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Discussion of the general form for light waves (†)

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1. In this first article on the theory of light waves, I propose to treat the most general form that such a wave can take in the interior of crystals that are endowed with double refraction, according to the illustrious Fresnel, and only in its geometric aspects. It is, by no means, my intention to occupy myself with the details of the optical questions here, but only to borrow the language. Before entering into that matter, I will develop some formulas that have a general usage, and which will, in turn, help is to either find the equation of the wave or to discuss it.

2. *If a surface is cut by a given plane then determine the curve of intersection in its proper plane analytically*. – Under the hypothesis of rectangular coordinates, let:

$$F(x, y, z) = 0$$

be the equation of the proposed surface. Let φ denote the angle that the intersecting plane makes with the xy-plane, and let α be the angle between the intersection of the two planes and the x-axis. The intersecting plane, which passes through the coordinate origin, moreover, will then be determined completely by the two angles φ and α. We propose to find the equation of the curve of intersection in the plane of that curve itself, while choosing the coordinate axes (v and w) to be two straight lines on that planes that are mutually perpendicular, and one of which (viz., the v-axis) will coincide with the intersection in the xy-plane.

In order to arrive at that, we first turn the x and y axes in their plane until one of the two agrees with the v-axis, while the other one is perpendicular to it. Upon distinguishing the new coordinates by primes, we will then have:

$$x = x'\cos\alpha - y'\sin\alpha, \qquad y = x'\sin\alpha + y'\cos\alpha, \qquad z = z'.$$

The intersecting plane will be perpendicular to the $x'y'$-plane, because it will pass through the x'-axis. The intersection of the two planes will then be the w-axis, whereas the v-axis will coincide with the x'-axis. The following relations will result from that for an arbitrary point of the intersecting plane:

(†) Translator's note: The accompanying figures that are alluded to in the text were not provided in either the journal article nor his collected works.

$$x' = v, \qquad y' = w \cos \varphi, \qquad z' = w \sin \varphi,$$

and upon combining these equations with the preceding equations we will obtain:

$$(1) \qquad x = v \cos \alpha - w \sin \alpha \cos \varphi, \quad y = v \sin \alpha + w \cos \alpha \cos \varphi, \quad z = w \sin \varphi.$$

Finally, in order to get the desired equation for the curve of intersection, we need only to substitute these values in the equation of the proposed surface.

If we represent the intersecting plane by the following equation:

$$Ax + By + Cz = 0$$

then we will have, from known formulas:

$$\sin^2 \alpha = \frac{A^2}{A^2 + B^2}, \qquad \cos^2 \alpha = \frac{B^2}{A^2 + B^2},$$

$$\cos^2 \varphi = \frac{C^2}{A^2 + B^2 + C^2}, \quad \sin^2 \varphi = \frac{A^2 + B^2}{A^2 + B^2 + C^2};$$

hence, we can further infer that:

$$\sin^2 \alpha \sin^2 \varphi = \frac{A^2}{A^2 + B^2 + C^2}, \qquad \cos^2 \alpha \sin^2 \varphi = \frac{B^2}{A^2 + B^2 + C^2}.$$

3. *Singularities of surfaces.* All of what I find in the treatises on the singular points of curved surface seems to me to be very unsatisfactory. In order to not lose sight of my main objective, I must be brief in what I will say, while reserving the exposition of a general theory for another occasion.

First, one must not confuse two types of singularities. The one type refers to points where the surface is touched by an infinite number of planes, and the other one, to planes that touch the surface at an infinite number of points. The tangent planes at a *singular point*, in general, envelop a *second*-degree cone, which will reduce to just one point if the tangent planes become imaginary. The singular point will then be an isolated point and will be conjugate to the surface. In the intermediate case where the cone reduces to a straight line, the singular point will constitute a *point* of the surface, in general. On the other hand, the points at which a surface is touched by a singular plane will form a conic section, in general. If that conic becomes imaginary then the tangent plane will be an isolated plane and conjugate to the surface. Before it disappears, the conic will reduce to a point, which indicates an inflection of the surface at that point.

4. We shall first occupy ourselves with *singular points*. In order for the surface whose equation is:

(1) $$V = F(x, y, z) = 0$$

to have such a point, it is necessary that the coordinates of that point must satisfy, along with the proposed equations, the following three equations, as well:

$$\frac{dV}{dx} = 0, \qquad \frac{dV}{dy} = 0, \qquad \frac{dV}{dz} = 0.$$

In this case, the coefficients of the equation of the tangent plane will present themselves in the indeterminate form 0 / 0.

Let:

(2) $$Ax + By + Cz + D = 0$$

be the equation of an arbitrary plane that passes through this point. This plane will cut the proposed surface along a curve that has a singular point that coincides with the singular point of the surface. In general, one can freely give this plane a position such that this point becomes either a double point, properly speaking, with two real tangents, or conjugate point, with respect to the curve of intersection. In the intermediate case, the singular point will become a point of regression of the curve, and at the same time, the intersecting plane will become one of the planes that are *tangent* to the surface at that point. Upon eliminating z from equations (1) and (2), one will obtain the following equation for the curve of intersection on the xy-plane:

(3) $$F\left[x, y, \left(-\frac{Ax + By + D}{C}\right)\right] = U = 0,$$

and since one does not lose the point of regression of a curve upon projecting it in an arbitrary manner, it will suffice that the double point of the curve that is represented by the latter equation be a point of regression in order for the plane (2) to touch the proposed surface. In my *System der analytischen Geometrie* (*), I saw that in the case of a point of regression of the curve (3), one will have the following equation:

(4) $$\left(\frac{d^2U}{dx\,dy}\right)^2 - \frac{d^2U}{dx^2} \cdot \frac{d^2U}{dy^2} = 0,$$

where one must refer the partial differential coefficients at this point, whose x and y coordinates are the same as the analogous coordinates of the singular point of the proposed surface. Upon remarking that one has:

$$\frac{dU}{dx} = \frac{dV}{dx} - \frac{dV}{dz} \cdot \frac{A}{C}, \qquad \frac{dU}{dy} = \frac{dV}{dy} - \frac{dV}{dz} \cdot \frac{B}{C},$$

(*) *System der analytischen Geometrie*, 1835. Third section, § 6.

$$\frac{d^2U}{dx^2} = \frac{d^2V}{dx^2} - 2\frac{d^2V}{dx\,dy}\cdot\frac{A}{C} + \frac{d^2V}{dz^2}\cdot\frac{A^2}{C^2},$$
$$\frac{d^2U}{dy^2} = \frac{d^2V}{dy^2} - 2\frac{d^2V}{dy\,dz}\cdot\frac{B}{C} + \frac{d^2V}{dz^2}\cdot\frac{B^2}{C^2},$$

$$\frac{d^2U}{dx\,dy} = \frac{d^2V}{dx\,dy} - \frac{d^2V}{dx\,dz}\cdot\frac{B}{C} - \frac{d^2V}{dy\,dz}\cdot\frac{A}{C} + \frac{d^2V}{dz^2}\cdot\frac{AB}{C^2},$$

equation (4) will transform into the following one:

(5)
$$\left[\left(\frac{d^2V}{dx\,dy}\right)^2 - \frac{d^2V}{dx^2}\cdot\frac{d^2V}{dy^2}\right]C^2 + \left[\left(\frac{d^2V}{dx\,dz}\right)^2 - \frac{d^2V}{dx^2}\cdot\frac{d^2V}{dz^2}\right]B^2$$
$$+\left[\left(\frac{d^2V}{dy\,dz}\right)^2 - \frac{d^2V}{dy^2}\cdot\frac{d^2V}{dz^2}\right]A^2 - \left[\frac{d^2V}{dx\,dy}\cdot\frac{d^2V}{dx\,dz} - \frac{d^2V}{dy\,dz}\cdot\frac{d^2V}{dx^2}\right]BC$$
$$-\left[\frac{d^2V}{dy\,dx}\cdot\frac{d^2V}{dy\,dz} - \frac{d^2V}{dx\,dz}\cdot\frac{d^2V}{dy^2}\right]AC - 2\left[\frac{d^2V}{dz\,dx}\cdot\frac{d^2V}{dz\,dy} - \frac{d^2V}{dx\,dy}\cdot\frac{d^2V}{dz^2}\right]AB$$
$$= 0.$$

If the coefficients A, B, C of the plane (2) satisfy that equation then the plane will touch the proposed surface. All of these tangent planes, which pass through the singular point, in addition, will envelop a second-degree cone that one can say is represented by equation (5), when one regards A, B, C as the independent variables. It will be the tangent cone to the surface at the singular point.

5. Now, suppose that the surface is determined by its tangent planes, and let:

$$Ax + By + Cz + D = 0$$

be one of its planes, which will depend upon only the three quantities A, B, C if we set $D = 1$. These quantities signify the reciprocal values, and taken with the opposite sign, of the three line segments that the plane cuts out on the three coordinate axes. We can then represent the surface by an equation of the form:

(1)
$$F(A, B, C) = W = 0$$

that will suffice for its determination just as completely as its equation in x, y, z. Likewise, when one regards A, B, and C as variables, the equation:

(2) $$Ax + By + Cz + 1 = 0$$

will represent a point. The coordinates of that point will be the three constants x, y, and z. Upon eliminating C from equations (1) and (2), one will obtain:

(3) $$F\left[A, B, -\left(\frac{Ax+By+1}{z}\right)\right] = S = 0.$$

This equation must be satisfied for all planes that pass through the point (2) that touches the proposed surface. It is the equation of the curve of intersection between the xy-plane and the cone that is circumscribed on that surface that has the point (2) for its summit, where that cone will be determined by its tangents. When one takes the point (2) to be in the singular plane of the surface, it will be obvious that, in general, that curve must have a double tangent and coincide with the intersection of the singular plane with the xy-plane. According to whether one takes the point to be external or internal to the curve of contact in the singular plane, the double tangent will touch either two real branches or two imaginary branches of the curve (3), respectively. In the latter case, it will be an isolated straight line and will be conjugate to that curve. If the point is found on the curve of contact itself then the curve (3) will have an inflection that is described by the following equation, which is entirely analogous to equation (4) in the preceding number:

$$4\left(\frac{d^2S}{dA\,dB}\right)^2 - \frac{d^2S}{dA^2}\cdot\frac{d^2S}{dB^2} = 0.$$

Upon developing this in the same manner, we will obtain:

(5) $$\left[\left(\frac{d^2W}{dA\,dB}\right)^2 - \frac{d^2W}{dA^2}\cdot\frac{d^2W}{dB^2}\right]z^2 + \left[\left(\frac{d^2W}{dA\,dC}\right)^2 - \frac{d^2W}{dA^2}\cdot\frac{d^2W}{dC^2}\right]y^2$$

$$+\left[\left(\frac{d^2W}{dB\,dC}\right)^2 - \frac{d^2W}{dB^2}\cdot\frac{d^2W}{dC^2}\right]x^2 - \left[\frac{d^2W}{dA\,dB}\cdot\frac{d^2W}{dA\,dC} - \frac{d^2W}{dB\,dC}\cdot\frac{d^2W}{dA^2}\right]yz$$

$$-\left[\frac{d^2W}{dB\,dA}\cdot\frac{d^2W}{dB\,dC} - \frac{d^2W}{dA\,dC}\cdot\frac{d^2W}{dB^2}\right]xz - 2\left[\frac{d^2W}{dC\,dA}\cdot\frac{d^2W}{dC\,dB} - \frac{d^2W}{dA\,dB}\cdot\frac{d^2W}{dC^2}\right]xy$$

$$= 0.$$

This is the equation for a cone whose summit is the coordinate origin. Its intersection with the singular plane will be the curve along which the proposed surface is touched by that plane.

6. If, upon taking $C = 1$, one determines a plane by the three constants A, B, and D from its equation:

$$z + AX + By + D = 0$$

then one can replace equation (1) with an equation of the form:

$$F(A, B, D) = W' = 0;$$

thus, upon eliminating D, it will become:

$$F[A_1\, B_1(-(Ax + By + z))] = S' = 0.$$

By an argument that is completely analogous to the one in the preceding number, one will obtain the following condition between the coordinates of the points at which the singular plane touches the surface:

$$\left(\frac{d^2S'}{dA\,dB}\right)^2 - \frac{d^2S'}{dA^2}\cdot\frac{d^2S'}{dB^2} = 0,$$

which will transform into the following one:

$$(6) \qquad \left[\left(\frac{d^2W'}{dA\,dB}\right)^2 - \frac{d^2W'}{dA^2}\cdot\frac{d^2W'}{dB^2}\right] + \left[\left(\frac{d^2W'}{dA\,dD}\right)^2 - \frac{d^2W'}{dA^2}\cdot\frac{d^2W}{dD^2}\right]y^2$$

$$+\left[\left(\frac{d^2W'}{dB\,dD}\right)^2 - \frac{d^2W'}{dB^2}\cdot\frac{d^2W'}{dD^2}\right]x^2 - 2\left[\frac{d^2W'}{dA\,dB}\cdot\frac{d^2W'}{dA\,dD} - \frac{d^2W'}{dB\,dD}\cdot\frac{d^2W'}{dA^2}\right]y$$

$$-\,2\left[\frac{d^2W'}{dB\,dA}\cdot\frac{d^2W'}{dB\,dD} - \frac{d^2W'}{dA\,dD}\cdot\frac{d^2W'}{dB^2}\right]x - 2\left[\frac{d^2W'}{dD\,dA}\cdot\frac{d^2W'}{dD\,dB} - \frac{d^2W'}{dA\,dB}\cdot\frac{d^2W'}{dD^2}\right]xy$$

$$= 0.$$

This is then the equation of the projection of the curve of contact in the singular plane onto the *xy*-plane.

