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On a direct determination of each individual reaction force for a constrained point-system by itself from the Lagrange equations of the second kind.

By

Dr. **Fr. Paulus** in Graz.

(With 4 text figures)

Of the mathem.-phys. department at the University of Graz.
Chairman: Prof. Dr. Anton Wassmuth.

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The main topic in theoretical mechanics is indeed defined by problems of the motion of constrained point-system, but it is precisely in the applications that the forces of constraint or reaction are often much more important (cf., **Voss**, *Enzyklopädie d. mathem. Wiss.*, IV, 1, pp. 82, **Stäckel**, *ibidem*, pp. 476 and 477).

When we cast a brief glance to the methods for determining such forces of constraint up to now, that will show that, e.g., the application of the Lagrange equations of the first kind alone would probably lead to that goal in only certain exceptional cases. However, if one uses a second method and first determines the motion of the system from the Lagrange equations of the second kind, i.e., its finite equations of motion, then transforms them in order to obtain the rectangular acceleration components $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ of each point, and only then employ the Lagrange equations of the first kind *for the determination of the reaction forces*, then that path will indeed be significantly simpler than the previous one, but will have the disadvantage that it requires a complicated transformation, as well as the fact that one cannot determine a certain reaction force independently of the other ones, but only along with all of the other ones (by way of the determinant!), such that it would be impossible to extend that method to continua with infinitely-many forces of constraint. Equations (6.a) and (8) will also produce λ_l as functions of time, and not as functions of the coordinates, as one ordinarily requires in the applications.

A third method is most advantageous, namely, instead of the so-called "generalized" coordinates, one chooses other coordinates that no longer fulfill all of the condition equations identically, but at least one of them will no longer fulfill them, namely, the one whose reaction force is to be determined. Moreover, the constraint equations in question will also lead to further calculations along with it, and as with the first method, only in *generalized* coordinates.

However, the practical value of this latter method will be very much compromised by the fact that one will not generally manage to get by with generalized coordinates, or even better, with coordinates that are best adapted to the problem and for which the Lagrange equations of the second kind will prove their true value, and other coordinates must be chosen.

That suggests a question that is very important for the practical applications, which is not, perhaps, also lacking in theoretical significance, namely, the question of whether it is therefore impossible to obtain an isolated, well-defined reaction force independently of the other ones from the Lagrange equations of the second kind "by adding a new parameter" and a "new assignment of the constant" in it. One would have then gained the advantage that one could also exploit the full use of those equations for the determination of the reaction force precisely when a skillful chose of coordinates for the Lagrange equations of the second kind would give the solution of the problem of motion.

The examination that I have made of that problem, which defines the content of the present work, has implied that this will actually happen under certain assumptions along a relatively-simple path.

The course of that investigation will split into three parts of itself:

I. The mechanical-theoretical part, which includes the derivation and establishment of the fundamental relations [equations (17) and (18)].

II. Discussion of the question: How can one base a method for the actual determination of constraint forces upon that?

III. Examples.

In addition, a fourth part was appended, namely, I.*a*, which will include the attempt at a geometric interpretation.

Part I.

Ansatz and assumptions.

n points with masses:

$$m_1 = m_2 = m_3 , \qquad m_4 = m_5 = m_6 , \qquad \ldots, \qquad m_{3n-2} = m_{3n-1} = m_{3n} ,$$

and rectangular coordinates:

$$x_1, x_2, x_3, \qquad x_4, x_5, x_6, \qquad \ldots, \qquad x_{3n-2}, x_{3n-1}, x_{3n} \,,$$

resp., shall define a system, i.e., the set of all mutually-connected ones, when they are coupled by τ (not by any means absolute) constraints:

$$\varphi_1 (x_1, x_2, \ldots, x_{3n}, t) = 0, \quad \varphi_2 (x_1, x_2, \ldots, x_{3n}, t) = 0, \quad \ldots, \quad \varphi_\tau (x_1, x_2, \ldots, x_{3n}, t) = 0, \qquad (1)$$

which can include time explicitly but should be holonomic. The system is under the influence of certain explicit (internal and external) forces. The resultant of all forces on the first point has the components X_1, X_2, X_3, the resultant that acts on the second point has components X_4, X_5, X_6, etc., and the one that acts upon the last point has the components X_{3n-2}, X_{3n-1}, X_{3n}.

The number $3n - \tau$ of x's gives the number of degrees of freedom of the system. Since we would like to employ the Lagrange equations of the second kind in their simplest form, we must next make:

Assumption 1: It is possible to describe the motion of the system with its s degrees of freedom by s variable parameters $p_1, p_2, \ldots, p_s$ that fulfill all constraint equations identically (generalized coordinates) and are connected to the rectangular coordinates by the formulas:

$$\begin{aligned} x_1 &= f_1 (p_1, p_2, \ldots, p_s, t) \,, \\ x_2 &= f_2 (p_1, p_2, \ldots, p_s, t) \,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{3n} &= f_{3n} (p_1, p_2, \ldots, p_s, t) \,. \end{aligned} \qquad (2)$$

Actually, in order to apply the Lagrange equations, it is additionally necessary that the parameters $p_1, \ldots, p_s$ should be independent of each other, and in fact geometrically, as well as mechanically, in order for each of them to be varied arbitrarily without necessarily implying variations of the remaining ones.

We would like to derive a consequence from the assumption that the coordinates $p_1, p_2, \ldots, p_s$ fulfill the constraint equations identically (characterize that assumption by a functional relationship, respectively): If one transforms the functions φ in (1) by means of the formulas (2) then they will go to functions of the $p_1, p_2, \ldots, p_s$. If the $p_1, p_2, \ldots, p_s$ are supposed to satisfy the constraint equations identically then when those functions, thus-transformed, are set equal to zero, they can be subject to no constraint, i.e., those equations must be fulfilled for every system of values $p_1, \ldots, p_s$, or each of the transformed functions must be independent of $p_1, p_2, \ldots, p_s$.

Now, the necessary condition for that (for the complete independence of the functions φ from the remaining p functions, it is also sufficient) is:

$$\frac{\partial \varphi_i}{\partial p_h} = \sum_{\kappa=1}^{3n} \frac{\partial \varphi_i}{\partial x_\kappa} \frac{\partial x_\kappa}{\partial p_h} = 0, \qquad i = 1, 2, \ldots, \tau; \; h = 1, 2, \ldots, s. \; (3)$$

Assumption 2: We would now like to exhibit the fact that after dropping the l^{th} constraint equation $\varphi_l = 0$, the motion of the point system can be described correctly by the now-present $s + 1$ degrees of freedom in terms of the $s + 1$ parameters $p_1, p_2, \ldots, p_s, p_{s+l}$, where p_{s+l} is added to the variables that are already present as a new variable coordinate.

Formulas (2.*a*) will now enter in place of formulas (2):

$$\begin{aligned} x_1 &= g_1 (p_1, p_2, \ldots, p_s, p_{s+l}, t) , \\ x_2 &= g_2 (p_1, p_2, \ldots, p_s, p_{s+l}, t) , \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{3n} &= g_{3n} (p_1, p_2, \ldots, p_s, p_{s+l}, t) . \end{aligned} \tag{2.a}$$

Since we would also like to employ the Lagrange equations of the second kind in this case, we must assume that $p_1, p_2, \ldots, p_s, p_{s+l}$ are also mutually independent and that the constraint equations (1), including the l^{th} one, i.e., the constraint equations:

$$\varphi_1 (x_1, x_2, \ldots, x_{3n}, t) = 0, \qquad \ldots, \qquad \varphi_{l-1} = 0, \qquad \varphi_{l+1} = 0, \qquad \ldots, \qquad \varphi_\tau = 0, \tag{1.a}$$

are fulfilled identically.

If we transform them by means of (2.*a*) then, as above, that assumption will imply the condition:

$$\frac{\partial \varphi_r}{\partial p_{s+l}} = \sum_{\kappa=1}^{3n} \frac{\partial \varphi_i}{\partial x_\kappa} \cdot \frac{\partial x_\kappa}{\partial p_{s+l}} = 0, \quad r = 1, 2, \ldots, l-1, l+1, \ldots, \tau, \tag{4}$$

and indeed at any arbitrary location p_{s+l} .

If we once more introduce $\varphi_l = 0$ then we will have the original case of s degrees of freedom before us, but the motion will be described by $s + 1$ parameters $p_1, p_2, \ldots, p_{s+l}$ now, in contrast to the previous case.

Therefore, if we also replace the $x_1, x_2, \ldots, x_{3n}$ in:

$$\varphi_1 (x_1, x_2, \ldots, x_{3n}, t) = 0$$

with the $p_1, p_2, \ldots, p_s, p_{s+l}$ by means of (2.*a*) then:

$$\varphi_l [g_1 (p_1, p_2, \ldots, p_{s+l}, t), g_2 (p_1, p_2, \ldots, p_{s+l}, t), \ldots, g_{3n} (p_1, p_2, \ldots, p_{s+l}, t)] = 0$$

will represent the connection that necessarily exists between $p_1, p_2, \ldots, p_s, p_{s+l}$ as non-generalized coordinates.

Assumption 3: We assume that the function:

$$\varphi_l [g_1 (p_1, p_2, \ldots, p_{s+l}, t), g_2 (p_1, p_2, \ldots, p_{s+l}, t), \ldots, g_{3n} (p_1, p_2, \ldots, p_{s+l}, t)]$$

decomposes into two factors $\psi_l\,(p_1, \ldots, p_s, t)$ and $\chi_l\,(p_{s+l}, t)$, only one of which, namely, χ_l, depends upon only p_{s+l} .

Since $p_1, p_2, \ldots, p_s$ are mutually-independent parameters, $\psi_l\,(p_1, p_2, \ldots, p_s, t)$ cannot vanish, but rather ([1]):

$$\chi_l\,(p_{s+l}, t) = 0$$

must be fulfilled.

It would then follow that the transformed condition:

$$\varphi_l\,(g_1, g_2, \ldots, g_s) = 0$$

will be fulfilled by only the roots of the equation $\chi_l\,(p_{s+l}, t) = 0$.

Let any one of them be denoted by $\bar{p}_{s+l}$:

$$\chi_l\,(\bar{p}_{s+l}) = 0 \;.$$

It is important to note that when Assumption 3 is fulfilled, upon further introducing the constraint $\varphi_l = 0$, the case of $s + 1$ degrees of freedom will go to the case of s degrees of freedom completely. In particular, one can always assume that formulas (2.*a*) will again be identical to formulas (2) in that way.

If we further transform the function φ_l in (1) by means of (2.*a*) in this case and compare it with the one that is transformed by means of (2) then they will differ by only the fact that p_{s+l} is variable in the one case, but constant in the other, just like formulas (2.*a*) and (2).

It will therefore be clear that one must look for a suitable p_{s+l} from among the constant quantities of the functions f in formulas (2).

The remaining assumptions, which are once more expressly emphasized, are:

1. holonomic constraint equations.
2. scleronomic and rheonomic, but likewise holonomic, coordinates.
3. generalized coordinates.

The actual method of proof.

Under those assumptions, we can now derive a relation upon the basis of which we will be in a position to determine the reaction force R_l that corresponds to the constraint $\varphi_l = 0$ from the Lagrange equations of the second kind directly.

([1]) The case in which the transformed function is *independent* of p_{s+l} cannot occur since otherwise it would follow that the original system already had $s + 1$ degrees of freedom.

I. True motion.

We shall first address the problem of the "motion" of the given point-system with the s degrees of freedom $p_1, p_2, \ldots, p_s$ when we determine the equations of motion by integrating either:

a) the Lagrange equations of the first kind:

$$
\begin{aligned}
m_1\,\ddot{x}_1 &= X_1 + \lambda_1 \frac{\partial\varphi_1}{\partial x_1} + \lambda_2 \frac{\partial\varphi_2}{\partial x_1} + \cdots + \lambda_\tau \frac{\partial\varphi_\tau}{\partial x_1}\,, \\
m_2\,\ddot{x}_2 &= X_2 + \lambda_1 \frac{\partial\varphi_1}{\partial x_2} + \lambda_2 \frac{\partial\varphi_2}{\partial x_2} + \cdots + \lambda_\tau \frac{\partial\varphi_\tau}{\partial x_2}\,, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
m_{3n}\,\ddot{x}_{3n} &= X_{2n} + \lambda_1 \frac{\partial\varphi_1}{\partial x_{3n}} + \lambda_2 \frac{\partial\varphi_2}{\partial x_{3n}} + \cdots + \lambda_\tau \frac{\partial\varphi_\tau}{\partial x_{3n}}\,,
\end{aligned} \tag{6.a}
$$

or

b) the Lagrange equations of the second kind:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_1}\right) - \frac{\partial L}{\partial p_1} &= P_1, \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_2}\right) - \frac{\partial L}{\partial p_2} &= P_2, \\
&\cdots\cdots\cdots \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_s}\right) - \frac{\partial L}{\partial p_s} &= P_s,
\end{aligned} \tag{6}
$$

namely:

$$x_1 = \Theta_1(t), \qquad x_2 = \Theta_2(t), \qquad \ldots, \qquad x_{3n} = \Theta_{3n}(t), \tag{7.a}$$

or

$$p_1 = \vartheta_1(t), \qquad p_2 = \vartheta_2(t), \qquad \ldots, \qquad p_s = \vartheta_s(t), \tag{7}$$

respectively.

Naturally, if those two systems of equations are supposed to describe the same motion then they must be arranged such that when the second one (7) is substituted in the formulas (2), that will produce the first one (7.*a*).

