

(66) $$\begin{cases} P^{\mu}_{\cdot\nu\rho\sigma} = \gamma_\nu^{\cdot\,k} \gamma_\rho^{\cdot\,l} \gamma_\sigma^{\cdot\,m} \gamma^{\mu}_{\cdot i} R^{i}_{\cdot\,klm} \\ \qquad = \tfrac{\varepsilon}{2} (-2 X_{\rho\sigma} X_\nu^{\cdot\,\mu} + X_\rho^{\cdot\,\mu} X_{\sigma\nu} - X_\sigma^{\cdot\,\mu} X_{\rho\nu}) \\ \qquad\quad + \tfrac{\varepsilon}{2} (X_\rho X_{\nu\;;\sigma}^{\cdot\,\mu} - X_\sigma X_{\nu\;;\rho}^{\cdot\,\mu}) \\ \qquad\quad - \tfrac{\varepsilon}{2} \left\{ X^\mu (X_{\rho\nu;\sigma} - X_{\sigma\nu;\rho}) - X_\nu (X_{\rho\;;\sigma}^{\cdot\,\mu} - X_{\sigma\;;\rho}^{\cdot\,\mu}) \right\} \\ \qquad\quad + \tfrac{1}{4} (X_\rho X_{\sigma\tau} - X_\sigma X_{\rho\tau}) (X^\mu X_\nu^{\cdot\,\tau} - X_\nu X^{\mu\nu}) \} . \end{cases}$$

Of particular interest to physics is the contracted curvature tensor:

$$P_{\nu\sigma} = P^{\mu}_{\cdot\nu\rho\sigma} ,$$

which is symmetric in $\nu$ and $\sigma$ in the case of a symmetric $\Gamma^{\nu}_{\lambda\mu}$ that satisfies (I). By the use of:

$$X^{\alpha} X_{\mu\nu;\alpha} = -\Pi_{\mu}^{\cdot\rho} X_{\rho\nu} - \Pi_{\nu}^{\cdot\rho} X_{\mu\rho} = -\frac{1}{2}(X_{\mu}^{\cdot\rho} X_{\rho\nu} - X_{\nu}^{\cdot\rho} X_{\mu\rho})$$

$$= -\frac{1}{2}(X_{\mu}^{\cdot\rho} X_{\rho\nu} + X_{\mu}^{\cdot\rho} X_{\nu\rho}) = 0 ,$$

we get:

$$(67) \quad \begin{cases} P_{\mu\nu} = P_{\nu\mu} = \gamma_{\mu}^{\cdot i} \gamma_{\nu}^{\cdot k} R_{ik} + \dfrac{\varepsilon}{2} X_{\mu}^{\cdot\alpha} X_{\nu\alpha} \\ \qquad -\dfrac{\varepsilon}{2}(X_{\mu} X_{\nu\ ;\alpha}^{\cdot\alpha} + X_{\nu} X_{\mu\ ;\alpha}^{\cdot\alpha}) + \dfrac{1}{4} X_{\mu} X_{\nu} X_{\rho\alpha} X^{\rho\sigma}. \end{cases}$$

We thus extract from this:

$$(67a) \qquad P_{;k} = \gamma^{\mu}_{\cdot i} \gamma^{\nu}_{\cdot k} P_{\mu\nu} = R_{ik} + \frac{\varepsilon}{2} X_{i}^{\cdot r} X_{kr} ,$$

$$(67b) \qquad P_{i(0)} = \gamma^{\mu}_{\cdot i} X^{\nu} P_{\mu\nu} = -\frac{1}{2} X_{i\ ;k}^{\cdot r} ,$$

Furthermore:

$$(68) \qquad P = g^{\mu\nu} P_{\mu\nu} = R + \frac{\varepsilon}{4} X_{\rho\sigma} X^{\rho\sigma},$$

$$(69) \qquad \bar{P} = P_{\mu\nu} X^{\mu} X^{\nu} = -\frac{1}{4} X_{\rho\sigma} X^{\rho\sigma}.$$