In order to see what it so unusual about the developments of the last two numbers, I shall refer to the "Note sur une théorie générale et nouvelle des surfaces courbes" that I published in the ninth volume of this journal.

7. *Determination of the axis of an ellipse.* – While assuming rectangular coordinates, let:

$$\mu y^2 + 2\nu xy + \rho x^2 = 1$$

be the equation of the ellipse, so one will obtain the following equation in order to determine the reciprocal values of the semi-axes:

$$V^4 - (\mu + \rho)\, V^4 + (\mu\rho - \nu^4) = 0. \tag{1}$$

I will content myself by simply transcribing this equation, which can be found in any work that treats conic sections, and I will cite just my own *Entwicklungen* (*).

Upon denoting the reciprocal values of the two semi-axes by $V_{,}$ and $V_{,,}$, we will obtain the following relation by solving equation (1):

$$V_{,}^2 - V_{,,}^2 = \pm\sqrt{(\mu+\rho)^2 - 4(\mu\rho - \nu^2)}\,. \tag{2}$$

8. *Reciprocal polar surfaces.* – One takes two surfaces to be reciprocal polar surfaces when the points of one are the poles of the tangent planes to the other with respect to an arbitrary second-order surface that I will call the *directrix*. Upon taking the directrix to be a sphere whose radius is equal to unity, the polar plane of an arbitrary given point whose three rectangular coordinates are x', y', z' (the origin being the center of the sphere) will have the equation:

$$x'x + y'y + z'z = 1.$$

If we determine the position of this plane by means of the three quantities A, B, and C, which are given the same meaning as in number 4, then we will have:

$$x' = -A, \qquad y' = -B, \qquad z' = -C.$$

It will then result that if either of the two reciprocal polar surfaces is given by the equation:

$$F(x, y, z) = 0$$

then one will obtain the following equation in order to determine the other one:

$$F(-A, -B, -C) = 0,$$

and conversely.

9. If one takes an arbitrary ellipsoid to be the directrix, which I will represent by the following equation:

$$\frac{x^2}{bc} + \frac{y^2}{ac} + \frac{z^2}{ab} = 1,$$

then the polar plane to an arbitrary point (x', y', z') will have the equation:

(*) *Analytische geometrische Entwicklungen*, 1828, vol. I, pp 251.

$$\frac{x'x}{bc} + \frac{y'y}{ac} + \frac{z'z}{ab} = 1,$$

from which, one will obtain, while preserving the preceding terminology:

$$\frac{x'}{bc} = -A, \qquad \frac{y'}{ac} = -B, \qquad \frac{z'}{ab} = -C.$$

Since one of the two reciprocal polar surfaces is given by the equation:

$$F(x, y, z) = 0,$$

the other one will be determined by the following equation:

$$F(-bcA, -acB, -abC) = 0,$$

and conversely.

10. Two concentric ellipsoids will be reciprocal polars with respect to a sphere that has the same center when the product of the corresponding semi-axes is equal to the square of the radius of the sphere. The two ellipses that are represented by the following two equations:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad a^2 x^2 + b^2 y^2 + c^2 z^2 = 1$$

will be polar reciprocal if the radius of the directrix sphere is equal to unity.

11. If the directrix surface is an arbitrary ellipsoid then the product of the two corresponding semi-axes of two reciprocal polar ellipsoids will be equal to the square of the corresponding semi-axis of the directrix surface. For example, the two ellipsoids:

$$\frac{(a^2+b^2)x^2}{(a^2+b^2)b^2} + \frac{2y^2}{(a^2+c^2)} + \frac{(c^2+b^2)z^2}{(a^2+c^2)b^2} = 1,$$

$$\frac{(a^2+c^2)x^2}{(a^2+b^2)c^2} + \frac{(a^2+c^2)y^2}{2a^2c^2} + \frac{(a^2+c^2)z^2}{(c^2+b^2)a^2} = 1$$

will then be two reciprocal polar surfaces with respect to the ellipsoid:

$$\frac{x^2}{abc} + \frac{y^2}{ac} + \frac{z^2}{ab} = 1.$$

12. Two *straight lines are reciprocal polars* with respect to an arbitrary directrix when one of them passes through the two points where the two tangent planes that pass through the other one touch the directrix surface. Two straight lines that are reciprocal polars with respect a sphere will have mutually perpendicular directions.

13. One calls two points such that one of them is the point where a straight line passes through the center of a second-order surface, and the other one is the point where it meets the polar plane of the latter, *conjugate poles* with respect to that surface. Two points that are situated on a diameter of a sphere such that the product of their distances is equal to the square of the radius will be two conjugate points with respect to that sphere. One obtains the one by dropping a perpendicular from the center to the polar plane of the other one.

14. *Circular sections of an ellipsoid.* – An ellipsoid whose three semi-axes a, b, and c is represented by the following equation:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \tag{1}$$

Upon combining this equation with that of a sphere of undetermined radius:

$$x^2 + y^2 + z^2 = r^2,$$

one can always arrive at an equation that contains only the squares of any two of the three variables. Such an equation will represent the system of two (real or imaginary) planes that are perpendicular to one of the coordinate planes and that will cut out two circles when they pass through the intersection of the sphere and the ellipsoid. Here, we shall determine only the two real planes. Upon supposing that $c > b$ and $b > a$ for this, we will only have to take $r = b$ and to then subtract the equation of the sphere from first equation, after having multiplied it by b^2. One will then obtain:

$$c^2 (b^2 - a^2)\, x^2 - a^2 (c^2 - b^2)\, z^2 = 0,$$

an equation that will reduce to the following two equations:

$$c\sqrt{(b^2-a^2)}\, x \pm a\sqrt{(c^2-b^2)}\, z = 0. \tag{2}$$

In order to determine the circular sections of the ellipsoid whose equation is:

$$a^2 x^2 + b^2 y^2 + c^2 z^2 = 1, \tag{3}$$

one needs only to replace a^2, b^2, c^2 with $1 / a^2$, $1 / b^2$, $1 / c^2$ in the preceding results, which will give:

$$\sqrt{(b^2 - a^2)}\, x \pm \sqrt{(c^2 - b^2)}\, z = 0. \tag{4}$$

for the plane of the sections in question.

One will likewise arrive at equation (2) by setting y equal to zero in the equations of the sphere and the ellipsoid before the elimination. From this, one will conclude that the two planes (2) are perpendicular to the xz-plane and that they will intersect along the two common diameters to the ellipse and the circle of intersection, or even in the directions that are perpendicular to these common diameters, if one first turns the ellipse in its plane in such a way that its c-axis falls upon the x-axis and its a-axis falls upon the z-axis.

In that position, the equation of the ellipse will become:

$$\frac{x^2}{c^2} + \frac{z^2}{a^2} = 1,$$

and the four tangents that are common with the circle in question:

$$x^2 + y^2 = b^2$$

will have the equations:

$$\sqrt{(b^2 - a^2)}\, x \pm \sqrt{(c^2 - b^2)}\, z \pm b\sqrt{(c^2 - a^2)} = 0.$$

They will then be parallel to the planes (4).

One will then see that all four of the circular sections (2) and (4) of the two ellipsoids (1) and (3) must pass through the y-axis, since they are perpendicular to the xz-plane, and that those of the ellipsoid (1) will cut that plane along two straight lines that are perpendicular to the common diameters $M'M_{,}$ and $M''M_{,,}$ (see Fig. 1), while those of the ellipsoid (3) will cut parallel to the two common tangents $T_{,}T_{,,}$ and $T'T''$ or $T_{,}T''$ or $T_{,,}T'$.

15. *Fresnel's elasticity surface.* – By considering molecular actions, *Fresnel* proved that in the most general case, the elasticity of the ether along the different directions inside of a crystal is completely determined by the elasticity along three fixed directions that are mutually perpendicular and will depend upon the nature of the crystal. He called them the *elasticity axes*, and then he constructed the *elasticity surface* by taking its radius vectors to be proportional to the squares of the elasticity along the same radii. For the equation of that surface, he found:

$$a^2 x^2 + b^2 y^2 + c^2 z^2 = (x^2 + y^2 + z^2)^2,$$

which can then be written:

$$a^2 x^2 + b^2 y^2 + c^2 z^2 = r^4,$$

if one calls its radius vector r.

In all of what follows, we will suppose that:

$$c^2 > b^2, \qquad b^2 > a^2,$$

in such a manner that the elasticity will be a *maximum* along the z-axis, and a *minimum* along the x-axis.

In crystals that have *just one* axis, two of the three elasticities a^2, b^2, c^2 will be equal. Here, there will be two cases to distinguish: In the crystals that one calls *positive*, one will have:

$$c^2 = b^2 > a^2,$$

and in the crystals that are called *negative*, one will have:

$$a^2 = b^2 < c^2.$$

If the three elasticities are all equal then there will be no double refraction.

16. *Relationship between the elasticity surface and two ellipsoids.* – The elasticity surface has an intimate relationship with the two ellipsoids that are represented by the following two equations:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{2}$$

$$a^2\,x^2 + b^2\,y^2 + c^2\,z^2 = 1. \tag{3}$$

The ellipsoid (2), which I will choose to be the *first one* throughout in all of what follows, has the three axes of the elasticity surface for its axes; the axes of the other one, which I will call the *second ellipsoid*, will have the inverse values.

By a very simple calculation, *Magnus* (*) has already proved that *if one drops perpendiculars from the center of the first ellipsoid onto the tangent planes then the geometric locus of the feet of these perpendiculars will be the elasticity surface.*

One has a second theorem: *The lengths of two radius vectors of the elasticity surface and the second ellipsoid, whose directions will coincide along the same arbitrary straight line, will be inverses of each other, in such a way that their product will be unity.*

In order to prove this, denote the two radius vectors by r and r', and write the equations of the elasticity surface and the second ellipsoid in the following forms:

$$\frac{a^2x^2}{r^2} + \frac{b^2y^2}{r^2} + \frac{c^2z^2}{r^2} = r^2,$$

$$\frac{a^2x^2}{r'^2} + \frac{b^2y^2}{r'^2} + \frac{c^2z^2}{r'^2} = \frac{1}{r'^2}.$$

(*) *Sammlung von Aufgaben und Lehrsätzen aus der analytischen Geometrie des Raumes*, by L. J. Magnus. First edition, 1837, pp. 402.

Since the two radii coincide in the same straight line, the three terms in the left-hand side of the first equation will be equal to the three corresponding terms in the second one, and upon equating the right-hand sides, one will get:

$$r^2\, r'^2 = 1,$$

which was to be proved. We can state the same theorem in the following manner:

The points of the elasticity surface are the poles that are congruent to the points of the second ellipsoid, and conversely, with respect to a sphere whose radius is equal to unity. [13]

If follows from this theorem that in the two curves of intersection of the elasticity surface and the second ellipsoid, the directions of the axes will coincide, but in such a way that the largest of one will correspond to the smallest of the other one. If one of the two curves is a circle then the other one will also be a circle.

From number 10, the two ellipsoids (2) and (3) will be reciprocal polars with respect to the same sphere. That theorem lies between the two theorems of the present number, because upon taking an arbitrary point on the surface of the second ellipsoid at will, the polar plane to that point will touch the first ellipsoid, and in order to obtain the same point of the conjugate pole, which belongs to the elasticity surface, one has only to drop a perpendicular from that center to the polar plane. We are then presented with a new proof of the first theorem in that number.

17. *Determination of the light wave by Fresnel*. – Upon drawing an arbitrary plane through the center of the elasticity surface, all of the vibratory motion in that plane can be decomposed into two rectilinear vibrations that exist along the two axes of the curve of intersection, which will be, in general, a fourth-degree oval. The motion of propagation will take place perpendicular to the sense of the vibration with a velocity that is proportional to the two semi-axes, while these axes will be themselves proportional to the squares of the elasticities. (cf., number 15) *Fresnel concluded from this that planes that are parallel to the intersecting plane and have distances from it that are equal to the axes of the curve of intersection will be tangent to the light wave.*

Two tangent planes that are parallel and situated on the same side of the center will correspond to vibrations in the planes that are perpendicular to these tangent planes and will be, at the same time, mutually perpendicular. The planes of the vibrations will be perpendicular to the planes of *polarization*.

18. *Analytical determination of the light wave by tangent planes.* – From number 16, the *Fresnel* construction can be translated thusly:

If one passes an arbitrary plane through the center of the *second ellipsoid:*

$$a^2\, x^2 + b^2\, y^2 + c^2\, z^2 = 1, \tag{3}$$

and one raises perpendiculars to that plane that equal the inverse values of the two semi-axes of the ellipse of intersection then planes that are parallel to the intersecting plane and pass through the extremities of these perpendiculars will be tangent to the light wave.

