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Extensions and examples of Hertzian mechanics

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(with 10 figures in the text)

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Hertzian mechanics ([1]) represented a certain completion of the development of theoretical mechanics that sought to remove the concept of force from mechanics as a fundamental concept that was "not further explainable" and was endowed with unavoidable ambiguities due to its metaphysical connotations. Naturally, in such a "force-less" mechanics, in the interests of simplicity of expression, one would also not be able to do without its subsequent introduction when one employs it, e.g., in Hertzian mechanics, that would be by definition ([2]), so one might introduce it in the form of an arbitrarily-constructed auxiliary concept.

Such an ambition already existed before **Hertz**, namely, **H. von Helmholtz**, as well as the English physicists **J. J. Thomson**, and above all, **W. Thomson** ([3]). In contrast to his predecessors, who only occasionally advanced to a kinetic explanation for the forces (potential energy, respectively) in the context of classical mechanics, Hertz's work is remarkable for the fact that it was the first time that someone sought to systematically construct force-less mechanics on the basis of only the three fundamental concepts of space, time, and mass. For **Hertz** ([4]), the reason for that was defined by the logical incompleteness in the "usual" presentations of it, namely, at its very foundations, and in connection with that, he cited the example of centrifugal force, in particular ([5]). Now, since there was also undoubtedly a misunderstanding on the part of **Hertz** ([6]) in just that cited case, one can agree with his critics, and even to this day, moreover. However, in any event, the ambition to eliminate those ambiguities was of prime importance to **Hertz**, and that explained why he concerned himself almost exclusively with the systematic foundation of a force-less mechanics, i.e., with the "logical, or if one prefers philosophical, aspects of the subject" ([7]).

([1]) **Heinrich Hertz**, *Die Prinzipien der Mechanik*, etc., Leipzig, 1894. This work will be cited as "H. M." in what follows.

([2]) H. M., *Einleitung*, pp. 33, line 7 from the bottom, *et seq.*

([3]) For the literature, cf., *Enz. d. math. Wiss.*, Bd. IV, I, **Voss**, *Prinzipien der rationale Mechanik*.

([4]) H. M., *Einleitung*, pp. 6 to 10.

([5]) H. M., *Einleitung*, pp. 6.

([6]) Cf., an article by the author that will appear soon: "Einige Bemerkungen über die Grundlagen der Mechanik."

([7]) H. M., pp. XXVII.

However, its conception obviously offers great practical interest as a physical hypothesis that is possibly quite powerful, which also defined the main incentive for his forerunners to take that viewpoint in their own investigations. However, that is especially because neither **Hertz** himself nor anyone since then gave any sort of examples of how to use Hertz's conception of things in any special cases ([1]). The examination of it that will be presented in what follows will attempt to fill that lacuna, at least in a preliminary way.

When one maintains the same general basic principles at all ([2]) and would like to eliminate the concept of force from mechanics, naturally, one cannot operate with constraints alone. From d'Alembert's principle ([3]), the only motions that one would be able to explain in a "force-less" way would then be the ones for which the external forces would do no work, in language of ordinary mechanics. Rather, one is compelled to assume that there are "invisible" or "hidden" masses that would be "coupled" with the visible ones in some way, and everything is arranged in such a way that one would (in essence) arrive at the same differential equations for the motion of the visible masses that would result from the influence of empirically-defined forces in the usual conception of things. Now, it might seem ([4]) that such an introduction of hidden masses would indeed be a useful tool for making those differential equations more mechanically intuitive, but that it would be irrelevant to the facts themselves, since it would not change the differential equations, which is all that is really definitive for the nature of a given problem. Now, that is not, by any means, an accurate opinion, as will be shown. Namely, when we, with **Hertz**, take, e.g., a hidden cyclic motion as our basis, we will get an apparent potential energy for the visible mass only *in the first approximation*, and that will raise the very interesting question: What would a completely-rigorous implementation of Hertz's picture imply? We will see that we can, in fact, infer conclusions from it that are more or less characteristic of the assumed mechanical picture, and perhaps they can also be expressed in such a form that they offer a handle on an experimental confirmation, say in terms of natural processes. They might then be the source of the discovery of new phenomena that were not known up to now.

In addition, examples of how one has to imagine such an "interposition of invisible masses" will be given in what follows, and indeed only ones that satisfy purely-formal mechanical requirements. Those examples, whose significance for the general investigation is not to be underestimated, due to their concrete and intuitive character, already allows one to see those noteworthy consequences, but they lack any physical interest. From a physical perspective, we would also increase the problem of somehow making it plausible that an explanation exists (and especially what form it would take) for the "rigid constraints" that are assumed to exist in the physical processes of nature. With such hypotheses that can also be assumed physically, one does not have to survey the domain of the mechanics of rigid bodies, but rather that of fluid bodies, in which one would have to start with "incompressible" ([5]) fluids that are equipped with special

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([1]) Cf., on this, the *Vorwort* by **Helmholtz** in H. M., pp. XXI.

([2]) An apparent potential energy can also appear for relative motion; cf., esp., the "quadrantal pendulum" of **Thomson** and **Tait**, *Treatise*, 2nd ed., no. 322, and **A. G. Webster**, *The dynamics of particles*, etc., 2nd ed., pps. 195 and 196.

([3]) Assuming that we allow only "fixed" constraints, in the spirit of **Hertz**.

([4]) Cf., H. M., pp. XXI.

([5]) Naturally, in Hertzian mechanics, there are also no elastic bodies as original elements. Cf., on this, also H. M., pp. 41, row 11 from the top, as well as 12 and 13.

properties. The equations of constraint would be non-holonomic, and given by, e.g., the continuity equation or the analytical expression for just those special properties, respectively.

However, for the time being, we would like to restrict ourselves to *holonomic* coordinates and constraint equations in order to not be inhibited by all-too-many complications from the outset. The extension of the present investigation to the (as already suggested) more important case of non-holonomic constraint equations would not prove to be all too difficult, moreover, any less than **Hertz** himself indeed kept to that restriction in regard to the equations of coupling. By contrast, the further assumption that the constraint equations cannot include time explicitly, i.e., the constraint that is represented by them should be "fixed" will be made, in principle. In fact, **Hertz** was of the opinion that he would have to explain the "unallowed" connections ([1]) on the basis of the once-admitted elements of his mechanics (viz., space, time, mass, rigid scleronomic constraints, and the basic law) ([2]). We shall not address that question here, since the assumption of such constraint equations already seems to be an abstraction that is useful for mainly-practical purposes in ordinary mechanics but is not intrinsically based in real relationships. For us, only the attempt to explain forces will come under consideration, i.e., the chapter "Systems influenced by forces" ([3]), on the one hand, and "Cyclic motion, hidden cyclic motion, conservative systems" ([4]), on the other. Of those two cases (the actual difference between them will be given later), it is, in fact, the latter that we shall base the following discussion upon, i.e., so adiabatic cyclic systems ([5]), since we must indeed exclude external forces from consideration now.

Hertz's book is still interesting, due to the proper, very clear, and elegant form of its presentation, which is rather obvious, insofar as he considered point-systems exclusively ([6]). We will make no use of that, but rather we shall appeal to the ordinary manner of expression in mechanics throughout, which is supported, above all, on the powerful tool of Lagrange equations of the second kind.

Developing the fundamental representation in general

We imagine a point-system S with n "visible" mass-points $m_\nu = m_{\nu+1} = m_{\nu+2}$ ($\nu \equiv 1 \bmod 3$) and their rectangular coordinates x_ν, in which $\nu = 1, 2, \ldots, 3n$. They are bound by l constraint equations $\varphi_\lambda(x_1, x_2, \ldots, x_{3n}) = 0$, $\lambda = 1, 2, \ldots, l$ ([7]). The system will then have $3n - l \equiv r$ degrees of freedom, and we let $p_1, \ldots, p_r$ be r independent parameters that determine the "configuration" of the system,

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([1]) H. M., pp. 90, def. 3. The usual expression for this is: rheonomic equation of constraint.

([2]) A "guided, constrained system." H. M., pp. 200 to 307.

([3]) H. M., pp. 207 to 235.

([4]) H. M., pp. 235 to 252, pp. 252, *et seq.*

([5]) H. M., pp. 240, def. 560.

([6]) I myself arrived at an entirely-similar (if not quite as systematically constructed) presentation independently of **Hertz** on the occasion of producing a different article (these Sitzungsberichte, 1910).

([7]) Thus, the tacit assumption will be made (which is not essential for what follows) that S, as well as $\mathfrak{S}$, does not decompose into autonomous parts as a result of the constraints. The special case in which the "invisible" system $\mathfrak{S}$ is coupled by two completely-separate systems with the same or different parameters cannot be considered in more detail for the time being. That was partially accomplished by **Hertz** by using the concept of isocyclic systems. Obviously, at least one x and one ξ must also appear in any coupling equations χ_ι (relative constraints).

with its r degrees of freedom, in an entirely unique way. We then refers to them as *generalized coordinates*. They are connected to the rectangular ones in some way: $x_\nu = f_\nu(p_1, \ldots, p_r)$. Obviously, when we then transform the functions φ_λ, the equations $\varphi_\lambda = 0$ must be fulfilled for all values of $p_1, \ldots, p_r$, i.e., identically ([1]).

Let $\mathfrak{S}$ be a second system with the "invisible" point-like masses $\mathfrak{m}_\kappa = \mathfrak{m}_{\kappa+1} = \mathfrak{m}_{\kappa+2}$ and the rectangular coordinates x_κ, $\kappa = 1, 2, \ldots, 3k$. Let the constraint equations be $\psi_\sigma(\xi_1, \xi_2, \ldots, \xi_{3k}) = 0$, $\sigma = 1, 2, \ldots, s$, so the number of degrees of freedom will be: $3k - s \equiv \tau$, and generalized coordinates $\alpha_1, \ldots, \alpha_\tau$ (which are defined geometrically or analytically in some way) are defined by the transformation formulas: $\xi_\kappa = f_\kappa(\alpha_1, \ldots, \alpha_\tau)$.

Now, both systems shall be coupled to each other by i constraint equations, namely, the so-called "coupling equations," say:

$$\bar{\chi}_\iota\ (x_1, \ldots, x_{3n}, \xi_1, \xi_2, \ldots, \xi_{3k}) = \chi_\iota(p_1, \ldots, p_r, \alpha_1, \ldots, \alpha_\tau) = 0\,, \qquad \iota = 1, 2, \ldots, i.$$

If $i > \tau$ then we can determine the $\alpha_1, \ldots, \alpha_\tau$ as functions of the $p_1, \ldots, p_r$ from τ of those equations, and the remaining $i - \tau$ equations would mean conditions on the parameters p_ρ alone, i.e., by the introduction of the invisible subsystem and the coupling equations, the number degrees of freedom in the total system $\Sigma^0 \equiv S + \mathfrak{S}$ would prove to be smaller than that of the visible one, which we can exclude ([2]). The case $i = \tau$ would be admissible, but it would imply merely an increase in the masses in the visible system. Moreover, it will not be excluded in the following. We therefore let $i < \tau$ and imagine that the i equations $\chi_\iota = 0$, which must include at least one parameter p and at least i different α, have been solved for any i of the parameters, say, for the first i:

$$\alpha_1 = \Phi_1(p_1, \ldots, p_r, \alpha_{i+1}, \ldots), \qquad \ldots, \qquad \alpha_i = \Phi_i(p_1, \ldots, p_r, \alpha_{i+1}, \ldots), \tag{1}$$

in which the Φ are functions of some or all p and the remaining $\alpha_{i+1}, \ldots, \alpha_\tau$.

Now, the $r + \tau - i$ mutually-independent coordinates $p_1, \ldots, p_r, \alpha_{i+1}, \ldots, \alpha_\tau$ are obviously generalized coordinates for the total system Σ^0 with the number of degrees of freedom equal to $3n$

([1]) If one knows r such generalized coordinates p_ρ for a system of r degrees of freedom and introduces s new constraints then one can always regard $r - s$ arbitrarily-chosen parameters p_ρ as also being generalized coordinates for the new case. That is because the newly-introduced constraint equations will generally define s of the original parameters as functions of the remaining ones. That also true, e.g., for rectangular coordinates. It is likewise clear that an arbitrary substitution with non-vanishing functional determinant and the same number of independent variables will again produce generalized coordinates when the original ones were such things. From the formula above, $x_\nu = f_\nu(p_1, \ldots, p_r)$, at least one subdeterminant of degree r in the matrix of first differential quotients $\partial x / \partial p$ must be non-zero, moreover.

([2]) An example of this would be the following one: Let the visible system be two mass-points that can move freely in a plane and are coupled to each other by a massless rod, while the invisible system is a fixed rigid body that can rotate about an axis that is normal to the plane. If one of the mass-points is constrained to always coincide with one and the same invisible mass during the motion then the visible system will lose one degree of freedom. Conversely, if the latter is the invisible system then the number of coupling equations, namely, $i = 2$, will be greater than the number $r = 1$ of degrees of freedom in the other, but smaller than the number $\tau = 3$ of degrees of freedom in the invisible ones. That case is also allowable in this article.

$- l + 3k - s - i = r + \tau - i \geq r$. That is because they satisfy its constraint equations identically: Relative to the φ_λ and ψ_σ, that is indeed an assumption, but in regard to the newly-added coupling equations, it will be an immediate consequence of the elimination process that was just applied.

The kinetic energy of the total system T is composed additively from those of S and $\mathfrak{S}$, i.e., T and $\mathfrak{T}$ together: $\mathsf{T} = T + \mathfrak{T}$.

