A NEW

GEOMETRY OF SPACE

BASED UPON THE CONSIDERATION OF

THE STRAIGHT LINE AS SPACE ELEMENT.

BY

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WITH A FOREWORD BY A. CLEBSCH.

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PART ONE.

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FOREWORD.

For quite some time, it was Plücker's goal to unite all of his research on the line structures that he introduced into geometry into one large work that could be published. Some previous papers (*) have been only partially reproduced in it, but for the most part, new and unpublished ideas were included in it. He was not granted his wish of seeing his objective fulfilled completely, but the greater part of his intended work had been published completely and checked by him personally by the time of his death. The esteemed publisher would not like to see the scientific public deprived of investigations of such profundity for any longer than is essential, and thus, whereas the continuation of the work should be accelerated as much as possible, here, only those parts whose publication were finalized under Plücker's own supervision will appear. It contains, along with the development of the general preliminary concepts, the theory of linear complexes, and then the beginnings of a comprehensive theory of second-degree complexes, which Plücker treated here for the first time (**). In the latter, he especially concerned himself with a class of remarkable surfaces of order 4 and class 4 that he called "complex surfaces," and his methods afforded him essential assistance in his research into their representation in terms of intuitive models (†).

For the continuation of the work, only a small part of the manuscript has been carried out completely, in general; however, it is fortunate that *Klein*, who was, up to now, Plücker's assistant in his physical lectures, which had already contributed to the dissemination of the work in many way, and who wished to make the spirit and methodology of the examinations his own, was put into a position of filling in the gaps in the manuscript in the spirit of Plücker through his verbal communications with the deceased. One may then hope to see that everything is completed in a way that is as close as possible to the way that Plücker himself would have indeed wished and foreseen, if – as has often happened for quite some time – the anticipation of death imposes the apprehension that it would not be possible for him to complete work himself. These continuations will be the subject of the further implementations of the theory of second-order complexes in a way that is analogous to Plücker's presentations on the theory of second-order surfaces. Plücker's methods will thus be preserved as faithfully as possible. It will be left to a younger generation to exploit and shape the rich abundance of thoughts that Plücker has generated in this, as in all of his geometric investigations, and in the sense of newer methods.

Thus, the scientific public will turn to the current book as the legacy of a great geometer, who, after his pioneering work in science in his younger years, again turned to

(*) Phil. Trans., (1865), pp. 725, translated in Liouv. Journal, series 2, v. XI; Proceedings of the Royal Soc. (1865); Les Mondes p. Moigno, Janvier, 1867, pp. 79; Annali di matematica, Ser. II, t, 1, pp. 160.

(**) Battaglini made investigations of these complexes as a consequence of Plücker's work on first-degree complex (Atti della Reale Accademia di Napoli, vol. III). A series of Plücker's results are included in this paper. Plücker found them by himself independently, moreover; his methods are completely different, and more geometric, than the newer algebraic methods that are employed by the Italian school.

(†) A large number of elegant models of this kind were constructed under Plücker's instruction by the engineer Epken in Bonn.

geometry at the end of his life, and developed new ideas with youthful vitality, as he was still gifted in old age with a new and large range of disciplines, which owed so much to his prior activities.

The publisher's wish, which made this project possible, namely, to give a true expression of his admiration for the deceased through his assistance in the publication, likewise afforded me the welcome opportunity to recognize graciously the usual liberality that the publisher has invested in the printing and endowment of the book.

Giessen, 8 June 1868.

A. Clebsch.

PART ONE

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Introductory considerations.

§ 1.

Coordinates of straight lines in space. Ray and axis.

1. We can regard lines from two different, but generally equivalent, viewpoints:

2. First, we can consider a line to be a geometric locus of points – that is, as being described by points; i.e., as a *ray*. In this way of looking at lines, we can work with point coordinates x, y, z, and, in a well-known way, represent a line by the equations of its projections onto two of three coordinates planes XZ and YZ:

$$\begin{aligned} x &= rz + \rho, \\ y &= sz + \sigma, \end{aligned} \tag{1}$$

from which, the equation for the projection onto the third coordinate plane XY will follow immediately:

$$ry - sx = (r\sigma - s\rho). \tag{2}$$

For the sake of brevity, we can set:

$$r\sigma - s\rho \equiv \eta, \tag{3}$$

and let:

$$r, s, \rho, \sigma, \eta \tag{4}$$

denote the five coordinates of a line that we consider to be a ray. As a consequence of the relation (3) that exists between them, these five coordinates will come down to four constants that are required for the determination of the line.

For a line that goes through a given point (x', y', z'), one has:

$$\begin{aligned} x' &= rz' + \rho, \\ y' &= sz' + \sigma. \end{aligned}$$

From this, one gets:

$$r = \frac{x - x'}{z - z'}, \qquad s = \frac{y - y'}{z - z'},$$

$$\rho = \frac{x'z - xz'}{z - z'}, \qquad s = \frac{yz' - y'z}{z - z'},$$

$$\eta = \frac{xy' - x'y}{z - z'}.$$

Instead of the five coordinates (4) for the line, we then take the following *six*, to which we temporarily give an arbitrary sign:

$$\left.\begin{array}{lll} \pm(x-x'), & \pm(y-y'), & \pm(z-z'), \\ \pm(yz'-y'z), & \pm(x'z-xz'), & \pm(xy'-x'y). \end{array}\right\} \tag{5}$$

Once we divide any five of the six coordinates by the sixth one, we will obtain values that have a definite relationship to the representation of lines, and by means of which, we can construct them. – In this way, the coordinate system will become symmetric with respect to the three coordinate axes. The condition equation:

$$(x-x')\,(yz'-y'z) + (y-y')\,(x'z-xz') + (z-z')\,(xy'-x'y) = 0, \tag{6}$$

exists between the six new equations, which is an identity in relation to x, y, z, x', y', z'.

When we consider x', y', z', as well as x, y, z to be variable, a ray through two points (x, y, z) and (x', y', z'), both of which are assumed to be arbitrary, will be determined. As a result of this arbitrariness, this assumption will reduce the six coordinates upon which the positions of two points depend to four, which will belong to the determination of a line.

3. Second, we consider a line to be enveloped by planes that rotate around it – viz., as an *axis* in which all enveloping lines intersect. In order to represent a line in this second sense by means of equations, we must make use of plane coordinates. If we take the following equation for the three constants that represent a plane in point coordinates:

$$tx + uy + vz + 1 = 0 \tag{7}$$

to be the coordinates of the plane then that will mean that we are employing the reciprocal values, with opposite signs, of the segments that are cut out from the three coordinate axes by the plane. The two equations:

$$\begin{aligned} t &= pv + \pi, \\ u &= qv + \chi, \end{aligned} \tag{8}$$

when taken individually, represent two points in the two coordinate planes XY and YZ. We can say that the system of both equations represents the line that connects the two points: i.e., a *ray*. The equation:

$$pu - qt = (p\chi - q\pi), \tag{9}$$

which derives from equations (8) when we eliminate the variable v, represents those points at which the third coordinate plane XY will be cut by the same line. In a completely analogous way to how we previously regarded r, s, σ, ρ, η as the five coordinates of a ray, when we set, for the sake of brevity:

$$p\chi - q\pi \equiv \omega, \tag{10}$$

we can now take:

$$p,\ q,\ \pi,\ \chi,\ \omega$$

to be the five coordinates of the line that is considered to be an axis.

If we denote the coordinates of a given plane that goes through the axis by t', u', v' then that will give:

$$t' = pv' + \pi,$$
$$u' = qv' + \chi,$$

and that will yield:

$$p = \frac{t-t'}{v-v'}, \qquad q = \frac{u-u'}{v-v'},$$
$$\pi = \frac{t'v-tv'}{v-v'}, \qquad \chi = \frac{uv'-u'v}{v-v'},$$
$$\omega = \frac{tu'-t'u}{v-v'}.$$

Thus, we can also take the following *six* coordinates:

$$\left.\begin{array}{lll} \pm(t-t'), & \pm(u-u'), & \pm(v-v'), \\ \pm(uv'-u'v), & \pm(t'v-tv'), & \pm(tu'-t'u) \end{array}\right\} \tag{12}$$

for the determination of axes, instead of the previous five (11), if we temporarily leave the sign undetermined. Once we divide any five of these six coordinates by the sixth one, we will obtain expressions that can serve for the construction of lines. The following identity regarding t, u, v, t', u', v' exists between the six new coordinates of an axis:

$$(t-t')\,(uv'-u'v) + (u-u')\,(t'v-tv') + (v-v')\,(tu'-t'u) = 0. \tag{13}$$

If we regard t', u', v', as well as t, u, v, as variables then a line – in the sense of an axis – will be determined by any two planes (t, u, v) and (t', u', v') that intersect in it.

4. If the same line is first determined as a *ray* and then as an *axis* then any of the two points (x, y, z) and (x', y', z') by which the ray is determined must lie in each of the planes (t, u, v) and (t', u', v') that serve to determine the axis, or, what means the same thing, each of the two planes must go through each of the two points. We will obtain the following four equations that correspond to them:

$$\left.\begin{array}{r} tx+uy+vz+1=0, \\ t'x+u'y+v'z+1=0, \\ tx'+uy'+vz'+1=0, \\ t'x'+u'y'+v'z'+1=0, \end{array}\right\} \tag{14}$$

which include the condition that the ray that is determined by the six coordinates (5) must coincide with the axis that is determined by the six coordinates (12).

From the first and last pairs of equations in (14), it follows that:

$$(t - t') \, x + (u - u') \, y + (v - v') \, z = 0,$$
$$(t - t') \, x' + (u - u') \, y' + (v - v') \, z' = 0,$$

and from this, when we eliminate $(v - v')$ and $(u - u')$ from them:

$$- (x'z - xz')(t - t') + (yz' - y'z)(u - u') = 0,$$
$$(xy' - x'z)(t - t') + (yz' - y'z)(v - v') = 0.$$

These equations may be solved as proportions, which are summarized in the following expressions:

$$(t - t') : (u - u') : (v - v') = (yz' - y'z) : (x'z - xz') : (xy' - x'y). \tag{15}$$

The second and fourth of equations (14) follows from the first and third ones:

$$(x - x') \, t + (y - y') \, u + (z - z') \, v = 0,$$
$$(x - x') \, t' + (y - y') \, u' + (z - z') \, v' = 0,$$

and from this, when we eliminate $(z - z')$ and $(y - y')$ from them:

$$- (t'v - tv') \, (x - x') + (uv' - u'v) \, (y - y') = 0,$$
$$(tu' - t'v) \, (x - x') + (uv' - u'v) \, (z - z') = 0.$$

These equations may be solved as the following proportions:

$$(x - x') : (y - y') : (z - z') = (uv' - u'v) : (t'v - tv) : (tu' - t'u). \tag{16}$$

If we finally eliminate x from, say, the first two equations in (14) and x' from the last two then that will give:

$$(tu' - t'u) \, y + (t'v - tv') \, z + (t - t') = 0,$$
$$(tu' - t'u) \, y' + (t'v - tv') \, z' + (t - t') = 0,$$

and if we then, in turn, perhaps eliminate $(t'v - tv')$ from these equations then that will give:

$$(tu' - t'u) \, (yz' - y'z) = (t - t') \, (z - z'),$$

from which:

$$(tu' - t'u) : (t - t') = (z - z') : (yz' - y'z). \tag{17}$$

This new proportion links the expressions (15) and (16), and thus leads to the following general summary of equal ratios:

$$(x - x') : (y - y') : (z - z') : (yz' - y'z) : (x'z - xz') : (xy' - x'y)$$

$$= (uv' - u'v) : (t'v - tv) : (tu' - t'u) : (t - t') : (u - u') : (v - v'). \tag{18}$$

We would like to take the signs of the six coordinates, which remain undetermined, in such a way that they appear in the foregoing proportions. This will be necessary for us later when we apply the same coordinates to the determination of forces and rotations (*). When we restrict ourselves to the consideration of forces here, this assumption will mean, in fact, that the six coordinates (5) are the three projections onto the coordinate axes and the three doubled rotational moments that relate to those same forces whose point of application is (x, y, z), and whose intensity equals the distance between the points (x, y, z) and (x', y', z'), and which is directed from the first point to the second one.

5. In the summary (18), the conditions are obtained by which a line (as a ray and an axis) will be represented in the double coordinate determination. If we go back to the original five ray coordinates and the original five axial coordinates then (18) will be converted into:

$$\begin{array}{lllllll} & r: & s: & 1: & -\sigma: & \rho: & ((r\sigma - s\rho) \equiv \eta) \\ = & -\chi: & \pi: & ((p\chi - q\pi) \equiv \omega): & p: & q: & 1. \end{array} \tag{19}$$

We retain the negative signs for σ and χ, since this is required for the symmetry of the coordinate determination that relates to OZ.

6. We can regard the proportions (19) as being derived from the proportions (18) by dividing the first terms in the one by $(z - z')$ and the last terms of the other by $(v - v')$. We can determine the two divisors in a way that is completely arbitrary and independent of each other. We can then, in turn, multiply the first terms of the proportions (19) by an arbitrary quantity h and the last terms by an arbitrary quantity l, and we can take these quantities to be imaginary (confer the following number). The five absolute coordinates will then be, on the one hand:

$$\frac{r}{h}, \frac{s}{h}, -\frac{\sigma}{h}, \frac{\rho}{h}, \frac{\eta}{h}, \tag{20}$$

and, on the other hand:

$$-\frac{\chi}{l}, \frac{\pi}{l}, \frac{\omega}{l}, \frac{p}{l}, \frac{q}{l}. \tag{21}$$

The equations of the three projections of the lines (1) and (2) will then be:

$$\begin{gathered} hx = rz + \rho, \\ hy = sz + \sigma, \\ h(ry - sx) = (r\sigma - s\rho) \equiv \eta. \end{gathered} \tag{22}$$

(*) Cf., "Fundamental views regarding mechanics," Phil. Transactions (1866), pp. 361, 369.

The equations of the three points at which the coordinate planes of the line are cut – viz., (8) and (9) – take on the form:

$$\begin{gathered} lt = pv + \pi, \\ lu = qv + \chi, \\ l\,(pu - qt) = (p\chi - q\pi) \equiv \omega. \end{gathered} \tag{23}$$

7. A real line may be determined by two imaginary points, as well as two real ones. In order to also include this manner of determination, we would like to determine the two points (x, y, z) and (x', y', z') in the following way:

$$\left.\begin{aligned} x &= x^0 + ix_0, & x' &= x^0 - ix_0, \\ y &= y^0 + iy_0, & y' &= y^0 - iy_0, \\ z &= z^0 + iz_0, & z' &= z^0 - iz_0, \end{aligned}\right\} \tag{24}$$

where we let i denote unity or $\sqrt{-1}$, according to whether the two points are real or imaginary, resp.. The six ray coordinates (5), when taken with the correct sign, will then become:

$$\left.\begin{matrix} 2ix_0, & 2iy_0, & 2iz_0, \\ 2i(y_0z^0 - y^0z_0), & 2i(x^0z_0 - x_0z^0), & 2i(x_0y^0 - x^0y_0). \end{matrix}\right\} \tag{25}$$

Since only the quotients of any two of their six coordinates come into consideration in the determination of a line, we can omit the real or imaginary factor $2i$ that appears in all of the foregoing expressions, and then obtain the following expressions for the six ray coordinates:

$$x_0, \quad y_0, \quad z_0, \quad (y_0\,z^0 - y^0\,z_0), \quad (x^0\,z_0 - x_0\,z^0), \quad (x_0\,y^0 - x^0\,y_0). \tag{26}$$

The determination of the line by means of the quantities x^0, y^0, z^0 and x_0, y_0, z_0 is therefore always real. The x^0, y^0, z^0 are the coordinates of the (always real) mean of the two real or imaginary points (x, y, z) and (x', y', z') through which the line goes. The distance from one point to another is $2i\sqrt{x_0^2 + y_0^2 + z_0^2}$, and the cosines of the angles that the line to be determined makes with the coordinate axes OX, OY, OZ behave like $x_0 : y_0 : z_0$, resp.

The considerations of the previous number are carried over immediately to the case in which we regard the line as an axis, instead of a ray, and thus determine it by planes. If we set:

$$\left.\begin{aligned} t &= t^0 + it_0, & t' &= t^0 - it_0, \\ u &= u^0 + iu_0, & u' &= u^0 - iu_0, \\ v &= v^0 + iv_0, & v' &= v^0 - iv_0 \end{aligned}\right\} \tag{27}$$

then we will obtain the following for the new axis coordinates that correspond to the ray coordinates (26):

$$t_0, \quad u_0, \quad v_0, \quad (u_0\, v^0 - u^0\, v_0), \quad (t^0\, v_0 - t_0\, v^0), \quad (t_0\, u^0 - t^0\, u_0). \tag{28}$$

8. If the new coordinate determinations (26) and (28) are to relate to the same line then one must have:

$$\begin{aligned} & x_0 : y_0 : z_0 : (y_0 z^0 - y^0 z_0) : (x^0 z_0 - x_0 z^0) : (x_0 y^0 - x^0 y_0) \\ = & (u_0 v^0 - u^0 v_0) : (t^0 v_0 - t_0 v^0) : (t_0 u^0 - t^0 u_0) : t_0 : u_0 : v_0. \end{aligned} \tag{29}$$

9. In the foregoing, we have determined lines by point-pairs and plane-pairs, and for them, we have taken conjugate imaginary points and planes, which leaves the coordinate determination real. However, we can also bring imaginary lines under consideration by means of their imaginary coordinates, which we will not go into here.

10. We can finally give the six coordinates of a line – whether we consider it to be a ray or an axis – a general form if we determine the points and planes upon which its construction depends, not, as before, by three coordinates, but by four coordinates now, in the well-known way. We would thus like to take the coordinates of the previous two points and planes to be:

$$\begin{matrix} x, y, z, \tau, & x', y', z', \tau' \\ t, u, v, w, & t', u', v', w', \end{matrix}$$

resp., which comes down to exchanging:

$$x, y, z, \qquad x', y', z'$$

with

$$\frac{x}{\tau}, \frac{y}{\tau}, \frac{z}{\tau}, \qquad \frac{x'}{\tau'}, \frac{y'}{\tau'}, \frac{z'}{\tau'},$$

resp., and

$$t, u, v, \qquad t', u', v',$$

with

$$\frac{t}{w}, \frac{u}{w}, \frac{v}{w}, \qquad \frac{t'}{w'}, \frac{u'}{w'}, \frac{v'}{w'},$$

resp., in the previous developments. After this exchange, we will obtain the ray coordinates for the determination of the line:

$$(x\tau' - x'\tau), \quad (y\tau' - y'\tau), \quad (z\tau' - z'\tau), \quad (y\, z' - y'z), \quad (x'\, z - x\, z'), \quad (x\, y' - x'y) \tag{30}$$

and the axial coordinates:

$$(uv' - u'\, v), \quad (t'\, v - t\, v'), \quad (t\, u' - t'\, u), \quad (t\, w' - t'\, w), \quad (u\, w' - u'\, w), \quad (v\, w' - v'\, w), \tag{31}$$

where we have dropped the factor 1 / $\tau\tau'$ from the first determination and the factor 1 / ww' from the second.

For the sake of the geometric construction of the line that we considered to be a spatial element in the foregoing investigations, we must return from its coordinates to the four constants upon which it depends in any case. For this, the new expressions for the coordinates offer a greater number of constants than are freely at our disposal, and herein lies their advantage over the coordinates (5) and (12), besides their greater degree of symmetry.

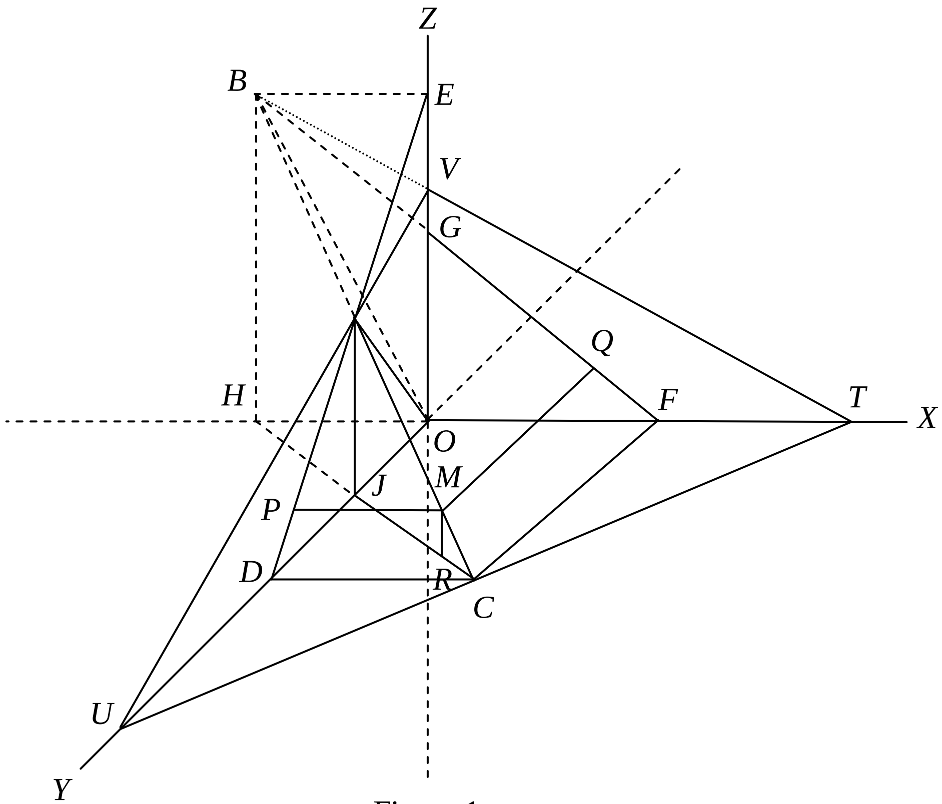

Figure 1.

11. For greater ease of imagination, we would like to clearly summarize everything that relates to the construction of a line in the double coordinate determination.

We would like to take the following equations for the three projections of the line to be determined onto *YZ*, *XZ*, *XY*:

$$hy = sz + \sigma,$$
$$hx = rz + \rho,$$
$$ry - sx = \frac{\eta}{h}.$$

Let them be represented in the accompanying Figure 1 by the lines *DE*, *FG*, *HI*. Let the equations of the three points at which this line cuts the coordinate planes be:

$$lu = qv + \chi,$$

$$lt \quad = rz + \pi,$$
$$pu - qt = \frac{\omega}{l}.$$

The three points that lie on the three projections *DE*, *FG*, *HI* are *A*, *B*, *C*. The coordinates of an arbitrary point *M* that lies in the line are:

$$x = MP, \qquad\qquad y = MQ, \qquad z = MR,$$

and the three coordinates of an arbitrary plane *TUV* that goes through the line are:

$$t = -\frac{1}{OT}, \qquad u = -\frac{1}{OU}, \qquad v = -\frac{1}{OV}.$$

We can determine the coordinates of the three points *A*, *B*, *C* in the double way: on the one hand, by its equations, and on the other hand, by the equations for the three projections *DE*, *FG*, *HI*, when we set the relevant point coordinates in them equal to zero. In this we way, we will come to:

$$\left.\begin{array}{l}
A\begin{cases} z = IA = OG = +\dfrac{q}{\chi} = -\dfrac{\rho}{r}, \\[2ex] y = GA = OI = -\dfrac{l}{\chi} = +\dfrac{\eta}{hr}, \end{cases} \\[4ex]
B\begin{cases} z = HB = OE = +\dfrac{p}{\pi} = -\dfrac{\sigma}{s}, \\[2ex] x = EB = OH = -\dfrac{l}{\pi} = -\dfrac{\eta}{hs}, \end{cases} \\[4ex]
C\begin{cases} y = FC = OD = -\dfrac{lp}{\omega} = +\dfrac{\sigma}{h}, \\[2ex] x = DC = OF = +\dfrac{lq}{\omega} = +\dfrac{\rho}{h}. \end{cases}
\end{array}\right\} \tag{32}$$

Likewise, we can determine the coordinates of the three projections *DE*, *FG*, *HI*, once, by their equations, and then when we set the relevant line coordinates equal to zero in the equations of the points *A*, *B*, *C* that lie on them, and thus obtain:

$$
\left.\begin{array}{l}
DE\begin{cases} v = -\dfrac{1}{OE} = +\dfrac{s}{\sigma} = -\dfrac{\pi}{p}, \\ u = -\dfrac{1}{OD} = -\dfrac{h}{\sigma} = +\dfrac{\omega}{lp}, \end{cases} \\
FG\begin{cases} v = -\dfrac{1}{OG} = +\dfrac{r}{\rho} = -\dfrac{\chi}{q}, \\ t = -\dfrac{1}{OF} = -\dfrac{h}{\rho} = -\dfrac{\omega}{lq}, \end{cases} \\
HI\begin{cases} u = -\dfrac{1}{OI} = -\dfrac{hr}{\eta} = +\dfrac{\chi}{l}, \\ t = -\dfrac{1}{OH} = +\dfrac{hs}{\eta} = +\dfrac{\pi}{t}. \end{cases}
\end{array}\right\} \tag{33}
$$

From the foregoing summary, we shall derive just the following relations here:

$$
\left.\begin{array}{ll}
+\dfrac{s}{h} = +\dfrac{l\pi}{\omega} = \tan DEZ, & +\dfrac{r}{h} = -\dfrac{l\chi}{\omega} = \tan FGZ, \\
+\dfrac{\eta}{h\rho} = -\dfrac{l}{q} = \tan AOZ, & +\dfrac{\eta}{h\sigma} = -\dfrac{l}{p} = \tan BOZ.
\end{array}\right\} \tag{34}
$$

12. The coordinates of a point and the coordinates of a plane will change when the coordinate axes that mediate their geometric construction change their position and direction. The old coordinates will be linear functions of the new ones, which include as constants those quantities by which the position of the new coordinate system is determined when compared to the old one. The same thing will be true for the coordinates of the line, whether we consider it to be a ray or an axis.

We would like to begin with the ray coordinates, for which, we would like to take the six quantities:

$$x - x', \quad x - x', \quad x - x', \quad yx' - y'z, \quad x'z - xz', \quad xy' - x'y.$$

After a parallel displacement of the coordinate axes, the first three coordinates will remain unchanged. We denote the coordinates of the new origin by x^0, y^0, z^0, and in order to distinguish the new coordinate values, we use bold-face script, which yields:

$$
\left.\begin{array}{l}
(\mathbf{yz}' - \mathbf{y}'\mathbf{z}) = (yz' - y'z) + y^0(z - z') - z^0(y - y'), \\
(\mathbf{x}'\mathbf{z} - \mathbf{xz}') = (x'z - xz') - x^0(z - z') + z^0(x - x'), \\
(\mathbf{xy}' - \mathbf{x}'\mathbf{y}) = (xy' - x'y) + x^0(z - z') - y^0(x - x'),
\end{array}\right\} \tag{35}
$$

and from this:

$$\left.\begin{aligned}(yz' - y'z) &= (\mathbf{yz}' - \mathbf{y}'\mathbf{z}) - y^0(\mathbf{z} - \mathbf{z}') + z^0(\mathbf{y} - \mathbf{y}'),\\ (x'z - xz') &= (\mathbf{x}'\mathbf{z} - \mathbf{xz}') + x^0(\mathbf{z} - \mathbf{z}') - z^0(\mathbf{x} - \mathbf{x}'),\\ (xy' - x'y) &= (\mathbf{xy}' - \mathbf{x}'\mathbf{y}) - x^0(\mathbf{y} - \mathbf{y}') + y^0(\mathbf{x} - \mathbf{x}'),\end{aligned}\right\}\tag{36}$$

where $(x - x')$, $(y - y')$, $(z - z')$ are identical with $(\mathbf{x} - \mathbf{x}')$, $(\mathbf{y} - \mathbf{y}')$, $(\mathbf{z} - \mathbf{z}')$. If we take r, s, σ, ρ, η to be the original coordinates and denote the new coordinates by r', s', σ', ρ', η' then we will obtain:

$$\left.\begin{aligned}&r = r', \qquad s = s',\\ &\sigma = \sigma' + y^0 - z^0 s',\\ &\rho = \rho' + x^0 - z^0 r',\\ &\eta = \eta' - x^0 s' + y^0 r'\end{aligned}\right\}\tag{37}$$

immediately from the last equations.

13. The transition from one coordinate system to another one in which the direction of the coordinate axis is different can be decomposed into three individual steps. For example, in the simplest case, where a rectangular coordinate system XYZ assumes any other attitude $X'Y'Z'$ by rotation around the origin, we would like to first rotate the original coordinate system XYZ around the axis OZ in such a way that, after rotation, the coordinate plane XZ will go through the position of the new axis OZ'. Second, after completing the rotation around OZ, we would like to rotate the coordinate system around the axis OY in its new attitude in such a way that the two axes OZ and OZ' will coincide in the XZ-plane. Third, all that remains is to rotate the system around OZ' in such a way that both axes OX and OY, which were brought into the coordinate plane $X'Y'$ by the first two rotations, will coincide with OX' and OY'. The three angles of rotation, from which the attitude of the new axes are determined with respect to the old one, appear as constants in the relevant conversion formulas for the coordinates of the points, plane, and lines. We would like to compute these angles once and for all in the sense that is appropriate to how things happen for rotational moments – i.e., from OX to OY, from OY to OZ, and from OZ to OX.

If OZ preserves its position, while the two axes OX and OY in the XY-plane rotate arbitrarily around OZ, and in their new positions OX' and OY' they define two angles α and α' with OX in the original position, then we will obtain the following relations between the old point coordinates x, y, z and x', y', z' and the new ones, which we would like to denote by $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ and $\mathbf{x}'$, $\mathbf{y}'$, $\mathbf{z}'$:

$$\begin{aligned}x &= \mathbf{x}\cos\alpha + \mathbf{y}\cos\alpha',\\ x' &= \mathbf{x}'\cos\alpha + \mathbf{y}\cos\alpha',\\ y &= \mathbf{x}\sin\alpha + \mathbf{y}\sin\alpha',\\ y' &= \mathbf{x}'\sin\alpha + \mathbf{y}'\sin\alpha',\\ z &= \mathbf{z}, \qquad z' = \mathbf{z}',\end{aligned}$$

and from this:

$$\left.\begin{aligned}(x-x') &= (\mathbf{x}-\mathbf{x}')\cos\alpha+(\mathbf{y}-\mathbf{y}')\cos\alpha',\\ (y-y') &= (\mathbf{x}-\mathbf{x}')\sin\alpha+(\mathbf{y}-\mathbf{y}')\sin\alpha',\\ (z-z') &= (\mathbf{z}-\mathbf{z}'),\\ (yz'-y'z) &= -(\mathbf{x}'\mathbf{z}-\mathbf{x}\mathbf{z}')\sin\alpha+(\mathbf{y}\mathbf{z}'-\mathbf{y}'\mathbf{z})\sin\alpha',\\ (x'z-xz') &= (\mathbf{x}'\mathbf{z}-\mathbf{x}\mathbf{z}')\cos\alpha-(\mathbf{y}\mathbf{z}'-\mathbf{y}'\mathbf{z})\cos\alpha',\\ (xy'-x'y) &= (\mathbf{x}\mathbf{y}'-\mathbf{x}'\mathbf{y})\sin\vartheta,\end{aligned}\right\} \tag{38}$$

if, for the sake of brevity, we set:

$$\alpha-\alpha'\equiv\vartheta.$$

If we take the five coordinates r, s, σ, ρ, η and r', s', σ', ρ', η' in place of the six ray coordinates in the two systems then we will immediately obtain the corresponding equations from the foregoing ones:

$$\left.\begin{aligned}r &= r'\cos\alpha+s'\cos\alpha',\\ s &= r'\sin\alpha+s'\sin\alpha',\\ \sigma &= \sigma'\sin\alpha+\sigma'\sin\alpha',\\ \rho &= \rho'\cos\alpha+\sigma'\cos\alpha',\\ \eta &= \eta'\sin\vartheta.\end{aligned}\right\} \tag{39}$$

In particular, if the new axes OX' and OY' are also perpendicular to each other then that will make:

$$\left.\begin{aligned}r &= r'\cos\alpha-s'\cos\alpha,\\ s &= r'\sin\alpha+s'\sin\alpha,\\ \sigma &= \sigma'\sin\alpha+\sigma'\sin\alpha,\\ \rho &= \rho'\cos\alpha-\sigma'\cos\alpha,\\ \eta &= \eta'.\end{aligned}\right\} \tag{40}$$

If, instead of rotating the two axes OX and OY, we rotate the two axes OX and OZ around O in their plane and let by γ and γ' denote the angles that these axes make in their new positions OX and OY with OZ in the original position then we will obtain the following equations in order to express the six old ray coordinates in terms of the new ones by a mere change of notation in equations (38):

$$\left.\begin{aligned}
(x-x') &= (\mathbf{x}-\mathbf{x}')\sin\gamma' + (\mathbf{z}-\mathbf{z}')\sin\gamma, \\
(y-y') &= (\mathbf{y}-\mathbf{y}'), \\
(z-z') &= (\mathbf{x}-\mathbf{x}')\cos\gamma' + (\mathbf{z}-\mathbf{z}')\cos\gamma, \\
(yz'-y'z) &= (\mathbf{yz}'-\mathbf{y}'\mathbf{z})\cos\gamma - (\mathbf{xy}'-\mathbf{x}'\mathbf{y})\cos\gamma', \\
(x'z-xz') &= (\mathbf{x}'\mathbf{z}-\mathbf{xz}')\sin\vartheta, \\
(xy'-x'y) &= -(\mathbf{yz}'-\mathbf{y}'\mathbf{z})\sin\gamma + (\mathbf{xy}'-\mathbf{x}'\mathbf{y})\sin\gamma',
\end{aligned}\right\} \tag{41}$$

where we have set:

$$\gamma' - \gamma \equiv \vartheta',$$

for the sake of brevity. From this, when we, in turn, go over to the five ray coordinates, we will get:

$$\left.\begin{aligned}
r &= \frac{r'\sin\gamma' + \sin\gamma}{r'\cos\gamma' + \cos\gamma}, \\
s &= \frac{s'}{r'\cos\gamma' + \cos\gamma}, \\
\sigma &= \frac{\sigma'\cos\gamma' + \eta'\cos\gamma'}{r'\cos\gamma' + \cos\gamma}, \\
\rho &= \frac{\rho'\sin\gamma'}{r'\cos\gamma' + \cos\gamma}, \\
\eta &= \frac{\sigma'\sin\gamma' + \eta'\sin\gamma}{r'\cos\gamma' + \cos\gamma},
\end{aligned}\right\} \tag{42}$$

from which, we will further have:

$$\frac{\rho}{s} = \frac{\rho'}{s'}\sin\vartheta'.$$

In particular, if the new coordinate axes *OX* and *OY* are perpendicular to each other then the foregoing equations will be converted into the following ones:

$$\left.\begin{aligned}
r &= \frac{r'\cos\gamma + \sin\gamma}{-r'\sin\gamma + \cos\gamma}, \\
s &= \frac{s'}{-r'\sin\gamma + \cos\gamma}, \\
\sigma &= \frac{\sigma'\cos\gamma - \eta'\cos\gamma}{-r'\cos\gamma + \cos\gamma}, \\
\rho &= \frac{\rho'}{-r'\cos\gamma + \cos\gamma}, \\
\eta &= \frac{\sigma'\sin\gamma + \eta'\sin\gamma}{-r'\sin\gamma + \cos\gamma}, \\
\frac{\rho}{s} &= \frac{\rho'}{s'}.
\end{aligned}\right\} \tag{43}$$

If we rotate the axes *OY* and *OZ* around *OX* then we will obtain the corresponding conversion formulas immediately by a change of notation, not only for the case of six, but also for that of five coordinates, when we start with formulas (42), as far as the latter is concerned. Thus, it would seem unnecessary to write down the new formulas. Meanwhile, it must be remarked that in this exchange the rotation of *OZ* to *OY* will thus be directed in the same sense as the angle whose trigonometric tangent was denoted by *s* in the basic equations (1). Should this rotation be taken in the sense established above – i.e., in the sense of the rotational moment about *OX* – then that would likewise yield the reduction to it.

14. We can also go directly from the five ray coordinates in the first system to the five ray coordinates in the second one. Let r, s, ρ, σ, η be the coordinates of a line in the first coordinate system, so:

$$\left.\begin{aligned}
x &= rz + \rho, \\
y &= sz + \sigma, \\
ry - sx &= \eta
\end{aligned}\right\} \tag{44}$$

are the equations of their projections. If r, s, ρ', σ', η' are the coordinates of that line in the second coordinate system then the equations of their three projections in this system will be:

$$\left.\begin{aligned}
\mathbf{x} &= r'\mathbf{z} + \rho', \\
\mathbf{y} &= s'\mathbf{z} + \sigma', \\
r'\mathbf{y} - s'\mathbf{x} &= \eta'.
\end{aligned}\right\} \tag{45}$$

If the new coordinate axes are parallel to the old ones and carry the displacement x^0, y^0, z^0 along *OX*, *OY*, *OZ*, resp., then one will have:

$$\mathbf{x} = x - x^0, \qquad \mathbf{y} = y - y^0, \qquad \mathbf{z} = z - z^0.$$

Hence, the last three equations will be converted into:

$$\begin{aligned} x &= r' z + (\rho' + x^0 - r' z^0), \\ y &= s' z + (\sigma' + y^0 - s' z^0), \\ r' y - s' x &= \eta' + r' y^0 - s' x^0, \end{aligned}$$

and thus these equations will become identical to equations (44), which will yield, as in number **12** (37):

$$\begin{aligned} r &= r', \qquad s = s', \\ \rho &= \rho' + x^0 - r' z^0, \\ \sigma &= \sigma' + y^0 - s' z^0, \\ \eta &= \eta' + r' y^0 - s' x^0. \end{aligned}$$

If, as in number **13**, we rotate the axes *OX* and *OY* around *O* in their plane then when we set:

$$\begin{aligned} z &= \mathbf{z}, \\ x &= \mathbf{x} \cos \alpha + \mathbf{y} \cos \alpha', \\ y &= \mathbf{x} \sin \alpha + \mathbf{y} \sin \alpha' \end{aligned}$$

the first two equations in (44) will go to the following ones:

$$\begin{aligned} \mathbf{x} \cos \alpha + \mathbf{y} \cos \alpha' &= r\, \mathbf{z} + \rho, \\ \mathbf{x} \sin \alpha + \mathbf{y} \sin \alpha' &= s\, \mathbf{z} + \sigma. \end{aligned}$$

Starting from these equations, if we, in turn, set $\alpha' - \alpha = \vartheta$ then that will yield:

$$\begin{aligned} \mathbf{x} &= \frac{r \sin \alpha' - s \cos \alpha'}{\sin \vartheta} \cdot \mathbf{z} + \frac{\rho \sin \alpha' - \sigma \cos \alpha'}{\sin \vartheta}, \\ \mathbf{y} &= -\frac{r \sin \alpha - s \cos \alpha}{\sin \vartheta} \cdot \mathbf{z} - \frac{\rho \sin \alpha - \sigma \cos \alpha}{\sin \vartheta}, \end{aligned}$$

which, when we make them identical to the first two of equations (45), will give the following relations:

$$\begin{aligned} r' \sin \vartheta &= r \sin \alpha' - s \cos \alpha', \\ -s' \sin \vartheta &= r \sin \alpha' - s \cos \alpha', \\ \rho' \sin \vartheta &= \rho \sin \alpha' - \sigma \cos \alpha', \\ -\sigma' \sin \vartheta &= \rho \sin \alpha' - \sigma \cos \alpha', \end{aligned}$$

and it will then follow, in agreement with the equations (39), that:

$$\begin{aligned} r &= r' \sin \alpha + s' \cos \alpha', \\ s &= r' \sin \alpha + s' \sin \alpha', \\ \rho &= \rho' \cos \alpha' + \sigma' \cos \alpha', \\ \sigma &= \rho' \sin \alpha' + \sigma' \cos \alpha', \end{aligned}$$

and:

$$\eta = \eta' \sin \vartheta.$$

Formulas (42) may be derived in the same way.

15. As a consequence of the proportions (19), we can likewise derive the conversion formulas for the axial coordinates of a given line from the formulas that were developed for the conversion of ray coordinates for it. If we denote the axial coordinates in the original system by:

$$p, q, \pi, \chi, \omega,$$

and in the new system by:

$$p', q', \pi', \chi', \omega',$$

then:

$$\begin{aligned}
p &= -\frac{\sigma}{\eta}, & p' &= -\frac{\sigma'}{\eta'}, \\
q &= \frac{\rho}{\eta}, & q' &= \frac{\rho'}{\eta'}, \\
\pi &= \frac{s}{\eta}, & \pi' &= \frac{s'}{\eta'}, \\
\chi &= -\frac{r}{\eta}, & \chi' &= -\frac{r'}{\eta'}, \\
\omega &= \frac{1}{\eta}, & \omega' &= \frac{1}{\eta'}.
\end{aligned}$$

If we then preserve the direction of the coordinate axes and put the origin at any point (x^0, y^0, z^0) then equations (37) will give:

$$\begin{aligned}
p &= \frac{p' - y^0\omega' + z^0\pi'}{1 - x^0\pi' - y^0\chi'}, \\
q &= \frac{q' + x^0\omega' + z^0\chi'}{1 - x^0\pi' - y^0\chi'}, \\
\pi &= \frac{\pi'}{1 - x^0\pi' - y^0\chi'}, \\
\chi &= \frac{\chi'}{1 - x^0\pi' - y^0\chi'}, \\
\omega &= \frac{\omega'}{1 - x^0\pi' - y^0\chi'}.
\end{aligned} \tag{46}$$

If the axes OX and OY are rotated in their plane in such a way that in their new position they define the angles α and α', resp., with OX in the original position then equations (39) will give:

$$\begin{aligned} p &= \frac{p'\sin\alpha' - q'\sin\alpha}{\sin\vartheta}, \\ q &= \frac{q'\cos\alpha' - p'\cos\alpha}{\sin\vartheta}, \\ \pi &= \frac{\pi'\sin\alpha' - \chi'\sin\alpha}{\sin\vartheta}, \\ \chi &= \frac{\chi'\cos\alpha' - \pi'\cos\alpha}{\sin\vartheta}, \\ \omega &= \frac{\omega'}{\sin\vartheta}. \end{aligned} \tag{47}$$

If we finally rotate OX and OZ around O in their plane then, if we preserve the previous notations, equations (42) will give:

$$\begin{aligned} p &= \frac{p'\cos\gamma - \cos\gamma'}{-p'\sin\gamma + \sin\gamma'}, \\ q &= \frac{q'\sin\vartheta'}{-p'\sin\gamma + \sin\gamma'}, \\ \pi &= \frac{\pi'}{-p'\sin\gamma + \sin\gamma'}, \\ \chi &= \frac{\chi'\sin\gamma' - \omega\sin\gamma}{-p'\sin\gamma + \sin\gamma'}, \\ \omega &= \frac{\omega'}{-p'\sin\gamma + \sin\gamma'}. \end{aligned}$$

§ 2.

On complexes and congruences in general.

16. If:

$$\begin{aligned} & (x - x') : (y - y') : (y - y') : (yz' - y'z) : (x'z - xy') : (xy' - x'y) \\ = \; & (uv - u'v) : (t'v - tv') : (tu' - t'u) : (t - t') : (u - u') : (v - v') \end{aligned}$$

then the ray coordinates:

$$(x - x') : (y - y') : (y - y') : (yz' - y'z) : (x'z - xy') : (xy' - x'y)$$

and the axial coordinates:

$$(uv - u'v) : (t'v - tv') \;\; : (tu' - t'u) : (t - t') \qquad : (u - u') \qquad : (v - v')$$

will belong to the same line. As a result, the same lines will also satisfy the following two equations in their ray and axial coordinates:

$$F[(x - x') : (y - y') : (y - y') : (yz' - y'z) : (x'z - xy') : (xy' - x'y)] \equiv \Omega_n = 0, \tag{1}$$
$$F[(uv - u'v) : (t'v - tv') : (tu' - t'u) \; : (t - t') \; : (u - u') \; : (v - v')] \equiv \Phi_n = 0, \tag{2}$$

if F denotes the same homogeneous function of the current six coordinates. We say that the totality of all lines whose coordinates satisfy such homogeneous equations defines a *complex*. We distinguish complexes by their *degrees n*, which we take to be the degrees of their equations. Any line of the complex can be regarded as a ray or axis; thus, the second type will necessitate that a line complex be represented by equations of the same degrees:

$$\Omega_n = 0, \qquad \Phi_n = 0,$$

which follow from each other immediately in a reciprocal way.

17. In equation (1), which might be homogeneous of degree n, in general, the lines of the complex are determined by any two of their points (x, y, z) and (x', y', z'). If we consider one of these points (x', y', z') to be given then equation (1) – when we regard x', y', z' as constants, but x, y, z as variable, as before – will henceforth represent only such lines that go through the given point and will thus define an n^{th}-order conic surface that has its vertex at the this point.