Since that motion will appear to be a special case of the one with $s + 1$ degrees of freedom, we would already like to add to (7) the condition that:

$$p_{s+l} = \dot{p}_{s+l} = \text{const.},$$

so

$$p_1 = \vartheta_1(t), \qquad p_2 = \vartheta_2(t), \qquad \ldots, \qquad p_s = \vartheta_s(t), \qquad p_{s+l} = \dot{p}_{s+l} = \text{const.} \tag{7}$$

The Lagrange equations (6.*a*) are produced by the argument that each point must feel the effect of a force from each constraint, such that each constraint φ_i, $i = 1, 2, \ldots, \tau$ will produce $3n$ components ξ_κ^i, $\kappa = 1, 2, \ldots, 3n$. From **d'Alembert**'s principle, they will be proportional to the derivatives $\partial\varphi_i / \partial x_\kappa$ and the proportionality factor λ_i will be the same for all κ, so:

$$\xi_\kappa^i = \lambda_i \cdot \frac{\partial \varphi_i}{\partial x_\kappa}, \qquad \kappa = 1, 2, \ldots, 3n.$$

The force of constraint:

$$R_i^\kappa = \lambda_i \cdot \sqrt{\left(\frac{\partial \varphi_i}{\partial x_\kappa}\right)^2 + \left(\frac{\partial \varphi_i}{\partial x_{\kappa+1}}\right)^2 + \left(\frac{\partial \varphi_i}{\partial x_{\kappa+2}}\right)^2} \tag{8}$$

will act upon a point $m_\kappa = m_{\kappa+1} = m_{\kappa+2}$ as a result of the constraint φ_I .

We cannot pose the problem of deriving the reaction force R_i^κ that is defined by (8) directly from the Lagrange equations of the second kind from the outset, at least not in general (cf., Part II) since distinguishing the individual points by means of their coordinates is no longer possible with the so-called "system coordinates." On the other hand, the remark that was made in the Introduction about calculating the reaction forces from the Lagrange equations of the first kind must extended by the fact that main difficulty naturally lies in the determination of the λ_i.

We can then consider the problem that was posed to have been solved when we have succeeded in determining the λ_i from the Lagrange equations of the second kind in a simple way.

As we have seen, the Lagrange equations of the first kind include the definition of the reaction forces, so to speak. They will thus serve as an important way of controlling the proof of the validity of equations (17) and (18) by direct transformation.

Let it be remarked in regard to the Lagrange equations of the second kind that the L in (6) is first defined in rectangular coordinates:

$$L \equiv \sum_{\kappa=1}^{3n} \tfrac{1}{2} m_\kappa \dot{x}_\kappa^2 ,$$

and it can be thought of as being transformed into generalized coordinates by means of the formulas that emerge from (2) upon differentiating with respect to time:

$$\dot{x}_\kappa = \frac{\partial x_\kappa}{\partial p_1} \cdot \dot{p}_1 + \frac{\partial x_\kappa}{\partial p_2} \cdot \dot{p}_2 + \cdots + \frac{\partial x_\kappa}{\partial p_s} \cdot \dot{p}_s + \frac{\partial x_\kappa}{\partial t} \qquad \kappa = 1, 2, \ldots, 3n. \tag{9}$$

One will get:

$$\left.\begin{aligned} L &= \tfrac{1}{2}\,\dot{p}_1^2 \sum_{\kappa=1}^{3n} m_\kappa \left(\frac{\partial x_\kappa}{\partial p_1}\right)^2 + \tfrac{1}{2}\,\dot{p}_2^2 \sum_{\kappa=1}^{3n} m_\kappa \left(\frac{\partial x_\kappa}{\partial p_2}\right)^2 + \cdots + \tfrac{1}{2}\,\dot{p}_s^2 \sum_{\kappa=1}^{3n} m_\kappa \left(\frac{\partial x_\kappa}{\partial p_s}\right)^2 \\ &+ \dot{p}_1\,\dot{p}_2 \cdot \sum_{\kappa=1}^{3n} m_\kappa \frac{\partial x_\kappa}{\partial p_1}\cdot\frac{\partial x_\kappa}{\partial p_2} + \dot{p}_1\,\dot{p}_3 \cdot \sum_{\kappa=1}^{3n} m_\kappa \frac{\partial x_\kappa}{\partial p_1}\cdot\frac{\partial x_\kappa}{\partial p_2} + \cdots + \dot{p}_{s-1}\,\dot{p}_s \cdot \sum_{\kappa=1}^{3n} m_\kappa \frac{\partial x_\kappa}{\partial p_{s-1}}\cdot\frac{\partial x_\kappa}{\partial p_s} \\ &+ \dot{p}_1 \cdot \sum_{\kappa=1}^{3n} m_\kappa \frac{\partial x_\kappa}{\partial p_1}\cdot\frac{\partial x_\kappa}{\partial t} + \cdots + \dot{p}_s \cdot \sum_{\kappa=1}^{3n} m_\kappa \frac{\partial x_\kappa}{\partial p_s}\cdot\frac{\partial x_\kappa}{\partial t} + \tfrac{1}{2}\sum_{\kappa=1}^{3n} m_\kappa \left(\frac{\partial x_\kappa}{\partial t}\right)^2 \\ &= \tfrac{1}{2}\sum_{h=1}^{s}\sum_{\mu=1}^{s} \dot{p}_h\,\dot{p}_\mu \cdot \sum_{\kappa=1}^{3n} m_\kappa \cdot \frac{\partial x_\kappa}{\partial p_h}\cdot\frac{\partial x_\kappa}{\partial p_\mu} + \sum_{h=1}^{s} \dot{p}_h \cdot \sum_{\kappa=1}^{3n} m_\kappa \frac{\partial x_\kappa}{\partial p_h}\cdot\frac{\partial x_\kappa}{\partial t} + \tfrac{1}{2}\sum_{\kappa=1}^{3n} m_\kappa \left(\frac{\partial x_\kappa}{\partial t}\right)^2 \end{aligned}\right\} \quad (10)$$

oe

$$L = \tfrac{1}{2}\sum_{h=1}^{s}\sum_{\mu=1}^{s} a_{h\mu}\,\dot{p}_h\,\dot{p}_\mu + \sum_{h=1}^{s} b_h\,\dot{p}_h + \gamma\,. \quad (10.a)$$

For the left-hand side of the Lagrange equations of the second kind (6), one will then get:

$$\left.\begin{aligned} \frac{\partial L}{\partial \dot{p}_h} &= \sum_{\mu=1}^{s} a_{h\mu}\,\dot{p}_\mu + b_h, \quad h = 1, 2, \ldots, s, \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_h}\right) &= \sum_{\mu=1}^{s} a_{h\mu}\,\ddot{p}_\mu + \sum_{\rho=1}^{s}\sum_{\mu=1}^{s} \frac{\partial a_{h\mu}}{\partial p_\rho}\cdot\dot{p}_\rho\cdot\dot{p}_\mu + \sum_{\rho=1}^{s}\frac{\partial b_h}{\partial p_\rho}\cdot\dot{p}_\rho + \sum_{\mu=1}^{s}\frac{\partial a_{h\mu}}{\partial t}\cdot\dot{p}_\mu + \frac{\partial b_h}{\partial t}, \\ \frac{\partial L}{\partial p_h} &= \sum_{\rho=1}^{s}\sum_{\mu=1}^{s} \frac{\partial a_{h\mu}}{\partial p_\rho}\cdot\dot{p}_\rho\cdot\dot{p}_\mu + \sum_{\rho=1}^{s}\frac{\partial b_\rho}{\partial p_h}\cdot\dot{p}_\rho + \frac{\partial \gamma}{\partial p_h}\,. \end{aligned}\right\} \quad (11)$$

The right-hand side is known to be:

$$P_h = \sum_{\kappa=1}^{3n} X_\kappa \frac{\partial x_\kappa}{\partial p_h}\ . \quad (12)$$

II. Imagined motion.

We now imagine the same system of n points. However, it is no longer subject to τ constraints, but to $\tau - 1$ of them:

$$\varphi_1 = 0, \quad \varphi_2 = 0, \quad \ldots, \quad \varphi_{l-1} = 0, \quad \varphi_{l+1} = 0, \quad \ldots, \quad \varphi_\tau = 0 \quad (1.a)$$

(in general, the points will no longer define a system; however, that will be entirely irrelevant in what follows), when we drop the l^{th} constraint $\varphi_l = 0$. From assumption (2), we can describe the motion of the system with those $s + 1$ degrees of freedom in terms of $p_1, p_2, \ldots, p_s, p_{s+l}$, and in particular, we can employ the Lagrange equations of the second kind:

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{p}_1}\right)-\frac{\partial \bar{L}}{\partial p_1} = P_1\,,\\
&\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{p}_2}\right)-\frac{\partial \bar{L}}{\partial p_2} = P_2\,,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{p}_s}\right)-\frac{\partial \bar{L}}{\partial p_s} = P_s\,,\\
&\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{p}_{s+l}}\right)-\frac{\partial \bar{L}}{\partial p_{s+l}} = P_{s+l}\,.
\end{aligned}
\tag{13}
$$

The *vis viva* is no longer the same as it was in I and will then be denoted by $\bar{L}$. $\bar{L}$ is defined from the same expression in rectangular coordinates as L, except that one uses the formulas that emerge from (2.*a*) upon differentiating with respect to time:

$$
\dot{x}_\kappa = \frac{\partial x_\kappa}{\partial p_1}\cdot\dot{p}_1+\frac{\partial x_\kappa}{\partial p_2}\cdot\dot{p}_2+\dots+\frac{\partial x_\kappa}{\partial p_s}\cdot\dot{p}_s+\frac{\partial x_\kappa}{\partial p_{s+l}}\cdot\dot{p}_{s+l}+\frac{\partial x_\kappa}{\partial t}\,. \tag{9.a}
$$

The quantities that appear here $\frac{\partial x_\kappa}{\partial p_1}$, …, $\frac{\partial x_\kappa}{\partial p_s}$ would generally be completely different from the ones with equation (9) that bear the same symbols. However, since the assumption (3) is also assumed to be applicable, the quantities $\frac{\partial x_\kappa}{\partial p_1}$, …, $\frac{\partial x_\kappa}{\partial p_s}$ will also differ from the ones in equations (9.*a*) and (9) by just the fact that p_{s+l} is variable in one case, while constant in the other. In a certain sense, they are then equal to each other, and that would become rigorous when the constraint φ_l = 0 is reintroduced. The same thing will be likewise true of L, $a_{\rho\mu}$, b_ρ, γ, etc., in equations (10.*b*), (11.*a*), (11.*b*).

Just as before, one derives from equation (9.*a*) that:

$$
\begin{aligned}
L = {} & \underline{\tfrac{1}{2}\,\dot{p}_1^2\sum_{\kappa=1}^{3n} m_\kappa\left(\frac{\partial x_\kappa}{\partial p_1}\right)^2+\dots+\tfrac{1}{2}\,\dot{p}_s^2\sum_{\kappa=1}^{3n} m_\kappa\left(\frac{\partial x_\kappa}{\partial p_s}\right)^2}+\tfrac{1}{2}\,\dot{p}_{s+l}^2\sum_{\kappa=1}^{3n} m_\kappa\left(\frac{\partial x_\kappa}{\partial p_{s+l}}\right)^2\\
& + \underline{\dot{p}_1\,\dot{p}_2\sum_{\kappa=1}^{3n} m_\kappa\,\frac{\partial x_\kappa}{\partial p_1}\cdot\frac{\partial x_\kappa}{\partial p_2}+\dots+\dot{p}_s\,\dot{p}_{s-1}\sum_{\kappa=1}^{3n} m_\kappa\,\frac{\partial x_\kappa}{\partial p_{s-1}}\cdot\frac{\partial x_\kappa}{\partial p_s}}\\
& + \dot{p}_1\,\dot{p}_{s+l}\sum_{\kappa=1}^{3n} m_\kappa\,\frac{\partial x_\kappa}{\partial p_1}\cdot\frac{\partial x_\kappa}{\partial p_{s+l}}+\dot{p}_2\,\dot{p}_{s-1}\sum_{\kappa=1}^{3n} m_\kappa\,\frac{\partial x_\kappa}{\partial p_2}\cdot\frac{\partial x_\kappa}{\partial p_s}+\dots+\dot{p}_s\,\dot{p}_{s+l}\sum_{\kappa=1}^{3n} m_\kappa\,\frac{\partial x_\kappa}{\partial p_s}\cdot\frac{\partial x_\kappa}{\partial p_{s+l}}\\
& + \underline{\dot{p}_1\sum_{\kappa=1}^{3n} m_\kappa\,\frac{\partial x_\kappa}{\partial p_1}\cdot\frac{\partial x_\kappa}{\partial t}+\dots+\dot{p}_s\sum_{\kappa=1}^{3n} m_\kappa\,\frac{\partial x_\kappa}{\partial p_s}\cdot\frac{\partial x_\kappa}{\partial t}}+\dot{p}_{s+l}\sum_{\kappa=1}^{3n} m_\kappa\,\frac{\partial x_\kappa}{\partial p_{s+l}}\cdot\frac{\partial x_\kappa}{\partial t}+\underline{\tfrac{1}{2}\sum_{\kappa=1}^{3n} m_\kappa\left(\frac{\partial x_\kappa}{\partial t}\right)^2}\,,
\end{aligned}
$$

or

$$L = \underline{\tfrac{1}{2}\sum_{\rho=1}^{s}\sum_{\mu=1}^{s} a_{\rho\mu}\,\dot{p}_{\rho}\,\dot{p}_{\mu}} + \sum_{\mu=1}^{s} a_{\mu s+l}\,\dot{p}_{\mu} + \tfrac{1}{2} a_{s+l\,s+l}\,\dot{p}_{s+l}^{2} + \underline{\sum_{\mu=1}^{s} b_{\rho}\,\dot{p}_{\mu}} + b_{s+l}\cdot\dot{p}_{s+l} + \underline{\gamma} = L + \Lambda\,, \qquad (10.b)$$

in which:

$$\Lambda = \dot{p}_{s+l}\sum_{\mu=1}^{s} a_{\mu s+l}\cdot\dot{p}_{\mu} + \tfrac{1}{2} a_{s+l\,s+l}\cdot p_{s+l}^{2} + b_{s+l}\cdot\dot{p}_{s+l}\ .$$

In regard to the left-hand sides of equations (13), we would like to distinguish between h = 1, 2, …, s and $s + l$:

1. h :

$$\left.\begin{aligned}
&\frac{\partial\bar{L}}{\partial\dot{p}_h} = \frac{\partial L}{\partial\dot{p}_h} + \frac{\partial\Lambda}{\partial\dot{p}_h} = \frac{\partial L}{\partial\dot{p}_h} + a_{hs+l}\cdot\dot{p}_{s+l}\,,\\
&\frac{d}{dt}\left(\frac{\partial\bar{L}}{\partial\dot{p}_h}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{p}_h}\right) + \dot{p}_{s+l}\sum_{\mu=1}^{s}\frac{\partial a_{h\mu}}{\partial\dot{p}_{s+l}}\,\dot{p}_{\mu} + \dot{p}_{s+l}\,\frac{\partial b_h}{\partial\dot{p}_{s+l}} + a_{hs+l}\cdot\ddot{p}_{s+l} + \dot{p}_{s+l}\sum_{\rho=1}^{s}\frac{\partial a_{hs+l}}{\partial\dot{p}_{\rho}}\,\dot{p}_{\rho}\\
&\qquad\qquad + \dot{p}_{s+l}^{2}\cdot\frac{\partial a_{hs+l}}{\partial\dot{p}_{s+l}} + \dot{p}_{s+l}\cdot\frac{\partial a_{hs+l}}{\partial t}\,,\\
&\frac{\partial\bar{L}}{\partial p_h} = \frac{\partial L}{\partial p_h} + \frac{\partial\Lambda}{\partial p_h} = \frac{\partial L}{\partial p_h} + \cdot\,\dot{p}_{s+l}\sum_{\mu=1}^{s}\frac{\partial a_{hs+l}}{\partial p_h}\cdot\dot{p}_{\mu} + \tfrac{1}{2}\,\dot{p}_{s+l}^{2}\cdot\frac{\partial a_{s+l\,s+l}}{\partial\dot{p}_h} + \dot{p}_{s+l}\cdot\frac{\partial b_h}{\partial\dot{p}_h}.
\end{aligned}\right\}\qquad (11.a)$$

2. $s + l$:

$$\left.\begin{aligned}
&\frac{\partial \bar{L}}{\partial \dot{p}_{s+l}} = \frac{\partial \Lambda}{\partial \dot{p}_{s+l}} = \sum_{\mu=1}^{s} a_{\mu s+l} \cdot \dot{p}_\mu + a_{s+l\,s+l} \cdot \dot{p}_\mu + b_{s+l}\,, \\
&\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{p}_{s+l}}\right) = \sum_{\mu=1}^{s} a_{\mu s+l} \cdot \ddot{p}_\mu + \sum_{\mu=1}^{s}\sum_{\rho=1}^{s} \frac{\partial a_{\mu s+l}}{\partial p_\mu} \cdot \dot{p}_\rho \cdot \dot{p}_\mu + \dot{p}_{s+l} \sum_{\mu=1}^{s} \frac{\partial a_{\mu s+l}}{\partial p_{s+l}} \cdot \dot{p}_\mu + \sum_{\mu=1}^{s} \frac{\partial a_{\mu s+l}}{\partial t} \cdot \dot{p}_\mu \\
&\qquad + \ddot{p}_{s+l} \cdot a_{s+l\,s+l} + \dot{p}_{s+l} \sum_{\rho=1}^{s} \frac{\partial a_{s+l\,s+l}}{\partial p_\rho} \cdot \dot{p}_\rho + \dot{p}_{s+l}^2 \cdot \frac{\partial a_{s+l\,s+l}}{\partial p_{s+l}} + \dot{p}_{s+l} \cdot \frac{\partial a_{s+l\,s+l}}{\partial t} \\
&\qquad + \sum_{\mu=1}^{s} \frac{\partial b_{s+l}}{\partial t} \cdot \dot{p}_\mu\,, \\
&\frac{\partial \bar{L}}{\partial p_{s+l}} = \frac{\partial L}{\partial p_{s+l}} + \frac{\partial \Lambda}{\partial p_{s+l}} \\
&\qquad = \underline{\tfrac{1}{2} \sum_{\rho=1}^{s}\sum_{\mu=1}^{s} \frac{\partial a_{\rho\mu}}{\partial p_{s+l}} \cdot \dot{p}_\rho \cdot \dot{p}_\mu + \sum_{\rho=1}^{s} \frac{\partial b_\rho}{\partial p_{s+l}} \cdot \dot{p}_\rho + \frac{\partial \gamma}{\partial p_{s+l}}} + \dot{p}_{s+l} \cdot \sum_{\mu=1}^{s} \frac{\partial a_{\mu s+l}}{\partial p_{s+l}} \cdot \dot{p}_\mu \\
&\qquad + \tfrac{1}{2}\, \dot{p}_{s+l}^2 \cdot \frac{\partial a_{s+l\,s+l}}{\partial p_{s+l}} + \dot{p}_{s+l} \cdot \frac{\partial b_{s+l}}{\partial p_{s+l}}.
\end{aligned}\right\} \tag{11.b}$$

In (12), one has:

$$\bar{P}_h = \sum_{\kappa=1}^{3n} X_\kappa \cdot \frac{\partial x_\kappa}{\partial p_h} = P_h\,, \tag{12.a}$$

$$\bar{P}_{s+l} = \sum_{\kappa=1}^{3n} X_\kappa \cdot \frac{\partial x_\kappa}{\partial p_{s+l}}\,, \tag{12.b}$$

and naturally one can infer $\frac{\partial x_\kappa}{\partial p_{s+l}}$ from (2.*a*).

The left-hand sides of each of first s of equations (13) then differ from the corresponding ones in (6) only by an additional term that appears *additively*. The right-hand sides of each of them remain unchanged in comparison to (6). The remark that was made in connection with equation (9.*a*) is also valid here.

Comparing two motions and conclusion.

A certain motion will be defined by Lagrange's equations (13) precisely as before, but it will not generally coincide with the latter since the finite equations of motion here look like:

$$p_1 = \omega_1(t)\,, \quad p_2 = \omega_2(t)\,, \quad \ldots, \quad p_s = \omega_s(t)\,, \quad p_{s+l} = \omega_{s+l}(t)\,. \tag{14}$$

However, we can regard (7) *phoronomically* as a special case of (14) since the specialized equations (14):

$$p_1 = \vartheta_1(t)\,, \quad \ldots, \quad p_s = \vartheta_s(t)\,, \quad p_{s+l} = \bar{p}_{s+l} = \text{const.} \tag{14.a}$$

define the same motion as (7).

How does it look from a mechanical standpoint then?

As long one assumes that there are accelerating forces, *it must be possible* to alter the explicit forces by the addition of supplementary forces such that the point-system will describe *the same* motion under the influence of the new explicit forces for the $s + 1$ degrees of freedom of the imagined case II as it would under the effect of the original (explicit) forces with the s degrees of freedom of the actual motion I, and those supplementary forces must agree in magnitude and direction with the forces of reaction of the mechanical device that realizes the constraint φ_l .

That conclusion (which can be extended to the case of equilibrium with no further discussion) is, as we see, essentially the same as the one that the Lagrange equations of the first kind implied, so the reaction forces that appear here are certainly essentially the same as the ones that were defined by (8).

However, as we have remarked before, we will not get R_l^κ [cf., (8)] here immediately, but we will get the λ_l directly (and with that we come to our actual problem) in the following way:

We imagine that we have found those supplementary forces, and let $\rho_1, \rho_2, \ldots, \rho_s, \rho_{s+l}$ be their generalized components, i.e., when we denote the rectangular components of the supplementary from now on by $\xi_1^l, \xi_2^l, \ldots, \xi_{3n}^l$ (corresponding to the X_l):

$$\rho_1 = \sum_{\kappa=1}^{3n} \xi_\kappa^l \frac{\partial x_\kappa}{\partial p_1}, \quad \rho_2 = \sum_{\kappa=1}^{3n} \xi_\kappa^l \frac{\partial x_\kappa}{\partial p_2}, \quad \ldots, \quad \rho_s = \sum_{\kappa=1}^{3n} \xi_\kappa^l \frac{\partial x_\kappa}{\partial p_s}, \quad \rho_{s+l} = \sum_{\kappa=1}^{3n} \xi_\kappa^l \frac{\partial x_\kappa}{\partial p_{s+l}} . \tag{15}$$

When we add those generalized components, by assumption, the Lagrange equations:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{p}_1}\right) - \frac{\partial \bar{L}}{\partial p_1} &= P_1 + \rho_1 , \\
\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{p}_2}\right) - \frac{\partial \bar{L}}{\partial p_2} &= P_2 + \rho_2 , \\
&\ldots\ldots\ldots\ldots \\
\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{p}_s}\right) - \frac{\partial \bar{L}}{\partial p_s} &= P_s + \rho_s , \\
\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{p}_{s+l}}\right) - \frac{\partial \bar{L}}{\partial p_{s+l}} &= P_{s+l} + \rho_{s+l}
\end{aligned} \tag{16}$$

will then determine the same motion as that the Lagrange equations (6), namely, the motion (7) [or (14.*a*)].

If we define the left-hand sides of equations (16) and (6) conversely by means of the functions (14.*a*) and their first and second derivatives and also introduce the coordinates on the right as functions of time, and possibly the velocities and accelerations that might appear in them, by means of (14.*a*) then it would be in the nature of the integral of a differential equation that every equation

(16) and every equation (6) would imply an identity with respect to time *t*. Now, since the left-hand sides of equations (16) go to the left-hand sides of equations (6) [cf., (11.*a*)] by means of (14.*a*), it would follow from a comparison of the right-hand sides of those *s* equations that all ρ_1, ρ_2, …, ρ_s must vanish.

This result, which is significant because it shows that it is just not possible to already determine the λ_l from ρ_1, …, ρ_s since they are all zero, can also be derived more simply from the argument that we indeed already know that ξ_1^l, ξ_2^l, …, ξ_{3n}^l are the components of the reaction forces R_l^κ. Therefore:

$$\rho_h = \sum_{\kappa=1}^{3n} \xi_\kappa^l \frac{\partial x_\kappa}{\partial p_h} = \lambda_l \cdot \sum_{\kappa=1}^{3n} \frac{\partial \varphi_l}{\partial x_\kappa} \frac{\partial x_\kappa}{\partial p_h} = \lambda_l \cdot \frac{\partial \varphi_l}{\partial p_h} = 0 \quad \text{for } h = 1, 2, \ldots, s,$$

cf., (3).

However, the last equation (16) yields the generalized constraint force components:

$$\rho_{s+l} \equiv \sum_{\kappa=1}^{3n} \xi_\kappa^l \frac{\partial x_\kappa}{\partial p_{s+l}} .$$

We can formulate that result as follows:

We will have:

$$\rho_{s+l} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_{s+l}}\right) - \frac{\partial L}{\partial p_{s+l}} - P_{s+l} \tag{17}$$

when we substitute the values of $p_1, p_2, \ldots, p_s$, $\dot{p}_1, \ldots, \dot{p}_s$, $\ddot{p}_1, \ldots, \ddot{p}_s$ *that correspond to the true motion* [*equation* (7), (14.*a*), *respectively*] *in the right-hand side, while setting* p_{s+l} *equal to* $\overline{p}_{s+l}$ *and* $\dot{p}_{s+l} = \ddot{p}_{s+l} = 0$. [On this subject, cf., (11.*b*) and (12.*b*).]

Naturally, for the case of equilibrium, one will have to set $\dot{p}_1 = \dot{p}_2 = \ldots = \dot{p}_s = \ddot{p}_1 = \ddot{p}_2 = \ldots = \ddot{p}_s = 0$ after the differentiation, while one substitutes those values $\overset{\circ}{p}_1$, $\overset{\circ}{p}_2$, …, $\overset{\circ}{p}_s$ of $p_1, p_2, \ldots, p_s$ that correspond to the equilibrium configuration.

Now, how does ρ_{s+l} relate to λ_l ? From the defining equations of ρ_{s+l}, that is equal to:

$$\rho_{s+l} = \sum_{\kappa=1}^{3n} \xi_\kappa^l \frac{\partial x_\kappa}{\partial p_{s+l}} = \lambda_l \sum_{\kappa=1}^{3n} \frac{\partial \varphi_l}{\partial x_\kappa} \frac{\partial x_\kappa}{\partial p_{s+l}}$$

because, for the aforementioned reasons, this ξ_κ^l is the same as the one in Lagrange's equations of the first kind, so it is equal to:

$$\xi_\kappa^l = \lambda_l \cdot \frac{\partial \varphi_l}{\partial x_\kappa} .$$

Therefore:

$$\rho_{s+l} = \lambda_l \cdot \frac{\partial \varphi_l}{\partial p_{s+l}}, \tag{18.a}$$

in which one must naturally likewise set $p_{s+l} = \overset{o}{p}_{s+l}$ after the differentiation, but $p_1, p_2, \ldots, p_s$ must be set equal to the expressions (14). It follows from (18.*a*) that:

$$\lambda_l = \rho_{s+l} \cdot \frac{1}{\dfrac{\partial \varphi_l}{\partial p_{s+l}}}. \tag{18}$$

In words: *One obtains* λ_l *from the* ρ_{s+l} *that is calculated from* (17) *upon dividing by* $\dfrac{\partial \varphi_l}{\partial p_{s+l}}$.

Second proof: We would now like provide a direct analytical proof of the result that was just derived, and therefore likewise proved, by transforming it into the one that was proved before.

In order to do that, we shall employ the identity:

$$\sum_{\kappa=1}^{3n} (m_\kappa \ddot{x}_\kappa - X_\kappa) \frac{\partial x_\kappa}{\partial p_h} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_h}\right) - \frac{\partial L}{\partial p_h} - P_h, \tag{19}$$

or the one that is equivalent to it, due to (12):

$$\sum_{\kappa=1}^{3n} m_\kappa \ddot{x}_\kappa \frac{\partial x_\kappa}{\partial p_h} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_h}\right) - \frac{\partial L}{\partial p_h}, \tag{19.a}$$

which is true for all values of the p and x that are associated by way of (2), as well as their first and second derivatives with respect to time, even when those associated values are the integrals of those Lagrange differential equations. It should be remarked in regard to the proof of (19) or (19.*a*) that these identities can be confirmed by performing the differentiations in formulas (2) directly. One will find a second, simpler, proof in **Boltzmann**, Part II of *Principe der Mechanik*, pp. 41.