In that expression:

$$T = \tfrac{1}{2} a_{11}\, \dot{p}_1^2 + \cdots + \tfrac{1}{2} a_{rr}\, \dot{p}_r^2$$

and

$$\mathfrak{T} = \tfrac{1}{2} \beta_{11}\, \dot{\alpha}_1^2 + \cdots + \tfrac{1}{2} \beta_{\tau\tau}\, \dot{\alpha}_\tau^2$$

are homogeneous quadratic functions in the $\dot{p}_\rho$ ($\dot{\alpha}_1, \ldots, \dot{\alpha}_\tau$, respectively). The coefficients:

$$a_{11}, \ldots, a_{rr} \qquad \left(\text{e.g., } a_{11} = \sum_{\nu=1}^{3n} \left(\frac{\partial x_\nu}{\partial p_1}\right)^2 \cdot m_\nu, \text{ etc.}\right)$$

are functions of the p and the masses m_ν , while the $\beta_{11}, \ldots, \beta_{\tau\tau}$ are functions of the α and $\mathfrak{m}_\kappa$. Obviously, the α and $\dot{\alpha}$ are no longer independent of the p and $\dot{p}$, due to the coupling. However, we can replace i of the α with functions of the remaining $\tau - i \equiv \mathfrak{r}$ of the α and p by means of equations (1), and in that way, T will go to:

$$\begin{aligned}\mathsf{T} = {} & \tfrac{1}{2} a_{11}\, \dot{p}_1^2 + \cdots + \tfrac{1}{2} a_{rr}\, \dot{p}_r^2 + \tfrac{1}{2} b_{11}\, \dot{p}_1^2 + \cdots + \tfrac{1}{2} b_{rr}\, \dot{p}_r^2 \\ & + c_{11} \cdot \dot{p}_1\, \dot{\alpha}_{i+1} + \cdots + c_{r\mathfrak{r}} \cdot \dot{p}_r\, \dot{\alpha}_\tau + \tfrac{1}{2} \mathfrak{a}_{11}\, \dot{\alpha}_{i+1}^2 + \cdots + \tfrac{1}{2} \mathfrak{a}_{\mathfrak{r}\mathfrak{r}}\, \dot{\alpha}_\tau^2 ,\end{aligned}$$

in which naturally the coefficients b, c, and $\mathfrak{a}$ still include only the $\mathfrak{m}_\kappa$, but they are now functions of the p and $\alpha_{i+1}, \ldots, \alpha_\tau$. We can now also regard the part:

$$\tfrac{1}{2} b_{11}\, \dot{p}_1^2 + \cdots + c_{11} \cdot \dot{p}_1\, \dot{\alpha}_{i+1} + \cdots + \tfrac{1}{2} \mathfrak{a}_{\mathfrak{r}\mathfrak{r}}\, \dot{\alpha}_\tau^2$$

as the kinetic energy $\mathfrak{T}$ of $\mathfrak{S}$, when taken by itself, at least in a formal analytical way. In particular, the important property of any kinetic energy of a system that it is a positive-definite quadratic function in the generalized velocities for the domain of variables in question will certainly remain untouched under the transformation (1) above, and even when we regard arbitrarily-many of the newly-introduced coordinates as arbitrary constants. However, the $p_1, \ldots, p_r, \alpha_{i+1}, \ldots, \alpha_\tau$ are no longer necessarily generalized coordinates, and indeed in general, they will no longer be well-defined coordinates at all for the invisible system when it is presented by itself, since i can indeed be smaller than the number of the parameters p that actually appear in equations (1) ([1]). If i is equal to that number then that would, however, generally be the case, and then after making an ultimate choice of the $p_1, \ldots, p_{r'}, \alpha_{r'+1}, \ldots, \alpha_\tau$ as the coordinates for $\mathfrak{S}$ in place of $\alpha_1, \ldots, \alpha_{r'}, \alpha_{r'+1}, \ldots, \alpha_\tau$, which would be possible according to (1), the coupling equations will read simply:

$$p_\rho - \bar{p}_\rho = 0\,, \qquad \rho = 1, 2, \ldots, r'\,, \qquad 0 < r' \leq r\,,$$

when we suggest that those p_ρ simultaneously mean parameters of S and $\mathfrak{S}$, respectively, by an overbar ([2]).

Now, it can happen that some or all of the $\alpha_{i+1}, \ldots, \alpha_\tau$ might no longer appear in the coefficients b, c, and $\mathfrak{a}$. We shall call such coordinates "cyclic," in contrast to the remaining ones, for which we shall preserve the terminology of "parameters." If all of them appear then we will have the general case of the system that is influenced by forces, which we would like to call "acyclic." If all of then are missing then we will get the other case that **Hertz** considered before that we shall call "pure cyclic," or more briefly "cyclic." Along with them, we can also regard a "mixed cyclic" ([3]) case as the most-general one since it includes the other two as special cases.

We shall first consider a pure-cyclic system and denote the cyclic coordinates (i.e., $\alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_\tau$ here) by $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_\mathfrak{r}$. The coefficients b, c, $\mathfrak{a}$ will be functions of only $p_1, \ldots, p_r, \ldots$ then. We introduce the notations:

$$\begin{aligned}
\mathfrak{T}_1 &\equiv \tfrac{1}{2} b_{11} \cdot \dot{p}_1^2 + b_{12}\, \dot{p}_1\, \dot{p}_2 + \cdots + \tfrac{1}{2} b_{rr} \cdot \dot{p}_r^2\,, \\
\mathfrak{T}_2 &\equiv c_{11} \cdot \dot{p}_1\, \dot{\mathfrak{p}}_1 + c_{12} \cdot \dot{p}_1\, \dot{\mathfrak{p}}_2 + \cdots + c_{21} \cdot \dot{p}_2\, \dot{\mathfrak{p}}_1 + \cdots + c_{r\mathfrak{r}} \cdot \dot{p}_r\, \dot{\mathfrak{p}}_\mathfrak{r}\,, \\
\mathfrak{T}_3 &\equiv \tfrac{1}{2} \mathfrak{a}_{11} \cdot \dot{\mathfrak{p}}_1^2 + \mathfrak{a}_{12}\, \dot{\mathfrak{p}}_1\, \dot{\mathfrak{p}}_2 + \cdots + \tfrac{1}{2} \mathfrak{a}_{\mathfrak{r}\mathfrak{r}} \cdot \dot{\mathfrak{p}}_r^2\,,
\end{aligned}$$

such that $\mathfrak{T} = \mathfrak{T}_1 + \mathfrak{T}_2 + \mathfrak{T}_3$. Obviously, $\mathfrak{T}_1$ and $\mathfrak{T}_3$ have an independent meaning: They represent the kinetic energy of the invisible system when the cyclic coordinates (the parameters, respectively) are kept constant and for that reason, they are positive-definite quadratic functions in the corresponding velocities of each "configuration" of the system that is allowable.

([1]) Example: For a visible mass of finite extent that moves in a plane, the coordinates x, y of a point that is fixed in it, and a locating angle φ with respect to the horizontal direction will be introduced as the "configuration coordinates." We imagine that this angle is defined to be, in particular, the direction of a straight (or curved) slit in that mass in which a point whose coordinates are ξ, η relative to an axis-system that is fixed in the plane and belongs to a second invisible mass with the generalized coordinates ξ, η, ζ is constrained to remain. If x, y are the coordinates of a point in that slit then the coupling equation can be written in the form: $y - \eta = \tan \varphi \cdot (x - \xi)$ or $\eta = \Phi\,(x, y, \varphi, \xi)$. Here, the x, y, φ would not at all define coordinates for the invisible system, when taken by itself.

([2]) "Direct coupling," H. M., pp. 207, nos. 450 to 454.

([3]) This is somewhat more general than in **Hertz**, H. M., pp. 255, no. 602.

First approximation.

With **Hertz**, we shall now make the further assumption that the kinetic energy $\mathfrak{T}$ can be assumed to be a homogeneous quadratic function of just the $\dot{\mathfrak{p}}_1, \dot{\mathfrak{p}}_2, \ldots, \dot{\mathfrak{p}}_r$ to a sufficiently-high degree of approximation ([1]): $\mathfrak{T} \equiv \mathfrak{T}_3$, so $\mathfrak{T}_1$ and $\mathfrak{T}_1$ can be neglected in comparison to $\mathfrak{T}_3$. Since the masses $\mathfrak{m}_\kappa$ appear only linearly in the coefficients b, c, and $\mathfrak{a}$, that assumption can always be made to agree with the fact that one assumes that the masses $\mathfrak{m}_\kappa$ are as small as possible, while the "cyclic intensities" $\dot{\mathfrak{p}}_1, \dot{\mathfrak{p}}_2, \ldots, \dot{\mathfrak{p}}_r$ are sufficiently large, to an arbitrarily high degree of precision.

We now apply the Lagrange equations of the second kind to the total system Σ^0 with the kinetic energy $\mathsf{T} = T + \mathfrak{T}_3$ and obtain the two groups of equations:

$$\frac{d}{dt}\left(\frac{\partial \mathsf{T}}{\partial \dot{p}}\right) - \frac{\partial \mathsf{T}}{\partial p} = 0 \qquad (r \text{ eqs.}), \tag{I}$$

$$\frac{d}{dt}\left(\frac{\partial \mathsf{T}}{\partial \dot{\mathfrak{p}}}\right) - \frac{\partial \mathsf{T}}{\partial \mathfrak{p}} = 0 \qquad (\mathfrak{r} \text{ eqs.}), \tag{II}$$

when we drop the superfluous indices. Since:

$$\mathsf{T} = T + \mathfrak{T}_3, \qquad \frac{\partial T}{\partial \mathfrak{p}} = \frac{\partial \mathfrak{T}_3}{\partial \mathfrak{p}} \equiv 0, \qquad \frac{\partial \mathfrak{T}_3}{\partial \dot{p}} \equiv 0, \qquad \frac{\partial T}{\partial \dot{\mathfrak{p}}} \equiv 0,$$

it will then follow that:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}}\right) - \frac{\partial T}{\partial p} = \frac{\partial \mathfrak{T}_3}{\partial p} \tag{I$'$}$$

$$\frac{d}{dt}\left(\frac{\partial \mathfrak{T}}{\partial \dot{\mathfrak{p}}}\right) = 0 \qquad \text{or} \qquad \frac{\partial \mathfrak{T}}{\partial \dot{\mathfrak{p}}} = \text{const.} \tag{II$'$}$$

When the last $\mathfrak{r}$ equations are written out in detail, they will read:

$$\begin{aligned}
\mathfrak{a}_{11}\,\dot{\mathfrak{p}}_1 + \mathfrak{a}_{12}\,\dot{\mathfrak{p}}_2 + \cdots + \mathfrak{a}_{1\mathfrak{r}}\,\dot{\mathfrak{p}}_1 &= C_1\,, \\
\mathfrak{a}_{21}\,\dot{\mathfrak{p}}_1 + \mathfrak{a}_{22}\,\dot{\mathfrak{p}}_2 + \cdots + \mathfrak{a}_{2\mathfrak{r}}\,\dot{\mathfrak{p}}_1 &= C_2\,, \\
\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots \\
\mathfrak{a}_{\mathfrak{r}1}\,\dot{\mathfrak{p}}_1 + \mathfrak{a}_{\mathfrak{r}2}\,\dot{\mathfrak{p}}_2 + \cdots + \mathfrak{a}_{\mathfrak{r}\mathfrak{r}}\,\dot{\mathfrak{p}}_1 &= C_{\mathfrak{r}}\,,
\end{aligned} \tag{2}$$

in which:

$$\mathfrak{a}_{ki} = \mathfrak{a}_{ik}\,.$$

([1]) H. M., pps. 235 and 236, no. 549. No. 550 is also essential for what follows.

If we let $\mathfrak{A}_{ik}$ denote the subdeterminants of the determinant $\mathfrak{A} \equiv |\, \mathfrak{a}_{ik} \,|$, divided by that determinant $\mathfrak{A}$, then the solution to (2) will read:

$$\dot{\mathfrak{p}}_i = \mathfrak{A}_{1i} \cdot C_1 + \ldots + \mathfrak{A}_{\mathfrak{r}i} \cdot C_{\mathfrak{r}} .$$

The solution assumes that the determinant $\mathfrak{A}$ does not vanish for the domain of variability for the variables p in question. However, that is excluded as the necessary condition for the quadratic function $\mathfrak{T}_3$ to be positive-definite.

If we now introduce those expressions for the cyclic rates of change in the kinetic energy $\mathfrak{T}_3$ then $\mathfrak{T}_3$ will take the form of a homogeneous quadratic function $\bar{\mathfrak{T}}_3$ in the constants C_i, $i = 1, 2, \ldots, \mathfrak{r}$, with coefficients that are once more functions of the $p_1, \ldots, p_r$. However, the dependency of the transformed function $\bar{\mathfrak{T}}_3$ on the parameters is different from the one that was at the basis for the differentiation in (I′), and we will no longer have $\frac{\partial \bar{\mathfrak{T}}_3}{\partial p} = \frac{\partial \mathfrak{T}_3}{\partial p}$, since the $\mathfrak{A}_{ik}$ are also differentiated with respect to the p when we construct $\frac{\partial \bar{\mathfrak{T}}_3}{\partial p}$. Rather, with the use of Euler's theorem for homogeneous functions, we will get:

$$\bar{\mathfrak{T}}_3 = \tfrac{1}{2}\sum_{i=1}^{\mathfrak{r}} \dot{\mathfrak{p}}_i \frac{\partial \mathfrak{T}}{\partial \dot{\mathfrak{p}}_i} = \tfrac{1}{2}\sum_{i=1}^{\mathfrak{r}} \dot{\mathfrak{p}}_i C_i = \tfrac{1}{2}\sum_{i=1}^{\mathfrak{r}} \sum_{k=1}^{\mathfrak{r}} \mathfrak{A}_{ik} C_i C_k , \tag{3}$$

and furthermore:

$$\begin{aligned}\frac{\partial \bar{\mathfrak{T}}_3}{\partial p} &= \frac{\partial \mathfrak{T}_3}{\partial p} + \sum_{i=1}^{\mathfrak{r}} \frac{\partial \mathfrak{T}_3}{\partial \dot{\mathfrak{p}}_i} \cdot \frac{\partial \dot{\mathfrak{p}}_i}{\partial p} = \frac{\partial \mathfrak{T}_3}{\partial p} + \sum_{i=1}^{\mathfrak{r}} C_i \cdot \frac{\partial \dot{\mathfrak{p}}_i}{\partial p} = \frac{\partial \mathfrak{T}_3}{\partial p} + \sum_{i=1}^{\mathfrak{r}} C_i \sum_{k=1}^{\mathfrak{r}} C_k \frac{\partial \mathfrak{A}_{ik}}{\partial p} \\ &= \frac{\partial \mathfrak{T}_3}{\partial p} + \sum_{i=1}^{\mathfrak{r}} \sum_{k=1}^{\mathfrak{r}} C_i C_k \frac{\partial \mathfrak{A}_{ik}}{\partial p} = \frac{\partial \mathfrak{T}_3}{\partial p} + 2\,\frac{\partial \bar{\mathfrak{T}}_3}{\partial p},\end{aligned}$$

so ([1]):

$$\frac{\partial \mathfrak{T}_3}{\partial p} = -\,\frac{\partial \bar{\mathfrak{T}}_3}{\partial p} .$$

The r equations (I′) can then be written in the form:

([1]) Formally, one will find this transformation, as well as the one that will be used later (pp. 22, *et seq.*) in **Webster**, *The dynamics of particles*, *etc.*, 2nd ed., pp. 176, which was first used by **Routh** and **Helmholtz**, and for an entirely-similar purpose by the latter, moreover. Our conception of it differs from his at a fundamental level, and his roughly overlaps with the "ignoration of coordinates" that was proposed by the Englishman **W. Thomson**, by the fundamental separation of visible and invisible masses.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}}\right) - \frac{\partial T}{\partial p} = -\frac{\partial \bar{\mathfrak{T}}_3}{\partial p}.$$

The T in that is the kinetic energy of the visible system S, but $\bar{\mathfrak{T}}_3$ includes only constant quantities in addition to the parameters p.