## § 7. **The form of the laws of Nature. – Variational principle.**

For physics, the ultimate goal for the application of the theory of the group of homogeneous coordinate transformations and its covariant structure is to derive the laws of gravitational fields and electromagnetic fields in a unified way. Thus, in the classical part of the theory, we restrict ourselves to the case of the absence of charge and mass. In order to reduce the number of field laws, we next propose the notion of *actual* tensors (projectors). This notion is such, that it is constructed only from the $g_{\mu\nu}$ and the $\Gamma^{\lambda}_{\mu\nu}$, as well as the derivatives of the $\Gamma^{\lambda}_{\mu\nu}$, *without the explicit inclusion of the* $X^{\nu}$ *and the* $\gamma^{\nu}_{\cdot k}$ *or the* $\gamma_{\nu}^{\cdot k}$, and also without anything explicitly entering into the $\Gamma^{\lambda}_{\mu\nu}$ but the derivatives of the $g_{\mu\nu}$.

The simplest form of the laws of Nature would then be the one that simply expresses the vanishing of an actual projector. However, one thus always obtains one equation too many, and it is thus necessary to cast the field laws in the following modified form: One must have:

$$(70) \qquad K_{\mu\nu} = F X_{\mu} X_{\nu} ,$$

in which $K_{\mu\nu}$ is an *actual* tensor, and indeed a *symmetric* tensor of the second rank, whereas $F$ is a yet-to-be-determined scalar, hence, a homogeneous function of null degree in the $X^\lambda$. From (70), one immediately finds that:

(71) $$F = K_{\mu\nu} X^\mu X^\nu \equiv \bar{K} ,$$

such that we can write:

(72) $$K_{\mu\nu} = \bar{K} X_\mu X_\nu .$$

Upon multiplying by $X^\mu$ and $X^\nu$ and then performing the associated contractions, one derives an identity, and now (72) includes only 14 independent equations instead of 15. The necessity of this identity is most closely connected with the relation (7) for the $g_{\mu\nu}$. In order to satisfy the general covariance and to allow for sufficient generality in the solution of the $g_{\mu\nu}$-field, (72) must satisfy five further differential identities that we can pose, by analogy with general relativity theory, in the form:

(73) $$K_{\mu\ ;\nu}^{\cdot\nu} \equiv 0 .$$

This is permissible, because:

$$(\bar{K} X_\mu X^\nu)_{;\,\nu} = \frac{\partial \bar{K}}{\partial X^\nu} X^\nu X_\mu + \bar{K} X_{\mu\,;\nu} X^\nu + \bar{K} X_\mu X^\nu{}_{;\,\nu} = 0 ,$$

since the individual terms all vanish.

We now decompose eq. (72) and the identities (73) into corresponding equations for the inhomogeneous tensors. With the fact that:

(74) $$K_{ik} = \gamma^\mu_{\cdot i} \gamma^\nu_{\cdot k} K_{\mu\nu} , \qquad K_{i(0)} = \gamma^\mu_{\cdot i} X^\nu K_{\mu\nu} ,$$

(72) becomes:

(75a) $$K_{ik} = 0 ,$$

(75b) $$K_{i(0)} = 0 .$$

Since (71) and (74) means the same thing as:

(74a) $$K_{\mu\nu} = \gamma_\mu^{\cdot i} \gamma_\nu^{\cdot k} K_{ik} + \varepsilon\, \gamma^\mu_{\cdot i} K_{i(0)} X_\nu + \varepsilon X_\mu K_{i(0)} \gamma_\nu^{\cdot i} + \bar{K} X_\mu X_\nu ,$$

then it follows that:

$$\begin{aligned} K_{\mu\ ;\nu}^{\cdot\nu} = {} & \gamma_\mu^{\cdot i}\ K_{i\ \ ;k}^{\cdot k} + X_\mu\ K^{\ k}_{\cdot(0);k} + (\gamma_{\mu\ ;\nu}^{\cdot i} \gamma^\nu_{\cdot k} + \gamma_\mu^{\cdot i} \gamma^\nu_{\cdot k;\nu}) K_i^{\cdot k} \\ & + \varepsilon X^\nu\ \gamma_{\mu\ ;\nu}^{\cdot i} K_{i(0)} + \varepsilon X_\mu K_{i(0)} \gamma^\nu_{\cdot k;\nu} + e\, X_\mu , K^{\ k}_{\cdot(0)} \gamma^\nu_{\cdot k} . \end{aligned}$$