We first determine the ellipse of intersection. As we did in the first number, upon fixing the position of the intersecting plane by the two angles α and φ, and upon setting:

$$x = v \cos \alpha - w \sin \alpha \cos \varphi, \quad y = v \sin \alpha + w \cos \alpha \cos \varphi, \quad z = w \sin \varphi$$

in equation (3), we will get the following equation for the ellipse in its plane:

$$\begin{aligned}(4) \qquad & [a^2 \cos^2\alpha + b^2 \sin^2\alpha]\, v^2 + 2(b^2 - a^2) \sin \alpha \cos \alpha \cos \varphi \cdot vw \\ & + [a^2 \sin^2\alpha \cos \varphi + b^2\cos^2\alpha \cos^2 \varphi + c^2 \sin^2 \varphi]\, w^2 = 1.\end{aligned}$$

If one then sets:

$$a^2 \cos^2\alpha + b^2 \sin^2\alpha = \mu, \qquad (b^2 - a^2) \sin \alpha \cos \alpha \cos \varphi = \nu,$$
$$a^2 \sin^2\alpha \cos \varphi + b^2\cos^2\alpha \cos^2 \varphi + c^2 \sin^2 \varphi = \rho$$

then we will have, after some simple trigonometric reductions:

$$(5) \qquad \begin{cases} (\mu+\rho) = (a^2+b^2)\cos^2\varphi + (a^2+c^2)\cos^2\alpha\sin^2\varphi + (b^2+c^2)\sin^2\alpha\sin^2\varphi, \\ (\mu\rho - \nu^2) = a^2b^2\cos^2\varphi + a^2c^2\cos^2\alpha\sin^2\varphi + b^2c^2\sin^2\alpha\sin^2\varphi, \end{cases}$$

and from number 7, in order to determine the inverse values of the two semi-axes of the ellipse of intersection (4), we will have the following equation:

$$(6) \qquad \begin{cases} V^4 - [(a^2+b^2)\cos^2\varphi + (a^2+c^2)\cos^2\alpha\sin^2\varphi + (b^2+c^2)\sin^2\alpha\sin^2\varphi]V^2 \\ \quad + a^2b^2\cos^2\varphi + a^2c^2\cos^2\alpha\sin^2\varphi + b^2c^2\sin^2\alpha\sin^2\varphi = 0. \end{cases}$$

Upon paying attention to the fact that:

$$\cos^2\varphi + \cos^2\alpha \sin^2\varphi + \sin^2\alpha \sin^2\varphi = 1,$$

one can write the same equation in the following manner:

$$\begin{aligned}(7) \qquad & (V^2 - b^2)(V^2 - c^2) \sin^2 \alpha \sin^2 \varphi + (V^2 - c^2)(V^2 - a^2) \cos^2 \alpha \sin^2 \varphi \\ & + (V^2 - a^2)(V^2 - b^2) \cos^2 \varphi = 0,\end{aligned}$$

or even in the form:

$$(8) \qquad \frac{\sin^2\alpha\sin^2\varphi}{V^2-a^2} + \frac{\cos^2\alpha\sin^2\varphi}{V^2-b^2} + \frac{\cos^2\varphi}{V^2-c^2} = 0.$$

When the intersecting plane is given by its equation:

$$Ax + By + Cz = 0,$$

equations (7) and (8) will transform right away into the following ones (cf., no. 1):

(9) $$(V^2 - b^2)(V^2 - c^2)\,A^2 + (V^2 - c^2)(V^2 - a^2)\,B^2 + (V^2 - a^2)(V^2 - b^2)\,C^2 = 0,$$

(10) $$\frac{A^2}{V^2 - a^2} + \frac{B^2}{V^2 - b^2} + \frac{C^2}{V^2 - c^2} = 0.$$

19. In these equations, V signifies the velocity of the light wave; i.e., that of its tangent planes. It is equal to the perpendicular that is dropped from the center of the wave onto its tangent planes, whose direction will be given by the values of A, B, and C. Should we desire that, conforming to number 4:

$$Ax + By + Cz + 1 = 0$$

should be the equation of the tangent planes then we would have, from known formulas:

$$V^2 = \frac{1}{A^2 + B^2 + C^2}.$$

Equation (6) is first changed by introducing A, B, and C into the following:

$$(A^2 + B^2 + C^2)\,V^4 - [(b^2 + c^2)\,A^2 + (c^2 + a^2)\,B^2 + (a^2 + b^2)\,C^2]\,V^2$$
$$+ [b^2 c^2 A^2 + c^2 a^2 B^2 + a^2 b^2 C^2] = 0,$$

and then upon seeking V^2 one will obtain:

(11) $$[b^2 c^2 A^2 + c^2 a^2 B^2 + a^2 b^2 C^2]\,[A^2 + B^2 + C^2]$$
$$- [(b^2 + c^2)\,A^2 + (c^2 + a^2)\,B^2 + (a^2 + b^2)\,C^2] + 1 = 0.$$

20. *In general, one can pass only one plane through an arbitrary radius vector of an ellipsoid in such a way that this radius vector will coincide with one of the two semi-axes of the ellipse of intersection.*

In order to prove this, we make the x-axis coincide with the given radius vector, and take the y-axis and z-axis at will, but perpendicular to each other and to the x-axis. The given ellipsoid will then be represented by an equation of the form:

$$Mx^2 + My^2 + Oz^2 + 2P\,zy + 2Q\,zx + 2\,R\,xy = 1.$$

Make an intersecting plane pass through the x-axis that makes an arbitrary angle φ with the xy-plane. By setting:

$$a = 0, \quad x = v, \quad y = w \cos \varphi, \quad z = w \sin \varphi$$

in formulas (1) of the first number, one will get the following equation:

$$Mv^2 + 2(Q \sin \varphi + R \cos \varphi)\, vw + (N \cos^2\varphi + O \sin^2\varphi + 2P \sin \varphi \cos \varphi)\, w^2 = 1$$

in order to determine the curve of intersection in its proper plane.

Since that equation is referred to rectangular coordinate axes, if one desires that the v-axis, which coincides with x-axis or the given radius vector, should contain one of the two axes of the ellipse of intersection then it is necessary that the left-hand side of the preceding equation must vanish, which will give the following equation:

$$\tan \varphi = -\frac{R}{Q}$$

for the determination of the arbitrary angle φ.

One then sees that there is always a plane that satisfies the required condition and that there is only one intersecting plane, provided that R and Q disappear simultaneously in the equation for the ellipsoid. In that particular case, the given radius will coincide with one of the three axes of the ellipsoid and one will then see that any plane that passes through that radius will have the required property.

21. *Construction of the plane in question.* – We first observe that in order for the xy-plane to be the desired plane it will suffice to choose the coordinate axes in such a manner that R disappears in the equation of the surface. We then construct the tangent plane to the extremity of the given radius vector, whose length we denote by x', and its equation will then be:

$$M(x - x') + Qz + Ry = 0.$$

Should one desire that R should disappear, then one would have to take the xz-plane to be perpendicular to that tangent plane, from which, one would then obtain the following construction by paying attention to the fact that the xy-plane is perpendicular to the xz-plane and that it passes through the radius vector, along with the former plane:

Construct the tangent plane to the extremity of the given radius vector, and drop a perpendicular to that plane from the center of the ellipsoid. The plane that one must construct will then be the one that passes through the radius vector, is perpendicular to the plane, and which contains both that radius and the perpendicular at the same time.

22. *The light wave* (see Fig. 4) *is the reciprocal polar surface with respect to a sphere whose radius is equal to unity to what one obtains by replacing the second ellipsoid with the first one in its construction.*

Let M be an arbitrary point of the second ellipsoid, and let P be the foot of the perpendicular that is dropped from the center onto the tangent plane at that point. The polar plane to M, which is likewise perpendicular to the plane of the figure, will touch the first ellipsoid at a point m that, being the pole of the tangent plane at M, will be found along the prolongation of *OP*, just as the prolongation of *OM* will be perpendicular to the tangent plane at *m*. Finally, $Op = 1 / OM$ and $OP = 1 / Om$. From the preceding number, the plane that passes through the radius vector *OM* and is perpendicular to the plane of the figure will cut the second ellipsoid in such a manner that in the ellipse of intersection, one of the semi-axes will coincide with the radius vector *OM*. Likewise, the plane that passes through the radius vector *Om* and is perpendicular to the plane of the figure will cut the first ellipsoid in such a manner that in the ellipse of intersection one of the semi-axes will coincide with the radius vector *Om*.

If one makes the two tangent planes and the two perpendiculars that are dropped from their centers turn around that point in an arbitrary manner then nothing will change in their reciprocal relationship with respect to the directrix sphere; *M* will always be the pole of the plane *mp*, while *m* will continue to be that of the plane *MP*. That will also be the case if, in particular, one actually turns the system of two planes a quarter of a revolution around an axis that is perpendicular to the plane of the figure, in such a manner that points *m*, *p*, *M*, and *P* will go to the positions *R*, *V*, *r*, and *v*, respectively. In the new position, the plane *RV* will be tangent to the surface of the light wave. As for the *rv*-plane, it will touch a second light wave that one obtains by replacing the two ellipsoids with each other, or – what amounts to the same thing – by replacing the three principal elasticities a^2, b^2, c^2 with their inverse values $1 / a^2$, $1 / b^2$, $1 / c^2$. If one gives all possible positions in the second ellipsoid to the point *M* then one will obtain all of the tangent planes to the surface of the first wave. These planes will all be related to the tangent planes to the surface of the second wave in such a manner that one will contain the poles of the other, and conversely. Thus, one sees that the two waves are both reciprocal polar surfaces with respect to the concentric sphere whose radius is equal to unity. That is what we proposed to prove.

23. *Equation of the wave surface in rectangular coordinates.* – Recall the final equation of number 19, which gave the determination of the wave by means of its tangent planes:

$$\begin{aligned}(11)\qquad & [b^2 c^2 A^2 + c^2 a^2 B^2 + a^2 b^2 C^2]\,[A^2 + B^2 + C^2] \\ & - [(b^2 + c^2)\, A^2 + (c^2 + a^2)\, B^2 + (a^2 + b^2)\, C^2] + 1 = 0.\end{aligned}$$

The quantities *A*, *B*, *C* have the same significance as in number (14), so we have only to put *x*, *y*, *z* in their place in order to obtain the equations of the reciprocal polar surface; one then gets:

$$\begin{aligned}(12)\qquad & [b^2 c^2 x^2 + c^2 a^2 y^2 + a^2 b^2 z^2]\,[x^2 + y^2 + z^2] \\ & - [(b^2 + c^2)\, x^2 + (c^2 + a^2)\, y^2 + (a^2 + b^2)\, z^2] + 1 = 0.\end{aligned}$$

This is therefore the second-order equation that was recorded in the preceding number, and in order to obtain that of the first one, which likewise relates to equation (11), it will suffice to write $1 / a^2$, $1 / b^2$, $1 / c^2$ in plane of a^2, b^2, c^2, which will give:

$$\begin{aligned}(13)\qquad & (a^2 x^2 + b^2 y^2 + c^2 z^2)\,(x^2 + y^2 + z^2) \\ & - (a^2 (b^2 + c^2)\, x^2 + b^2 (c^2 + a^2)\, y^2 + c^2 (a^2 + b^2)\, z^2] + a^2 b^2 c^2 = 0.\end{aligned}$$

24. From the theorem in number 22, one will obtain the points of one of the two waves by looking for the poles of the tangent planes to the other one. Thus, for example, the point *R*, which is the pole to the tangent plane *vr* to the second wave, will be the one where the first one is touched by the plane *RV*. The construction that is due to *Fresnel* for finding the points of the surface of the light wave directly will follow from this right away:

After having cut the first ellipsoid with a diametral plane, draw two lines that start from the center and are perpendicular to that plane, and are equal to the largest and the smallest semi-diameter of the ellipse of intersection, respectively; the locus of the extremities of these perpendiculars will be the surface of the light wave.

The necessary calculation for inferring the equation of the wave from this theorem was already completed almost entirely in number 18. Upon letting r denote the length of the perpendicular that is raised from the center to the section, we will have only to substitute r for $1 / V$ in the equations of that number, and at the same time, a^2, b^2, c^2, in place of $1 / a^2$, $1 / b^2$, $1 / c^2$. If we make that substitution in equation (10):

$$\frac{A^2}{V^2 - a^2} + \frac{B^2}{V^2 - b^2} + \frac{C^2}{V^2 - c^2} = 0$$

then we will find:

$$\frac{a^2 A^2}{a^2 - r^2} + \frac{b^2 B^2}{b^2 - r^2} + \frac{c^2 C^2}{c^2 - r^2} = 0.$$

Then, upon taking x, y, z to be the coordinates of the extremities of the perpendicular r, one will get:

$$\frac{A^2 r^2}{A^2 + B^2 + C^2} = x^2, \qquad \frac{B^2 r^2}{A^2 + B^2 + C^2} = y^2, \qquad \frac{C^2 r^2}{A^2 + B^2 + C^2} = z^2,$$

which are equations that make the latter equation transform into the following one:

$$\frac{a^2 x^2}{a^2 - r^2} + \frac{b^2 y^2}{b^2 - r^2} + \frac{c^2 z^2}{c^2 - r^2} = 0. \tag{14}$$

That is the desired equation for the light wave. Upon clearing the denominators and dividing by r^2 or $(x^2 + y^2 + z^2)$, we will get back to equation (13) of the preceding number.

25. *Construction of the plane of vibration.* – Recall figure 4, for the moment. The vibrations that take place along *OM* will produce a plane wave (viz., the envelope of the surface of the light wave in question) that is perpendicular to the plane of the paper and will propagate along *OV*. The point *R* where it will touch that surface will give the corresponding light ray. The following theorem will result from that:

The plane of vibration for an arbitrary light ray is the one that passes through that ray and is perpendicular to the plane that touches the surface at the point where it meets the light ray.

Upon referring to the figure, the other ray, which corresponds to vibrations along the axis perpendicular to *OM*, and which produces a plane wave that is parallel to the one that we just considered, will be found in a plane that is perpendicular to the plane of the figure and will cut it along *OV*. This same plane will contain the point of contact.