We shall now make a special use of that identity for the Case II and take the particular equation for the coordinate p_{s+l}:

$$\sum_{\kappa=1}^{3n} (m_\kappa \ddot{x}_\kappa - X_\kappa) \frac{\partial x_\kappa}{\partial p_{s+l}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_{s+l}}\right) - \frac{\partial L}{\partial p_{s+l}} - P_{s+l}$$

from the associated system of equations (13) and only now consider equations (14.*a*) = (7) [(7.*a*), resp.], so, from (17), the right-hand side will be equal to ρ_{s+l} for those special values of $p_1, p_2, \ldots, p_s$, and their derivatives. On the left-hand side, all of the coordinates, velocities, and accelerations refer to the case of the motion I in any event. We can then think of that part as arising from the Lagrange equations of the first kind (6.*a*) and therefore set them equal to:

$$\sum_{\kappa=1}^{3n}(m_\kappa \ddot{x}_\kappa - X_\kappa)\frac{\partial x_\kappa}{\partial p_{s+l}} = \lambda_1 \sum_{\kappa=1}^{3n}\frac{\partial \varphi_1}{\partial x_\kappa}\cdot\frac{\partial x_\kappa}{\partial p_{s+l}} + \cdots + \lambda_l \sum_{\kappa=1}^{3n}\frac{\partial \varphi_l}{\partial x_\kappa}\cdot\frac{\partial x_\kappa}{\partial p_{s+l}} + \cdots + \lambda_\tau \sum_{\kappa=1}^{3n}\frac{\partial \varphi_\tau}{\partial x_\kappa}\cdot\frac{\partial x_\kappa}{\partial p_{s+l}}$$
$$= \lambda_1 \cdot \frac{\partial \varphi_1}{\partial p_{s+l}} + \cdots + \lambda_l \cdot \frac{\partial \varphi_l}{\partial p_{s+l}} + \cdots + \lambda_\tau \cdot \frac{\partial \varphi_\tau}{\partial p_{s+l}}.$$

According to (4), all terms will vanish except for $\lambda_l \cdot \frac{\partial \varphi_l}{\partial p_{s+l}}$, such that we have once more confirmed that:

$$\rho_{s+l} = \lambda_l \cdot \frac{\partial \varphi_l}{\partial p_{s+l}},$$

in which:

$$\rho_{s+l} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_{s+l}}\right) - \frac{\partial L}{\partial p_{s+l}} - P_{s+l}.$$

The derivation here also implies that the differential quotient $\frac{\partial \varphi_l}{\partial p_{s+l}}$ refers to the location $\bar{p}_{s+l}$.

Concluding remark.

This would now be the place to prove that not all of the assumptions are necessary. In regard to the first and second ones, namely, holonomic constraint equations and coordinates, that cannot be decided with no further analysis since it is very debatable whether the individual analytical relations can be given the mechanical meaning that they now have at all, or at least the same simple one. The case of constraint inequalities has been likewise left uninvestigated. By contrast, the last assumption of generalized coordinates is certainly unnecessary since if σ constraint equations exist between the s variable parameters $p_1, p_2, \ldots, p_s$, and the reaction force that acts on an arbitrary point m_κ is to be determined from each of the $\tau - \sigma$ constraints, which are fulfilled identically by the $p_1, p_2, \ldots, p_s$, and then eliminated, then this case will not really be essentially different from that of generalized coordinates: We can just as well regard the reaction forces that originate in the constraints that expressly carried and which we assume to be known (we assume to have been already calculated, respectively) as explicit forces, like the ones that were given originally, and think of those reaction forces as being combined with the latter. That will explain the fact that all of the results that were derived before can also be extended to the present case with no further discussion.

I would now like to follow through a line of reasoning in connection with this rigorously-followed path of the investigation up to now whose main results were (17) and (18) that will define an extension of it that is indeed unnecessary, but still worth mentioning. For me, it was additionally of great heuristic value and will provide us with relations that can simplify the actual calculation of a constraint force R_i^κ considerably in some special cases.

However, in regard to the "fictitious" geometric interpretation that was employed in it, it was proved right at the beginning that we could make use of only a picture in it, while the analytical relations will take on an actual meaning that is completely independent of whether that picture is admissible.

Part I.a.

Geometric interpretation.

1. Fundamental, purely-geometric considerations.

We start from the formulas:

$$\begin{aligned} x_1 &= f_1\,(p_1, p_2, \ldots, p_s, p_{s+l}, t_0)\,, \\ x_2 &= f_2\,(p_1, p_2, \ldots, p_s, p_{s+l}, t_0)\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{3n} &= f_{3n}\,(p_1, p_2, \ldots, p_s, p_{s+l}, t_0)\,, \end{aligned} \tag{2.a}$$

in which we once more write f in place of the symbol g in order to suggest that those functions g will go to the functions f in formulas (2) with no further discussion upon reintroducing that constraint $\varphi_l = 0$, and we assume only that the assumption 3 is fulfilled. We imagine a location $\overset{\circ}{p}_1$, $\overset{\circ}{p}_2, \ldots, \overset{\circ}{p}_s, \overset{\circ}{p}_{s+l}$ that simultaneously corresponds to a value t_0. Naturally, we take the associated values of t_0 and $(p)_0$ for the true motion.

They belong to a system of values $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{3n}$ that defines the same configuration of the point-system by way of formulas (2.*a*). All of the following considerations refer to such a location as the starting point, in which the coordinates are still variable, but time t will preserve the constant value t_0.

We would like to think of a point P_0 as being defined by that location:

$$(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{3n}) = (\overset{\circ}{p}_1, \overset{\circ}{p}_2, \ldots, \overset{\circ}{p}_s, \overset{\circ}{p}_{s+l}).$$

We can now regard the set of all systems of values of $x_1, x_2, \ldots, x_{3n}$ for which a certain one of all the parameters $p_1, p_2, \ldots, p_s, p_{s+l}$, say, p_h, $h = 1, \ldots, s, s + l$, remains constant by assuming that a function Φ_h of $x_1, x_2, \ldots, x_{3n}$ is given for each h, such that when it is transformed by means of (2.*a*) and set equal to zero, only p_h will be subject to a constraint:

$$\Phi_h\,(x_1, x_2, \ldots, x_{3n}) = \Phi_h\,(p_h) = 0\,, \qquad h = 1, 2, \ldots, s, s + l. \tag{20}$$

We have learned about one such function in the special case of the coordinate p_{s+l} in φ_l. Despite the fact that we are mainly interested in only that case, we would still like to develop the following more general one.

We imagine that there are more such functions ψ_h $(x_1, x_2, \ldots, x_{3n})$ that will be equal to p_h identically under the given transformation:

$$\psi_h (x_1, x_2, \ldots, x_{3n}) = p_h + c_h , \qquad h = 1, 2, \ldots, s, s + l , \quad (21)$$

in which the constant c_h should only be independent of all p.

Any function Φ_h of p_h will then seem to be obviously a function of ψ_h, and in addition the differential quotients:

$$\frac{\partial p_h}{\partial x_1}, \quad \frac{\partial p_h}{\partial x_2}, \quad \ldots, \quad \frac{\partial p_h}{\partial x_{3n}},$$

which we would like to consider mainly at the location $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{3n}$, can be defined in terms of the new functions.

We can now interpret ψ_h $(x_1, x_2, \ldots, x_{3n}) = 0$, which is equivalent to $p_h + c_h = 0$, as a type of "surface" and consistently interpret:

$$\frac{\partial p_h}{\partial x_1}\cdot\frac{1}{W_h^h}, \qquad \frac{\partial p_h}{\partial x_2}\cdot\frac{1}{W_h^h}, \qquad \ldots, \qquad \frac{\partial p_h}{\partial x_{3n}}\cdot\frac{1}{W_h^h}, \tag{22}$$

when

$$W_h^h = \sqrt{\left(\frac{\partial p_h}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial p_h}{\partial x_{3n}}\right)^2},$$

as the direction cosines of their normal. The special surface:

$$\bar{\psi}_h (x_1, x_2, \ldots, x_{3n}) \equiv p_h - \overset{\circ}{p}_h = 0 ,$$

which one derives from the general case for $c_h = - \overset{\circ}{p}_h$, goes through the location:

$$(\overset{\circ}{p}_1, \ldots, \overset{\circ}{p}_s, \overset{\circ}{p}_{s+l}) = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{3n}),$$

or as we can say, through the point P_0, since $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{3n}$ fulfill the equation $\bar{\psi}_h = 0$ according to (2.*a*) and (21).

If we leave all p and $t = t_0$ constant in (2.*a*), while p_μ is variable then those formulas will assume the simpler form:

$$\begin{aligned} x_1 &= f_1 (p_\mu) , \qquad \mu = 1, 2, \ldots, s, s + l, \\ x_2 &= f_2 (p_\mu) , \\ &\ldots\ldots\ldots\ldots \\ x_{3n} &= f_{3n} (p_\mu) . \end{aligned} \tag{23}$$

We would likewise prefer to interpret (23), in which all coordinates x will generally vary when p_μ varies, and therefore *all* points of the system will describe certain paths that are the parametric representation of a *curve*, as it were, and the differential quotients of those functions f:

$$\frac{\partial x_1}{\partial p_\mu} \cdot \frac{1}{W_t^\mu}, \qquad \frac{\partial x_2}{\partial p_\mu} \cdot \frac{1}{W_t^\mu}, \qquad \ldots, \qquad \frac{\partial x_{3n}}{\partial p_\mu} \cdot \frac{1}{W_t^\mu}, \tag{24}$$

when

$$W_i^\mu = \sqrt{\left(\frac{\partial x_1}{\partial p_\mu}\right)^2 + \cdots + \left(\frac{\partial x_{3n}}{\partial x_\mu}\right)^2},$$

will be the direction cosines of its normal.

According to the constant values of the remaining p, that curve will have a different position since the functional relationship (23) would then become a different one. If we denote the functions f for the special values of the remaining p :

$$\overset{\circ}{p}_1, \ \overset{\circ}{p}_2, \ \ldots, \ \overset{\circ}{p}_{\mu-1}, \ \overset{\circ}{p}_{\mu+1}, \ \ldots, \ \overset{\circ}{p}_s, \ \overset{\circ}{p}_{s+1}$$

by $\bar{f}$ then the curve that is defined by:

$$\begin{aligned} x_1 &= \bar{f}_1(p_\mu), \\ x_2 &= \bar{f}_2(p_\mu), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{3n} &= \bar{f}_{3n}(p_\mu), \end{aligned} \tag{25}$$

for $p_\mu = \overset{\circ}{p}_\mu$ will go through the location:

$$(\overset{\circ}{p}_1, \ldots, \overset{\circ}{p}_s, \overset{\circ}{p}_{s+l}) = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{3n}),$$

i.e., through the point P_0, or in other words: (25) will yield the system of values $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{3n}$ for $p_\mu = \overset{\circ}{p}_\mu$.

We would now like to relate the curve $(\bar{f}_\mu)$ that is defined by (25) with the surface $\bar{\psi}_h$, and we must then distinguish between $h \neq \mu$ and $h = \mu$.

We shall first assume that $h = \mu$, and we have just found that for this case, the surface $\bar{\psi}_h$ and the curve $(\bar{f}_\mu)$ will certainly have a point in common with each other, namely, P_0. Since the surface $\bar{\psi}_h$ is defined by:

$$p_h - \overset{\circ}{p}_h = 0 ,$$

but p_h is a variable parameter in the representation of the curve $(\bar{f}_\mu)$, so they will have that and only that point P_0 in common with each other. If $h \neq \mu$ then it will be obvious that P_0 will still be a common point of $\bar{\psi}_h = 0$ and $(\bar{f}_\mu)$. However, all points of the curve $(\bar{f}_\mu)$ will now lie upon the surface $\bar{\psi}_h = 0$, in addition, since $\bar{\psi}_h = 0$ means that $p_h = \overset{\circ}{p}_h$, and that is also assumed of all curves $(\bar{f}_\mu)$. [cf., Fig. 1]

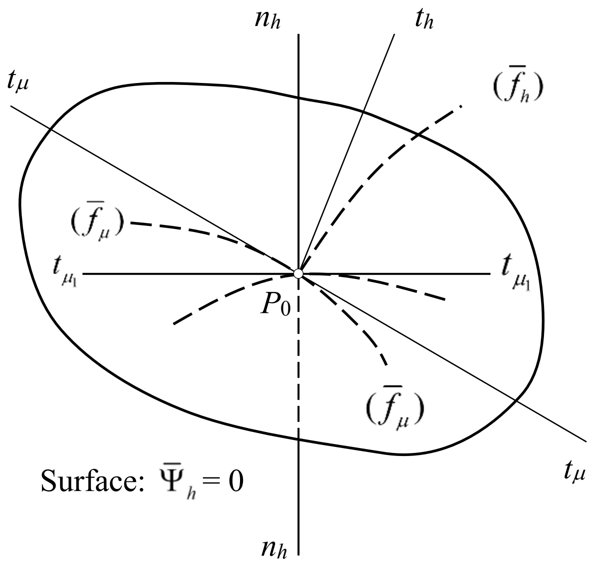


Figure 1.

Fig. 1 is drawn under the assumption that $n = 1$, i.e., a single point and ordinary three-dimensional space (see Appendix 4).