18. Equation (2), which we, in turn, would like to take to be the general homogeneous equation of degree n, will determine the lines of those complexes by way of any two planes (t, u, v) and (t', u', v') that intersect in them. If we consider one of these planes (t', u', v') to be given then equation (2), which represented the complex up to now, will henceforth represent – when we consider t', u', v' to be constant, but t, u, v to be variable now – only the lines of the complex that lie inside of the given plane, and thus envelop a curve of class n in it.

19. In the previous two numbers, we have proved the following theorem:

For a complex of degree n, the lines that go through a given point of space define a conic surface of order n.

For a complex of degree n, the lines that lie in a given plane that is drawn through space envelop a curve of class n.

These two theorems each include the general geometric definition of a line complex of degree n. Either of the two theorems is a necessary consequence of the other one.

We can thus group the lines of a complex together in a double way: Once, in such a way that they define conic surfaces and each point of space is the vertex of such a conic surface, and then, in such a way that they envelop curves and each plane through space includes such a curve. The degree of the complex is the order of the conic surface, as well as the class of a plane curve. Therefore, a line complex of degree n will also be regarded as a complex of n^{th}-order conic surfaces and as a complex of plane curves of class n.

20. The lines of two given complexes that coincide define a *congruence*. Their coordinates simultaneously satisfy the equations of both complexes, which we, by the application of five ray coordinates, would like to represent by the general equations:

$$\Omega_m = 0, \qquad \Omega_n = 0, \tag{3}$$

and by the application of five axis coordinates, in the form:

$$\Phi_m = 0, \qquad \Phi_n = 0, \tag{3}$$

where m and n denote the degree of the two complexes.

mn lines of a congruence go through each point of space, which are the lines of intersection of two cones of order m and n, resp. mn lines of the congruence lie in each plane drawn through space, which are the common tangents to two curves of class m and n, resp.

The lines of a congruence belong to infinitely many complexes, which, when we denote an undetermined coefficient by μ, will all be represented by either the equation:

$$\Omega_m + \mu \cdot \Omega_n = 0 \tag{5}$$

or by the equation:

$$\Phi_m + \mu \cdot \Phi_n = 0. \tag{6}$$

We say that all such complexes define a two-parameter group of complexes. Each of the latter equations that represent such a group is the symbol of a congruence and, in a certain sense, the equation itself.

21. Congruences are classified by the number of their lines that go through a given point, or which lie in a given plane. This number is, in the foregoing:

$$mn \equiv k.$$

All complexes that belong to a given congruence are, in general, of equal degree. However, when the degrees of these complexes do not conform to the general case, one can find one of them whose degree is lower. This will take place in the case of equations (5) and (6), in which, when $m > n$ the degree of the complex will be m, in general, but for special case in which m becomes infinitely large, it will reduce to n.

The congruences in which the number of lines that go through a given point or lie in a given plane – which amounts to k – define as many types of coordinates as the number of ways that the number k can be decomposed into factors m and n; thus, when k is a prime number there will be only one of them. Therefore, we denote the type of a congruence by the symbol:

$$[m, n]. \tag{7}$$

22. The ray or axial coordinates of those lines that simultaneously belong to three complexes will simultaneously satisfy the corresponding equations for the three complexes, which we would like to represent by either:

$$\Omega_m = 0, \qquad \Omega_n = 0, \qquad \Omega_g = 0, \tag{8}$$

or by:

$$\Phi_m = 0, \qquad \Phi_n = 0, \qquad \Phi_g = 0. \tag{9}$$

They will thus be subject to three conditions. Since a line is determined by its five coordinates, it will follow that each of these coordinates is a function of the other three, or – what amounts to the same thing – each of the coordinates is a function of a variable that is assumed to arbitrary. Having later developments in mind, we take that variable to be time, such that the foregoing can be expressed by saying that the line in question will generate a surface when we let time vary continuously. We would like to call such a surface that is generated by the motion of a line – ignoring the trivial case of skew surfaces – a *ray surface* or an *axial surface*, and when we consider these expressions to be synonymous such a surface will also refer to a ruled surface.

The coincident lines of three complexes define a ray or axial surface.

A ray or axial surface simultaneously belongs to all complexes that are represented by each of the two equations:

$$\Omega_m + \mu\,\Omega_n + \mu'\Omega_g = 0, \tag{10}$$
$$\Phi_m + \mu\,\Phi_n + \mu'\Phi_g = 0, \tag{10}$$

when μ and μ' mean undetermined coefficients; it belongs to each congruence that is determined by any two of these complexes. We say that all of the complexes that belong to a given ray surface define a three-parameter group of complexes that is represented by the foregoing two equations.

If we consider the Φ_m, Φ_n, Φ_g to be functions of the five ray coordinates r, s, ρ, σ, η then we will obtain the equation of the ray surface in point coordinates x, y, z when eliminate the five ray coordinates from the three equations (8) and the following three equations:

$$\eta = r\sigma - s\rho,$$
$$x = r\,z + \rho,$$
$$y = s\,z - \sigma.$$

The resultant equation in x, y, z is of degree $2mng$, in general.

If we consider the Φ_m, Φ_n, Φ_g to be functions of the five axis coordinates p, q, π, χ, ω then we will obtain the equation of the axial surface in plane coordinates t, u, v when we eliminate the five ray coordinates from the three equations (9) and the following three equations:

$$\omega = p\chi - q\pi,$$
$$t = p\, z + \pi,$$
$$u = q\, v + \chi.$$

The resulting equation will be of degree $2mng$, in general.

A ray or axial surface is of equal order and class, in general.

Ray surfaces of a given order and class may be arranged into different coordinate types. These types are obtained from the degree of the complex that determines the surface. If we denote the order and class of the surface by 2λ then the number of such types will be equal to the number of possible decompositions of λ into three factors. If we take m, n, g to be any such functions then we can denote the type of the surface more precisely by the symbol:

$$[m, n, g].$$

23. Four complexes have only a finite number of lines in common. If the degree of the four complexes is m, n, g, h then this number will amount to:

$$2mngh,$$

which will follow immediately when we determine the five coordinate values from the four equations of the complex and either the equation:

$$\eta = r\sigma - s\rho$$

or:

$$\omega = p\chi - q\pi,$$

resp.

24. Plane curves are determined by either their points or their tangents. Two such curves have a certain number of intersection points and common tangents. If we go from the two dimensions of the plane to the three dimensions of space then we will elevate ourselves from plane curves to surfaces, which are determined by either their points or their tangential planes. Two surfaces intersect in a spatial curve and will be enveloped by a developable surface; three surfaces have a certain number of intersection points and common tangential planes. From surfaces, we ascend to complexes that consist of lines, which we can, on the one hand, consider to be rays, and, on the other hand, as axes. The lines that agree in two complexes – in which the two complexes intersect in some fashion – define a congruence, and those that belong to three complexes simultaneously define a ray or axial surface. Four complexes likewise correspond to only a certain number of rays or axes.

There is an analysis of two variable quantities that can be represented in a plane and an analysis of three variables that can be pictured in space. The analysis of four variables finds its visual representation when we give these variables the meaning of line coordinates.

25. With this, we have reached the limits of the development in the present volume. However, the path to new generalizations has been initiated. We can add a fifth independent coordinate for the line to the four. Here, we once more encounter coordinate relationships that correspond to the way that we first considered the line to be a ray and then as an axis. If we take the fifth coordinate in the former way of looking at things in four coordinates to be one of the quantities for a given segment that we will either apply arbitrarily or at a given point then we will have thus determined a force. Its five coordinates are its intensity and the four ray coordinates of the line along which it acts. The symmetry and simplicity of the representation require that here, instead of taking the four independent ray coordinates, we also take the five coordinates r, s, ρ, σ, η, between which the relation exists that:

$$\eta = r\,\sigma - s\,\rho,$$

and which we derive from the six coordinates of the line when we divide five of them by the other one. However, we have already occasionally stressed that these six coordinates refer to the projections X, Y, Z of an arbitrary force that acts along the line onto the coordinate axes and the twice the moments L, M, N of this force in relation to the same axis. If the magnitude of the force is given then these quantities, between which, the relation exists:

$$X \cdot L + Y \cdot M + Z \cdot N = 0,$$

can be regarded as the six coordinates of the force. The same coordinates that take on only relative values for rays will take on absolute values for forces. Ray complexes will be represented by homogeneous equations in the six complexes, and force complexes, by general equations.

Just as we can represent a force, when it is considered to be a ray, by a line and by two points that lie on it, so can we represent a rotation (expressed more precisely, the other type of *force* that brings about a rotation) by a line, considered to be an axis, and two planes that go through it. When we then exchange the point coordinates with plane coordinates and correspondingly, ray coordinates with axial coordinates, the six force coordinates:

$$X, Y, Z, L, M, N$$

will go to other expressions:

$$\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}, \mathfrak{L}, \mathfrak{M}, \mathfrak{N},$$

between which, the relation will exist:

$$\mathfrak{X} \cdot \mathfrak{L} + \mathfrak{Y} \cdot \mathfrak{M} + \mathfrak{Z} \cdot \mathfrak{N} = 0.$$

These six expressions determine a rotation and are to be regarded as the six coordinates of this rotation. As a consequence of the latter condition equation, they reduce to the five independent coordinates for it. These coordinates, which possess only relative values for axes, take on absolute values for a rotation. Homogeneous equations between the six coordinates of a rotation represent axis complexes, while non-homogeneous equations between the coordinate represent rotation complexes.

However, whereas rays and axes are identical, in and of themselves, forces and rotations, in turn, are placed next to each other in a way that is analogous to points and planes. The principle of reciprocity finds the same application to forces and rotations as it does to points and planes. However, the transition from the three coordinates of points and planes to the four coordinates of lines is entirely similar to the transition that we make when we go from the five independent coordinates of forces and rotations to the six independent coordinates of *dynames*.

By the term "dyname," I am referring to the cause of an arbitrary motion of a rigid system, or, since the nature of this cause, like the nature of a force itself, eludes our understanding, the motion itself: i.e., not the cause, but the effect. Since both are proportional, in the mathematical representation this will come down to replacing an ideal unit with a concrete one. – Arbitrary forces and rotations, when they act simultaneously, may be reduced to two forces, as well as two rotations, in an infinitude of ways. We can therefore regard a dyname in two ways, as well as determining it in two ways: On the one hand, by two forces and on the other hand, by two rotations, and this corresponds to representing, on the one hand, the coordinates of two forces, and, on the other hand, the coordinates of two rotations, respectively.

However, the six coordinates of a dyname are the same six quantities:

$$X, Y, Z, L, M, N,$$

or:

$$\mathfrak{X}, \mathfrak{Y}, \mathfrak{Y}, \mathfrak{Z}, \mathfrak{L}, \mathfrak{M}, \mathfrak{N}$$

that originally served for the determination of lines for us, when we assigned only relative values to them and between which a condition equation was imposed. They will then serve for the determination of forces and rotations when give them absolute values, under the restricting assumption of the condition equation. When this condition equation is removed, they will become the coordinates of a dyname. For a given dyname, the six coordinates will take on absolute values, and conversely, if we assign arbitrary values to these values, they will determine a dyname in a linear way.

Just as the reciprocity between points and planes is true for a line, so is the reciprocity between forces and rotations true for a dyname. We can represent a line complex by one equation in a two-fold coordinate system, just as we can represent a dyname complex by one equation in a two-fold coordinate system. The properties of both complexes are dual, in an analogous sense.

In the foregoing reasoning on coordinates, an intermediate possibility still remained unconsidered, that involved the case in which the six coordinates in question are not subject to the restricting condition, are assigned only relative values, and correspondingly, we let homogeneous equations enter in place of the general equations that represent dyname complexes. Mechanics, in particular, would then disappear, and, to confine myself to a brief suggestion: Geometric structures would appear that would have the same relationship to dynames that lines do to forces and rotations.

The foregoing considerations find their completion in dynames.

Chapter One

First-degree line complexes and their congruences.

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§ 1.

First-degree line complexes.

26. If we take the general homogeneous equation of the first degree in terms of the six ray coordinates:

$$(x - x'), \quad (y - y'), \quad (z - z'), \quad (yz' - y'z), \quad (x'z - xz'), \quad (xy' - x'y) \tag{1}$$

to be the following one:

$$A(x - x') + B(y - y') + C(z - z') + D(yz' - y'z) + E(x'z - xz') + F(xy' - x'y) = 0, \tag{2}$$

in order to represent a complex of first degree, then we will simultaneously obtain the representation of the same complex in terms of the axial coordinates:

$$(t - t'), \quad (u - u'), \quad (v - v'), \quad (uv' - u'v), \quad (t'v - tv'), \quad (tu' - t'u) \tag{3}$$

in the following equation:

$$D(t - t') + E(u - u') + F(v - v') + (uv' - u'v) + B(t'v - tv') + C(tu' - t'u) = 0. \tag{3}$$

In order to go from one of these equations to the other one, we must only exchange the point coordinates x, y, z, x', y', z' with the plane coordinates t, u, v, t', u', v', resp., and likewise exchange A,B, C with D, E, F, resp.

If we take the five coordinates:

$$r, s, \sigma, \rho, \eta, \tag{5}$$

instead of six coordinates (1) and (3), then, from nos. **2** and **3**, equations (2) and (4) will go to:

$$Ar + Bs + C - D\sigma + E\rho + F\eta = 0 \tag{7}$$

and (*):

(*) We cannot avoid distinguishing one of the three coordinate axes in the analytical representation of the lines. In establishing equations (1) and (2) as the fundamental ones, we have chosen OZ for this axis, in order to make everything that relates to this axis symmetric in the angle between the two planes XZ and YZ.

$$Dp + Eq + F - A\chi + B\pi + C\omega = 0. \tag{8}$$

27. We can develop both equations (3) and (4), which represent the same complex, in the following way:

$$\begin{aligned} &(A + Fy' - Ez')\,x \\ &+ (B - Fx' + Dz')\,y \\ &+ (C + Ex' - Dy')\,z \\ &+ (Ax' + By' + Cz') = 0, \end{aligned} \tag{9}$$

and:

$$\begin{aligned} &(D + Cu' - Bv')\,t \\ &+ (E - Ct' + Av')\,u \\ &+ (F + Bt' - Au')\,z \\ &- (Dt' + Eu' + Fv') = 0, \end{aligned} \tag{10}$$

resp.

If we first let (x', y', z') be a given point, and we then consider x', y', z' to be constant in (9), while x, y, z are variable, then this equation will represent a plane, namely, the geometric locus of arbitrary points of the rays that go through the given point; in other words, the geometric locus of these rays themselves. The equation will be satisfied if we replace the variable quantities with the coordinates of the given point; the respective plane will go through that point. Each point of space will then correspond to a plane that contains all of the lines of the complex that go through this point.

If we next let the t', u', v' in (10) refer to a given plane (t', u', v'), and thus regard them as constant, while t, u, v remain variable, then this equation will represent a point in plane coordinates that will envelop the axes of the complex that lie in the given plane; that is, the point at which these axes intersect. Thus, in every plane there are infinitely many lines of the complex that are united into a point of that plane, which we will describe by saying that it corresponds to the plane.

At each point of space, there are infinitely many lines of the complex that lie in a plane that goes through that point.

In each plane that goes through space, there are infinitely many lines of the complex that intersect at a point of the plane.

The two parts of the theorem imply each other. The relationship between points and planes is a reciprocal one. For any arbitrary point of space, there is a plane that includes

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This is in contradiction to the manner in which the rotational moment with respect to the three coordinate axes is defined in mechanics.

For the sake of later investigations in mechanics, we decide to establish that the three doubled moments are represented by the last three coordinates (1), with their signs. The desired symmetry in relation to OZ will then achieved. However, in order to further present the analytical examination of complexes in the case where we take (7) and (8) to be the general equations, we must consider the positive σ and, corresponding to it, the positive χ to be coordinates, although we must introduce the terms that include σ and χ in odd powers with negative signs. Corresponding to them, $D\sigma$ and $A\chi$ will occur in the two equations (7) and (8) with negative signs.

the lines of the complex that go through this point, and conversely, for that plane there is, in turn, a point at which all lines of the complex that lie in this plane intersect.

28. The plane that corresponds to a given point is determined by any two lines of the complex that intersect at that point, and the point that corresponds to a given plane is determined by two lines of the complex that lie in the plane.

Let P and P' be two points, through which the line (PP') passes, and let p and p' be the two planes that correspond to these points, which intersect in a second line (pp'). All lines that go through P or P' and intersect the line (pp') will then belong to the complex. If two lines that go through P and P', respectively, intersect at any point of (pp') then the plane that contains these two lines, will be the plane that corresponds to their point of intersection on (pp'), and this plane will go through (PP'). One also likewise proves that not only the planes that correspond to the two points P and P', but, in fact, all of the planes that correspond to all of the points of the line (PP'), will intersect in the line (pp'). We call the two lines (PP') and (pp'), whose relationship to each other is reciprocal, two *conjugate polars relative to the complex.*

1. *Any line in space has a conjugate polar.*
2. *Any line in space can be regarded as a ray.*
3. *If that ray is described by a point then the planes that correspond to this point will envelop an axis that is conjugate to the ray.*
4. *Any line in space can be regarded as an axis.*
5. *If a line in space is enveloped by the planes that rotate around it then the point that corresponds to this plane will describe a ray that is conjugate to the axis.*
6. *Any two conjugate lines can be regarded as a ray and an axis.*
7. *Any line that intersects two conjugate polars is a line of the complex.*
8. *Any line of the complex can be regarded as two coincident conjugate lines.*

29. A complex is completely determined by five of its lines. Each of the lines produces a linear equation for the determination of the five independent constants of the general complex equation. Four of the five constants can thus be replaced in such a way that any two associated polars of the complex are given. Namely, since any given line has only one associated line, which is determined in a linear way by four constants, we will then obtain four linear condition equations between the constants of the general equation when any two associated polars of a complex are given. Two given associated polars of a complex are thus equivalent to four of its lines for its determination, such that the complex is completely determined whenever we know one other line of it, in addition to the two associated polars.

This yields a fifth simple construction of a complex when any five of its lines are given: If we select any four of these five lines then the two lines that intersect these four lines will be two associated polars of the complex, and any new line that intersects these two associated polars is a new line of the complex. We can proceed in this manner, by appending the lines thus found in order to define new combinations of any four of them.

Thus, we must not overlook the fact that four real lines will not always be intersected by two real lines, since these two lines can also be imaginary (*).

If a complex is given by five of its lines then for each given point we can construct the corresponding plane, and for each given plane, the corresponding point. A pair of associated polars of the complex is determined by any four given lines. A single line can be drawn through a given point that intersects the polars of each pair. The line that is thus determined lies in the plane that corresponds to the point and is completely determined by two of these lines. A given plane cuts the two polars of each pair in two points. The lines that connect the two intersection points of each pair intersect in the point that corresponds to the plane that is determined by two of these lines.

The foregoing remarks conclude with the general theory of reciprocity. The equation of the complexes (2) and (4) can be regarded as special cases of the general equation in point coordinates x, y, z, x', y', z' and the plane coordinates t, u, v, t', u', v', by which the reciprocity of two systems is expressed, to begin with. If these equations are symmetric with respect to x, y, z and x', y', z', as well as with respect to t, u, v and t', u', v', then the same point will correspond to the same plane in each of the two systems relative to the polar plane to that point and its pole in the other system. This happens in the case of complex equations. However, one must add the condition that the pole of a given plane must lie in that plane itself. By this new condition, it is no longer possible to construct poles and polar planes in the desired way by means of surfaces of order and class two. (**) Whereas, in general, the polar plane of a point is determined by three of its points, and the pole of a plane, by three planes that intersect at it, here, it suffices to know two points and two planes for this determination. If a line that rotates around a fixed point describes a conic surface of order n then the associated polar will envelop a curve of class n in which the fixed point corresponds to the planes that go through this plane. The n lines in which the planes that correspond to the vertex of the cone intersect will likewise be the n tangents that go from the vertex (which reciprocally corresponds to the planes) to

(*) Three of the five given lines can always be regarded as three lines that include the two generators of a hyperboloid. If a fourth line cuts the hyperboloid then one may lay a line of the second generator of the hyperboloid through each of the two intersection points that intersects all four lines. If the two intersection points are imaginary then both lines will be the corresponding ones.

(**) The analytical basis for this lies in the following remark: In the general case, the basic equation for the reciprocity is (we restrict ourselves here to the case of point coordinates and make the equations homogenous by the introduction of τ):

$$(ax' + by' + cz' + d\tau')\,x + (bx' + b_1y' + c_1z' + d_1\tau')\,y + (cx' + c_1y' + c_2z' + d_2\tau')\,z + (dx' + d_1y' + d_2y' + d_3\tau')\,\tau = 0.$$

If we write x', y', z', τ' for x, y, z, τ, resp., in the left-hand side of this equation then it will become a homogeneous function of second degree:

$$\Pi = ax'^2 + 2bx'y' + 2cx'z' + 2dx'\tau' + b_1y'^2 + 2c_1y'z' + 2d_1y'\tau' + c_2z'^2 + 2d_2z'\tau' + d_3\tau'^2.$$

By means of this function, we can write the reciprocity equation in the following way:

$$\frac{d\Pi}{dx'}\cdot x + \frac{d\Pi}{dy'}\cdot y + \frac{d\Pi}{dz'}\cdot z + \frac{d\Pi}{d\tau'}\cdot \tau = 0.$$

the curve in this plane. Namely, since these lines belong to the complex, they will be their own associated polars.

30. We now return to the purely analytical path of investigation.

The ordinary coordinates of points that correspond to the given plane (t', u', v') and are represented by equation (10) in plane coordinates are:

$$\begin{aligned} x &= -\frac{D + Cu' - Bv'}{Dt' + Eu' + Fv'}, \\ y &= -\frac{E - Ct' + Av'}{Dt' + Eu' + Fv'}, \\ z &= -\frac{F + Bt' - Au'}{Dt' + Eu' + Fv'}. \end{aligned} \tag{11}$$

If the given plane is displaced parallel to itself then the points that correspond to it and lie in it will describe a geometric locus. If we distinguish the coordinates of the corresponding points of those parallel planes that go through the coordinate origin by x_0, y_0, z_0 then that will give, when we set t', u', v' equal to ∞:

$$\begin{aligned} x_0 &= -\frac{Cu' - Bv'}{Dt' + Eu' + Fv'}, \\ y_0 &= -\frac{-Ct' + Av'}{Dt' + Eu' + Fv'}, \\ z_0 &= -\frac{Bt' - Au'}{Dt' + Eu' + Fv'}, \end{aligned} \tag{12}$$

and, from this:

$$\begin{aligned} x - x_0 &= -\frac{D}{Dt' + Eu' + Fv'}, \\ y - y_0 &= -\frac{E}{Dt' + Eu' + Fv'}, \\ z - z_0 &= -\frac{F}{Dt' + Eu' + Fv'}. \end{aligned} \tag{13}$$

From this, we infer that:

$$(x - x_0) : (y - y_0) : (z - z_0) = D : E : F, \tag{14}$$

so the geometric locus in question, which is represented by the double equation:

$$\frac{x - x^0}{D} = \frac{y - y^0}{E} = \frac{z - z^0}{F}, \tag{15}$$

is a line. The direction of this line is independent of the direction of the parallel planes. We call it a *diameter of the first-degree line complex*, and say that the parallel planes are associated with the diameter, and conversely, or that the parallel planes are associated with the diameter.

All diameters of a first-degree line complex are parallel to each other. One such diameter goes through any point of space.

31. Among the diameters of the complex, there is just one of them that is perpendicular to the plane that it is associated with. Should the double equation (15) represent this *axis* then one would find that:

$$t', u', v' = D : E : F$$

expresses the notion that it is perpendicular to the parallel plane (t', u', v'), so equations (12) would then give the following values for x_0, y_0, z_0 :

$$x_0 = \frac{BF - CE}{D^2 + E^2 + F^2}, \quad y_0 = \frac{CD - AF}{D^2 + E^2 + F^2}, \quad z_0 = \frac{AE - BD}{D^2 + E^2 + F^2}. \tag{16}$$

In particular, these coordinate values will be equal to zero, and the axis will go through the origin when:

$$A : B : C = D : E : F. \tag{17}$$

We would like to call the planes that are perpendicular to the axis the *principal intersections of the complex.* The principal intersection that goes through the origin has the equation:

$$Dx + Ey + Fz = 0. \tag{18}$$

If F vanishes then the diameters of the complex (among which one also finds the axis itself) will be parallel to the plane XY. When F and C vanish simultaneously, x_0 and y_0 will be equal to zero. The axis of the complex will cut the coordinate axis OZ; z_0 will take on the value above for the point of intersection. The coordinates $(x - x')$, $(y - y')$, $(yz' - y'z)$, $(x'y - xy')$ will be equal to zero on the axis OZ. This axis will then be a line of the complex when F and C vanish, and indeed, one that is cut by the axis itself. The principal intersection:

$$Dx + Ey = 0$$

will go through it.

32. We would like to treat equation (9), which represents the planes that contain all of the lines of the complex that go through a given point (x', y', z') – in other words, the ones that correspond to this point – in the same manner. If we call the coordinates of this plane t, u, v then that will make:

$$t = -\frac{A + Fy' - Ez'}{Ax' + By' + Cz'},$$
$$u = -\frac{B - Fx' + Dz'}{Ax' + By' + Cz'}, \tag{19}$$
$$v = -\frac{C + Ex' - Dy'}{Ax' + By' + Cz'}.$$

If we assume that the point (x', y', z') moves along a fixed line that goes through the origin then the ratio of the coordinates of the point $x' : y' : z'$ will remain constant. The point at infinity on a fixed line will corresponds to a certain plan, which we denote by:

$$t_0 = -\frac{Fy' - Ez'}{Ax' + By' + Cz'},$$
$$u_0 = -\frac{-Fx' + Dz'}{Ax' + By' + Cz'}, \tag{20}$$
$$v_0 = -\frac{Ex' - Dy'}{Ax' + By' + Cz'}.$$

It will follow from this that:

$$t - t_0 = -\frac{A}{Ax' + By' + Cz'},$$
$$u - u_0 = -\frac{B}{Ax' + By' + Cz'}, \tag{21}$$
$$v - v_0 = -\frac{C}{Ax' + By' + Cz'},$$

so:

$$(t - t_0) : (u - u_0) : (v - v_0) = A : B : C.$$

If we regard t, u, v as variables then the double equation:

$$\frac{t - t_0}{A} = \frac{u - u_0}{B} = \frac{v - v_0}{C} \tag{22}$$

will represent a line that is enveloped by the planes that correspond to the points of a fixed line that goes through the origin. Since the origin of the coordinates, by the arbitrary assumption itself, has no special relationship to the complex, the general theorem on conjugate polars in the foregoing will be proved (no. **28**).

Equation (19) shows that if the point (x', y', z') lies in the plane that is represented by the equation:

$$Ax + By + Cz = 0 \tag{23}$$

then the coordinates t', u', v' of the corresponding plane will become infinitely large, so the plane itself will go through the origin. From this, it will follow that the plane (23) is the one that corresponds to the origin and that, as a consequence, it will be the geometric locus of all lines that are conjugate to the ones that go through the origin. Lines that lie in the plane and likewise go through the origin will be their own conjugates and will thus belong to the complex.

If we consider, among those lines that go through the origin, the diameter of the complex that goes through this point, in particular, then, as a result of the double equation (15) for each point (x', y', z') of it, one will have:

$$x' : y' : z' = D : E : F. \tag{24}$$

It will then follow from equations (20) that:

$$t_0 = 0, \qquad u_0 = 0, \qquad v_0 = 0,$$

and the double equation (21), which reduces to:

$$\frac{t}{A} = \frac{u}{B} = \frac{v}{C}$$

for them, will give the polar that is conjugate to the diameter as a line that lies in the plane at infinity (23).

By the arbitrary nature of the origin of the coordinate system, we have thus expressed the idea that a line that is associated with an arbitrary diameter of the complex will lie at infinity in any plane that corresponds to a point of it. However, a line that lies at infinity in a given plane will admit no closer approximation, since it has lost its direction, and will remain the same when the plane that contains it is displaced parallel to itself. A line at infinity will always be parallel to a given line and will assume all possible positions at infinity when the plane rotates around one of its points. In each such position, it will correspond to a diameter of the complex. All lines infinity in space define a plane at infinity whose corresponding point is itself at infinity in the given direction, because it lies in that plane. A consequence of this is that the diameters that converge to this point will be parallel to each other.

Distinguished among the lines that go through the origin are finitely many of them that are perpendicular to the plane:

$$Ax + By + Cz = 0 \tag{25}$$

that corresponds to the origin, and thus, they will be perpendicular to any line in this plane; i.e., to any line that is associated with one that goes through the origin, and, in particular, to the ones that are associated with themselves. The line in question is thus characterized by the fact that for each of its points:

$$x' : y' : z' = A : B : C, \tag{26}$$

from which, t_0, u_0, v_0 will take on the following values:

$$t_0 = -\frac{BF - CE}{A^2 + B^2 + C^2},\quad u_0 = -\frac{CD - AF}{A^2 + B^2 + C^2},\quad v_0 = -\frac{AE - BD}{A^2 + B^2 + C^2}. \tag{27}$$

If we substitute these values in the double equation (21) then in the plane (23) this equation will represent the lines that are associated with the line (26) that goes through the origin.

If the axis of the complex goes through the origin of the coordinates axes then it will be the one that is perpendicular to its associated line. However, from (15), one will then have:

$$x' : y' : z' = D : E : F,$$

from which:

$$A : B : C = D : E : F,$$

in agreement with (17).

33. One finds from equations (10) that:

$$\begin{aligned} Cu - Bv + D &= 0, \\ -Ct + Av + E &= 0, \\ Bt - Au + F &= 0 \end{aligned} \tag{28}$$

are the equations for the three points that correspond to the coordinate planes *YZ, XZ, XY*, while:

$$Dt + Eu + Fv = 0 \tag{29}$$

represents those points that correspond to the plane at infinity and are themselves at infinity in the given direction.

One finds from equations (9):

$$\begin{aligned} Fy - Ez + A &= 0, \\ -Fx + Dz + B &= 0, \\ Ex - Dy + C &= 0, \end{aligned} \tag{30}$$

are the equations for the three planes that correspond to the points at infinity in the directions of the three coordinate axes *OX, OY, OZ*, while, as we already pointed out (23):

$$Ax + By + Cz = 0$$

represents the plane that corresponds to the origin.

34. From the three equations (9) and the three equations (10), we obtain correspondingly, the condition equation:

$$(Ax + By + Cz)\,(Dt + Eu + Fv) + (AD + BE + CF) = 0 \tag{31}$$

if (x, y, z) is a point and (t, u, v) is a plane that mutually correspond to each other relative to the complex.

The foregoing equation includes, as a special case:

$$AD + BE + CF = 0. \tag{32}$$

This special case corresponds to a particularization of the first-degree complex.

35. The two general equations:

$$\begin{gathered} A(x - x') + B(y - y') + C(z - z') + D(yz' - y'z) + E(x'z - xz') + F(xy' - x'y) = 0, \\ D(t - t') + E(u - u') + F(v - v') + A(uv' - u'v) + B(t'v - tv') + C(tu' - t'u) = 0, \end{gathered}$$

which represent the first-degree complex in the doubled coordinate determination, will simplify when we let one of the rectangular coordinate axes coincide with one of the axes of the complex, from which, the other two will lie in a principal intersection of it. If we choose the coordinate axis that coincides with the axis of the complex to be OZ, OY, OX, in sequence, then, by the vanishing of:

$$\begin{gathered} A, B, \text{ and } D, E, \\ A, C \text{ and } D, F, \\ B, C \text{ and } E, F, \end{gathered}$$

respectively, the foregoing two equations will assume the following forms:

$$\begin{aligned} &(xy' - x'y) + k(z - z') = 0, \\ &(x'z - xz') + k(y - y') = 0, \qquad (33) \\ &(xy' - x'y) + k(x - x') = 0, \end{aligned} \qquad \begin{aligned} &(v - v') + k(tu' - t'u) = 0, \\ &(u - u') + k(t'v - tv') = 0, \qquad (34) \\ &(t - t') + k(uv' - u'v) = 0. \end{aligned}$$

In this form, they include only one constant (k), and it is the same in all equations. This is obvious from the outset. This value does not change when we go from one of the two equations in the same row to the other one. This follows from the double determination of the line by means of point and plane coordinates, from which, for example:

$$\frac{xy' - x'y}{z - z'} = \frac{v - v'}{tu' - t'u}.$$

However, the value of the constant k will also remain unchanged under the transition from one of the three equations of the complex to one of the other ones. The expressions:

$$\frac{xy' - x'y}{z - z'}, \quad \frac{x'z - xz'}{y - y'}, \quad \frac{yz' - y'z}{x - x'},$$

for example, have an absolute geometric meaning when they are referred to an arbitrary line of the complex, which is mediated by the current choice of coordinate system, but independent of it. Under the transition from the one coordinate system to another, the foregoing three expressions will go into each other by means of the corresponding coordinate permutation; however, their geometric meaning, which must always be the same, will not change, and, as a result, k will not change, either.

We would like to call the quantity k, which represents the length of a line, the *parameter* of the complex. The complex is completely determined by its parameter, when we neglect its position in space.

36. The general equation of a first-degree line complex includes five mutually independent constants in each of the two coordinate determinations. Equations (33) and (34) involve just one constant. The number of constants has thus been reduced by four. However, since we have six constants at our disposal for the determination of a new coordinate system, the coordinate system that is at the basis of the latter equations will be determined only incompletely. We have two constants available for position, without which these equations could change in any manner. We will confirm this in the following number.

37. The first of the three equations (33):

$$(xy' - x'y) + k(z - z') = 0,$$

which we can choose arbitrarily, does not change when the origin of the coordinates moves arbitrarily along OZ, which is the axis of the complex. The same equation will also remain unchanged when the coordinate system rotates arbitrarily around OZ. Then, on the one hand, z and z' will remain unchanged, and, on the other hand, $xy' - x'y$ will also preserve its value. This expression, in fact, represents the projection onto XY of twice the area of the triangle whose three vertices are the origin of the coordinate system and the two points (x, y, z) and (x', y', z'), through which the lines of the complex are determined, and this projection will not change when the complex rotates around its axis OZ. Thus, equations (33), and as a result, equations (34) themselves, will remain unchanged when the origin of the axis of the complex moves along the axis of the complex and the coordinate system rotates around this axis. In other words:

A first-degree line complex remain unchanged whenever it is displaced parallel to its axis and when it is rotated around it.

All of the lines of the complex in the original position go to other lines of it after the translation and rotation.

38. We can transform the general complex equations (2), (4) into the six equations (33), (34) step-wise by changing the coordinate system. Since the single constant that enters into these equations has the same value (viz., k), in this transformation one will be dealing with only the determination of k, and it will suffice to carry out the transformation in a single case. If, for the sake of brevity, we use the equations:

$$Ar + Bs + C - D\sigma + E\sigma + F\eta = 0, \tag{7}$$
$$Dp + Eq + F - A\chi + B\pi + C\omega = 0 \tag{8}$$

for our basis, instead of equations (2), (4), then equations (33) and (34) will assume the following form:

$$\begin{aligned} \eta + k &= 0, \\ \rho + ks &= 0, \\ -\sigma + kr &= 0, \end{aligned} \tag{35}$$
$$\begin{aligned} \tau + \frac{1}{\chi} &= 0, \\ \pi + \frac{q}{\chi} &= 0, \\ -\chi + \frac{p}{\chi} &= 0. \end{aligned} \tag{36}$$

We shall confine ourselves to deriving the first of equations (35) from equation (7).

If we displace the original coordinate system, to which equation (7) is referred, parallel to itself, and the coordinates of the new origin are x^0, y^0, z^0, then, by an application of the conversion formulas (37) of no. 12, this equation will go to the following one:

$$(A + Fy^0 - Ez^0)\, r' + (B - Fx^0 + Dz^0)\, s' + (C + Ex^0 - Dy^0) - D\sigma' + E\rho' + F\eta' = 0. \tag{37}$$

In particular, when:

$$\frac{x^0}{D} = \frac{y^0}{E} = \frac{z^0}{F},$$

the form of the original equation will do not change under the displacement of the coordinate system. The complex will then remain the same when it is displaced parallel to the direction of those lines that are represented by the last equation when we regard the x^0, y^0, z^0 in them as variable – i.e., parallel to the direction of the diameter [cf., (15)].

We obtain the cosines of the angles that this diametral direction makes with the three coordinate axes OX, OY, OZ in the form of:

$$\frac{D}{\sqrt{D^2+E^2+F^2}}, \qquad \frac{E}{\sqrt{D^2+E^2+F^2}}, \qquad \frac{F}{\sqrt{D^2+E^2+F^2}},$$

resp.

We would like to rotate the original coordinate system around OZ through an angle α in the sense that was established in no. 13. From the conversion formulas (40) of no. 13, the general equation (7) will go to the following one:

$$\begin{aligned}&(A\cos\alpha+B\sin\alpha)r' +(-A\sin\alpha+B\cos\alpha)s' +C\\ &-(D\cos\alpha+E\sin\alpha)\sigma'+(-D\sin\alpha+E\cos\alpha)\rho'+F\eta'=0.\end{aligned} \tag{38}$$

We would like to determine α in such a way that:

$$-D\sin\alpha+E\cos\alpha=0, \tag{39}$$

from which:

$$\cos^2\alpha = \frac{D^2}{D^2+E^2}. \tag{40}$$

We can then write the equation of the complex in the following way, if we omit the prime:

$$A'r + B's + C' - D's + F'\eta = 0, \tag{41}$$

an equation that, since ρ is missing, characterizes the complex in question as one whose diameters are parallel to the XZ plane. Since C' and F' keep their previous values, one will find that:

$$\begin{aligned}A' &= (AD \quad +BE)\frac{\cos\alpha}{D},\\ B' &= (-AE+BE)\frac{\cos\alpha}{D},\\ D' &= (D^2 \quad +E^2)\frac{\cos\alpha}{D},\end{aligned} \tag{42}$$

and from this:

$$\begin{aligned}A'D' &= AD+BE,\\ D'^2 &= D^2+E^2.\end{aligned} \tag{43}$$

After performing the first rotation of the coordinate system, we would like to rotate it through an angle γ around OY, which, as in no. 13, may be measured from OZ to OX. The conversion formulas (43) of no. 13 will then give:

$$\begin{aligned}&(A'\cos\gamma-C'\sin\gamma)r'+B's'+(A'\sin\gamma+C'\cos\gamma)\\ &-(D'\cos\gamma-F'\sin\gamma)\sigma' \qquad +(-D'\sin\gamma+F'\cos\gamma)\eta'=0\end{aligned} \tag{44}$$

for the equation of the complex. In order for the new axis OZ to coincide with the diameter of the complex that runs through the origin, σ' must drop out of equation (44). We correspondingly set:

$$D'\cos\gamma - F'\sin\gamma = 0, \tag{45}$$

from which:

$$\cos^2\gamma = \frac{F'^2}{D'^2+F'^2}. \tag{46}$$

For the sake of brevity, we can then write the complex equation (44) in the following way:

$$A''r + B''s + C'' + F''\eta = 0. \tag{47}$$

Since B'' keeps its previous value, one will have:

$$\begin{aligned} A'' &= (A'F' - C'D')\frac{\cos\gamma}{F'}, \\ C'' &= (A'D' + C'F')\frac{\cos\gamma}{F'}, \\ -F'' &= (D'^2 + F'^2)\frac{\cos\gamma}{F'}, \end{aligned} \tag{48}$$

and from this:

$$\begin{aligned} \frac{C''}{F''} &= \frac{A'D' + C'F'}{D'^2 + F'^2} \\ &= \frac{AD + BE + CF}{D^2 + E^2 + F^2}. \end{aligned} \tag{49}$$

If we finally displace the coordinate axes parallel to themselves, as in no. **12**, then equation (47) will become:

$$(A'' + F''\, y^0)\, r' + (B'' - F''\, x^0)\, s' + C'' + F''\, \eta' = 0, \tag{50}$$

and will reduce, when we take:

$$y^0 = -\frac{A''}{F''}, \qquad x^0 = \frac{B''}{F''}, \tag{51}$$

to:

$$\eta + k = 0, \tag{52}$$

when we, for the sake of brevity, set:

$$k \equiv \frac{AD + BE + CF}{D^2 + E^2 + F^2}. \tag{53}$$

The complex will then have its axis along the OZ coordinate axis, while, as usual, the other two coordinate axes, OX and OY, which are perpendicular to each other and OZ, will intersect OZ at an arbitrary point.

39. We immediately obtain the interpretation for the form of the equations:

$$\eta + k = 0, \qquad (xy' - x'y) + k(z - z') = 0.$$

If we imagine a force of arbitrary intensity that acts in the direction of any line of the complex then we can regard the expression $(xy' - x'y)$ as the double moment of this force relative to the axis of the complex and $(z - z')$ as proportional to the force on this axis. Thus:

If an arbitrary force acts along the lines of a linear complex then the ratio of the projection of this force onto the axis of the complex to the moment of this force relative to the axis will be constant and equal to the parameter of the complex (*).

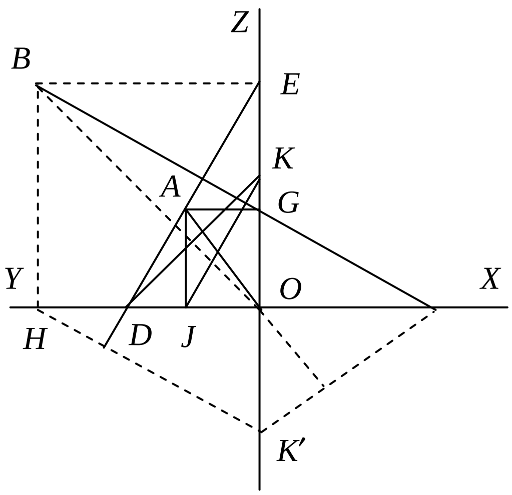

Figure 2.

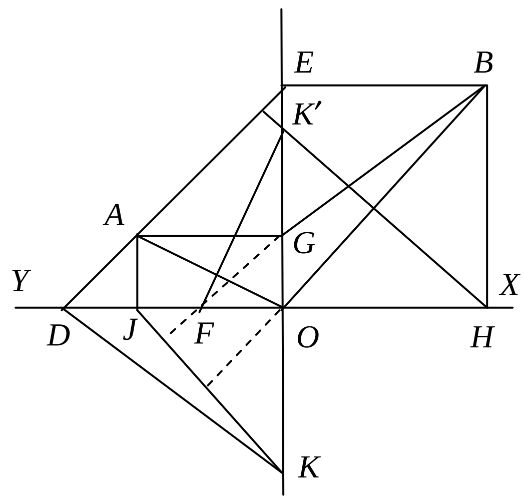

Figure 3.

If we take the points (x, y, z) and (x', y', z'), by which the lines of the complex are determined, to be, in particular, the two points A and B at which the coordinate planes are be cut by them then the values of x and y' will vanish. One will then have:

$$x'y = K(z - z'), \tag{54}$$

so, referring to the figures (Figs. 2, 3) (**), one will have:

$$k = \frac{OH \cdot OJ}{EG}, \tag{55}$$

which is an immediate consequence of the foregoing theorem, which also follows immediately from the determination of the constants for the line (confer number 11). We will get:

(*) This theorem will enter into the later discussion of the mechanical aspects in its natural connection with other things.

(**) For our purposes, there is a certain advantage to the sort of projection under which the three mutually-perpendicular coordinate axes OZ, OY, OX are represented in the same plane in such a way that – say – OZ and OX keep their natural position, but the positive extension of OY coincides with the negative extension of OX.

$$-\,\eta = OJ \cdot \frac{OH}{EG} = -\,OJ \text{ cotan } BFX = OJ \tan OJK = OK = k, \tag{56}$$

$$-\,\eta = OH \cdot \frac{OJ}{EG} = OH \text{ cotan } ADY = -\,OH \tan OHK' = -\,OK' = k \tag{57}$$

for an arbitrary line of the complex. Therefore, JK will be drawn perpendicular to GF and HK' will be perpendicular to ED. In the case of the first figure, k will be positive, and in the case of the second figure, k will be negative.

From the aforementioned number **11**, if we employ the axial coordinates of a line instead of its ray coordinates, and then start with the equations:

$$\omega = \frac{1}{k} = 0, \qquad (v - v') + k(tu' - t'u) = 0$$

then we will have:

$$-\,\frac{1}{\omega} = \frac{OF \cdot OJ}{OG} = OF \tan AOZ = -\,OF \tan OFK' = -\,OK' = k, \tag{58}$$

$$-\,\frac{1}{\omega} = -\,\frac{OD \cdot OH}{OE} = -\,OD \tan BOZ = OD \tan ODK = OK = k. \tag{59}$$

Thus, FK' will be drawn perpendicular to OA and DK will be perpendicular to OB.