26. *Equation of the plane of vibrations.* – If we differentiate the wave equation:

$$(a^2x^2 + b^2y^2 + c^2z^2)\,(x^2 + y^2 + z^2) - (a^2(b^2 + c^2)\,x^2 + b^2(c^2 + a^2)\,y^2 + c^2(a^2 + b^2)\,z^2] + a^2b^2c^2 = V = 0$$

then, upon setting:

$$a^2x^2 + b^2y^2 + c^2z^2 = E, \qquad x^2 + y^2 + z^2 = r^2,$$

to abbreviate, we will have the following equations:

$$\frac{dV}{dx} = 2[E + a^2\,(r^2 - b^2 - c^2)]\,x,$$

$$\frac{dV}{dy} = 2[E + b^2\,(r^2 - a^2 - c^2)]\,y,$$

$$\frac{dV}{dz} = 2[E + c^2\,(r^2 - a^2 - b^2)]\,z,$$

from which, one will infer:

$$\begin{aligned}\frac{dV}{dx}\cdot x + \frac{dV}{dy}\cdot y + \frac{dV}{dz}\cdot z &= 4\,Er^2 - 2[a^2\,(b^2 + c^2)\,x^2 + b^2\,(a^2 + c^2)\,y^2 + c^2\,(a^2 + b^2)\,z^2],\\ &= 2[Er^2 - a^2\,b^2\,c^2]\\ &= 2[a^2\,(b^2 + c^2)\,x^2 + b^2\,(a^2 + c^2)\,y^2 + c^2\,(a^2 + b^2)\,z^2 - 2\,a^2\,b^2\,c^2].\end{aligned}$$

Now, let x', y', z' be the coordinates of an arbitrary point of the surface of the light wave, and let E' and r' be the corresponding values of E and r; the equation of the tangent plane to that point will be:

$$\begin{aligned}&[E' + a^2\,(r'^2 - b^2 - c^2)\,x'x + [E' + b^2\,(r'^2 - a^2 - c^2)]\,y'y + [E' + c^2\,(r'^2 - a^2 - b^2)]\,z'z\\ &\qquad = a^2(b^2 + c^2)\,x'^{\,2} + b^2\,(a^2 + c^2)\,y'^{\,2} + c^2\,(a^2 + b^2)\,z'^{\,2} - 2\,a^2\,b^2\,c^2,\end{aligned}$$

an equation that we can write as follows:

$$Ax'x + By'y + Cz'z = D,$$

to abbreviate. The equation of the plane of vibration that passes through the origin will be of the form:

$$A'x + B'y + C'z = 0.$$

In order to determine it completely, one has the two condition equations:

$$A'x' + B'\,y' + C'\,z' = 0, \qquad A'A\,x' + B'B\,y' + C'C\,z' = 0,$$

one of which expresses the idea that the plane in question passes through the point (x', y', z') and the other one, that it is perpendicular to the tangent plane. If one subtracts the last equation from the first one, after having previously multiplied it by A, then that will give:

$$(A - B)\,B'\,y' + (A - C)\,C'\,z' = 0.$$

Thus:

$$\frac{B'}{C'} = -\,\frac{(A-C)z'}{(A-B)y'} = \frac{A-C}{y'} : \frac{B-A}{z'},$$

and upon substituting A', A, and x' in place of B', B, and y':

$$\frac{A'}{C'} = \frac{B-C}{x'} : -\frac{B-A}{z'}.$$

The equation for the plane of vibration will then become the following one:

$$\frac{B-C}{x'}\cdot x + \frac{C-A}{y'}\cdot y + \frac{A-B}{z'}\cdot z = 0;$$

upon introducing the values of A, B, C, one will get:

$$\text{(15)} \qquad \frac{(b^2-c^2)(r'^2-a^2)}{x'}\cdot x + \frac{(c^2-c^2)(r'^2-b^2)}{y'}\cdot y + \frac{(a^2-b^2)(r'^2-c^2)}{z'}\cdot z = 0.$$

27. From the considerations of number 20, one concludes that a given direction of vibration will correspond to only one unique light ray. Meanwhile, there will be an exception when the vibrations take place parallel to one of the coordinate axes. It is obvious from the cited number that the corresponding points of the surface of the light wave will constitute three circles that are described in the three coordinate planes by radii that are equal to the three semi-axes of the first ellipsoid, respectively, and perpendicular to those planes.

Upon setting y, x, and z equal to zero in the equation of the wave surface, successively, one will find the following three equations:

$$(a^2 x^2 + c^2 z^2 - a^2 c^2)(x^2 + y^2 - b^2) = 0,$$
$$(b^2 x^2 + c^2 z^2 - b^2 c^2)(x^2 + y^2 - a^2) = 0,$$
$$(c^2 x^2 + b^2 z^2 - a^2 c^2)(x^2 + y^2 - c^2) = 0,$$

each of which will represent a system that consists of a circle and an ellipse. The light rays that start from the points of each of the three circles will provoke vibrations that are mutually-parallel and perpendicular to the plane of the figure. One obtains the three ellipses that are situated in the same planes when one gives each of the ellipses of intersection of the first ellipsoid (which are drawn more faintly in the figures) a quarter revolution in its plane around the center. For all of the light rays that start from the points of each of these three ellipses, the plane of vibration will be the plane of the curve, in such a manner that all of the vibrations will take place in that plane, and perpendicular to the various rays.

28. *Singular points of the wave.* – It suffices to observe that the equation of the light wave (13) and equation (11), which serves to determine its tangent planes, are of the same degree, which is greater than two, in order to assure that the surface will have a greater number of *singular points*, and at the same time, *singular planes*, with the restriction that these singular points and singular planes can be imaginary or situated at infinity. We shall first occupy ourselves with the singular points.

If there exists a singular point then its coordinates must satisfy both the equation of the surface and three equations that are deduced from it by successively differentiating it with respect to the three variables. That will give the following four equations:

$$V = Er^2 - (a^2 (b^2 + c^2) x^2 + b^2 (a^2 + c^2) y^2 + c^2 (a^2 + b^2) z^2) + a^2 b^2 c^2 = 0,$$

$$\tfrac{1}{2}\frac{dV}{dx} = (E + a^2 (r^2 - b^2 - c^2))\, x = 0,$$
$$\tfrac{1}{2}\frac{dV}{dy} = (E + b^2 (r^2 - a^2 - c^2))\, y = 0,$$
$$\tfrac{1}{2}\frac{dV}{dz} = (E + c^2 (r^2 - a^2 - b^2))\, z = 0.$$

In order to satisfy these equations, it will suffice to set:

$$(16) \qquad y = 0, \qquad E + a^2 (r^2 - b^2 - c^2) = 0, \qquad E + c^2 (r^2 - a^2 - b^2) = 0,$$

because upon subtracting the last two of these three equations from each other, one will get:

$$r^2 = b^2,$$

and upon subtracting them, after having multiplied them by c^2 and a^2, respectively, one will obtain:

$$E = a^2 c^2.$$

These two new equations are those of the circle and the ellipse along which the surface is cut by the *xz*-plane. Therefore, four points of intersections of these two curves will determine four singular points that will all be real, under our assumption that the value of b^2 is intermediate between a^2 and c^2.

In this manner, we will obtain four singular points of the surface in each of the coordinate planes, but among these twelve singular points, eight of them will be imaginary.

In order to discuss the nature of the four real singular points M', $M_{\prime}$, M'', and $M_{\prime\prime}$, whose coordinates are:

$$x = \pm c\sqrt{\frac{b^2 - a^2}{c^2 - a^2}}, \qquad z = \pm a\sqrt{\frac{c^2 - b^2}{c^2 - a^2}}, \qquad y = 0,$$

we shall differentiate once more, while simultaneously paying attention to the three equations (16). That will give:

$$\tfrac{1}{2}\frac{d^2V}{dx^2} = 4\,a^2\,x^2 = \frac{4a^2c^2(b^2 - a^2)}{c^2 - a^2},$$

$$\tfrac{1}{2}\frac{d^2V}{dy^2} = E + b^2\,(r^2 - a^2 - c^2) = -\,(b^2 - a^2)(c^2 - b^2),$$

$$\tfrac{1}{2}\frac{d^2V}{dz^2} = 4\,c^2\,z^2 = \frac{4a^2c^2(c^2 - b^2)}{c^2 - a^2},$$

$$\tfrac{1}{2}\frac{d^2V}{dx\,dy} = 0,$$

$$\tfrac{1}{2}\frac{d^2V}{dx\,dz} = 2(a^2 + c^2)\,xz = 2ac\,(a^2 + c^2)\cdot\frac{\sqrt{(b^2 - a^2)(c^2 - b^2)}}{c^2 - a^2},$$

$$\tfrac{1}{2}\frac{d^2V}{dy\,dz} = 0,$$

and if one substitutes these values in equation (5) of number 4, and after having divided both sides by:

$$\frac{4a^2c^2(b^2 - a^2)(c^2 - b^2)}{c^2 - a^2},$$

one will find the following equation:

$$(17) \qquad (b^2 - a^2)\,C^2 + (c^2 - a^2)\,B^2 + (c^2 - b^2)\,A^2 \mp \frac{a^2 + c^2}{ac}\sqrt{(b^2 - a^2)(c^2 - b^2)}\cdot AC = 0,$$

which one can then write as:

$$a^2\, x''^2\, C^2 + a^2\, c^2\, B^2 + c^2\, z''^2\, A^2 - (a^2 + c^2)\, x''\, z''\, AC = 0, \tag{18}$$

upon distinguishing the coordinates of the singular point by double primes.

It follows from this that if one passes an arbitrary plane through that point then that plane will touch the wave surface at the same point, as long as the three constants in its equation:

$$A\,(x - x'') + B\,y + C\,(z - z'') = 0$$

satisfy the preceding equation. All of these planes will envelop a second-degree conic surface that is tangent to the wave surface at the singular point. One will find its equations in rectangular coordinates with no effort.

29. *Outgoing light cone.* – If one raises perpendiculars to the planes that touch the cone in the preceding number at the singular point then these perpendiculars will constitute a new cone of the same order. It will be composed of light rays that all correspond to just one ray that has a perpendicular incidence and traverses the interior of the crystal along one of the two straight lines $M'M_{,}$ and $M''M_{,,}$, which, from number 14, will be perpendicular to the two circular sections of the first ellipsoid.

Upon denoting the coordinates of an arbitrary point of such a perpendicular by x, y, z, one will have:

$$(x - x'') : y : (z - z'') = A : B : C.$$

Thus, since equation (18) is homogeneous with respect to A, B, and C, we will immediately obtain, upon replacing these quantities with $(x - x'')$, y, and $(z - z'')$, respectively:

$$a^2\, x''^2\, (z - z'')^2 + a^2\, c^2\, y^2 + c^2\, z''^2\, (x - x'')^2 - (a^2 + c^2)\, x''\, z''\, (x - x'')(z - z'') = 0. \tag{19}$$

This is the equation of the light cone in question; it simplifies as follows:

$$a^2\, x''^2\, z^2 + a^2\, c^2\, y^2 + c^2\, z''^2 x^2 - (a^2 + c^2)\, x''\, z''\, xy + (c^2 - a^2)\, x''z''(x''z - z''x) = 0. \tag{20}$$

30. *Planes of vibration for outgoing rays.* – In order to determine the direction of the vibrations that correspond to the various rays of the outgoing light cone, from number 21, we have only to drop some perpendiculars from the center onto the planes that touch the surface at the singular points. The straight lines in these tangent planes that join the bases of the perpendiculars to the singular points will indicate the directions of the corresponding vibrations. The plane of vibration will, at the same time, contain that straight line, the singular ray (which starts from the singular point), and the outgoing ray.

We therefore look for the curve that is the geometric locus of the bases of the perpendiculars in question, whose coordinates we denote by x, y, z. We obtain, in turn, two surfaces that contain that curve. First, there is the cone that is defined by these same perpendiculars, and which is nothing but the light cone, when transported parallel to itself in such a manner that its center will coincide with that of the wave. Consequently, its equation will then be the one that one finds when one puts x, y, z into (19), in place of $(x - x'')$, y, $(z - z'')$. One will then immediately obtain the following equation:

$$a^2 x''^2 z^2 + a^2 c^2 y^2 + c^2 z''^2 x^2 - (a^2 + c^2) x'' z'' xz = 0,$$

which can also be written as follows:

$$a^2 c^2 (x^2 + y^2 + z^2) - (x'' x + z'' z)(a^2 x'' x + c^2 z'' z) = 0. \tag{21}$$

In the second place, the same curve must belong to a sphere that has one diameter that is the radius vector that ends at the singular point. Its equation will then be:

$$\left(x - \frac{x''}{2}\right)^2 + y^2 + \left(z - \frac{z''}{2}\right)^2 = \left(\frac{b}{2}\right)^2,$$

or, upon reducing:

$$x^2 + y^2 + z^2 = x'' x + z'' z. \tag{22}$$

From equations (21) and (22), one infers that:

$$(x'' x + z'' z)(a^2 x'' x + c^2 z'' z - a^2 c^2) = 0.$$

Upon neglecting the factor $(x'' x + z'' z)$, which is foreign to the question, the equation:

$$a^2 x'' x + c^2 z'' z = a^2 c^2$$

will be that of the tangent plane that is perpendicular to the xz-plane and will cut that plane along the tangent at the singular point to the ellipse of intersection with the wave surface whose equation is:

$$a^2 x^2 + c^2 z^2 = a^2 c^2.$$

The desired locus is therefore the circle along which the sphere is cut by that plane. In order to obtain it, one needs only to drop a perpendicular OP from the center O onto the tangent to the ellipse at the singular point M' (see Figs. 1, 5) and to then construct a circle that is perpendicular to the plane of the ellipse and has $M'P$ for its diameter.

We further remark that the factor that we neglected indicates that in order to get the other direction, in which the cone (21), as well as the cone (20), is cut along a circle, it is necessary to take the intersecting planes to be perpendicular to the singular ray OM', which one can see immediately, since $M'P$ is perpendicular to OQ.