We infer from the geometric picture that was developed up to now that the normal to $\bar{\psi}_h = 0$ must also be perpendicular to all curves $(\bar{f}_\mu)$, $\mu \neq h$ (their tangents at the point P_0, respectively). Proceeding consistently, we will regard the expression:

$$\left(\frac{\partial p_h}{\partial x_1} \cdot \frac{1}{W_n^h}\right)\left(\frac{\partial x_1}{\partial p_\mu} \cdot \frac{1}{W_t^\mu}\right) + \cdots + \left(\frac{\partial p_h}{\partial x_{3n}} \cdot \frac{1}{W_n^h}\right)\left(\frac{\partial x_{3n}}{\partial p_\mu} \cdot \frac{1}{W_t^\mu}\right) = \cos\,(n_h, t_\mu) ,$$

in which the differential quotients all refer to the location P_0, as the cosine of an angle (n_h, t_μ), and we now have to show that:

$$\frac{1}{W_n^h} \cdot \frac{1}{W_t^\mu}\left(\frac{\partial p_h}{\partial x_1} \cdot \frac{\partial x_1}{\partial p_\mu} + \cdots + \frac{\partial p_h}{\partial x_{3n}} \cdot \frac{\partial x_{3n}}{\partial p_\mu}\right) = 0 , \qquad h \neq \mu . \tag{26}$$

The expression in parentheses vanishes, as one will see when one differentiates the identity (21) with respect to p_μ and in so doing observes that $\mu \neq h$.

By contrast, one has that:

$$\cos(n_h, t_\mu) = \frac{1}{W_n^h} \cdot \frac{1}{W_t^\mu} \left(\frac{\partial p_h}{\partial x_1} \cdot \frac{\partial x_1}{\partial p_\mu} + \cdots + \frac{\partial p_h}{\partial x_{3n}} \cdot \frac{\partial x_{3n}}{\partial p_\mu} \right) = \frac{1}{W_n^h} \cdot \frac{1}{W_t^\mu} \tag{27}$$

is generally non-zero. Namely, the expression in parentheses is equal to 1 here, which can be deduced by differentiating (21) with respect to p_h.

2. Orthogonality of a coordinate p_h with respect to the remaining ones.

We say that the coordinate p_h is *orthogonal to the remaining ones* when the tangent direction t_h coincides with the normal direction n_h, when the proportion:

$$\frac{\partial p_h}{\partial x_1} : \frac{\partial p_h}{\partial x_2} : \ldots : \frac{\partial p_h}{\partial x_{3n}} = \frac{\partial x_1}{\partial p_\mu} : \frac{\partial x_2}{\partial p_\mu} : \ldots : \frac{\partial x_{3n}}{\partial p_\mu} \tag{28}$$

or

$$\frac{\partial p_h}{\partial x_\nu} = k \cdot \frac{\partial x_\nu}{\partial p_\mu}, \qquad \nu = 1, 2, \ldots, 3n \tag{28.a}$$

exists.

It follows from (28.*a*) that:

$$W_n^h = \sqrt{\left(\frac{\partial p_h}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial p_h}{\partial x_{3n}}\right)^2} = k \cdot \sqrt{\left(\frac{\partial x_1}{\partial p_h}\right)^2 + \cdots + \left(\frac{\partial x_{3n}}{\partial p_h}\right)^2},$$

so

$$W_n^h = k \cdot W_t^h .$$

On the other hand, the expression in parentheses in (27) is equal to:

$$k \cdot (W_t^h)^2 .$$

Thus, as we expected, when p_h is orthogonal to the remaining p, we will have, in fact:

$$\cos(n_h, t_\mu) = \frac{1}{k \cdot W_t^h} \cdot \frac{1}{W_t^h} \cdot k \cdot (W_t^h)^2 = 1 . \tag{29.a}$$

We can also easily confirm another closely-related consequence analytically: Namely, along with n_h, t_h must also be normal to all curves (their tangents at the point P_0, respectively) when $\mu \neq h$, i.e., they must fulfill the equation:

$$\cos(t_h, t_\mu) = \frac{1}{W_t^h} \cdot \frac{1}{W_t^\mu} \left(\frac{\partial x_1}{\partial p_h} \cdot \frac{\partial x_1}{\partial p_\mu} + \cdots + \frac{\partial x_{3n}}{\partial p_h} \cdot \frac{\partial x_{3n}}{\partial p_\mu} \right) = 0 \qquad \mu \neq h. \tag{29}$$

We can easily prove that as follows:

From (28.*a*), the expression in parentheses is equal to:

$$\frac{1}{k} \left(\frac{\partial p_h}{\partial x_1} \cdot \frac{\partial x_1}{\partial p_\mu} + \cdots + \frac{\partial p_h}{\partial x_{3n}} \cdot \frac{\partial x_{3n}}{\partial p_\mu} \right)$$

and will then vanish on the same grounds as in (26).

Based upon that explanation, the coefficients of $\dot{p}_\mu \, \dot{p}_\rho$ in the expression for L [cf., (10)] can also be given a geometric interpretation: Namely, if one sets all masses:

$$m_1 = m_2 = \ldots = m_{3n} = 1$$

then one will have:

$$\sum_{\kappa=1}^{3n} \frac{\partial x_\kappa}{\partial p_\mu} \cdot \frac{\partial x_\kappa}{\partial p_\rho} = \cos(t_h, t_\mu) \cdot W_t^\mu \cdot W_t^\rho . \tag{30}$$

At the same time, we infer from this that (30) will vanish when p_μ or p_ρ is orthogonal to the remining ones, from (29).

3. Forces.

We would like to follow through with that line of reasoning consistently for the force vectors and their components, as well.

We define a new concept of the *reaction force of the constraint* φ_l by combining all components ξ_κ^l. We define it by the expression:

$$R_l = \lambda_l \cdot \sqrt{\left(\frac{\partial \varphi_l}{\partial x_1}\right)^2 + \left(\frac{\partial \varphi_l}{\partial x_2}\right)^2 + \cdots + \left(\frac{\partial \varphi_l}{\partial x_{3n}}\right)^2} \tag{31}$$

and ascribe the $3n$ rectangular components to it:

$$\xi_1^l = \lambda_l \cdot \frac{\partial \varphi_l}{\partial x_1}, \qquad \xi_2^l = \lambda_l \cdot \frac{\partial \varphi_l}{\partial x_2}, \qquad \ldots, \qquad \xi_{3n}^l = \lambda_l \cdot \frac{\partial \varphi_l}{\partial x_{3n}}, \tag{32}$$

or since (20) says that one has:

$$\varphi_l\,(x_1, x_2, \ldots, x_{3n}) = \varphi_l\,(p_{s+l}) \tag{33}$$

for $h = s + l$, the components will be:

$$\xi_1^l = \lambda_l \cdot \frac{\partial \varphi_l}{\partial p_{s+l}} \cdot \frac{\partial p_{s+l}}{\partial x_1}, \qquad \xi_2^l = \lambda_l \cdot \frac{\partial \varphi_l}{\partial p_{s+l}} \cdot \frac{\partial p_{s+l}}{\partial x_2}, \qquad \ldots, \qquad \xi_{3n}^l = \lambda_l \cdot \frac{\partial \varphi_l}{\partial p_{s+l}} \cdot \frac{\partial p_{s+l}}{\partial x_{3n}}. \tag{34}$$

We will regard the relation:

$$\xi_1^l : \xi_2^l : \ldots : \xi_{3n}^l = \frac{\partial \varphi_l}{\partial x_1} : \frac{\partial \varphi_l}{\partial x_2} : \ldots : \frac{\partial \varphi_l}{\partial x_{3n}} = \frac{\partial p_{s+l}}{\partial x_1} : \frac{\partial p_{s+l}}{\partial x_2} : \ldots : \frac{\partial p_{s+l}}{\partial x_{3n}} \tag{35}$$

as an analogous extension, or the one that is derived from it when one recalls (31), and is equivalent to it:

$$\left.\begin{aligned} \frac{\xi_\kappa^l}{R_l} &= \frac{\lambda_l \cdot \dfrac{\partial \varphi_l}{\partial x_\kappa}}{\lambda_l \cdot \sqrt{\left(\dfrac{\partial \varphi_l}{\partial x_1}\right)^2 + \cdots + \left(\dfrac{\partial \varphi_l}{\partial x_{3n}}\right)^2}} \\ &= \frac{\lambda_l \cdot \dfrac{\partial \varphi_l}{\partial x_{s+l}} \cdot \dfrac{\partial p_{s+l}}{\partial x_\kappa}}{\lambda_l \cdot \dfrac{\partial \varphi_l}{\partial p_{s+l}} \cdot \sqrt{\left(\dfrac{\partial p_{s+l}}{\partial x_1}\right)^2 + \cdots + \left(\dfrac{\partial p_{s+l}}{\partial x_{3n}}\right)^2}} \\ &= \frac{\dfrac{\partial p_{s+l}}{\partial x_\kappa}}{W_n^{s+l}}, \quad k = 1, 2, \ldots, 3n \end{aligned}\right\} \tag{36}$$

as the analytical expression of the idea that *the direction of R_l coincides with the direction of the normal n_{s+l} to the surface* $\varphi_l\,(p_{s+l}) = 0$ (*the surface* $p_{s+l} = \overset{\text{o}}{p}_{s+l}$, *respectively*). [cf., (22) on this]

If we proceed similarly then we will arrive at the concept of the new explicit force:

$$R = \sqrt{X_1^2 + X_2^2 + \cdots + X_{3n}^2}, \tag{37}$$

with the $3n$ rectangular components:

$$X_1, X_2, \ldots, X_{3n}.$$

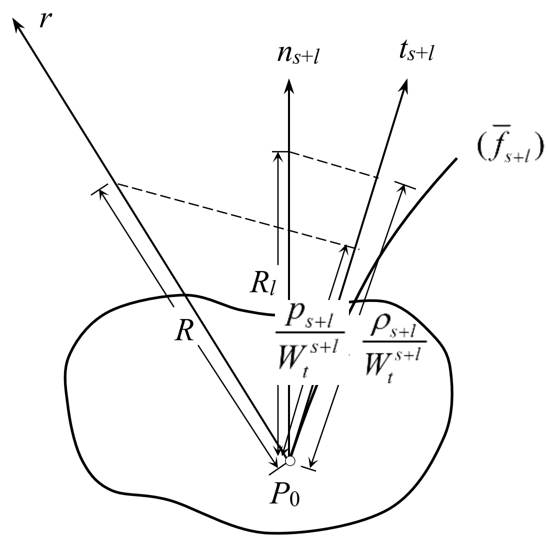
When the numbers:

$$\frac{X_1}{R}, \frac{X_2}{R}, \ldots, \frac{X_{3n}}{R} \tag{38}$$

are interpreted as direction cosines, they will determine the direction r of R, and the expression:

$$\frac{X_1}{R} \cdot \frac{\frac{\partial x_1}{\partial p_h}}{W_t^h} + \frac{X_2}{R} \cdot \frac{\frac{\partial x_2}{\partial p_h}}{W_t^h} + \cdots + \frac{X_{3n}}{R} \cdot \frac{\frac{\partial x_{3n}}{\partial p_h}}{W_t^h} = \cos(r, t_h) \tag{39}$$

will take the form of the cosine of the angle between the direction r and the tangent t_h to the curve $(\bar{f}_h)$, $h = 1, 2, \ldots, s, s + l$.



Surface $\varphi_l = 0$ ($p_{s+l} = \overset{\circ}{p}_{s+l}$, respectively)

Figure 2.

It will then follow from this, when one recalls (12.*b*), that:

$$P_h = \sum_{\kappa=1}^{3n} X_\kappa \frac{\partial x_\kappa}{\partial p_h} = R \cdot W_t^h \cdot \cos(r, t_h), \tag{40}$$

or

$$\frac{P_h}{W_t^h} = R \cdot \cos(r, t_h),$$

and in particular, for $h = s + l$:

$$\frac{P_{s+l}}{W_t^{s+l}} = R \cdot \cos(r, t_{s+l}). \tag{41}$$

In words:

$\frac{P_h}{W_t^h}$ *and in particular,* $\frac{P_{s+l}}{W_t^{s+l}}$ *is, as it were, the component of R that falls along the direction t (and indeed* t_h *or* t_{s+l}*, respectively) or the projection of R onto* t_h *(*t_{s+l}*, respectively).*

Since, from (20):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_h}\right) - \frac{\partial L}{\partial p_h} = \sum_{\kappa=1}^{3n} m_\kappa \, \ddot{x}_\kappa \cdot \frac{\partial x_\kappa}{\partial p_h}$$

is always fulfilled, when we set all masses equal to 1 and define an acceleration G with the direction g by:

$$G = \sqrt{\ddot{x}_1^2 + \ddot{x}_2^2 + \cdots + \ddot{x}_{3n}^2}\,,$$

due to the fact that:

$$\sum_{\kappa=1}^{3n} \ddot{x}_\kappa \cdot \frac{\partial x_\kappa}{\partial p_h} = G \cdot W_t^h \cos(g, t_h), \tag{42}$$

the left-hand side of each Lagrange equation of the second kind, divided by W_t^h, will take the form of the normal projection of an acceleration G onto the direction t_h .

If we then divide the Lagrange equation of the second kind:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_h}\right) - \frac{\partial L}{\partial p_h} = P_h$$

by W_t^h then the equation:

$$G \cdot \cos(g, t_h) = R \cdot \cos(r, t_h) \tag{43}$$

will follow from (40) and (42), in which all masses are set equal to 1. It includes a very remarkable mechanical-geometric meaning for the Lagrange equations of the second kind:

When all masses are set equal to 1, *from* (43), *the Lagrange equations of the second kind will prove to say nothing but that the acceleration component in the direction of motion that belongs to the varying coordinate* p_h *is equal to the force component that falls along that direction*.

Since all developments can be interpreted in ordinary Euclidian space for $n = 1$, a result of true mechanical significance will have been achieved with equation (43) in the case of a single point (application to the "inclined plane," etc.!).

Since it is always permissible to consider the reaction force R_l to be an explicit force, the ρ_1, ρ_2, …, ρ_s, ρ_{s+l} in (16) must have a meaning that corresponds entirely to that of P_1, P_2, …, P_s, P_{s+l}, namely, they represent the projections of R_h onto t_h, multiplied by W_t^h. That is confirmed by the defining formulas (15) for the ρ:

$$\rho_h = \sum_{\kappa=1}^{3n} \xi_\kappa \cdot \frac{\partial x_\kappa}{\partial p_h} = R_l \cdot W_t^h \cos(g, t_h) .$$

Now, r_l means the same direction as n_{s+l}, i.e., R_l is normal to the surface $p_{s+l} = \overset{o}{p}_{s+l}$, and therefore to all tangent direction $t_1, t_2, \ldots, t_s$, so from (26):

$$\rho_1 = \rho_2 = \ldots = \rho_s = 0 ,$$

while:

$$\rho_{s+l} = R_l \cdot W_t^{s+l} \cos(r_l, t_{s+l}) \neq 0 ,$$

from (27).