The following construction may also be mentioned (Figs. 4, 5):

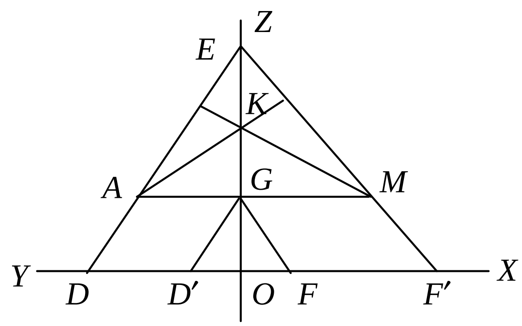

Figure 4.

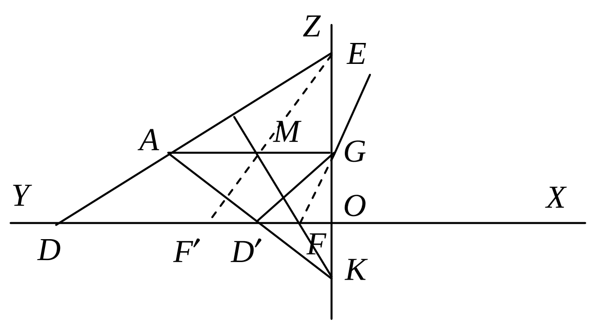

Figure 5.

Through the points DE and FG, which are the projections of a given line of the complex, we can, in a single way, lay a system of two parallel plane: EDF' and GDF', which intersect the coordinate axes OZ, OY, OX in the points E and G, D and D', F and F', respectively. We then get:

$$EG = \frac{\eta}{rs}, \qquad DD' = -\frac{\eta}{r}, \qquad F'F = -\,\frac{\eta}{s},$$

and from this:

$$\eta = \frac{DD' \cdot F'F}{EG}. \tag{60}$$

In order to construct this expression, we draw a plane parallel to *XY* through *G* that cuts *DE* at *A* and *F*′*E* at *M*. We will then have:

$$-\frac{DD'\cdot F'F}{EG} = \frac{AG\cdot GM}{EG} = GK = k, \tag{61}$$

if *K* is the point of the triangle *AME* at which the perpendiculars from the vertices to the opposite sides intersect. In the first of the two figures, *k* is positive, while in the second one, it is negative.

40. Since the equation for the complex contains only one constant when the axis is given and assumed to be a coordinate axis, it will also be true that this complex is determined completely when a line of the complex is likewise given, along with the axis. Just as the constant was determined to be *k* in the latter developments, as long as a line of the complex was given, so can we also conversely construct all lines of the complex when *k* is given. We can subject the line that we would like to determine to three linear conditions from the outset, and thus arrive at a series of problems that I will not discuss further here.

41. We again take the equation of the complex to be:

$$(xy' - x'y) + k(z - z') = 0. \tag{33}$$

If we regard *x*, *y*, *z* as the variables then:

$$y'\cdot x - x'\cdot y + k\cdot z - k\cdot z' = 0 \tag{62}$$

will be the equation of the plane that corresponds to any point (x', y', z'). If we call the angle that this plane makes with the axis of the complex λ then we will have:

$$\sin^2\lambda = \frac{k^2}{y'^2 + x'^2 + K^2}, \tag{63}$$

and, as a result:

$$y'^2 + x'^2 = \frac{k^2}{\tan^2\lambda}. \tag{64}$$

The interpretation of the foregoing equations gives the following geometric relations:

If any point P is given then the same associated plane will go through any line that can be drawn from that point perpendicular to the axis of the complex.

The associated planes of all points that have the same distance from the axis of the complex all define the same angle with this axis.

While a point describes a circle when it rotates around the axis, the associated planes will envelop a cone of rotation that has that point at which the plane of the circle cuts the axis for its vertex.

If the points of the circle describe a diameter when they move parallel to the axis then the cone will be displaced parallel to itself, such that its vertex will always remain on the axis.

Diameters that have the same distance from the axis of the complex define the same angle with their associated planes.

42. If we take a line through a point P that is drawn perpendicular to the axis to be the OX axis then the coordinates z' and y' will vanish. The equation of the associated plane will then be:

$$x'y = kz. \tag{65}$$

For the points of those lines that cut this YZ plane, one has:

$$\frac{y}{z} = \tan\lambda = \frac{k}{x'}, \tag{66}$$

so for the line that is perpendicular to it and goes through the origin, one will have:

$$\frac{y}{z} = -\frac{x'}{k} \qquad \text{or} \qquad \frac{z}{y} = -\frac{k}{x'}. \tag{67}$$

If k is given then we can then likewise determine the plane that is associated with an arbitrary point P, and conversely, if any point and its associated plane are given then we can determine the parameter k of the complex.

Let a perpendicular PK to this axis be erected by the aforementioned manner of projection at the point P, which is assumed to be on OX, and let its length be taken to be equal to k. Any line OL that is drawn through O perpendicular to OK will then be the intersection line of the plane that corresponds to P with the coordinate plane YZ. The plane itself is found with that. Likewise, when the plane LOX and k are given that will immediately yield the point P that is associated with this plane.

If the point P is at a distance from the axis then $\tan\lambda$ will decrease in proportion to the distance.

The foregoing gives an intuitive picture for a complex. All lines that go through the arbitrary point P and cut the line OL belong to the complex, and this will still be true when the point P and the line OL are displaced parallel to the axis, and also when the point rotates around the axis with OL. The circle that the point describes under this rotation will correspond to a cone of rotation whose axis goes through the center of the circle and is perpendicular to its plane, in such a way that any line of the complex that goes through a point of the circle will contact this cone. By the converse of this theorem, one must consider that, to some extent, equations (63), (64) correspond to the same cone of the same circle for two different complexes whose parameters have the same absolute values, but opposite signs. Of the two tangential planes that can be laid through a point of the circle to the cone, if the sign of the parameter k is given then only one of them can be chosen whose intersection line with the YZ-plane defines an angle with the axis of the complex whose trigonometric tangent (66) is equal to $+ k/x'$. Due to the fact that when the circle moves parallel to itself along the axis OZ, thus describing a cylinder of rotation, the cone that corresponds to the circle will move parallel to itself in a manner that is similar to the one that was already discussed in the previous number.

43. The lines in space arrange themselves into pairs with respect to a given complex, such that each line will have its associate and the relationship of any two associated lines to each other will be reciprocal and will be mediated by the complex in a linear way. From the discussion above, we would like to base our analysis, in turn, upon the simplest equation for the complex:

$$(xy' - x'y) + k(z - z') = 0,$$

where the axis of the complex is taken to be the OZ axis.

Let (x', y', z') and (x'', y'', z'') be any two points in space, and let the line that connects them be one of two conjugate polars. The equations of this line are:

$$\begin{aligned}(z' - z'')x &= (x' - x'') : -(x'z'' - x''z'),\\ (z' - z'')y &= (y' - y'') : -(y'z'' - y''z'),\end{aligned} \tag{68}$$

and their five ray coordinates, which we would like to distinguish by r_0, s_0, ρ_0, σ_0, η_0, are:

$$r_0 = \frac{x' - x''}{z' - z''}, \quad s_0 = \frac{y' - y''}{z' - z''},$$

$$\rho_0 = -\frac{x'z'' - x''z'}{z' - z''}, \quad \sigma_0 = -\frac{y'z'' - y''z'}{z' - z''}, \tag{69}$$

$$\eta_0 = \frac{x'y'' - x''y'}{z' - z''}.$$

We can construct the second of the two associated polars as the line of intersection of those two planes that correspond to the two points (x', y', z') and (x'', y'', z''), which lie on the former plane, and which are the following ones:

$$\begin{aligned} y'x - x'y + kz - kz' &= 0,\\ y''x - x''y + kz - kz'' &= 0.\end{aligned} \tag{70}$$

From these two equations, one obtains, by successively eliminating y and x:

$$\begin{aligned}(x'y''-x''y')x+k[-(x'-x'')z+(x'z''-x''z')]&=0,\\(x'y''-x''y')y+k[-(y'-y'')z+(y'z''-y''z')]&=0,\end{aligned} \tag{71}$$

and therefore the first four of the five coordinates of the second line, which we would like to distinguish by r^0, s^0, ρ^0, σ^0, η^0, will be:

$$\begin{aligned}r^0&=k\cdot\frac{x'-x''}{x'y''-x''y'}, &\qquad s^0&=k\cdot\frac{y'-y''}{x'y''-x''y'},\\ \rho^0&=-k\cdot\frac{x'z''-x''z'}{x'y''-x''y'}, &\qquad \sigma^0&=-k\cdot\frac{y'z''-y''z'}{x'y''-x''y'}.\end{aligned} \tag{72}$$

From the combination of the foregoing four equations with equations (69), one obtains a series of relations between the five coordinates of the two conjugate polars:

$$\begin{aligned}\frac{\rho_0}{r_0}&=\frac{\rho^0}{r^0}, &\quad \frac{\sigma_0}{s_0}&=\frac{\sigma^0}{s^0},\\ \frac{r_0}{s_0}&=\frac{r^0}{s^0}, &\quad \frac{\rho_0}{\sigma_0}&=\frac{\rho^0}{\sigma^0},\end{aligned} \tag{73}$$

and furthermore:

$$\begin{aligned}\frac{r_0}{\eta_0}&=\frac{r^0}{k}, &\quad \frac{s_0}{\eta_0}&=\frac{s^0}{k},\\ \frac{\rho_0}{\eta_0}&=\frac{\rho^0}{k}, &\quad \frac{\sigma_0}{\eta_0}&=\frac{\sigma^0}{k},\end{aligned} \tag{74}$$

and from this, when we consider that:

$$\eta^0 = r^0\,\sigma^0 - s^0\,\rho^0,$$

it will follow that:

$$\eta_0\eta^0 = k^2. \tag{75}$$

We can summarize all of the relations in the following equations:

$$\frac{r_0}{r^0}=\frac{s_0}{s^0}=\frac{\rho_0}{\rho^0}=\frac{\sigma_0}{\sigma^0}=\frac{\eta_0}{k}=\frac{k}{\eta^0}. \tag{76}$$

The reciprocal relationship between the two associated lines is expressed in these equations collectively. In order to go from the second of the two conjugate polars to the first one, we obtain:

$$\frac{r^0}{\eta^0} = \frac{r_0}{k}, \qquad \frac{s^0}{\eta^0} = \frac{s_0}{k},$$
$$\frac{\rho^0}{\eta^0} = \frac{\rho_0}{k}, \qquad \frac{\sigma^0}{\eta^0} = \frac{\sigma_0}{k}. \tag{77}$$

If we consider that any two mutually perpendicular planes that go through the OZ axis can be taken to be the XY, YZ coordinate planes without changing the equation of the complex in any way then we can deduce from the first two of equations (73) that any plane that goes through the axis of the complex will be cut by any two associated lines in such a way that the two points of intersection will lie on a straight line that is perpendicular to the axis.

The square of the distance to the point at which one of the two associated lines cuts the plane perpendicular to the axis OZ, which is determined by any value of z, is:

$$(s_0\, z + \sigma_0)^2 + (r_0\, z + \rho_0)^2.$$

The value of z for which this distance is a minimum is:

$$z = -\,\frac{s_0\sigma_0 + r_0\rho_0}{s_0^2 + \rho_0^2}\,. \tag{78}$$

If we draw the XY-plane through the shortest distance (which will not change the equation of the complex) then we will obtain the condition equation:

$$s_0\, \sigma_0 + r_0\, \rho_0 = 0, \tag{79}$$

and the shortest distance itself will be:

$$\sigma_0^2 + \rho_0^2\,.$$

The condition equation (79) brings with it the corresponding expression for the other conjugate polar:

$$s^0\, \sigma^0 + r^0\, \rho^0 = 0. \tag{80}$$

The shortest distances from any two associated polars to the axis of the complex lie in the same plane perpendicular to this axis and coincide in the same line in this plane.

The last part of this theorem follows immediately from the foregoing theorem. The direct proof of it is based in the fact that when we let the OZ-axis coincide with the shortest distance to one of the associated lines, s_0 will vanish, which brings with it the fact that σ^0 will also vanish (74). As a result of equation (79), r_0 will then vanish, so, from (74), r^0 will, as well. Since equation (80) will be satisfied by this, the proof is complete.

The shortest distances themselves are ρ_0 and ρ^0. Thus:

$$\eta_0 = - s_0\, \rho_0 = \frac{k^2}{\eta^0} = - \frac{k^2}{s^0 \rho^0}. \tag{81}$$

44. There are infinitely many complexes that have a given line for their axis. Each of them is determined completely by one of its lines. Each of two conjugate lines thus determines a complex that also has the given axis for its own axis, and on which that line lies. The parameter of the given complex is proportional to the mean of the two parameters of the two new complexes. All of the lines of one of them have the lines that lie on the other one for their conjugates. We can call the two complexes two *polar complexes relative to the given one.*

The foregoing delivers a series of simple constructions in the manner of representation that we are using. If the complex:

$$\eta + k = 0$$

is given then we can construct the associated line to any given line, and the complex will be determined completely when the axis of the complex is given, along with a system of two conjugate polars that fulfill the conditions that, from the previous number, must be satisfied by a given complex axis.

Let D_0E and F_0G be the projections of a given line onto YZ and XZ (Fig. 6). If OZ is the axis of the complex then we will know that the corresponding projection of the conjugate line likewise goes through E and G, and all that will still remain for the determination of this line is to find two points D^0 and F^0 at which its two projections OY and OX intersect. In a manner analogous to the earlier one, we would like to let A_0 and B_0 denote the points at which the given line meets the YZ and XZ coordinate planes, resp., and their projections onto OY (OX, resp.) by J_0 (H_0, resp.). Finally, let the complex parameter k be equal to $OK = - OK'$.

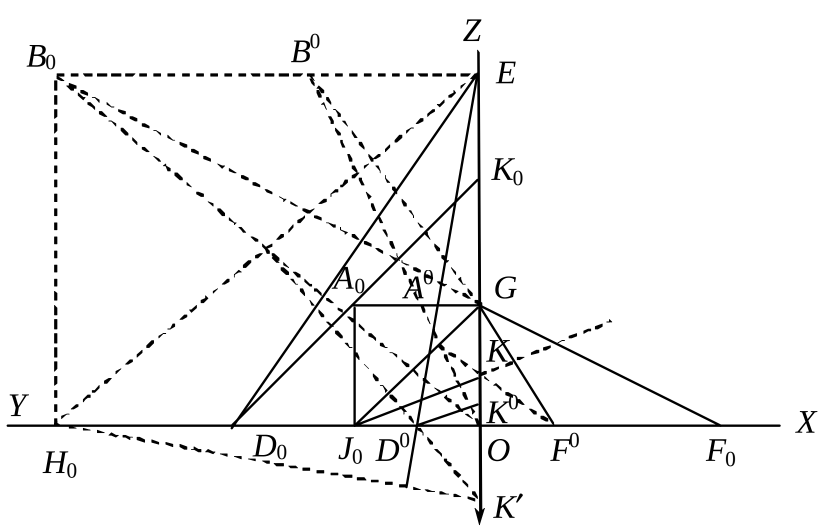

Figure 6.

In the figure, we have dropped perpendiculars from K' to H_0E and from K to J_0G. The first perpendicular cuts OY at D^0 and the second one cuts OX at F^0.

One draws two lines from K and K' to J_0 and H_0 and drops two perpendiculars from these two lines to G and E, respectively. These two perpendiculars will cut OX and OY at F^0 and D^0, resp.

The two foregoing constructions are immediately linked with equations (74).

On the one hand, one essentially has the construction [§ 1, (34)]:

$$\rho^0 = k\,\frac{\rho_0}{\eta_0} = \frac{-k}{\tan A_0OZ} = -\,OK \cdot \tan A_0OJ_0 = OK \tan OKF^0 = OF^0, \tag{82}$$

$$\sigma^0 = k\,\frac{\sigma_0}{\eta_0} = \frac{k}{\tan B_0OZ} = -\,OK' \cdot \tan B_0OJ_0 = OK' \tan OKD^0 = OD^0; \tag{83}$$

on the other [§ 1, (33)]:

$$\tan F^0GO = -\,r^0 = -\,k\,\frac{r_0}{\eta_0} = \frac{OK}{OJ_0} = \frac{OF^0}{OG}, \tag{84}$$

$$\tan D^0EO = -\,s^0 = -\,k\,\frac{s_0}{\eta_0} = \frac{OK'}{OH_0} = \frac{OD^0}{OE}. \tag{85}$$

From the foregoing constructions, we can likewise derive other ones that immediately give, instead of the projections of the lines that are to be determined, the points at which they cut the coordinate planes.

We likewise obtain the complex parameter $k = OK = -\,OK'$ immediately when the two associated polars are given. K and K' will then be the crossing points of the perpendiculars that can be dropped from the vertices of the triangles J_0GF^0 and H_0ED^0 to the opposite sides.

The parameters of the two polar complexes, which we shall distinguish by k_0 and k^0, are given immediately from a discussion in an earlier number. One drops perpendiculars from D_0 and D^0 to OB_0 and OB^0, resp., which cut OZ at K_0 and K^0, resp. One will then have (no. 39):

$$k_0 = OK_0\,, \qquad k^0 = OK^0, \tag{86}$$

from which:

$$OK_0 \cdot OK^0 = \overline{OK}^2. \tag{87}$$

45. If the parameter of the complex k vanishes then that will specialize the complex. Its equation:

$$xy' - x'y = 0 \tag{88}$$

shows that all lines of the complex cut its axis. The general geometric definition of a first-degree complex preserves its validity, such that infinitely many lines of it will go through each point of space, all of them will lie in the same plane, and correspondingly each plane that goes through space will contain infinitely many lines of it that intersect at the same point, even with the specialization. Only those planes that are associated with arbitrary points will intersect in the axis of the complex, just as points that are associated with arbitrary planes will all lie on this axis. All diameters of this complex will coincide on its axis. Any arbitrary line will be associated with the axis.

If we represent the complex by the general equation:

$$Ar + Bs + C - D\sigma + E\rho + F\eta = 0$$

then in order to express the idea that it is specialized in the manner in question (cf., also no. 34) we will obtain the equation (*):

$$AD + BE + CF = 0. \tag{89}$$

For the determination of those lines of the complex that go through any given point (x, y, z), we can eliminate r, s, and h between the general equation of the complex and the equations:

$$\begin{aligned} x &= rz + \rho, \\ y &= sz + \sigma, \\ ry - sx &= \eta, \end{aligned}$$

which express the idea that the given point lies on the line (r, s, ρ, σ, η). In the resulting equation:

$$(A + Fy - Ez)\, r + (B - Fx + Dz)\, s + (C + Ex - Dy) = 0, \tag{90}$$

r and s determine the direction of the plane that is associated with the point (x, y, z). If the three equations:

$$\begin{aligned} A + Fy - Ez &= 0, \\ B - Fx + Dz &= 0, \\ C + Ex - Dy &= 0 \end{aligned} \tag{91}$$

(*) Here, we omit the case in which D, E, and F vanish simultaneously, and thus:

$$k \equiv \frac{AD+BE+CF}{D^2+E^2+F^2}$$

becomes infinitely large. The basis for this is as follows: A complex with infinitely large parameters, whose equation we shall take to be:

$$(xy' - x'y) + k(z - z') = 0,$$

includes only those lines that are parallel to the XY plane or lie at infinity. The foregoing equation will, in fact, be satisfied only when one either has:

$$z - z' = 0$$

or:

$$\frac{xy' - x'y}{z - z'} = \infty.$$

For such a complex, the concept of axis as a completely determined line will then fall away since every line parallel to OZ will have an equal to right to this name.

However, we can also consider the same complex to be a complex of a special type whose parameter equals zero and whose axis lies at infinity in the XY plane. The justification for this lies in the fact that as long as the condition:

$$AD + BE + CF = 0$$

is fulfilled, one can generally speak of a complex of special type whose parameter is equal to zero (cf., also equation (91) in the text).

are satisfied simultaneously (which the condition equation (89) assumes) then the direction of the associated plane will be indeterminate. The point (x, y, z) will then be assumed to be on a line whose three projections are represented by the last three equations, if we consider x, y, z to be variable. This line will be the axis of the complex.

Without the restricting condition equation (89), the foregoing three equations, when taken individually, will represent planes that correspond to points that lie at infinity along the directions of the coordinate axes OX, OY, OZ.

In a similar way, the three equations:

$$\begin{aligned} D + Cu - Bv &= 0, \\ E - Ct + Av &= 0, \\ F + Et - Au &= 0 \end{aligned} \tag{92}$$

represent those points in the coordinate planes YZ, XZ, XY that are associated with these planes. These three equations will be valid simultaneously when the condition equation (89) is satisfied. The three points will then lie along a line and will be the ones at which the three coordinate planes are cut by the axis of the complex.

The condition equation (89) remains unchanged when we consider the complex to be an axial complex and correspondingly represent it by the equation:

$$Dp + Eq + F - A\chi + B\pi + C\omega = 0.$$

However, we remark that this equation will then be illusory in the special case that we consider if we take the axis of the complex to be one of the three coordinate axes, as we did in the case of ray coordinates.

We can satisfy the condition equation (89) in such a way that we set three of the constants of the general complex equation equal to zero, and will obtain four essentially different cases, when we choose the vanishing constants to be, in sequence:

$$D, E, F, \qquad A, B, C, \qquad C, D, E, \qquad A, B, F.$$

These four cases correspond to the following equations in ray and axis coordinates:

$$\left.\begin{aligned} Ar + Bs + C &= 0, \\ -A\chi + B\pi + C\omega &= 0, \\ -D\sigma + E\rho + F\eta &= 0, \\ Dp + Eq + F &= 0, \\ Ar + Bs + F\eta &= 0, \\ -A\chi + B\pi + F &= 0, \\ C - D\sigma + E\rho &= 0, \\ C\omega + Dp + Eq &= 0. \end{aligned}\right\} \tag{93}$$

In the case of equations (93), the axis of the complex lies at infinity on all of the lines that intersects it. It is, like all lines of the complex, parallel to the plane that is represented by the equation:

$$Ax + By + Cz = 0 \tag{97}$$

[cf., the note about (89)].

In the case of equations (94), the axis of the complex is perpendicular at the origin to the plane that is represented by the equation:

$$Dx + Ey + Fz = 0. \tag{98}$$

In the case of equations (95), the axis of the complex is parallel to the coordinate axis *OZ* and cuts the plane *XY* at a point that is represented in this plane by the equation:

$$Bt - Au + Fv = 0. \tag{99}$$

In the case of equations (96), the axis in the *XY*-plane in question is finite and will be represented in this plane by the equation:

$$C + Ex - Dy = 0. \tag{100}$$

46. In the foregoing, when we took the axis of a complex to be one of the three coordinates axes *OZ*, *OY*, *OX*, we were led to present its equation in the following simple forms:

$$\eta + k = 0, \qquad \rho + ks = 0, \qquad \sigma - kr = 0,$$

in which k means the parameter of the complex. The origin can thus have an arbitrary position on the axis of the complex and the remaining two coordinate axes can be assumed to be arbitrary, under the condition that they remain perpendicular to each other and to the axis of the complex. From now on, we would like to take an arbitrary diameter of the complex that is parallel to it to be the *OZ*-axis, instead of the axis. With no loss of generality, we can lay the *YZ*-plane through the diameter and the axis. We would like to denote the distance from the diameter to the axis by y^0. When we switch (cf., no. **14**):

$$\eta \qquad \text{with} \qquad \eta + y^0 \cdot r,$$

the same complex that was previously represented by the equation:

$$\eta + k = 0$$

will be represented by the equation:

$$\eta + y^0\, r + k = 0 \tag{101}$$

from now on. Since the *OZ* and *OY* axes will remain unchanged if we then rotate the *OX* axis in the *XZ*-plane in such a way that after the rotation it forms an angle of δ with *OZ*, the conversion formulas (42) of number **13**, if we write $\gamma' = \delta$, $\gamma = 0$, will express:

$$r \text{ and } \eta$$

in the following forms:

$$\frac{r\sin\delta}{r\cos\delta+1}, \qquad \frac{\eta\sin\delta}{r\cos\delta+1},$$

resp. If we substitute these expressions in the equation of the complex then that will give:

$$\eta \sin \delta + y^0 r \sin \delta + k(r \cos \delta + 1) = 0. \tag{102}$$

δ means the inclination of the XY-plane with respect to the OZ-axis, and thus with respect to the diameter of the complex. If we determine this inclination by the equation:

$$y^0 \sin \delta + k \cos \delta = 0 \tag{103}$$

then that will simplify the equation of the complex into the following form:

$$\eta + \frac{k}{\sin\delta} = 0, \tag{104}$$

or:

$$\eta + k' = 0, \tag{105}$$

when we set:

$$\frac{k}{\sin\delta} \equiv k'. \tag{106}$$

We have called the constant k the "parameter of the complex," but we can also call it the "parameter of the axis of the complex," and in this sense, speak of the parameter of any arbitrary diameter whatsoever, and, in particular, let k' denote the parameter of the diameter that is taken to be the OZ-axis. Among the diameters of a complex, the axis has the smallest parameter.

When we represent the complex by the foregoing equation, we refer it to any of its diameters as the OZ axis and take an arbitrary associated plane through the diameter to be the XY-plane. The two axes OX and OY in this plane will be perpendicular to each other and OY will be the projection of the diameter onto the plane associated with it.

In order to then go from an arbitrary diameter to the axis, we merely need to displace this diameter along the line in the conjugate plane, and indeed, along a line segment of length:

$$-y^0 = k' \cos \delta.$$

The equation of the complex, which we can write in the form:

$$(xy' - x'y) + k'(z - z') = 0, \tag{107}$$

will remain unchanged when we rotate the rectangular coordinate axes arbitrarily inside the XY-plane. However, if we rotate them independently of each other in such a way that after the rotation they make an angle ε then we will have to switch:

$$(xy' - x'y) \qquad \text{and} \qquad \eta$$

with:

$$(xy' - x'y) \sin \varepsilon \qquad \text{and} \qquad \eta \sin \varepsilon,$$

resp. Thus, the form of the complex equation will also still remain the same:

$$\eta + k'' = 0, \tag{108}$$

in which we set:

$$\frac{k'}{\sin \varepsilon} = \frac{k}{\sin \varepsilon \sin \delta} \equiv k''. \tag{109}$$

This will be the equation of the complex when we take the *OZ*-axis to be an arbitrary diameter that makes an angle δ with its plane and choose two arbitrary axes *OX* and *OY* in the plane that subtend an angle ε.

We obtain the corresponding forms for the equations:

$$\rho + \frac{ks}{\sin \delta \sin \varepsilon} = 0, \qquad \sigma - \frac{kr}{\sin \delta \sin \varepsilon} = 0, \tag{110}$$

when we let *OX* and *OY*, in sequence, coincide with the axis of the complex, instead of *OZ*.

47. Up to now, in our discussion of complexes we have left unmentioned the influence that the sign of the parameter has on the nature of a complex. We correspondingly obtain *two essentially different types of complex of the first degree* for the two signs that this value can have.

If we select any line from among the lines of the complex and translate it parallel to the axis of the complex while rotating it arbitrarily around this axis then in all of its new positions it will coincide with other lines of the complex. It thus continually contacts a cylinder of rotation whose axis is the axis of the complex and whose circle of intersection has the shortest distance for the lines from the axis of the complex for its radius. The line that contacts the cylinder can, in agreement with the statement, move around the cylinder in such a way that it envelops a curve. *This curve will then be a helix that lies on the cylinder*. If we displace the helix through the height of loop on the cylinder then the tangents to the helix in the various positions of the latter will all give complex lines that contact the cylinder.

Let:

$$\eta + k \equiv (r\sigma - s\rho) + k = 0$$

be the equation of the complex, and let:

$$y^2 + x^2 = R^2 \tag{111}$$

be the equation of a cylinder of rotation that has the axis of the complex for its own axis, and whose circular basis has radius *R*. Any line whose three coordinates are:

$$r = 0, \quad \rho = R, \quad \sigma = 0 \tag{112}$$

will then be a tangent to the cylinder. In order to express the fact that it belongs to the complex, we obtain:

$$Rs = k. \tag{113}$$

The line lies in a plane that is parallel to the *YZ* coordinate plane. If its projection onto *YZ* defines an angle λ with *OZ* that has a positive trigonometric tangent then it will be the tangent to a *right-wound* helix that is described on the cylinder. As a result of the following equation:

$$Rs - R \tan \lambda = k, \tag{114}$$

the parameter of the complex k will be positive. Conversely, if $\tan \lambda$ is negative then the line will be tangent to a *left-wound* helix that is described on the cylinder and *the parameter of the complex k will be negative.* From the last equation, however, it follows, when we set *R* equal to all positive values in it, that all lines of a complex will be tangent to right-wound helices when one line of it contacts a right-wound helix, just as all lines of a complex will be tangent to left-wound helices when one line of it contacts a left-wound helix. We thus have two essentially different types of first-degree complexes, which we would like to distinguish as *right-wound* and *left-wound* complexes.

We can regard a first-degree complex as the totality of tangents to helices that are inscribed in a cylinders of rotation and whose circular intersections have radii that increase from 0 *to* ∞. *All helices are wound the same way for the same complex.*

For every cylinder, the pitch *h* of the helix is determined by the equation:

$$h = \frac{2\pi R}{\tan \lambda}. \tag{115}$$

If we eliminate λ between this equation and the foregoing one then that will give:

$$h \cdot k = 2\pi R^2; \tag{116}$$

that is: *for any cylinder, the product of the pitch of the helix with the parameter of the complex is equal to twice the area of its circular section.*

48. If we represent a complex by the general equation:

$$Ar + Bs + C - D\sigma + E\rho + F = 0$$

then we have:

$$k = \frac{AD + BE + CF}{D^2 + E^2 + F^2}$$

for its parameter. The complex will then be *right-wound* when:

$$AD + BE + CF > 0 \tag{117}$$

and *left-wound* when:

$$AD + BE + CF < 0. \tag{118}$$

The transitional case:

$$AD + BE + CF = 0 \tag{119}$$

corresponds to the notion that *the axis of the complex is cut by all of its lines* (cf., no. 45).

The values of the constant k are equal, but of opposite sign, for two conjugate complexes. The helices of both complexes are oppositely wound. For the sake of visualization, we can regard two complexes as the mirror images of each other, if we think of the plane of the mirror as being perpendicular to the common axis of the complex.

Each point of space is cut by two oppositely wound helices that are inscribed on the same cylinder and each belong to one of two conjugate complexes. The tangents to the two helices at this point are lines of the two complexes that go through it. The angle that they make with each other is $2(\pi - \lambda)$. However, one has:

$$\tan(\pi - \lambda) = \frac{R}{k}, \tag{120}$$

so the tangent of the angle at which the lines of the two complexes intersect each other is:

$$\tan 2(\pi - \lambda) = \frac{2Rk}{k^2 - R^2}. \tag{121}$$

This angle of intersection decreases with the distance from the point to the axis of the complex. When:

$$k = R,$$

it will go through a right angle, and it will become ever larger when R increases further, such that for $R = \infty$, it will approach the limit π.

Only one helix of a complex goes through each given point. The tangent to this helix at the given point is a line of the complex and thus lies in the plane corresponding to this point. This plane is determined completely by a second line that goes through this point and belongs to the complex. We find such lines in the consecutive tangents to the same helix. The plane that contains the two tangents is the osculating plane of the helix at the given point.

The osculating plane to a complex helix at each of its points is the plane that corresponds to this point.

We find the confirmation of this theorem in the fact that, on the one hand, both planes go through the tangents to the helix at the given point, and, on the other hand, they both go through the perpendicular that can be dropped from this point to the axis.

If a point moves along one of two associated polars then at each point they will correspond to a helix and an osculating plane to it that always goes through the other polar. If the line is a side (*Seite*) of a cylinder that the helix is inscribed in – in other words, a diameter of the complex – then the corresponding osculating planes will be parallel and will be the planes associated with the diameters, such that the polar associated with the diameter lies at infinity (*).

49. I shall conclude this investigation of first-degree complexes with some general remarks.

Just as we can construct polygons whose vertices lie in a given plane from lines, and solid angles whose planes go through a given point, we can also simultaneously construct spatial polygons and polyhedra from the lines of a first-degree complex that correspond to them. The sides of these spatial polygons are the edges of the polyhedra. At the vertices of the polygon, two successive sides of it will intersect. The plane that goes through two such sides will be the plane that corresponds to the vertex in the complex and a face of the polyhedron. The mutual relationships between polygons and polyhedra are the ones that we already spoke of in the note in number **29**.

We would like to call a spatial polygon whose sides are lines of the complex a *complex polygon* and the corresponding polyhedron, a *complex polyhedron*.

In order to describe a complex polyhedron, we choose a line of the complex and an initial vertex of the polygon that lies in it. Just as infinitely many lines go through any plane through a point, so do infinitely many lines of a complex go through a point. On a complex line that goes through the first vertex of a polygon, we choose the second vertex, then choose the third one on one of the complex lines that goes through it, and so on. In order to close the polygon, through the last point that is determined in this way, we draw the plane that corresponds to this point under the complex. It will intersect the first complex line at a point. The line that connects both points will be a line of the complex and close the polygon. We can derive a complex polyhedron from the corresponding complex polygon, or also construct it directly in a manner that is analogous to this one. To that end, we consider a given complex line to be the edge of a polyhedron and lay the first face of it through this edge, through an arbitrary complex line in this plane, we lay the second face, through an arbitrary complex line in the latter, the third, and so forth. In the last-determined polyhedral face, we determine the point that corresponds to it in the complex. The plane that goes through this point and the first complex line closes the polyhedron.

The sides of a complex polygon are oriented the same – that is, they are tangents to equally-wound helices – and that will define a characteristic sequence of vertices of that polygon as a consequence. The faces of a complex polyhedron through which two oriented edges of it go will be rotated in the same sense. We can refer to polygons and polyhedra as right-wound or left-wound in their own right according to whether they belong to right-wound or left-wound complexes, respectively. The mirror image (we revert to our previous manner of visualization and, in turn, take the reflecting surface to

(*) We have always taken the constants in the general equation of a complex to be real and thus k, as well. However, if we combine several complexes then the generality of the discussion will extend in such a way that we must now consider complexes with imaginary constants.

be perpendicular to the complex axis) of a complex polygon or complex polyhedron will belong to the conjugate complex and will be wound the opposite way.

50. A plane curve will be enveloped by a line that constantly moves in the plane, and a cone surface will be described by a line that rotates around one of its points. A spatial curve will be enveloped by a continuously moving line that belongs to a given complex at all of its points, while a developable surface will be simultaneously described by it. We refer to any of the former as *curves* and any of the latter as *developable surfaces of the first-degree complex.*

We can inscribe infinitely many curves of a given complex in any given surface. Such a complex curve will go through each given point of the given surface. The tangent to the complex curve at this point will be the line in which the plane that corresponds to the given point in the complex intersects the tangential plane to the surface at this point. (*)

In order to summarize everything in a word: Just as there is a *geometry of the plane*, there is also a *geometry of the first-degree complex.*

§ 2. Congruences of two linear complexes.

51. The coincident lines of two first-degree linear complexes will define a *line congruence.* We can consider the lines of a congruence to be rays and axes, and correspondingly, congruences can be represented in two ways: first, by a system of equations in ray coordinates:

$$\left.\begin{aligned} \Omega &\equiv Ar + Bs + C - D\sigma + E\rho + F\eta = 0, \\ \Omega' &\equiv A'r + B's + C' - D'\sigma + E'\rho + F'\eta = 0, \end{aligned}\right\} \tag{1}$$

and secondly, by a system of two equations in axial coordinates:

$$\left.\begin{aligned} \Phi &\equiv Dp + Eq + F - Ak + B\pi + C\omega = 0, \\ \Phi' &\equiv D'p + E'q + F' - A'k + B'\pi + C'\omega = 0. \end{aligned}\right\} \tag{2}$$

52. In each of the two complexes that determine the congruence, infinitely many lines will go through a given point that will lie in the plane that corresponds to the point. The line of intersection of the two planes that correspond to the given point is the single

(*) In order to clarify the general reasoning of the text by a simple example, we would like to consider the outer surface of a sphere whose center falls on the axis of the complex, and whose radius R is arbitrary. The complex curves inscribed on this sphere define a system of *loxodromes* that intersect the meridian of the sphere at an angle of λ, which is given by the equation:

$$\tan\lambda = \frac{k}{R}.$$

line that simultaneously goes through that point and both complexes, and will thus belong to the complex. *A single straight line* will go through every point of space, which we will say *corresponds to the point* under the congruence.

In each of the two complexes, infinitely many lines will lie in a given plane that cuts the point that corresponds to the plane. The straight line that links the two corresponding points in the given planes will be the only one that simultaneously lies in that plane and the two complexes, and thus belongs to the congruence. *A single straight line* will lie in any plane that is drawn through space, of which we will say that it *corresponds to the plane* under the congruence.

We can regard the two given relations, one of which will necessarily be a consequence of the other one, as the *geometric definition* of a congruence of linear complexes.

53. For the determination of a congruence, we can replace the two given complexes:

$$\Omega = 0, \qquad \Omega' = 0$$

with two other ones that will be represented by the following equation:

$$\Omega + \mu\,\Omega' = 0 \tag{3}$$

for an arbitrary choice of the undetermined coefficient μ, and we can then also replace Ω and Ω' with Φ and Φ', resp. We say that all complexes that can be represented by the foregoing equations, and two of which will determine the congruence, define a *two-parameter group of linear complexes.*

54. In number **31**, we obtained the following equation for the principal section of the complex (3) that goes through the coordinate origin:

$$Dx + Ey + Fz + \mu\,(D'x + E'y + F'z) = 0, \tag{4}$$

and this equation will be satisfied for arbitrary values of μ when one simultaneously has:

$$\begin{aligned} Dx + \ Ey + \ Fz &= 0, \\ D'x + E'y + F'z &= 0. \end{aligned} \tag{5}$$

Since the coordinate origin will be assumed to be arbitrary from now on, this will therefore express the idea that in any complex a two-parameter group that belongs to the congruence will intersect the principal section that goes through a given point in the same straight line. Since the diameter of a complex is perpendicular to its principal section, that will yield the following theorem:

The diameter of any complex in a two-parameter group will be parallel to its plane.

55. In order to determine the direction of the diameter, we will get the following double equation:

$$\frac{x}{D+\mu D'} = \frac{y}{E+\mu E'} = \frac{z}{F+\mu F'} \tag{6}$$

when we introduce the direction constants r and s:

$$r = \frac{D+\mu D'}{F+\mu F'}, \qquad s = \frac{E+\mu E'}{F+\mu F'}. \tag{7}$$

If we eliminate μ from these equations then we will find that:

$$(E'F - E\,F')\,r - (D'F - D\,F')\,s + (D'E - D\,E') = 0. \tag{8}$$

The direction of the plane that is parallel to the diameter of all the complexes is determined by this equation. When we replace r and s with x / z and y / z, we will get the equation:

$$(E'F - E\,F')\,x - (D'F - D\,F')\,y + (D'E - D\,E')\,z = 0, \tag{9}$$

which will represent the plane that goes through the origin in the chosen direction in ordinary point-coordinates, and we will then get the double equation for the determination of the direction that is perpendicular to that plane:

$$\frac{x}{E'F - EF'} = \frac{y}{D'F - ED'} = \frac{z}{D'E - DE'}. \tag{10}$$

If we make this direction the OZ-axis then F and F' will vanish, and the complex equations (1) will become:

$$\begin{aligned} \Omega &\equiv Ar + Bs + C - D\sigma + E\rho = 0, \\ \Omega' &\equiv A'r + B's + C' - D'\sigma + E'\rho = 0. \end{aligned} \tag{11}$$

The diameters of all complexes of the group (3) will then be parallel to the XY-plane.

56. If one of the three following condition equations:

$$D'E - D\,E' = 0, \qquad D'F - D\,F' = 0, \qquad E'F - E\,F' = 0 \tag{12}$$

is true then the straight line along which the principal section that goes through the origin intersects all complexes of the group will lie in one of the three coordinate planes XY, XZ, YZ, resp. The diameters of all the complexes will then be parallel to a plane that goes through OZ, OY, OX, resp. If the three condition equations (12) were satisfied simultaneously then that would raise the contradiction that a plane would have to be simultaneously parallel to the three coordinate axis, which would prevent any determination of that plane. One would then correspondingly find one of the complexes of the group (3):

$$\mu = -\frac{D}{D'} = -\frac{E}{E'} = -\frac{F}{F'},$$

whose equation we could take to be the following one:

$$(A'D - AD')\, r + (B'D - BD')\, s + (C'D - C\,D') = 0. \tag{13}$$

All of the lines in this complex would be parallel to the plane that is represented by the equation:

$$(A'D - AD')\, x + (B'D - BD')\, y + (C'D - C\,D')\, z = 0. \tag{14}$$

The congruence would be specialized in this case by the fact that since its lines all belong to this complex, they would all be parallel to the plane that was just determined. Therefore, the axes of all complexes in the two-parameter group would be parallel to each other, which could be seen in the double equation (10). We would like to call such a congruence *parabolic*. We will exclude them from the following considerations, and will subject them to a special discussion later on (no. **75**).

This case will also come about especially when F and F' vanish, and:

$$D'E - DE' = 0, \tag{15}$$

moreover.

57. If we shift the origin of the coordinates to any point (x^0, y^0) then the constant term in the equation of the complex group (3) will become:

$$(C + Ex^0 - Dy^0) + \mu\,(C' + E'x^0 - D'y^0).$$

This term will then drop out when the new origin in the XY-plane is assumed to be on the straight line that is represented by:

$$(C + Ex - Dy) + \mu\,(C' + E'x - D'y) = 0. \tag{16}$$

If we take this point to be the intersection of the two straight lines:

$$\left.\begin{array}{r} C + E\,x - D\,y = 0, \\ C' + E'x - D'y = 0 \end{array}\right\} \tag{17}$$

then the constant term will vanish from the equation of all complexes of the group. One will then get:

$$\left.\begin{array}{r} \Omega \equiv A r + B s - D\sigma + E\rho = 0, \\ \Omega' \equiv A' r + B' s - D'\sigma + E'\rho = 0. \end{array}\right\} \tag{18}$$

We have:

$$Ax + By + Cz + \mu\,(A'X + B'y + C'z) = 0 \tag{19}$$

for the general equation of those planes that correspond to the origin in the various complexes of the group (3) (no. **32**), and this equation will be satisfied, independently of the particular value of μ, when one simultaneously has:

$$\left.\begin{aligned} A\,x + B\,y + C\,z &= 0, \\ A'x + B'y + C'z &= 0. \end{aligned}\right\} \tag{20}$$

All of the planes that correspond to the origin will then intersect on the straight line that is represented by the two equations, and since the origin will assumed to be arbitrary from now on, we will arrive at the following theorem:

In the complexes of a two-parameter group, a given point will correspond to planes that intersect in the same line.

This theorem is given immediately by combining the ones in numbers **52** and **53**.

When C and C' vanish, the planes that correspond to the origin under the various complexes of the two-parameter group will intersect along the OZ-axis.

58. When F and F', as well as C and C', vanish, OZ will become a common line to all complexes, and thus a line of the congruence that is intersected by the axes of all complexes (cf., no. **31**).

In any congruence, there is, in general, a single and completely-determined straight line that will be intersected by the axes of all complexes of the two-parameter group by which the congruence is determined.

We would like to call the straight line that has an exclusive relationship with the congruence the *axis of the congruence.* When we base the determination of the function on equation (18), we will take the axis of the congruence to be the OZ-axis.

When the condition equation:

$$D'E - D\,E' = 0$$

is verified, the congruence can generally no longer be represented by the system of two equations (18). F and F' cannot both drop out. The principal sections of all complexes of the two-parameter groups that go through the origin will then intersect (when the condition equation above is satisfied) on one of the straight lines that lie in the coordinate planes, and whose equation will be the following one:

$$y + \frac{Dx}{E} \equiv y' + \frac{D'x}{E} = 0.$$

From number **54**, the diameter – and especially the axes – of all complexes of the group will be parallel to a plane that is perpendicular to that line. Therefore, the *XY*-plane cannot be parallel to the diameter, in general, and therefore one will get the impossibility of F and F' vanishing. This possibility first will arise when:

$$\frac{D}{D'} = \frac{E}{E'} = \frac{F}{F'};$$

that is, in the case of a parabolic congruence, such that equations (5) become identical, and corresponding to that, all of the principal sections that go through the origin will coincide. All of the complex axes will then be perpendicular to this plane, and will then be mutually parallel, in agreement with number **56**. In order for F and F' to drop out, one will then need only to choose the *XY*-plane such that it is perpendicular to the plane in question, or – what amounts to the same thing – the *OZ*-axis must lie in that plane, although it can have any direction within that plane.