Now, consider an arbitrary ray of the light cone that meets two circular sections, which are perpendicular to $M'Q'$ and $M'T$, and N and N'. The corresponding plane of vibration will pass through that ray and the singular ray $OM'T$. Its inclination above the plane of the figure will have a measure that equals one-half the arc length $Q'N'$ of the second circular section. It will be zero for the ray $M'Q'$ and equal to 90o for the ray $M'T$, which is the prolongation of the singular ray OM'.

If one projects the image onto a screen *EE* that is perpendicular to the singular ray then one will obtain a light circle $Q'N''N'T$ that passes through the ray. It follows from the preceding that the plane of vibration will describe an arc of the circle around the singular ray OM' that is one-half of the one that the light ray describes in the light circle, or, to appeal to the terminology of physics, when one regards the light circle that traverses a slab of tourmaline that is cut parallel to the axis, just one ray will disappear, whereas, upon turning the slab through an arbitrary angle in its plane, the ray that disappears will describe twice the angle around the center of the light circle, in such a manner that it will traverse the entire periphery if the tourmaline turns through just 180° (*).

31. *Analytic determination of the singular planes.* – We look for these planes among the ones that are perpendicular to the coordinate planes. One has the two equations:

$$V = 0, \qquad \frac{dV}{dx} = 0$$

for determining the points of the wave surface at which it is touched by planes perpendicular to the *xz*-plane, while preserving the notation of number 26. The last of these equations decomposes into the two following ones:

$$(23) \qquad y = 0, \qquad (a^2 + b^2)\,x^2 + 2\,b^2\,y^2 + (c^2 + b^2)\,z^2 - (a^2 + c^2)\,b^2 = 0,$$

the first of which – which is obvious, moreover – expresses the idea that the tangent planes to the points of the wave surface that are situated in the *xz*-plane will be perpendicular to that plane. The other one represents an ellipsoid, from which, it follows that upon conveniently combining the equation of the wave surface:

$$(24) \qquad \begin{aligned} V = (a^2\,x^2 + b^2\,y^2 + c^2\,z^2)(x^2 + y^2 + z^2) \\ - (a^2\,(b^2 + c^2)\,x^2 + b^2\,(a^2 + c^2)\,y^2 + c^2\,(a^2 + b^2)\,z^2 + a^2\,b^2\,c^2 = 0, \end{aligned}$$

and the following equation:

$$[(a^2 + b^2)\,x^2 + 2\,b^2\,y^2 + (c^2 + b^2)\,z^2 - (a^2 + c^2)\,b^2]^2 = 0,$$

(*) Mr. Lloyd was the first to arrive at this result by experiment. He expressed it thusly:

> "…I discovered the remarkable law…that the angle between the planes of polarization of any two rays of the cone is half the angle between the planes containing the rays themselves in the axis." (On conical refraction, pp. 7)

The planes of polarization are perpendicular to those of vibration (17), and the axis in question will be the optical axis, which is perpendicular to one of the common tangents to the circle and the ellipse of intersection in the *xz*-plane. One sees that this theorem is true only approximately; it is proved, as such, by analysis in the cases of Aragonite, where one has to satisfy the necessary condition $b^2 = ac$, roughly.

one must obtain that of a cylinder that is perpendicular to the xz-plane and enveloped by the tangent planes to the wave surface at the points where it (like the wave) is cut by the ellipsoid in question. Since the equation of this cylinder does not contain y, one will achieve this objective by subtracting the latter equation from the wave equation, after having previously multiplied it by $4b^2$. We can give the following form to the resulting equation:

$$[(b^2 - a^2)\, x^2 + (c^2 - b^2)\, z^2 - (c^2 - a^2)\, b^2]^2 - 4(b^2 - a^2)(c^2 - b^2)\, x^2 z^2 = 0,$$

and then decompose it into the two equations:

$$(b^2 - a^2)\, x^2 + (c^2 - b^2)\, z^2 - (c^2 - a^2)\, b^2 \pm 2\sqrt{(b^2 - a^2)(c^2 - b^2)}\; x\, z = 0.$$

After writing these equations thusly:

$$\left[\sqrt{b^2 - a^2}\; x \pm \sqrt{c^2 - b^2}\; z\right]^2 - (c^2 - a^2)\, b^2 = 0,$$

one will decompose them once more and one will obtain the following four equations:

$$\sqrt{b^2 - a^2}\; x \pm \sqrt{c^2 - b^2}\; z \pm \sqrt{c^2 - a^2}\; b = 0.$$

The cylinder in question will then degenerate into a system of four planes that are perpendicular to the xz-plane. Since each point of the curve of intersection of any of these planes with the ellipsoid (23) will be a point where the wave surface is touched by that plane, the four planes will be singular planes to the wave. It will follow from the developments in number 14 that these singular planes will cut the xz-plane along the four common tangents to the circle and the ellipse that are curves of intersection of the wave in that plane. From the same number, the four singular planes will be parallel to the circular sections of the *second* ellipsoid:

$$a^2 x^2 + b^2 y^2 + c^2 z^2 = 1.$$

Since their directions will depend uniquely upon the differences between the three coefficients of x^2, y^2, and z^2, when taken two at a time, and changing nothing if the constant member changes, it is obvious that they will be the same as those of the circular sections of the ellipsoid (23). Therefore, the contact curves in the four singular planes will be *circles*. In order to find them, one needs only to construct the common tangents to the circle and the ellipse of intersection, and to then describe, for each of them, a circle FGH (Fig. 1 and 6) that is perpendicular to the plane of the figure and has a diameter that is equal to the line segment of the tangent that is found between the two contact points (viz., H and I). The square of that diameter will be found to equal $\left(\dfrac{-(c^2 - b^2)(a^2 - b^2)}{b^2}\right)$.

An arbitrary ray OG that goes from the interior of the crystal and ends at an arbitrary point of the contact curve in one of the singular planes will leave it perpendicular to that plane along GG'. All of the rays that leave will thus constitute a circular light cylinder that is perpendicular to the singular plane, or – what amounts to the same thing – a cylinder whose axis is parallel to one of the optical axes of the crystal. On the other hand, an external light ray that falls upon the crystal with perpendicular incidence will produce a light cone in the interior.

For an arbitrary light ray OGG', the plane of vibration will be the one that contains both that ray and the optical axis OHH', that axis being perpendicular to the singular plane (26). The angle that this plane makes with the xz-plane will have a measure that equals one-half of the angle IG, in such a manner that this angle will vanish for the ray OII' (which refracts extraordinarily in the xz-plane), and which will be equal to 90° for the (ordinary) ray OHH'. It is then proved that *the angle that is defined by the planes of vibration for two arbitrary rays will be one-half the one that is defined by the two planes that contain both the two rays and the axis of the cylinder.*

The results of this number were predicted theoretically by *Hamilton* and then verified experimentally by *Lloyd*.

I do not need to add that the four singular planes that are perpendicular to each of the xy and zy-planes will become imaginary.

32. *Second way of determining the singular planes.* We are far too accustomed to regarding the equation of a curve or surface as only the one that expresses a relation between the coordinates of their points. I can find nowhere that one regards the equation:

$$\frac{A^2}{V^2-a^2}+\frac{B^2}{V^2-b^2}+\frac{C^2}{V^2-c^2} = 0 \tag{10}$$

as representing the surface of a light wave. Nevertheless, one can deduce all of the properties of the surface from this equation in a manner that is just as complete and easy as when one deduces them from the following equation in x, y, z:

$$\begin{gathered}(a^2 x^2 + b^2 y^2 + c^2 z^2)\,(x^2 + y^2 + z^2)\\ - (a^2 (b^2 + c^2)\, x^2 + b^2 (c^2 + a^2)\, y^2 + c^2 (a^2 + b^2)\, z^2] + a^2 b^2 c^2 = 0.\end{gathered} \tag{13}$$

One must stress, in addition, that the illustrious *Fresnel* did not succeed in transforming the one of these equations into the other one, and that *Ampère* was the first do that in 1828, and by an enormous amount of calculation. Theoretical considerations compel me to repeat the discussion of the singularities of the wave surface in what follows.

Start with equation (10), in the following form, which we gave it in number 19:

$$\begin{aligned}W = {}&[b^2 c^2 A^2 + c^2 a^2 B^2 + a^2 b^2 C^2]\,[A^2 + B^2 + C^2]\\ &- [(b^2 + c^2)\, A^2 + (c^2 + a^2)\, B^2 + (a^2 + b^2)\, C^2] + 1 = 0,\end{aligned} \tag{11}$$

where A, B, and C will have the same significance as they did in number 5. If there exists a singular plane then it will be necessary that the three constants A, B, C in its equation:

$$Ax + By + Cz = 1$$

will satisfy, in addition to the proposed equation, the following three equations:

$$\tfrac{1}{2}\frac{dW}{dA} = [2b^2 c^2 A^2 + c^2 (a^2 + b^2) B^2 + b^2 (a^2 + c^2) C^2 - (b^2 + c^2)] A = 0,$$

$$\tfrac{1}{2}\frac{dW}{dB} = [c^2 (a^2 + b^2) A^2 + 2a^2 c^2 B^2 + a^2 (b^2 + c^2) C^2 - (a^2 + c^2)] B = 0,$$

$$\tfrac{1}{2}\frac{dW}{dC} = [b^2 (a^2 + c^2) A^2 + a^2 (b^2 + c^2) B^2 + 2a^2 b^2 C^2 - (a^2 + b^2)] C = 0.$$

Upon setting $B = 0$, the second of these three equations will be satisfied. In order for the first and third ones to be likewise satisfied, it is necessary that one have:

$$b^2 [2 c^2A^2 + (a^2 + c^2) C^2] - (b^2 + c^2) = 0,$$
$$b^2 [(a^2 + c^2) A^2 + 2 a^2C^2] - (a^2 + b^2) = 0.$$

If one subtracts these equations from each other then one will obtain, upon neglecting the common factor of $(c^2 - a^2)$:

(25) $$b^2 (A^2 + C^2) = 1,$$

and if one subtracts them, after having previously multiplied the first equation by a^2 and the second one by c^2, then after dividing by $b^2 (a^2 - c^2)$, one will get:

(26) $$c^2A^2 + a^2 C^2 = 1.$$

The last two equations are thus equivalent to the ones from which they were derived. They give:

$$B = 0, \qquad b^2 B^2 = \frac{b^2 - a^2}{c^2 - a^2}, \qquad b^2 C^2 = \frac{c^2 - b^2}{c^2 - a^2}$$

for the determination of the singular planes.

One will easily prove that these same values will satisfy the equation of the surface. Upon setting $B = 0$ in it, one will obtain the following equation:

$$[b^2 (A^2 + C^2) - 1][c^2 A^2 + a^2 C^2 - 1] = 0,$$

which will decompose into the two equations (25) and (26). One again concludes from this that the singular planes will cut the *xz*-plane along the common tangents to the circle and ellipse of intersection that are represented by equations (25) and (26).

In order to determine the contact curve in the singular planes whose positions we just determined, one must differentiate again. If one refers everything to the singular plane then one will get:

$$\tfrac{1}{2}\frac{d^2W}{dA^2} = 4b^2\,c^2\,A^2 = 4\cdot\frac{c^2(b^2-a^2)}{c^2-a^2},$$

$$\tfrac{1}{2}\frac{d^2W}{dB^2} = c^2\,(a^2+b^2)\,A^2 + a^2\,(b^2+c^2)\,C^2 - (a^2+b^2)$$

$$= b^2\,(c^2A^2 + a^2C^2) - a^2\,c^2\,(A^2+C^2) - (a^2+c^2) = -\frac{(c^2-b^2)(b^2-a^2)}{b^2},$$

$$\tfrac{1}{2}\frac{d^2W}{dC^2} = 4a^2\,b^2\,C^2 = 4\cdot\frac{a^2(c^2-b^2)}{c^2-a^2},$$

$$\tfrac{1}{2}\frac{d^2W}{dA\,dB} = 0,$$

$$\tfrac{1}{2}\frac{d^2W}{dA\,dC} = 2b^2\,(a^2+c^2)\,AC = 2\,(a^2+c^2)\,\frac{\sqrt{(c^2-b^2)(b^2-a^2)}}{c^2-a^2},$$

$$\tfrac{1}{2}\frac{d^2W}{dB\,dC} = 0,$$

and if one substitutes these values into equation (5) of number 5 then, after dividing by $\dfrac{4(b^2-a^2)(c^2-b^2)}{b^2(c^2-a^2)}$, it will then result that:

$$(27)\qquad c^2(b^2-a^2)\,z^2 + b^2(c^2-a^2)\,y^2 + a^2(c^2-b^2)\,x^2 \mp (a^2+c^2)\sqrt{(c^2-b^2)(b^2-a^2)}\;xz = 0.$$

This is the equation of the cone that has its center at the origin, and which gives the curve of contact in the singular plane by its intersection with that plane. The cone in question will be the interior light cone in Figure 6.