With that, the quantities ρ_h, $h = 1, 2, \ldots, s, s + l$ will admit the mechanical-geometrical interpretation:

ρ_h / W_t^h *can be regarded as the normal projection of R_l onto the tangent direction t_h*, and therefore the announced attempt at a geometric interpretation, i.e., a consistent and natural extension of the geometric relationships that actually exist in the case of a single point to a system of n points, has been developed somewhat more thoroughly. It offers us an intuitive geometric picture, and for that reason, as was mentioned before in *loc. cit.*, it will also have heuristic value. However, in regard to the examination that was carried out in Part I, it was restricted to the derivation of equation (18), and indeed the following must then be remarked:

Originally, we treated the relationship of the concept of R_l that is defined (31) to ρ_{s+l} and the discovery of the relevant relations:

$$\rho_{s+l} = \frac{R_l}{W_n^{s+l}} \qquad (R_l = \rho_{s+l} \cdot W_n^{s+l} \text{, respectively}), \tag{44}$$

which one derives immediately from the equation that will exist:

$$\rho_{s+l} = R_l \cdot W_n^{s+l} \cdot \frac{1}{W_t^{s+l}} \cdot \frac{1}{W_n^{s+l}}$$

when one recalls (27), has contributed to the geometric picture that was just sketched out to an exceptional degree, although now since R_l is only a fictitious mechanical concept, it has only a somewhat loose connection with the main topic.

However, we do see that:

If R_l, as it was defined by (31), were to define a true mechanical concept then the relations (44) would take on a more proper (self-evident, respectively) meaning in comparison to equation (18) since the quantities R_l have an immediate relationship to ρ_{s+l}, whereas they would otherwise indeed say that the same thing as (18), due to the fact that:

$$R_l = \lambda_l \cdot \frac{\partial \varphi_l}{\partial p_{s+l}} \cdot W_n^{s+l} \, .$$

In the case that was assumed, the investigation into the orthogonality of a coordinate would also take on a true meaning since the equation that is implied by (27) and (29.*a*):

$$W_n^{s+l} = \frac{1}{W_t^{s+l}} \, , \tag{45}$$

or the one that follows from when one recalls (44):

$$R_l = \rho_{s+l} \cdot \frac{1}{W_t^{s+l}} \, , \tag{46}$$

respectively, in which $\frac{1}{2}(W_t^{s+l})^2$ will emerge from the coefficient of $\dot{p}_{s+l}^2$ in $\bar{L}$ when one sets all masses equal to 1, namely, that R_l can be determined from ρ_{s+l} by means of a quantity W_t^{s+l} that is already given along with $\bar{L}$ when p_{s+l} is orthogonal to the remaining p.

4. Appendix.

We now ask: When can we make use of the computational advantage that is included in formulas (44) and (46)?

The possibility of applying the simplification that is based upon formula (44) is, as we showed thoroughly in Part II, not connected with the fact that R_l is the concept that was defined by (31). Rather, in formula (46) (should we be able to make use of it), R_l means the ordinary concept of force R_l^κ, and we then address the question of the circumstances under which that would be the case. As would emerge from a glimpse at (31), that would apply to only two cases:

1. Only a single point is present.

2. The constraint $\varphi_l = 0$ imposes a restriction upon on a single point $m_\kappa = m_{\kappa+1} = m_{\kappa+2}$, i.e., it is a so-called "absolute" constraint.

Since the case 1 is includes as a special case of 2, it would suffice to treat the latter *in extenso:*

By assumption, φ_l has the form:

$$\varphi_l \, (x_\kappa, x_{\kappa+1}, x_{\kappa+2}) = 0 \, .$$

Thus, from (31):

$$R_l = \lambda_l \cdot \sqrt{\left(\frac{\partial \varphi_l}{\partial x_\kappa}\right)^2 + \left(\frac{\partial \varphi_l}{\partial x_{\kappa+1}}\right)^2 + \left(\frac{\partial \varphi_l}{\partial x_{\kappa+2}}\right)^2} = R_l^\kappa .$$

Along with φ_l :

$$p_{s+l} \equiv \psi_{s+l} (x_\kappa, x_{\kappa+1}, x_{\kappa+2})$$

likewise includes only the three κ-coordinates of the one point: φ_l will then be a function of p_{s+l}, and when it is otherwise set equal to zero, along with ψ_{s+l}, it would then restrict more than those three κ-coordinates.

We now have to include in the calculations the fact that with formula (46), we have indeed assumed that p_{s+l} is orthogonal to the other parameters $p_1, p_2, \ldots, p_s$. When the definition of orthogonality that is given by (28) is applied to the case here, that will yield the condition:

$$\frac{\partial p_{s+l}}{\partial x_\kappa} : \frac{\partial p_{s+l}}{\partial x_{\kappa+1}} : \frac{\partial p_{s+l}}{\partial x_{\kappa+2}} = \frac{\partial x_\kappa}{\partial p_{s+l}} : \frac{\partial x_{\kappa+1}}{\partial p_{s+l}} : \frac{\partial x_{\kappa+2}}{\partial p_{s+l}} . \tag{47}$$

It has an immediate geometric meaning: Since p_{s+l} occurs in only $f_\kappa, f_{\kappa+1}, f_{\kappa+2}$, equations (23) will reduce to the three equations:

$$\begin{aligned} x_\kappa &= f_\kappa (p_{s+l}) , \\ x_{\kappa+1} &= f_{\kappa+1} (p_{s+l}) , \\ x_{\kappa+2} &= f_{\kappa+2} (p_{s+l}) , \end{aligned} \tag{48}$$

and will then represent an actual curve, just as the equation:

$$\varphi_l (x_\kappa, x_{\kappa+1}, x_{\kappa+2}) = 0 ,$$

or

$$p_{s+l} - \overset{\circ}{p}_{s+l} \equiv \psi_{s+l} (x_\kappa, x_{\kappa+1}, x_{\kappa+2}) = 0$$

respectively, will define an actual surface.

Thus, there will also be an actual tangent t_{s+l} and normal n_{s+l} here, and the condition (47) will, in fact, mean that the tangent direction t_{s+l} must coincide with the normal n_{s+l} for each point on the surface $\varphi_l = 0$ ($\psi_{s+l} = 0$, respectively).

We can then deduce a criterion for whether the coordinate p_{s+l} is or is not orthogonal that is very useful in the applications. The condition:

$$\varphi_l (x_\kappa, x_{\kappa+1}, x_{\kappa+2}) = 0$$

defines the surface on which the associated point is constrained to remain as a result of this l^{th} condition. However, (48) is the analytical expression for that space curve that gives us the path of the point $m_\kappa = m_{\kappa+1} = m_{\kappa+2}$ that would actually be described when the parameter p_{s+l} varies. It would then prove to be easy to decide whether the tangent to that curve did or did not coincide with the normal to the surface.

However, the most reliable and convenient characterization of that is included in the expression for $\bar{L}$: Namely, if p_{s+l} is orthogonal to the other p, and φ_l is an "absolute" condition then no term of the form:

$$a_{hs+l}\ \dot{p}_{s+l}\ \dot{p}_h\,, \qquad h \neq s+l$$

will enter into L since the coefficient a_{hs+l} will generally have the form:

$$\sum_{\nu=1}^{3n} m_\nu \cdot \frac{\partial x_\nu}{\partial p_h} \cdot \frac{\partial x_\nu}{\partial p_{s+l}}\,, \qquad h = 1, 2, \ldots, s,$$

and on grounds that were mentioned before in regard to (48), it will reduce to:

$$m_\kappa \left(\frac{\partial x_\kappa}{\partial p_h} \cdot \frac{\partial x_\kappa}{\partial p_{s+l}} + \frac{\partial x_{\kappa+1}}{\partial p_h} \cdot \frac{\partial x_{\kappa+1}}{\partial p_{s+l}} + \frac{\partial x_{\kappa+2}}{\partial p_h} \cdot \frac{\partial x_{\kappa+2}}{\partial p_{s+l}} \right).$$

When one recalls (47) [(28.*a*), respectively], it will then be equal to zero:

$$\sum_{\nu=1}^{3n} m_\nu \cdot \frac{\partial x_\nu}{\partial p_h} \cdot \frac{\partial x_\nu}{\partial p_{s+l}} = m_\kappa \cdot \frac{1}{K} \cdot \frac{\partial p_{s+l}}{\partial p_h} = 0\,,$$

in which K means a proportionality factor that is inserted into (47) [cf., the concluding consequences of (29), or (26), respectively].

On the other hand, the coefficient of $\frac{1}{2}\dot{p}_{s+l}^2$ is equal to:

$$\sum_{\nu=1}^{3n} m_\nu \left(\frac{\partial x_\nu}{\partial p_{s+l}} \right)^2 = m_\kappa \left[\left(\frac{\partial x_\kappa}{\partial p_{s+l}} \right)^2 + \left(\frac{\partial x_{\kappa+1}}{\partial p_{s+l}} \right)^2 + \left(\frac{\partial x_{\kappa+2}}{\partial p_{s+l}} \right)^2 \right] = m_\kappa \cdot (W_l^{s+l})^2\,, \tag{49}$$

which is a fact that demands the special value of the formula that is true along with (46):

$$R_l^\kappa = R_l = \rho_{s+l} \cdot \frac{1}{W_l^{s+l}}\,. \tag{50}$$

Part II.

Development of a new method.

This method is based upon the use of relations (17) and (18) as a way of solving the following problem:

Let a constrained point-system be given that consists of n points with masses m_κ *, external forces* X_κ, $k = 1, 2, \ldots, 3n$, *etc.* (cf., Ansatz, Part I). *Determine the individual reaction force that acts upon the point* $m_\kappa = m_{\kappa+1} = m_{\kappa+2}$ *as a result of the constraint* φ_l .

It would already emerge from the introduction to Part I that the method to be explained is based, from the outset, upon the assumption that one has found the solution to the problem of motion with the help of the Lagrange equations of the second kind (6).

We must then distinguish between two types of assumptions:

a) Ones that necessarily bear upon the way that the problem of motion is presented (its solution, respectively), and in particular, by means of Lagrange's equation of the second kind.

b) Ones that we have to make especially in regard to the application of the relations (17) and (18).

The exhibition and solution of the question of pure motion is resolved in the following way:

Suppose that one is given:

[1] the masses m_κ, $\kappa = 1, 2, \ldots, 3n$ of all n points.
[2] the forces X_κ, $\kappa = 1, 2, \ldots, 3n$, and indeed as functions x_κ and possibly their derivatives, as well as time t.
[3] the equations of constraint φ_l , $l = 1, 2, \ldots, \tau$ in rectangular coordinates.

Now, if:

[4] Assumption 1, pp. 3, is fulfilled, and one knows:
[5] the transformation formulas (2) then one can transform L into generalized coordinates [equation (10)] derive the Lagrange equations of the second kind (6) by performing the required differentiations on the expression (10), from which one will obtain the finite equations of motion (7) [(14.*a*), respectively] by integration and considering
[6] the initial conditions, which are likewise assumed to be given.

In connection with that, we now need to find the force of constraint R_l^κ that originated in the constraint $\varphi_l = 0$ and acts upon the point:

$$m_\kappa = m_{\kappa+1} = m_{\kappa+2},$$

but under the two assumptions that:

[7] Assumptions 2 and 3 are fulfilled, and that one knows
[8] formulas (2.*a*).

One exhibits the expression for $\overline{\mathrm{L}}$ *in generalized coordinates, possibly by means of formulas* (2.*a*), *and derives* ρ_{s+l} *from it by performing the differentiations that are given by* (17). *One then differentiates the function* φ_l *that is transformed by means of* (2.*a*) *with respect to* p_{s+l} *and substitutes the constant value* $\overline{\mathrm{p}}_{s+l}$ *for it after the differentiation. One will then find the* λ_l *from* (18), *and indeed initially as functions of the coordinates* p_h, *the velocities* $\dot{\mathrm{p}}_h$, *the accelerations* $\ddot{\mathrm{p}}_h$, *and time t* (h = 1, 2, …, s).

In order to obtain R_l^κ, *one will only have to carry out the square root that appears in* (8) *by means of formulas* (2) *in terms of a function of the* p_h, h = 1, 2, …, s.

λ_l, as well as R_l^κ, can be represented as functions of time alone by means of (7) [(14.*a*), respectively].

As was mentioned before (pp. 2), the representation of λ_l [R_l^κ, respectively] as a function of the coordinates p_h is much more important for the practical applications, as well as for theoretical purposes. That is because we will get a clearer picture of the functional variation of a reaction force R_l^κ for a constrained point-system when we know that force, which is a function of the relevant configuration of that system, as a function of time.

If the arrangement defines the solution to the problem that was posed is useful only in practice then it will be, on the other hand, definitive of the practical value of the relevant methods. Therefore, it will point to a special advantage of the new method in that it is especially adapted to that case to an extraordinary extent.