However, when the condition equation above is satisfied, C and C' cannot also drop out simultaneously, in general. In fact, inside of *XY*, the shift of the coordinate origin that demands this dropping out is illusory, and indeed in such a way that the two lines (17), at whose intersection the new origin lies, will be parallel. (The values of F and F' do not come into consideration in this.) Only when one has:

$$\frac{C}{C'} = \frac{D}{D'} = \frac{E}{E'}$$

simultaneously, and as a consequence, the two straight lines (17) coincide in a single one, can C and C', in turn, be removed by a shift of the origin, and indeed, to that end, we can take any arbitrary point of the straight line:

$$1 + \frac{Ex}{C} - \frac{Dy}{C} \equiv 1 + \frac{E'x}{C'} - \frac{D'y}{C'} = 0$$

to be the new coordinate origin. Later, we will encounter the case in which the two lines (17) coincide in a single one, and we will see that this coincidence is based upon a special position of the congruence relative to the coordinate system.

59. If one of the three conditions:

$$A'B - AB' = 0, \qquad A'C - AC' = 0, \qquad B'C - BC' = 0 \tag{21}$$

is satisfied then any straight line in which all of the planes that correspond to the origin intersect will lie in the coordinate planes, *XY*, *XZ*, *YZ*, respectively. If two of these equations – and consequently, all three of them – are satisfied then among the complexes of the group (3), corresponding to:

$$\mu = -\frac{A}{A'} = -\frac{B}{B'} = -\frac{C}{C'},$$

there will be one of them whose lines all intersect on its axis, and this axis will go through the origin. We can take:

$$-(A'D - AD')\,\sigma + (A'E - AE')\,\rho + (A'F - AF')\,\eta = 0 \tag{22}$$

to be its equation. These conditions will correspond when C and C' vanish and one simultaneously has:

$$A'D - AD' = 0. \tag{23}$$

The axis of the complex that goes through the origin will then lie in the YZ-plane.

If C and C', F and F' vanish at the same time, and the condition (23) is likewise fulfilled then the axis of the complex will coincide with the coordinate axis OY. The discussion of this case will find its completion later (in no. **76**).

60. The complexes Ω and Ω' are any two that we have selected arbitrarily from the complex group (3). However, among the infinitude of complexes in the group, in general, one will find ones that depend upon one less constant, and whose lines will all cut the axis (cf., no. **45**). For the determination of these complexes, we would like to start with the function determination (18), which, from the foregoing, is always permitted, except in the case where the condition equations (12) are likewise valid. All axes of the complex, which is, moreover, represented by the equation:

$$(Ar + Bs - D\sigma + E\rho) + m\,(A'r + B's - D'\sigma + E'\rho) = 0, \tag{24}$$

will then intersect OZ in a right angle.

If the complex that corresponds to an arbitrary value of μ is of the type referred to then, from number **45**, we will obtain the following equations for the three projections of its axis:

$$(A - Ez) + \mu\,(A' - E'z) = 0, \tag{25}$$
$$(B - Dz) + \mu\,(B' - D'z) = 0, \tag{26}$$
$$(Ex - Dy) + \mu\,(E'x - D'y) = 0. \tag{27}$$

In such a complex, the plane that corresponds to a point in space is the one that can be drawn through the point and the axis of the complex in which all of its diameters coincide (no. **55**). The equation of the plane that corresponds to the origin is, by our assumption on the coordinate axes, the following one:

$$(Ax + By) + \mu\,(A'x + B'y) = 0. \tag{28}$$

The axis of the complex will then lie in this plane.

If we express any point on the axis of a to-be-determined complex by (x, y, z) then the foregoing four equations will exist between these coordinates simultaneously. If we eliminate Z from (25) and (26) then we will get:

$$(A + \mu A')(D + \mu D') + (B + \mu B')(E + \mu E') = 0. \tag{29}$$

We obtained the same equation by eliminating y / x between (27) and (28). It expresses the idea that the parameter of the complex vanishes (no. **38**). We could have posed it from the outset.

61. When we develop the last equation, it will become:

$$(A'D' + B'E')\,\mu^2 + [(A'D + AD') + (B'E + EB')]\,\mu + (AD + EB) = 0. \tag{30}$$

If we denote the roots of this equation by μ^0 and μ_0 then that will give:

$$\mu^0 + \mu_0 = -\frac{(A'D + AD') + (B'E + BE')}{A'D' + B'E'}, \tag{31}$$

$$(\mu^0 - \mu_0)^2 = -\frac{[(A'D - AD') + (B'E - BE')]^2 - 4(A'B - AB')(D'E - DE')}{(A'D' + B'E')^2}. \tag{32}$$

There are then two complexes of a special kind in the complex group:

$$\Omega + \mu\Omega' = 0,$$

such that the lines in each of them intersect along a fixed line – viz., the axis. According to whether the two values of μ^0 and μ_0 are real or imaginary, the same will be true for the two complexes and their axes. We would like to call the axes of the two complexes thus determined the two *directrices of the congruence.*

All lines of a congruence will cut its directrices.

62. From the result that we achieved in the previous number, we can, moreover, define a congruence geometrically by saying that it is the totality of all lines that cut two given fixed straight lines. The straight line in the congruence that *corresponds to a given point* is thus the one that goes through the given point and intersects the two directrices, while the straight line that *corresponds to a given plane* is the one that connects the intersection points of the given plane with the two directrices.

63. If we eliminate μ from the two equations (25) and (26) then that will give:

$$\frac{A - Ez}{A' - E'z} = \frac{B + Dz}{B' + D'z}, \tag{33}$$

and when we develop this, we will get:

$$(D'E - DE')\, z^2 + [(A'D - AD') + (B'E - BE')]\, z + (A'B' - AB') = 0. \tag{33}$$

The roots of this equation determine the planes in which the directrices of the congruence lie, and thus the points at which OZ will be cut by the two directrices.

If we eliminate μ from the two equations (27) and (28) then that will give:

$$\frac{Ex - Dy}{E'x - D'y} = \frac{Ax + By}{A'x + B'y}. \tag{35}$$

The two values that this equation gives for y / x are the trigonometric tangents of the angles that the two directrices of the congruence make with the direction of OX in the plane that was just determined. If we set:

$$\frac{y}{x} = \tan \vartheta,$$

when we call that angle ϑ, then when we develop (35), that will give:

$$(B'D - BD') \tan^2 \vartheta + [(A'B - AD') - (B'E - BE')] \tan \vartheta - (A'E - AE') = 0. \tag{36}$$

64. Due to the coincidence of the OZ coordinate axis with the straight line that cuts the two directrices of the congruence that is determined by (3) at right angles, the equations of the two complexes Ω and Ω', which we choose arbitrarily from the two-parameter group, will assume the following form:

$$\left.\begin{aligned} Ar + Bs - D\sigma + E\rho &= 0, \\ A'r + B's - D'\sigma + E'\rho &= 0. \end{aligned}\right\} \tag{37}$$

We can remove even more constants from the system of two equations.

The point that lies on the OZ axis at the midpoint between the two directrices shall be called the *center of the congruence*, and one half-the distance between the two directrices shall be called its *constant*. If we then lay the XY-plane through the midpoint of the congruence then equation (34) will give:

$$(A'D - AD') + (B'E - BE') = 0, \tag{38}$$

and thus, if we denote the constant of the congruence by Δ:

$$\Delta = \sqrt{-\frac{A'B - AB'}{D'E - DE'}}. \tag{39}$$

Because the case:

$$D'E - DE' = 0$$

under discussion is temporarily excluded, Δ will always take on a finite value.

Up to now, the directions of the two coordinate axes have remained undetermined. We now additionally determine these directions in such a way that it bisects the angle that the directions of the two directrices define with each other – which can happen in two ways – so equation (36) will give:

$$(A'D - AD') - (B'E - BE') = 0, \tag{40}$$

and the trigonometric tangent of the angle that the directions of the two directrices make with OX will become:

$$\tan\vartheta = \pm\sqrt{\frac{A'E - AE'}{B'D - BD'}}. \tag{41}$$

If we give the OX and OY axes the directions that were just referred to, and simultaneously assume that the origin is at the center of the congruence then the two condition equations (38) and (40) will be likewise valid, and can then be replaced by the following two:

$$A'D - AD' = 0, \tag{42}$$
$$B'E - BE' = 0. \tag{43}$$

We would like to call the two coordinate axes in the position thus determined the *two auxiliary axes* of the congruence. They lie in the *central plane* of the congruence and bisect the angle that the two projections of the directrices onto this plane define with each other.

65. This coordinate system yields:

$$\Delta = \sqrt{-\frac{A'B'}{D'E'}} = \sqrt{-\frac{AB}{DE}}, \tag{44}$$

$$\tan\vartheta = \pm\sqrt{-\frac{A'E'}{B'D'}} = \pm\sqrt{-\frac{AE}{BD}}. \tag{45}$$

A congruence is determined by its two directions in a linear way, and will thus depend upon *eight* mutually-independent constants. *Six* of these are again related to the choice of coordinate system, which is determined completely when we take the principal axis and the two auxiliary axes of the congruence to be coordinate axes. The two complexes (37) that serve to determine the congruence depend upon six independent constants that enter into their equations. Since there are two condition equations (42) and (43), the number of them will reduce to four. Two of these four constants will be still superfluous, which finds its explanation in the fact that we did not choose two distinguished complexes of the two-parameter group:

$$\Omega + \mu\,\Omega' = 0$$

for the determination of the congruence, but two arbitrary ones – namely, Ω and Ω' – corresponding to $\mu = 0$ and $\mu = \infty$, respectively. However, two distinguished complexes of the group are the two that have the two directrices for axes; that is, the ones whose parameters are equal to zero. If we take these two complexes to be Ω and Ω' then that will yield the two new condition equations:

$$A\,D + B\,E = 0, \tag{46}$$

$$A'D' + B'E' = 0. \tag{47}$$

Along with the six constants of the position, two constants will then remain for the determination of the congruence. The number of constants will then necessarily be reduced to eight.

66. In the new coordinate determination, the expressions (31) and (32) that were developed above will become:

$$\mu^0 + \mu_0 = -\,2\,\frac{AD' + BE'}{A'D + B'E'}, \tag{48}$$

$$(\mu^0 - \mu_0)^2 = -\,4\,\frac{(A'B - AB')(D'E - DE')}{(A'D + B'E')^2}. \tag{49}$$

The two roots μ^0 and μ_0 are real when:

$$(A'B - AB')(D'E - DE') < 0, \tag{50}$$

and imaginary when:

$$(A'B - AB')(D'E - DE') > 0. \tag{51}$$

The foregoing expression can be written in accordance with the conditions equations (42) and (43) in the following form:

$$A'B'D'E'\left[\frac{B}{B'} - \frac{A}{A'}\right]^2, \qquad ABDE\left[\frac{B'}{B} - \frac{A'}{A}\right]^2.$$

The reality of the two roots will thus depend upon whether the products $A'B'D'E'$ and $ABCE$, which agree with each other in sign, are negative or positive. In the former case, μ^0 and μ_0 will be real, and with them, in accordance with (44) and (45), A and the two values of tan ϑ will be real, as well; in the latter case, μ^0, μ_0, Δ, and the two values of tan ϑ will be simultaneously imaginary.

67. The two values of μ^0 and μ_0 will be equal to each other when one of the two condition equations:

$$A'B - AB' = 0, \qquad D'E - DE' = 0 \tag{52}$$

is satisfied. However, in this case, with consideration given to equations (42) and (43), one will generally get:

$$\frac{A}{A'} = \frac{B}{B'} = \frac{D}{D'} = \frac{E}{E'}.$$

The two complexes Ω and Ω' of the two-parameter group, and consequently all complexes of that group, will then be identical to them. The determination of the congruence becomes illusory.

The apparent contradiction is resolved by this.

68. However, there are also special cases in which the equation form (18) keeps its meaning even when the two values μ^0 and μ_0 are equal to each other. In general, the two equations (42) and (43), in conjunction with one of the two equations (53), demand the second of the latter equations. However, if – say – A and A' are equal to zero then this will no longer be the case; we will then be dealing with an actual congruence that is of a special type.

In fact, (44) and (45) will vanish in this case, as well as Δ and $\tan \vartheta$. *The two directrices of the congruence coincide in a straight line.* In agreement with this, the numerator in the value (49) for $(\mu^0 - \mu_0)^2$ will vanish, while the denominator will keep a finite value.

In our case, we can take the equation of the two-parameter group (37), with consideration given to equation (43):

$$B'E - BE' = 0,$$

to be the following one:

$$(Bs + E\rho) - D\sigma + \mu\,[(Bs + E\rho) - D'\sigma] = 0, \tag{53}$$

and select the distinguished complexes from that group to be the following two whose equations are:

$$Bs + E\rho = 0, \qquad \sigma = 0,$$

and these equations will also be written in the following way in homogeneous coordinates:

$$B\,(y - y') + E\,(x'z - xz') = 0, \qquad yz' - y'z = 0. \tag{54}$$

The first of the foregoing two complexes has the OY coordinate axis for its axis; its parameter is B / E. The second complex is of a special kind that has a parameter that is equal to zero. Its axis, which will then cut all of its lines, will fall along the OX coordinate axis. In agreement with (44) and (45), OX will then be the directrix of the congruence.

Whereas a congruence must generally be determined by its two directrices, in the special cases in which the two directrices coincide with a straight line, in addition to that straight line, yet another new complex of the two-parameter group that is determined by the congruence will be given (*).

In the complex whose equation is the following one:

$$B\,(y - y') + E\,(x'z - xz') = 0,$$

any point of the OX coordinate axis will correspond to the plane:

$$By + Ex'z = 0,$$

where x' refers to the distance to the point from the origin. For any other complex of the two-parameter group (53), we will find *the same* plane, since y' and z' will vanish for all points that lie along OX. This plane goes through OX. We then conclude:

If the two directrices in a congruence coincide with a straight line then that line itself will be a common line of all complexes; that is, a line of the congruence.

We thus obtain a congruence of the kind in question when we take all lines in a complex that cut a fixed line in it. When a point advances along a line of a complex, the plane that corresponds to it will rotate around that line (no. **28**). Infinitely many lines of the congruence will then go through each point of that straight line in which the two directrices coincide, which will all belong to a plane that goes through that straight line, in its own right. If the point advances along a straight line then the plane will rotate around it. The relationship between points and planes is completely reciprocal.

We further specify that A, A', B, and B' be equal to zero. From (44), Δ will then vanish, while, from (45), tan ϑ will take the form 0 / 0, and since no relation exists between the vanishing coefficients, it will be indeterminate. Thus, to be consistent, the numerator and denominator in the expression (49) for $(\mu^0 - \mu_0)^2$ will likewise take on the value zero.

Any line that goes through the origin in the XY coordinate plane is a directrix of the congruence.

We take the equation of the two-parameter group that the congruence determines to be the following one, into which only mutually-independent constants enter:

$$-D\sigma + E\rho + \mu\,(-D'\sigma + E'\rho) = 0. \tag{55}$$

In particular, we can select the following two complexes from this group:

(*) The basis for this lies in the fact that a straight line represents *four* constants, while a congruence – which, like the present one, is specified by one condition – will depend upon *seven*. We find the *three* remaining constants in the second given complex, which then depends upon only *three* arbitrary constants, since it is coupled to the *two* conditions that its axis *cuts* the straight line in which the two directrices of the congruence coincide, and in fact at *right angles*.

$$\rho = 0, \qquad \sigma = 0,$$

which will have the following equations, when expressed in homogeneous coordinates:

$$x'z - xz' = 0, \quad yz' - y'z = 0. \tag{56}$$

Therefore, the congruence encompasses, on the one hand, all lines that lie in the *XY* plane, as well as all lines that go through the origin.

Every line of the congruence cuts all of its directrices. The directrices are themselves lines of the congruence.

In general, a single line of the congruence in question will go through a given point: viz., the line that connects it to the origin. In particular, if the given point lies in the *XY* plane then infinitely many lines of the congruence will go through it that all belong to the aforementioned coordinate plane. If one shifts the point that is assumed to be in the *XY* plane to the origin, in particular, then each of the straight lines that go through it will belong to the congruence.

On the other hand, in general, one line of the congruence lies in each given plane: viz., its intersection line with the *XY* coordinate plane. If the plane that goes through the origin coincides with the *XY* coordinate plane, in particular, then each of the straight lines that lie in it will belong to the congruence.

69. In the foregoing, we took the principal axis of a congruence and its two auxiliary axes to be coordinate axes, and thus represented the congruence by the following two complex equations:

$$\left.\begin{aligned} \Omega &\equiv Ar + Bs - D\sigma + E\rho = 0, \\ \Omega' &\equiv A'r + B's - D'\sigma + E'\rho = 0, \end{aligned}\right\} \tag{37}$$

under the assumption that:

$$A'D - AD' = 0, \tag{42}$$

$$B'E - B\,E' = 0, \tag{43}$$

and obtained:

$$\Delta^2 = -\frac{AB}{DE}, \tag{44}$$

$$\tan^2 \vartheta = -\frac{AE}{BD} \tag{45}$$

for the geometric determination of the congruence. The last two equations leave undecided whether the directrix of the congruence that corresponds to + tan ϑ cuts its principal axis at a point for which $z = + \Delta$ or $z = -\Delta$, so the other directrix, which corresponds to – tan ϑ, will cut the axis at the point whose z has the opposite sign. Therefore, in agreement with the above, we will arrive at the same equations (44) and (45) when we simultaneously change the signs of A and B and A' and B' in equations (37), or – what amounts to the same thing – the signs of D and E and D' and E'. If we denote the complexes that corresponding to this exchange by the symbols Ω_1 and Ω_1' then the following equations will enter in place of the previous ones:

$$\left.\begin{aligned} \Omega_1 &\equiv A r + B s + D\sigma - E\rho = 0, \\ \Omega_1' &\equiv A' r + B' s + D'\sigma - E'\rho = 0. \end{aligned}\right\} \tag{57}$$

The two congruences that are represented by the two equation-pairs (37) and (57) have the same center and the same central plane, and perpendicular to it, the same principal axis and the same auxiliary axes. The distances between the two directrices and the angles that their directions make with each other are equal in the two congruences. The relationship between the two congruences is a reciprocal one; when we once more appeal to the previous image of a reflection and consider the *XY*-plane (or some other coordinate plane in its place) to be a plane of reflection, the one will be the mirror image of the other. We would like to call two congruences that have this relationship to each other two *conjugate* congruences.

70. In the foregoing, we showed that a congruence simultaneously belongs to all complexes in a two-parameter family, and that among these complexes, in general, two of them will be of the special kind whose parameter is equal to zero. The axis of each of the two complexes will be cut by all of its lines, from which, it will follow that all lines of the congruence will cut its two axes, since they must also belong to these two complexes. Correspondingly, we have defined these axes to be the two directrices of a congruence. However, we can also regard the two directrices from a different viewpoint.

71. A congruence is determined in such a way that its lines simultaneously belong to two complexes that are taken arbitrarily from a two-parameter group of complexes. The complex will be represented by the equation:

$$\Omega + \mu\,\Omega' = 0,$$

when we set μ equal to two successive, arbitrary values in it. In that way, however, the number of independent constants in that equation will be reduced by one unit. The number of constants upon which the congruence depends will then amount to:

$$2\,(5 - 1) = 8,$$

which is the sum of the constants in two complexes when their constants have been reduced from five to four. If a line that belongs to the congruence is given then we will obtain a linear condition equation between the four constants of each of the two complexes by which the congruence will be given. *Four* given straight lines of the congruence are necessary and sufficient to determine the two complexes, and thus, the congruence. Two straight lines that do not belong to the congruence are determined by four given straight lines of the congruence if they intersect the four given ones. These lines depend upon *eight* constants; they mutually determine the four given lines and *all* lines of the congruence.

Four lines of a complex will determine a congruence that the complex belongs to. If the congruence is given by *four* of its lines then a complex that belongs to the congruence will be determined by a *fifth* line. The two lines that cut the four given lines are, on the one hand, the two directrices of the congruence, and on the other hand, two associated polars of any complex that belongs to the congruence.

Two associated polars of a given complex are the two directrices of a congruence that the complex belongs to.

The two directrices of a given congruence are two associated polars of any complex that belongs to the congruence.

If the two directrices coincide in a straight line then this common line to all complexes will then itself be a line of the congruence (cf., no. **68**).

72. In general, only one line of a given congruence will go through a given point, just as only one of its lines will lie in any plane. We can consider the two directrices to be the locus of points through which infinitely many lines of the congruence go, as well as, on the other hand, the locus of all points that are enveloped by planes in which infinitely many lines of the congruence lie. Namely, if a point is assumed to be on one of two associated polars of a complex in the two-parameter group then the plane that goes through the point and the other polar will be the plane that corresponds to the point in the complex. Therefore, if the two associated polars belong to all complexes of a two-parameter group then each point of one of the two common polars in all complexes will correspond to the same plane that is determined by the fact that it goes through the other polar. The relationship between the two polars to the complexes of the group is completely reciprocal. Conversely, we can also start with a plane that is drawn through one of the two polars; the point that corresponds to that plane in all complexes of the two-parameter group will then be the same point, and indeed, it will be the intersection of that plane with the other polars. Thus, whereas a given point will correspond to *one* straight line in a congruence, if it is assumed to be on one of the two directrices, in particular, then it will correspond to *one* plane that goes through the other directrix, just as any plane that generally corresponds to *a single* straight line, will correspond to *one* point that lies on one of the directrices when it goes through the other directrix.

73. We can thus paraphrase the foregoing definition of the directrices in the following way: They are the geometric loci of those points that correspond to *the same plane* in the various complexes of the relevant two-parameter family, or also the loci of points that are enveloped by those planes that correspond to *the same point* in the various complexes. This is immediately linked to a new analytical determination of the two directrices of a congruence whether we make use of ray coordinates or axial coordinates.

As before (3), we would like to take:

$$\Omega + \mu\,\Omega' = 0$$

to be the equation of the complex group, in which we generally set:

$$\Omega \equiv A\,r + B\,s + C \;-D\,\sigma + E\,\rho + F\,\eta,$$
$$\Omega' \equiv A'r + B's + C' - D'\sigma + E'\rho + F'\eta.$$

The equation of the plane that corresponds to a given point x', y', z' in any complex of the group that is referred to an arbitrary choice of undetermined coefficients will then be the following one (no. **27**):

$$\begin{aligned}(A + Fy' - Ez')\,x + (B - Fx' + Dz')\,y + (C + Ex' - Dy')\,z + (Ax' + By' + Cz')\\ + \mu\,[(A' + F'y' - E'z')\,x + (B' - F'x' + D'z')\,y + (C' + E'x' - D'y')\,z + (A'x' + B'y' + C'z')]\\ = 0.\end{aligned} \tag{58}$$

This equation will always be satisfied, no matter what the value of μ might be, when one has:

$$(A + Fy' - Ez')\,x + (B - Fx' + Dz')\,y + (C + Ex' - Dy')\,z + (Ax' + By' + Cz') = 0, \tag{59}$$

$$\begin{aligned}(A' + F'y' - E'z')\,x + (B' - F'x' + D'z')\,y + (C' + E'x' - D'y')\,z + (A'x' + B'y' + C'z')\\ = 0,\end{aligned} \tag{60}$$

simultaneously. The two planes that are represented by these equations will correspond to the given point in the complexes Ω and Ω'; they will have a common line of intersection with the planes that correspond to the same point in the various complexes of the group. In particular, if the two planes (59) and (60) coincide then all of the planes (58) that correspond to that point will coincide. In order for this to happen, the last two equations must be zero identically, which immediately yields the following *six* relations:

$$\frac{A+Fy'-Ez'}{A'+F'y'-E'z'} = \frac{B-Fx'+Dz'}{B'-F'x'+D'z'} = \frac{C+Ex'-Dy'}{C'+E'x'-D'y'} = \frac{A+By'+Cz'}{A'+B'y'+C'z'}. \tag{61}$$

The point (x', y', z') that is determined by (61) lies on the two directrices of the congruence. We would like to consider its coordinates to be variable and correspondingly drop the prime that they are endowed with from now on.

74. In order to interpret equations (61) geometrically, we would like to let P and P', Q and Q', R and R' denote those planes that are associated with the axes OX, OY, OZ, resp., in the two complexes Ω and Ω' – in other words, they correspond to the points that lie at infinity on these axes – and denote the planes that correspond to the origin in the two complexes by S and S'. The equations of these planes will then be:

$$\left.\begin{array}{ll} A+Fy-Ez=p=0, & A'+F'y-E'z=p'=0, \\ B-Fx+Dz=q=0, & B'-F'x+D'z=q'=0, \\ C+Ex-Dy=r=0, & C'+E'x-D'y=r'=0, \\ Ax+By+Cz=s=0, & A'x+B'y+C'z=s'=0. \end{array}\right\} \tag{62}$$

The following two identities exist between the linear functions p, q, r, s and p', q', r', s' :

$$\left.\begin{array}{l} p\,x+q\,y+r\,z \equiv s, \\ p'x+q'y+r'z \equiv s'. \end{array}\right\} \tag{63}$$

Reciprocally, the special form of the eight linear functions is determined by these two identities.

The straight lines PP', QQ', RR', SS' are four straight lines that correspond in the congruence to those four points that have a distinguished position relative to the chosen coordinate system, namely, the three points that lie at infinity in the directions of the three coordinate axes and the origin. The four lines belong to the congruence. The two directrices of the congruence are determined completely by saying that they cut these four straight lines. The two directrices will be *real* or *imaginary* according to whether the ruled surface that has any three of the four straight lines PP', QQ', RR', SS' as the lines of its generators is or is not cut by the fourth of these lines, respectively.

After introducing the eight symbols, the four-part equation (61) will become:

$$\frac{p}{p'} = \frac{q}{q'} = \frac{r}{r'} = \frac{s}{s'}. \tag{64}$$

As a consequence of the two identities (62), this will immediately yield the three-part equation:

$$\frac{p}{p'} = \frac{q}{q'} = \frac{r}{r'}. \tag{65}$$

This equation will then be sufficient for the determination of the two directrices. It will resolve into the following three equations:

$$\left.\begin{array}{l} pq'=p'q, \\ pr'=p'r, \\ qr'=q'r, \end{array}\right\} \tag{66}$$

which represent three second-order ruled surfaces that go through the two directrices. As a consequence of the four-part equation (64), three new ruled surfaces get added to these three ruled surfaces, which likewise contain the two directrices, and which will be represented by the equations:

$$\left.\begin{aligned} ps' &= p's, \\ ps' &= p's, \\ qs' &= q's. \end{aligned}\right\} \tag{67}$$

The two directrices are determined in this way such that they intersect on any two of the six hyperboloids (66) and (67) (*).

(*) It should be stressed especially that the four-part equation (64) represents a *system of two real or imaginary straight lines* in exactly the same way that the three-part equation:

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

represents a *single* straight line. The foregoing equation then contains five independent constants, one of which is superfluous and comes down expressing to the fact that (x_0, y_0, z_0) is an arbitrary point of the line that is represented. By the determination of the functions that was made, equation (64) contains ten independent constants, including two superfluous ones that are required by the fact that the two complexes Ω and Ω' can be replaced with two other ones in the two-parameter group:

$$\Omega + \mu\,\Omega' = 0.$$

It would seem appropriate to derive this result directly, as well.

Equation (65) will be satisfied when the three equations:

$$\begin{aligned} p &= \lambda\, p', \\ q &= \lambda\, q', \\ r &= \lambda\, r' \end{aligned}$$

happen to be satisfied simultaneously, which, when developed, will go to the following ones:

$$\left.\begin{aligned} (A-\lambda A')+(F-\lambda F')y-(E-\lambda E')z&=0, \\ (B-\lambda B')-(F-\lambda F')x+(D-\lambda D')z&=0, \\ (C-\lambda C')+(E-\lambda E')x-(D+\lambda D')y&=0. \end{aligned}\right\} \tag{68}$$

Those points that simultaneously lie in the planes that are represented by these equations will belong to the locus that is represented by equation (65). However, for a given value of λ, the foregoing three equations will generally contradict each other. This contradiction will be eliminated only when x, y, z become infinitely large, and thus the relevant point goes to infinity. However, it would then not be permissible to derive the fourth equation:

$$s = \lambda s'$$

from the foregoing three.

However, if one has:

$$(A - \lambda A')(D - \lambda D') + (B - \lambda B')(E - \lambda E') + (C - \lambda C')(F - \lambda F') = 0, \tag{69}$$

in particular, then one of the three equations in question will be an algebraic consequence of the other two; the three respective planes will intersect in a straight line. Since the last equation generally gives two values to λ, there will also be two such straight lines. The points of these two straight lines will then be points that lie at infinity whose coordinates satisfy equation (65), and thus (64). *Two straight lines will then be represented by equation* (64), *namely, the two directrices.*

In order to add some clarifications, we would like to start with the theorem that *two ruled surfaces of order and class two that go through two straight lines will intersect in two other straight lines, in addition.*

If C and F, C' and F' are equal to zero, in particular, then setting the four expressions (61) equal to each other will yield the two equations (34) and (36) by which we previously determined the two directrices ([*]).

This theorem will also preserve its meaning when one of the two given straight lines lies at infinity in a given plane. The surfaces will then no longer be two one-sheeted hyperboloids, but two hyperbolic paraboloids whose lines will be parallel to a generator of the given plane. Thus, if the six surfaces (66) and (67) have two fixed straight lines for the lines of one of their two generators then, when composed pair-wise, they will have two other lines for the common lines of their other generator. Conversely, it must then be verified that any two of the six ruled surfaces go through the same two straight lines.

If we return to the functions that are represented by the symbols that enter into equations (66), and set, for the sake of brevity:

$$(E'F - EF')\,x - (D'F - DF')\,y + (D'E - DE')\,z \equiv g,$$
$$(A'B - AB') - (A'F - AF')\,x + (B'F - BF')\,y + [(A'D - AD') - (B'E - BE')]\,z \equiv h_2$$

then the first of the three equations (66) will assume the form:

$$h_2 + g\,z = 0, \tag{70}$$

with which, the last two of equations (65) will go to the following ones:

$$\left.\begin{aligned} h_2 + gz &= 0, \\ h + gz &= 0. \end{aligned}\right\} \tag{71}$$

The functions g are the same in the three equations. The expressions h_1 and h will be obtained immediately when we first switch B and B' with C and C', E and E' with F and F' in h_2, as well as switching y with z with a change of sign and changing the sign of x; we then switch A and A' with C and C', D and D' with F and F', as well as switching x with z, with a change of sign, and changing the sign of y.

The original form of the three equations (66) shows that the three ruled surfaces that are represented by these equations, when taken pair-wise, have PP', QQ', RR' for a common generator. The new form of these equations shows that these three surfaces are hyperbolic paraboloids and have a second common generator that lies at infinity in the plane that is represented by the equation:

$$(E'F - EF')\,x - (D'F - DF')\,y + (D'E - DE')\,z \equiv h = 0.$$

This plane is parallel to the three lines PP', QQ', RR'.

We can then develop equations (67) in the following way:

$$\left.\begin{aligned} (pq' - p'q)\,y + (pr' - p'r)\,z &= 0, \\ (qr' - q'r)\,z + (pq' - p'q)\,x &= 0, \\ (pr' - p'r)\,x + (qr' - q'r)\,y &= 0, \end{aligned}\right\} \tag{72}$$

and then, from the foregoing, we can also write them as follows:

$$\left.\begin{aligned} h_1 z + h_2 y &= 0, \\ h\,z + h_2 x &= 0, \\ h\,y + h_1 x &= 0. \end{aligned}\right\} \tag{73}$$

([*]) Confer "On a New Geometry of Space," Phil. Trans. (1865), pp. 750.

75. Here, we would like to append to equations (61) only the discussion that is connected with those two cases that were left out of the previous discussion as a result of the special coordinate determination. In the one case, one has:

$$\frac{D}{D'} = \frac{E}{E'} = \frac{F}{F'}, \tag{74}$$

while in the other case, one has:

$$\frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'}. \tag{75}$$

In the first case, when we set:

$$\mu = -\frac{D}{D'} = -\frac{E}{E'} = -\frac{F}{F'}, \tag{76}$$

we will get a complex whose equation we can take to be the following three identities:

$$\left.\begin{aligned} (A'D - AD')\,r + (B'D - BD')s + (C'D - CD') = 0, \\ (A'E - AE')\,r + (B'E - BE')s + (C'E - CE') = 0, \\ (A'F - AF')\,r + (B'F - BF')s + (C'F - CF') = 0. \end{aligned}\right\} \tag{77}$$

All lines of the congruence will then be parallel to a plane whose equation we will get when we replace r and s with x / z and y / z, resp., in the foregoing equations. A directrix lies at infinity in the same plane: viz., the line of intersection of parallel planes. We have called a congruence whose one directrix lies at infinity a *parabolic* congruence. The first expressions in (61) give the means to determine the directrix that does not lie at infinity when one sets them equal in pairs:

$$\left.\begin{aligned} (A'B - AB') - (A'F - AF')\,x - (B'F - BF')\,y + [(A'D - AD') + (B'E - BE')]\,z = 0, \\ (A'C - AC') + (A'E - AE')\,x - [(A'D - AD') + (C'F - CF')]\,y + (C'E - CE')\,z = 0, \\ (B'C - BC') + [(B'E - BE') + (C'F - CF')]\,x - (B'D - BD')\,y - (C'D - CD')\,z = 0 \end{aligned}\right\} \tag{78}$$

These three equations represent three planes that go through the directrix. In particular, when F and F' vanish, the conditions in question will reduce to:

$$D'E - D\,E' = 0.$$

We then get the following mutually identical equations for the plane that is parallel to one directrix:

$$\left.\begin{aligned} (A'D - AD')x + (B'D - BD')\,y + (C'D - CD')z = 0, \\ (A'E - AE')x + (B'E - BE')\,y + (C'E - CE')z = 0, \end{aligned}\right\} \tag{79}$$

and the following equations for the other one:

$$\left.\begin{array}{l}(A'B - AB') + [(A'D - AD') + (B'E - BE')]\,z = 0, \\ (A'C - AC') + (A'E - AE')x - (A'D - AD')\,y + (C'E - CE')\,z = 0, \\ (B'C - BC') + (B'E - BE')x - (B'D - BD')\,y + (C'D - CD')\,z = 0.\end{array}\right\} \tag{80}$$

The directrix that does not lie at infinity will then be parallel to the *XY*-plane.

The conditions in question will also be satisfied, in particular, when D and D', E and E' vanish simultaneously. One directrix will then lie at infinity in the previous plane, which will now be represented by the equation:

$$(A'F - AF')\,x + (B'F - BF')\,y + (C'F - CF')\,z = 0. \tag{81}$$

For the one directrix, one gets:

$$\left.\begin{array}{c}(A'B - AB') - (A'F - AF')x - (B'F - BF')y = 0, \\ y = \dfrac{A'C - AC'}{C'F - CF'}, \\ z = -\dfrac{B'C - BC'}{C'F - CF'}.\end{array}\right\} \tag{82}$$

It will then be parallel to the *OZ* axis and cut the *XY* plane at a point whose coordinates are determined by the last two equations. If we substitute these coordinate values in the first of the last three equations then that equation will be satisfied as a consequence of the identity:

$$(A'B - AB')(C'F - CF') + (B'C - BC')(A'F - AF') - (A'C - AC')(B'F - BF') \equiv 0.$$

In particular, if:

$$(A'D - AD') + (B'E - BE') + (C'F - CF') = 0 \tag{83}$$

then that will specify a parabolic congruence. Thereupon, the three planes (78), by whose intersection the finite directrix of the congruence was determined, will become parallel to each other and to the plane in which the second directrix at infinity lies.

The two directrices of a parabolic congruence coincide in a straight line at infinity.

We shall not go further into this special kind of congruence, since it is completely analogous to the case that was treated in number **68**.

The foregoing condition equation (83) will be satisfied due to (62), in particular, when:

$$\left.\begin{array}{r}AD + BE + CF = 0, \\ A'D' + B'E' + C'F' = 0.\end{array}\right\} \tag{84}$$

Thereupon, all complexes of the two-parameter group that is determined by the congruence will be of the particular kind whose parameters vanish. In agreement with that, the three planes (78) will coincide in a single one. Since the axes of all complexes that belong to a parabolic congruence are parallel to each other, we conclude that:

The congruence has infinitely many mutually-parallel directrices that lie in the same plane. The directrix at infinity also lies in that plane.

This case corresponds to the second case of number **68**. It is merely the common intersection of the directrices, which has been shifted to infinity.

76. When the condition equations (63) are fulfilled, the special values of the undetermined coefficient:

$$\mu = -\frac{A}{A'} = -\frac{B}{B'} = -\frac{C}{C'} \tag{85}$$

will correspond to a complex of the two-parameter group that is represented by one of the three following mutually identical equations:

$$\left.\begin{aligned} -(A'D - AD')\sigma + (A'E - AE')\rho + (A'F - AF')\eta = 0, \\ -(B'D - BD')\sigma + (B'E - BE')\rho + (B'F - BF')\eta = 0, \\ -(C'D - CD')\sigma + (C'E - CE')\rho + (C'F - CF')\eta = 0. \end{aligned}\right\} \tag{86}$$

The axis of the complex will be perpendicular at the origin to the plane that will be represented by the last equation when we switch – σ, ρ, η with x, y, z, resp. Since the parameter of the complex is equal to zero, this axis will be one of the two directrices of the congruence. In agreement with that, equations (61) will be satisfied when x, y, z vanish simultaneously. In the present case, these equations will reduce to:

$$\frac{A + Fy - Ez}{A' + F'y - E'z} = \frac{B - Fx + Dz}{B' - F'x + D'z} = \frac{C + Ex - Dy}{C' + E'x - D'y} = \frac{A}{A'} = \frac{B}{B'} \equiv \frac{C}{C'}. \tag{87}$$

If we set the first three of their four terms equal to the fourth one then that will give:

$$\begin{aligned} (C'F - CF')\,y - (C'E - CE')\,z &= 0, \\ (C'F - CF')\,x - (C'D - CD')\,z &= 0, \\ (C'E - CE')\,x - (C'D - CD')\,y &= 0. \end{aligned}$$

These equations, in which we can write B and A in place of C, can be consolidated in the following way:

$$\frac{x}{C'D - CD'} = \frac{y}{C'E - CE'} = \frac{z}{C'F - CF'}, \tag{88}$$

and represent the directrix that goes through the origin.

The condition equations (63) will be satisfied, in particular, when A and A', B and B' vanish. If we set the first three expressions in (61) equal to each other pair-wise then we will get:

$$\left.\begin{array}{r} [(E'F-EF')x-(D'F-DF')y+(D'E-DE')z]\,z=0, \\ [(C'F-CF')+(E'F-EF')x-(D'F-DF')y+(D'E-DE')z]\,y=0, \\ [(C'F-CF')+(E'F-EF')x-(D'F-DF')y+(D'E-DE')z]\,x=0. \end{array}\right\} \tag{89}$$

In order to satisfy the foregoing three equations simultaneously, it will suffice to set:

$$\left.\begin{array}{c} z=0, \\ (C'F-CF')+(E'F-EF')x-(D'F-DF')y=0. \end{array}\right\} \tag{90}$$

The straight line that is represented by these two equations is the second directrix of the congruence. It lies in the XY coordinate plane.

77. If one has:

$$\frac{A}{A'}=\frac{B}{B'}=\frac{C}{C'}, \qquad \frac{D}{D'}=\frac{E}{E'}=\frac{F}{F'}$$

simultaneously then the congruence will be a parabolic one whose directrix that does not lie at infinity will go through the coordinate origin. The equation of the plane that is drawn through the origin and parallel to the lines of the congruence will then be the following one:

$$Ax+By+Cz=0.$$

The directrix that goes through the origin will have the equations:

$$\frac{x}{D}=\frac{y}{E}=\frac{z}{F},$$

and the plane that goes through the origin and is perpendicular to it will have the equation:

$$Dx+Ey+Fz=0.$$

78. If we specialize by setting:

$$A'B-AB'=0, \qquad \frac{D}{D'}=\frac{E}{E'}=\frac{F}{F'} \tag{91}$$

then all of the lines of the parabolic congruence will be parallel to a plane that is perpendicular to the XY coordinate plane, while its directrix will go through the origin.

If:

$$A, A', B, B' = 0, \qquad \frac{D}{D'} = \frac{E}{E'} = \frac{F}{F'} \tag{92}$$

then all lines of the parabolic congruence will be parallel to the *XY* plane.

If:

$$D'E - DE' = 0, \qquad F, F' = 0, \qquad \frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'} \tag{93}$$

then the directrix of the parabolic congruence will lie in the *XY* plane.

If:

$$D, D', E, E' = 0, \qquad \frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'} \tag{93}$$

then the directrix of the parabolic congruence will fall upon the *OZ* coordinate axis.

If:

$$A, A', B, B', D, D', E, E' = 0 \tag{95}$$

then the lines of the parabolic congruence will be parallel to the *XY* coordinate plane, and its directrix will coincide with the *OZ* coordinate axis. With that assumption, the equation of the two-parameter complex group will become:

$$(C + F\eta) + \mu\,(C' + F'\eta) = 0, \tag{96}$$

and for an arbitrary choice of k all complexes of the group will be represented by (*):

$$\eta + k = 0. \tag{97}$$

If:

$$\frac{C}{C'} = \frac{D}{D'} = \frac{E}{E'} \tag{98}$$

then

$$\mu = -\frac{C}{C'} = -\frac{D}{D'} = -\frac{E}{E'} \tag{99}$$

will give the following equation for a complex of the two-parameter group:

$$(A'C - AC')\, r + (B'C - BC')\, s - (C'F - CF')\, \eta = 0.$$

(*) If we set η equal to the expression $\frac{xy' - x'y}{z - z'}$ then the equation in the text will go to the following one:

$$\frac{xy' - x'y}{z - z'} = k,$$

and when k is undetermined, it will give:

$$x'y - xy' = 0, \qquad z - z' = 0$$

simultaneously.

This complex will be of a special kind whose lines all intersect its axis, and that axis, which is a directrix of the congruence, will be parallel to the *OZ* coordinate plane here and will cut the *XY* plane at a point whose equation in line coordinates of that plane will be the following one:

$$(B'C - BC')\,t - (A'C - AC')\,u - (C'F - CF')\,w = 0.$$

[Cf., no. **45** (95)].

In particular, if:

$$\frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'} = \frac{E}{E'} = \frac{F}{F'} \tag{100}$$

then one directrix of the congruence will coincide with the *OZ* axis.

In order to give one last example, we would like to set:

$$\frac{A}{A'} = \frac{B}{B'} = \frac{F}{F'}, \qquad \frac{C}{C'} = \frac{D}{D'} = \frac{E}{E'}. \tag{101}$$

If we then take:

$$\left.\begin{aligned} \mu &= -\frac{A}{A'} = -\frac{B}{B'} = -\frac{F}{F'}, \\ \mu &= -\frac{C}{C'} = -\frac{D}{D'} = -\frac{E}{E'} \end{aligned}\right\} \tag{102}$$

in succession then we will obtain the following equations for two complexes of the group:

$$\Omega + \mu\,\Omega' = 0,$$

namely:

$$\left.\begin{aligned} &(A'C - AC') - (A'D - AD')\,\sigma + (A'E - AE')\,\rho = 0, \\ &(A'C - AC')\,r - (B'C - BC')\,s - (C'F - CF')\,\eta = 0. \end{aligned}\right\} \tag{103}$$

These two equations will reduce to:

$$\left.\begin{aligned} C - D\sigma + E\rho &= 0, \\ Ar + Bs + F\eta &= 0, \end{aligned}\right\} \tag{104}$$

and will represent two complexes of the special kind whose parameters vanish. The axes will be the directrices of the congruence. One of them will lie in the *XY* plane and will be represented in that plane by the equation:

$$C + Ex - Dy = 0. \tag{105}$$

The other will be parallel to the *OZ* axis and will cut the *XY* plane at a point that will be represented by the equation:

$$Bt - Au + F = 0 \tag{106}$$

in that plane.

79. Up to now, we have appealed to rectangular coordinate axes for the analytic representation of a two-parameter complex group and the congruence that it determines, and thereby chose the center of the congruence to be the coordinate origin and the *OZ* axis to be the line that intersects the two directrices at right angles. When we then let the two *OX* and *OY* axes coincide with the two auxiliary axes of the congruence, we obtained the general determination of them in number **69** in the simplest way.

However, we can also take any arbitrary line of the congruence to be the *OZ* axis and the point at which it cuts the central plane to be the origin. If we then displace the two auxiliary axes in that plane parallel to themselves in such a way that they intersect at the new origin then, as before, they will bisect the angle that the two directrices define with each other when projected onto the central plane along *OZ*. It is clear that the equation of the complex group will keep its previous form in the new coordinate determination. If γ is the angle of inclination of the *OZ* axis with respect to the *XY* plane then Δ / sin γ will enter in place of Δ, moreover; that is, the distance from the intersection point of the *OZ* axis with the two directrices to the coordinate origin.

Finally, we can also choose the two *OX* and *OY* axes in the central plane arbitrarily without changing the form of the equation above, in such a way that they will define four harmonics with the projections of the two directrices. We can refer to the *OZ* axis as a principal diameter and the *OX* and *OY* axes as two conjugate auxiliary diameters of the congruence. The conjugate auxiliary diameters will also remain real when the two directrices are imaginary.