With that, we have obtained the equations of motion for Σ^0 in a form whose left-hand side coincides with the left-hand side of the Lagrange differential equations for the system S, as we see that they are the same equations that would be true for the *free* visible system S in the presence of a potential energy $\bar{\mathfrak{T}}_3$ in the ordinary conception of things. Initially, we cannot decide between the two pictures at all since the "invisible" system, along with the couplings, is not accessible to observation.

The next question to ask will be: Can every potential function in ordinary mechanics be explained in this "kinetic" way, or in other words, can a cyclic system that simulates each such function be constructed? We can answer that question *in the affirmative* for a very general class of functions, but with a certain restriction ([1]). However, as one sees immediately, it is not possible to determine the cyclic system uniquely. That is because we do not even know which constants in $U = \bar{\mathfrak{T}}_3$ should be regarded as constant momenta C_i, i.e., whether an empirical potential function is generated by a *monocyclic* or *polycyclic* system. However, that would be the only way that we could produce the function $\mathfrak{T}_3$, which now belongs to infinitely-many systems with the same approximate expression for their kinetic energy. We could proceed systematically with its determination only in special cases where more details about the possible cyclic systems would be given from the outset on physical grounds. If, on the one hand, the status of Hertz's picture as a principle for research seems to be compromised by its all-too-pervasive arbitrariness and indeterminacy then, on the other hand, it is for just that reason that it enables one to subsume a good number of *natural* phenomena for which a basis for their explanation is lacking in ordinary mechanics. Nonetheless, it excludes cases from consideration that are allowable in ordinary mechanics, but do not correspond to any natural processes, and in an entirely well-defined way ([2]).

One can then make, e.g., the following remark: From our explanation of the potential function as the kinetic energy of hidden masses, it can assume also assume only *positive* values, including zero, while its sign remains otherwise completely arbitrary. This will also be verified in the examples in a remarkable way.

Before that, let us briefly treat an important special case in general, namely, that of the monocyclic system, i.e., one with a single cyclic coordinate $\mathfrak{p}$. To the same order of approximation as with polycycles, we have:

([1]) Cf., on this, **Hertz**, *Mech.*, pp. 44.

([2]) Cf., H. M., pps. 2 to 3, 23, 42, and pp. 284, no. 602. See also pp. 21 of this treatise. In addition, the question remains open of the dynamical explanation for those forces that are indeed functions of the coordinates, but not the negative derivatives of a potential functions.

$$\mathsf{T} = T + \mathfrak{T}_3 = \tfrac{1}{2}\, a_{11}\, \dot{p}_1^2 + \cdots + \tfrac{1}{2}\, a_{rr}\, \dot{p}_r^2 + \tfrac{1}{2}\, \mathfrak{a}\, \dot{\mathfrak{p}}^2 ,$$

in which $\mathfrak{a}$ is once more regarded as a function of the parameters $p_1, \ldots, p_r$. In an entirely-analogous way, we will obtain the r equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}}\right) - \frac{\partial T}{\partial p} = \frac{\partial \mathfrak{T}_3}{\partial p} = \frac{1}{2}\frac{\partial \mathfrak{a}}{\partial p}\cdot \dot{\mathfrak{p}}^2 ,$$

and in addition, an equation:

$$\frac{\partial \mathfrak{T}}{\partial \dot{\mathfrak{p}}} \equiv \mathfrak{a}\, \dot{\mathfrak{p}} = C ,$$

so

$$\dot{\mathfrak{p}} = \frac{C}{\mathfrak{a}} .$$

Upon substituting that in the previous one, it will follow that:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}}\right) - \frac{\partial T}{\partial p} = \frac{1}{2}\frac{\partial \mathfrak{a}}{\partial p}\cdot \frac{C^2}{\mathfrak{a}^2} = - \frac{\partial}{\partial p}\left(\frac{1}{2}\frac{C^2}{\mathfrak{a}}\right),$$

so

$$\frac{1}{2}\frac{C^2}{\mathfrak{a}} = \bar{\bar{\mathfrak{T}}}_3 ,$$

which can also be obtained directly:

$$\bar{\bar{\mathfrak{T}}}_3 = \frac{1}{2}\mathfrak{a}\cdot \frac{C^2}{\mathfrak{a}^2} = \frac{1}{2}\frac{C^2}{\mathfrak{a}} .$$

It can be useful in the applications to state that result in the form: The kinetic energy $\tfrac{1}{2}\mathfrak{a}\,\dot{\mathfrak{p}}^2$ corresponds to the apparent potential $\dfrac{1}{2}\dfrac{C^2}{\mathfrak{a}}$, and conversely.

For the examples, we shall initially assume only monocyclic systems for which the visible one is “one-parameter,” in addition, i.e., it depends upon only a single coordinate p. One can obtain any number of such things by the various models for a centrifugal regulator ([1]). If we select any one of them – say, the one in the accompanying diagram (Fig. 1) – then we can easily follow through the general line of reasoning.

([1]) As I noticed only later, some examples of an entirely-similar nature that touch upon this principle were already given by **Brill**: *Vorles. z. Einf. i. d. Mech. raumerf. Massen*. Cf., moreover, **Boltzmann**, *Vorles. über Maxwell's Theorie d. Elekt., etc.*, Bd. I.

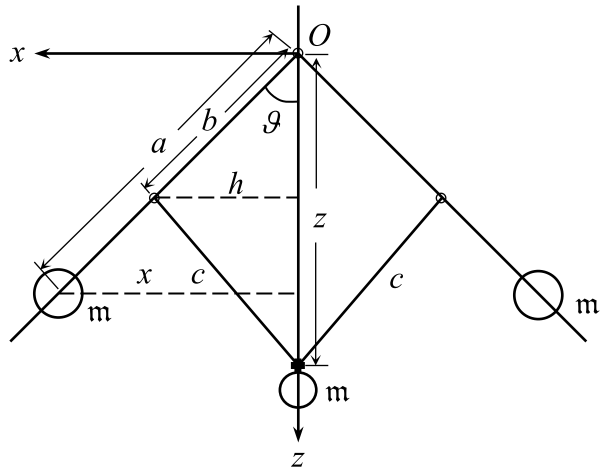


Figure 1.

The constraint equations of the visible system, which consists of the point-like mass m, are $x = 0$, $y = 0$, while z is a general coordinate. The invisible system has only one constraint equation: $x^2 + y^2 + z^2 - a^2 = 0$. General coordinates are ϑ and the angle of rotation φ around the z-axis. The rods c establish the coupling between both systems, and since:

$$h = b \sin \vartheta$$

and

$$h\,z = \tfrac{1}{2}\sqrt{[(b+c)^2 - z^2][z^2 - (b-c)^2]}\,,$$

the coupling equation will read:

$$4b^2 \sin^2 \vartheta \cdot z^2 = [(b+c)^2 - z^2][z^2 - (b-c)^2]\ .$$

Corresponding to equations (1), that will imply ϑ as a function of z : $\vartheta = f(z)$.

When we assume that the connecting rods a, b, c are massless, the expressions for the kinetic energies are ([1]):

$$T = \tfrac{1}{2} m \cdot \dot{z}^2$$

for the visible system and:

$$\mathfrak{T} = \mathfrak{m}\, a^2\, \dot{\vartheta}^2 + \mathfrak{m}\, a^2 \sin^2 \vartheta \cdot \dot{\varphi}^2$$

for the invisible one, and in $\mathsf{T} = T + \mathfrak{T}$, the ϑ would now have to be replaced by z :

$$\mathsf{T} = \tfrac{1}{2}\mathfrak{m}\,\dot{z}^2 + \mathfrak{m}\,a^2 \cdot f'(z)^2 \cdot \dot{z}^2 + \mathfrak{m}\,a^2 \cdot \frac{[(b+c)^2 - z^2][z^2 - (b-c)^2]}{4b^2 z^2} \cdot \dot{\varphi}^2\ .$$

([1]) The consideration of those masses will require no essential alterations, see pp. 33.

We see that φ is a cyclic coordinate, as we can choose it to be large enough, while $\mathfrak{m}$ is small enough that:

$$\mathfrak{T}_1 = \mathfrak{m}\, a^2 \cdot f'(z)^2 \cdot \dot{z}^2$$

can be neglected. We will then have:

$$\mathsf{T} = T + \mathfrak{T}_3 = \tfrac{1}{2} m \dot{z}^2 + \frac{1}{2}\frac{\mathfrak{m}\, a^2}{2b^2 z^2}[(b+x)^2 - z^2][z^2 - (b-c)^2]\cdot \dot{\varphi}^2 = \tfrac{1}{2} m \dot{z}^2 + \tfrac{1}{2}\mathfrak{a}\cdot\dot{\varphi}^2 ,$$

in which:

$$\mathfrak{a} \equiv \frac{\mathfrak{m}\, a^2}{2b^2 z^2}[(b+x)^2 - z^2][z^2 - (b-c)^2] .$$

From what was said before, we can then obtain the potential energy in the form of:

$$\bar{\mathfrak{T}}_3 = \frac{1}{2}\cdot\frac{C^2}{\mathfrak{a}} .$$

It can assume only positive values, which would follow from the structure of the function $\mathfrak{a}$.

In order to have yet another example, we imagine two massive balls $\mathfrak{m}$ that move along a smooth rod, and each of them is coupled to m by a massless inextensible string of length l (see Fig. 2). The cyclic subsystem shall again rotate around the z-axis, and m shall be constrained to remain along it. Here, as in all of the examples that will be discussed, friction is excluded. The constraint equations for the masses $\mathfrak{m}$ are then: z = const. = l, when the z-points positive upwards, and O is at a distance of l from A and defines the coordinate origin. The distance x from the balls to the axis and the angle of rotation φ around the z-axis will serve as general coordinates. The constraint equations for m are once more $x = 0$, $y = 0$. z, when measured from O, is to be used as a general parameter, with no further discussion. The coupling that is defined by the string has the equation $z = x$. Now, one has:

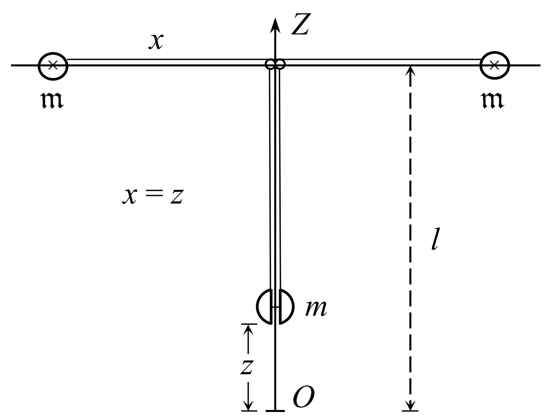


Figure 2.

$$T = \tfrac{1}{2} m \dot{z}^2 , \qquad \mathfrak{T} = \mathfrak{m}\, \dot{x}^2 + \mathfrak{m}\, x^2\, \dot{\varphi}^2 = \mathfrak{m}\, \dot{z}^2 + \mathfrak{m}\, z^2 \cdot \dot{\varphi}^2 ,$$

so the visible potential energy will be:

$$\bar{\mathfrak{T}}_3 = \frac{1}{4}\frac{C^2}{\mathfrak{m}\, z^2} ,$$

or the force that acts upon m in the positive z-direction will be:

$$Z = -\frac{\partial \overline{\mathfrak{T}}_3}{\partial z} = \frac{1}{4}\frac{C^2}{\mathfrak{m}}\cdot\frac{1}{z^3}.$$

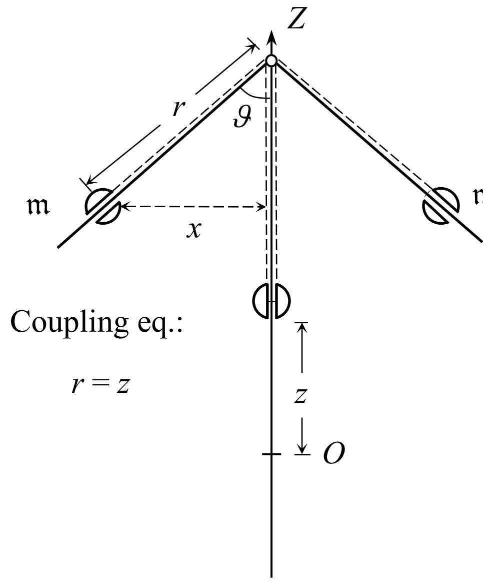


Figure 3.

With a slight generalization, we will get (see Fig. 3):

$$T = \tfrac{1}{2}\, m\dot{z}^2, \qquad \mathfrak{T} = \tfrac{1}{2}\, 2\mathfrak{m}\cdot(\dot{r}^2 + r^2\sin^2\vartheta\cdot\dot{\varphi}^2),$$

for fixed ϑ, with the coupling equation $r = z$, so:

$$\mathsf{T} = \tfrac{1}{2}\, m\dot{z}^2 + \mathfrak{m}\dot{z}^2 + \mathfrak{m}z^2\sin^2\vartheta\cdot\dot{\varphi}^2,$$

or approximately:

$$\mathsf{T} = \tfrac{1}{2}\, m\dot{z}^2 + \mathfrak{m}z^2\sin^2\vartheta\cdot\dot{\varphi}^2.$$

If ϑ were not fixed, but freely-varying, then the invisible system would have $\tau = 3$ degrees of freedom, and its kinetic energy reads:

$$\mathfrak{T} = \tfrac{1}{2}\, 2\mathfrak{m}\cdot(\dot{r}^2 + r^2\sin^2\vartheta\cdot\dot{\varphi}^2 + r^2\cdot\dot{\vartheta}^2).$$

However, not all $\alpha_{i+1}, \ldots, \alpha_\tau$ (i.e., ϑ and φ here) will be cyclic coordinates then (namely, ϑ is not), and we would be dealing the mixed-cyclic case.

In the examples up to now, $\mathfrak{T}_3$ had the form:

$$\mathfrak{T}_3 = \tfrac{1}{2}\,\mathfrak{m}\,x^2\dot{\varphi}^2,$$

when we denote the distance from $\mathfrak{m}$ to the rotational axis by x in each case. However, that is also true in general since $\mathfrak{T}_3$ does indeed mean the rigorous expression for the kinetic energy of the subsystem $\mathfrak{S}$ for fixed parameters. Thus, we will also always have:

$$\bar{\mathfrak{T}}_3 = \frac{1}{2}\frac{C^2}{\mathfrak{m}\,x^2} = \gamma_0\cdot\frac{1}{x^2},$$

in which γ_0 is an essentially-positive constant.