From (64), one has:

$$\gamma^{\mu}_{\cdot k;\rho} = \frac{\varepsilon}{2}(X_\rho X_\nu^{\cdot\mu} - X^\mu X_{\rho\nu})\gamma^{\nu}_{\cdot k}\,,$$

$$\gamma_{\mu;\rho}^{\cdot k} = \frac{\varepsilon}{2}(-X_\rho X_\mu^{\cdot\nu} - X_\mu X^{\nu}_{\cdot\rho})\gamma_\nu^{\cdot k}\,.$$

Thus, one has:

$$K_i^{\cdot k}\gamma_{\mu;\nu}^{\cdot i}\gamma^{\nu}_{\cdot k} = \frac{\varepsilon}{2}X_\mu X^{\nu}_{\cdot\rho}\gamma_\nu^{\cdot i}\gamma^{\rho}_{\cdot k}K_i^{\cdot k} = \frac{\varepsilon}{2}X_\mu X^{i}_{\cdot k}K_i^{\cdot k} = 0\,,$$

$$\gamma^{\nu}_{\cdot k;\nu} = 0\,, \qquad X^\nu\,\gamma_{\mu;\nu}^{\cdot i} = -\frac{1}{2}X_\mu^{\cdot\nu}\gamma_\nu^{\cdot i}\,,$$

and it follows that:

(75) $$K_{\mu;\nu}^{\cdot\nu} = \gamma_\mu^{\cdot i}(K_i^{\cdot k}{}_{;k} - \varepsilon X_{ik}K^{k}_{\cdot(0)}) + \varepsilon X_\mu K^{k}_{\cdot(0);k}\,.$$

Thus, (73) splits into two identities:

(76a) $$K_i^{\cdot k}{}_{;k} - \varepsilon X_{ik}K^{k}_{\cdot(0)} \equiv 0\,,$$

(76b) $$K^{k}_{\cdot(0);k} \equiv 0\,.$$

In the case of the presence of matter, as we shall see part II, these identities lead to the theorem of the conservation of energy, impulse, and charge, which are summarized in *one* tensor equation in the homogeneous coordinates.

It is now worth pointing out that one obtains equations of the form (72) with the identities (73) when one starts with a variational principle:

(77) $$\delta\cdot L\sqrt{|g|}\;dX^{(1)}\ldots dX^{(5)} = 0$$

with the supplementary condition:

(78) $$\delta(g_{\mu\nu}X^\mu X^\nu) = \delta g_{\mu\nu}X^\mu X^\nu = 0\,,$$

in which $L$ refers to an actual scalar and $|g|$ refers to the absolute value of the determinant of the $g_{\mu\nu}$. As usual, the variations of the $g_{\mu\nu}$ and the $\frac{\partial g_{\mu\nu}}{\partial X^\alpha}$ shall vanish on the boundary.

Next, if one lets:

(79) $$\delta\cdot L\sqrt{|g|}\;dX^{(1)}\ldots dX^{(5)} \equiv \pm K\,\delta g_{\mu\nu}\sqrt{|g|}\;dX^{(1)}\ldots dX^{(5)}\,,$$

without regard for the supplementary condition, then from the fact that:

$$X^\mu X^\nu\,\delta g_{\mu\nu} = -X_\mu X_\nu\,\delta g^{\mu\nu}\,,$$

the field equations read like:

(70) $$K_{\mu\nu} = F\,X_\mu X_\nu\,,$$

in which $F$ is a yet-to-be-determined Lagrange multiplier, which would have been previously introduced into (70); furthermore, $K_{\mu\nu}$ is an actual tensor when $L$ is an actual scalar. Finally, $K_{\mu\nu}$ also satisfies the identities (73), as one knows, in which one performs the variations of the $g^{\mu\nu}$ under an infinitesimal coordinate transformation:

$$X'^{\nu} = X^{\nu} + \varepsilon\,\xi^{\nu} ,$$

from which:

$$\delta g^{\mu\nu} = e\left( g^{\rho\nu}\frac{\partial\xi^{\mu}}{\partial X^{\rho}} + g^{\mu\rho}\frac{\partial\xi^{\nu}}{\partial X^{\rho}} - \frac{\partial\xi^{\rho}}{\partial X^{\rho}}\, g^{\mu\nu} \right)$$

and the variation of the action integral vanishes identically.