Previously, we defined two conjugate congruences in such a way that the axes and auxiliary axes were the same for both, except that the directrices that went through the vertex of the axis had their directions switched. We can replace the axis of the congruence in this definition with an arbitrary diameter. A congruence will then have infinitely many conjugates: Each of its diameters will correspond to one.

80. The foregoing includes the complete discussion of the two congruences that are determined by two-parameter complex groups. In the sequel, we would like to link this discussion with some new considerations that are determined in order to give an intuitive picture to the nature of such congruences.

In connection with number **69**, with the assumption of rectangular coordinates:

$$Ar + Bs - D\sigma + E\rho = 0 \tag{107}$$

represents one of the two complexes of a special kind that have one of the two directrices of the congruence for an axis. We then get the following equation for the determination of the constants in this equation:

$$AD + BE = 0, \tag{108}$$

along with the two equations (44) and (45). The first two condition equations yield:

$$\frac{A^2}{D^2} = \Delta^2 \tan^2 \vartheta, \tag{109}$$

and (44) and (108) give:

$$\frac{B^2}{D^2} = \Delta^2. \tag{110}$$

If we divide the last two equations and consider (108) then that will give:

$$\frac{A^2}{B^2} = \frac{E^2}{D^2} = \tan^2 \vartheta. \tag{111}$$

If we set $D = 1$, while only taking absolute values of A, B, E, and we consider that, from number **66**, the product:

$$ABC = \Delta^2 \tan^2 \vartheta$$

must have a positive value for the case of real directrices then that will give the four following possible determinations of the constants:

$$A = -\Delta \tan \vartheta, \qquad B = +\Delta, \qquad E = -\tan \vartheta, \tag{112}$$
$$A = -\Delta \tan \vartheta, \qquad B = -\Delta, \qquad E = +\tan \vartheta, \tag{113}$$
$$A = +\Delta \tan \vartheta, \qquad B = +\Delta, \qquad E = +\tan \vartheta, \tag{114}$$
$$A = +\Delta \tan \vartheta, \qquad B = -\Delta, \qquad E = -\tan \vartheta. \tag{115}$$

The first two combinations, and likewise the last two, can be derived from each other when one simultaneously changes the signs of Δ and $\tan \vartheta$. The first two combinations thus determine the complexes in question of one of two conjugate congruences, while the last two determine those of the other one. When we set:

$$\left.\begin{aligned} \Xi &\equiv \sigma - \Delta \tan \vartheta \cdot r - \tan \vartheta \cdot \rho + \Delta s = 0, \\ \Xi' &\equiv \sigma - \Delta \tan \vartheta \cdot r + \tan \vartheta \cdot \rho - \Delta s = 0, \end{aligned}\right\} \tag{116}$$

we can then represent the complex group of one congruence by:

$$\Xi + \mu \, \Xi' = 0, \tag{117}$$

and when we set:

$$\left.\begin{aligned} \Xi_1 &\equiv \sigma + \Delta \tan \vartheta \cdot r + \tan \vartheta \cdot \rho + \Delta s = 0, \\ \Xi_1' &\equiv \sigma + \Delta \tan \vartheta \cdot r - \tan \vartheta \cdot \rho - \Delta s = 0, \end{aligned}\right\} \tag{118}$$

we can represent the conjugate congruence by:

$$\Xi_1 + \mu \, \Xi_1' = 0. \tag{119}$$

81. Perhaps it is not inappropriate to also derive the foregoing equations in a direct way. While preserving the coordinate determination up to now, let the equations of the two directrices of a congruence, which are regarded as given, be:

$$\left.\begin{array}{ll} y = \tan\vartheta\cdot x, & z = \Delta, \\ y = -\tan\vartheta\cdot x, & z = -\Delta. \end{array}\right\} \tag{120}$$

One will then be dealing with the determination of two complexes of a special kind whose axes coincide with the two directrices. If we displace the two complexes with their axes in such a way that the latter shift into the *XY* coordinate plane, which coincides with the central plane of the congruence, then we will get the equations of the two complexes in the new position immediately when we switch the given x and y directions with ρ and σ in the equations, resp. In this way, we will get:

$$\left.\begin{array}{l} \sigma = \tan\vartheta\cdot\rho, \\ \sigma = -\tan\vartheta\cdot\rho. \end{array}\right\} \tag{121}$$

If we then return the complex to its original position then we will have to switch (no. **12**) ρ and σ with:

$$\rho + \Delta\cdot r \qquad \text{and} \qquad \sigma + \Delta\cdot s,$$

resp., in the equation of the first one and with:

$$\rho - \Delta\cdot r \qquad \text{and} \qquad \sigma - \Delta\cdot s,$$

resp., in the equation of the second one. After this exchange, we will get:

$$\left.\begin{array}{l} \sigma - \Delta\tan\vartheta\cdot r - \tan\vartheta\cdot\rho + \Delta\cdot s = 0, \\ \sigma - \Delta\tan\vartheta\cdot r + \tan\vartheta\cdot\rho - \Delta\cdot s = 0. \end{array}\right\} \tag{116}$$

These equations are the same as the ones that we just found for the first of the two conjugate congruences; we will get the equations of the second one by changing the sign of $\tan\vartheta$ in (116), (118).

82. We can take the two complexes Ξ and Ξ', instead of the two complexes Ω and Ω', for the determination of the congruence, and thus represent the same complex group that we previously represented by the equation:

$$\Omega + \mu\,\Omega' = 0 \tag{3}$$

by the equation:

$$\Xi + \mu\,\Xi' = 0 \tag{117}$$

from now on. When we develop this equation and set:

$$\frac{1-\mu}{1+\mu} = \lambda, \tag{122}$$

for the sake of brevity, it will become:

$$\sigma - \Delta \tan \vartheta \cdot r - \lambda (\tan \vartheta \cdot \rho - \Delta \cdot s) = 0. \tag{123}$$

When we substitute all possible values for λ, it will represent all of the complexes of the two-parameter group by which the congruence is determined.

Two complexes, in particular, belong to these complexes, which correspond to the values $\lambda = 0$ and $\lambda = \infty$, and when we set:

$$\left.\begin{aligned} \Delta \tan \vartheta &\equiv k^0, \\ \frac{\Delta}{\tan \vartheta} &\equiv k_0, \end{aligned}\right. \tag{124}$$

for the sake of brevity, they can be represented by the two equations:

$$\left.\begin{aligned} \Omega^0 &= +\sigma - k^0 \cdot r = 0, \\ \Omega_0 &= +\rho - k_0 \cdot s = 0. \end{aligned}\right\} \tag{61}$$

The parameters of the two complexes are k^0 and k_0 . Their axes coincide with the two auxiliary axes of the congruence. Their point of intersection is the center of the congruence. Due to their distinguished relationship to the congruence, we would like to emphasize them especially, and call them its *two central complexes.*

When the conjugate congruences enter in place of the given one, the axes of the two central complexes, which coincide with the common auxiliary axes of the two congruences, will remain the same. The absolute values of their two parameters will not change, but only as a result of changing the sign of tan ϑ simultaneously with the sign of both parameters.

If we set:

$$\lambda \tan \vartheta \equiv \lambda_0 ,$$

for the sake of brevity, then the equation of the complex group will assume the following simple form:

$$\Omega^0 + \lambda_0\, \Omega_0 \equiv (\sigma - k^0\, r) + \lambda_0\, (\rho + k_0\, s) = 0. \tag{126}$$

One has:

$$\left.\begin{aligned} k^0 k_0 &= -\Delta^2, \\ \frac{k^0}{k_0} &= -\tan^2 \vartheta, \end{aligned}\right\} \tag{127}$$

and therefore:

$$\left.\begin{aligned} k^0 - k_0 &= \frac{\Delta}{\sin\vartheta\cos\vartheta} = \frac{2\Delta}{\sin 2\vartheta}, \\ k^0 + k_0 &= \frac{-2\Delta}{\tan 2\vartheta}, \\ \frac{k^0 - k_0}{k^0 + k_0} &= -\cos 2\vartheta. \end{aligned}\right\} \tag{128}$$

Here, we get the *two* parameters of the central complex of the congruence, in addition to the *six* constants of its position, for the determination of that congruence.

83. If we start with the two equations:

$$\begin{aligned} \Omega &\equiv Ar + Bs - D\sigma + E\rho = 0, \\ \Omega' &\equiv A'r + B's - D'\sigma + E'\rho = 0, \end{aligned} \tag{129}$$

by which we previously determined the congruence, and between whose coefficients the relations will exist:

$$\begin{aligned} A'D - AD' &= 0, \\ B'E - BE' &= 0, \end{aligned}$$

when the auxiliary axes of the congruence are chosen to be the OX and OY coordinate axes, then we can easily derive the equation of the two central complexes from this. To that end, we merely need to subtract the two equations, after we first multiply the first one by B' and the second one by B, and then multiply the first one by A' and the second one by A. In that way, when we consider the foregoing condition equations, we will get:

$$\begin{aligned} (B'D - BD')\,\sigma + (A'B - AB')\,r &= 0, \\ (A'E - AE')\,\rho + (A'B - AB')\,s &= 0, \end{aligned}$$

from which:

$$\left.\begin{aligned} k^0 &= -\frac{A'B - AB'}{B'D - BD'}, \\ k_0 &= \frac{A'B - AB'}{A'E - AE'}. \end{aligned}\right\} \tag{129}$$

84. In order to determine one of the complexes in the two-parameter group that we would like to represent by the equation:

$$Ar + Bs - D\sigma + E\rho = 0,$$

we must know its parameter k, the value of z for the point at which its axis cuts the OZ axis, and the angle ω that the direction of this axis defines with the direction of the OX

axis. For the determination of these constants, we can start in the same simple way, once by assuming that the two central complexes are known, and then by assuming that the two directrices of the congruence are known. Corresponding to that, if we first set the last equation to equation (126) identically, and then set it equal to equation (123) identically then that will yield the following relations:

$$\left.\begin{aligned} A &= -\;k^0 &&= -\Delta\tan\vartheta, \\ B &= \;\;\lambda_0 k_0 &&= \;\;\lambda\Delta, \\ &\;\;\;\;D = -1, && \\ E &= \;\;\lambda_0 &&= -\lambda\tan\vartheta. \end{aligned}\right\} \tag{130}$$

If we set C and F equal to zero then the general equations of the previous paragraphs (15), (16), and (53) will yield

$$\left.\begin{aligned} \tan\omega &= \frac{E}{D}, \\ z &= \frac{AE - BD}{E^2 + D^2}, \\ k &= \frac{AD + BE}{E^2 + D^2} \end{aligned}\right\} \tag{131}$$

for the complex in question. If we introduce k^0, k_0, and λ_0 then that will give:

$$\tan\omega = -\lambda_0, \tag{132}$$

$$z = -\frac{\lambda}{1+\lambda_0^2}(k^0 - k_0), \tag{133}$$

$$k = \frac{k^0 + \lambda_0^2 k_0}{1+\lambda_0^2}, \tag{134}$$

and thus, when we eliminate λ_0:

$$z = (k^0 - k_0)\sin\omega\cos\omega, \tag{135}$$

$$k = k^0\cos^2\omega + k^0\sin^2\omega, \tag{136}$$

and finally, after eliminating ω:

$$z^2 + (k - k^0)(k - k_0) = 0. \tag{137}$$

When we introduce the constants of the two directrices, equations (135) and (136) will go to the following ones:

$$z = \Delta\cdot\frac{\sin 2\omega}{\sin 2\vartheta}, \tag{138}$$

$$k = -2\Delta\cdot\frac{\sin(\omega+\vartheta)\cdot\sin(\omega-\vartheta)}{\sin 2\vartheta}. \tag{139}$$

Each value of ω corresponds to a single value of z in (135), (138), and a value of the complex parameter in (136), (139). However, since each value of z corresponds to two directions of the complex axis and two values of k, which can be real or imaginary, there will be a maximum distance between the complex axes and the central plane. Equation (138) will give this maximum immediately, corresponding to the angle $\omega = \pi / 4$:

$$z = \frac{\Delta}{\sin 2\vartheta} = \tfrac{1}{2}(k^0 - k_0), \tag{140}$$

and simultaneously, from (136), one will have:

$$k = -\frac{\Delta}{\tan 2\vartheta} = \tfrac{1}{2}(k^0 + k_0). \tag{141}$$

85. The discussion of the foregoing analytic developments yields a series of geometric results.

In number **64**, we gave the OX axis one of the two directions that bisect the acute and obtuse vertex angles that are defined by the two directrices of a given congruence, and choose the positive half of this axis arbitrarily. We calculate the angle between the positive half of the OX axis and the positive half of the OY axis. The positive half of OY is determined when we denote that one of the two directions that corresponds to a positive Z by ϑ. $(\tfrac{1}{2}\pi - \vartheta)$ enters the two-fold coordinate system in place of ϑ, and thus (124) reciprocally switches the values of k^0 and k_0 with a change of sign. We would like to choose the coordinate system in such a way that OX bisects the *acute* vertex angle, which is defined by the projections of the two directrices in the central plane of the congruence. k^0 is then positive in (128), k_0 is negative, and since $\tan 2\vartheta > 0$:

$$k^0 + k_0 < 0.$$

The parameter of the central complex whose axis lies along OX is k^0 and positive, while the parameter of the central complex whose axis lies along OY is k_0 and negative. The absolute value of the second parameter is taken to be greater than that of the first.

Previously, along with the given congruence, we constructed a second one that we called its conjugate (no. **69**), and which we obtained when we simultaneously changed the signs of the two parameters k^0 and k_0 of the given central complex, or – what amounts to the same thing – when Δ remained the same and ϑ changed its sign. Along with the given congruence, one can define yet a third one, which we would like to call its *adjoint*, and which one obtains when k^0 and k_0 are switched with each other with their signs changed, as well. It will emerge from (124) by letting $\left(\frac{\pi}{2} - \vartheta\right)$ enter in place of ϑ. Finally, we obtain yet a fourth congruence that depends upon the given one immediately when we first take the conjugate of the given one and then take the adjoint of that one, which amounts to switching k^0 and k_0 without changing sign, or – what amounts to the same thing – replacing ϑ with $\left(\frac{\pi}{2}-\vartheta\right)$.

Under our assumption, 2ϑ is an acute angle for the given congruence; the corresponding angle $(\pi - 2\vartheta)$ will be obtuse for the adjoint congruence. If we denote the parameters of the two central complexes of the adjoint congruence by (k^0) and (k_0), to distinguish them, then we will have:

$$(k^0) + (k_0) > 0,$$

and since (k^0) is positive and (k_0) is negative, (k^0) will have a larger absolute value than (k_0).

The axis, the central plane and the two auxiliary axes in it, as well as the distance between the two directrices, will remain the same for all four congruences.

86. If we denote the coordinates of any point on the axis of any complex in the two-parameter group by x, y, z then we will have:

$$\cos^2 \omega = \frac{x^2}{x^2+y^2}, \qquad \sin^2 \omega = \frac{y^2}{x^2+y^2},$$

with which, equation (135) will go to the following one:

$$(x^2 - y^2)\, z \pm (k^0 - k_0)\, xy = 0. \tag{142}$$

This equation represents the ruled surface that is defined by the axes of the complexes of the two-parameter group that determines the congruence.

According to whether we take one or the other of the two signs in the foregoing equations, it will refer to the given congruence or its conjugate. Should it refer to the given one, then from the coordinate determination that we chose, according to which, $(k^0 - k_0)$ is positive when take y / x to be equal to the tangent of the angle ϑ – and thus, positive – the value of z will also be positive and equal to $+\Delta$. We must then choose the lower sign, and thus obtain:

$$(x^2 - y^2)\, z - (k^0 - k_0)\, xy = 0. \tag{143}$$

The only constant that enters into this equation – viz., $(k^0 - k_0)$ – is the sum of the absolute values of the parameters of the central complex. However, from (140), this sum will also be twice the maximum of z, and thus equal to the height h of the surface that is included by two planes, through whose midpoint the central plane goes. The surface will be cut by each intermediate plane along two straight lines that are perpendicular to each other in the central plane, in which coincide with the two axes of the central complex. When the intersecting planes of the central plane move away on the positive side, the angle that they define with each other will always become smaller, until it vanishes for $\omega = \pi / 4$ in the limit plane, and the two lines then coalesce into a single one. When the intersecting planes of the central plane move away on the negative side, the angle that the two lines of intersection define with each other will become an obtuse one, until it becomes equal to π in the other limit plane, which corresponds to $\omega = -\pi / 4$, and then the two lines of intersection will again coalesce. Equation (135) shows that lines that bisect the angle of the two lines of intersection in an arbitrary plane that is parallel to the central plane will lie in those two planes that define the same angle with the *XY* and *YZ* coordinate planes (*).

Since the given congruence depends upon *two* constants k^0 and k_0, but the surface in question depends upon just *one* constant that is the difference of the latter two, this surface will have the same relationship to infinitely many congruences, so it will be the geometric locus of the relevant complex axes. Among these congruences, one also finds the adjoint of the given one; we can then exchange k^0 and k_0, while changing their signs, without changing the equation of surface. This surface thus has the same relationship to the given congruence and its adjoint.

87. We would like to determine the complex in the two-parameter group geometrically in such a way that we apply the parameters (with consideration given to their signs) that correspond to its axes that all intersect *OZ* to the axes. We will then get a curve that is inscribed in the ruled surface that was considered in the previous number, by which the entire two-parameter complex group was determined. We would like to call this curve the *characteristic curve of the congruence*. It will suffice to know the projection of that curve onto the *XY* coordinate plane; each point of the projection will then correspond to a single real point on the surface.

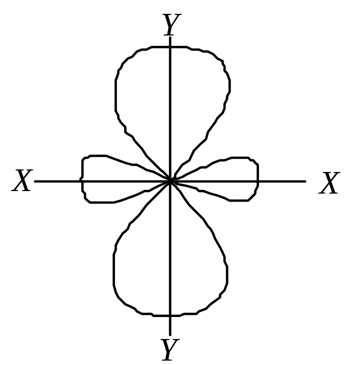

Figure 7.

Equation (136) will be the equation of that projection in polar coordinates when we consider k to be the guiding ray in it and simultaneously consider ω to be variable. When we set:

$$k = \pm\sqrt{x^2 + y^2}\,, \qquad \cos\omega = \frac{x}{k}, \qquad \sin\omega = \frac{y}{k},$$

this equation will go to the following one:

$$(x^2 + y^2)^3 = (k^0 x^2 + k_0 y^2)^2, \tag{144}$$

and will thus represent the projected curve in ordinary point coordinates. This equation will remain the same when we simultaneously change k^0 and k_0 . The curve that is represented by the equation will then have the same relationship to the given congruence and its conjugate. It consists (Fig. 7) of four pair-wise equal loops, which lie inside of four of the vertex angles that are defined by the projection of the two directrices.

(*) Which is a great simplification for the models that I can construct for this and similar surfaces, and that appeal to geometric intuition.

88. When we treat equation (137) in the same way that we treated equation (136) in the previous number, it will produce a new surface that goes through the curve of double curvature that was just determined. This equation will be converted into the following one:

$$(x^2 + y^2 + z^2 + k^0\, k_0)^2 = (k^0 + k_0)^2\, (x^2 + y^2), \tag{145}$$

and will represent a fourth-order surface. This surface will be a surface of revolution whose axis is OZ. For its meridian curves in the XZ plane, we get:

$$(x^2 + z^2 + k^0\, k_0)^2 = (k^0 + k_0)^2\, x^2,$$

when we let y vanish, and:

$$z^2 + \left(x \pm \frac{k^0 + k_0}{2}\right)^2 = \left(\frac{k^0 - k_0}{2}\right)^2 .$$

This equation represents a system of two circles whose two-sided radius is:

$$\tfrac{1}{2}(k^0 - k_0) \equiv h, \tag{146}$$

and whose center on the OX axis has the distance:

$$-\tfrac{1}{2}(k^0 + k_0) \equiv c \tag{147}$$

from the OZ axis, on the opposite side. The two circles intersect along OZ in those two points at which that axis is cut by the two directrices (*).

The new surface will then be generated by rotating a circle around the axis of the congruence. Its radius is equal to one-half the height of the ruled surface (142). Its center lies in the central plane, and its distance from the OZ axis is equal to the parameter of the complex whose axis falls in the limit plane of the ruled surface (142). The surface of rotation lies completely between those planes and will be contacted by each of them along the circumference of a circle.

Equation (145) will remain unchanged when k^0 and k_0 are exchanged reciprocally, as well as when both constants change their signs simultaneously. The surface of rotation thus refers simultaneously to the given congruence, its conjugate, its adjoint, and the one that is conjugate adjoint to it.

89. To summarize, we get the following determination of the axes of the two-parameter complex group by which the given congruence is determined: We have assumed that $\vartheta < \pi / 4$. We would like to start with the value $\omega = 0$, for which, the complex axis will lie in the central plane and the complex parameter will attain its

(*) We can remark, in passing, that the intersection points of the two directrices with the axis of the congruence are the two focal points of an ellipsoid of rotation whose center coincides with the center of the congruence whose rotational axis lies along OZ and is equal to h, while the radius of its equatorial circle has the value c.

positive maximum k. When ω increases from 0 to $+ \vartheta$, the complex axis will move away from the central plane on its positive side, while the complex parameter will decrease. When ω then increases from ϑ to $\pi / 4$, z (which is the distance from the central plane that goes through Δ, where the complex axis will coincide with one of the directrices of the congruence) will increase until it attains its maximum of $\frac{1}{2}(k^0 - k_0) \equiv h$, while the complex parameter will go through zero and take on negative values, and the limit will be equal to $\frac{1}{2}(k^0 + k_0) \equiv c$. If the complex axis advances in such a way as to rotate around OZ from $\omega = \pi / 4$ to $\omega = \pi / 2$ then it will again approach the central plane, while the negative value of the complex parameter will increase until it attains the maximum of k_0 in that plane. If the rotation continues from $\omega = \pi / 2$ to $\omega = 3\pi / 4$ then the axis will again move away from the central plane on its negative side until it attains its negative maximum (viz., $- h$) in the limit z, while the negative value of the complex parameter will decrease and take the value c in the limit. Under the rotation from $\omega = 3\pi / 4$ to $\omega = \pi - \vartheta$, the axis will again approach the central plane until it coincides with the second axis of the congruence, corresponding to $z = - \Delta$, while the negative complex parameter will decrease until it vanishes. If the axis completes its rotation around OZ when ω increases from $(\pi - \vartheta)$ to π then it will once more approach the central plane until it again assumes the position from which we started, while the complex parameter, which changes its sign, will increase and, in turn attain its positive maximum in the central plane.

90. In order to achieve symmetry in this investigation, we must consider the given congruence simultaneously with the aforementioned other three that depend upon it immediately. That will first demand that we reconsider the ruled surfaces that are determined by equation (142) with the double sign. We can represent the system of these two surfaces by the single equation:

$$(x^2 + y^2)^2 z^2 = (k^0 - k_0)^2 x^2 y^2. \tag{148}$$

The complete intersection of the surface of rotation (145) with the two ruled surfaces decomposes into two algebraic space curves, one of which lies on each of the two surfaces. The projections of the two spatial intersection curves onto the central plane cover it, and thus resolve into two sixth-degree curves, one of which will be represented by the previous equation:

$$(x^2 + y^2)^3 = (k^0 x^2 + k_0 y^2)^2, \tag{149}$$

while the other one will be represented by the following one:

$$(x^2 + y^2)^3 = (k_0 x^2 + k^0 y^2)^2. \tag{150}$$

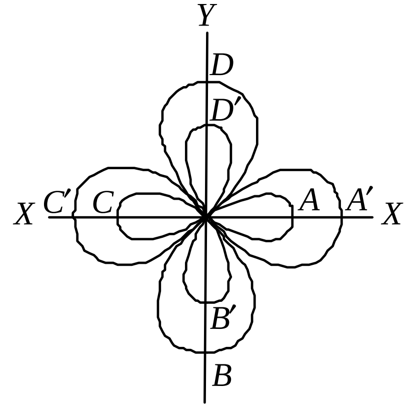

Figure 8.

Under the assumption of real directrices that we have used up to now, each of the two curves (Figure 8) will consist of four loops that define a four-fold point at the coordinate origin. When one rotates one of the two curves in its plane around the origin through an angle of $\pi / 2$, one will obtain the other one.

The characteristic curve of the given congruence lies upon the first ruled surface (142), but it does not define a closed path in it. Its projection onto the central plane of the congruence defines only one-half *AOBOC* of the curve (149). It is limited by two points that lie on *OX* on both sides of the origin and at equal distances from it.

The characteristic curve of the *adjoint* congruence lies on the same ruled surface, and its projection is the *A′OB′OC′* half of the curve (150). Analogously to before, it breaks into two points on *OX*.

The characteristic curve of the *conjugate* congruence lies in the second ruled surface (142). Its projection defines the *CODOA* half of the curve (149), which extends the projection of the characteristic curve of the given congruence to the complete curve (149). The two characteristic curves break into the same two points *A* and *D* on *OX*.

The characteristic curve of the *conjugate-adjoint* congruence lies in the second ruled surface, and its projection defines the second half *C′OD′OA′* of the curve (150), which extends the projection of the characteristic curve of the adjoint congruence to the complete, algebraic curve.

The second projecting cylinder, which cuts the central plane in the curve (150), cuts the first ruled surface along a closed curve that consists of two components: viz., the characteristic curve of the adjoint congruence and the mirror image of the characteristic curve of the conjugate-adjoint one.

Likewise, the characteristic curve of the conjugate-adjoint congruence and the mirror image of the characteristic curve of the adjoint congruence define a closed curve on the second ruled surface, which, like the foregoing one, has the curve (150) for its projection.

The four closed curves thus determined define the complete real part of an algebraic space curve. Each of them has two double points, which fall upon the common axis of the four congruences in those two points at which that axis is cut by the directrices of the congruences. Each space curve will be divided into four branches at these two points, such that we obtain sixteen such curve branches, in total, which all emanate from the two points on the axis. The eight curve branches on one ruled surface and the eight curve branches on the second ruled surface have the eight loops of the two curves (149) and (150) for common projections. Those curve branches that have the large loops for their projections cut the limiting lines of the two ruled surfaces in points that have the same distance from the axis; those curve branches whose projections are the small ovals do not cut the axis.

The four closed curves lie completely on the surface of rotation that is represented by equation (145).

91. If we start with a given ruled surface (143) then we can carry the characteristic curves of infinitely many congruences on it. Each of these curves is determined by an intersection with a surface of rotation that is included with the ruled surface between limit planes. These limit planes contact the ruled surface in a straight line and the surface of rotation in a circle. The directions of the two contact lines that cut the *OZ* axis are perpendicular to each other; the two contact circles have their centers on the axis and their radii equal to each other. The individual surface of rotation is determined completely by this radius. This radius is equal to the distance from the center of the circle that is generated by rotating the surface of rotation around OZ to that axis. If we successively give the center of this circle in the central plane, whose radius always remains the same, all possible distance from the OZ axis then we will get all possible surfaces of rotation and a congruence that corresponds to each of them.

If we preserve the previous notation then the difference of the parameters of the two central complexes will not change; it will be:

$$k^0 - k_0 = 2h, \tag{151}$$

while the sum of these constants will vary from one congruence to another in such a way that:

$$k^0 + k_0 = -2c. \tag{152}$$

Thus:

$$k^0 = h - c, \qquad k_0 = -(h + c), \tag{153}$$

$$\Delta^2 = -k^0\, k_0 = h^2 - c^2, \tag{154}$$

$$\tan^2 \vartheta = -\frac{k^0}{k_0} = \frac{h-c}{h+c}. \tag{155}$$

If we then successively take the constant c, by which the instantaneous surface of rotation is determined, to have all possible positive values then each value of that constant will correspond to a characteristic curve on the given ruled surface. The curves that correspond to the instantaneous adjoint congruences will possess the same absolute values of c, but with the opposite signs.

92. If $c = 0$ then one will get:

$$k^0 = -k_0 = h = \Delta, \qquad \tan^2 \vartheta = 1. \tag{156}$$

The two directrices will then lie in the planes that limit the ruled surface and have the largest possible distance from the central plane. Their two directions will be perpendicular to each other and will be the same for both adjoint congruences. The equation of the surface of rotation in this case will be:

$$x^2 + y^2 + z^2 = h^2. \tag{157}$$

When c increases, the absolute value of the negative k^0 will increase, while that of the positive k_0 will decrease. The distance of the two directrices from the central plane will decrease, and the angle that their two directions make with each other will always increase beyond a right angle. Within the limits of $2h$ and 0, we can choose the distance between the two directrices of a congruence arbitrarily. The circle that generates the surface of rotation will then cut the axis of rotation at two real points.

At the limit $c = h$, one has:

$$k^0 = 0, \qquad k_0 = -2h, \qquad \Delta = 0, \qquad \tan \vartheta = 0. \tag{158}$$

The parameter of one of the two central complexes is equal to zero. The two directrices of the congruence coincide with the OX axis. The circle that generates the surface of rotation contacts the axis of rotation OZ at the coordinate origin O. The equation of the surface of rotation becomes:

$$(x^2 + y^2 + z^2)^2 = h^2 (x^2 + y^2). \tag{159}$$

As before, the characteristic curve determines the parameter and the axis position of infinitely many complexes. This is the first case that was treated in no. **68**.

When $c > h$, k^0 will become negative, as is k_0 . The two directices of the congruence, like their adjoints, will become imaginary; either their directions or their intersections with OZ will remain real. The surface of rotation will define a complete circuit, as long as the generating circle does not cut the OZ axis, and its intersection curve with the ruled surface will be drawn around that axis without cutting it.

When the absolute value of the negative k^0 increases, c (the distance to the center of the generating circle) will always grow larger, while the ratio of the two parameters of the central complex of the congruence will approach unity. In the limit, one will have:

$$\tan^2 \vartheta = -1. \tag{160}$$

93. We can infer an uncommon, simple process for carrying the characteristic curves of all congruence on the given ruled surface from the equation:

$$z^2 + (k - k^0)(k - k_0) = 0. \tag{137}$$

Under the transition from one characteristic curve to another, the two constants k^0 and k_0 will increase by the same quantity, which might also be the value of z. The foregoing equation will always be satisfied during it when the variables k themselves take on the same increases.

Therefore, let any characteristic curve that is inscribed on a ruled surface be given, and we can take it to be, in particular, the one along which the ruled surface is cut by a sphere that has the same altitude above its diameter and also the same center as it. We will then successively obtain all characteristic curves when we approach all intersection points of the given curves with the generators of the ruled surface on these generators of the axis by a constant increment or go away from it.

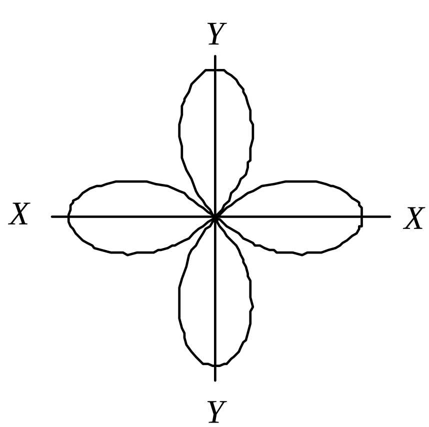

Figure 9.

We will arrive at the same construction in a geometric way when we ponder the fact that a characteristic curve is the geometric locus of those points at which the generators of the surface is cut by the circle that describes the surface of rotation, and the fact that from one characteristic curve to another, the center of the circle (whose plane goes through *OZ*) will approach the *OZ* axis or move away from it.

94. When we project the characteristic curves of two conjugate congruences onto the central plane for the case in which the surface of rotation coincides with a spherical outer surface, we will obtain the equation:

$$(x^2 + y^2)^2 = h^2 (x^2 - y^2)^2 \tag{161}$$

for the projection. Like the general curves (149) or (150), the projected curve will have a four-fold point at the origin; the four loops that it consists of are equal. With our assumption, the two curves (149) and (150) will coincide in the one (161) (Fig. 9).

Under the second transition (Fig. 10), where the two directrices coincide with *OZ*, the two equations (149) and (150) will go to the following ones:

$$(x^2 + y^2)^3 = 4h^2 y^4, \tag{162}$$
$$(x^2 + y^2)^3 = 4h^2 x^4. \tag{163}$$

When the value of ϑ that corresponds to increasing c by $\pm h$ gradually gets larger by π / 4 in such a way that in the one case it decreases until it vanishes and in the other case it approaches π / 2, two loops of the curve (161) will gradually vanish when the points at which, in one case, the *OX* axis and in the other case, the *OY* axis are cut by it, always move closer to the point *O*, while likewise its tangents that intersect at *O* always approach the respective coordinate axes and coincide with them in the limit. The curve will then consist of two equal ovals that contact one of the two auxiliary axes on opposite sides.

Finally, when c grows beyond h and ϑ becomes imaginary, the curve will surround the origin *O*, at which four isolated points of it will coincide (Fig. 11), moreover.

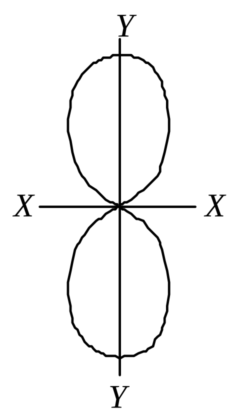

Figure 10.

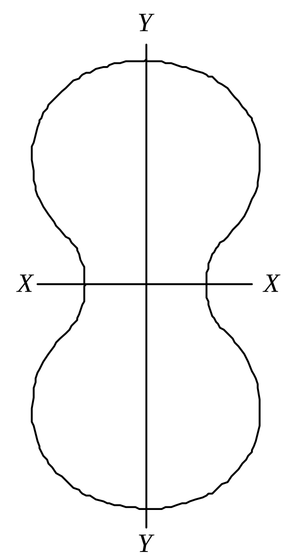

Figure 11.

The curves that are represented by each of the two equations (149) and (150) for different choices of constants, like the space curves whose projections they are, will all be obtained when one of them is given. When we consider k to be a guiding ray in the equation:

$$k = k^0 \cos^2 \omega + k_0 \sin^2 \omega, \tag{136}$$

this equation will be the equation of the same curve in polar coordinates that we previously represented by equation (149). One of these curves is given by definite values of k^0 and k_0, and we will get all of the remaining ones when we let these constants increase by the same quantity δ. However, we will then have:

$$k + \delta = (k^0 + \delta) \cos^2 \omega + (k_0 + \delta) \sin^2 \omega ; \tag{164}$$

i.e., all guiding rays will increase by δ from one curve to another.

The equation of the curve (162) becomes:

$$k = 2h \sin^2 \omega \tag{165}$$

in polar coordinates, so this curve can be constructed in an exceptionally simple way with the help of a curve with a diameter $2h$. The construction of all curves (149) and (150) is then given by that.

95. The discussion of the complexes of a two-parameter group:

$$\Omega + \mu \, \Omega' = 0$$

is still lacking for the case in which a parabolic congruence is determined by this group. For the determination of such a congruence, it is sufficient to know its single directrix and a plane that is parallel to all of its lines. We would like to take the directrix to be the *OX* coordinate axis. Among the complexes of the group, one will then find one whose equation is:

$$\sigma = 0. \tag{166}$$

We would further like to draw the *ZX* coordinate plane through *OX* in such a way that it is perpendicular to the plane that all lines of the congruence are parallel to. We can then give the equation of that plane the following form:

$$x + \lambda z = 0, \tag{167}$$

in which λ means a given constant. We will then get:

$$r + \lambda = 0, \tag{168}$$

in order to express a complex that consists of lines that are all parallel to the plane in question, and which will then likewise belong to the congruence.

When take Ω and Ω' to be the two complexes thus determined we will get:

$$s + \mu \, (r + \lambda) = 0 \tag{169}$$

for the equation of the group. This equation says that all lines of the parabolic congruence cut the *OX* axis and are parallel to the plane (167).

The axes of the various complexes that define the parabolic congruence all lie in the *XY* plane and cut out a piece:

$$y = -\mu\,\lambda \tag{170}$$

from the *OY* axis in that plane (no. **31**). The respective parameter is:

$$k = -\mu, \tag{171}$$

and therefore:

$$y = \lambda\,k \tag{172}$$

will be the equation of the characteristic curve of the parabolic congruence. When we lay k along the complex axis from *OY* outward, and thus as x, this equation will represent a straight line in *XY* that defines the same angle with *OX* as the plane (167) does with the *YZ* coordinate plane.

96. In connection with the geometric considerations of number **79**, in analogy to what happened in number **46** for a single complex, we can deduce some analytic developments that aim to put the equation for a congruence into its simplest expression in oblique coordinate, as well. Let:

$$\sigma - k^0\,r = 0, \quad \rho + k_0\,s = 0 \tag{173}$$

be the two central complexes by which a congruence is determined in rectangular coordinates. We would like to place the origin in the central plane at an arbitrary point (x^0, y^0). To that end, if we first displace the coordinate system parallel to itself in the direction of *OY* through an increment y^0 then the equation of the second complex, which has *OY* for its axis, will remain unchanged, while the equation of the first complex will go to the following one:

$$s - \frac{k^0}{\sin\delta^0}\cdot r = 0, \tag{174}$$

in which:

$$y^0 \sin\delta^0 - k^0 \cos\delta^0 = 0. \tag{175}$$

The *OX* axis will remain a diameter of the first congruence under this displacement. In order for us to rotate the *OZ* axis in the *XZ* plane around *OY* in such a way that *OX* defines the angle δ^0 with *OZ* in the new position, *YZ* will have to be the plane that is associated with the diameter *OX*, and δ^0 will have to be the angle of inclination of the diameter out of its associated plane. The angle *YOZ* will remain a right angle.

If we then displace the axis system parallel to *OX* through an increment x_0 then the equation of the first complex (173) will remain unchanged, while the equation of the second complex will go to the following one:

$$\rho + \frac{k_0}{\sin\delta_0}\cdot s = 0, \tag{176}$$

in which:

$$x_0 \sin\delta_0 + k_0 \cos\delta_0 = 0. \tag{177}$$

Here, the angle δ_0 is the inclination angle of *OY* with respect to *XZ*, so it is the inclination angle of the diameter of the complex that falls in *OY* with respect to its associated plane. The angle *XOZ* remains a right angle.

The equations of the two planes that are associated in the two complexes with *OX* – the diameter of the first one – and *OY* – the diameter of the second complex have the equations:

$$\left.\begin{aligned} x &= \cot\delta^0 \cdot z, \\ y &= \cot\delta^0 \cdot z. \end{aligned}\right\} \tag{178}$$

Finally, if we take the *OZ* axis to be the line of intersection of the two associated planes then the *YZ* and *XZ* planes, which are conjugate to *OX* and *OY*, resp., will no longer be perpendicular to each other in the analytic representation of the two axes. If we denote the angles *YOZ* and *XOZ* by ε^0 and ε_0, resp. then we will have:

$$\sin\delta^0 \sin\varepsilon^0 = \sin\delta_0 \sin\varepsilon_0 = \sin\delta, \tag{179}$$

when we let δ denote the inclination angle of the new *OZ* axis with respect to *XY*.

If we then take any two diameters of the central complex to be *OX* and *OY*, instead of its two axes, when we displace the original coordinate axes parallel to themselves, and take *OZ* to be the intersection of two planes that are associated with those diameters, then the equations of that complex will become:

$$\sigma - \frac{k^0}{\sin\delta}\cdot r = 0, \qquad \rho + \frac{k_0}{\sin\delta}\cdot s = 0, \tag{180}$$

and the same congruence that was previously determined by the equation:

$$(\sigma - k^0\, r) + \mu\,(\rho + k_0\, s) = 0$$

will now be determined by the equation of entirely the same form:

$$\left(\sigma - \frac{k^0}{\sin\delta}\cdot r\right) + \mu\left(\rho + \frac{k_0}{\sin\delta}\cdot s\right) = 0 \tag{181}$$

in the new coordinate system.

97. If we eliminate $\cot\delta^0$ and $\cot\delta_0$ from (178) using (175) and (177) then we will get:

$$\frac{y}{x} = -\frac{k^0 x_0}{k_0 y^0},$$

or

$$\tan\alpha\cdot\tan\alpha' = -\frac{k^0}{k_0}, \tag{182}$$

if α and α' are the angles that, on the one hand, the line that links the new origin to the old one, and on the other hand, the projection of the new principal diameter of the congruence, define with the *OX* axis. In particular, if $k^0 = k_0$ then the two lines will be perpendicular to each other for any change of origin.

We have:

$$\sin^2 \delta = \frac{1}{1 + \cot^2 \delta^0 + \cot^2 \delta_0},$$

so:

$$\frac{1}{\sin^2 \delta} = 1 + \cot^2 \delta^0 + \cot^2 \delta_0,$$

and with consideration given to (175) and (177):

$$\frac{x_0^2}{k_0^2} + \frac{y^{02}}{k^{02}} = \frac{1}{\tan^2 \delta}, \tag{183}$$

or:

$$k^{02} x_0^2 + k_0^2 y^{02} = \Delta^2 \cdot \frac{1}{\tan^2 \delta}. \tag{184}$$

It follows from this that δ will be constant when the new origin is chosen to be on an ellipse in the central plane whose axes fall on *OX* and *OY* in the way that k_0 relates to k^0, resp.

The principal diameters of a congruence that have the same inclination with respect to the central plane cut that plane in the points of an ellipse.

98. Let an *imaginary congruence* be given by two imaginary complexes. Under the assumption of rectangular coordinate axes, we would like to take the equation:

$$(\sigma - k_1 r\sqrt{-1}) + \mu(\rho + k_2 s\sqrt{-1}) = 0 \tag{185}$$

to be the symbol of such a congruence. When we simultaneously change the signs of k_1 and k_2, this equation will go to the following one:

$$(\sigma + k_1 r\sqrt{-1}) + \mu(\rho - k_2 s\sqrt{-1}) = 0 \tag{186}$$

It will then refer to yet a second imaginary congruence. In analogy with the above, we will refer to the two congruences as two conjugate imaginary congruences. The equations of the two congruences can be combined into the following quadratic equation:

$$(\sigma + \mu\rho)^2 + (k_1\, r - \mu\, k_2\, s)^2 = 0. \tag{187}$$

The two central complexes of the two congruences are:

$$\sigma \mp k_1 r\sqrt{-1} = 0, \qquad \rho \pm k_2 s\sqrt{-1} = 0.$$

The two congruences have a real common principal axis and two common real auxiliary axes. The distance between the two directrices and the angle that the two directrices define will be equal for both of them. If we call that distance Δ and that angle ϑ then, from number **82**, we will have:

$$\begin{aligned} k_1\, k_2 &= \Delta^2, \\ \frac{k_1}{k_2} &= -\tan^2 \vartheta. \end{aligned} \tag{188}$$

If k_1 and k_2 agree in sign then Δ will be real and tan ϑ will be imaginary. The two directrices of the one congruence will then intersect the two directrices of the other one at two real points on the *OZ* axis. The directions of the two directrices will be imaginary. When projected onto *XY*, they will be represented by the two equations:

$$\sqrt{k_1} \cdot x \pm \sqrt{-k_2} \cdot y = 0,$$

which can be combined into the following one:

$$k_1\, x^2 + k_2\, y^2 = 0.$$

When k_1 and k_2 have opposite signs, Δ will become imaginary and tan ϑ will remain real. The projection of the two directrices onto *XY* will then be real, but the points at which the *OZ* axis is cut by them will be imaginary.

In summary, we have encountered a four-fold distinction between congruences:

1. The two directrices are *real*.

2. The two directrices are *imaginary*, and indeed in such a way that either they go through a real point or they have a real direction.

3. The two directrices are imaginary, but they cut the axis of the congruence in two real points through which the two directrices of the conjugate congruence also go.

4. The two directrices are imaginary and go through no real point on the axis of the congruence, but they have real directions.

In the first two cases, the complexes of the two-parameter group that the congruence determines will be real, and in the last two, they will be imaginary.

§ 3.

Congruences of three linear complexes. Ruled surfaces.

99. Let:

$$\left.\begin{aligned} \Omega &\equiv A\,r + B\,s + C\; - D\,\sigma + E\,\rho + F\,\eta = 0, \\ \Omega' &\equiv A'r + B's + C' - D'\sigma + E'\rho + F'\eta = 0, \\ \Omega'' &\equiv A''r + B''s + C'' - D''\sigma + E''\rho + F''\eta = 0 \end{aligned}\right\} \qquad (1)$$

be the general equations of three given first-degree complexes. The straight lines whose coordinates satisfy these three equations will simultaneously belong to three given complexes. They will simultaneously belong to *all* complexes of the three-parameter group that is represented by the following equation:

$$\Omega + \mu\,\Omega' + \mu'\,\Omega'' = 0, \qquad (2)$$

when we denote two undetermined coefficients by μ and μ'. From number **22**, such lines will define a surface of *order and class two*, so if we first direct our attention to only real straight lines, it will be a one-sheeted hyperboloid that can degenerate into a hyperbolic paraboloid. We must then not overlook the fact that only the lines of one of its two generators will be determined by the complex group. We would then like to refer to this generator as the *first generator* of the surface.