We can make advantageous use of that fact when we would now like to find, conversely, a monocyclic system for a given potential function $U(z)$ in an arbitrary way, and indeed in this way. We set:

$$U(z) = \gamma_0\cdot\frac{1}{x^2},$$

and must now actually realize the connection between x and z that is established in that way by construction. Moreover, the constraint equations of $\mathfrak{S}$, or also the couplings, can still be given as arbitrary then. If we start from, e.g., a well-defined coupling then that would imply the adaptability of the construction to different potential functions $U(z)$ due to the arbitrariness in the constraint equations that is left open at a suitable point in it. We shall try to determine, e.g., the curve $Z = f(x)$ that lies in a meridian plane and to which the mass m is constrained in such a way that the given function $U(z)$ will arise from $\gamma_0\cdot\frac{1}{x^2}$ by means of the coupling with the equation $Z - Z_0 + x - x_0 = z$ (let Z_0, x_0 be the initial values that correspond to the value $z = 0$).

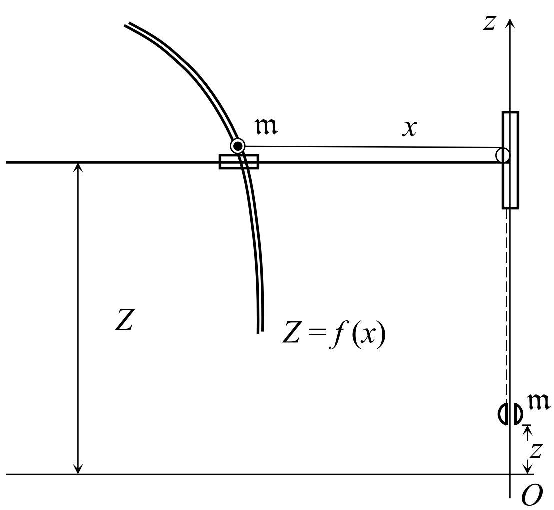


Figure 4.

One can also drop the string and establish m directly on the sleeve.

x can be used as the general parameter for $\mathfrak{m}$.

If we solve the equation:

$$\gamma_0\cdot\frac{1}{x^2} = U(z)$$

for z, $z = F(x)$, then the determination of Z as a function of x with the desired property will already follow from the coupling relation above. However, the reality of the curve $Z = f(x)$ must clearly be proved here, and in order to do that, it is now essential to assume that $U(z)$, as a kinetic energy, can assume only positive values, in addition to zero. Then and only then will it always give real values of x for *real* values of z as roots of the equation:

$$\gamma_0 \cdot \frac{1}{x^2} = U(z) ,$$

and even positive values, which likewise come under consideration.

Now, we still have to further show that a (positive or negative) real value of Z will be defined by the coupling equation for that associated pair of values (z, x). However, that is guaranteed by its form above.

We can just as well decide upon a suitable constraint equation once and for all and leave the appropriate form of coupling undetermined (see Fig. 5).

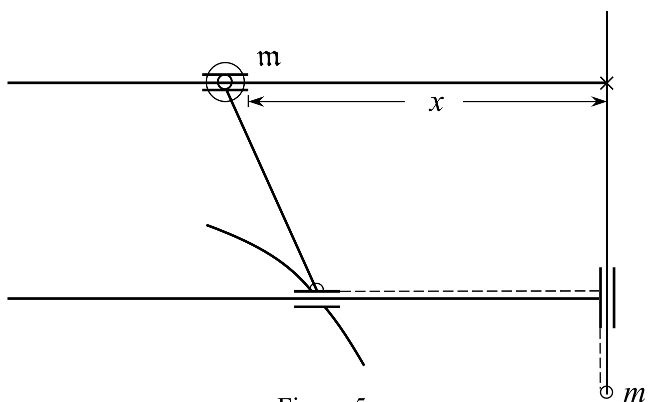


Figure 5.

Supported by this general argument, which is related to a question that **Hertz** posed ([1]), one can also succeed in constructing a model for the important case of attraction that is inversely proportional to the square of the distance $U(r) = -\gamma / r + h$ [$U(r) = \gamma / r + h$ for repulsion], $\gamma > 0$, with the center of attraction at the point $r = 0$ by determining a curve $Z = f(x)$. In that way, it would once more be the case that one would need to have $U(z) \geq 0$ since the otherwise-arbitrary addition of the here-necessary positive constant h would become essential once it is chosen to be fixed, and that would also imply a lower limit for the positive values of r. That will become especially clear when one would like to achieve an actual implementation along a somewhat-different path. In the equation:

$$\gamma_0 \cdot \frac{1}{x^2} = -\frac{\gamma}{r} + h ,$$

or

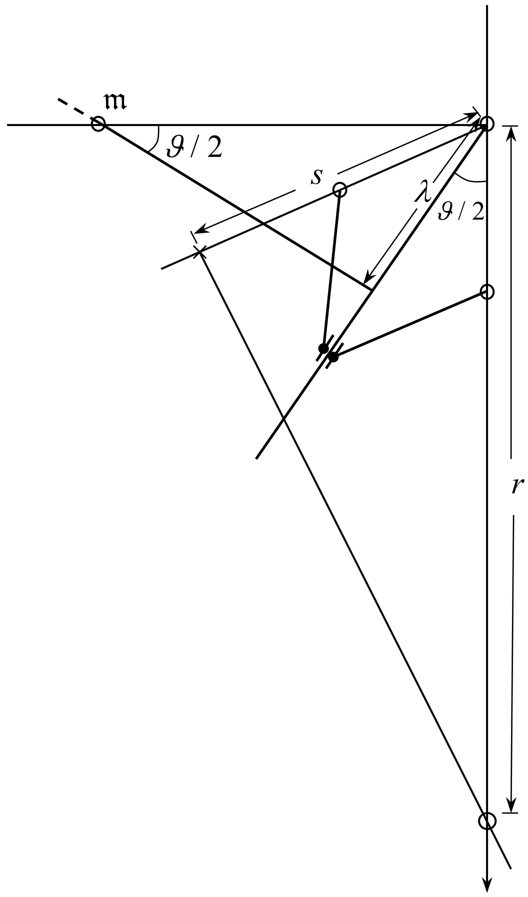
$$x^2 \cdot \left(r - \frac{\gamma}{h} \right) = \frac{\gamma_0}{h} \cdot r ,$$

γ / h and γ_0 / h must have the dimensions:

([1]) H. M., pp. 44, row 20 from the top, *et seq.*

$$\left[\frac{\gamma}{h}\right] = \text{cm}, \qquad \left[\frac{\gamma_0}{h}\right] = \text{cm}^2,$$

which would also follow immediately from the meaning of the individual quantities.



(For attraction!)

Figure 6.

We then set $\gamma / h = s$, $\gamma_0 / h = l^2$ and will then have:

$$x^2 (r - s) = l^2 \cdot r$$

or

$$x = l\sqrt{\frac{r}{r-s}} = l\sqrt{\frac{1}{1-\frac{s}{r}}},$$

in which we can only ascribe any meaning to the absolute value of the square root. As long as $s \leq r$, s / r can be set equal to cos ϑ:

$$\frac{s}{r} = \cos\,\vartheta,$$

and when $\lambda = l / \sqrt{2}$, it will then follow that:

$$x = \lambda \frac{1}{\sin \vartheta / 2} .$$

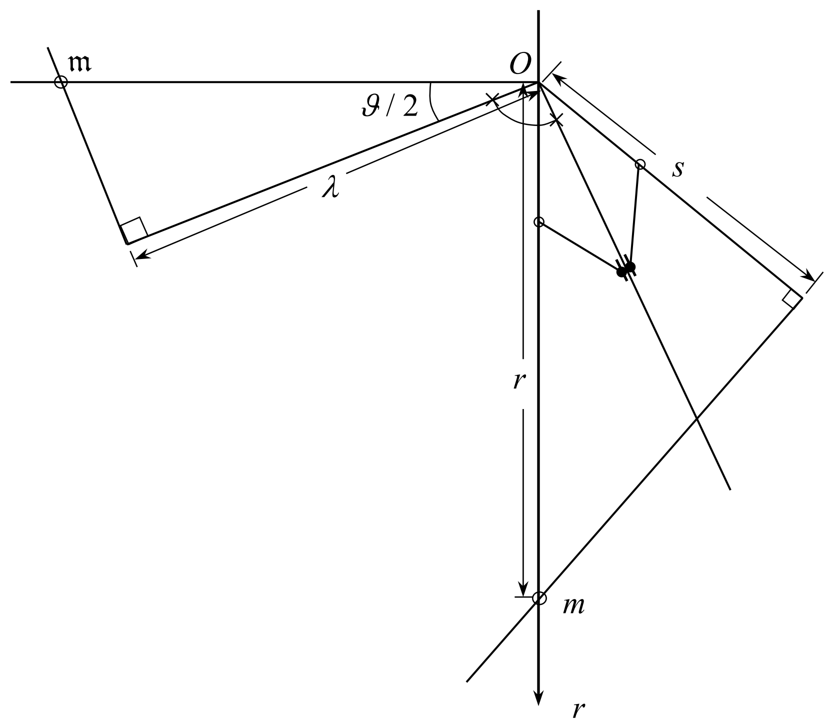


Figure 7.

For repulsion, one has:

$$r \geq s , \qquad \frac{s}{r} = \cos \vartheta , \quad x = \lambda \frac{1}{\cos \vartheta / 2} ,$$

with the same upper limit for r.

If one sets $h = 0$ here then the construction can be simplified considerably (Fig. 8).

In the case of attraction, as h decreases indefinitely, s, as well as r, will become infinitely large, and at the same time, λ, as well as x, which is the distance from 𝔪 to the axis.

One can also make **Hertz**'s theorems on adiabatic cyclic systems (H. M., pp. 242) more intuitive with the examples that were given. We have generally only given such things for $r = 1$ and $\mathfrak{r} = 1$. One can easily produce models for multiparametric or polycyclic systems by generalizing the second example on pp. 12.

If we attach several mutually-independent balls to each of the two rods, instead of just one of them, then all of those 𝔪 will correspond to one and the same cyclic coordinate φ, while the various strings can be coupled by just as many independent parameters of the visible system. However, if we again let several such horizontal rods a rotate around the same axis b independently of each other then we will obtain just as many independent cyclic coordinates, and we can also combine both cases arbitrarily by a variety of couplings.

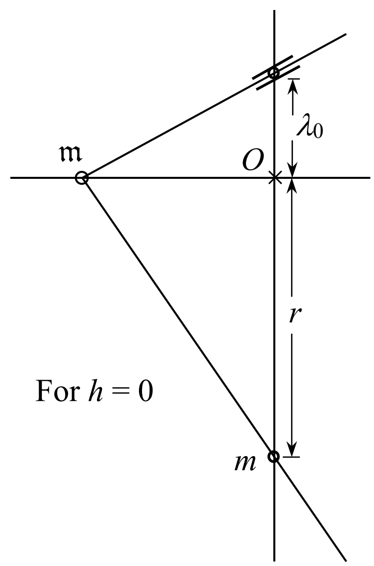


Figure 8.

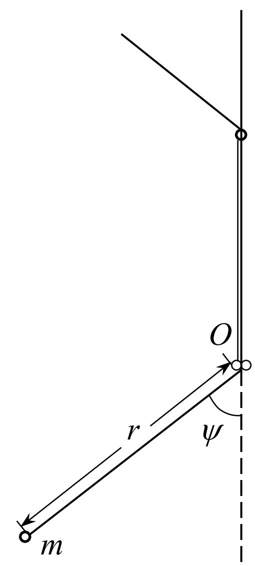


Figure 9.

If we now treat central motion in space ($r = 3$) then our given construction will already suffice for one-dimensional motion, moreover, when we merely cease to demand that the mass *m* must slide along the *z*-axis. The center of rotation is chosen most simply such that $r = 0$ for $x = 0$, e.g., by means of rolling.

Rigorous formulation.

The potential energy of ordinary mechanics can then be explained in terms of the kinetic energy of hidden masses, as in the foregoing, but as we have likewise remarked at the time, only in the first approximation. Now, from the suggestions that were made in the introduction, it is just that fact that imparts a special interest to such an attempt at explanation, and above all, to the one that is the most developed, namely, Hertz's mechanics. That is because it follows from that theory that, on the one hand, strictly speaking, there are no motions at all in Hertzian mechanics that can be represented in the usual way by a potential function. On the other hand, when we assert the Hertzian standpoint, so we regard such potential motions as merely approximation to reality, the question will arise: What would follow from a rigorous consideration? Such conclusions would need to have a new significance that would go beyond the scope of the "first picture," and the next question would be: Do our experiments or observations given a reference point for those consequences in any way?

Part One

In what follows, we shall then start from the rigorous expression for the kinetic energy $\mathsf{T} = T + \mathfrak{T}$, where in which $\mathfrak{T}$ is equal to $\mathfrak{T} = \mathfrak{T}_1 + \mathfrak{T}_2 + \mathfrak{T}_3$, in general. We shall initially assume that $\mathfrak{T}_2$ is identically zero, or in other words, that the cyclic coordinates should be *orthogonal* to the parameters. In that form, the assumption will also take on an immediate geometric meaning in special cases. We can then combine the terms T and $\mathfrak{T}_1$ in the kinetic energy T that have the same $\dot{p}$ pairwise and introduce the notation $\alpha_{\rho\mu} = a_{\rho\mu} + b_{\rho\mu}$. In that way, this case will be reduced to the previous one, in which it is not the kinetic energy T of the visible system that appears on the left-hand side of the Lagrange equations, as would otherwise be essential for our considerations, but the energy $T + \mathfrak{T}_1$. Therefore, its motion can actually result as if the kinetic energy T were increased by a certain amount $\mathfrak{T}_1$. Since $\mathfrak{T}_1$ is always present (cf., H. M., pp. 236, no. 550), and since it is a more-rigorous expression for the *vis viva* of $\mathfrak{S}$ for fixed cyclic coordinates, it is also greater than zero, and for the parameter velocities that are accessible to observation, it would necessarily seem to be an increase in the mass in the observed increase in kinetic energy to someone whose knows nothing about the existence of the subsystem $\mathfrak{S}$, assuming that he has established the empirically-determined analytical formula for the forces (potential, respectively), which would give him no reason to leave the domain of applicability of the first picture ([1]). The further evaluation of that situation would depend very much upon the sort of influence that the observer would have on the process in question. If we assume that he can still detect the moving visible masses in some way (see footnote) by relying upon observations alone then he would already see from two exact determinations of them from the $\dot{p}$ and the nature of the forces that he assumed the remarkable fact that he would obtain masses from the second one that were different from the masses that were obtained from first one. We can see that most simply with a one-parameter system S with:

$$T = \tfrac{1}{2} a\, \dot{p}^2 \qquad \text{and} \qquad \mathfrak{T}_1 = \tfrac{1}{2} b\, \dot{p}^2 .$$

a and b are generally functions of p and a, moreover, so that is a function that is known completely to the observer and whose non-geometric constants mean the masses or concepts that are derived from them, such as moments of inertia, etc. He can infer the magnitude of the *vis viva* of the visible system from the law of conservation of energy $T + \mathfrak{T}_1 + \bar{\mathfrak{T}}_3 = \text{const.}$, but since he is ignorant of the cyclic subsystem, he would incorrectly infer it to be $T + \mathfrak{T}_1$, rather than just the amount T, which would seem to be correct from our standpoint. However, he would once more calculate the masses from that incorrect magnitude for the *vis viva* $\tfrac{1}{2}(a+b)\cdot\dot{p}^2$ in precisely the

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([1]) We understand the term "ordinary mechanics" or the "first picture" to mean a theory of mechanics that recognizes space, time, mass, and force as its basic concepts, but does not define them. Newton's fundamental law and d'Alembert's principle (or something similar to it) are valid as actual facts of experience. We must, and will accordingly, actually determine force and mass in concrete cases without going into the details of the epistemological complexities of such an assumption, assuming they nonetheless make some tools available to us. In regard to mass, that is also true for Hertz's mechanics.

same way that we would from the magnitude T, and that is why he would necessarily get a mass that depends upon the configuration of the system since b is a completely different function of the constants and the p. For multiparameter systems S, it must also generally prove to depend upon the velocity ([1]).