33. By following a completely analogous path, one can immediately obtain the equation for the contact curve if one replaces equation (11) with the following equation:

$$(28)\qquad \begin{aligned} W' = {} & (b^2\,c^2\,A^2 + a^2\,c^2\,B^2 + a^2\,b^2)(A^2+B^2+1) \\ & - [(b^2+c^2)\,A^2 + (c^2+a^2)\,B^2 + (a^2+b^2)]\,D^2 + D^4 = 0, \end{aligned}$$

which amounts to determining a tangent plane to the wave surface by means of the three constants A, B, and D in its equation, when it is presented in the form:

$$z + Ax + By + D = 0.$$

One will then have:

$$\tfrac{1}{2}\frac{dW'}{dA} = [2\, b^2\, c^2\, A^2 + c^2\,(a^2 + b^2)\, B^2 - (b^2 + c^2)\, D^2]\, A = 0,$$

$$\tfrac{1}{2}\frac{dW'}{dB} = [c^2\,(a^2 + b^2)\, A^2 + 2\, a^2\, c^2\, B^2 - (a^2 + c^2)\, D^2]\, B = 0,$$

$$\tfrac{1}{2}\frac{dW'}{dD} = 2D^2 - [(b^2 + c^2)\, A^2 + (c^2 + a^2)\, B^2 + (a^2 + b^2)]\, D = 0,$$

which will then give:

$$B = 0, \qquad A^2 = \frac{b^2 - a^2}{c^2 - b^2}, \qquad D^2 = b^2 \cdot \frac{c^2 - a^2}{c^2 - b^2}$$

for the singular plane. Then, after differentiating one more time and keeping these values in mind, one gets:

$$\tfrac{1}{2}\frac{d^2W'}{dA^2} = 4\, b^2\, c^2\, A^2 = 4\, b^2\, c^2 \cdot \frac{b^2 - a^2}{c^2 - b^2},$$

$$\tfrac{1}{2}\frac{d^2W'}{dB^2} = [c^2\,(a^2 + b^2)\, A^2 + 2a^2\, c^2 B^2 - a^2 + c^2 D^2)] = -\,(b^2 - a^2)(c^2 - a^2),$$

$$\tfrac{1}{2}\frac{d^2W'}{dD^2} = 4D^2 = 4\, b^2 \cdot \frac{c^2 - a^2}{c^2 - b^2},$$

$$\tfrac{1}{2}\frac{d^2W'}{dA\,dB} = 0,$$

$$\tfrac{1}{2}\frac{d^2W'}{dA\,dD} = -\,2(b^2 + c^2)\, AD = \mp\frac{2(b^2 + c^2)\, b\sqrt{(b^2 - a^2)(c^2 - a^2)}}{c^2 - b^2},$$

$$\tfrac{1}{2}\frac{d^2W'}{dR^2} = 0.$$

Upon substituting this in equation (6) of number 6, and dividing by $b^2\,(c^2 - a^2)(b^2 - a^2)$, one will get:

$$\frac{c^2(b^2 - a^2)}{c^2 - b^2} + y^2 + \frac{c^2 - a^2}{c^2 - b^2} \cdot x^2 \pm \frac{b^2 + c^2}{b} \cdot \frac{\sqrt{(b^2 - a^2)(c^2 - a^2)}}{c^2 - b^2} = 0,$$

an equation that one can put into the following form:

$$\frac{c^2(b^2-a^2)}{c^2-b^2}+y^2+\frac{c^2-a^2}{c^2-b^2}\left[x\pm\frac{b^2+c^2}{2b}\cdot\sqrt{\frac{b^2-a^2}{c^2-a^2}}\right]^2=\frac{(c^2-b^2)(b^2-a^2)}{4b^2}.$$

This is the equation of the projection of the contact curve onto the xy-plane. Since A^2 is the square of the tangent of the angle that the singular plane makes with the xy-plane, the square of the cosine of that angle will be $\frac{c^2-b^2}{c^2-a^2}$, so one will see that the contact curve is a *circle* whose radius is equal to:

$$\frac{\sqrt{(c^2-b^2)(b^2-a^2)}}{2b}.$$

34. *Singular points.* The equation:

(29) $$c^2 (a^2 + b^2)(A^2 + 2a^2 c^2 B^2 + a^2 (b^2 + c^2) C^2 - (a^2 + c^2) = 0$$

will determine an ellipsoid by its tangent planes. It will be easy to obtain the equation of that ellipsoid in rectangular coordinates. It is obvious that B and C will disappear for the tangent planes that are parallel to the yz-plane, while $1 / C$ will become equal to the semi-axis of the ellipsoid that coincides with the x-axis. One will obtain the other two semi-axes in the same manner. The preceding equation gives:

$$B = 0, \qquad C = 0, \qquad \frac{1}{A^2}=\frac{c^2(a^2+b^2)}{a^2+c^2},$$

$$A = 0, \qquad C = 0, \qquad \frac{1}{B^2}=\frac{2a^2c^2}{a^2+c^2},$$

$$A = 0, \qquad B = 0, \qquad \frac{1}{C^2}=\frac{a^2(b^2+c^2)}{a^2+c^2},$$

so the desired equation will be the following one:

$$\frac{(a^2+c^2)\,x^2}{(a^2+b^2)\,c^2}+\frac{(a^2+c^2)\,y^2}{2a^2c^2}+\frac{(a^2+c^2)\,z^2}{(b^2+c^2)\,a^2}=1.$$

After that digression, I now return to my objective. If we algebraically combine the equation:

(13) $$(a^2 x^2 + b^2 y^2 + c^2 z^2)\,(x^2 + y^2 + z^2) \\ - (a^2 (b^2 + c^2)\,x^2 + b^2 (c^2 + a^2)\,y^2 + c^2 (a^2 + b^2)\,z^2] + a^2 b^2 c^2 = 0.$$

which gives the light wave by its tangent planes, with the following equation:

$$[c^2 (a^2 + b^2) A^2 + 2 a^2 c^2 B^2 + a^2 (b^2 + c^2) C^2 - (a^2 + c^2)]^2 = 0$$

then the resulting equation will determine a new surface, in general. That surface will have the property that it touches the wave surface along a curve, in such a manner that the common tangent planes to the two surfaces will envelop the ellipsoid (29). Subtract equation (13) from the latter equation, after having multiplied it by $4a^2 c^2$; the resulting equation can be put into the form:

$$[c^2 (b^2 - a^2) A^2 + a^2 (c^2 - b^2) C^2 - (c^2 - a^2)]^2 - 4a^2 c^2 (b^2 - a^2)(c^2 - a^2) A^2 C^2 = 0,$$

and will decompose into the following two:

$$c^2 (b^2 - a^2) A^2 + a^2 (c^2 - b^2) C^2 - (c^2 - a^2) \pm 2ac \sqrt{(b^2 - a^2)(c^2 - a^2)}\; A\, C = 0,$$

These two equations can thus be written:

$$\left[c\sqrt{b^2 - a^2}\, A \pm a\sqrt{c^2 - b^2}\, C\right]^2 - (c^2 - a^2) = 0,$$

which decomposes once more in the following manner:

$$c\sqrt{b^2 - a^2}\, A \pm a\sqrt{c^2 - b^2}\, C \pm \sqrt{c^2 - a^2} = 0.$$

All of the planes whose constants A and C satisfy one of the four equations will pass through one of four points whose coordinates will be:

$$y = 0, \quad x = \pm c\sqrt{\frac{b^2 - a^2}{c^2 - a^2}}\,, \qquad z = \pm a\sqrt{\frac{c^2 - b^2}{c^2 - a^2}}\,.$$

These four points will thus replace the surface in question. It will follow from this that they will be singular points of the wave surface, and that the four cones that are circumscribed on the ellipse (29) and have their centers at these points will be the ones that touch the wave along each of their edges.

35. *Geometric considerations.* One can arrive at the complete determination of two sorts of singularities on the surface of the light wave by a purely geometric, and very simple, argument.

To an arbitrary point of the first ellipsoid there will always correspond just one point p of the elasticity surface (Fig. 4), in such a way that the latter point p will be the base of the perpendicular that is dropped from the center O onto the tangent plane to the ellipsoid at the first point m. From number 25, the diametral plane that passes through the two points m and p will be the plane of vibration. When one draws two straight lines OM and OV in that plane that are perpendicular and equal to Om and Op, respectively, the light ray that corresponds to the vibrations that take place along *Op* will be *OM*, while the wave front will be perpendicular to *OV* at *V*.

It will likewise correspond to a diametral section of the elasticity surface, and *vice versa*. Here, we shall confine ourselves to considering, more particularly, the two circular sections of one and the other of the two surfaces. It is easy to see that these four sections belong to the same sphere that has the mean semi-axis of the ellipsoid for its radius; that was proved in number 14 in regard to two circular sections of it. The radius of the circular sections of the second ellipsoid will thus be equal to $1 / b$, and that of the corresponding – likewise circular – sections of the surface of elasticity will be equal to b, whose inverse value is $1 / b$. This was to be proved.

We first occupy ourselves with two circular sections of the elasticity surface. The right cylinder that has either of these two sections $H_0\ B\ H^0$ for its base will envelop the first ellipsoid. The contact curve $F_0\ B\ I^0$, which is necessarily planar, will be its corresponding section, such that every edge of the cylinder will cut the two sections of the elasticity surface and the ellipsoid at two corresponding points; in the figure, these will be the points h and g. The section of the ellipsoid will be an ellipse whose minor axis, which equals b, will coincide with a diameter of the circular section on the y-axis. The plane of vibration will always pass through the center O and two corresponding points – such as h and g – in such a way that it is perpendicular to the circular section $H_0\ B\ H^0$. Turn the triangle hOg in its plane around the center O until OH becomes perpendicular to Oh. HG will then be perpendicular to hg and parallel to Oh. Therefore, no matter what the two corresponding points h and g might be in the new position, the point H will the same fixed point, whereas the point G will always be found in the same plane that is parallel to the circular section and distant from it by an amount $OH = OB = b$. In order to determine the geometric locus of the various positions of the point G in this plane, we observe that one has:

$$hg = H_0\, I_0 \cos H_0\, OH,$$

or even, upon substituting:

$$HG = HI \cos GHI.$$

One concludes from this that the angle HGI is a right angle. Therefore, the desired locus will be a circle that is perpendicular to the xz-plane and has HI for one of its diameters. The light rays that correspond to the various points of the section $I_0\ B\ I^0$ will then define a second-order light cone. The wave front will be the same for all rays. All of their planes of vibration will pass through the point H.

36. In the second place, consider two circular sections of the first ellipsoid. Along either of these two sections $M_0\ B\ M^0$ (see Fig. 8), the ellipsoid will be touched by a second-degree circumscribed cylinder. If one drops perpendicular from the center to the tangent planes to that cylinder then the base of these perpendiculars will constitute the corresponding curve of the elasticity surface. In order to determine it, draw a diametral plane through the center that is perpendicular to the edges of the circumscribed cylinder. That plane will cut the cylinder along an ellipse whose major axis – which equal b – will coincide with the y-axis. It will contain the desired curve that one constructs by dropping perpendiculars from the center to the tangents to the ellipse of intersection. Let *m* be a point of the circular section of the ellipsoid, so the edge of the cylinder that passes through *m* will cut the ellipse at *n*, where it is touched by the line *nv*. *Ov*, which is perpendicular to *nv*, will be likewise perpendicular to the plane *mnv*, which touches the ellipsoid at *m*. Rotate the triangle *vOm* through a quarter of a revolution around the center *O* in its own plane, which is that of vibration. No matter what the two corresponding points *m* and *v* might be in the new position, *M′* will be the same fixed point. Since *Ovm* is a right angle, *M′V* will become parallel to *Ov*, in such a way that the point *V* must remain constantly in a plane that passes through *M′* and is parallel to the oval section of the elasticity surface. Since the angle *OVM′* is likewise a right angle, *V* must be found on the surface of a sphere that is described on *OM′* as its diameter. One will conclude from this that the geometric locus of the points *V* will be a circle that is perpendicular to the *xz*-plane and will have *MP* for one of its diameters.

Therefore, all of the various points of the circular section of the ellipsoid will correspond to just one unique light ray, whereas the wave front, which depends upon the points of the corresponding section of the elasticity surface, will vary from one point to another. At its various positions, it will envelop a second-degree cone whose tangent planes will touch the circle *M′ VP*, and will be perpendicular to the radius vectors that end on the points of the circle, respectively. The planes of vibration will all pass through the light ray *OM′*.