100. Three of complexes Ω, Ω', Ω'' that are chosen arbitrarily from the three-parameter group, when taken pair-wise, will define three congruences (Ω Ω'), (Ω Ω''), (Ω', Ω''). The lines of the surface will then also belong to these three congruences, and as a result, will intersect the two directrices of each of the three congruences. Three of the six directrices will be sufficient for the determination of the surface, from which, we will get the usual construction of the hyperboloid. However, along with the first generator of the surface, we will also encounter its second generator. The lines of the first generator are the ones that belong to all of the complexes of the three-parameter group, while the lines of the second generator will be the directrices of all congruences that we obtain when we combine the complexes of the group pair-wise.

101. In order to construct the ruled surface, we can also return to the complexes of the three-parameter group, and for that purpose, choose the three complexes Ω, Ω', Ω'', in turn. Let A^0B^0 be any given straight line, and let AB, $A'B'$, $A''B''$ be the three associated polars of this line relative to the three complexes. Those lines that cut A^0B^0 and AB, A^0B^0 and $A'B'$, A^0B^0 and $A''B''$, respectively, will belong to the complexes Ω, Ω', Ω''. In general, there will be two straight lines that cut four given ones. Thus, the two straight lines that cut A^0B^0 and AB, $A'B'$, $A''B''$ will simultaneously belong to all three complexes, and thus, to the ray surface. We will obtain the same two rays of the surface when we let any other complexes of the group (2) enter in place of the complexes Ω, Ω', Ω''. The polars of the given straight line with respect to all complexes of the group will define a congruence that has the two rays of the surface for its directrices.

The two points at which the given straight line is met by the rays of the surface can be real or imaginary and coincide. In the latter case, the surface will be contacted by that line.

In particular, we can choose the straight line A^0B^0 in such a way that it is one of the two directrices of the congruence that belongs to the complexes Ω and Ω', so the polar $A'B'$ will coincide with AB. Any ray of the surface will then cut the two lines A^0B^0 and AB; these lines belong to its second generator. However, the rays of the surface will also cut $A''B''$, as well as the polars of A^0B^0 relative to all complexes of the group.

In summary, we obtain the following general theorems:

A one-sheeted hyperboloid simultaneously belongs to three mutually independent complexes, and as a result of that, to all complexes of a three-parameter group. The straight lines that are common to all complexes are the rays of its first generator, while the directrices of the congruences of any two of these complexes define the lines of its second generator.

The polars of a given straight line relative to all complexes of a three-parameter group define a congruence whose two directrices are those two rays of the surface that cut the given straight line. The polars of an arbitrary line of the second generator of the surface relative to all complexes of the group are lines of that generator.

102. The central planes of any three congruences that belong to the surface intersect in a point at which three diameters of the congruences – viz., those three straight lines that go through that point and cut the two directrices of the three congruences – mutually bisect each other. These diameters are likewise three *diameters of the surface.* Their vertices are their intersections with the directrices that are lines of the second generator of the surface.

The central planes of all congruences of a three-parameter group:

$$\Omega + \mu\,\Omega' + \mu'\Omega'' = 0$$

intersect at the same point: viz., the center of the surface that is given by the group (*).

(*) Since a direct proof of this theorem might seem desirable, I will add the following: The expression:

$$A'B - AB',$$

when we replace the two complexes Ω and Ω' with any other two complexes in the two-parameter group:

$$\Omega + \lambda\,\Omega' = 0$$

(perhaps when we take corresponding values of λ^0 and λ_0), will be converted into the following one:

$$(A + \lambda_0 A')\,(B + \lambda^0 B') - (A + \lambda^0 A')\,(B + \lambda_0 B') \equiv (\lambda_0 - \lambda^0)\,(A'B - AB').$$

We can consider two arbitrary lines of the second generator of the surface to be directrices of a congruence that the lines of its first generator belong to, and likewise consider two arbitrary lines of its first generator to be directrices of a congruence that the lines of its second generator belong to.

The foregoing expression – and the same thing will be true for all expressions $A'C - AC'$, $B'C - BC'$, … that are constructed in the same way from two pairs of corresponding coefficients of the equations of the two complexes Ω and Ω' – will then change its value under a permutation of the complexes only in such a way that a factor of $(\lambda_0 - \lambda^0)$ appears, which merely depends upon the choice of the two complexes in the two-parameter group.

The central plane of the congruence that corresponds to the two-parameter group, whose equation we would like to take to be the following one:

$$p'' = 0,$$

is independent of the choice of the two complexes that we make for the determination of the congruence. As a consequence of that, the coefficients of its equation must be homogeneous functions of the same degree of $(A'B - AB')$ and analogously-defined expressions: $(A'C - AC')$, $(B'C - BC')$, …

Similar statements are true for the two congruences:

$$\Omega + \lambda\,\Omega'' = 0, \qquad \Omega' + \lambda\,\Omega'' = 0,$$

whose central planes might have the following equations:

$$p' = 0, \quad p = 0.$$

In our special case, the expressions of the form in question are contained in the three equations only in a linear way.

If we take any congruence of the three-parameter complex group and represent it by:

$$(\Omega + \mu_0\,\Omega' + \mu_0'\Omega'') + \lambda\,(\Omega + \mu^0\,\Omega' + \mu_1'\Omega'') = 0$$

and its central plane by:

$$q = 0$$

then when we set:

$$\pi \equiv \mu_0'\mu^0 - \mu_1^0\mu_0, \qquad \pi' \equiv \mu_0' - \mu_1^0, \qquad p'' \equiv \mu_0 - \mu^0,$$

we will easily deduce from the foregoing that:

$$q \equiv \pi\,p + \pi'\,p' + \pi''\,p'',$$

from which, one will get the proof that all central planes intersect in the same point.

We can express this theorem in the following way:

The central planes of the congruence of a three-parameter complex group define a three-parameter group of planes in their own right.

Just as the equation of the complex group is the symbol of a ray surface, the last equation will be the equation of the symbol of a point, namely, the center of the surface, at which infinitely many central planes will intersect.

Here, I must content myself by saying that I can give an extended interpretation to the theorem in this book, when it is expressed in the new form, that is associated with a far-reaching viewpoint. Should I be allowed to extend the developments that are restricted to straight lines here to forces, rotations, dynames, later on, this theorem would find its modest place in a systematic whole.

Any plane that is parallel to any two lines of the same generator of a hyperboloid and bisects the distance between them will go through the center of the surface.

The locus of the centers of all surfaces that go through two non-intersecting lines is a plane. The locus of the centers of all surfaces that go through a spatial rectangle is a straight line.

If a third line of a generator is added to two lines of the same generator then pair-wise combination of the three lines will yield three congruences that have these line-pairs for directrices. The surface is determined completely by these congruences. The intersection point of the three central planes of the congruences is the center of the surface; the three straight lines that go through the center and cut the two directrices are three of its diameters.

103. A plane that cuts a second-degree surface in one straight line will intersect it in a second one, in addition. The two lines of intersection will belong to the two different generators of the surface. Each such surface will be a tangential plane, and the point at which the two generators intersect on it will be the contact point. Any line that goes through the intersection of two lines of different generators and lies in the plane that goes through these lines will be a tangent to the surface. A plane that goes through a given generator and the center of the surface will be a tangential plane in which the contact point lies at infinity in the direction of the given generators when the second generator is parallel to the given one.

The planes that one can draw through each of the two directrices in each of three congruences whose lines belong to the first generator of a surface and parallel to the central plane will be tangential planes at the vertices of the relevant diameters. The central plane is associated with the diameter relative to the surface. The two directrices are lines of the second generator in the tangential planes; one obtains the lines of the first generator when one draws a straight line through the vertex of the diameter in any tangential plane that is parallel to the directrix of the other one.

104. From the foregoing, a plane that is parallel to any two lines of the same generator and bisects the distance between them will go through the center of the surface. If we let the two lines coincide then the plane in question will go through that line itself, and will thus be a tangential plane that goes through the center. The contact point will go to infinity. If the straight line is generated by a continuous motion of the surface then the plane in question will envelop a conic surface that will likewise be described by a straight line that goes through the center and will remain parallel to the straight line that generates the surface in all of its positions. We will obtain the same conic surface when the straight line that describes the surface belongs to the other generator. This conic surface, which will thus contact every plane that goes through the center and any line of one of the two generators, and whose sides will be those lines that are drawn through the center parallel to any line of one of the two generators, is called the *asymptotic cone* of the surface. The sides of the asymptotic cone are not the only straight lines that contact the surface at infinity. Any straight line that lies in a tangential plane to the asymptotic cone and is parallel to those sides along which that cone will be contacted is an *asymptote of the surface.* Two such asymptotes can be drawn through each point outside the cone that are parallel to two of its sides.

105. The two lines of the second generator of a surface that go through the two vertices of any of its diameters are the two directrices of a congruence that belongs to the surface. The central plane of the congruence is the diametral plane that is associated with the diameter relative to the surface. If we project the two directrices onto the central plane along the diameter then we will obtain the asymptotes of the intersection curve of the surface with the central plane. Any two associated diameters of the intersection curve will fall on two associated auxiliary diameters of the congruence. Any diameter of the congruence and two associated auxiliary diameters in its central plane shall be called three *associated diameters of the surface.*

According to whether the diameter does or does not encounter the surface, the directrices of the relevant congruence will be real or imaginary, respectively, and corresponding to that, the two asymptotes of the intersection curve in the central plane will also be real or imaginary, respectively. This curve will be a *hyperbola* in one case and an *ellipse* in the other. The intersection curves in planes that are parallel to the central plane will be equally-oriented hyperbolas or ellipses. The hyperbolas will degenerate into systems of straight lines in the planes that go through the endpoints of the diameters and are tangential planes. The ellipses will always keep finite dimensions, since the corresponding tangential planes are imaginary. If we consider one side of the asymptotic cone to be the diameters then the directrices of the relevant congruence will coincide (cf., no. **68**), and the plane that contacts the asymptotic cone along that side will be its central plane. The intersection curve of the surface with the central plane degenerates into a system of two parallel lines whose diameter is a side of the cone. The intersection curves in parallel planes are *parabolas* whose diameters are parallel to sides of the cone.

106. Every diameter of the surface corresponds to two different congruences whose directrices intersect at the endpoints of the diameters and are lines of the two different generators of the surface. We have called two such congruences (no. **79**) two conjugate congruences relative to the diameter. Those of these two congruences that have two lines of the second generator for directrices belong to the lines of the first generator; the other one, which has two lines of the first generator for directrices, belongs to the lines of the second generator.

107. The associated polars of a given line A_0B_0 of space relative to the various complexes of the three-parameter group:

$$\Omega + \mu\,\Omega' + \mu'\Omega'' = 0$$

by which a ruled surface is determined define a congruence whose two directrices are lines of the first generator of the first surface (no. **101**). The given straight line cuts the two directrices at two points. Let these two intersection points be A_0 and B_0 ; they are likewise the two intersection points of the given line with the surface. Let the two directrices be $A_0\,A^0$ and $B_0\,B^0$. The planes that go through $A_0\,B_0$ and $A_0\,A^0$ will contact the surface because $A_0\,A^0$ is a line of the first generator; let the contact point that lies on this line be A^0. Likewise, the plane that goes through $A_0\,B_0$ and $B_0\,B^0$ will contact the surface at a point of $B_0\,B^0$; let that point be B^0. We would like to connect the two contact points A^0 and B^0, which lie on the two directrices, and thus on the surface, with a straight line A^0B^0.

If a tangential plane of the surface is drawn through a line of its first generator, $A_0\,A^0$ or $B_0\,B^0$, respectively, then the line of the second generator that goes through the contact point, A^0 or B^0, respectively, will be determined in such a way that it will cut any other line of the first generator, $B_0\,B^0$ or $A_0\,A^0$, respectively. In the construction above, $A_0\,B^0$ and $A^0\,B_0$ will then be lines of the second generator. $A_0A^0B^0B_0$ will be a rectangle that is described on the surface whose two pairs of opposite sides belong to the second generator. The two diagonals of the rectangle will be $A_0\,B_0$ and A^0B^0. The sides of the rectangle will likewise be four of the six edges of a tetrahedron; the surface itself, which will contain each of two successive sides, will contact the ruled surface at the four corners of the rectangle. $A_0\,B_0$ and A^0B^0 will be the two remaining mutually opposite edges of the trihedron.

108. It follows immediately from the foregoing that the relationship between the two lines $A_0\,B_0$ and A^0B^0 on the surface is completely reciprocal. The two tangential planes to the surface that can be drawn through each of them will contact the surface at the two intersection points of the other ones; the tangential planes at the intersection points of each of them with the surface will intersect on the other one. We call the two lines *two associated polars relative to the surface.* Any line in space is associated with a second one as its associated polar.

If we determine a congruence in such a way that we can take any two lines of a surface to be its directrices then the congruence will associate lines pair-wise in such a way that each of these lines will correspond to another one, with which, it will define two associated polars relative to the surface. Those of these lines that coincide with their associates polars will belong to the surface.

Any two lines of the one generator, along with any two lines of the other one, define a rectangle that is inscribed on the surface, as the four edges of a tetrahedron that is circumscribed on it; the two diagonals of this rectangle, or – what amounts to the same thing – the two opposite edges of the tetrahedron, will be two conjugate polars relative to the surface.

109. Three lines of the one generator of a ruled surface and three lines of the other generator will intersect each other at nine points that belong to the surface. These three points can be arranged into three groups:

$$P,\ Q,\ R, \qquad P',\ Q',\ R', \qquad P'',\ Q'',\ R'' \tag{3}$$

in such a way that at the three points of the same group the three lines of one generator will intersect the three lines of the other generator. The nine points will correspond to nine planes that contact the surface at these points:

$$p,\ q,\ r, \qquad p',\ q',\ r', \qquad p'',\ q'',\ r''. \tag{4}$$

The three lines of the one generator will contain the points:

$$P,\ Q'',\ R', \qquad R,\ P'',\ Q', \qquad Q,\ R'',\ P',$$

and the lines of the generator will contain the points:

$$P,\ R'',\ Q', \qquad Q,\ P'',\ R', \qquad R,\ Q'',\ P'.$$

The nine points will determine three hexagons that are inscribed on the surface:

$$\left.\begin{array}{l} P'Q''R'P''Q'R'', \\ P\,Q''\,R\,P''\,Q\,R'', \\ P\ Q'\,R\,P'\ Q\ R'. \end{array}\right\} \tag{5}$$

In a similar way, we obtain three six-faced bodies that are defined by the tangential planes at the corners of the three hexagons. The entire geometric structure is determined just the same regardless of whether we start with the three points of the three groups (3), or the three tangential planes at three such points, or finally with one of the three hexagons (5), and correspondingly, arbitrarily choose three points of the surface, or three of its tangential planes, or a hexagon that is inscribed on the surface from the outset.

If we start with three points P, Q, R of the surface then a plane (P, Q, R) will be determined by these three points and a point (p, q, r) will be determined by the three tangential planes at these points. The three lines of intersection of the three tangential planes will be the three diagonals of the third hexagon:

The three diagonals of a hexagon that is inscribed on a ruled surface intersect at the same point.

The first inscribed hexagon has P' and P'', Q' and Q'', R' and R'' as its opposite vertices; the tangential planes at the three pairs of opposite vertices intersect in the three lines (P, Q), (P, R), (Q, R) that connect the given points P, Q, R with each other pair-wise, and thus lie in the same plane.

The tangential planes at any two opposite vertices of a hexagon that is inscribed on a ruled surface intersect in three straight lines that lie in the same plane.

110. An inscribed hexagon, which we would like to take to be the first one, determines three inscribed rectangles. The sides of each rectangle are those four sides of the hexagon that meet in two opposite vertices of it when taken pair-wise. The three diagonals of the hexagon (R', R''), (Q', Q''), (P', P''), which intersect at the point (p, q, r), are three diagonals of the three rectangles; the second three diagonals of these rectangles are (P, Q), (P, R), (Q, R), which lie in the plane (P, Q, R). Consistent with number **104**, three straight lines that go through the same point will then have three straight lines as associated points that lie in the same plane. Thus:

The associated polars of all lines that intersect in the same point lie in the same plane.

Therefore, any point in space will correspond to a plane, and any plane, to a point. The plane is the *polar plane of the point*, *and the point is the pole of the plane*, relative to the surface. From the foregoing, the tangential planes of the surface to the points of planar intersection curve will envelope a conic surface that goes through that curve has the pole of the intersecting plane as its center. Conversely, all tangential planes of the surface that go through a point will contact the surface in a plane curve whose plane is the polar plane of the given point (*).

111. We could deduce some analytic developments that are determined to further support and extend the foregoing geometric ideas, although this would not be the place to pursue them further. We would like to represent a ray surface that is given by the equations of three first-degree complexes, which we can choose arbitrarily from a three-parameter group:

$$\Omega + \mu\,\Omega' + \mu'\,\Omega'' = 0,$$

by an equation in ordinary point-coordinates.

We will next take the three complexes of the group to be three complexes whose lines all intersect its axis. If we determine the origin arbitrarily and draw the three coordinate planes through the three axes of the complex then we will obtain the following equations for the three complexes:

$$\Omega \equiv C \ \ - D \ \ \sigma + E \ \ \rho = 0, \tag{6}$$
$$\Omega' \equiv C' \ - D' \ \sigma + E' \ \rho = 0, \tag{7}$$
$$\Omega'' \equiv C'' - D''\sigma + E''\rho = 0. \tag{8}$$

We have the following two relations:

$$x = rz + \rho, \qquad y = sz + \sigma$$

(*) I have already considered the three associated hexagons that are inscribed on the surface some time ago in *System der Geometrie des Raumes* (cf. no. **87-93**), and carried out the proof in analytical symbols that, on the one hand, the three points at which the diagonals of the three hexagons intersect will lie on a straight line, and on the other hand, the three planes that contain the lines of intersection of the tangential planes at the opposite vertices of the three hexagons will intersect on a second straight line, and finally that these two straight lines will be two associated polars relative to the surface.

between the coordinates of any point x, y, z that lies on any ray, and the four coordinates r, s, ρ, and σ of the ray, from which, the determination of the fifth coordinate will follow:

$$r\,y - s\,x = \eta.$$

If we eliminate the five ray coordinates from the foregoing six equations then the resulting equation in x, y, z will represent the ray surface in point coordinates.

If we first eliminate η then we will get:

$$\begin{aligned}(B' - F'x)\,s + F'y \cdot r - D'\,\sigma &= 0,\\ (A'' + F''\,y)\,r + F''\,x \cdot s - E''\rho &= 0,\end{aligned}$$

instead of the last two complex equations (7) and (8), and when we then eliminate ρ and σ, we will get:

$$\begin{aligned}Ez \cdot r - Dz \cdot s - C - E\,x + D\,y &= 0,\\ F'z \cdot r + (B' - F'x + D'z)\,s - D'\,y &= 0,\\ (A'' + F''y - E''z)\,r - F''x \cdot s + E''\,x &= 0.\end{aligned}$$

If we determine the values of s and r from the last two of the foregoing three equations and substitute them into the first of these equations then that will give:

$$\begin{aligned}&E\,x\,z\,[E''\,(B' - F'x + D'\,z) - D'\,F''\,y]\\ &+ D\,y\,z\,[D'\,(A'' + F''y - E''\,z) - E''\,F'x]\\ &+ (C + Ex - Dy)\,[(A'' + F''y - E''z)(B' - F'x + D'z) + F\,F''\,xy] = 0.\end{aligned}$$

The higher powers of x, y, z will vanish from this equation, and we will get:

$$\begin{aligned}&A''B'C + A''\,(B'E - CF')\,x + B'\,(CF'' - A''D)\,y + C\,(A''D' - B'E'')\,z\\ &- A''E\,F' \cdot x^2 - B'D\,F'' \cdot y^2 - C\,D'E''\,z^2\\ &+ (CD'F'' + B'DE'')\,y\,z + (CE''F + A''D'E)\,x\,z + (B'EF'' + A''DF')\,x\,y = 0.\end{aligned}$$

If we divide this equation by ABC and write:

$$\frac{E}{C}, \quad -\frac{D}{C}, \quad -\frac{F'}{B'}, \quad \frac{D'}{B'}, \quad \frac{F''}{A''}, \quad -\frac{E''}{A''}$$

as:

$$t', \quad u'', \quad t'', \quad v', \quad u', \quad v'',$$

respectively, then that will give the following equation:

$$\begin{aligned}&1 + (t' + t'')\,x + (u' + u'')\,y + (v' + v'')\,z\\ &+ t'\,t''\,x^2 + u'\,u''\,y^2 + v'\,v''\,z^2\\ &+ (u'\,v' + u''\,v'')\,y\,z + (t'\,v' + t''\,v'')\,x\,z + (t'\,u' + t''\,u'')\,x\,y = 0. \qquad (9)\end{aligned}$$

This equation will represent the same surface in point coordinates that was originally represented by the three complex equations (6), (7), and (8). When we introduce the six new constants, those three equations will become:

$$\left.\begin{aligned} t'\rho + u''\sigma + 1 = 0, \\ -v'\sigma - t''\eta + s = 0, \\ u'\eta - v''\rho + r = 0, \end{aligned}\right\} \tag{10}$$

and as a result, when μ and μ' denote two undetermined coefficients, the general equation of the three-parameter complex group by which the ray surface is determined will be the following one:

$$(t'\rho + u''\,\sigma + 1) + \mu\,(v'\sigma + t''\,\eta - s) + \mu'\,(u'\,\eta - v''\,\rho + r) = 0. \tag{11}$$

112. If we set z, y, and x equal to zero in equation (9) in succession then we will obtain:

$$\left.\begin{aligned} (t'x + u''y + 1)\cdot(t''x + u'y + 1) = 0, \\ (v'z + t''x + 1)\cdot(v''z + t'x + 1) = 0, \\ (u'y + v''z + 1)\cdot(u''y + v'z + 1) = 0. \end{aligned}\right\}$$

The intersection curve of the surface with the three coordinate planes will thus degenerate into a system of two straight lines. The surface will be contacted by the XY, XZ, YZ coordinate planes; the lines of the second generator of the surface in these planes will be:

$$\left.\begin{aligned} t'x + u''y + 1 = 0, \\ v'z + t''x + 1 = 0, \\ u'y + v''z + 1 = 0, \end{aligned}\right\} \tag{12}$$

and the lines of the first generator will be:

$$\left.\begin{aligned} t''x + u'y + 1 = 0, \\ v''z + t'x + 1 = 0, \\ u''y + v'z + 1 = 0. \end{aligned}\right\} \tag{13}$$

The contact points in the three coordinate planes will be the intersections of the lines of the first and second generator in each of the three planes. The three lines of the second generator will be the axes of three complexes of the three-parameter group on all of whose lines the complexes intersect, or in other words, three directrices of three congruences of the group. If we exchange the three lines of the second generator with the three lines of the first generator then the following three complexes will enter in place of the three complexes (10):

$$\left.\begin{aligned} t''\rho + u'\sigma + 1 &= 0, \\ -v''\sigma - t'\eta + s &= 0, \\ u''\eta - v'\rho + r &= 0, \end{aligned}\right\} \tag{14}$$

and for the determination of its ray surface, we will obtain the new three-parameter complex group:

$$(t''\rho + u'\sigma + 1) + \mu_1\,(v''\sigma + t'\eta - s) + \mu_1'(u''\eta - v'\rho + r) = 0. \tag{15}$$

Any congruence of one of the two three-parameter complex groups (11) and (15) will correspond to a conjugate congruence in the other one.

113. In particular, if:

$$t' + t'' = 0, \qquad u' + u'' = 0, \qquad v' + v'' = 0 \tag{16}$$

then the equation of the surface will assume the following simpler form:

$$1 - t'^2\,x^2 - u'^2\,y^2 - v'^2\,z^2 + 2\,u'\,v' \cdot y\,z + 2\,t'\,v'\,x\,z + 2\,t'\,u'\,x\,y = 0. \tag{17}$$

The two lines of different generators in each of the three coordinate planes will then be parallel to each other, and will be equally distant from the coordinate origin. It will be the center of the surface. The three coordinate planes will contact the asymptotic cone of the surface.

114. If the surface is a *hyperbolic paraboloid*, in particular, then the three straight lines (12) will remain lines of the same of its generators, but will be subject to the condition that they be parallel to a given plane. If we take the equation of this plane to be:

$$ax + by + cz = 0 \tag{18}$$

then that will give:

$$\frac{a}{b} = \frac{t'}{u''}, \qquad \frac{b}{c} = \frac{u'}{v''}, \qquad \frac{c}{a} = \frac{v'}{t''}, \tag{19}$$

which yields the following condition equation between the six constants upon which the surface depends:

$$t'\,u'\,v' = t''\,u''\,v''. \tag{20}$$

As a result of this condition equation, the lines of the second generator will be parallel to a second given plane. If we take:

$$a'x + b'y + c'z = 0 \tag{21}$$

to be the equation of that plane then that will give:

$$\frac{a'}{b'} = \frac{t''}{u'}, \qquad \frac{b'}{c'} = \frac{u''}{v'}, \qquad \frac{c'}{a'} = \frac{v''}{t'}. \tag{22}$$

If we develop equations (18) and (21) then that will give:

$$\left.\begin{array}{l} t''v''x + u'v'y + v'v''z = 0, \\ t'v'x + u''v''y + v'v''z = 0. \end{array}\right\} \tag{23}$$

The line of intersection of those two planes that the lines of the first and second generators of the paraboloid are parallel to will determine the direction of its *diameter.*

We will find this by considering the condition equation (20):

$$\frac{t't''x}{t'-t''} = \frac{u'u''y}{u'-u''} = \frac{v'v''z}{v'-v''}. \tag{24}$$

115. Up to now, we have mainly considered straight lines to be *rays*, because this manner of presentation lies closer to our viewpoint, and we require brevity. The concept of a straight line as an *axis* is, however, equivalent. Ray congruences will then appear to be axial congruences and ray surfaces will appear to be axial surfaces. Here, we would like to consider the same surface that we have just regarded as a ray surface as an axial surface from now on. It will be determined by the previous complexes Ω, Ω', Ω'', which will be represented by the following equations:

$$\Phi \equiv C\,\omega + D\,p + E\,q = 0, \tag{25}$$
$$\Phi' \equiv B'\,\pi + D'\,p + F' = 0, \tag{26}$$
$$\Phi'' \equiv -A''\kappa + E''\,q + F'' = 0. \tag{27}$$

We will obtain the equation of this surface in plane coordinates t, u, v when we eliminate the five axial coordinates from the foregoing three equations and the equations:

$$\begin{aligned} t &= p\,v + \pi, \\ u &= q\,v + \kappa, \\ p\,u - q\,t &= \omega. \end{aligned}$$

If we then eliminate ω, π, κ from the first and sixth, second and fourth, and the third and fifth of the foregoing six equations, respectively, then that will give:

$$\begin{aligned} (C\,u + D\,)\,p &= (C\;\,t - E\;\,)\,q, \\ (B'\,t + F'\,) &= (B'\,v - D'\,)\,p, \\ (A''v + E'')\,q &= (A''\,u - F''), \end{aligned}$$

and thus, when we multiply these three equations together, we will get:

$$\frac{(Ct-E)(A''u-F'')(B'v-D')}{(B't+F')(Cu+D)(A''v+E'')} = 1.$$

If we divide the numerator and denominator of this fraction on the left-hand side of this equation by $A'' \cdot B' \cdot C$ then, when we, in turn, introduce the previous constants t' and t'', u' and u'', v' and v'', for the sake of brevity, that will give:

$$\frac{(t-t')(u-u')(v-v')}{(t-t'')(u-u'')(v-v'')} = 1. \tag{28}$$

When we develop this equation, the product of the three variables will drop out. It will represent the same surface in plane coordinates that we previously represented by equation (9) in point coordinates.

116. The foregoing equation will be satisfied when one has simultaneously:

$$\left.\begin{aligned} t-t' &= 0 \\ u-u' &= 0, \\ v-v' &= 0, \end{aligned}\right\} \qquad \left.\begin{aligned} u-u'' &= 0 \\ v-v'' &= 0, \\ t-t'' &= 0, \end{aligned}\right\} \tag{29}$$

and likewise, when one has simultaneously:

$$\left.\begin{aligned} t-t' &= 0 \\ u-u' &= 0, \\ v-v' &= 0, \end{aligned}\right\} \qquad \left.\begin{aligned} v-v'' &= 0 \\ t-t'' &= 0, \\ u-u'' &= 0. \end{aligned}\right\} \tag{30}$$

Equations (29) and (30) will reduce to six distinct ones, and when taken individually, they will represent six points, two of which will lie on each of the three coordinate axes. When they are combined pair-wise, as we did in the foregoing, the axes, which lie in the three coordinate planes and likewise on the surface, will represent, on the one hand, the three lines of the second generator of the surface (12), and on the other hand, the three lines of the first generator of the surface (13). The surface will be contacted by the three coordinate planes.

We can represent the surface, corresponding to its double generator, by each of the two following three-parameter groups of linear axial complexes:

$$(\omega - u''\, p + t'\, q) + \lambda\,(\pi + v'\, p - t'') + \lambda'\,(k + v''\, q - u') = 0, \tag{31}$$
$$(\omega - u'\, p + t''\, q) + \lambda_1\,(\pi + v'' p - t') + \lambda_1'\,(k + v'\, q - u'') = 0, \tag{32}$$

in which we denote the undetermined coefficients by λ, λ', λ_1, λ_1'.

117. If we take any three tangential planes of the asymptotic cone of the surface to be the three coordinate planes then as a consequence of the relations (16):

$$t' + t'' = 0, \qquad u' + u'' = 0, \qquad v' + v'' = 0,$$

the equation of the surface in plane coordinates will assume the following form:

$$\frac{(t-t')(u-u')(v-v')}{(t+t')(u+u')(v+v')} = 1. \tag{33}$$

In the case of the hyperbolic paraboloid, the general equation (28) can be specialized in such a way that the constant term drops out in the development, which will once more lead to the previous condition equation (20).

118. In the last numbers, we have represented the same generator of the same surface, in one case by three linear equations in ray coordinates, and in the other case by three linear equation in axial coordinates, and derived the equation of that surface from three linear equations in point coordinates, in one case, and in plane coordinates, in the other.

As a second example, we would like to determine a ruled surface by three complexes of a special kind when we take their equations to be three that emerge from the previous ones when we permute the constants with their reciprocal values and reciprocally switch:

$$r, s, \rho, \sigma, \eta \qquad \text{with} \qquad p, q, \pi, \kappa, \omega,$$

resp.

In that way, when we denote the reciprocal values to t', t'', u', u'', v', v'' by x', x'', y', y'', z', z'', respectively, we will get the following complex equations in place of the three in (10):

$$\left.\begin{aligned} x'\pi + y''\kappa + 1 &= 0, \\ -z'\kappa - x''\omega + q &= 0, \\ y'\omega - z''\pi + p &= 0. \end{aligned}\right\} \tag{34}$$

119. If we eliminate the five axial coordinates p, q, π, κ, ω from these equations and the three equations:

$$t = pv + \pi, \qquad u = qv + \kappa, \qquad pu - qt = \omega$$

then we will get the following equation for the surface in plane coordinates:

$$\begin{gathered} 1 + (x' + x'')\,t + (y' + y'')\,u + (z' + z'')\,v \\ + x'\,x''\,t^2 + y'\,y''\,u^2 + z'\,z''\,v^2 \\ + (y'\,z' + y''\,z'')\,uv + (x'\,z' + x''\,z'')\,tv + (x'y' + x''y'')\,tu = 0. \end{gathered} \tag{35}$$

We will get this equation immediately when we replace t', t'', u', u'', v', v'' with x', x'', y', y'', z', z'' and x, y, z with t, u, v, resp., in equation (9).

In order to express this complex (34) in ray coordinates, we get:

$$\left.\begin{aligned} \eta - y''\cdot r + x'\cdot s &= 0, \\ \rho + \; z'\cdot r - x'' \quad &= 0, \\ -\sigma - z''\cdot s + y' \quad &= 0, \end{aligned}\right\} \tag{36}$$

and when we eliminate the ray coordinates r, s, ρ, σ, η from these three equations and the following three:

$$x = rz + \rho, \qquad y = sz + \sigma, \qquad ry - sx = \eta,$$

we will get:

$$\frac{(x-x')(y-y')(z-z')}{(x-x'')(y-y'')(z-z'')} = 1. \tag{37}$$

We will obtain this equation immediately when we exchange t', t'', u', u'', v', v'' with x', x'', y', y'', z', z'' and t, u, v with x, y, z, resp., in equation (28).

The two equations (35) and (37) represent the same surface in plane and point coordinate that was represented by the system of linear equations (34) and (36) in axial and ray coordinates.

120. If we set v, u, and t equal to zero in equation (35) in succession then we will get:

$$\left.\begin{aligned} (x't + y''u + 1)(x''t + y'u + 1) &= 0, \\ (z'v + x''t + 1)\,(z''v + x't + 1) &= 0, \\ (y'u + z''v + 1)(y''u + z'v + 1) &= 0. \end{aligned}\right\} \tag{38}$$

Whereas the tangential planes of a surface of order and class two that are parallel to a given straight line will envelop a cylinder, in general, this cylinder will degenerate into a system of two parallel straight lines when we take the given straight line to be the three coordinate axes in succession. The coordinate axes will then be parallel to any three generators of the ruled surface, or – what amounts to the same thing – any three sides of the asymptotic cone. All planes that go through any line of the surface will then be tangential planes to the surface. Three lines of the first generator that are taken to be parallel to the three coordinates axes will then be parallel to the three lines of the second generator. The two point-pairs at which the *YZ*, *XZ*, *ZY* planes are met by the lines of both generators, which are parallel to the *OZ*, *OY*, *OX* axes, respectively, will be represented by equations (38).

121. In agreement with that, in order to satisfy equation (37), we will get, in the one case, the three equations-pairs:

$$\left.\begin{array}{r} x-x'=0, \\ y-y'=0, \\ z-z'=0, \end{array}\right\} \qquad \left.\begin{array}{r} y-y''=0, \\ z-z''=0, \\ x-x''=0, \end{array}\right\} \tag{39}$$

which represent the three lines of the second generator, and in the other case, the three equation-pairs:

$$\left.\begin{array}{r} x-x'=0, \\ y-y'=0, \\ z-z'=0, \end{array}\right\} \qquad \left.\begin{array}{r} z-z''=0, \\ x-x''=0, \\ y-y''=0, \end{array}\right\} \tag{40}$$

which represent the three lines of the first generator, which are parallel to the three coordinate axes *OZ*, *OX*, *OY* and *OY*, *OZ*, *OX*, respectively.

123. The three complexes of a special kind by which the surface is determined – in one case, by equations (34) and in the other case by equations (36) – have those lines of the second generator that are represented by the equation-pair (39) for their axes, and on the other hand, are determined by the fact that they are parallel to the *OX*, *OY*, *OZ* coordinate axes and cut the *YZ*, *XZ*, *XY* coordinate planes, respectively, which will be represented by the equations:

$$\left.\begin{array}{r} y'u+z''v+1=0, \\ z'v+x''t+1=0, \\ x't+y''u+1=0 \end{array}\right\} \tag{41}$$

in these planes.

If we take the three sides of the asymptotic cone itself to be the coordinate axes then that will give:

$$x' + x'' = 0, \qquad y' + y'' = 0, \qquad z' + z'' = 0. \tag{42}$$

The equation of the surface in plane coordinates will then assume the following form:

$$1 - x'^2\, t^2 - y'^2\, u^2 - z'^2\, v^2 + 2y'\, z' \cdot uv + 2\, x'z' \cdot tv + 2x'\, y' \cdot tu = 0, \tag{43}$$

and the equation of that surface in point coordinates will assume the following one:

$$\frac{(x-x')(y-y')(z-z')}{(x+x')(y+y')(z+z')} = 1. \tag{44}$$

124. Finally, first under the assumption of rectangular coordinate axes, we would like to take the following three equations for the complexes of a three-parameter group:

$$\Omega + \mu\,\Omega' + \mu'\Omega'' = 0$$

that determine a ruled surface, namely:

$$\left.\begin{aligned} \Omega &\equiv \sigma - k_1 r = 0, \\ \Omega' &\equiv \rho + k_2 s = 0, \\ \Omega'' &\equiv \eta + k_3 \quad = 0. \end{aligned}\right\} \tag{45}$$

The axes of the three complexes then will fall upon the three *OX*, *OY*, *OZ* coordinate axes, so like them they will be mutually-perpendicular and intersect each other at the coordinate origin. The parameters will be k_1, k_2, k_3 . If we combine the three complexes pair-wise then we will get three congruences (Ω', Ω''), (Ω, Ω''), (Ω, Ω'), whose principal axes will fall upon *OX*, *OY*, *OZ*, resp., and whose pairs of auxiliary axes will fall upon *OY* and *OZ*, *OX* and *OZ*, and *OX* and *OY*, resp.

If we, as before, eliminate σ, ρ, and η by means of the three equations:

$$x = rz + \rho, \qquad y = sz + \sigma, \qquad rx - sy = \eta$$

then that will give:

$$\begin{aligned} y - sz - k_1 r &= 0, \\ x - rz + k_2 s &= 0, \\ ry - sx + k_3 &= 0. \end{aligned}$$

It follows from the first two of the foregoing equations that:

$$\begin{aligned} (k_1 k_2 + z^2)\, r &= \quad x z + k_2 y, \\ (k_1 k_2 + z^2)\, s &= - y z + k_1 x, \end{aligned}$$

and when we eliminate r and s from these two equations and the third of the foregoing three equations, it will follow that:

$$k_1 x^2 + k_2 y^2 + k_3 z^2 + k_1 k_2 k_3 = 0,$$

or:

$$\frac{x^2}{k_2 k_3} - \frac{y^2}{k_1 k_3} + \frac{z^2}{k_1 k_2} + 1 = 0. \tag{46}$$

125. If the parameters of the three original complexes are all positive, and thus the parameters of the three new complexes will be negative, or conversely, when the former is negative and the latter is positive, then the surface will be imaginary. In every remaining case, as long as the values of the parameters remain real, the surface will be a one-sheeted hyperboloid. The values of two of the three parameters of the two groups of complexes will then agree in sign, and the values of the present three parameters will have the opposite sign. According to whether the parameters of the first, second, or third complexes of the group do or do not deviate in sign from the parameters of the two remaining complexes, the imaginary axis of the hyperboloid will fall upon the *OX*, *OY*, *OZ* coordinate axes, respectively. In the first case, we will get:

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = -1, \tag{47}$$

when we set:

$$\left.\begin{aligned} k_2 k_3 &= \ \ a^2, \\ k_1 k_3 &= -b^2, \\ k_1 k_2 &= -c^2. \end{aligned}\right\} \tag{48}$$

If we base the foregoing developments on plane coordinates, in place of point coordinates, and thus consider the lines of the complex to be axes, instead of rays, then we will get the following equation for the same ruled surface, which will now become an axial surface from now on:

$$k_2\, k_3\, t^2 + k_1\, k_3\, u^2 + k_1\, k_2\, v^2 + 1 = 0. \tag{49}$$

126. Any given straight line in space is parallel to of the infinitude of diameters of a three-parameter group. We can then consider the three coordinate axes to be diameters of three complexes by which a ruled surface is determined. In the case of rectangular coordinate axes, equations (45) will represent three complexes whose three axes fall upon the three axes of the ruled surface. The principal parameters of the three complexes are k_1, k_2, k_3 . When we take any three associated diameters of that ruled surface to be the coordinate axes, equation (45) will always represent three complexes of the three-parameter group, except that k_1, k_2, k_3 will then no longer represent the principal parameters of the three complexes, but the parameters of their three diameters that coincide with the three coordinate axes. We would like to denote these three parameters by, k_1^0, k_2^0, k_3^0, to distinguish them, and let the three constants above keep their meanings as principal parameters of the three complexes. We would like to call three complexes of the three-parameter group whose diameters are parallel to any three associated diameters of the ruled surface that is determined by this group, *three conjugate complexes*, relative to the ruled surface. Let ε'', ε', ε be the three angles *XOY*, *XOZ*, *YOZ*, resp., that the three coordinate axes define with each other, when taken pair-wise, and let δ'', δ', δ be the inclination angles of *OZ* with respect to *XY*, *OY* with respect to *XZ*, and *OX* with respect to *YZ*, respectively. The three expressions:

$$\sin \varepsilon'' \sin \delta'', \quad \sin \varepsilon' \sin \delta', \quad \sin \varepsilon \sin \delta$$

will then be equal to each other. If we denote that value by γ, for the sake of brevity, then we will get:

$$k_1^0 = \frac{k_1}{\gamma}, \qquad k_2^0 = \frac{k_2}{\gamma}, \qquad k_3^0 = \frac{k_3}{\gamma},$$

and thus:

$$k_1^0 : k_2^0 : k_3^0 = k_1 : k_2 : k_3 . \tag{50}$$

If, corresponding to equations (48), we set:

$$\left.\begin{aligned} k_2^0\, k_3^0 &= a_0^2\,, \\ k_1^0\, k_3^0 &= -b_0^2\,, \\ k_1^0\, k_2^0 &= -c_0^2 \end{aligned}\right\} \tag{51}$$

then we will get:

$$\left(\frac{x}{a_0}\right)^2 - \left(\frac{y}{b_0}\right)^2 - \left(\frac{z}{c_0}\right)^2 = -1 \tag{52}$$

for the equation of the ruled surface in oblique coordinates. $a_0\sqrt{-1}$, b_0, c_0 mean those radii of the surface that coincide with the *OX*, *OY*, *OZ* coordinate axes, respectively. If we set:

$$\gamma^2 \cdot a_0^2\, b_0^2\, c_0^2 \equiv \Theta^2 \tag{53}$$

then it is known that Θ will be a quantity that does not change when we take any other three associated diameters of the surface to be the coordinate axes instead of the three given associated diameters.

127. If we multiply the last two equations (51) with each other term-by-term and divide by the first of these equations then that will give:

$$k_1^{02} = \frac{b_0^2\, c_0^2}{a_0^2}\,,$$

and if we introduce k_1 in place of k_1^0 then:

$$k_1^2 = \gamma^2\,\frac{b_0^2\, c_0^2}{a_0^2} = \frac{\Theta^2}{a_0^2}\,.$$

Thus:

$$k_1 = \pm\,\frac{\Theta}{a_0}\,. \tag{54}$$

k_1 is the principal parameter of the complex of the three-parameter group whose diameter is parallel to the *OX* axis, and a_0^2 (when taken with the opposite sign) is the square of the radius of the surface that falls upon that axis. Since we can take any arbitrary diameter of the ruled surface to be a coordinate axis from the outset (whereby, a_0^2 must be taken to have a positive or negative sign according to whether the new diameter cuts the surface or not), this will immediately yield the following theorem:

The principal parameters of the complexes of a three-parameter group whose diameter is parallel to any diameter of the ruled surface that is determined by the group are, conversely, proportional to the square of the length of the diameter of the surface.

128. An arbitrary plane that is drawn through the origin is simultaneously associated with a diameter of the ruled surface, the principal diameter of a congruence that belongs to that surface, and a diameter of a complex of the three-parameter group by which the surface is determined. The planes that are associated with the diameter of the surface are parallel to the central plane of the congruence and are likewise associated with the diameter of the complex that coincides with the diameter of the surface. This will follow immediately from the equations of the three conjugate complexes (45) by which a ruled surface is determined, also under the assumption of oblique coordinates. The diameter of one of the three complexes that falls upon one of the three coordinate axes is associated with the coordinate plane that goes through the other two coordinate axes, and that plane will be, on the one hand, the central plane of the congruence that is determined by the remaining two complexes and, on the other hand, the diametral plane of the surface that is conjugate to its diameter that coincides with the diameter of the complex.

A complex of a three-parameter group is determined completely when the direction of its diameter is given. In the previous number, we obtained its parameters in the simplest way by means of the corresponding ruled surface. The foregoing discussion gives us the associated planes to its diameter. The construction of its axis therefore reverts back to the one in number **46**.

One applies the parameters of the complex to the diameter of the surface that has the given direction of the diameter of the complex from the center outward and projects them onto the diametral plane of the surface that is associated with the diameter. From the previous number, the parameter will be equal to Θ / r_0^2, if we denote the length of the radius of the surface by r_0^2, and its projection will be equal to:

$$\frac{\Theta}{r_0^2} \cos \delta_0,$$

if we call the angle that the diameter of the surface defines with its conjugate plane δ_0. If we then displace the diameter of the surface parallel to itself on the projecting plane through an increment that is equal to that projection then the displaced diameter of the surface in the new position will be the axis of the complex.

This displacement can be performed in the opposite direction. Corresponding to that, we will get the two axes of two different complexes that are parallel to each other and equally distant from the center. These two complexes belong to the two different generators of the ruled complex.