In that case, he would already arrive at a decision between rest masses and moving masses. It would impose itself upon him even more clearly when he is able to introduce external forces on the system S in such a way that it would be in equilibrium and at rest in a certain configuration. If we assert the standpoint that was given in the footnote on pp. 19, viz., that he can actually determine the forces that produce that equilibrium, then he would get precisely the values for them that would correspond to the right-hand sides of the Lagrange equations, i.e., the $-\frac{\partial \bar{\mathfrak{T}}_3}{\partial p}$. That is because if the total system Σ^0 is in motion then from the customary conception of those equations for Σ^0, namely:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}}\right) - \frac{\partial T}{\partial p} = -\frac{d}{dt}\left(\frac{\partial \mathfrak{T}_1}{\partial \dot{p}}\right) + \frac{\partial \mathfrak{T}_1}{\partial p} - \frac{\partial \bar{\mathfrak{T}}_3}{\partial p},$$

the right-hand sides will, however, represent the forces that are exerted upon the visible system S by its coupling with $\mathfrak{S}$. For the special case of rest under the influence of well-defined, newly-added forces P, it will simply follow for them that:

$$P = +\frac{\partial \bar{\mathfrak{T}}_3}{\partial p},$$

([1]) We can easily see that in the following example: Let:

$$T = \tfrac{1}{2} a_{11}\,\dot{p}_1^2 + \tfrac{1}{2} a_{22}\,\dot{p}_2^2, \quad \mathfrak{T}_1 = \tfrac{1}{2} b_{11}\,\dot{p}_1^2 + b_{12}\,\dot{p}_1\,\dot{p}_2 + \tfrac{1}{2} b_{22}\,\dot{p}_2^2,$$

so

$$T + \mathfrak{T}_1 = \tfrac{1}{2}\alpha_{11}\,\dot{p}_1^2 + b_{12}\,\dot{p}_1\,\dot{p}_2 + \tfrac{1}{2}\alpha_{22}\,\dot{p}_2^2.$$

Based upon the structure of T alone, that total amount for the *vis viva* will give:

$$T + \mathfrak{T}_1 = \tfrac{1}{2}\left[\alpha_{11} + 2b_{12}\frac{\dot{p}_2}{\dot{p}_1}\right]\dot{p}_1^2 + \tfrac{1}{2}\alpha_{22}\,\dot{p}_2^2,$$

corresponding to the combination, or:

$$= \tfrac{1}{2}\alpha_{11}\,\dot{p}_1^2 + \tfrac{1}{2}\left[\alpha_{22} + 2b_{12}\frac{\dot{p}_1}{\dot{p}_2}\right]\dot{p}_2^2,$$

or also:

$$= \tfrac{1}{2}\left[\alpha_{11} + b_{12}\frac{\dot{p}_2}{\dot{p}_1}\right]\dot{p}_1^2 + \tfrac{1}{2}\left[\alpha_{22} + b_{12}\frac{\dot{p}_1}{\dot{p}_2}\right]\dot{p}_2^2.$$

These are masses that apparently depend upon the ratio of the velocities $\dot{p}_1 / \dot{p}_2$.

since T, as well as $\mathfrak{T}_1$, is equal to zero them. (Naturally, the p must first be replaced with the values that correspond to the selected "configuration" after the differentiation.) It will now occur to him that he has no grounds for assuming that there is any variation of the forces in the case of motion. However, since there are, in fact, other forces at work for that same "configuration," when considered from our standpoint, the only way around that for him that still remains is the assumption that the mass varies with the configuration and the state of motion.

Furthermore, he must also be able to decide between "longitudinal" and "transverse" masses, as we can see immediately in the example of planar central motion. The kinetic energy T is:

$$\mathsf{T} = T + \mathfrak{T}_1 + \mathfrak{T}_3 = \tfrac{1}{2} m(\dot{r}^2 + r^2 \dot{\psi}^2) + \tfrac{1}{2} \mathfrak{m}\, f(r) \cdot \dot{r}^2 + \tfrac{1}{2} \mathfrak{m} \cdot \Phi(r) \cdot \dot{\varphi}^2 ,$$

in which $f(r)$ proves to vary according to the special type of subsystem $\mathfrak{S}$ that is employed, while $\Phi(r)$ must satisfy the condition that:

$$\frac{1}{2} \frac{C^2}{\mathfrak{m}\,\Phi(r)} = -\frac{\gamma}{r} + h = \bar{\mathfrak{T}}_3 .$$

If we assume a circular motion, so $\dot{r} = 0$, then $\mathfrak{T}_1$ will drop out, and the equations of motion will read:

$$r = \text{const.}, \qquad m r^2 \dot{\psi}^3 = 0 ,$$

so

$$\dot{\psi} = 0 , \qquad \psi = \text{const.},$$

and

$$- m r \dot{\psi}^2 = -\frac{\partial \mathfrak{T}_3}{\partial r} = -\frac{\gamma}{r^2} ,$$

or

$$m r \dot{\psi}^2 = \frac{\gamma}{r^2} .$$

That equation determines the centripetal acceleration, and that would give the "natural" mass m as the "transverse" mass. However, if one assumes that $\dot{\psi} = 0$ for the total motion then it will follow that:

$$\mathsf{T} = \tfrac{1}{2} m \dot{r}^2 + \tfrac{1}{2} \mathfrak{m} \cdot f(r) \cdot \dot{r}^2 + \tfrac{1}{2} \mathfrak{m} \cdot \Phi(r) \cdot \dot{\varphi}^2 ,$$

and the energy equation will become:

$$\tfrac{1}{2}[m + \mathfrak{m} \cdot f(r)] \dot{r}^2 - \frac{\gamma}{r} + h = \text{const.},$$

and if he considers just the coefficient of $\dot{r}$ to be a mass then that would show that it depends upon the configuration by way of the increase $\mathfrak{m} \cdot f(r)$, and indeed, the “longitudinal” mass, which differs from the transverse one that was previously obtained. Here as well, he would assume that same force would be in effect for the second case that applied to the first case when the mass m is at a distance of r from the center of attraction.

It will follow from our argument with no further discussion that such a difference between the masses will, in fact, emerge clearly for large parameter velocities $\dot{p}$. In fact, one would be inclined to adopt that conceptual picture mainly in phenomena in the electrical (optical, resp.) context, which take place with very large velocities.

If one ponders the fact that the modern theory of electricity is based upon equations to which **Maxwell** arrived by brilliantly drawing upon dynamical models (cf., on this, e.g., **Boltzmann**, *Vorlesungen über Maxwell’s Theorie*, etc.), as well as its great analogy with the hydrodynamical equations of vortex motion, then perhaps that might give expression to the hope that we have before us in the Hertzian picture the foundation for an explanation that would subsume a great many phenomena that might also possibly allow us to solve the “relativistic” problems of that domain, which are already very relevant to mechanics, in a natural way ([1]).

Yet another remark needs to be made. The law of conservation of energy is indeed true for the total system Σ^0 in the form:

$$T + \mathfrak{T}_1 + \bar{\mathfrak{T}}_3 = \text{const.},$$

but not for the sum of kinetic and potential energy $T + \bar{\mathfrak{T}}_3$ of the visible ones. It would then be apparently a non-conservative system, and indeed forever, strictly speaking (cf., also the examples) ([2]).

Second Part.

Nonetheless, our previous considerations in regard to the apparent variability of masses are not meant to be understood to mean that they must always enter into every phenomenon. In fact, the effect of $\mathfrak{T}_1$ can be cancelled by the appearance of $\mathfrak{T}_2$ ([3]), or at least weakened. Whether or not $\mathfrak{T}_2$ vanishes identically can not be decided from the outset, but its presence would be excluded by some special effects.

When we combine $T + \mathfrak{T}_1 \equiv \mathfrak{L}$ into a single homogeneous quadratic function of the $\dot{p}$, the kinetic energy T will now become:

$$\mathsf{T} = \mathfrak{L} + \mathfrak{T}_2 + \mathfrak{T}_3 = \tfrac{1}{2}\alpha_{11}\,\dot{p}_1^2 + \cdots + \tfrac{1}{2}\alpha_{rr}\,\dot{p}_r^2 + c_{11}\,\dot{p}_1\,\dot{\mathfrak{p}}_1 + \cdots + c_{r\mathfrak{r}}\,\dot{p}_1\,\dot{\mathfrak{p}}_\mathfrak{r} + \tfrac{1}{2}\mathfrak{a}_{11}\,\dot{\mathfrak{p}}_1^2 + \cdots + \tfrac{1}{2}\mathfrak{a}_{\mathfrak{r}\mathfrak{r}}\,\dot{\mathfrak{p}}_\mathfrak{r}^2 .$$

([1]) See also **Helmholtz**’s Foreword to H. M., pp. XXII, last paragraph.
([2]) Cf., H. M., nos. 664 and 665.
([3]) Cf., on this, the footnote on pp. 28.

Lagrange's equations would then read:

$$\frac{d}{dt}\left(\frac{\partial \mathsf{T}}{\partial \dot{p}}\right) - \frac{\partial \mathsf{T}}{\partial p} = 0$$

and

$$\frac{d}{dt}\left(\frac{\partial \mathsf{T}}{\partial \dot{\mathfrak{p}}_\sigma}\right) \equiv \frac{d}{dt}\left(\frac{\partial (\mathfrak{T}_2 + \mathfrak{T}_3)}{\partial \dot{\mathfrak{p}}_\sigma}\right) = 0 \,,$$

or

$$\frac{\partial \mathsf{T}}{\partial \dot{\mathfrak{p}}_\sigma} = \frac{\partial}{\partial \dot{\mathfrak{p}}_\sigma}(\mathfrak{T}_2 + \mathfrak{T}_3) = C\sigma \,. \tag{4}$$

When the last one is written out in detail, it will read:

$$\begin{aligned}
\mathfrak{a}_{11}\,\dot{\mathfrak{p}}_1 + \mathfrak{a}_{12}\,\dot{\mathfrak{p}}_2 + \cdots + \mathfrak{a}_{1\mathfrak{r}}\,\dot{\mathfrak{p}}_\mathfrak{r} &= C_1 - (c_{11}\,\dot{p}_1 + c_{21}\,\dot{p}_2 + \cdots + c_{r1}\,\dot{p}_r) \equiv C_1 - S_1 \,, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\mathfrak{a}_{\mathfrak{r}1}\,\dot{\mathfrak{p}}_1 + \mathfrak{a}_{\mathfrak{r}2}\,\dot{\mathfrak{p}}_2 + \cdots + \mathfrak{a}_{\mathfrak{r}\mathfrak{r}}\,\dot{\mathfrak{p}}_\mathfrak{r} &= C_\mathfrak{r} - (c_{1\mathfrak{r}}\,\dot{p}_1 + c_{2\mathfrak{r}}\,\dot{p}_2 + \cdots + c_{r\mathfrak{r}}\,\dot{p}_r) \equiv C_\mathfrak{r} - S_\mathfrak{r} \,.
\end{aligned}$$

Solving those equations will yield:

$$\begin{aligned}
\dot{\mathfrak{p}}_1 &= \mathfrak{A}_{11}\,(C_1 - S_1) + \mathfrak{A}_{12}\,(C_2 - S_2) + \cdots + \mathfrak{A}_{1\mathfrak{r}}\,(C_\mathfrak{r} - S_\mathfrak{r}) \,, \\
&\cdots\cdots\cdots\cdots\cdots\cdots, \\
\dot{\mathfrak{p}}_\mathfrak{r} &= \mathfrak{A}_{\mathfrak{r}1}\,(C_1 - S_1) + \mathfrak{A}_{\mathfrak{r}2}\,(C_2 - S_2) + \cdots + \mathfrak{A}_{\mathfrak{r}\mathfrak{r}}\,(C_\mathfrak{r} - S_\mathfrak{r}) \,,
\end{aligned}$$

in which the $\mathfrak{A}_{ik}$ have the same meaning as before.

If we now reintroduce the expressions for the $\dot{\mathfrak{p}}$ into the kinetic energy T then it will become a function of the velocities $\dot{p}_1, \ldots, \dot{p}_r$, and the constant "generalized momenta" $C_1, \ldots, C_\mathfrak{r}$. It will be a homogeneous quadratic function of all those variables, but not of the C_σ or the $\dot{p}_\sigma$ when taken by themselves, since products like $C_\sigma \cdot \dot{p}_\rho$ would appear. We shall denote the transform of the function T by T', and we will then have:

$$\mathsf{T}(p_1, \ldots, p_r, \dot{p}_1, \ldots, \dot{p}_r, \dot{\mathfrak{p}}_1, \ldots, \dot{\mathfrak{p}}_\mathfrak{r}) \equiv \mathsf{T}'(p_1, \ldots, p_r, \dot{p}_1, \ldots, \dot{p}_r, \dot{\mathfrak{p}}_1, \ldots, \dot{\mathfrak{p}}_\mathfrak{r}) \tag{5}$$

identically.