We further remark that from (18a) for fixed coordinates $Xn$ and under the assumption that the supplementary condition (78) is valid, one has:

(80) $$\delta g^{\mu\nu} = \gamma^{\mu}_{\cdot i}\gamma^{\nu}_{\cdot k}\,\delta g^{ik} + 2 g^{ik}\gamma^{\mu}_{\cdot i}\,\delta\gamma^{\nu}_{\cdot k} .$$

If one also fixes the coordinates $x^k$ then one has, moreover:

(81a) $$\delta\gamma_{\mu}^{\cdot\, i} = \delta\left( \frac{\partial x^{i}}{\partial X^{\mu}} \right) = 0 ,$$

hence, from the fact that:

$$\gamma_{\nu}^{\cdot\, i}\gamma^{\nu}_{\cdot k} = \delta^{i}_{\cdot k} , \qquad \gamma^{\nu}_{\cdot k} X_{\nu} = 0 ,$$

$$\gamma_{\nu}^{\cdot\, i}\,\delta\gamma^{\nu}_{\cdot k} = 0 , \qquad X_{\nu}\,\delta\gamma^{\nu}_{\cdot k} = -\gamma^{\nu}_{\cdot k}\,\delta X_{\nu} ,$$

(81b) $$\delta\gamma^{\nu}_{\cdot k} = -\,\varepsilon X^{\nu}\,(\gamma^{\rho}_{\cdot k}\,\delta X_{\rho}) ,$$

it follows, from the validity of (78), on account of (74), that:

(82) $$\delta\cdot L\sqrt{|\,g\,|}\; dX^{(1)} \dots dX^{(5)} = \pm \left\{ K_{ik} - 2\varepsilon K^{k}_{\cdot(0)}\gamma^{\rho}_{\cdot k}\,\delta X_{\rho} \right\} \sqrt{|\,g\,|}\; dX^{(1)} \dots dX^{(5)} .$$

We still have to establish the scalar $L$ and thus, the $K_{\mu\nu}$. If we demand that $L$ shall include no derivatives of the $\Gamma^{\lambda}_{\mu\nu}$ higher than the first ones, then $L$ is uniquely identified with the curvature scalar.

(83) $$L = P .$$

Therefore, by means of the contracted curvature tensor $P_{\mu\nu}$ that is defined by (67), one has:

(84) $$K_{\mu\nu} = P_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} P ,$$

in analogy with the relativistic theory of gravitation. Independently of the variational principle, this $K_{\mu\nu}$ is the only actual tensor that includes no derivatives of the $\Gamma^{\lambda}_{\mu\nu}$ that are higher than the first ones, and also satisfies the identities (73). One thus has:

(84a) $$K_{ik} = R_{ik} - \frac{1}{2} g_{ik} R + \frac{\varepsilon}{2}\left( X_{i}^{\cdot k} X_{kr} - \frac{1}{4} g_{ik} X_{rs} X^{rs} \right)$$

(84b) $$K_{i(0)} = -\frac{1}{2} K_{i\ ;k}^{\cdot k} .$$

If we equate this with the field equations from the combined Einstein theory of gravitation and Maxwell theory of electrodynamics, which in the present case of non-existent charge and matter, read like:

(85a) $$R_{ik} - \frac{1}{2} g_{ik} R + \frac{\kappa}{c^2}\left( F_{i}^{\cdot k} F_{kr} - \frac{1}{4} g_{ik} F^{rs} F_{rs} \right) = 0$$

and:

(85b) $$F_{i\ ;k}^{\cdot k} = 0 ,$$

then we see that get agreement between these equations when the number $r$ that was introduced in (44) according to:

$$X_{ik} = r\, f_{ik} = r \frac{\sqrt{\kappa}}{c} F_{ik} ,$$

satisfies the equation:

(86) $$\frac{\varepsilon r^2}{2} = 1 .$$

Since $r$ is real, it likewise follows from this that:

(86a) $$\varepsilon = +1 ,$$

(86b) $$r = \pm\sqrt{2} .$$

We further remark that (84a) can then also be written as:

(84c) $$K^{i}_{\cdot(0)} = -\frac{1}{r} \frac{\sqrt{\kappa}}{c} F^{ik}_{\ ;k} .$$

The determination of *e* and *r can* also come about by the requirement, which is equivalent to (85a), that the expression:

$$L = P = R + \frac{\varepsilon}{4} X_{rs} X^{rs}$$

be identical with the ordinary form:

(87) $$L = R + \frac{\kappa}{c^2}\frac{1}{2}F_{rs}\,F^{rs}.$$

We have achieved our goal. The second system of Maxwell equations and the gravitational equations melt together into a single system that is directly connected with the curvature. The first system of Maxwell equations [eq. (39)] follows directly from the assumed structure of space. Furthermore, the law of motion for a charged mass point can be interpreted as the generalized equation for geodetic lines.

With this, we can close part one, but for the sake of completeness, we would like to refer to a generalization of the Ansatz for the $\Gamma^{\lambda}_{\mu\nu}$ that is due to Schouten and van Dantzig.

## Appendix: Generalization of the Ansatz for the $\Gamma^{\lambda}_{\mu\nu}$.

Schouten and van Dantzig have shown that the most general Ansatz for the $\Gamma^{\lambda}_{\mu\nu}$ whose consequences are in harmony with physics is:

In place of (III) and (35), use the postulate:

(III′) $$\Pi_{\mu}^{\cdot\nu} = p\frac{1}{2}X_{\mu}^{\cdot\nu}, \qquad Q_{\mu}^{\cdot\nu} = q\frac{1}{2}X_{\mu}^{\cdot\nu},$$

in which $p$ and $q$ are numerical coefficients. From (29), one further has:

(88) $$S_{\mu\lambda}^{\cdot\,\cdot\,\nu}\,X^{\lambda} = \frac{1}{4}(p-q)\,X_{\mu}^{\cdot\nu}.$$

Therefore, from (29′) and (36), we infer from (31) that:

$$Q_{\mu\nu} = q\frac{1}{2}X_{\mu\nu} = \frac{1}{2}X_{\mu\nu} + (S_{\lambda\mu,\nu} + S_{\lambda\nu,\mu} + S_{\mu\nu,\lambda})\,X^{\lambda}$$

or:

$$(q-1)\frac{1}{2}X_{\mu\nu} = -\frac{1}{4}(p-q)(X_{\mu\nu} + X_{\nu\mu}) + S_{\mu\nu,\lambda}\,X^{\lambda},$$

hence:

(89) $$S_{\mu\nu,\lambda}\,X^{\lambda} = (q-1)\frac{1}{2}X_{\mu\nu}.$$

From (II′) and (32a), it then follows that:

(90) $$S_{\mu\nu,\lambda}\;\gamma^{\mu}_{\cdot m}\gamma^{\nu}_{\cdot n}\gamma^{\lambda}_{\cdot l} = 0,$$

hence:

(91) $$S_{\mu\nu,\lambda} = \varepsilon\,\frac{1}{4}(p-q)\,(X_{\nu}\,X_{\mu\lambda} - M\,X_{\nu\lambda}) + \varepsilon\,(q-1)\frac{1}{2}X_{\lambda}\,X_{\mu\nu}.$$

For the geodetic lines, it further follows that:

$$\frac{\delta a^\nu}{d\tau} = \frac{\delta}{d\tau}\left(\gamma^{\nu}_{\cdot k}\frac{dx^k}{d\tau} + \varepsilon a X^\nu\right) = 0, \qquad a = \text{const.}$$

and:

$$\frac{d^2 x^l}{d\tau^2} + \begin{Bmatrix} l \\ mn \end{Bmatrix}\frac{dx^m}{d\tau}\frac{dx^n}{d\tau} = -\,\varepsilon\, a\, Q_k^{\cdot l}\,\frac{dx^k}{d\tau}$$

$$= \varepsilon\, a\, Q^l_{\cdot m}\,\frac{dx^m}{d\tau} = \frac{1}{2}\,\varepsilon\, a\, q\, r\,\frac{\sqrt{\kappa}}{c}\,F^l_{\cdot m}\,\frac{dx^m}{d\tau},$$

hence:

(92) $$\varepsilon\, a = \frac{c}{\sqrt{\kappa}}\,\frac{2}{rq}\,\frac{e}{mc}$$

and with:

(93) $$p_\mu = m\gamma_\mu^{\cdot k} g_{kl}\,\frac{dx^l}{d\tau} + \frac{e}{m}\,\frac{c}{\sqrt{\kappa}}\cdot\frac{2}{rq}\,X_\mu$$

(94) $$\frac{\delta}{d\tau}\,p_\mu = 0\,.$$

Consequently:

(95) $$\delta\left(\gamma_\mu^{\cdot k} m g_{kl}\,\frac{dx^l}{d\tau} + \frac{2}{q}\,\frac{e}{c}\,\Phi_\mu + \frac{1}{F}\,\frac{\partial F}{\partial X^\mu}\right) = 0\,,$$

for a certain choice of $F$. Schouten posed the particular requirement that $\Phi_\mu$ shall appear in this expression with the coefficient 1, which also entails that:

$$q = 2\,.$$

However, we would like do without this completely, since such a demand seems to us to be in no way imperative.

One can define the curvature tensor $P^\mu_{\cdot\nu\rho\sigma}$ by way of (60) when one preserves the order of the indices in the $\Gamma^\lambda_{\mu\nu}$. By a lengthy calculation, one then finds for the curvature tensor, instead of (68), the expression:

(96) $$P = R + \frac{\varepsilon}{4}\,(q^2 + 2p - 2pq)\,X_{\rho\sigma}X^{\rho\sigma}\,.$$

On the basis of the variational principle:

$$\delta\cdot\int P\,\sqrt{|\,g\,|}\;dX^{(1)}\,\ldots\,dX^{(5)} = 0\,,$$

with the supplementary condition:

$$\delta(g_{\mu\nu}X^\mu X^\nu) = 0\,,$$

one finds the field equations:

$$K_{ik} = 0 \,, \qquad K_{i(0)} = 0 \,,$$

and, in place of (84a) and (84b), the following expressions appear:

(97a) $$K_{ik} = R_{ik} + \frac{1}{2} g_{ik} R + \frac{\varepsilon}{4}(q^2 + 2p - 2pq)?\left( X_{i}^{\cdot\nu} X_{kr} - \frac{1}{4} g_{ik} X_{rs} X^{rs} \right)$$

and:

(97b) $$K_{i(0)} = -\frac{1}{2}(q^2 + 2p - 2pq)\, X_{i\;\;;k}^{\cdot k} \,.$$

The latter follows from (82); one should observe that (84) is no longer valid. (96) implies the following condition:

(98) $$\frac{\varepsilon r^2}{2}(q^2 + 2p - 2pq) = 1 \,,$$

fro which one infers that $\varepsilon$ has the same sign as $(q^2 + 2p - 2pq)$. From (98), one can also write (97b) as:

(99) $$K_{i(0)} = \varepsilon \frac{1}{r} \frac{\sqrt{\kappa}}{c} F_{i\;\;;k}^{\cdot k} \,.$$

As a critique of the generalized Znsatz for the $\Gamma^{\lambda}_{\mu\nu}$, it must be remarked that the curvature scalar in this case is not the only actual scalar, since:

$$J = g^{\mu\nu}\, S_{\mu\rho}^{\cdot\cdot\sigma} S_{\nu\sigma}^{\cdot\cdot\rho}$$

can also come into consideration, which agrees with:

$$X_{\rho\sigma} X^{\rho\sigma} \,,$$

up to a generally non-vanishing numerical factor. An arbitrary Lagrangian function can be represented as a linear combination of $J$ and $P$. For that reason, we would like to propose that the original Ansatz $S_{\mu\nu}^{\cdot\cdot\lambda} = 0$ is the most natural one. On the other hand, the Einstein-Mayer theory can be characterized by $p = 0$, since only $a^{\mu}{}_{;k} = \gamma^{k}_{\cdot\nu}\, a^{\mu}{}_{;\nu}$ applies to it.

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