129. We obtain:

$$k_1^0 = \pm \frac{b_0\, c_0}{a_0}, \qquad k_2^0 = \mp \frac{a_0\, c_0}{b_0}, \qquad k_3^0 = \pm \mp \frac{a_0\, b_0}{c_0} \tag{55}$$

for the parameters of the diameters of the three conjugate complexes that fall upon the *OX*, *OY*, *OZ* axes, resp., and:

$$k_1 = \pm \frac{\Theta}{a_0^2}, \qquad k_2 = \mp \frac{\Theta}{b_0^2}, \qquad k_3 = \mp \frac{\Theta}{c_0^2} \tag{56}$$

for the principal parameters of that complex. In accordance with equations (58), k_2^0 and k_3^0 have the same signs and k_1^0 deviates from them in sign, from which, it will follow, with consideration given to the proportions (50), that k_3 and k_2 have the same sign, while k_1 has the opposite sign to them. We must then take the three upper or the three lower signs together in equations (55), as well as in equations (56).

If we multiply the three equations (55) by each other term-by-term then that will give:

$$k_1^0 k_2^0 k_3^0 = \pm\, a_0\, b_0\, c_0 \,. \tag{57}$$

The product of the parameters of the diameters of three conjugate complexes that coincide with the three associated diameters of the ruled surface is equal to the product of the three radii of the surface.

If we multiply the three equations (56) by each other term-by-term then that will give:

$$k_1\, k_2\, k_3 = \pm \frac{\Theta^3}{a_0^2\, b_0^2\, c_0^2} = \pm\, \gamma^2 \cdot a_0\, b_0\, c_0 = \pm\, \gamma^2 \cdot a\, b\, c,$$

so:

$$\frac{k_1 k_2 k_3}{abc} = \pm\, \gamma^2. \tag{58}$$

From the same three equations (56), we will further get:

$$(a_0^2 - b_0^2 - c_0^2) = \pm\, \Theta \left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \right),$$

and from this:

$$\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = \pm \frac{a^2 - b^2 - c^2}{abc}\,. \tag{59}$$

The sum of the reciprocal values of the parameters of any three complexes that are associated with a given ruled surface is constant.

130. We would like to displace the axes of the complexes of a three-parameter group by which a ruled surface is determined parallel to themselves until they go through the center of the surface, and then, when we consider the diameter of the surface to be a guiding ray, apply to each of the from the center outwards the principal parameters of the complexes that also have that diameter for their own. We will then get a new surface that plays the same role in relation to the ruled surface that the characteristic curve of a ruled surface does in relation to it. We would like to call the new surface the *characteristic surface* of the ruled surface.

If we denote any guiding ray of the characteristic surface by r and the corresponding guiding ray of the ruled surface by r_1 then we will have:

$$r = \pm \frac{\Theta}{r_1^2}.$$

We would like to refer the ruled surface (47) to its three axes as coordinate axes. When we then call the three angles that an arbitrary guiding ray r_1 defines with the three axes α, β, γ then we can write the equation of that surface in the way below:

$$\frac{\cos^2\alpha}{a^2} - \frac{\cos^2\beta}{b^2} - \frac{\cos^2\gamma}{c^2} = -\frac{1}{r_1^2},$$

and obtain the equation of the characteristic surface when we replace $1/r_1^2$ with its value $\pm r/\Theta$ in this equation. In this way, we will get:

$$\Theta\left\{\frac{\cos^2\alpha}{a^2} - \frac{\cos^2\beta}{b^2} - \frac{\cos^2\gamma}{c^2}\right\} = \pm r,$$

and if we revert to the rectangular point coordinates:

$$\Theta\left\{\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right\} = \pm r^3. \tag{60}$$

If we square both sides of the last equation and write $(x^2 + y^2 + z^2)$ and $a^2\,b^2\,c^2$ for r^2 and Θ^2, respectively, then that will give:

$$a^2\,b^2\,c^2\left\{\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right\}^2 = (x^2 + y^2 + z^2)^3, \tag{61}$$

or:

$$(b^2\,c^2\,x^2 - a^2\,c^2\,y^2 - a^2\,b^2\,z^2)^2 = a^2\,b^2\,c^2\,(x^2 + y^2 + z^2)^3. \tag{62}$$

131. The complete characteristic surface decomposes into two parts that are individually represented by equation (60) when we take r with both signs in succession. There will always be two three-parameter groups of complexes that have the geometric relationship to each other that the complexes of the two groups differ from each other merely by the fact that their parameters have opposite signs. Two such complex groups will correspond to the two generators of the same surface. In particular, there will thus be two systems of three conjugate complexes whose diameters are parallel to any three associated diameters of the ruled surface, as well, and whose respective parameters are equal, but have the opposite signs. The two generators of the ruled surface will be determined by the two groups of associated complexes. *The characteristic surface will be related equivalently to both generators.*

132. In accordance with the relations (48), we can introduce the principal parameters of three conjugate complexes whose diameters are parallel to the axes of the ruled surface into the equation of the characteristic surface in place of the three semi-axes of the ruled surface. In that way, we will find that:

$$(k_1 x^2 + k_2 y^2 + k_3 z^2)^2 = (x^2 + y^2 + z^2)^3, \tag{63}$$

while the equation of the ruled surface itself will become the following one:

$$k_1 x^2 + k_2 y^2 + k_3 z^2 = k_1 k_2 k_3 , \tag{64}$$

after the introduction of those constants.

If we set x, y, z equal to zero in equation (63) in sequence then we will obtain the following equations for the intersection curve of the characteristic surface with the three coordinate planes:

$$\left.\begin{aligned} (k_1x^2 + k_2y^2)^2 &= (x^2 + y^2)^3, \\ (k_1x^2 + k_3z^2)^2 &= (x^2 + z^2)^3, \\ (k_2y^2 + k_3z^2)^2 &= (y^2 + z^2)^3. \end{aligned}\right\} \tag{67}$$

Three congruences are determined by three conjugate complexes, when taken pair-wise, which we can refer to as three *conjugate congruences* of the ruled surface, in their own right. The two systems of three conjugate complexes whose diameters are parallel to the three axes of the ruled surface correspond to two systems of three conjugate congruences, each of which has an axis of the ruled surface for its principal axis and its other two axes for auxiliary axes. The three congruences of one of the two systems correspond to one of the three congruences of the other system of the other generator of the ruled surface. The first of the three equations (67) agrees completely with equation (149) of the previous paragraph. From that, we infer the following theorem:

The three intersection curves of the characteristic surface of a given ruled surface with the three principal sections XY, XZ, YZ of the latter surface are, in these principal *sections, the projections of the characteristic curves of three conjugate congruences that have OZ, OY, OX for their principal axes, respectively* (*).

We confirm what the discussion in the previous paragraphs of the intersection curve (67) asserted, and remark merely that the intersection curve that lies in *XY* and *XZ* consists of four loops and has a four-fold point at the center of the surface, while that point will be an isolated point for the intersection curve that lies in *YZ*.

133. In order to satisfy the equation of the characteristic surface, we can simultaneously set:

$$\left.\begin{aligned} k_1x^2 + k_2y^2 + k_3z^2 &= \pm\,\kappa\cdot k_1k_2k_3, \\ x^2 + \quad y^2 + \quad z^2 &= \kappa_0\sqrt{k_1^2\,k_2^2\,k_3^2}\,, \end{aligned}\right\} \tag{68}$$

when we denote two arbitrary constants by κ and κ_0, between which the following relationship exists:

$$\kappa^2 = \kappa_0^3.$$

In particular, this relation will be satisfied when the two constants are equal to unity. A characteristic surface can be described by a space curve that is *the intersection of a sphere with two second-order surfaces.* In each of its positions, this curve will determine complexes whose parameters are equal, up to sign.

In our case, the given ruled surface is a one-sheeted hyperboloid. If we once more introduce the square of the semi-axes into the last two equations, instead of the parameters, then we will obtain:

$$\left.\begin{aligned} \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} &= \pm\,\kappa, \\ x^2 + y^2 + z^2 &= \kappa_0\sqrt{a^2b^2c^2}\,. \end{aligned}\right\} \tag{69}$$

When we set κ and κ_0 equal to unity, the first of the two foregoing equations will represent the given one-sheeted hyperboloid when we take the lower sign and a two-sheeted hyperboloid when we take the upper sign. The two hyperboloids will have the same asymptotic cones, and the squares of any two equally-directed diameters of them will be equal and of opposite signs. We would like to call a one-sheeted hyperboloid and a two-sheeted one that have this reciprocal relationship to each other two *associated* hyperboloids.

The characteristic surface of a given one-sheeted hyperboloid goes through the curve along which intersect the given one-sheeted hyperboloid and the two-sheeted hyperboloid that is associated with a sphere whose radius is equal to the cube root of the product of three semi-axes of the given hyperboloid.

(*) We can extend the theorem in the text to three arbitrary associated complexes of a given ruled surfaces and the three corresponding associated diameters.

If we let the linear dimensions of the two hyperboloids increase in a quadratic way and let the radius of the sphere thus determined increase in a cubic way then the intersection curves will describe the characteristic surface.

134. The complete characteristic surface divides into two parts that are separated from each other by the asymptotic cone. The one part consists of path curves that lie on the one-sheeted hyperboloid. They determine the parameters of complexes whose diameters are parallel to the real diameters of the given one-sheeted hyperboloid. The other part consists of path curves that lie on the two-sheeted hyperboloids. They determine the (always real) parameters of the complexes whose diameters are parallel (along their lengths) to the imaginary diameters of the given one-sheeted hyperboloid. When the diameter of the surface through which the sides of the asymptotic cone goes becomes infinitely large, the corresponding parameters will become zero. This transition will correspond to the transitions between real and imaginary diameters of the surface and between positive and negative parameters of the complexes.

135. Up to now, we have considered merely the one-sheeted hyperboloid whose generators are real straight lines. The imaginary ruled surfaces, which we would like to call *imaginary ellipsoids*, correspond to the case where the parameters of the three central complexes have the same sign. If we correspondingly set:

$$\begin{aligned} k_2\, k_3 &= a^2, \\ k_1\, k_3 &= b^2, \\ k_1\, k_2 &= c^2 \end{aligned} \tag{70}$$

then we will get the following equation for the imaginary ellipsoid:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+1=0. \tag{71}$$

However, equation (64), which represents the ruled surface, will also remain real when the parameters of the three central complexes simultaneously become imaginary. If we replace k_1, k_2, k_3 with the imaginary values $k_1'\sqrt{-1}$, $k_2'\sqrt{-1}$, $k_3'\sqrt{-1}$ then that equation will go to the following one:

$$k_1'x^2+k_2'y^2+k_3'z^2=-\,k_1'k_2'k_3'. \tag{72}$$

Here, we have, in turn, two cases to distinguish: Either only two of the three new constants have the same sign and the third one has the opposite sign, or the signs of all of them coincide. In former case, we can set:

$$\begin{aligned} k_2'k_3' &= a^2, \\ k_1'k_3' &= -\,b^2, \end{aligned} \tag{73}$$

$$k_1' k_2' = -c^2,$$

and obtain:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \tag{74}$$

The surface is then a *two-sheeted hyperboloid*. In the second case, we can set:

$$\begin{aligned} k_2' k_3' &= a^2, \\ k_1\, k_3 &= b^2, \\ k_1\, k_2 &= c^2 \end{aligned} \tag{75}$$

and get:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{74}$$

The surface is then an *ellipsoid*.

Two imaginary generators will intersect at each point of the two-sheeted hyperboloid and the ellipsoid. The surfaces will be generated by imaginary lines in two ways, since the two imaginary straight lines that intersect at each point of the surfaces will belong to their two generators.

136. The considerations concerning characteristic surfaces in number **133** will be first completed when we consider the characteristic surface of the imaginary ruled surface (which remains real), and the imaginary characteristic surfaces of the two-sheeted hyperboloid and the ellipsoid, in addition to the characteristic surface of the one-sheeted hyperboloid.

We have called the one-sheeted and two-sheeted hyperboloid, which are represented by equations (47) and (74), two associated hyperboloids in the case where $k_1 = k_1'$, $k_2 = k_2'$, $k_3 = k_3'$. Under the same assumption, we will say that the imaginary and real ellipsoids that are represented by equations (72) and (76), resp., are associated.

In order to get the equation of the characteristic surface for the imaginary ellipsoid (72), we merely need to change the signs of b^2 and c^2 in the equation of this surface for the one-sheeted hyperboloid (47). In place of (61), one will then get:

$$a^2\, b^2\, c^2 \left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right\}^2 = (x^2 + y^2 + z^2)^3, \tag{77}$$

and in place of (69), we will get the following two equations:

$$\left.\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= \pm\kappa, \\ x^2 + y^2 + z^2 &= \kappa_0 \sqrt[3]{a^2 b^2 c^2}, \end{aligned}\right\} \tag{78}$$

in which the previous condition equations persist for κ and κ_0. Here, the characteristic surface consists of a real and an imaginary component. These two components will be generated by curves, along which real and imaginary ellipsoids that are associated with spheres will intersect. Among the imaginary ellipsoids, one finds the given one, in particular. The real ellipsoid that is associated with it will always be intersected by the characteristic surface along a real curve that simultaneously lies on a sphere whose radius is equal to the third root of the product of the three semi-axes of that ellipsoid.

If the equation of the real characteristic surface (63) for the case of the one-sheeted hyperboloid and the imaginary ellipsoid is to be related to the case of the two-sheeted hyperboloid and the real ellipsoid then we must switch k_1, k_2, k_3 with $k_1'\sqrt{-1}$, $k_2'\sqrt{-1}$, $k_3'\sqrt{-1}$. The square of the guiding ray of the characteristic surface will then become negative, so the surface itself will be imaginary. However, we will get a new real surface when we take $r\sqrt{-1}$ for the imaginary guiding ray. We will then get:

$$(k_1' x^2 + k_2' y^2 + k_3' z^2)^2 = (x^2 + y^2 + z^2)^3,$$

and when $k_1 = k_1'$, $k_2 = k_2'$, $k_3 = k_3'$, in particular, this equation will be the same as the one that we started with.

The characteristic surface of a one-sheeted hyperboloid then likewise determines the imaginary parameters of all complexes of the associated two-sheeted hyperboloid, as well as the characteristic surface of an imaginary ellipsoid, and the imaginary parameters of all complexes of the associated real ellipsoid.

137. In number **98**, we distinguished between four different types of congruences. Any diameter of a surface of order and class two that has a center will coincide, in direction and magnitude, with the principal diameter of a congruence that belongs to the surface. It will thus correspond to the diameter of the *one-sheeted hyperboloid* of a congruence of the first or second kind, according to whether this diameter does or does not cut the hyperboloid. The transition refers to the case in which the two directrices of the congruence coincide in an asymptote of the surface.

Any diameter of an *imaginary ellipsoid* will correspond to a congruence of the second kind.

Any diameter of a *two-sheeted hyperboloid* will correspond to a congruence of the third or fourth kind according to whether it does or does not cut the surface, resp.

Any diameter of a *real ellipsoid* will correspond to a congruence of the third kind.

138. The surfaces of order and class two that have no center and, in one case, are generated by real straight lines and in the other case, by imaginary ones are excluded from the foregoing developments; viz., the hyperbolic and elliptic paraboloids, resp.

139. In number **111**, we already determined a second-order surface by three complexes whose parameters were equal to zero. It emerged from this that the axes of three such complexes should be considered to be lines of the second generator of the surface that are cut by the lines of the first generator. We drew three arbitrary planes through the three lines of the second generator and took these planes to be coordinate planes. The surface then contacted these three planes. These planes cut the surface in three lines of the first generator, in addition to the three lines of the second generator. In each of these planes, the intersection of the lines was the two-fold generator of the points at which they contacted the surface. With analogous assumptions, we can also determine the two-sheeted hyperboloid and the ellipsoid. We take any three tangential planes of one of these surfaces to be coordinate planes. Each of these planes will then go through two conjugate imaginary lines of the surface, and these lines will intersect in the real points at which the planes contact the surface. Consistent with that, we would like to set:

$$\left.\begin{aligned} t', t'' &\equiv t_0 \pm t_0'\sqrt{-1}, \\ u', u'' &\equiv u_0 \pm u_0'\sqrt{-1}, \\ v', v'' &\equiv v_0 \pm v_0'\sqrt{-1}, \end{aligned}\right\} \tag{79}$$

when we return to the cited number. Equations (12) and (13) of that number, which represent the three lines of the second generator and the three lines of the first one that lie in the three coordinate planes, will then go to the following ones:

$$\left.\begin{aligned} (t_0 + t_0'\sqrt{-1})x + (u_0 - u_0'\sqrt{-1})y + 1 &= 0, \\ (v_0 + v_0'\sqrt{-1})z + (t_0 - t_0'\sqrt{-1})x + 1 &= 0, \\ (u_0 + u_0'\sqrt{-1})y + (v_0 - v_0'\sqrt{-1})z + 1 &= 0, \end{aligned}\right\} \tag{80}$$

and

$$\left.\begin{aligned} (t_0 - t_0'\sqrt{-1})x + (u_0 + u_0'\sqrt{-1})y + 1 &= 0, \\ (v_0 - v_0'\sqrt{-1})z + (t_0 + t_0'\sqrt{-1})x + 1 &= 0, \\ (u_0 - u_0'\sqrt{-1})y + (v_0 + v_0'\sqrt{-1})z + 1 &= 0. \end{aligned}\right\} \tag{81}$$

The coordinates of the three contact points with the three *XY*, *XZ*, *YZ* coordinate planes are (*):

$$x_1 = \frac{-u_0'}{t_0u_0' + t_0'u_0}, \qquad y_2 = \frac{-t_0'}{t_0u_0' + t_0'u_0},$$

(*) Equations (81) immediately give:

$$x_1\, y_1\, z_1 = x_2\, y_2\, z_2,$$

which is a geometric relationship between any three tangential planes of a given surface of order and class two, but this is not the place for a discussion of that.

$$x_2 = \frac{-v_0'}{t_0 v_0' + t_0' v_0}, \qquad z_1 = \frac{-t_0'}{t_0 v_0' + t_0' v_0}, \tag{82}$$
$$y_1 = \frac{-v_0'}{u_0 v_0' + u_0' v_0}, \qquad z_2 = \frac{-u_0'}{u_0 v_0' + u_0' v_0}.$$

We get the equation of the ruled surface from (9):

$$\begin{aligned} 1 + 2t_0\, x + 2u_0\, y + 2v_0\, z + (t_0^2 + t_0'^2)\, x^2 + (u_0^2 + u_0'^2)\, y^2 + (v_0^2 + v_0'^2)\, z^2 \\ + 2\,(u_0\, v_0 - u_0'\, v_0')\, y\, z + 2\,(t_0\, v_0 - t_0'\, v_0')\, x\, z + 2(t_0\, u_0 - t_0'\, u_0')\, x\, y = 0. \end{aligned} \tag{83}$$

According to whether:

$$(t_0^2 + t_0'^2)^2 > (t_0 v_0' - t_0' v_0)(t_0 v_0' - t_0' v_0)$$

or

$$(t_0^2 + t_0'^2)^2 < (t_0 v_0' - t_0' v_0)(t_0 v_0' - t_0' v_0),$$

this equation will represent a *two-sheeted hyperboloid* or an *ellipsoid*, respectively (*).

If we set t_0, u_0, v_0 equal to zero in this equation then that will give:

$$1 + t_0'^2 x^2 + u_0'^2 y^2 - v_0'^2 z^2 - 2u_0'\, v_0' y\, z - 2t_0'\, v_0' x\, z - 2t_0'\, u_0' x\, y = 0. \tag{84}$$

The surface will then be a two-sheeted hyperboloid that is referred to its center as the origin of the coordinates.

140. The determination of the elliptic paraboloid is completely analogous to the determination of the hyperbolic one in number **114**. The condition equation (20) goes to the following one:

$$\frac{t_0}{t_0'} + \frac{u_0}{u_0'} + \frac{v_0}{v_0'} = 1. \tag{85}$$

Along with the lines of the two generators, the two planes that are parallel to them will also be conjugate imaginary. We get the following equations for them, which we combine into a single one:

$$\begin{aligned} [(t_0 v_0 + t_0' v_0') \mp (t_0 v_0' - t_0' v_0)\sqrt{-1}]\, x + [(u_0 v_0 + u_0' v_0') \mp (u_0 v_0' + u_0' v_0)\sqrt{-1}]\, y + (v_0^2 + v_0'^2)\, z \\ = 0, \end{aligned} \tag{86}$$

and for the determination of the direction of the real intersection, we will get:

(*) *Geometrie des Raumes*, no. **26**.

$$\frac{t_0^2 + t_0'^2}{t_0'}\,x = \frac{u_0^2 + u_0'^2}{u_0'}\,y = \frac{v_0^2 + v_0'^2}{v_0'}\,z \tag{87}$$

from (24).

141. With that, we have represented all of the real surfaces of order and class two by equations of three linear complexes whose parameters vanish, and derived their equations in point coordinates from a system of three such equations: the one-sheeted hyperboloid (9), the same thing, referred to its center (17), the hyperbolic paraboloid (9), under the assumption of the condition equation (20), the two-sheeted hyperboloid, and the ellipsoid (83), the former referred to its center (84), and finally, under the assumption of the condition equation (85), the elliptic paraboloid (86). The assumption that the coordinate origin lies inside of the stated surfaces remains excluded here for the cases of the two-sheeted hyperboloid, the elliptic paraboloid, and the ellipsoid. There is no inside and no outside for the case of the one-sheeted hyperboloid. The imaginary surfaces are excluded completely. The coordinate system will become illusory when the origin is chosen to be on the surface.

142. The same surfaces of order and class two that we have represented by three linear equations in ray coordinates have also been represented by us in an analogous way by three equations in axial coordinates, and just as we have derived the equation of the surface in point coordinates, we have also derived the equation of the that surface in plane coordinates. For real surfaces that are not generated by real straight lines, equation (28), which we obtained in number **115**, will go to the following one:

$$\frac{((t-t_0)-t_0'\sqrt{-1})((u-u_0)-u_0'\sqrt{-1})((v-v_0)-v_0'\sqrt{-1})}{((t-t_0)+t_0'\sqrt{-1})((u-u_0)+u_0'\sqrt{-1})((v-v_0)+v_0'\sqrt{-1})} = 1. \tag{88}$$

If we develop this then the imaginaries will vanish from this equation.

143. Real and imaginary conic surfaces, as well as real and imaginary plane curves, can be represented by three linear equations, either in ray coordinates or axial coordinates. These are not to be regarded as surfaces of class two or as surfaces of order two.

144. However, the question of whether the ruled surfaces that we have represented by the symbol:

$$\Omega + \mu\,\Omega' + \mu'\Omega'' = 0$$

might degenerate into other geometric structures by specializing the complexes Ω, Ω', Ω'' is not resolved with that.

We would like to let one of the three complexes retain its full generality, but assume that the other complexes are of the special sort that all of the lines in each of them encounter a fixed straight line, and that the two fixed straight lines *intersect*, or – what amounts to the same thing – *lie in the same plane.* We would like to let the two *OZ* and *OY* coordinate axes coincide with them. The equations:

$$\Omega' \equiv \eta \equiv r\sigma - s\rho = 0, \qquad\qquad \Omega'' \equiv \rho = 0$$

will then represent the two complexes in question. These two equations have the consequence that *either* σ *or* ρ will be equal to zero. In agreement with this, on the one hand, all lines whose coordinates satisfy the three equations:

$$\left.\begin{array}{c} Ar + Bs + C = 0, \\ \rho = 0, \;\; \sigma = 0, \end{array}\right\} \tag{89}$$

and, on the other hand, all lines whose coordinates satisfy the three equations:

$$\left.\begin{array}{c} Bs + C - D\sigma = 0, \\ \rho = 0, \;\; r = 0, \end{array}\right\} \tag{90}$$

will belong to the ruled surface that is represented by the three-parameter complex group. All lines that simultaneously belong to the three complexes (89) will lie in the plane that is represented by the equation:

$$Ax + By + Cz = 0 \tag{91}$$

and will go through the coordinate origin in that plane. All lines that simultaneously belong to the three complexes (90) will lie in the *YZ* coordinate plane and will go through the point in that plane that is represented by the equation:

$$Cu - Bv + Dw = 0. \tag{92}$$

The plane (91) will remain the same for all complexes of the three-parameter group:

$$(Ar - Bs + C) + \mu\rho + \mu's = 0$$

as it is for *all* of the planes that correspond to the coordinate origin. The point (92) will remain the same for all complexes of the three-parameter group:

$$(Bs + C - D\sigma) + \mu\rho + \mu'r = 0$$

as it is for *all* of the points that correspond to the *YZ* coordinate plane.

The lines that belong to the ruled surfaces thus determined then *lie in two planes and go through a fixed point of the line of intersection of the two planes in each of these two planes. The two planes and the two points correspond to each other in all complexes of the three-parameter group.*

We can represent the ruled surface by a second-order equation in point coordinates. We will then obtain the two planes that we just determined; however, each trace of the generators of these planes will vanish along a straight line that rotates around a fixed point inside of them. We will get the two points when we appeal to plane coordinates for the representation of the ruled surface; however, each trace of the envelope of these points will vanish along a straight line that lies in a fixed plane.

145. In this case, the geometric determination of the ruled surface will come down to the determination of the straight lines that intersect two given, mutually-intersecting, straight lines and belong to a given complex, moreover. These lines will either lie in the plane of the two given straight lines and go through the intersection point or they will go through the intersection point of the two given straight lines and likewise lie in the planes that correspond to that point in the complex.

The foregoing geometric considerations can be extended in such a way that a complex of the special kind can be found amongst the complexes of the group. The fixed lines that are cut by all lines of this complexes and which do not encounter the two given straight lines, in general, will, like them, be cut by the lines of the ruled surface. This ruled surface is, in general, a one-sheeted hyperboloid whose lines cut a generator of the three given straight lines, but degenerates when two of the three given lines intersect into a system of two planes or a system of two points, respectively.

Nothing essential will change in the foregoing relationships when the fixed line encounters one of the two given intersecting straight lines,. One of the three lines of the same surface generator will then be cut by the remaining two at two points, or – what amounts to the same thing – the three generators will lie in two planes. These two planes, on the one hand, and the two intersection points, on the other, will be the ones into which the ruled surface will degenerate.

146. However, if the intersection point of the two given straight lines corresponds to the plane that goes through that line in the complexes of the three-parameter group then the constants B and C will vanish in the foregoing analytic developments; the plane (91) will then coincide with the YZ coordinate plane and the point (92), with the coordinate origin.

In this case, the ruled surface will degenerate into *two coincident planes* (a system of two *coincident points* in these planes, respectively).

147. In the last cases considered, one must carefully distinguish between the one for which:

$$\sigma = 0, \quad \rho = 0, \quad \eta = 0, \tag{93}$$

or

$$r = 0, \quad \rho = 0, \quad \eta = 0, \tag{93}$$

with no further conditions. In the first case, the coordinates of every straight line that goes through the origin will satisfy the three-parameter complex equation, while in the second case, the coordinates of every straight line that lies in the *YZ* plane will satisfy it.

Just as two coincident directrices will first determine a congruence (no. **68**) when one adds the condition that its lines belong to a given complex that has a line that coincides with the directrices, so will three generators that go through the same point first determine a ruled surface when one adds the condition that its lines belong to a given complex. When these two conditions are absent, we will obtain, in the first case, a complex of the special kind whose lines cut the two coincident directrices instead of the congruence. In the second case, two of the three complex equations:

$$\rho = 0, \qquad \eta \equiv r\,\sigma - s\,\rho = 0 \tag{95}$$

require that $r\,\sigma$ must be equal to zero, and this condition can correspond to the vanishing of σ, as well as the vanishing of r. Thus, the two foregoing equations will be equivalent, in the one case, to the three equations (93), and in the other, to the three equations (94). The congruences of the special kind that are represented by the two equations (95) and have the *OZ* and *OY* coordinate axes for their directrices will encompass, in the one case, all lines that lie in the *YZ* – viz., the plane of the two directrices – and in the other case, all lines that go through *O* – viz., the intersection of the two directrices. If one adds the condition $\sigma = 0$ to equation (93) then all of the lines that lie in the plane of the two directrices and do not go through their intersection will be excluded from the congruence. If one adds the condition $r = 0$ to equation (94) then all lines will be excluded from the congruence that go through the intersection of the two directrices and do not lie in the same plane as them. We can then say that the two equations (93) and (94) together will represent the congruences of the special kind. The lines of the one component of the congruence will envelop a point that we will consider to be a ruled surface of class one that can be represented by an equation in plane coordinates. The lines of the other component of the congruence will lie in a plane that we will consider to be a ruled surface of order one and that can be represented by an equation in point coordinates (*).

148. In the present paragraphs – in which we introduced the straight line, in its double geometric meaning as a ray and an axis, as a *space element*, instead of the point and the plane – we determined a surface of order and class two by three linear equations in such a way that each of its two generators were represented by three such equations. While the surface, and thus its tangential planes and their contact points are real, the two lines of intersection of the tangential planes with the surface – viz., the two generators that go through the contact point – can be real, as well as imaginary. From that viewpoint, thus extended, we can regard all surfaces of order and class two as ruled surfaces. All of the properties of such surfaces, including the path that is taken in the foregoing, can be derived in the same way from the discussion of the three linear

(*) In order to prevent possible mistakes in the analytical discussion of the particular cases in question, it is generally advisable to base it upon homogeneous equations in the six line coordinates. For example, if we switch the *OZ* and *OX* coordinate axes with each other in the foregoing analytical discussion then we can easily be led to hasty conclusions.

equations in ray and axial coordinates as they have been derived up to now from the discussion of a quadratic equation in point or plane coordinates.

Chapter Two

Second-degree complexes.

Part I.

Two-fold analytic representation of a complex of degree two. Complex curves of class two that are enveloped by lines of the complex. Complex cones of order two that are described by lines in it. Complex surfaces of order and class four, one of which is described by complex curves, and the other of which is enveloped by complex cones.

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§ 1.

The general equation of the second-degree line complex in ray and axial coordinates.

149. Of the four ray coordinates:

$$r,\ s,\ \rho,\ \sigma,$$

r and s mean the trigonometric tangents of the angles that the two projections of the rays onto the XZ and YZ coordinate planes define with the OZ coordinate axis, while ρ and σ mean the line segments that these two projections cut out of the OX and OY coordinate axes. The fifth ray coordinate:

$$\eta \equiv r\sigma - s\rho$$

is derived from them.

Let the general second-degree equation in the five coordinates be the following one (*):

$$\text{(I)}\qquad \begin{aligned} &Ar^2 + Bs^2 + C + D\sigma^2 + E\rho^2 + F\eta^2 \\ &+ 2Gs + 2Hr + 2Jrs + 2K\rho\eta - 2L\sigma\eta - 2M\rho\sigma \\ &- 2Nr\sigma + 2Os\rho \\ &+ 2Pr\rho + 2Qr\eta + 2Rs\eta - 2Ss\sigma - 2T\sigma + 2U\rho = 0. \end{aligned}$$

(*) The same considerations that allowed us to take negative signs for the coefficients of σ and κ in the general equation of the first-degree complex in five ray or axial coordinates, respectively (confer the note in number **26**), allow us to do the same thing in the corresponding equations for the second-degree complex.

This equation contains nineteen mutually independent constants. It is unnecessary to add a last term of $+ 2V\eta$; that would have the effect of reducing the absolute values of the constants N and O by V. Indeed, the introduction of such a superfluous term would succeed in making things symmetric, in general. It is not advisable to preserve such a term for special examinations, and all the less since we can add it into special cases with no further analysis.

150. From this general equation, we can immediately go on to the following one, in which x', y', z' and x, y, z appear as the coordinates of any two points of a line of the complex (*)(**):

$$
\text{(II)}\quad
\begin{aligned}
&A(x-x')^2+B(y-y')^2+C(z-z')^2\\
&+D(yz'-y'z)^2+E(x'z-xz')^2+F(xy'-x'y)^2\\
&+2G(y-y')(z-z')+2H(x-x')(z-z')+2J(x-x')(y-y')\\
&+2K(xy'-x'y)(x'z-xz')+2L(xy'-x'y)(yz'-y'z)+2M(x'z-xz')(yz'-y'z)\\
&+2N(x-x')(yz'-y'z)+2O(y-y')(x'z-xz')\\
&+2P(x-x')(x'z-xz')+2Q(x-x')(xy'-x'y)\\
&+2R(y-y')(xy'-x'y)+2S(y-y')(yz'-y'z)\\
&+2T(z-z')(yz'-y'z)+2U(z-z')(x'z-xz')=0.
\end{aligned}
$$

If we regard x', y', z' as the coordinates of any fixed point and then take them to be constant, while we let x, y, z vary, then this general complex equation will be the equation of a *second-order conic surface. This conic surface will have the fixed point for its center, and its lines will be the lines of the complex that go through the center.*

151. Of the four axial coordinates:

$$p,\ q,\ \pi,\ \kappa;$$

the last two π and κ; when taken to be reciprocal and negative, mean that x and y are the two points at which the straight line cuts the XZ and YZ planes, respectively. If one connects these two points with the coordinate origin with straight lines then these lines will define two angles with the OZ axis in the XZ and YZ coordinate planes whose

(*) Introductory considerations in no. **2**.

(**) The two terms:

$$2N(x-x')(yz'-y'z) + 2O(y-y')(x'z-xz')$$

would combine with the superfluous term:

$$2V\eta \equiv 2\,V(z-z')(xy'-x'y).$$

However, the three terms could then combine into the following two:

$$2(N-V)(x-x')(yz'-y'z) + 2(O-V)(y-y')(x'z-xz').$$

trigonometric tangents, which are taken to be reciprocal and negative, will be p and q. The fifth coordinate:

$$\omega \equiv p\kappa - q\pi$$

is derived from them.

The equation of the same second-degree complex that we represented in ray coordinates by equation (I) will become the following one with the use of axis coordinates (*):

$$\text{(III)} \quad \begin{aligned} &Dp^2 + Eq^2 + F + A\kappa^2 + B\pi^2 + C\omega^2 \\ &+ 2Kq + 2Lp + 2Mpq + 2G\pi\omega - 2H\kappa\omega - 2J\pi\kappa \\ &- 2Np\kappa + 2Oq\pi \\ &+ 2Sp\pi + 2Tp\omega + 2Uq\omega - 2Pq\kappa - 2Q\kappa + 2R\pi = 0. \end{aligned}$$

152. We can immediately go from this general equation to one in which t', u', v' and t, u, v appear as the coordinates of any two planes that intersect in the line in question (**):

$$\text{(IV)} \quad \begin{aligned} &D(t-t')^2 + E(u-u')^2 + F(v-v')^2 \\ &+ A(uv'-u'v)^2 + B(t'v-tv')^2 + C(tu'-t'u)^2 \\ &+ 2K(u-u')(v-v') + 2L(t-t')(v-v') + 2M(t-t')(u-u') \\ &+ 2G(tu'-t'u)(t'v-tv') + 2H(tu'-t'u)(uv'-u'v) + 2J(t'v-tv')(uv'-u'v) \\ &+ 2N(t-t')(uv'-u'v) + 2O(u-u')(t'v-tv') \\ &+ 2S(t-t')(t'v-tv') + 2T(t-t')(tu'-t'u) \\ &+ 2U(u-u')(tu'-t'u) + 2P(u-u')(uv'-u'v) \\ &+ 2Q(v-v')(uv'-u'v) + 2R(v-v')(t'v-tv') = 0. \end{aligned}$$

When we let t', u', v refer to an arbitrary, fixed plane and correspondingly consider them to be constant, equation (IV), which is the general equation of a second-degree complex, will represent *a curve of class two that will be enveloped by the lines of the complex that lie in the fixed plane.*

153. The exchange of:

$$r,\ s,\ 1,\ -\sigma,\ \rho,\ \eta$$

and

$$-\kappa,\ \pi,\ \omega,\ p,\ q,\ 1,$$

as well as the corresponding exchange of:

$$(x-x'),\ (y-y'),\ (z-z'),\ (yz'-y'z),\ \ (x'z-xz'),\ (xy'-x'y)$$

(*) Introductory considerations from no. **5**.

(**) Intro. cons., no. **3**.

and

$$(uv' - u'v),\ (t'v - tv'),\ (tu' - t'u),\ (t - t'),\ (u - u'),\ (v - v'),$$

which we must make in order for equations (I) and (III) and equations (II) and (IV) to be consistent with each other, come from the exchange of:

$$r,\ s,\ \sigma,\ \rho,\ \eta$$

with

$$p,\ q,\ \kappa,\ \pi,\ \omega,$$

resp., on the one hand, and the exchange of:

$$x,\ y,\ z,\ x',\ y',\ z'$$

with

$$t,\ u,\ v,\ t',\ u',\ v',$$

resp., on the other, as well as the reciprocal exchange of:

$$A, B, C, \qquad G, H, J, \qquad P, Q, R$$

with

$$D, E, F, \qquad K, L, M, \qquad S, T, U$$

resp., in both cases.

154. Equation (I) will first become symmetric when we make it homogeneous by the introduction of a sixth variable into it, as was suggested already (intro. cons., no. **6**). If h is the new variable, and we preserve the superfluous constant V, moreover, then (I) will go to (*):

(*) The introduction of h amounts to the replacement of the first of the three projections of the straight line (r, ρ, s, σ):

$$x = rz + \rho, \qquad y = sz + \sigma, \qquad ry = sx + \eta$$

with the following two:

$$hx = rz + \rho, \qquad hy = sz + \sigma.$$

We can thus represent the straight line in a symmetric way in terms of the two equations of any two of its projections, such as the last two, in the following way:

$$sx = ry - \eta, \qquad sz = hy - \sigma,$$

and the following:

$$ry = sx + \eta, \qquad rz = hx - \rho,$$

in which the condition equation will be fulfilled:

$$r\sigma - s\rho = h\eta.$$

It is hardly necessary to remark here that when we write $(z - z')$ for h, equation (V) will go to equation (I).

$$\text{(V)}\quad \begin{gathered} Ar^2 + Bs^2 + Ch^2 + D\sigma^2 + E\rho^2 + F\eta^2 \\ + 2Gsh + 2Hrh + 2Jrs + 2K\rho\eta - 2L\sigma\eta - 2M\rho\sigma \\ - 2Nr\sigma + 2Os\rho + 2Vh\eta \\ + 2Pr\rho + 2Qr\eta + 2Rs\eta + 2Ss\sigma - 2Th\sigma + 2Uh\rho = 0. \end{gathered}$$

155. A permutation of the three coordinate axis with each other corresponds to a permutation of the constants in the general equation of the second-degree complex. We would thus like to use equation (II) as a basis, but for the sake of symmetry, we will add the term:

$$2V(z - z')(xy' - x'y),$$

when we exchange N and O with N' and O', resp., and set:

$$N = N' - V', \qquad O = O' - V'.$$

If we then first exchange the two coordinate axes OX and OY with each other then $(x - x')$ and $(y - y')$ will switch reciprocally, while $(z - z')$ will remain unchanged, as well as exchanging $(x'z - xz')$ and $-(yz' - y'z)$, while $(xy' - x'y)$ will change sign. In that way, the exchange will by no means affect the coefficients:

$$C, F, J, M,$$

while

$$A, D, G, K$$

will switch with

$$B, E, H, L,$$

respectively, with no change of sign, and:

$$N', P, R, T$$

will switch with

$$O', S, Q, U,$$

resp., with a simultaneous change of sign, and V' will change its sign.

Thus, in equation (V):

$$r \text{ and } \rho$$

will switch reciprocally with

$$s \text{ and } \sigma,$$

respectively, while η will change it sign.

If we secondly permute the OX and OZ with each other then the expressions $(x - x')$ and $(z - z')$ will switch in (II), while $(y - y')$ will remain unchanged, and likewise $(yz' - y'z)$ will switch with $-(xy' - x'y)$, while $(x'z - xz')$ will change its sign. Correspondingly, one exchanges:

$$r \text{ and } \rho$$

with

$$h \text{ and } -\eta$$

resp., in (V), while s changes its sign. The exchange will therefore not affect the coefficients:

$$B, E, H, L,$$

while the coefficients:

$$A, D, G, K$$

will switch with:

$$C, F, J, M,$$

resp., with no change of sign, but:

$$N', P, R, T$$

will switch with:

$$V', U, S, Q,$$

resp., with simultaneous sign changes, and O' will change its sign.

If we thirdly permute the two axes OY and OZ with each other then the expressions $(y - y')$ and $(z - z')$ will switch with each other in (II), as well as $(xy' - x'y)$ and $-(x'z - xz')$, while $(x - x')$ will remain unchanged, and $(yz' - y'z)$ will change its sign. One exchanges:

$$s \text{ and } \sigma$$

with

$$h \text{ and } \eta,$$

resp., in (V), while ρ changes its sign. The exchange will not affect:

$$A, D, G, K,$$

while the coefficients:

$$B, E, H, L$$

will switch with:

$$C, F, J, M,$$

resp., with no change of sign, but:

$$O', P, R, T$$

will switch with:

$$V', Q, U, S,$$

resp., with a simultaneous change of sign, and N' will change its sign.

It remains for us discuss the modifications that come about when we let the superfluous term drop away.

If we set V' equal to zero in the first permutation then the coefficients:

$$N \text{ and } O$$

will switch simultaneously with the exchange of the OX and OY coordinate axes with a change of sign.

If we set O' equal to zero in the second permutation then that will make $V' = -O$, $N' = N - O$. With the exchange of the OX and OZ coordinate axes, V' and N' will switch with a change of sign, or – what amounts to the same thing:

$$O \text{ and } N - O$$

will switch with no change of sign.

Finally, in the third case, the coefficients:

$$N \text{ and } N - O$$

will switch with a change of sign simultaneously with the OY and OZ coordinate axes.

One should not overlook the fact that in all equations the rotational moments of OX with respect to OY, of OY with respect to OZ, and of OZ with respect to OX are to be taken after the permutation.

156. Since equation (III) contains the same constants – but in a different sequence – as equation (I), if it is to represent the same second-degree complex when referred to the same coordinate axes then the permutation rules that were developed in the previous number will also preserve their complete and immediate validity for the equation of the complex in axis coordinates.