However, on the same grounds as before, $\partial \mathsf{T} / \partial \dot{p}$ is not equal to, say, $\partial \mathsf{T}' / \partial \dot{p}$; similarly, $\partial \mathsf{T} / \partial p$ is different from $\partial \mathsf{T}' / \partial p$. From the rules of differentiation, we will find that we have:

$$\frac{\partial \mathsf{T}'}{\partial p_\rho} = \frac{\partial \mathsf{T}}{\partial p_\rho} + \sum_{\sigma=1}^{\mathfrak{r}} \frac{\partial \mathsf{T}}{\partial \dot{\mathfrak{p}}_\sigma} \cdot \frac{\partial \dot{\mathfrak{p}}_\sigma}{\partial p_\rho}, \qquad \rho = 1, 2, \ldots, r$$

and (1)

$$\frac{\partial \mathsf{T}'}{\partial \dot p_\rho} = \frac{\partial \mathsf{T}}{\partial \dot p_\rho} + \sum_{\sigma=1}^{\mathfrak{r}} \frac{\partial \mathsf{T}}{\partial \dot{\mathfrak{p}}_\sigma} \cdot \frac{\partial \dot{\mathfrak{p}}_\sigma}{\partial \dot p_\rho}, \qquad \rho = 1, 2, \dots, r.$$

Now, from equation (4), one has:

$$\frac{\partial \mathsf{T}}{\partial \dot{\mathfrak{p}}_\sigma} = C_\sigma ,$$

so:

$$\frac{\partial \mathsf{T}'}{\partial p_\rho} = \frac{\partial \mathsf{T}}{\partial p_\rho} + \sum_{\sigma=1}^{\mathfrak{r}} C_\sigma \frac{\partial \dot{\mathfrak{p}}_\sigma}{\partial p_\rho}$$

and

$$\frac{\partial \mathsf{T}'}{\partial \dot p_\rho} = \frac{\partial \mathsf{T}}{\partial \dot p_\rho} + \sum_{\sigma=1}^{\mathfrak{r}} C_\sigma \frac{\partial \dot{\mathfrak{p}}_\sigma}{\partial \dot p_\rho},$$

or upon rearranging that so the differentiation is outside of the summation sign:

$$\frac{\partial \mathsf{T}'}{\partial p_\rho} = \frac{\partial}{\partial p_\rho}\left(\mathsf{T}' - \sum_{\sigma=1}^{\mathfrak{r}} C_\sigma \dot{\mathfrak{p}}_\sigma\right) \tag{6}$$

and

$$\frac{\partial \mathsf{T}'}{\partial \dot p_\rho} = \frac{\partial}{\partial \dot p_\rho}\left(\mathsf{T}' - \sum_{\sigma=1}^{\mathfrak{r}} C_\sigma \mathfrak{p}_\sigma\right). \tag{7}$$

If we then replace T with the function:

$$\Phi \equiv \mathsf{T}' - \sum_{\sigma=1}^{\mathfrak{r}} C_\sigma \dot{\mathfrak{p}}_\sigma ,$$

which **Helmholtz** called the *kinetic potential* (up to sign), then the Lagrange equations for Σ^0 will read:

$$\frac{d}{dt}\left(\frac{\partial \Phi}{\partial \dot p_\rho}\right) - \frac{\partial \Phi}{\partial p_\rho} = 0, \qquad r = 1, 2, \dots, r.$$

The function T' already includes only constants in addition to the p and $\dot p$. In order to also put Φ into such a form, we shall now construct all of the $\partial \mathsf{T} / \partial \dot p_\rho$ and $\partial \mathsf{T} / \partial \dot{\mathfrak{p}}_\rho$ in the equations that were written down originally:

(1) In this article, we shall formally follow the presentation in **A. G. Webster**, *The dynamics of particles*, etc., pps. 176-179, as well as pps. 182 to 184. However, the interpretation of it is entirely different. Cf., footnote pp. 8. Everything up to now is also true for non-cyclic coordinates $\alpha_{i+1}, \dots, \alpha_{\mathfrak{r}}$, moreover.

$$\frac{\partial \mathsf{T}}{\partial \dot{\mathfrak{p}}_1} = \mathfrak{a}_{11}\,\dot{\mathfrak{p}}_1 + \cdots + \mathfrak{a}_{1\mathfrak{r}}\,\dot{\mathfrak{p}}_{\mathfrak{r}} + c_{11}\,\dot{p}_1 + c_{21}\,\dot{p}_2 + \cdots + c_{r1}\,\dot{p}_r = C_1 ,$$

$$\frac{\partial \mathsf{T}}{\partial \dot{\mathfrak{p}}_2} = \mathfrak{a}_{21}\,\dot{\mathfrak{p}}_1 + \cdots + \mathfrak{a}_{2\mathfrak{r}}\,\dot{\mathfrak{p}}_{\mathfrak{r}} + c_{12}\,\dot{p}_1 + c_{22}\,\dot{p}_2 + \cdots + c_{r2}\,\dot{p}_r = C_2 ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial \mathsf{T}}{\partial \dot{\mathfrak{p}}_{\mathfrak{r}}} = \mathfrak{a}_{\mathfrak{r}1}\,\dot{\mathfrak{p}}_1 + \cdots + \mathfrak{a}_{\mathfrak{r}\mathfrak{r}}\,\dot{\mathfrak{p}}_{\mathfrak{r}} + c_{1\mathfrak{r}}\,\dot{p}_1 + c_{2\mathfrak{r}}\,\dot{p}_2 + \cdots + c_{r\mathfrak{r}}\,\dot{p}_r = C_{\mathfrak{r}} ,$$

$$\frac{\partial \mathsf{T}}{\partial \dot{p}_1} = c_{11}\,\dot{\mathfrak{p}}_1 + c_{12}\,\dot{\mathfrak{p}}_2 + \cdots + c_{1\mathfrak{r}}\,\dot{\mathfrak{p}}_{\mathfrak{r}} + \alpha_{11}\,\dot{p}_1 + \alpha_{12}\,\dot{p}_2 + \cdots + \alpha_{1r}\,\dot{p}_r ,$$

$$\frac{\partial \mathsf{T}}{\partial \dot{p}_2} = c_{21}\,\dot{\mathfrak{p}}_1 + c_{22}\,\dot{\mathfrak{p}}_2 + \cdots + c_{2\mathfrak{r}}\,\dot{\mathfrak{p}}_{\mathfrak{r}} + \alpha_{21}\,\dot{p}_1 + \alpha_{22}\,\dot{p}_2 + \cdots + \alpha_{2r}\,\dot{p}_r ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial \mathsf{T}}{\partial \dot{p}_r} = c_{r1}\,\dot{\mathfrak{p}}_1 + c_{r2}\,\dot{\mathfrak{p}}_2 + \cdots + c_{r\mathfrak{r}}\,\dot{\mathfrak{p}}_{\mathfrak{r}} + \alpha_{r1}\,\dot{p}_1 + \alpha_{r2}\,\dot{p}_2 + \cdots + \alpha_{rr}\,\dot{p}_r .$$

Now, from Euler's theorem, one has:

$$2\mathsf{T} = \sum_{\rho=1}^{r} \dot{p}_\rho \frac{\partial \mathsf{T}}{\partial \dot{p}_\rho} + \sum_{\sigma=1}^{r} \dot{p}_\sigma \cdot C_3 .$$

For the first sum, when we recall our previous notation, namely:

$$S_\sigma \equiv c_{1\sigma}\,\dot{p}_1 + c_{2\sigma}\,\dot{p}_2 + \cdots + c_{r\sigma}\,\dot{p}_r , \qquad \sigma = 1, 2, \ldots, \mathfrak{r}, \tag{8}$$

the last r rows and the first $\mathfrak{r}$ columns will initially yield:

$$\sum_{\sigma=1}^{\mathfrak{r}} \dot{\mathfrak{p}}_\sigma\, S_\sigma ,$$

and from the last r of the remaining columns will yield $2\,\mathfrak{L}$, such that:

$$2\,\mathsf{T} = 2\mathfrak{L} + \sum_{\sigma=1}^{\mathfrak{r}} \dot{\mathfrak{p}}_\sigma\,(C_\sigma + S_\sigma) .$$

We shall now address the quadratic functions S and C :

$$S \equiv \tfrac{1}{2}\sum_{\sigma=1}^{\mathfrak{r}}\sum_{\mu=1}^{\mathfrak{r}} \mathfrak{A}_{\sigma\mu}\, S_\sigma S_\mu \tag{9}$$

and

$$C \equiv \tfrac{1}{2}\sum_{\sigma=1}^{\mathfrak{r}}\sum_{\mu=1}^{\mathfrak{r}} \mathfrak{A}_{\sigma\mu}\, C_\sigma C_\mu \;=\; \overline{\mathfrak{T}}_3 \qquad \text{[pp. 8, equation (3)].}$$

Our previous equations for the $\dot{\mathfrak{p}}$, pp. 23, can then be written:

$$\dot{\mathfrak{p}}_\sigma \;=\; \frac{\partial C}{\partial C_\sigma} - \frac{\partial S}{\partial S_\sigma}$$

and furthermore:

$$2\,\mathsf{T} = \;2\,\mathfrak{L} + \sum_{\sigma=1}^{\mathfrak{r}} (C_\sigma + S_\sigma)\left(\frac{\partial C}{\partial C_\sigma} - \frac{\partial S}{\partial S_\sigma}\right).$$

From Euler's theorem, since C and S are homogeneous function of degree two, we will again have:

$$\sum_{\sigma=1}^{\mathfrak{r}} C_\sigma \frac{\partial C}{\partial C_\sigma} = 2\,C$$

and

$$\sum_{\sigma=1}^{\mathfrak{r}} S_\sigma \frac{\partial S}{\partial S_\sigma} = 2\,S\,,$$

such that:

$$2\,\mathsf{T} = \;2\,\mathfrak{L} + 2C - 2S + \sum_{\sigma=1}^{\mathfrak{r}} S_\sigma \frac{\partial C}{\partial C_\sigma} - \sum_{\sigma=1}^{\mathfrak{r}} C_\sigma \frac{\partial S}{\partial S_\sigma}\;.$$

However, we have:

$$\sum_{\sigma=1}^{\mathfrak{r}} S_\sigma \frac{\partial C}{\partial C_\sigma} \;=\; \sum_{\sigma=1}^{\mathfrak{r}} S_\sigma \sum_{\mu=1}^{\mathfrak{r}} \mathfrak{A}_{\sigma\mu} C_\mu \;=\; \sum_{\mu=1}^{\mathfrak{r}} C_\mu \sum_{\sigma=1}^{\mathfrak{r}} \mathfrak{A}_{\sigma\mu}\, S_\sigma \;=\; \sum_{\mu=1}^{\mathfrak{r}} C_\mu \frac{\partial S}{\partial S_\mu}\,.$$

The sums in the last expression for T that was written out will cancel then, and what will remain is:

$$\mathsf{T}' = \mathfrak{L} - S + C = T + \mathfrak{T}_1 - S + C\,, \tag{10}$$

and the kinetic potential Φ will become:

$$\Phi \equiv \mathsf{T}' - \sum_{\sigma=1}^{\mathfrak{r}} C_\sigma \dot{\mathfrak{p}}_\sigma \;= \mathfrak{L} - S + C - \sum_{\sigma=1}^{\mathfrak{r}} C_\sigma \left(\frac{\partial C}{\partial C_\sigma} - \frac{\partial S}{\partial S_\sigma}\right)$$

or

$$\Phi = \mathfrak{L} - S + C - \sum_{\sigma=1}^{\mathfrak{r}} C_\sigma \frac{\partial S}{\partial S_\sigma} = T + \mathfrak{T}_1 - S - C + \Gamma, \tag{11}$$

when one sets:

$$\Gamma \equiv \sum_{\sigma=1}^{\mathfrak{r}} C_\sigma \frac{\partial S}{\partial S_\sigma}.$$

With that, we have achieved our next goal and can even make the following important remarks about that: First of all, when we form the Lagrange equations of motion, we must indeed regard the function Φ as then expression for the *vis viva* and on the other hand use T' in the energy equation [cf., also the identity (5), as well as equations (6) and (7)]. However, we can also see immediately that Γ, which includes the $\dot{p}$ only linearly, will make no contribution to the energy of motion when we note that the Lagrange equations for Σ^0:

$$\frac{d}{dt}\left(\frac{\partial \Phi}{\partial \dot{p}_\rho}\right) - \frac{\partial \Phi}{\partial p_\rho} = 0$$

can also be written:

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{p}_\rho}(\mathfrak{L} - S)\right) - \frac{\partial}{\partial p_\rho}(\mathfrak{L} - S) + \frac{\partial C}{\partial p_\rho} = -\frac{d}{dt}\frac{\partial \Gamma}{\partial \dot{p}_\rho} + \frac{\partial \Gamma}{\partial p_\rho} \tag{12}$$

since the $\dot{p}$ do not occur at all in C. If we then multiply by dp_ρ and sum over $\rho = 1, \dots, r$ then we will get derivatives of a homogeneous function $\mathfrak{L} - S + C = \mathsf{T}'$ on the left-hand side (cf., e.g., **Webster**, *Dynamics*, pp. 125), while the right-hand side is zero. That is because if we introduce the notations:

$$g_{\rho\sigma} \equiv c_{\rho 1}\, \mathfrak{A}_{\sigma 1} + c_{\rho 2}\, \mathfrak{A}_{\sigma 2} + \dots + c_{\rho \mathfrak{r}}\, \mathfrak{A}_{\sigma \mathfrak{r}}$$

and

$$\sum_{\sigma=1}^{\mathfrak{r}} C_\sigma\, g_{\rho\sigma} \equiv G_\rho$$

then we can write Γ in the form:

$$\Gamma = G_1 \cdot \dot{p}_1 + G_2 \cdot \dot{p}_2 + \dots + G_r \cdot \dot{p}_r\,,$$

and it follows for the right-hand side of (12) that:

$$-\frac{d}{dt}\frac{\partial \Gamma}{\partial \dot{p}_\rho} + \frac{\partial \Gamma}{\partial p_\rho} = -\sum_{i=1}^{r} \frac{\partial G_\rho}{\partial p_i} \cdot \dot{p}_i + \sum_{i=1}^{r} \frac{\partial G_i}{\partial p_\rho} \cdot \dot{p}_i = \sum_{i=1}^{r} \left(\frac{\partial G_i}{\partial p_\rho} - \frac{\partial G_\rho}{\partial p_i}\right) \cdot \dot{p}_i\,.$$

The coefficient of $\dot{p}_\rho$ in that equation is:

$$\frac{\partial G_\rho}{\partial p_\rho} - \frac{\partial G_\rho}{\partial p_\rho} = 0\,.$$

The coefficient of $\dot p_i$ is:

$$\frac{\partial G_i}{\partial p_\rho} - \frac{\partial G_\rho}{\partial p_i}$$

and that of $\dot p_\rho$ in the equation that corresponds to the i^{th} parameter p_i is:

$$\frac{\partial G_\rho}{\partial p_i} - \frac{\partial G_i}{\partial p_\rho}\,,$$

which is equal and opposite to the previous one. If we write $dp_\rho = \dot p_\rho \cdot dt$ instead then we will see immediately that the summation must give zero.