37. *Velocity of plane waves in the interior of a crystal*. We have previously let $V_{\prime}$ and $V_{\prime\prime}$ denote the perpendiculars that were dropped from the center *O* onto the plane wave fronts that envelop the wave surface, which starts from an arbitrary point *O* of the interior of the crystal, which is taken to be the center, and propagates in all directions. Two corresponding values of $V_{\prime}$ and $V_{\prime\prime}$ will refer to two plane waves that will always accompany each other along the same direction. If we develop equation (2) of number 7:

$$V_{\prime}^2 - V_{\prime\prime}^2 = \pm\sqrt{(\mu+\rho)^2 - 4(\mu\rho - \nu^2)}$$

by using the two equations (5) of number 18, which we write in the following manner, in regard to the last equation of number 2:

$$(\mu + \rho) = \frac{(a^2+b^2)\,C^2 + (a^2+b^2)\,B^2 + (b^2+c^2)\,A^2}{A^2+B^2+C^2},$$

$$(\mu\rho - \nu^2) = \frac{a^2b^2C^2 + a^2b^2B^2 + b^2c^2A^2}{A^2+B^2+C^2},$$

then it will result, after some simple reductions, that:

$$(V_{,}^2 - V_{,,}^2)^2 = \frac{[(b^2-a^2)C^2 + (c^2-a^2)B^2 + (c^2-b^2)A^2]^2 - 4(b^2-a^2)(c^2-a^2)A^2C^2}{(A^2+B^2+C^2)^2}.$$

Let ζ, η, ϑ denote the three angles that the perpendicular to the fronts that are parallel to the two plane waves makes with the three x, y, and z axes, respectively. We will have:

$$\cos^2\zeta = \frac{A^2}{A^2+B^2+C^2}, \qquad \cos^2\eta = \frac{B^2}{A^2+B^2+C^2}, \qquad \cos^2\vartheta = \frac{C^2}{A^2+B^2+C^2}.$$

Upon introducing these values into the preceding equation, one will get:

(1) $(V_{,}^2 - V_{,,}^2)^2$

$$= [(b^2-a^2)\cos^2\vartheta + (c^2-a^2)\cos^2\eta + (c^2-b^2)\cos^2\zeta] - 4(b^2-a^2)(c^2-a^2)\cos^2\zeta\cos^2\vartheta,$$

and if one decomposes its right-hand side into two factors:

$$(V_{,}^2 - V_{,,}^2)^2 = \left[\{\sqrt{(b^2-a^2)}\cos\vartheta + \sqrt{(c^2-b^2)}\cos\zeta\}^2 + (c^2-a^2)\cos^2\eta\right]$$
$$\times\left[\{\sqrt{(b^2-a^2)}\cos\vartheta - \sqrt{(c^2-b^2)}\cos\zeta\}^2 + (c^2-a^2)\cos^2\eta\right].$$

Let $\alpha_{,}$ and $\alpha_{,,}$ (see Fig. 1) denote the angles that the two tangents $T_{,}T''$ and $T_{,}T_{,,}$ that are common to the circle and the ellipse of intersection in the xz-plane make with the x-axis, and let $\gamma_{,}$ and $\gamma_{,,}$ denote the angles that the same tangents define with the z-axis, in such a way that one has:

$$\cos\alpha_{,,} = \cos\alpha_{,} = +\sqrt{\frac{c^2-b^2}{c^2-a^2}}, \qquad -\cos\gamma_{,,} = \cos\gamma_{,} = +\sqrt{\frac{b^2-a^2}{c^2-a^2}}.$$

The preceding equation can then be written thusly:

$$(V_{,}^2 - V_{,,}^2)^2 =$$
$$(c^2-a^2)^2\,[(\cos\gamma_{,}\cos\vartheta + \cos\alpha_{,}\cos\zeta)^2 + \cos^2\eta][(\cos\gamma_{,,}\cos\vartheta + \cos\alpha_{,,}\cos\zeta)^2 + \cos^2\eta].$$

However, according to the known formula that gives the angle between two lines by means of the three angles that each of them make with the three coordinate axes, one will have:

$$\cos\gamma_{,}\cos\vartheta + \cos\alpha_{,}\cos\zeta = \cos\varepsilon, \qquad \cos\gamma_{,,}\cos\vartheta + \cos\alpha_{,,}\cos\zeta = \cos\varepsilon',$$

if we let ε and ε' denote the angles that the perpendicular to the wave front makes with the directions of the tangents $T_{,}T''$ and $T_{,}T_{,,}$, resp. Finally, let ψ and ψ' denote the angles that the same perpendicular makes with the two optical axes. Since these axes are perpendicular to the two tangents, and at the same time, to the y-axis, one will have:

$$\cos^2 \eta + \cos^2 \varepsilon + \cos^2 \psi = 1, \qquad \cos^2 \eta + \cos^2 \varepsilon' + \cos^2 \psi' = 1,$$

so

$$\cos^2 \eta + \cos^2 \varepsilon = \sin^2 \psi, \qquad \cos^2 \eta + \cos^2 \varepsilon' = \text{asin}^2 \psi'.$$

After having made the substitutions thus indicated and taking the square roots, one will finally get the equation:

(2) $$V_{\prime}^2 - V_{\prime\prime}^2 = (c^2 - a^2) \sin \psi \sin \psi'.$$

One sees that the two velocities are never equal, except in the case where one of the angles ψ and ψ' disappears. The front of two waves that move in the same direction will then be perpendicular to one of the two optical axes HH' and $H_{\prime\prime}H''$.

If the front of two waves is perpendicular to the xz-planes then one of the two velocities will be constant and equal to b. Should one wish that this be $V_{\prime\prime}$, and one increases the angles ψ and ψ' up to π then one will get:

(3) $$V_{\prime\prime}^2 = b^2 \pm (c^2 - a^2) \sin \psi \sin \psi',$$

where one can take the + sign when the perpendicular that is dropped from the center to the wave front falls between OH and $OH_{\prime\prime}$ or their prolongations, whereas one must take the – sign when it falls between OH and OH'', or their prolongations. In the former case, the extraordinary wave will lead the ordinary wave, while in the latter case, the opposite will be true.

If the front of two waves is perpendicular to one of the two planes that bisect the angles of two optical axes – i.e., if it is perpendicular to the xy and yz-planes – then one of the two velocities will be constant. Consider the wave that is perpendicular to the xy-plane and consequently set:

$$\cos \vartheta = 0, \qquad \cos^2 \rho + \cos^2 \eta = 1$$

in equation (1), so one will get:

$$V_{\prime}^2 - V_{\prime\prime}^2 = \pm [c^2 - (a^2 \cos^2 \eta + b^2 \cos^2 \zeta)],$$

where one must take the – sign when one desires that $V_{\prime\prime}$ should be constant. It would then result that:

$$V_{\prime\prime}^2 = c^2, \qquad V_{\prime}^2 = a^2 \cos^2 \eta + b^2 \cos^2 \zeta,$$

in such a way that one independently obtains the velocity of the ordinary wave and that of the extraordinary wave from one or the other, respectively. The ordinary wave will always lead the extraordinary wave. The construction of the velocity of the extraordinary wave is linked by the latter equation to the following property of the ellipse (viz., that of intersection with the xy-plane) that if one projects the two semi-axes onto any of its tangents then the sum of the squares of the projections will be equal to the square of the perpendicular that is dropped from the center to that same tangent.

While the ordinary wave constantly leads the extraordinary one perpendicular to the xy-plane, when the waves are perpendicular to the yz-plane, it will be the extraordinary wave that leads the ordinary one.

38. According to equation (1) of number 7, one has:

$$V_{\prime}^2 + V_{\prime\prime}^2 = \mu + \rho = \frac{(a^2+b^2)C^2 + (a^2+c^2)B^2 + (b^2+c^2)A^2}{A^2+B^2+C^2}$$

$$= (a^2 + b^2)\cos^2\vartheta + (a^2 + c^2)\cos^2\eta + (b^2 + c^2)\cos^2\zeta .$$

Upon noting that:

$$\cos^2\vartheta + \cos^2\eta + \cos^2\zeta = 1,$$

that equation can be written thusly:

$$V_{\prime}^2 + V_{\prime\prime}^2 = (a^2 + b^2) + (b^2 - a^2)\cos^2\zeta - (c^2 - b^2)\cos^2\vartheta$$

$$= (a^2 + c^2) + (c^2 - a^2)\left[\frac{b^2-a^2}{c^2-a^2}\cos^2\zeta - \frac{c^2-b^2}{c^2-a^2}\cos^2\vartheta\right].$$

If one lets α' and α'' denote the two angles that the two optical axes OH' and OH'' make with the x-axis, and lets γ' and γ'' denote the two angles that they makes with the z-axis then one will have:

$$\cos\alpha'' = \cos\alpha' = +\sqrt{\frac{b^2-a^2}{c^2-a^2}}, \qquad -\cos\gamma'' = \cos\gamma' = +\sqrt{\frac{c^2-b^2}{c^2-a^2}}.$$

It will follow from this that:

$$V_{\prime}^2 + V_{\prime\prime}^2 = (a^2 + c^2) + (c^2 - a^2)[\cos\alpha'\cos\zeta + \cos\gamma'\cos\vartheta][\cos\alpha''\cos\zeta + \cos\gamma''\cos\vartheta],$$

and finally:

(4) $$V_{\prime}^2 + V_{\prime\prime}^2 = (a^2 + c^2) + (c^2 - a^2)\cos\psi\cos\psi'.$$

If one successively adds equation (2) of the preceding number to that equation and then subtracts it then one will get:

$$V_{\prime}^2,\ V_{\prime\prime}^2 = \tfrac{1}{2}[(a^2 + c^2) + (c^2 - a^2)(\cos\psi\cos\psi' \pm \sin\psi\sin\psi')],$$

and by simple trigonometric transformations:

$$(5) \qquad V_{,}^2,\ V_{,,}^2 = \tfrac{1}{2}[(a^2 + c^2) + (c^2 - a^2)\cos(\psi \mp \psi')] = c^2 - (c^2 - a^2)\sin^2 \tfrac{1}{2}(\psi \mp \psi').$$

This equation will give the velocities of each of the two waves separately; its discussion would offer no difficulty.

39. *Velocity of light rays inside of a crystal.* – Upon replacing a^2, b^2, c^2 with their inverse values $1 / a^2$, $1 / b^2$, $1 / c^2$ in the developments of the two preceding numbers, one must replace the direction of the perpendicular that is dropped from the center to the wave front with that of the light ray, unity divided by the velocity of the wave with the velocity of the ray, and finally, the two optical axes with the two straight lines $M_{,}M'$ and $M_{,,}M''$, which are the common diameters to the circle and ellipse of intersection in the *xz*-plane. If one lets $P_{,}$ and $P_{,,}$ denote the velocities of two rays that coincide in the same direction and lets φ and φ' denote the two angles that this direction makes with the two straight lines $M_{,}M'$ and $M_{,,}M''$ then one will get:

$$(6) \qquad \frac{1}{P_{,}^2} - \frac{1}{P_{,,}^2} = \pm\left(\frac{1}{c^2} - \frac{1}{a^2}\right)\sin\varphi\sin\varphi'.$$

This theorem is due to *Biot.* I found it stated without proof in *Herschel's Traité de la lumière*, no. 1018, as well as in *Airy's Philosophical tracts*, 2nd ed., pp. 352.

In each of the coordinate planes, the velocity of one of the rays is constant and equal to the velocity of the corresponding wave; that will be the one that constitutes the ordinary wave. The velocity of the other wave is determined by the radius vectors of the ellipse of intersection in the same plane. One then has, for example, for the light rays, when referred to the plane of the optical axes:

$$P_{,}^2 = b^2, \qquad \frac{1}{P_{,,}^2} = \frac{1}{b^2} \pm \left(\frac{1}{c^2} - \frac{1}{a^2}\right)\sin\varphi\sin\varphi',$$

or even:

$$\frac{1}{P_{,,}^2} = \frac{\cos^2\eta}{a^2} + \frac{\cos^2\zeta}{b^2}.$$

Finally, we remark that it can happen that the velocity of the extraordinary ray exceeds that of the ordinary ray that follows the same route, while the corresponding extraordinary wave proceeds more slowly than the ordinary wave. That is what happens for the rays that are situated between *OI* and *OM″*. On the other hand, it can happen that when two waves follow the same route, the ordinary one surpasses the extraordinary one in velocity, while the corresponding ordinary ray lags behind the extraordinary ray. That is what happens when the perpendicular that is dropped from the center to the two parallel waves is situated between *OQ* and *OH″*.

40. The formulas that were obtained in number 38 are transformed by the same substitutions into the following ones:

$$\frac{1}{P_{\prime}^{2}}+\frac{1}{P_{\prime\prime}^{2}}=\left(\frac{1}{a^{2}}+\frac{1}{c^{2}}\right)-\left(\frac{1}{a^{2}}-\frac{1}{c^{2}}\right)\cos\varphi\cos\varphi',$$

$$\frac{1}{P_{\prime}^{2}},\frac{1}{P_{\prime\prime}^{2}}=\frac{1}{c^{2}}+\left(\frac{1}{a^{2}}-\frac{1}{c^{2}}\right)\sin^{2}\tfrac{1}{2}(\varphi\pm\varphi').$$

The latter result is due to the illustrious *Fresnel* (*Biot*, *Précis élémentaire de physique*, Third ed., vol. II, pp. 259).

41. *Given the wave front, determine the planes of vibration by means of two optical axes.* – Let the oval of Fig. 9 be an arbitrary diametral section of the elasticity surface, *VV*, its *maximum* diameter, and *V′V′*, its *minimum* diameter. The two planes of vibration will then be the ones that intersect along *VV* and *V′V′*, since they are perpendicular to the plane of the figure. The two circular sections of the surface will cut the oval along two diameters *KK* and *K′K′*, being both equal to 2*b*, will necessarily make equal angles with each of the two straight lines *VV* and *V′V′*. The optical axes will be perpendicular to these circular sections, whose inclination from the plane of the oval will depend upon the position of that plane. If one constructs two planes that are perpendicular to the diameters *KK* and *K′K′*, respectively, then each of these planes will pass through the normal to the plane of the figure and one of the two optical axes, in addition. It will cut the oval, to which it is perpendicular, along the two straight lines *UU* and *U′U′*, which make equal angles with each of the two diameters *VV* and *V′V′*.

When a wave front is given, the two corresponding planes of vibration will divide the angles into two equal parts between the two planes that will each contain one of the two optical axes, in addition to the normal to the front.

This elegant theorem is due to the sagacity of *Biot*. One finds a proof of it at the cited location in *Airy's* book.

42. *Given a light ray, determine the planes of vibration by means of the two normals to the circular sections of the first ellipsoid.* – In the preceding number, we say that the plane of vibration simultaneously passes through light ray and the perpendicular that is dropped from center to the wave front, and that, in the second place, it cuts the two corresponding sections of the elasticity surface and the first ellipsoid along one of their *maximum-minimum* diameters in either case, while being perpendicular to both of them. Upon replacing the diametral section of the elasticity surface with a diametral section of the first ellipsoid in the considerations of the preceding number, one gets this other theorem:

Given a light ray, the two corresponding planes of vibrations will divide into two equal parts the angles that are defined by two planes that will each contain one of the two normals to the circular sections of the first ellipsoid, in addition to the ray.

Conforming to the preceding number, the planes of vibration of the two rays, in which an external ray is divided, will intersect with equal angles upon entering the crystal with perpendicular incidence. From the present number, the same thing will happen when two rays into which an internal ray has divided leave the crystal.