Instead of the complete integration of the differential equations (6), *i.e., instead of the finite equations of motion* (7), *we actually need to assume only a* ***first*** *integral of those differential equations. We can imagine that the accelerations are expressible in terms of the velocities by means of the differential equations themselves, and the velocities, and therefore also the accelerations are expressible in terms of the coordinates. As a result,* ρ_{s+l}, *along with* λ_l *and* R_l^κ, *can be represented as functions of only the parameter* p_h. *The time t that might possibly appear explicitly can be replaced with the best-suited coordinate by inverting one of the functions* (7). *Moreover, that explicit appearance of time t is, in fact, less important in the case of practical application than before when one is dealing with only the constraints* φ_l, l = 1, 2, …, τ. *On the other hand, it will imply a complication that is in the very nature of the problem itself, because in order for time t to not occur explicitly in the result, it would be necessary that* ρ_{s+l} *would have to be free of it, and one therefore assumes scleronomic constraints, as well as pure forces of motion* X_κ. *The first assumption is necessarily connected with the fact that time t is also missing from formulas* (2), *and therefore* (2.*a*), *as well.*

The situation will take an especially simple form when a force function exists. In that case, the principle of the conservation of energy in generalized coordinates will present itself as a first integral. That is because the expressions for the concepts of *vis viva* and *force function* that appear in it must already be defined by exhibiting the Lagrange equations (6):

$$\left(P_h = -\frac{\partial V}{\partial p_h} \right).$$

However, in most of the cases that occur in applications, many simplifications will present themselves, such as having direct knowledge of L, *et al.*, such that the actual calculation will often proceed more simply than it does in general. Namely, that is true of the transformation of the φ_l into generalized coordinates, which can ordinarily be done with no formulas (it is already achieved when one discovers a coordinate p_{s+l} that corresponds to the assumptions, respectively).

One will get a far-reaching simplification of a general type when one knows how to invert formulas (2.*a*), i.e., the functions ψ_h, $h = 1, 2, \ldots, s, s + l$ in equation (21). In that case, we can make use of the formula:

$$R_l^\kappa = \lambda_l \cdot \frac{\partial \varphi_l}{\partial p_{s+l}} \cdot \sqrt{\left(\frac{\partial p_{s+l}}{\partial x_\kappa}\right)^2 + \left(\frac{\partial p_{s+l}}{\partial x_{\kappa+1}}\right)^2 + \left(\frac{\partial p_{s+l}}{\partial x_{\kappa+2}}\right)^2}$$

that corresponds to (44), or:

$$R_l^\kappa = \rho_{s+l} \cdot \sqrt{\left(\frac{\partial p_{s+l}}{\partial x_\kappa}\right)^2 + \left(\frac{\partial p_{s+l}}{\partial x_{\kappa+1}}\right)^2 + \left(\frac{\partial p_{s+l}}{\partial x_{\kappa+2}}\right)^2}\,.$$

Here, as well, in order to obtain a consistent representation for R_l^κ, one must either transform the square root expression into general coordinates p_h or transform ρ_{s+l} into rectangular coordinates.

However, the most-direct determination of R_l^κ will bring about the exceptional circumstances under which one can decide which assumption to apply to formula (50):

One must first see whether φ_l is an "absolute" constraint, i.e., a restriction that is imposed upon only a single point. In the applicable case, one begins, as one does in general, with the construction of $\bar{L}$ and determines from the way that it was constructed whether the coordinate is orthogonal in the way that was given on pps. 20 and 21 Then and only then will the formula by which one obtains R_l^κ from ρ_{s+l} upon dividing by W_t^{s+l} be true, namely:

$$R_l^\kappa = \rho_{s+l} \cdot \frac{1}{W_t^{s+l}}\,. \tag{50}$$

However, one finds W_t^{s+l} *from the expression for* $\bar{L}$ *that is known already: Namely, if one sets the masses:*

$$m_\kappa = m_{\kappa+1} = m_{\kappa+2} = 1$$

then $\frac{1}{2}(W_t^{s+l})^2$ *will be the coefficient of* $\dot{p}_{s+l}^2$.

That case will become very important in a different context, and it offers some essential advantages. When a force function exists, it can happen that one can succeed in exhibiting the Lagrange equations (6) without appealing to rectangular coordinates on the basis of a geometric argument. Might one be given *L*and *V* directly in terms of generalized coordinates then nothing more would be required. Now, if that is likewise true of $\bar{L}$ then a return to rectangular coordinates will obviously be no longer necessary in the calculation of R_l^κ from equation (50).

I believe that I have then found a method that is, first of all, new ([1]), and secondly, considerably simpler than the one that has been used up to now, even in the most general case. With the simplifications that it admits in special cases, one will be in possession of its true meaning for the practical calculation of reaction forces and that might perhaps make it possible to solve some problems that were either insoluble or only by *indirect* means up to now. In particular, for systems with few degrees of freedom, but numerous constraint equations (so for *continua*, in particular), it will allow one to determine each reaction force individually, i.e., independently of the other, which can seem quite useful in practice.

Part III.

Examples.

First example: Two massive points m and m' are coupled with an inclined plane by an inextensible string.

a) The brief solution in rectangular coordinates by the ordinary method by means of Lagrange's equations of the first kind:

Coordinates:

$$m, \quad \ldots, x, \quad z,$$
$$m', \ldots, x', z'.$$

Explicit forces:

$$X = 0, \qquad Z = m g ,$$
$$X' = 0 , \qquad Z' = m' g .$$

Constraints:

$$\varphi_1 \equiv z - x \tan\alpha = 0 ,$$
$$\varphi_2 \equiv z' - x' \tan\alpha' = 0 ,$$
$$\varphi_3 \equiv x\cos\alpha + z\sin\alpha + x'\cos\alpha' + z'\sin\alpha' - l = 0 .$$

([1]) **H. K. Hollefreund** has also determined forces of constraint, among other things, in a similar manner, but only in some special examples, in a program lecture that was first made known to me after long after the completion of my work, "Die Elemente der Mechanik, etc.," Berlin, 1903/6.

The Lagrange equations of the first kind:

$$m\,\ddot{x} = \lambda_1 \frac{\partial \varphi_1}{\partial x} + \lambda_3 \frac{\partial \varphi_3}{\partial x} = -\,\lambda_1 \tan\,\alpha + \lambda_3 \cos\,\alpha\,,$$

$$m\,\ddot{z} = \lambda_1 \frac{\partial \varphi_1}{\partial z} + \lambda_3 \frac{\partial \varphi_3}{\partial z} + m\,g = \lambda_1 + \lambda_3 \sin\,\alpha + m\,g\,,$$

$$m'\,\ddot{x}' = \lambda_1 \frac{\partial \varphi_1}{\partial x'} + \lambda_3 \frac{\partial \varphi_3}{\partial x'} = -\,\lambda_1 \tan\,\alpha' + \lambda_3 \cos\,\alpha'\,,$$

$$m'\,\ddot{z}' = \lambda_1 \frac{\partial \varphi_1}{\partial z'} + \lambda_3 \frac{\partial \varphi_3}{\partial z'} + m'\,g = \lambda_1 + \lambda_3 \sin\,\alpha' + m'\,g\,,$$

$$\frac{d\varphi_1}{dt} = \dot{z} - \dot{x}\tan\alpha = 0\,,$$

$$\frac{d\varphi_2}{dt} = \dot{z}' - \dot{x}'\tan\alpha' = 0\,,$$

$$\frac{d\varphi_3}{dt} = \dot{x}\cos\alpha + \dot{z}\sin\alpha + \dot{x}'\cos\alpha' + \dot{z}'\sin\alpha' = 0\,,$$

and

$$\frac{d^2\varphi_1}{dt^2} = \ddot{z} - \ddot{x}\tan\alpha = 0\,,$$

$$\frac{d^2\varphi_2}{dt^2} = \ddot{z}' - \ddot{x}'\tan\alpha' = 0\,,$$

$$\frac{d^2\varphi_3}{dt^2} = \ddot{x}\cos\alpha + \ddot{z}\sin\alpha + \ddot{x}'\cos\alpha' + \ddot{z}'\sin\alpha' = 0\,.$$

Therefore:

$$\lambda_1\,(1 + \tan^2\,\alpha) = -\,m\,g\,,$$

$$\lambda_2\,(1 + \tan^2\,\alpha') = -\,m'\,g\,,$$

$$\frac{\lambda_3}{m} + \frac{\lambda_3}{m'} + g\,(\sin\alpha + \sin\alpha') = 0\,,$$

which makes:

$$\lambda_1 = -\,m\,g\cos^2\,\alpha\,,$$

$$\lambda_2 = -\,m'g\cos^2\,\alpha'\,,$$

$$\lambda_3 = -\,\frac{m\,m'g\,(\sin\alpha + \sin\alpha')}{m + m'}\,.$$

Thus, from formula (8) ([1]):

$$R_1^1 \equiv R_1 = m\,g\cos\,\alpha\,,$$

$$R_2^2 \equiv R_2 = -\,m'g\cos\alpha'\,,$$

([1]) Cf., e.g., **Budde**, *Mechanik*, 1890, pp. 380.

$$R_3^1 = -\frac{m\,m' g\,(\sin\alpha + \sin\alpha')}{m+m'} = R_3^2 .$$

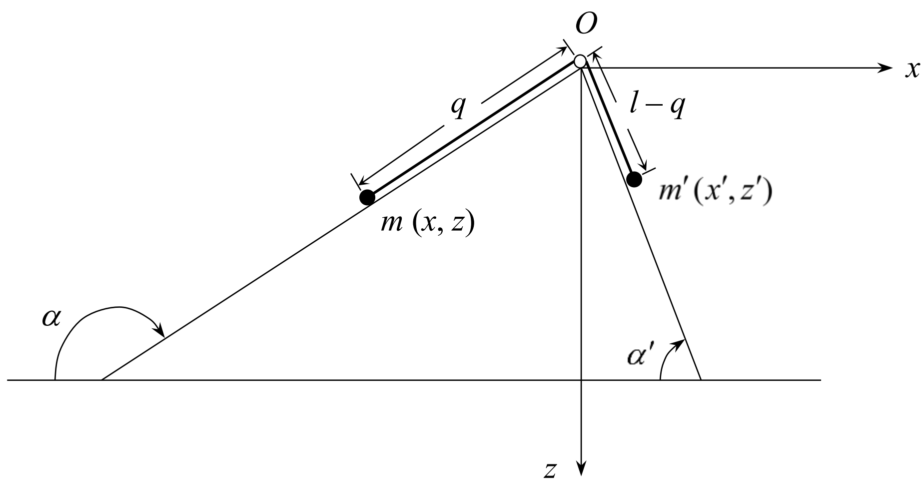


Figure 3.

b) Solution in general coordinates using the new method by means of Lagrange's equations of the second kind.

Assumption 1, pp. 3, is fulfilled for the parameter q, which corresponds to the only degree of freedom in the system. Formulas (2) are then:

$$\begin{aligned} x &= q\cos\alpha , \\ z &= q\sin\alpha , \\ x' &= (l-q)\cos\alpha' , \\ z' &= (l-q)\sin\alpha' . \end{aligned}$$

By means of them, or directly, one will find that:

$$L = \tfrac{1}{2} m\cdot\dot{q}^2 + \tfrac{1}{2} m'\cdot\dot{q}^2 = \tfrac{1}{2}(m+m')\cdot\dot{q}^2 ,$$

and likewise, that the force function that exists here is:

$$V = -\,m\,g\,q\sin\alpha - m' g\,(l-q)\sin\alpha' .$$

Thus, the Lagrange equations of the second kind (6) will be:

$$\frac{\partial L}{\partial\dot{q}} = (m+m')\cdot\dot{q} , \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}}\right) = (m+m')\cdot\ddot{q} ,$$

$$\frac{\partial L}{\partial \dot{q}} = 0 , \qquad P_q = -\frac{\partial V}{\partial q} = g\,(m \sin\alpha - m' \sin\alpha') \ .$$

Therefore:

$$\ddot{q} = \frac{g}{m+m'}(m \sin\alpha - m' \sin\alpha') \ .$$

The motion of the system is then found with that.

1. Determining the reaction forces R_3^1 and R_3^2 that are exerted upon m and m' by the constraint φ_3.

Assumptions 2 and 3 are fulfilled relative to the parameter l. For example, l = const. is equivalent to $\varphi_3 = 0$. Formulas (2.a) will emerge from (2) when one thinks of l as variable. As a result of that, or directly, one will get:

$$\bar{L} = \tfrac{1}{2} m \cdot \dot{q}^2 + \tfrac{1}{2} m' \cdot (\dot{l} - \dot{q})^2 = \tfrac{1}{2}(m+m') \cdot \dot{q}^2 + \tfrac{1}{2}\, m' \cdot \dot{l}^2 - m' \cdot \dot{l}\,\dot{q} ,$$

$$V = -\, m\, g\, q \sin\,\alpha - \, m' g\,(l - q) \sin\alpha' ,$$

which is naturally the same as before. Thus:

$$\frac{\partial \bar{L}}{\partial \dot{l}} = m' \cdot \dot{l} - m' \cdot \dot{q} , \quad \frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{l}}\right) = m'(\ddot{l} - \ddot{q}) , \qquad \frac{\partial \bar{L}}{\partial l} = 0 ,$$

$$P_l = -\frac{\partial V}{\partial l} = m' g \sin\alpha' .$$

Thus, when one now sets $\dot{l} = \ddot{l} = 0$:

$$\rho_{s+l} \equiv \rho_l = -\, m' \ddot{q} - m' g \sin\alpha' ,$$

so when one recalls the differential equation for q above:

$$\rho_{s+l} \equiv \rho_l = -\frac{m\,m'}{m+m'} \cdot g\,(\sin\alpha + \sin\alpha') .$$

The transform of φ_3 is obviously:

$$l - \underline{l} = 0 ,$$

so:

$$\frac{\partial \varphi_3}{\partial l} = 1 ,$$

so from equation (18):

$$\lambda_l = \rho_{s+l} ,$$

and corresponding to equation (8), one has moreover:

$$\left.\begin{aligned} R_3^1 &= \lambda_l \cdot \sqrt{\left(\frac{\partial \varphi_3}{\partial x}\right)^2 + \left(\frac{\partial \varphi_3}{\partial z}\right)^2} = \lambda_l = -\frac{m\,m'}{m+m'} \cdot g\,(\sin\alpha + \sin\alpha'), \\ R_3^2 &= \lambda_l \cdot \sqrt{\left(\frac{\partial \varphi_3}{\partial x'}\right)^2 + \left(\frac{\partial \varphi_3}{\partial z'}\right)^2} = \lambda_l = -\frac{m\,m'}{m+m'} \cdot g\,(\sin\alpha + \sin\alpha') \end{aligned}\right\} \quad R_3^1 = R_3^2 .$$