The phrase *gyroscopic term* is often used for the terms that appear in Γ. That expression was chosen by **Thomson** and **Tait** (*Treatise*, pp. 393) because it will also appear in Φ when a rotating rigid body defines the subsystem. As is known, **H. von Helmholtz** had inferred some far-reaching conclusions from that.

For one-parameter systems, Γ will again drop out of the Lagrange equations (12) entirely.

The second remark refer to the homogeneous quadratic function S that enters into both formulas (10) and (11) and is certainly positive-semidefinite (¹): From (9), it is positive-definite in the S_σ, but it can never become indefinite under a transformation like (8), and semidefinite in the $\dot p$. When $\mathfrak{T}_2$ appears at all, it would then, in fact, counteract the effect of $\mathfrak{T}_1$.

It should be further remarked that some interesting conclusions can be inferred from the first of equations in the set on pp. 25 in their applications.

As an example, we initially consider a one-parameter system, and we will easily obtain such a thing by altering one of the previous examples. In Fig. 10, the xy-plane is folded over into the xz-plane, i.e., the reference plane. g is a rod that is constrained to be fixed along the rotational axis and along which $\mathfrak{m}$ can slide, while $O'A\mathfrak{m}$ is once more a length of string that runs horizontally. We need to have the connection between the coordinates x, y of $\mathfrak{m}$ relative to the system x, y, z that is fixed in space and the general parameters s, φ, and we will find them most simply by imagining a rectangular system ξ, η that is fixed in the rod g, so it will participate in the rotation.

(¹) In fact, it can vanish without all $\dot p_\rho$ being zero since one can have $C_{\rho\sigma} = 0$. By contrast, the function $\mathfrak{T}_1 - S$ is also still positive-definite because it arises from $\mathfrak{T}$ by transformation of the original variables $\dot p_1, \dots, \dot p_r, \dot{\mathfrak{p}}_1, \dots, \dot{\mathfrak{p}}_r$, (corresponding to equations on pp. 23 above, to which one must add the identities $\dot p_\rho = \dot p_\rho$, $\rho = 1, \dots, r$, such that the determinant of the substitution will also be non-zero) to the new ones $\dot p_1, \dots, \dot p_r$, $C_1, \dots, C_{\mathfrak{r}}$, when we set the last ones $C_1, \dots, C_{\mathfrak{r}}$ equal to zero since we regard then as variables for the moment. On this, cf., the remark about non-conservative systems on pp. 22.

Now, one has:

$$\mathfrak{T} = \tfrac{1}{2}\mathfrak{m}(\dot{x}^2 + \dot{y}^2)\,, \qquad x = \xi \cos\varphi - \eta \sin\varphi\,, \qquad y = \xi \cos\varphi - \eta \sin\varphi\,.$$

Upon differentiating the last two equations with respect to time, it will follow that:

$$\mathfrak{T} = \tfrac{1}{2}\mathfrak{m}[\dot{\xi}^2 + \dot{\eta}^2 + 2(\dot{\eta}\,\xi - \eta\,\dot{\xi})\cdot\dot{\varphi} + (\xi^2 + \eta^2)\cdot\dot{\varphi}^2]\ .$$

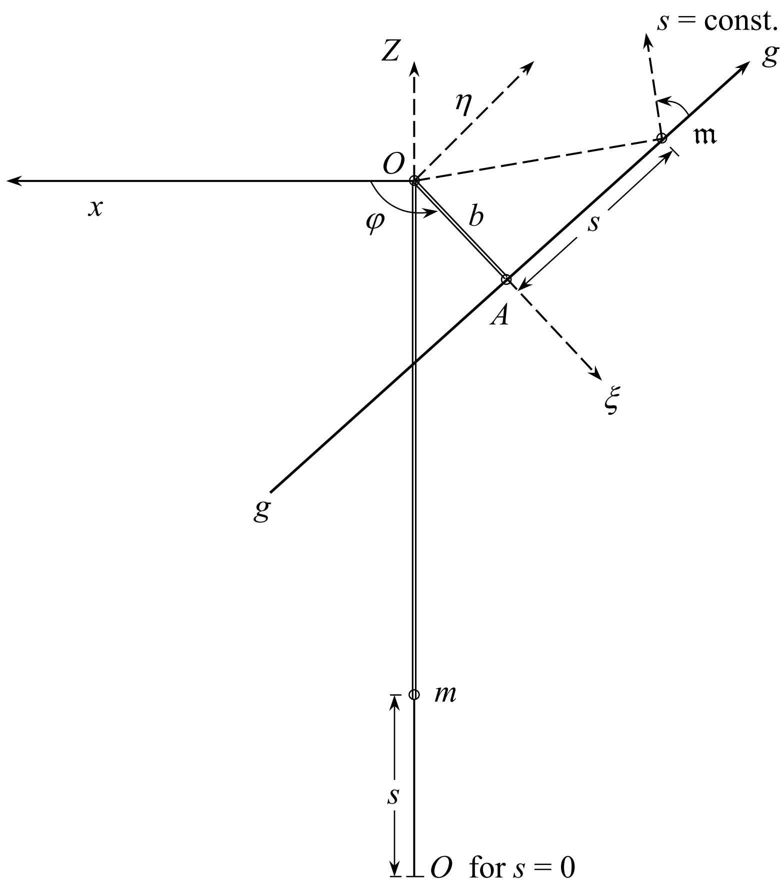

Figure 10.

However, $\xi = b$, $\eta = s$, so:

$$\mathfrak{T} = \tfrac{1}{2}\mathfrak{m}[\dot{s}^2 + 2b\,\dot{s}\,\dot{\varphi} + (s^2 + b^2)\,\dot{\varphi}^2]\ ,$$

and since $T = \frac{1}{2}m\,\dot{s}^2$, one will ultimately have:

$$\mathsf{T} = \tfrac{1}{2}\mathfrak{m}\,\dot{s}^2 + \tfrac{1}{2}\mathfrak{m}\,\dot{s}^2 + \mathfrak{m}\,b\,\dot{s}\,\dot{\varphi} + \tfrac{1}{2}\mathfrak{m}(s^2 + b^2)\cdot\dot{\varphi}^2\,.$$

$\mathfrak{T}_2$ does, in fact, appear here, and a glance at Fig. 10 will show that φ is not orthogonal to s here, i.e., when $\mathfrak{m}$ moves at constant s (constant φ, respectively). Thus, the two directions of motion will subtend an angle that is different from 90º.

It will follow from an easy calculation that:

$$\mathfrak{a}_{11} = \mathfrak{m}\,(s^2 + b^2)\ , \qquad \mathfrak{A}_{11} = \frac{1}{\mathfrak{m}\,(s^2 + b^2)}\ , \qquad C = \frac{1}{2}\frac{C_1^2}{\mathfrak{m}\,(s^2 + b^2)}\,,$$

$$S_1 = c_{11} \cdot \dot{s}\,, \qquad c_{11} = \mathfrak{m}\,b\,, \qquad S_1 = \mathfrak{m}\,b\,\dot{s}\,,$$

$$S = \tfrac{1}{2}\mathfrak{A}_{11} \cdot S_1^2 = \frac{1}{2}\frac{\mathfrak{m}\,b^2\,\dot{s}^2}{s^2 + b^2}\,, \qquad \Gamma = C_1\,\mathfrak{A}_{11}\,S_1 = \frac{b\,\dot{s}}{s^2 + b^2} \cdot C_1\,,$$

$$\mathsf{T}' = \tfrac{1}{2} m\,\dot{s}^2 + \tfrac{1}{2}\mathfrak{m}\,\dot{s}^2 - \frac{1}{2}\frac{\mathfrak{m}\,b^2\,\dot{s}^2}{s^2 + b^2} - \frac{1}{2}\frac{C_1^2}{\mathfrak{m}\,(s^2 + b^2)}\,,$$

$$\Phi = \tfrac{1}{2} m\,\dot{s}^2 + \tfrac{1}{2}\mathfrak{m}\,\dot{s}^2 - \frac{1}{2}\frac{\mathfrak{m}\,b^2\,\dot{s}^2}{s^2 + b^2} - \frac{1}{2}\frac{C_1^2}{\mathfrak{m}\,(s^2 + b^2)} + \frac{b\,C_1 \cdot \dot{s}}{s^2 + b^2}\,.$$

We easily verify that:

$$\frac{d}{dt}\left(\frac{\partial \Gamma}{\partial \dot{s}}\right) - \frac{\partial \Gamma}{\partial s} = 0\,.$$

In order to have a two-parameter example in which Γ also appears in equations (12), we consider two rigid bodies K_1 and K_2, of which, K_1 can rotate around a horizontal axis o_1, but K_2, along with K_1, is once more coupled by an articulated link with an axis o_2 that is normal to the previous one. Let a third body K_3 be rotatable around an axis o_3 that is fixed in K_2 and once more normal to o_2, and which should give us the "invisible" system, while K_1 and K_2 define the visible one. We then address the expression for the kinetic energy T. We then introduce the angles ψ and φ as the general coordinates of the system S: ψ is the angle of rotation around o_1, while ϑ is the angle of inclination of a distinguished direction s that is fixed in the body K_2 with respect to a plane that is normal to o_1: That line shall run through the center of mass of K_2 and a point on the axis o_2 that is normal to it. If we then likewise consider the total system Σ^0 then ψ and ϑ can also be employed as parameters for K_3, to which the angle of rotation φ around o_3 can be added.

The *vis viva* T_1 of K_1 can be written down immediately:

$$T_1 = \tfrac{1}{2} O_1 \cdot \dot{\psi}^2\,,$$

in which O_1 is the moment of inertial of K_1 with respect to o_1 . In order to find T_2 from K_2, we next use the theorem of **König**: T_2 = energy of translation of the center of mass S_2 of K_2 + energy of rotation of K_2 around the instantaneous axis of rotation around it. We find, e.g., from the kinematics of relative motion, that the former is:

$$\tfrac{1}{2} M_2 (l^2 \dot{\psi}^2 + s_2^2 \cdot \dot{\vartheta}^2),$$

in which l is the distance from the point of intersection of s with o_2 to o_1, and s_2 is the distance from the center of mass S_2 to o_2 . In addition, l shall be normal to o_2 and o_1 .

That will likewise be the formula for the center of mass energy of K_3 when we just take the distance s_3 from the center of mass S_3 from K_3 to o_2 in place of s_2, and take the mass M_3 instead of M_2, assuming that the rotational axis o_3 coincides with the direction s :

$$\tfrac{1}{2} M_3 (l^2 \dot{\psi}^2 + s_3^2 \cdot \dot{\vartheta}^2) \ .$$

In regard to the rotational energy, we would similarly like to calculate that of K_3 :

$$R_3 = \tfrac{1}{2} J_3 \cdot \omega_3^2 .$$

The J_3 in that means the moment of inertia of K_3 about the instantaneous rotational axis through the center of mass, and ω_3 is the angular velocity around it. In order to get a moment of inertia that does not depend upon the motion, we introduce a fixed coordinate system x, y, z in the body K_3 whose axes shall be the principal axes of inertia at the center of mass ([1]). It is then known that:

$$R_3 = \tfrac{1}{2} (L_3 \cdot \omega_{x3}^2 + M_3 \cdot \omega_{y3}^2 + N_3 \cdot \omega_{z3}^2) .$$

L_3, M_3, N_3 are now constant moments of inertia for the axes x, y, z, and ω_{x3}, ω_{y3}, ω_{z3} are the angular velocities around them. We still need to have the connection between the ω_{x3}, etc., and the $\dot{\psi}$, $\dot{\vartheta}$, $\dot{\varphi}$. One sees from a vectorial combination of the angular velocities that ω has the components: $\dot{\psi}$, $\dot{\vartheta}$, $\dot{\varphi}$ with respect to o_1, o_2, o_3 . Now, if x, as the principal axis of inertia for K_3, as well as K_2, falls along the axis o_3, and φ already measures the angle between the y-axis and o_2, then $\dot{\psi}$, $\dot{\vartheta}$, $\dot{\varphi}$ will give the following components relative to x, y, z :

	$\dot{\varphi}$	$\dot{\psi}$	$\dot{\vartheta}$
x	$\dot{\varphi}$	$\dot{\psi}\sin\vartheta$	0
y	0	$\dot{\psi}\cos\vartheta\sin\varphi$	$\dot{\vartheta}\cos\varphi$
z	0	$\dot{\psi}\cos\vartheta\cos\varphi$	$-\dot{\vartheta}\sin\varphi$

([1]) This is assumed merely for the same of simplicity. If the coordinate axes x, y, z that are established in what follows are not principal axes of inertia then the mixed products of ω_{x3}, ω_{y3}, ω_{z3} will appear in R_3 . The expressions for them will not change in that way.

such that:

$$\omega_{x3} = \dot{\varphi} + \dot{\psi} \sin \vartheta ,$$
$$\omega_{y3} = \dot{\psi} \cos \vartheta \sin \varphi + \dot{\vartheta} \cos \varphi ,$$
$$\omega_{z3} = \dot{\psi} \cos \vartheta \cos \varphi - \dot{\vartheta} \sin \varphi .$$

Therefore, one will have:

$$R_3 = \tfrac{1}{2}[L_3 (\dot{\varphi} + \dot{\psi} \sin \vartheta)^2 + M_3 (\dot{\psi} \cos \vartheta \sin \varphi + \dot{\vartheta} \cos \varphi)^2 + N_3 (\dot{\psi} \cos \vartheta \cos \varphi - \dot{\vartheta} \sin \varphi)^2].$$

We will then get R_2 from that when change the index on L, M, N to 2 and set $\dot{\varphi} = 0$ and $\varphi = 0$, i.e., o_2 shall be the principal axis of inertia for K_2 (otherwise, we would have to introduce an angle $\varphi = \varphi_1$):

$$R_2 = \tfrac{1}{2}[L_3 \dot{\psi}^2 \sin^2 \vartheta + M_3 \dot{\vartheta}^2 + N_3 \dot{\psi}^2 \cos^2 \vartheta].$$

Thus:

$$\begin{aligned}
\mathsf{T} = {} & \tfrac{1}{2} O_1 \cdot \dot{\psi}^2 + \tfrac{1}{2} M_2 (l^2 \dot{\psi}^2 + s_2^2 \dot{\vartheta}^2) + \tfrac{1}{2} M_3 (l^2 \dot{\psi}^2 + s_3^2 \dot{\vartheta}^2) \\
& + \tfrac{1}{2}[L_2 \cdot \dot{\psi}^2 \sin \vartheta^2 + M_2 \cdot \dot{\vartheta}^2 + N_2 \cdot \dot{\psi}^2 \cos^2 \vartheta] \\
& + \tfrac{1}{2}[L_3 \cdot \dot{\varphi}^2 + 2L_3 \cdot \dot{\varphi} \dot{\psi} \sin \vartheta + L_3 \cdot \dot{\psi}^2 \sin^2 \vartheta \\
& + M_3 \cdot \dot{\psi}^2 \cos^2 \vartheta \sin^2 \varphi + 2M_3 \cdot \dot{\psi} \dot{\vartheta} \cos \vartheta \sin \varphi + M_3 \cdot \dot{\vartheta}^2 \cos^2 \varphi \\
& + N_3 \cdot \dot{\psi}^2 \cos^2 \vartheta \cos^2 \varphi - 2N_3 \cdot \dot{\psi} \dot{\vartheta} \cos \vartheta \sin \varphi \cos \varphi + N_3 \cdot \dot{\vartheta}^2 \sin^2 \varphi] .
\end{aligned}$$

We shall assume that $M_3 = N_3$. The terms in the last bracketed expression with $\dot{\psi}^2 \cdot \cos^2 \vartheta$ will then collectively give $N_3 \cdot \dot{\psi}^2 \cos^2 \vartheta$, the ones with $\dot{\vartheta}^2$ will give $M_3 \dot{\vartheta}^2$, and the one with $\dot{\psi} \dot{\vartheta} \cos \vartheta \sin \varphi \cos \varphi$ will drop out.