43. *Double refraction, according to Huyghens's principle.* – When a light wave that propagates in air strikes the surface of a crystal, it will be excited at all of the points where its front meets the partial waves, either outside of the crystal in air or inside of it. The one – which is a spherical form – will be enveloped by a plane in the same arbitrary instant; that will be the corresponding front of the reflected wave. The other one will take a form that depends upon the position of the three elasticity axes in the crystal and the magnitude of the elasticity along these axes. At an arbitrary instant, the surfaces of these waves will be enveloped by two different planes, which will constitute the corresponding fronts of the two refracted waves. It will follow from this principle – which is originally due to *Huyghens* – that at the same arbitrary instant, the incident wave front and those of the reflected and refracted waves must cut the surface of the crystal (which we assume to be planar) along the same straight line.

Let *AA* (cf., Fig. 10) be the surface of the crystal, *FO*, the incident wave front, both of which are perpendicular to the plane of the paper, in such a way that this plane will be the plane of incidence for the light ray *RO*, which is perpendicular to the wave front at *O*. Let $\tau_1 - \tau$ denote the time interval that it takes for the wave front to pass from *FO* to the new position $F'O'$. After that interval – at the epoch τ_1 – the partial waves that are excited in air around each point of the straight line that is perpendicular to the plane of the paper at *O* will be spheres whose radii are equal to *OG* – i.e., equal to $\tau_1 - \tau$ – if we take the unit of length to be the velocity with which the propagation takes place in air. During this same interval, a light wave will form inside of the crystal around each point of the perpendicular at *O* that has the velocities $a(\tau_1 - \tau)$, $b(\tau_1 - \tau)$, $c(\tau_1 - \tau)$ along the three elasticity axes. The dimensions of the two series of partial waves – viz., external and internal – that exist at the epoch τ_1 will then diminish in proportion to $\tau_1 - \tau$ (i.e., as the original wave front approaches from the position $F'O'$) in such a manner that the waves that are excited at the various points of the perpendicular at O' will reduce to simple points. It will then result that since the wave equation is homogeneous, at the epoch τ_1, all of these waves will be enveloped by planes that pass through the perpendicular O'. In order to obtain them, one needs only to construct the planes that touch the surface of one arbitrary partial wave, while passing through that perpendicular.

Upon limiting ourselves to internal waves, we take the one that is described around the point *O*. At the epoch τ_1, the two tangent planes $O' E\ D$, $O' E' D'$, which are perpendicular to the plane of incidence, will be the fronts of the two refracted waves, while the two radius vectors *OE*, *OE′* that pass through the two contact points *E*, *E′* will be the two rays into which the incident ray *RO* will divide under refraction.

In the general case, none of the two refracted rays will be found in the plane of incidence. The two planes of vibration *ODE*, *OD′E′* will be the ones that pass between the two rays *OE*, *OE′* and the two perpendiculars *OD*, *OD′* that are dropped from the center *O* (in the plane of incidence) onto the fronts of the two refracted waves.

The two refracted rays will both be found in the plane of incidence when the plane of incidence either contains the two optical axes or makes equal angles with these axes (i.e., when that plane coincides with one of the coordinate planes). The velocity of one of them will be constant, in such a manner that one will incline the incident ray to the surface of the crystal; the corresponding plane of vibration will be perpendicular to that of incidence. The velocity of the other ray will change with the inclination; the corresponding plane of vibration will coincide with the plane of incidence. The planes of vibration will then be mutually perpendicular for the two refracted rays.

As for perpendicular incidence, the preceding construction will reduce to that of the two tangent planes to the partial wave surface that is described around the point *O*, which will be parallel to each other and to the incident wave front. In this case, the two planes of vibration for the two refracted rays will be mutually perpendicular, no matter what the surface of the crystal might be.

44. *Surface of wave slowness.* – Above, we found the equation:

$$\begin{gathered}(b^2 c^2 A^2 + a^2 c^2 B^2 + a^2 b^2 C^2)(A^2 + B^2 + C^2) \\ - [(b^2 + c^2) A^2 + (a^2 + c^2) B^2 + (a^2 + b^2) C^2] + 1 = 0\end{gathered}$$

for determining the wave surface by its tangent planes, in the most general case, where *A*, *B*, *C* denote the inverse values of the line segments that such a plane will cut out along the three coordinate axes. According to number 8, upon replacing these variables with *x*, *y*, *z*, the resulting equation in rectangular coordinates:

$$\begin{gathered}(b^2 c^2 x^2 + a^2 c^2 y^2 + a^2 b^2 z^2)(x^2 + y^2 + z^2) \\ - [(b^2 + c^2) x^2 + (a^2 + c^2) y^2 + (a^2 + b^2) z^2] + 1 = 0\end{gathered}$$

will represent the reciprocal polar surface to that of the wave with respect to sphere whose radius is equal to unity. One will arrive at the same equation if one replaces the three constants a^2, b^2, c^2 with $1 / a^2$, $1 / b^2$, $1 / c^2$ in the wave equation:

$$\begin{gathered}(a^2 x^2 + b^2 y^2 + c^2 z^2)(x^2 + y^2 + z^2) \\ - [a^2 (b^2 + c^2) x^2 + b^2 (a^2 + c^2) y^2 + c^2 (a^2 + b^2) z^2] + a^2 b^2 c^2 = 0.\end{gathered}$$

From this, one infers the following theorem:

The surfaces of two waves that are such that the three axes (principal velocities) of one of them are inverses of the axes of the other one will be polar reciprocal with respect to a sphere whose radius is equal to unity.

Whereas the radius vectors of one of the two surfaces will represent the velocity of the various light rays, those of the other one will be equal to unity divided by the velocity of the corresponding plane waves. Whereas the perpendiculars that are dropped from the center onto the tangent planes of one of them will represent the velocity of the plane waves, those that are dropped onto the fronts of the other one will be equal to unity divided by the velocity of the corresponding rays. These relationships are entirely reciprocal (*).

45. *Hamilton's construction.* – From number 43, the determination of the two refracted rays demands that one must construct the two planes that pass through the straight line that is perpendicular to the plane of incidence at O' (cf., Fig. 10) and touch the wave surface. In general, it will be simpler to construct the poles of these tangent planes. One obtains these poles by looking for the points where the straight line that is the reciprocal polar to the perpendicular that is contained in the tangent planes cuts the reciprocal polar surface to the wave. Take the unit to be the ray $OG = OL$ of the spherical wave that is described around the point of incidence O. With respect to that sphere, the straight line GL that is contained in the plane of incidence and passes through the two points G and L where the incident ray RO and the OL cut the sphere will be the reciprocal polar of the one that is perpendicular to the plane of incidence (number 12) at O'. The following construction will result from that:

Construct the spherical surface of the wave that propagates in air around the point of incidence and the reciprocal polar surface, with respect to that sphere, of the wave that is described around the same point inside the crystal. The straight line that passes through the two points G and L where the incident ray ROG and the reflected ray OL meet the sphere will, in general, cut the polar surface at two points K and K'. The fronts of the two refracted waves will be perpendicular to the radius vectors OK, OK'; their velocities OD,

(*) I must conclude from the suggestions that were made by Mr. *Lloyd*, in his excellent "*Report on the progress and present state of physical optics*" (London, 1835) that the reciprocal polar surface of a light wave with respect to a sphere whose radius is equal to unity will be precisely the one that Mr. *Hamilton* called the "*surface of wave slowness*." While having no knowledge of *Hamilton's* paper, I will confine myself to borrowing the following construction from the *Report*, where I found it simply stated:

> "…they lead to a very elegant construction for the reflected or refracted ray, which is, in most cases, more convenient than that of *Huygens*. That when a ray proceeds from air into any crystal, we have only to construct the *surfaces of wave slowness* belonging to the two media and having their common centre at the point of incidence. Let the incident ray be then produced to meet the sphere, which represents the normal slowness of the wave in air, and from the point of intersection, let a perpendicular be drawn to the reflecting or refracting surface. This will cut the surface of slowness of the reflected or refracted waves, in general, in two points. The lines connecting these points with the centre will represent the direction and normal slowness of the *waves*, while the perpendiculars from the centre on the tangent planes at the same points will represent the direction and slowness of the *rays* themselves."

One will confirm that this construction amounts to that of the following number if one intends *slowness* to mean unity divided by the velocity.

OD´ will be equal to unity divided by these radius vectors. The perpendiculars that are dropped from the center onto the tangent planes to the polar surface to the wave will give the directions of the two refracted rays whose velocities *OE*, *OE´* will be equal to unity divided by these perpendiculars.

46. *The wave surface is its own reciprocal polar.* If we eliminate *A*, *B*, *C* from the equation that was referred to in number 44:

$$(b^2 c^2 A^2 + a^2 c^2 B^2 + a^2 b^2 C^2)(A^2 + B^2 + C^2)$$
$$- [(b^2 + c^2) A^2 + (a^2 + c^2) B^2 + (a^2 + b^2) C^2] + 1 = 0$$

by means of the following three equations:

$$(1) \qquad b^2 c^2 A^2 = x^2, \qquad a^2 c^2 B^2 = y^2, \qquad a^2 b^2 C^2 = z^2$$

then, from number 9, the resulting equations in *x*, *y*, and *z* will represent the polar surface to that of the wave with respect to the ellipsoid:

$$(2) \qquad \frac{x^2}{bc} + \frac{y^2}{ac} + \frac{z^2}{ab} = 1,$$

whose three semi-axes are equal to $\sqrt{bc}$, $\sqrt{ac}$, $\sqrt{ab}$, respectively. However, the equation that is obtained from the indicated elimination, namely:

$$(a^2 x^2 + b^2 y^2 + c^2 z^2)(x^2 + y^2 + z^2)$$
$$- [a^2 (b^2 + c^2) x^2 + b^2 (a^2 + c^2) y^2 + c^2 (a^2 + b^2) z^2] + a^2 b^2 c^2 = 0,$$

is nothing but that of the surface of the wave itself. This presents us with the following elegant theorem:

The wave surface is its own reciprocal with respect to an ellipsoid whose three axes are proportional to the means of the products of the three axes of the first ellipsoid, when taken pair-wise.

As far as the optics of crystals is concerned, this theorem constitutes a type of redundancy (or *duality*, if I may appeal to *Gergonne's* word, without, however, attributing any metaphysical significance to it): The phenomena present themselves in pairs, in such a way that one of each pair is deduced from the other one by means of the director ellipsoid (2). It will respond to each point of the wave surface with a unique plane that touches it, and consequently, each internal light ray that ends at that point there will be a unique internal wave front, or furthermore, a unique external ray that is perpendicular to that front at the contact point. The singular points of the wave will produce its singular planes. One sees directly from this that the points in question will be determined by the equations:

$$c\sqrt{b^2 - a^2}\cdot A \pm a\sqrt{c^2 - b^2}\cdot C + \sqrt{c^2 - a^2} = 0,$$

which transforms into the following ones:

$$\sqrt{b^2 - a^2}\cdot x \pm \sqrt{c^2 - b^2}\cdot y + b\sqrt{c^2 - a^2} = 0,$$

by means of equations (1), which are those of the singular planes. The perpendiculars to the singular planes will thus play the same role with respect to the exiting light rays that the diameters that pass through the singular points play with respect to the entering rays. The one and the other, which are frequently confused, compete for the name of "optical axes." The tangent cones at the four singular points will produce the curves of contact in the four singular planes, just as the unique ellipsoid that is enveloped by the four cones will produce the ellipsoid that contains the four curves of contact (number 11). The two beautiful experiments of *Hamilton* and *Lloyd* determine the one in terms of the other by means of the director ellipsoid (2).

47. *Third construction of the two refracted rays.* I will conclude this article by deducing a new construction of the refracted rays from the theorem in the preceding number that seem to me to be the simplest possible one among the ones that relate to *Huygens's principle.*

Construct the polar straight line with respect to the director ellipsoid to the one that is perpendicular to the plane of incidence at O′. It will cut the wave surface that is described around the point O at two points. The two straight lines that go from the point O to these points will be the two refracted rays, while the two planes that contain the perpendicular to O′ and pass through these same two points will be the fronts of the two corresponding plane waves. Finally, it was proved in the foregoing that the two planes of vibration will be the ones that one obtains by drawing planes through the light rays that are perpendicular to the fronts of the corresponding waves.

The article that you just read will give rise to other ones, having itself been born in the beautiful article that was entitled "*On the phenomena presented by light in its passage along the axes of biaxial crystals,*" by the rev. *Humphrey Lloyd*, etc., from the seventeenth volume of the Transactions of the royal Irish Academy, Dublin, 1833, for which I must acknowledge the extreme goodwill of the author.

No physical experiment has made a bigger impression on me than that of conical refraction. A unique light ray that enters a crystal and leaves it with the character of a light cone: that is something unheard-of that has no analogy. *Hamilton* announced it by starting from the form of the wave that must be deduced from an abstract theory by means of long calculations. I confess that I would have despaired to see a result that extraordinary confirmed by experiment when it was predicted by just a theory that the genius of *Fresnel* has recently created. However, Mr. *Lloyd* has proved that the experiments are in perfect agreement with *Hamilton's* predictions, so any prejudice against a theory that is so marvelously supported is bound to disappear. At the same time, the general form for light waves that was indicated by *Fresnel* is bound to take on especial importance. It is only in recent times that I could be occupied with it. From the *Report* that was cited above, other than *Hamilton*, only one other Irish geometer is likewise occupied with it; I have not been able to procure the work of either of these geometers. I have cited everything that I am aware of.

Bonn, April and May 1838.