The assumptions on the parameter p_{s+l} correspond to α here. In formulas (2), we must now think of q and α as variable. $\bar{L}$ can also be exhibited directly here again:

$$\bar{L} = \tfrac{1}{2}\,m\cdot(\dot{q}^2 + \dot{\alpha}^2\,q^2) + \tfrac{1}{2}\,m'\cdot\dot{q}^2 = \tfrac{1}{2}\,m\cdot\dot{q}^2 + \tfrac{1}{2}\,m'\cdot\dot{q}^2 + \tfrac{1}{2}\,m'\,\dot{\alpha}^2\,q^2 .$$

V is the same as before.

$$\frac{\partial \bar{L}}{\partial \dot{\alpha}} = m\,\dot{\alpha}\,q^2 , \qquad \frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{\alpha}}\right) = m\,\ddot{\alpha}\,q^2 + 2\,m\,\dot{\alpha}\,q\cdot\dot{q} ,$$

$$\frac{\partial \bar{L}}{\partial \alpha} = 0 , \qquad P_\alpha = -\frac{\partial V}{\partial \alpha} = m\,g\,q\cos\alpha .$$

If one now sets:

$$\dot{\alpha} = \ddot{\alpha} = 0$$

then it will follow that:

$$\rho_{s+l} = \rho_\alpha = -\,m\,g\,q\cos\alpha .$$

The case of orthogonality is posed as an example here. For $m = 1$, one gets the coefficient of $\dot{\alpha}^2$ from $\bar{L}$ as:

$$\tfrac{1}{2}(W_l^\alpha)^2 = \tfrac{1}{2}q^2 , \qquad W_l^\alpha = q ,$$

so from formula (50):

$$R_1^1 = \rho_\alpha \cdot \frac{1}{W_l^\alpha} = -\,m\,g\cos\alpha .$$

If one calculates with that as a test of the general method then that will yield the result:

$$\rho_\alpha = -\,m\,g\,q\cos\alpha ,$$

$$\lambda_\alpha = \frac{\rho_\alpha}{\dfrac{\partial\varphi_1}{\partial\alpha}},$$

$$\varphi_1 = z - x \tan \underline{\alpha} = q\,(\sin \alpha - \cos \alpha \cdot \tan \underline{\alpha}),$$

$$\frac{\partial\varphi_1}{\partial\alpha} = q\,(\cos \alpha + \sin \alpha \tan \underline{\alpha}),$$

and when one sets $a = \underline{\alpha}$:

$$\frac{\partial\varphi_1}{\partial\alpha} = \frac{q}{\cos\alpha},$$

and therefore:

$$\lambda_\alpha = \frac{\rho_\alpha}{\dfrac{\partial\varphi_1}{\partial\alpha}} = \frac{-m\,g\,q\cos\alpha}{\dfrac{q}{\cos\alpha}} = -\,m\,g\cos^2 \underline{\alpha},$$

and thus, from equation (8), one will have:

$$R_1^1 = \lambda_1 \cdot \sqrt{\left(\frac{\partial\varphi_1}{\partial x}\right)^2 + \left(\frac{\partial\varphi_1}{\partial z}\right)^2} = -\,m\,g\cos^2 \cdot \sqrt{\tan^2\alpha + 1} = -\,m\,g\cos\alpha,$$

as before.

3. Determining R_2^2:

In place of α, simply α' will appear, while everything else is just as it was in 2. One has:

$$\bar{L} = \tfrac{1}{2}\, m \cdot \dot{q}^2 + \tfrac{1}{2} m' [\dot{q}^2 + \dot{\alpha}'^2 \cdot (l-q)^2].$$

Thus, α' is also orthogonal:

$$\frac{\partial\bar{L}}{\partial\dot{\alpha}'} = m'\,\dot{\alpha}'\,(l-q)^2,\quad \frac{d}{dt}\left(\frac{\partial\bar{L}}{\partial\dot{\alpha}'}\right) = m'\,\dot{\alpha}'\,(l-q)^2 - 2\,m'\,\dot{\alpha}'\,(l-q)\cdot\dot{q},\quad \frac{\partial\bar{L}}{\partial\alpha'} = 0,$$

$$P_{\alpha'} = -\frac{\partial V}{\partial\alpha'} = m'\,g\,(l-q)\cos\alpha',$$

so

$$\rho_{s+l} \equiv \rho_{\alpha'} = -\,m'\,g\,(l-q)\cos\alpha'.$$

$\bar{L}$ implies that:

$$W_l^{\alpha'} = l - q,$$

so once more, from formula (50):

$$R_2^2 = \rho_{\alpha'} \cdot \frac{1}{W_l^{\alpha'}} = - \, m' g \cos\alpha' \, .$$

Second example: Let F be a rigid, massive, planar surface with a center of mass S, total mass M, and moment of inertia K relative to the suspension point O' in the xz-plane of a rectangular coordinate system with its origin at O and lies vertically. The point O' has the coordinates x, z in this system, but it will be, in addition the coordinate origin of an axis-cross x', z' that is always parallel to the previous one, i.e., a system whose z'-axis should always point vertically, no matter how O' displaces. S has the coordinates ξ, ζ in the latter.

Determine the reaction force R that acts upon the carrier of O' when F performs oscillations about the point O' in the xz-plane.

Here, it is irrelevant whether we regard the surface F as a continuum or a manifold of discrete points. In the latter case, we assume that there are n points with masses $m_1, m_2, \ldots, m_n$ that should have the rectangular coordinates $(x_1, z_1), (x_2, z_2), \ldots, (x_n, z_n)$, resp., in the O-system.

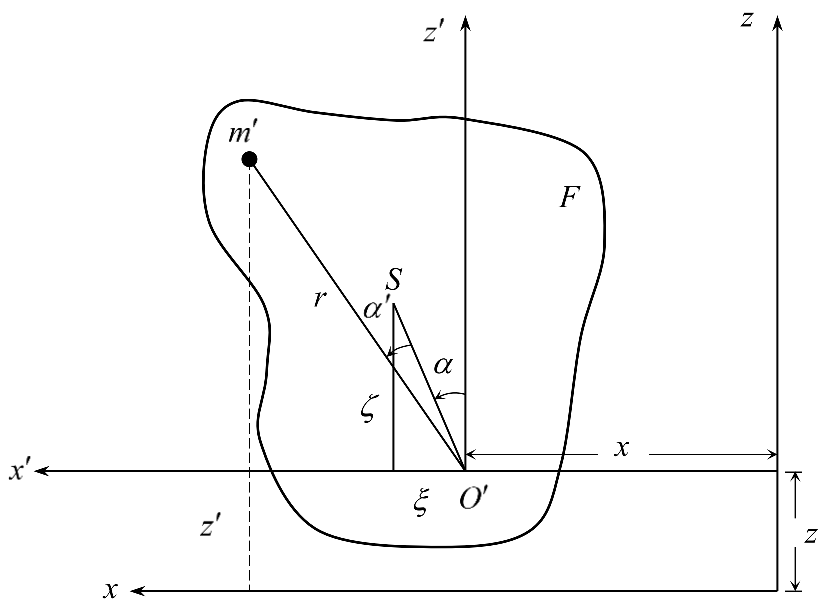


Figure 4.

All points of the system are coupled by relative constraints. In addition, the point O' is coupled by two absolute ones, because the condition that O' should be immobile can be represented analytically by only two constraints. The relative constraints are given by the *rigidity* of the surface. We need only to concern ourselves with the absolute constraints and choose them to be, most simply, x = const., z = const., such that it will read:

$$\varphi_1 \equiv x - \underline{x} = 0 \, , \qquad \varphi_2 \equiv z - \underline{z} = 0 \, ,$$

in rectangular coordinates.

The constraint forces $R_x = X$, $R_z = Z$ that originate in those constraints are likewise the rectangular components of the resultant R that one seeks, and which acts directly upon O', and only indirectly on the remaining points by means of the rigidity of the surface.

The quantities:

$$\vartheta_\nu \equiv \sphericalangle(O'm_\nu, O'S)$$

and

$$r_\nu \equiv O'm_\nu$$

(cf., Fig. 4) are constant as a result of the relative constraints, while:

$$\varphi \equiv \sphericalangle(O'S, Z')$$

is variable and corresponds to the single degree of freedom in the system. The assumption (1) is also fulfilled for this φ. Formulas (2) are obviously:

$$\left.\begin{array}{l} \left.\begin{array}{l} x_\nu = x + \sin(\vartheta_\nu + \varphi)\cdot r_\nu, \\ z_\nu = z + \cos(\vartheta_\nu + \varphi)\cdot r_\nu, \end{array}\right\} \quad \nu = 1, 2, \dots, n \\ x = x, \\ z = z, \end{array}\right\} \tag{2}$$

with the help of (2) or directly:

$$L = \sum \tfrac{1}{2} m_\nu \cdot (r_\nu \dot{\varphi})^2 = \tfrac{1}{2}\dot{\varphi}^2 \sum m_\nu \cdot r_\nu^2 = \tfrac{1}{2}\dot{\varphi}^2 K,$$

$$\frac{\partial L}{\partial \dot{\varphi}} = K\dot{\varphi}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = K\ddot{\varphi}, \qquad \frac{\partial L}{\partial \varphi} = 0,$$

$$P_\varphi = \sum m_\nu \cdot g \frac{\partial z_\nu}{\partial \varphi} = -\sum m_\nu \cdot g\, r_\nu \sin(\vartheta_\nu + \varphi) = -g\,\xi M.$$

Therefore, the Lagrange equations of the second kind for the degree of freedom φ will become:

$$K\ddot{\varphi} = -g\,\xi M. \tag{I}$$

Since:

$$V = -\sum m_\nu z_\nu = -\sum m_\nu (z_\nu - z) - g\,z M = -g\,\zeta M + \text{const.}, \tag{II}$$

const.

the energy principle will yield:

$$L + V = \tfrac{1}{2}\dot{\varphi}^2 \cdot K - g\,\zeta M = \text{const.} = c\,. \tag{III}$$

1. Determining R_x .

Clearly, Assumptions 2 and 3 are fulfilled for x. One will get formulas (2.a) from (2) when one also lets x be variables. It will yield:

$$L = \sum \tfrac{1}{2} m_\nu\,(\dot{x}_\nu^2 + \dot{z}_\nu^2) = \tfrac{1}{2}\dot{x}^2 M + \dot{x}\,\dot{\varphi}\,\zeta\,M + \tfrac{1}{2}\dot{\varphi}^2 K\,,$$

$$\frac{\partial \bar{L}}{\partial \dot{x}} = \dot{x}\,M + \dot{\varphi}\,\zeta\,M\,, \qquad \frac{\partial \bar{L}}{\partial x} = 0\,,$$

$$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{x}}\right) = \ddot{x}\,M + \ddot{\varphi}\,\zeta\,M + \dot{\varphi}\,\dot{\zeta}\,M\,,$$

or since:

$$\dot{\zeta}\,M = -\,\xi\,M\,\dot{\varphi}\,,$$

one will have:

$$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{x}}\right) = \ddot{x}\,M + \ddot{\varphi}\,\zeta\,M - \dot{\varphi}^2\,\xi\,M\,,$$

$$P_x = -\,\frac{\partial V}{\partial x} = \sum X_\nu\,\frac{\partial x_\nu}{\partial x} = 0\,,$$

such that:

$$\rho_x = -\,\ddot{\varphi}\,\zeta M - \dot{\varphi}^2 \xi M\,,$$

or when one eliminates the accelerations by means of (I):

$$\rho_x = -\,\dot{\varphi}^2 \xi M - g\,\frac{\xi\,\zeta\,M^2}{K}\,.$$

[ρ_x can also be represented as a function of φ by means of (III)!]

Since:

$$\frac{\partial \varphi_1}{\partial x} = 1\,, \qquad \frac{\partial \varphi_1}{\partial z} = 0\,,$$

from equation (18), one will have:

$$\lambda_x = \rho_x\,,$$

and from (8):

$$R_x = \lambda_x = \rho_x = -\,\dot{\varphi}^2 \cdot \xi M - g\,\xi\,\zeta\,\frac{M^2}{K}\,.$$

2. Determining R_z:

After removing $\varphi_2 = 0$, z will fulfill the assumptions (2) and (3), and the same thing will be true for z here that was true for x before in regard to formulas (2.a). That will make $\bar{L}$ equal to:

$$\bar{L} = \tfrac{1}{2}\dot{z}^2 \cdot M + \tfrac{1}{2}\dot{\psi}^2 K - \dot{z}\,\dot{\varphi}\,\xi M,$$

$$\frac{\partial \bar{L}}{\partial \dot{z}} = \dot{z} M - \dot{\varphi}\,\xi M, \quad \frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{z}}\right) = \ddot{z} M - \ddot{\varphi}\,\xi M - \dot{\varphi}\,\dot{\xi} M.$$

Now, one again has:

$$\dot{\xi} M = \dot{\varphi}\,\zeta M,$$

so:

$$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{z}}\right) = \ddot{z} M - \ddot{\varphi}\,\xi M - \dot{\varphi}^2\,\zeta M.$$

Since:

$$\frac{\partial \bar{L}}{\partial z} = 0, \qquad P_z = \sum m_\nu \cdot g \frac{\partial z_\nu}{\partial z} = g M = -\frac{\partial V}{\partial z},$$

it will ultimately follow that:

$$\rho_z = -\ddot{\varphi}\,\xi M - \dot{\varphi}^2 \zeta M - g M,$$

or after eliminating the acceleration $\ddot{\varphi}$ by means of (I):

$$\rho_z = -g\,\xi^2 \cdot \frac{M^2}{K} - \dot{\varphi}^2 \zeta M - g M.$$

Since one also has:

$$\frac{\partial \varphi_2}{\partial x} = 0, \qquad \frac{\partial \varphi_2}{\partial z} = 1,$$

here, one will have:

$$\rho_z = \lambda_z$$

and

$$R_z = \lambda_z = \rho_z = g\,\xi^2 \cdot \frac{M^2}{K} - \dot{\varphi}^2 \zeta M - g M,$$

and ultimately:

$$R^2 = R_x^2 + R_z^2.$$

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