We write out clearly and in detail:

$$T = \left[\tfrac{1}{2} O_1 + \tfrac{1}{2} M_2 l^2 + \tfrac{1}{2} L_2 \sin^2 \vartheta + \tfrac{1}{2} N_2 \cos^2 \vartheta\right] \cdot \dot{\psi}^2 + \left[\tfrac{1}{2} M_2 s_2^2 + \tfrac{1}{2} M_2\right] \cdot \dot{\vartheta}^2 ,$$

$$\mathfrak{T}_1 = \left[\tfrac{1}{2} M_3 l^2 + \tfrac{1}{2} L_3 \sin^2 \vartheta + \tfrac{1}{2} N_3 \cos^2 \vartheta\right] \cdot \dot{\psi}^2 + \left[\tfrac{1}{2} M_3 s_2^2 + \tfrac{1}{2} M_3\right] \cdot \dot{\vartheta}^2 ,$$

$$\mathfrak{T}_2 = L_3 \cdot \dot{\varphi} \dot{\psi} \sin \vartheta ,$$

$$\mathfrak{T}_3 = \tfrac{1}{2} L_3 \cdot \dot{\varphi}^2 .$$

It then follows in succession from this that ([1]):

([1]) The independence of the function $\bar{\mathfrak{T}}_3 = C$ from the p expresses a more general fact here. A rigorous examination must also show that a *non*-constant potential function *cannot* be created from a structure composed of tops unless such a thing is already present anyway due to the remaining connections.

$$\mathfrak{a}_{11} = L_3\,, \qquad \mathfrak{A}_{11} = \frac{1}{L_3}\,, \qquad \bar{\mathfrak{T}}_3 = C = \frac{1}{2}\frac{1}{L_3}C_1^2 = \text{const.},$$

$$S_1 = L_3 \cdot \dot{\psi}\sin\vartheta\,, \qquad S = \frac{1}{2}\frac{1}{L_3}\cdot L_3^2\cdot\dot{\psi}^2\sin^2\vartheta = \frac{1}{2}L_3\cdot\dot{\psi}^2\sin^2\vartheta\,,$$

$$\Gamma = C_1\,\mathfrak{A}_{11}\,S_1 = C_1\cdot\dot{\psi}\sin\vartheta\,,$$

$$\mathsf{T}' = T + \mathfrak{T}_1 - \frac{1}{2}L_3\cdot\dot{\psi}^2\sin^2\vartheta + \frac{1}{2}\frac{C_1^2}{L_3}\,,$$

$$\Phi = T + \mathfrak{T}_1 - \frac{1}{2}L_3\cdot\dot{\psi}^2\sin^2\vartheta + \frac{1}{2}\frac{C_1^2}{L_3} + C_1\cdot\dot{\psi}\sin\vartheta\,.$$

The potential function C, as a constant, merely changes the amount of energy, but has no effect on the motion of the visible system. The right-hand sides of equations (12) are:

$$-\frac{d}{dt}\left(\frac{\partial\Gamma}{\partial\dot{\psi}}\right) + \frac{\partial\Gamma}{\partial\psi} = -\,C_1\cdot\dot{\vartheta}\cos\vartheta\,,$$
$$-\frac{d}{dt}\left(\frac{\partial\Gamma}{\partial\dot{\vartheta}}\right) + \frac{\partial\Gamma}{\partial\vartheta} = +\,C_1\cdot\dot{\psi}\cos\vartheta\,.$$

The first group of equations on pp. 25 will become:

$$\frac{\partial\mathsf{T}}{\partial\dot{\varphi}} = L_3\cdot\dot{\psi}\sin\vartheta + L_3\cdot\dot{\varphi} = C_1$$

here.

There will then exist an interaction between $\dot{\psi}$ and $\dot{\varphi}$ (effect of the ship gyro!), which will also be present when $\dot{\vartheta} = 0$. Such a thing will be lacking only when one simultaneously has $\dot{\vartheta} = 0$ and $\vartheta = 0$. For that special case, cf., **A. Föppl**, *Vorles. über techn. Mechanik*, Bd. VI, 1910, pp. 210, *et seq.*

Moreover, our example would give us an opportunity to ascertain the resistance to deviation for ϑ = const. on the basis of my method of determining the constraint forces from the Lagrange equations of the second kind (see footnote 6, pp. 3)

If the body K_2 were already counted as an invisible one then we would once more have an example of the mixed-cyclic case before us.

In conclusion, we would like to briefly consider the first example on pp. 10 of the mass of the connecting rods and also allow, in a somewhat more general way, a rotation of the body K_3 that consists of the rod a and $\mathfrak{m}$ around a. Now, if ψ is the rotation angle of the latter, and the center of mass of K_3 remains at rest in that way, then one will get a rotational energy of the following form, which proves to have the same structure as the kinetic energy $\mathfrak{T}$ of $\mathfrak{m}$ before:

$$R_3 = \tfrac{1}{2}[L\dot{\vartheta}^2 + M\cdot\dot{\varphi}^2\sin^2\vartheta + N(\dot{\varphi}^2\cos^2\vartheta + 2\dot{\varphi}\dot{\psi}\cos\vartheta + \dot{\psi}^2)]$$
$$= \tfrac{1}{2}L\cdot\dot{\vartheta}^2 + \left(\tfrac{1}{2}M\sin^2\vartheta + \tfrac{1}{2}N\cos^2\vartheta\right)\cdot\dot{\varphi}^2 + N\cdot\dot{\varphi}\dot{\psi}\cos\vartheta + \tfrac{1}{2}N\dot{\psi}^2 .$$

The L, M, N in that are constant moments of inertia, and one assumes that $L = M$.

ϑ can again be replaced with z here by means of the coupling equation $\vartheta = f(z)$. One will obtain an entirely-similar expression for the rod b, except that ϑ will be replaced with the angle η that can, however, be likewise expressed in terms of z.

The system is one-parameter, but bi-cyclic (φ and ψ are cyclic coordinates).

If only K_3 were regarded as the invisible system, e.g., as a built-in top, then we would have two parameters, namely, ϑ or z and φ, and we could then discuss the gyroscopic term in equations (12) (the connection between $\dot{\varphi}$ and $\dot{\psi}$ that is established by the first group of equations on pp. 25, respectively).

General systems that are influenced by forces.

(Acyclic and mixed-cyclic case)

We shall restrict ourselves to merely stating the problem. We again have $\mathsf{T} = T + \mathfrak{T}$. Now, one has a general expression for $\mathfrak{T}$ in terms of $\dot{\alpha}_1, \ldots, \dot{\alpha}_\mathfrak{r}$, namely:

$$\mathfrak{T} \equiv \tfrac{1}{2}\beta_{11}\cdot\dot{\alpha}_1^2 + \cdots + \tfrac{1}{2}\beta_{\mathfrak{rr}}\cdot\dot{\alpha}_\mathfrak{r}^2 ,$$

in which the β_{ik} are functions of the $\alpha_1, \ldots, \alpha_\mathfrak{t}$. However, we can also use equations (1) to express $\mathfrak{T}$ in the form:

$$\mathfrak{T} = \tfrac{1}{2}b_{11}\dot{p}_1^2 + \cdots + \tfrac{1}{2}b_{rr}\dot{p}_r^2 + c_{11}\,\dot{p}_1\,\dot{\alpha}_{i-1} + \cdots + c_{r\mathfrak{r}}\,\dot{p}_r\,\dot{\alpha}_\mathfrak{r} + \tfrac{1}{2}\mathfrak{a}_{11}\dot{\alpha}_{i-1}^2 + \cdots + \tfrac{1}{2}\mathfrak{a}_{\mathfrak{rr}}\dot{\alpha}_\mathfrak{r}^2 .$$

The coefficients b, c, and $\mathfrak{a}$ are now functions of the p and $\alpha_{i-1}, \ldots, \alpha_\mathfrak{t}$.

When $\mathfrak{T}$ is taken to have the second form, the Lagrange equations of motion will be:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}}\right) - \frac{\partial T}{\partial p} = -\frac{d}{dt}\left(\frac{\partial \mathfrak{T}}{\partial \dot{p}}\right) + \frac{\partial \mathfrak{T}}{\partial p}$$

and

$$\frac{d}{dt}\left(\frac{\partial \mathfrak{T}}{\partial \dot{\alpha}_\mu}\right) - \frac{\partial \mathfrak{T}}{\partial \alpha_\mu} = 0\,, \qquad \mu = i + 1, \ldots, \mathfrak{r}.$$

We can imagine that we can determine the p and $\alpha_{i-1}, \ldots, \alpha_\mathfrak{r}$ as functions of time from those $r + \mathfrak{r}$ equations. In any event, those expressions are substituted in the first group of the equations above, and they can then be discussed.

As far as the forces that $\mathfrak{S}$ exerts up S are concerned (i.e., *kinetostatics*), when we have solved the problem of motion for Σ^0, we can determine them from the first group:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}_\rho}\right) - \frac{\partial T}{\partial p_\rho} = P_\rho\,,$$

in which the P_ρ are the generalized force components. That is, in essence, the same idea that is at the basis for my method of 1910 (see footnote 6, pp. 3). It is even a special case of the one that was applied here. Namely, in that article, we assume that a single coupling equation defined the single new coordinate α, not as a function of the ones that were present already $p_1, \ldots, p_r$, but as a constant, whereas here, in addition to the larger number of allowable new coordinates, that number can also be still greater than the number of coupling equations, the $\alpha_1, \ldots, \alpha_i$ are regarded as completely general as functions of the p and the $\alpha_{i+1}, \ldots, \alpha_\mathfrak{r}$ by means of (1).

It should be further noted that one could also base $\mathfrak{T}$ upon the first form and employ the coupling equations (1) with that. Naturally, one would then have to use undetermined multipliers.

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Appendix.

It should be pointed out that Hertz's mechanics needs to be extended for only those potential functions C that depend upon just the coordinates essentially, in such a way that the total energy will not be set equal to $T + C$, but to $T + \mathfrak{T}_1 - S + C$. However, it might be the case that *more general* potential functions already correspond to the total part $\mathfrak{T}_1 - S + C$, or the forces that are derived from them by the formula:

$$-\frac{d}{dt}\frac{\partial(\mathfrak{T}_1 - S)}{\partial \dot{p}} + \frac{\partial(\mathfrak{T}_1 - S)}{\partial p} - \frac{\partial C}{\partial p} - \frac{d}{dt}\frac{\partial \Gamma}{\partial \dot{p}} + \frac{\partial \Gamma}{\partial p}\,,$$

respectively. According to our previous considerations, it will follow from this that an actual distinction between rest masses, moving masses, etc. can once more go away, and even in ordinary mechanics, when one assumes a suitable force law. From the usual standpoint, the appearance of such a distinction would then serve as a hint that the force law or potential function should be *corrected*.

Let us examine **Weber**'s fundamental law of electrodynamics as an example of that, and indeed initially in the form of the potential: $V = \frac{e_1 e_2}{r}\left[1 - \frac{1}{a^2}\dot{r}^2\right]$, in which e_1 should be at rest, which agrees with $\mathfrak{T}_1 - S + C$ only partially since $S = -\frac{e_1 e_2}{a^2}\cdot\frac{1}{r}\dot{r}^2$, $C = \frac{e_1 e_2}{r}$, while nothing in it would correspond to $\mathfrak{T}_1$. However, it is interesting that one will get exactly the Weber expression from this by using the formula that was given above, cf., also *Enz. d. math. Wiss.*, Bd. V, art. 21, pp. 49.

However, in regard to the aforementioned distinction between masses, that shows that when one assumes the validity of that law and an erroneous application of Coulomb's law along the same lines as on pp. 21, one must, in fact, obtain different apparent longitudinal and transversal masses.

Furthermore, the assertion on pp. 5, that the transformed quadratic function $\mathfrak{T}$ is positive-definite is correct under the general assumptions only in the sense that it can assume only positive values, including zero (i.e., it is positive semi-definite). That would also correct the remark that was made on the footnote on pp. 28, in which $\mathfrak{T}$ was assumed to be positive-definite. The determinant $\mathfrak{A}$ (pp. 8) can then be equal to zero for some special configurations, in general. Here, the considerations in this article require a certain extension since otherwise the assumptions would have to be somewhat more restricted (e.g., the number r' of parameters p that actually appear in (1) on pp. 4 should be equal to i, and the functional determinant of the $\alpha_1, \ldots, \alpha_i$ with respect to the $p_1, \ldots, p_i$ should be non-zero).

Finally, it should also be added that one can convince oneself of the fact that was expressed in the footnote on pp. 32 in the following way: C means the *vis viva* of the top for fixed parameters. Therefore, C can only be the rotational energy itself. However, as such, it will depend upon only the angular velocity, but it will be entirely independent of the special configuration of the total system, i.e., it will be independent of the parameters.

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