157. If we place the coordinate origin at any point (x_0, y_0, z_0) then the following expressions will appear in place of ρ, σ, η (intro. cons. no. **14**):

$$\begin{aligned} &\rho + r\,z_0 - x_0\,, \\ &\sigma + s\,z_0 - y_0\,, \\ &\eta + s\,x_0 - ry_0\,, \end{aligned}$$

while r and s remain unchanged, with which, equation (I) will go to (*):

$$\begin{aligned} &(A + Ez_0^2 + Fy_0^2 - 2Ky_0z_0 + 2Pz_0 - 2Qy_0)r^2 \\ &+ (B + Dz_0^2 + Fx_0^2 - 2Lx_0z_0 + 2Rx_0 - 2Sz_0)s^2 \\ &+ (C + Dy_0^2 + Ex_0^2 - 2Mx_0y_0 + 2Ty_0 - 2Ux_0) \\ &\qquad + D\sigma^2 + E\rho^2 + F\eta^2 \end{aligned}$$

(*) If we introduce the three terms:

$$-2Nr\sigma + 2Os\rho + 2Vh\eta\,,$$

in place of the two terms:

$$-2Nr\sigma + 2Os\rho\,,$$

then we can write the values of these terms that we obtain after conversion as:

$$-2(N - Ly_0 + Mz_0)\,r\sigma + 2(O + Kx_0 - Mz_0)\,s\rho + 2(V - Kx_0 + Ly_0)\,\eta\,.$$

$$
\begin{aligned}
&+2(G - Dy_0 z_0 - Kx_0^2 + Lx_0 y_0 + Mx_0 z_0 - Ox_0 + Sy_0 - Tz_0)s \\
\text{(VI)}\qquad &+2(H - Ex_0 y_0 + Kx_0 y_0 - Ly_0^2 + My_0 z_0 + Ny_0 - Px_0 + Uz_0)r \\
&+2(J - Fx_0 y_0 + Kx_0 z_0 + Ly_0 z_0 - Mz_0^2 - (N - O)z_0 + Qx_0 - Ry_0)rs \\
&\qquad + 2K\rho\eta - 2L\sigma\eta - 2M\rho\sigma \\
&- 2(N + Kx_0 - 2Ly_0 + Mz_0)\, r\sigma + 2(O + 2Kx_0 - Ly_0 - Mz_0)\, s\rho \\
&\quad + 2(P + Ez_0 - Ky_0)\, r\rho + 2(Q - Fy_0 + Kz_0)\, r\eta \\
&\quad + 2(R + Fx_0 - Lz_0)\, s\eta - 2(S - Dz_0 + Lx_0)\, s\sigma \\
&\quad + 2(T + Dy_0 - Mx_0)\, \sigma + 2(U - Ex_0 + My_0)\, \rho = 0.
\end{aligned}
$$

158. In order to refer the equation of the complex in axial coordinates to the new origin, we merely need to carry out the same permutations by which we derived the complex equation (III) from (I) in number **153** in the present equations. In this way, we immediately get:

$$
\begin{aligned}
&\qquad Dp^2 + Eq^2 + F \\
&+(A + Ez_0^2 + Fy_0^2 - 2Ky_0 z_0 + 2Pz_0 - 2Qy_0)\kappa^2 \\
&+(B + Dz_0^2 + Fx_0^2 - 2Lx_0 z_0 + 2Rx_0 - 2Sz_0)\pi^2 \\
&+(C + Dy_0^2 + Ex_0^2 - 2Mx_0 y_0 + 2Ty_0 - 2Ux_0)\omega^2 \\
&\qquad + 2Kq + 2Lp + 2Mpq \\
&+2(G - Dy_0 z_0 - Kx_0^2 + Lx_0 y_0 + Mx_0 z_0 - Ox_0 + Sy_0 - Tz_0)\pi\omega \\
\text{(VII)}\qquad &-2(H - Ex_0 z_0 + Kx_0 y_0 - Ly_0^2 + My_0 z_0 + Ny_0 - Px_0 + Uz_0)\kappa\omega \\
&-2(J - Fx_0 y_0 + Kx_0 z_0 + Ly_0 z_0 - Mz_0^2 - (N - O)z_0 + Qx_0 - Ry_0)\pi\kappa \\
&-2(N + Kx_0 - 2Ly_0 + Mz_0)p\kappa + 2(O + 2Kx_0 - Ly_0 - Mz_0)q\pi \\
&\quad + 2(S - Dz_0 + Lx_0)\, p\pi + 2(T + Dy_0 - Mx_0)\, p\omega \\
&\quad + 2(U - Ex_0 + My_0)\, q\omega + 2(P + Ez_0 - Ky_0)\, q\kappa \\
&\quad + 2(Q - Fy_0 + Kz_0)\, \kappa + 2(R + Fx_0 - Lz_0)\, \pi = 0.
\end{aligned}
$$

159. We would like to further replace the coordinate system to which the complex (I) was originally referred with another one whose axes intersect at the original origin, but whose directions have changed arbitrarily. In the introductory considerations, this coordinate conversion was performed in three successive operations, in which each time one of the three coordinate axes was preserved, while the other two were rotated in their plane arbitrarily. We thus satisfy ourselves with writing the result of these three analogous operations. Starting from a rectangular coordinate system, we would like to preserve the *OZ* coordinate axis and rotate the other *OY* and *OZ* coordinate-axes in such a way that in their new positions they define the angles α and α' with *OX* in the original position, so that the angle that the *OX* and *OY* axes define with each other in the new position becomes $(\alpha' - \alpha) \equiv \vartheta$. When we switch (intro. cons., no. **14**):

$$
\begin{aligned}
&r \quad \text{with} \quad r\cos\alpha + s\cos\alpha', \qquad && s \quad \text{with} \quad r\sin\alpha + s\sin\alpha', \\
&\rho \quad \text{with} \quad \rho\cos\alpha + \sigma\cos\alpha', \qquad && \sigma \quad \text{with} \quad \rho\sin\alpha + \sigma\sin\alpha',
\end{aligned}
$$

$$\eta \quad \text{with } \eta \sin \vartheta,$$

equation (I) will go to the following one:

$$\begin{aligned}
&(A \cos^2 \alpha + B \sin^2 \alpha + 2J \sin \alpha \cos \alpha)\, r^2 + (B \sin^2 \alpha' + A \cos^2 \alpha' + 2J \sin \alpha' \cos \alpha')\, s^2 + C \\
&+ (D \sin^2 \alpha' + E \cos^2 \alpha' - 2M \sin \alpha' \cos \alpha')\, \sigma^2 + (E \cos^2 \alpha + D \sin^2 \alpha - 2M \sin \alpha \cos \alpha)\, \rho^2 + F \sin^2 \vartheta \cdot \eta^2 \\
&+ 2(G \sin \alpha' + H \cos \alpha')\, s + 2(H \cos \alpha + G \sin \alpha)\, r \\
&+ 2(J (\sin \alpha \cos \alpha' + \sin \alpha' \cos \alpha) + A \cos \alpha \cos \alpha' + B \sin \alpha \sin \alpha')\, r\, s \\
\text{(VIII)}\quad &+ 2(K \cos \alpha - L \sin \alpha) \sin \vartheta \cdot \rho\eta - 2(L \sin \alpha' - K \cos \alpha') \sin \vartheta \cdot \sigma\eta \\
&- 2(M (\sin \alpha \cos \alpha' + \sin \alpha' \cos \alpha) - D \sin \alpha \sin \alpha' - E \cos \alpha \cos \alpha')\, \rho\sigma \\
&- 2(N \sin \alpha' \cos \alpha - O \sin \alpha \cos \alpha' - P \cos \alpha \cos \alpha' + S \sin \alpha \sin \alpha')\, r\sigma \\
&- 2(O \sin \alpha' \cos \alpha - N \sin \alpha \cos \alpha' + P \sin \alpha \sin \alpha' - S \cos \alpha \cos \alpha')\, s\rho \\
&+ 2\, (P \cos^2 \alpha - (N - O) \sin \alpha \cos \alpha - S \sin^2 \alpha)\, r\rho + 2\, (Q \cos \alpha + R \sin \alpha) \sin \vartheta \cdot r\, \eta \\
&+ 2\, (R \sin \alpha' + Q \cos \alpha') \sin \vartheta \cdot s\eta - 2\, (S \sin^2 \alpha' + (N - O) \sin \alpha' \cos \alpha' - P \cos^2 \alpha')\, s\sigma \\
&- 2\, (T \sin \alpha' - U \cos \alpha')\, \sigma + 2\, (U \cos \alpha - T \sin \alpha)\, \rho = 0.
\end{aligned}$$

If we set:

$$\vartheta = \frac{\pi}{2}, \quad \sin \alpha' = \cos \alpha, \quad \cos \alpha' = -\sin \alpha$$

then the coordinate system will remain rectangular, and will be merely rotated through an angle of α around the OZ axis.

Under the same change of coordinate system, equation (III) will go to the following one:

$$\begin{aligned}
&(D \sin^2 \alpha' + E \cos^2 \alpha' - 2M \sin \alpha' \cos \alpha')\, p^2 + (E \cos^2 \alpha + D \sin^2 \alpha - 2M \sin \alpha \cos \alpha)\, q^2 + F \sin^2 \vartheta \\
&+ (A \cos^2 \alpha + B \sin^2 \alpha + 2J \sin \alpha \cos \alpha)\, \kappa^2 + (B \sin^2 \alpha' + A \cos^2 \alpha' + 2J \sin \alpha' \cos \alpha')\, \pi^2 + C\, \omega^2 \\
&+ 2\, (K \cos \alpha - L \sin \alpha) \sin \vartheta \cdot q + 2\, (L \sin \alpha' - K \cos \alpha') \sin \vartheta \cdot p \\
&- 2\, (M (\sin \alpha \cos \alpha' + \sin \alpha' \cos \alpha) - D \sin \alpha \sin \alpha' - E \cos \alpha \cos \alpha')\, pq \\
\text{(IX)}\quad &+ 2\, (G \sin \alpha' + H \cos \alpha')\, \pi\omega - 2\, (H \cos \alpha + G \sin \alpha)\, \kappa\omega \\
&- 2(J (\sin \alpha \cos \alpha' + \sin \alpha' \cos \alpha) + A \cos \alpha \cos \alpha' + B \sin \alpha \sin \alpha')\, \pi\kappa \\
&- 2(N \sin \alpha' \cos \alpha - O \sin \alpha \cos \alpha' - P \cos \alpha \cos \alpha' + S \sin \alpha \sin \alpha')\, p\kappa \\
&- 2(O \sin \alpha' \cos \alpha - N \sin \alpha \cos \alpha' + P \sin \alpha \sin \alpha' - S \cos \alpha \cos \alpha')\, q\pi \\
&+ 2\, (S \sin^2 \alpha' + (N - O) \sin \alpha' \cos \alpha' - P \cos^2 \alpha')\, p\pi + 2\, (T \sin \alpha - U \cos \alpha')\, p\omega \\
&+ 2\, (U \cos \alpha - T \sin \alpha)\, q\omega - 2\, (P \cos^2 \alpha - (N - O) \sin \alpha \cos \alpha - S \sin^2 \alpha)\, p\kappa \\
&- 2\, (Q \cos \alpha + R \sin \alpha) \sin \vartheta \cdot \kappa + 2\, (R \sin \alpha' + Q \cos \alpha') \sin \vartheta \cdot \pi = 0.
\end{aligned}$$

§ 2.

Equatorial surfaces that are described by a complex curve whose plane moves parallel to itself.

160. Due to the great complexity of a second-degree complex, we must try to find some means of easing our overview, and thus our understanding, of that subject. The two theorems that we already gave in the previous paragraphs as the immediate geometric expression of equations (II) and (IV), which represent the second-degree complex in the two-fold coordinate determination, will serve as a means to that end for us. Namely, when we, on the one hand, combine the infinitely many lines of the complex that lie in the same plane into a single group, we can introduce the *curve of class two that is enveloped* by them in place of it. On the other hand, when we unite the infinitely many lines of the complex that go through that point into a group, we can, in an analogous way, introduce that *second-order conic surface* that the complex defines in place of it.

Since all lines in space lie with a given point in some plane, in order to encompass all lines of the complex, we will then need, on the one hand, to consider only those complex curves whose planes go through the given point. On the other hand, since all lines in space cut a given plane, we will obtain all lines of the complex when we consider only those cones whose centers lie in the given plane. Thus, infinitely many (∞^2) complex curves (infinitely many (∞^2) complex cones, resp.) will appear in place of infinitely many (∞^3) complex lines.

161. We can go a step further. If a plane moves then the varying curve of class two that is enveloped in it by lines of the complex will describe a surface. If a point moves then a surface will be enveloped by the varying complex cones that has that point for their vertices. In the determination of the complex, infinitely many (∞) complex curves (infinitely many (∞) complex cones, resp.) will replace these surfaces. The simplest surfaces of this kind will correspond, on the one hand, to the case in which the plane of the curve thus described rotates around a fixed axis or moves parallel to itself, and on the other hand, to the case in which the center of the enveloping cone describes a fixed straight line, or, when the fixed line goes to infinity, to the case in which the cone degenerates into enveloping cylinders whose axes are parallel to a given plane.

We would like to call all such surfaces thus determined *complex surfaces.*

When we introduce these complex surfaces, we can replace the infinitely many (∞^3) complex lines with infinitely many (∞) complex surfaces whose fixed axes lie in a given plane and intersect in a given point of this plane.

We would like to subject each of the given generators of the complex surface to an analytical discussion, in succession.

162. We would like to start with the general equation:

$$\begin{aligned}
&D(t-t')^2 + E(u-u')^2 + F(v-v')^2 \\
&+ A(uv' - u'v)^2 + B(t'v - tv')^2 + C(tu' + t'u)^2 \\
&+ 2K(u-u')(v-v') + 2L(t-t')(v-v') + 2M(t-t')(u-u') \\
&+ 2G(tu' - t'u)(t'u - tv') + 2H(tu' - t'u)(uv' - u'v) + 2J(t'v - tv')(uv' - u'v) \\
&+ 2N\,(t-t')\,(uv' - u'v) \quad + 2O\,(u-u')(t'v - tv') \\
&+ 2S\,(t-t')\,(t'v - tv') \quad + 2T\,(t-t')\,(tu' - t'u) \\
&+ 2U\,(u-u')\,(tu' - t'v) \quad + 2P\,(u-u')\,(uv' - u'v) \\
&+ 2Q\,(v-v')\,(uv' - u'v) \quad + 2R\,(v-v')\,(t'v - tv'),
\end{aligned} \tag{IV}$$

which represents the second-degree complex in axial coordinates. If we consider t', u', v' to be constant in this equation then it will represent a curve of class two in space that will contact all planes whose coordinates t, u, v satisfy the equation. This curve will lie in the plane (t', u', v') and will be enveloped by lines of the complex in it.

The projection of this curve onto one of the three coordinates planes YZ, XZ, XY is deduced immediately when we set t¸ u, v equal to zero, respectively. In that way, if we only consider the projection onto YZ and likewise make the equation homogeneous by the introduction of w and w' then we will obtain:

$$\begin{aligned}
&(Dt'^2 + Eu'^2 + Fv'^2 + 2K\,u'\,v' + 2L\,t'\,v' + 2\,M\,t'\,u')\,w^2 \\
&- 2\,(F\,v'w' + K\,u'w' + L\,t'w' - (N-O)\,t'u' - P\,u'^{\,2} - Q\,u'v' + R\,t'\,v' + S\,t'^{\,2})\,vw \\
&+ (Au'^2 + Bt'^2 + Fw'^2 - 2J\,t'\,u' - 2Q\,u'\,w' + 2R\,t'\,w')\,v^2 \\
&- 2\,(E\,u'\,w' + K\,v'\,w' + M\,t'\,w' + N\,t'\,v' + P\,u'\,v' + Q\,v'^{\,2} - T\,t'^{\,2} - U\,t'\,u')\,u\,w \\
&- 2\,(A\,u'\,v' + G\,t'^{\,2} - H\,t'\,u' - J\,t'\,v' - Kw'^{\,2} - O\,t'\,w' + P\,u'\,w' - Q\,v'\,w')\,u\,v \\
&+ (A\,v'^{\,2} + C\,t'^{\,2} + E\,w'^{\,2} - 2H\,t'\,v' + 2P\,v'\,w' - 2\,U\,t'\,w')\,u^2 = 0.
\end{aligned} \tag{1}$$

163. If we take t', u', v' to be constant in the foregoing equation and let w' vary then the plane (t', u', v', w') that contains the complex curve will move parallel to itself. In particular, if we make the assumption that this plane is parallel to the YZ-plane then we will get:

$$u' = 0, \qquad v' = 0, \qquad \frac{w'}{t'} = -\,x',$$

in which x' means the distance from the instantaneous plane of the complex curve to the YZ-plane. If we likewise divide by t'^2 then equation (1) will be converted into the following one:

$$\begin{aligned}
&Dw^2 + 2(Lx' - S)vw + (Fx'^2 - 2Rx' + B)v^2 \\
&+ 2(Mx' + T)uw + 2(Kx'^2 - Ox' - G)uv + (Ex'^2 + 2Ux' + C)u^2 = 0.
\end{aligned} \tag{2}$$

Once the distance x to a plane that is parallel to YZ has been determined, this equation will give the projection of the complex curve that lies in that plane onto YZ in ordinary line coordinates u, v, w, or also this curve itself in its own plane when we displace the YZ coordinate plane parallel to itself in such a way that it coincides with the plane of the instantaneous complex curve. If we then also consider x' to be variable and then drop the prime then that equation:

$$\begin{aligned} Dw^2 + 2(Lx - S)vw + 2(Fx^2 - 2Rx + B)v^2 \\ + 2(Mx + T)uw + 2(Kx^2 - Ox - G)uv + (Ex^2 + 2Ux + C)u^2 = 0 \end{aligned} \tag{3}$$

will represent *the totality of all complex curves whose planes are parallel to YZ in mixed point and line coordinates x, u, v, w.*

Complex curves in planes that are parallel to each other define a complex surface that we would like to call an *equatorial surface,* while the individual complex curves might be called *latitude curves.*

Equation (3) includes *thirteen* mutually-independent constants. Since the coordinate determination has no further relationship to the equatorial surface than the fact that the direction of the *YZ* coordinate plane is a distinguished one, the equatorial surface will depend upon *fifteen* constants, in all.

164. We obtain the determination of the center of the latitude curve in a plane that is determined by x' in a well-known way by its equation in line coordinates. The coordinates of this point are:

$$z = \frac{Lx' - S}{D}, \qquad\qquad y = \frac{Mx' + T}{D}, \tag{4}$$

and when we drop the prime that will yield:

$$\left.\begin{aligned} Dz - Lx + S = 0, \\ Dy - Mx - T = 0. \end{aligned}\right\} \tag{5}$$

When we consider the x to be variable, these two equations will represent a straight line, and this straight line will be the geometric locus of the centers of the complex curves that define the equatorial surface. We would like to call this straight line the *diameter of the equatorial surface* and the planes of the latitude curves the *associated planes of this diameter.*

Any system of parallel planes corresponds to an equatorial surface in the complex with a diameter that is associated with the planes that are parallel to its own plane.

165. Equation (3) gives any latitude curve in its plane in line coordinates u, v, w after this plane has been determined by the value of x. However, we can also represent this same curve in its plane by the ordinary point coordinates y and z. We will then find its equation in a known way (*):

(*) If *the same* conic section in the *YZ* plane is represented, in one case by means of point coordinates y, z, and in the other case by means of line coordinates u, v, w, by the two equations:

$$ay2 + 2byz + 2dy + 2ez + f = 0,$$

$$
\begin{aligned}
&[(Lx-S)^2 - D(Fx^2 - 2Rx + B)]\, y^2 \\
&+2[(D(Kx^2 - Ox - G) - (Lx-S)(Mx+T)]\, y^2 \\
&+[(Mx+T)^2 - D(Ex^2 - 2Ux + C)]\, z^2 \\
&+2[(Mx+T)(Fx^2 - 2Rx + B) - (Lx-S)(Kx^2 - Ox - G)]\, y \\
&+2[(Lx-S)(Ex^2 - 2Ux + C) - (Mx+T)(Kx^2 - Ox - G)]\, z \\
&+2[(Kx^2 - Ox - G)^2 - (Fx^2 - 2Rx + B)(Ex^2 - 2Ux + C)] = 0.
\end{aligned} \tag{6}
$$

If we consider not just y and z, but also x, to be variable in this equation then it will represent the equatorial surface *in ordinary point coordinates.*

Equatorial surfaces are therefore fourth-order surfaces. They will be cut by the planes that are conjugate to their diameter in second-order curves, *since a double ray of the surface will lie at infinity in these planes.*

166. We obtain the following three equations:

$$
\left.
\begin{aligned}
&Dw^2 + 2(Lx-S)vw + (Fx^2 - 2Rx + B)v^2 \\
&+2(Mx+T)uw + 2(Kx^2 - Ox - G)uv + (Ex^2 + 2Ux + C)u^2 = 0, \\[1ex]
&Ew^2 + 2(My-U)tw + (Dy^2 - 2Ty + C)t^2 \\
&+2(Ky+P)vw + 2(Ly^2 + Ny - H)tv + (Fy^2 + 2Qy + A)v^2 = 0, \\[1ex]
&Fw^2 + 2(Kz-Q)uw + (Ez^2 - 2Pz + A)u^2 \\
&+2(Lz+R)tw + 2(Mz^2 - (N-O)z - J)tu + (Dz^2 + 2Sz + B)t^2 = 0
\end{aligned}
\right\} \tag{7}
$$

for the equations of the equatorial surface whose latitude curves are parallel to YZ, XZ, XY, respectively, in mixed point and line coordinates. The first of the foregoing three equations is equation (3) of number **163**, and the other two are derived from it by the permutation rules of number **155**. When we substitute all possible values for the three variables x, y, z, the equations will represent the individual latitude curves in their planes.

$$Aw^2 + 2Bvw + Cv^2 + 2Duw + 2Euv + Fu^2 = 0,$$

then we can determine the constants of the one equation in terms of the constants of the other one in the following way:

$$
\begin{aligned}
a &= B^2 - AC, & A &= b^2 - ac, \\
b &= AE - BD, & B &= ae - bd, \\
c &= D^2 - AF, & C &= d^2 - af, \\
d &= CD - BE, & D &= cd - be, \\
e &= BF - DE, & E &= bf - de, \\
f &= E^2 - CF, & F &= e^2 - cf.
\end{aligned}
$$

I have abstracted these expressions from second group of the "developments" in nos. **484** and **552**.

In particular, if we set x, y, z equal to zero then we will obtain the equations of the three complex curves in the three coordinate planes:

$$\left.\begin{aligned} Dw^2 - 2Svw + Bv^2 + 2Tuw - 2Guv + Cu^2 &= 0, \\ Ew^2 - 2Utw + Ct^2 + 2Pvw - 2Htv + Av^2 &= 0, \\ Fw^2 - 2Ouw + Au^2 + 2Rtw - 2Jtu + Bt^2 &= 0. \end{aligned}\right\} \tag{8}$$

§ 3.

Meridian surfaces that are described by a complex curve whose plane rotates around a fixed straight line.

167. Equatorial surfaces, which were the subject of examination in the previous paragraphs, are the geometric loci of curves that will be enveloped by lines of the complex in parallel planes, or, in other words, ones whose planes intersect in infinitely-distant straight lines. They are to be regarded as a specialization of complex curves that are the geometric loci of complex curves whose planes go through a fixed axis. We would like to refer to such complex surfaces as *meridian surfaces*, while we likewise call the complex curves that define a meridian surface *meridian curves*, and the plane in which they lie, *meridian planes.*

The determination of the meridian surfaces is connected with the equation:

$$\begin{gathered} (Dt'^2 + Eu'^2 + Fv'^2 + 2Ku'v' + 2Lt'v' + 2Mt'u')\,w^2 \\ -2(Fv'w' + Ku'w' + Lt'w' - (N-O)t'u' - Pu'^2 - Qu'v' + Rt'v' + St'^2)\,vw \\ +(Au'^2 + Bt'^2 + Fw'^2 - 2Jt'u' - 2Qu'w' + 2Rt'w')\,v^2 \\ -2(Eu'w' + Kv'w' + Mt'w' + Nt'v' + Pu'v' + Qv'^2 - Tt'^2 - Ut'u')\,uw \\ -2(Au'v' + Gt'^2 - Ht'u' - Jt'v' - Kw'^2 - Ot'w' + Pu'w' - Qv'w')\,uv \\ +(Av'^2 + Ct'^2 + Ew'^2 - 2Ht'v' + 2Pv'w' - 2Ut'w')\,u^2 = 0, \end{gathered} \tag{1}$$

by which we determined the equatorial surface in the previous paragraphs.

168. For an arbitrary choice of coordinate system, we can – with no loss of generality – take the fixed axis around which the planes of the complex curve rotate to be one of the three coordinate axes. If we choose it to be the OZ-axis then we must set v' and w' equal to zero in the foregoing equations. It will then go to the following one:

$$\begin{gathered} (Dt'^2 + Eu'^2 + 2Mt'u')\,w^2 + 2((N-O)\,t'u' + Pu'^2 - St'^2)\,vw + (Au'^2 + Bt'^2 - 2Jt'u')\,v^2 \\ + 2(Tt' + Uu')\,t'\cdot uw - 2(Gt' - Hu')\,t'uv + Ct'^2u^2 = 0. \end{gathered} \tag{9}$$

The position of the meridian plane is determined by t' / u'; if y and x are two of the three coordinates of an arbitrary point of the plane then we can determine them in the same way by:

$$\frac{y}{x} = -\frac{t'}{u'}.$$

The last equation will then be the following one, when we likewise order it in powers of x and y:

$$\begin{gathered}(Ex^2 - 2Mxy + Dy^2)\,w^2 + 2(Px^2 - (N - O)xy - Sy^2)\,vw + (Ax^2 + 2Jxy + By^2)\,v^2 \\ -2(Ux - Ty)\,y \cdot uw - 2(Hx + Gy)\,y \cdot uv + Cy^2u^2 = 0.\end{gathered} \tag{10}$$

When we permute the OZ and OY with each other according to the permutation rules of the first paragraph, this equation will go to:

$$\begin{gathered}(Fx^2 - 2Lxy + Dy^2)\,w^2 - 2(Qx^2 - Nxy - Tz^2)\,uw + (Ax^2 + 2Hxz + Cz^2)\,u^2 \\ +2(Rx - Sz)z \cdot vw - 2(Jx + Gz)z \cdot uv + Bz^2 \cdot v^2 = 0,\end{gathered} \tag{11}$$

and this equation, in turn, will go to the following one when we switch the two axes OY and OX with each other:

$$\begin{gathered}(Fy^2 - 2Kyz + Ez^2)\,w^2 + 2(Ry^2 - Oyz - Uz^2)\,tw + (By^2 + 2Gyz + Cz^2)\,t^2 \\ -2(Qy - Pz)\,z \cdot vw - 2(Jy + Hz)\,z \cdot tv + Az^2 \cdot v^2 = 0.\end{gathered} \tag{12}$$

Equation (11) represents the projection onto YZ of those complex curves whose planes go through OY, while equation (12) represents the projection onto XZ of the complex curves whose planes go through OX. We would like to regard the latter as *the general equation of the meridian surfaces in mixed point and line coordinates.*

Like the general equation of the equatorial surface (3), it contains *thirteen* mutually-independent constants. However, whereas in the case of equatorial surfaces, the coordinate system depended upon the surface only insofar as the direction of the YZ coordinate plane was given by it, here, the meridian surface will be determined by the OX axis. A meridian surface will then depends upon *four new constants,* in addition to the thirteen constants above, so *seventeen* constants, in all.

169. We would like to base the following discussion upon the latter equation.

If we denote the angle that an arbitrary meridian plane defines with XZ by φ then:

$$\tan \varphi = \frac{y}{x}.$$

We then obtain the equation of the projection of the relevant meridian surface onto XZ when replace the coordinates:

$$y \text{ and } z$$

in the last equation with the trigonometric functions:

$$\sin \varphi \quad \text{and} \quad \cos \varphi,$$

resp. It will then go to the following one:

$$\begin{aligned}
&(F\sin^2\varphi - 2K\sin\varphi\cos\varphi + E\cos^2\varphi)\,w^2 \\
&+2(R\sin^2\varphi - O\sin\varphi\cos\varphi - U\cos^2\varphi)\,tw \\
&+(B\sin^2\varphi + 2G\sin\varphi\cos\varphi + C\cos^2\varphi)\,t^2 \\
&-2(Q\sin\varphi - P\cos\varphi)\cos\varphi\cdot vw - 2(J\sin\varphi + H\cos\varphi)\cos\varphi\cdot tv + A\cos^2\varphi\cdot v^2 = 0,
\end{aligned} \tag{13}$$

and, when we divide by $\cos^2 \varphi$, that will give:

$$\begin{aligned}
&(F\tan^2\varphi - 2K\tan\varphi + E)\,w^2 \\
&+2(R\tan^2\varphi - O\tan\varphi - U)\,tw \\
&+(B\tan^2\varphi + 2G\tan\varphi + C)\,t^2 \\
&-2(Q\tan\varphi - P)\,vw - 2(J\tan\varphi + H)\,tv + Av^2 = 0.
\end{aligned} \tag{14}$$

Finally, if we rotate the XZ coordinate plane around OX through an angle of φ, such that after the rotation XZ' will coincide with the relevant meridian plane in the new position, then the new coordinate determination t / w will remain unchanged, while one will get v / w for $v / w \cdot \cos \varphi$, which is constructed in ordinary line coordinates on OZ'. In order to then obtain the equation of the meridian curve *in its own plane*, we have to write v, in place of $v \cdot \cos \varphi$, in equation (13), with which, it will go to the following one:

$$\begin{aligned}
&(F\sin^2\varphi - 2K\sin\varphi\cos\varphi + E\cos^2\varphi)\,w^2 \\
&+2(R\sin^2\varphi - O\sin\varphi\cos\varphi - U\cos^2\varphi)\,tw \\
&+(B\sin^2\varphi + 2G\sin\varphi\cos\varphi + C\cos^2\varphi)\,t^2 \\
&-2(Q\sin\varphi - P\cos\varphi)vw - 2(J\sin\varphi + H\cos\varphi)tv + Av^2 = 0.
\end{aligned} \tag{15}$$

When we consider φ to be variable, the last equation will represent the totality of all meridian curves; in other words, the meridian surface itself.

In the case of equations (13) and (14), this will happen in such a way that once φ has taken on a definite value by the choice of the meridian plane, these equations will represent the projection of the meridian curve onto XZ in line coordinates t, v, w, with which the curve itself will be given. The same curve will be represented in its own plane by means of the latter equation (15) by the choice of φ. If the meridian plane rotates around OX then the meridian curve in it that generates the meridian surface will change, independently of φ. In each of its positions, it is referred to the unchanged remaining axis OX and a variable axis OZ', which rotates around OX with it and in the meridian plane that contains it.

We thus arrive at an analytical representation and construction of the meridian surfaces that is completely analogous to the representation and construction of the equatorial surfaces.

170. The equation of the pole of the OX axis relative to the curve of class two that was represented by equation (14) that corresponds to the instantaneous values of φ is the following one:

$$(Q \tan \varphi - P)\, w + (J \tan \varphi + H)\, t - Av = 0.$$

The curve (14) is the projection onto XZ of the relevant meridian curve, and thus the pole in question is likewise the projection of the pole of the OX axis relative to the meridian curve itself. Two of the three coordinates of that point will then be:

$$x = \frac{J \tan \varphi + H}{Q \tan \varphi - P}, \qquad z = \frac{-A}{Q \tan \varphi - P},$$

and the third one will be:

$$y = z \cdot \tan \varphi = \frac{-A \tan \varphi}{Q \tan \varphi - P}.$$

In order to determine the geometric locus of the pole of OY relative to the various meridian curves, we have to eliminate φ from the foregoing three equations. To that end, we set $\tan \varphi$ equal to its value y / z in the second equation, which will give:

$$Qy - Pz + A = 0. \tag{16}$$

The first equation gives:

$$x = \frac{Jy + Hz}{Qy - Pz} = -\frac{Jy + Hz}{A},$$

from which, it will follow that:

$$Ax + Jy + Hz = 0. \tag{17}$$

We have thus arrived at the following result:

When a plane rotates around a fixed axis that lies in it, the geometric locus of the poles of that fixed axis relative to all complex curves that the plane contains during its rotation will be a straight line.

We would like to call this straight line the *polar* of the meridian surface.

171. The foregoing equation (15) is, in turn, regarded as the equation of the complex surface in mixed coordinates. $\tan \varphi$ is then to be considered as one of the three linear coordinates y / z, x / z, $1 / z$ of a point (x, y, z), while t, u, w mean the line coordinates of the plane.

In order to represent the meridian surface that we speak of in point coordinates x, y, z, we return to equation (12), which is equivalent to (15). We merely need to introduce the two point coordinates x and z in place of the line coordinates t, v, w by which that equation expresses the projections of the meridian curves in that plane onto YZ. The known transformation (no. **165**, Note), when applied to the present case, will give the following equation when we likewise divide by z^2:

$$\begin{aligned}
&[(Ry^2 - Oyz - Uz^2)^2 - (Fy^2 - 2Kyz + Ez^2)(By^2 + 2Gyz + Cz^2)] \\
&- 2[(Jy + Hz)(Fy^2 - 2Kyz + Ez^2) - (Qy - Pz)(Ry^2 - Oyz - Uz^2)]\,x \\
&+ [(Qy - Pz)^2 - A(Fy^2 - 2Kyz + Ez^2)]\,x^2 \\
&- 2[(Qy - Pz)(By^2 + 2Gyz + Cz^2) - (Jy + Hz)(Ry^2 - Oyz - Uz^2)] \\
&+ 2[A(Ry^2 - Oyz - Uz^2) - (Qy - Pz)(Jy + Hz)]\,x \\
&+ [(Jy + Hz)^2 - A(By^2 + 2Gyz + Cz^2)] = 0.
\end{aligned} \tag{18}$$

The meridian surfaces, like the equatorial surfaces, are then of order four.

172. Any straight line that goes through the OX axis cuts the meridian surface in four points, two of which will coincide on that axis. *The axis is then a double ray of the meridian surface.* An arbitrary plane cuts the meridian surface in a fourth-order curve that has a double point on its double ray. That point will go to infinity when the intersecting plane is parallel to the double ray. If the plane goes through the double ray then it will also be a double line of the intersection curve. As a consequence, the order of the curve will reduce to two, with which it will become a complex curve.

§ 4.

Meridian surfaces that are enveloped by complex cones whose centers lie upon a straight line.

173. All lines of a second-degree complex that encounter a given straight line can be grouped together in two ways: On the one hand, they define the totality of all tangents to infinitely many complex curves of class two whose planes go through the straight line, and on the other hand, they define the totality of all lines of infinitely many complex cones of order two whose vertices lie along the given straight line. We can thus consider the same complex surface that we regarded as being described complex curves in the previous two paragraphs as being enveloped by complex cones from now on.

In agreement with that, one can draw two tangents to the complex curve that lie in each plane that goes through a given straight line that goes through an arbitrary point. These two lines are lines of the complex, and when the plane rotates around the given straight line as an axis, they will generate a conic surface that belongs to the complex, which has the given point for its vertex and which circumscribes the complex surface in question. Each point of the given straight line will then correspond to a complex cone that will be of order two, since it will be cut by each of the planes that go through its vertex along two straight lines. The curve in which such a cone contacts the complex surface is not a plane curve, in general, nor do the tangential planes to the surface at the points of a complex curve generally envelop a conic surface.

174. We would like to start with the general equation:

$$\begin{aligned} & A\,(x-x')^2 + B\,(y-y')^2 + C\,(z-z')^2 \\ & + D\,(yz'-y'z)^2 + E\,(x'z-xz')^2 + F\,(xy'-x'y)^2 \\ & + 2G\,(y-y')(z-z') + 2H\,(x-x')(z-z') + 2J\,(x-x')\,(y-y') \\ & + 2K\,(xy'-x'y)\,(x'z-xz') + 2L\,(xy'-x'y)\,(yz'-y'z) + 2M\,(x'z-xz')(yz'-y'z) \\ & + 2\,N\,(x-x')(yz'-y'z) + 2O\,(y-y')(x'z-xz') \\ & + 2\,P\,(x-x')(x'z-xz') + 2\,Q\,(x-x')(xy'-x'y) \\ & + 2\,R\,(y-y')(xy'-x'y) + 2\,S\,(y-y')(yz'-y'z) \\ & + 2\,T\,(z-z')(yz'-y'z) + 2\,U\,(z-z')(x'z-xz') = 0, \end{aligned} \tag{II}$$

which represents the second-degree complex in ray coordinates. If we consider x', y', z' to be constant in this equation then it will represent a second-order cone that goes through all of the points in space whose coordinates x, y, z satisfy the equation. This cone will have the point (x', y', z') for its vertex and will be the geometric locus of the lines of the complex that go through that point.

The intersection of this cone with one of the three coordinate planes YZ, XZ, XY is obtained immediately when we set x, y, z, resp., equal to zero in the foregoing equation. In this way, when we consider only the intersection with YZ, we will get the following equation:

$$\begin{aligned} & (Ax'^2 + By'^2 + Cz'^2 + 2Gy'z' + 2Hx'z' + 2Jx'y') \\ & - 2(Cz' + Gy' + Hx' - (N-O)x'y' + Px'^2 - Sy'^2 - Ty'z' + Ux'z')\,z \\ & + (C + Dy'^2 + Ex'^2 - 2Mx'y' - 2Ty' + Ux')\,z^2 \\ & - 2(By' + Gz' + Jx' + Nx'z' - Qx'^2 - Rx'y' + Sy'z' + Tz'^2)\,y \\ & - 2(Dy'z' - G + Kx'^2 - Lx'y' - Mx'z' - Ox' + Sy' - Tz')\,yz \\ & + (B + Dz'^2 + Fx'^2 - 2Lx'z' - 2Rx' + 2Sz')\,y^2 = 0. \end{aligned} \tag{19}$$

This equation is analogous to the one that we derived in no. **162** from the equation of the complex in axial coordinates in order to represent the projection of the complex curve that lies in the (t', u', v', w') onto the coordinate plane YZ. In order to derive the new equation from the earlier one (1) directly, we have only to set w and w' equal to 1 in it and then proceed in accordance with the permutation rules in number **153**.

175. Equation (19) represents a second-order curve in the coordinate plane *YZ*, namely, the locus of intersection points of lines of the complex that go through the given point (x', y', z') with that plane. The cone is thus determined completely.

If we consider x', y', z' to be variable in this equation, along with y and z, and regard them as the coordinates of the vertex of a complex cone then we can say that *the foregoing equation* (19) *represents the totality of all complex cones, and therefore also the complex itself.*

We would like to let the point (x', y', z') move along a straight line. The complex cones in question will then envelop a complex surface. If we take that straight line to be the *OX* coordinate axis, in particular, then the enveloped surface will be the same meridian surface that we determined in the previous paragraphs as the geometric locus of those complex curves whose planes intersect in that same axis.

176. When we set y' and z' equal to zero, consistent with the assumption that was made, the latter equation will go to the following one:

$$\begin{gathered}(Fx'^2 - 2Rx' + B)\,y^2 - 2(Kx'^2 - Ox' - G)\,yz + (Ex'^2 + 2Ux' + C)\,z^2 \\ + 2\,(Qx' - J)\,x'y - 2(Px' + H)\,x'z + Ax'^2 = 0.\end{gathered} \tag{20}$$

When consider x' to be variable, along with y and z, this equation will then represent the totality of all conic surfaces of the complex whose vertices lie on the *OX* axis, and is then to be regarded as the equation of the complex surface that is enveloped by them, in the sense that was established above. The equation gives the base for such a conic surface in *YZ* in point coordinates once its vertex is determined by x'. Any straight line that connects this point with a point of the base will then be a line of the cone.

We can construct the tangential planes to the cone directly, and indeed in such a way that we draw planes through its center and the tangents to the base in *YZ*. A coordinate of one such tangential plane is:

$$\frac{t}{w} = -\frac{1}{x},$$

so we can write the latter equation in the following form:

$$\begin{gathered}(Fw^2 + 2Rtw + Bt^2)\,y^2 - 2(Kw^2 + Otw - Gt^2)\,yz + (Ew^2 - 2Utw + Ct^2)\,z^2 \\ + 2(Qw + Jt)\,wy - 2(Pw - Ht)\,wz + Aw^2 = 0.\end{gathered} \tag{21}$$

The foregoing equation represents the meridian surface in mixed point and plane coordinates.

177. The tangential planes of the enveloping cone are likewise tangential planes of the enveloping complex surface. The vertex of the corresponding enveloping cone is determined by the choice of t / w as the coordinate of such a plane. Since that plane goes through a tangent to the base of the cone in *YZ*, the other two coordinates of such a tangential plane will be identical with the two coordinates of that tangent in its plane. If we then introduce the line coordinates u / w and v / w in place of the two point coordinates y and z, with which this equation goes to the following one when one applies the transformation formulas (no. **165**, Note), after division by w^2:

$$\begin{gathered}
[(Kw^2+Otw-Gt^2)^2-(Fw^2+2Rtw+Bt^2)(Ew^2-2Utw+Ct^2)] \\
-2[(Pw-Ht)(Fw^2+2Rtw+Bt^2)-(Qw+Jt)(Kw^2+Otw-Gt^2)]v \\
+[(Qw+Jt)^2-A(Fw^2+2Rtw+Bt^2)]v^2 \\
+2[(Qw+Jt)(Ew^2-2Utw+Ct^2)-(Pw-Ht)(Kw^2+Otw-Gt^2)]u \\
-2[A(Kw^2+Otw-Gt^2)-(Qw+Jt)(Pw-Ht)]uv \\
+[(Pw-Ht)^2-A(Ew^2-2Utw+Ct^2)]u^2=0
\end{gathered} \tag{22}$$

then this equation will represent *the same meridian surface in plane coordinates* that we represented in *point coordinates* in the previous paragraphs by equation (18).

The meridian surfaces are surfaces of order four, as well as surfaces of class four.

178. In order to obtain the polar plane to the OX axis with respect to an arbitrary conic surface whose vertex lies in that axis, we simply need to draw a plane through its instantaneous vertex and the polar to the coordinate origin relative to the intersection curve in YZ. If we take equation (20) to be the equation of this intersection curve, once x' has been chosen, then, as is known, we will obtain the equation:

$$(Qx' - J)\, y - (Px' + H)\, z + Ax' = 0$$

for the polar in question after omitting the common factor of x'. With that, the equation of the polar plane will become:

$$-Ax + (Qx' - J)\, y - (Px' + H)\, z + Ax' = 0. \tag{23}$$

In particular, this equation will be satisfied, independently of x', when one has simultaneously:

$$\begin{aligned} Ax + Jy + Hz &= 0, \\ Qy - Pz + A &= 0. \end{aligned}$$

The polar planes of the OX coordinate axis relative to the complex cone whose vertex lies upon OX intersect in the same *straight line* that was represented by the last two equations.

These two equations are, however, the same ones that we obtained earlier (no. **170**) for the polar of the meridian surface.

The polar of a meridian surface then has the double relationship to it that, on the one hand, it is the geometric locus of the poles of the double ray of the surface relative to all *meridian curves, and on the other hand, it is enveloped by the polar planes to the same straight line relative to all enveloping complex cones.*

179. One can draw four tangential planes to the complex surface through any straight line that cuts the *OX* axis, two of which will go through the axis. This axis will then be *a double axis of the meridian surface.* One can draw a cone of class four that has a double plane that goes through *OX* from an arbitrary point on the surface. In particular, if the point is chosen to be on the double axis of the complex surface then it will also be a double line of the contact cone of class four – that is, a straight line that goes through its center that will be enveloped by infinitely many tangential planes. In that way, the cone will reduce to class two, with which it will be a complex cone.

When we combine these results with the ones in the foregoing paragraphs, we will arrive at the consequence that the *OX* coordinate axis is simultaneously a double ray and a double axis of the same meridian surface. *We can then speak of the double line of the meridian surface and regard it as a double ray, in one case, and a double axis, in the other.*

§ 5.

Equatorial surfaces that are enveloped by cylindrical surfaces of the complex whose lines are parallel to a fixed plane.

180. A cylinder whose center on the double line of a meridian surface lies at infinity belongs to the set of complex cones that envelop the meridian surface. There are infinitely many such cylindrical surfaces. Any given direction is a line of one such cylinder, as well as parallel to its axis. It is obvious that not every two cylinders have a common line, and that all lines of all cylinders will define the totality of all lines of the complex. We can group together infinitely many of the cylinders whose axes are parallel to a given plane. Such cylinders will then envelop a surface. In order to ease the understanding of a complex, we can then also group together its infinitely many (∞^3) lines into infinitely many (∞^2) groups, each of which will consist of the lines of a cylinder, and in turn, introduce infinitely many (∞) surfaces instead of the infinitely many (∞^2) cylinders, each of which will be enveloped by infinitely many (∞) cylinders.

The surface that is enveloped by infinitely many complex cylinders whose axes are parallel to a given plane is nothing but the equatorial surface that is defined by complex curves in planes that are parallel to the given one. The equatorial surface should be regarded as one of the previously-considered complex surfaces whose double line lies at infinity and whose polar is its diameter.

181. In order to represent the totality of all complex cylinders by a single equation, we merely need to take the x', y', z' in equation (19) of the previous paragraphs to be infinitely large. We then obtain the following equation, which is homogeneous in these quantities:

$$\begin{aligned}&[Fx'^2 - 2Lx'z' + Dz'^2]\,y^2 - 2[Kx'^2 - Lx'y' - Mx'z' + Dy'z']\,yz + [Ex'^2 - 2Mx'y' + Dy'^2]\,z^2\\ &+ 2[Qx'^2 + Rx'y' - Nx'z' - Sy'z' - Tz'^2]\,y - 2[Px'^2 - (N-O)x'y' - Sy'^2 + Ux'z' + Ty'z']\,z \quad (24)\\ &+ [Ax'^2 + 2Jx'y' + By'^2 + 2Hx'z' + 2Gy'z' + Cz'^2] = 0.\end{aligned}$$

If we let α, β, γ denote the angles that the instantaneous direction of the axis of the cylinder defines with the three coordinate axes (assuming rectangular coordinate axes) then we will have:

$$x' : y' : z' = \cos\alpha : \cos\beta : \cos\gamma;$$

we can then introduce $\cos\alpha$, $\cos\beta$, $\cos\gamma$, in place of x', y', z' in the last equation. Once these three cosines are determined, the foregoing curve will represent that second-order curve along which the relevant cylinder cuts the *YZ* coordinate plane. If we likewise regard the three cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$, between which, the known relation exists:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

as variable then *we can also regard the same equation* (24), which now assumes the following form:

$$\begin{aligned}&[F\cos^2\alpha - 2L\cos\alpha\cos\gamma - D\cos^2\gamma]\,y^2\\ &- 2[K\cos^2\alpha - L\cos\alpha\cos\beta - M\cos\alpha\cos\gamma + D\cos\beta\cos\gamma]\,yz\\ &+ [E\cos^2\alpha - 2M\cos\alpha\cos\beta + D\cos^2\beta]\,z^2\\ &+ 2[Q\cos^2\alpha + R\cos\alpha\cos\beta - N\cos\alpha\cos\gamma - S\cos\beta\cos\gamma - T\cos^2\gamma]\,y \qquad (25)\\ &- 2[P\cos^2\alpha - (N-O)\cos\alpha\cos\beta - S\cos^2\beta + U\cos\alpha\cos\gamma - T\cos\beta\cos\gamma]\,z\\ &+ [A\cos^2\alpha + 2J\cos\alpha\cos\beta + B\cos^2\beta + 2H\cos\alpha\cos\gamma + 2G\cos\beta\cos\gamma + C\cos^2\gamma] = 0,\end{aligned}$$

as also being the equation of the complex itself. All of the constants of the general complex equation enter into it. The quantities:

$$y, \quad z, \quad \frac{\cos\beta}{\cos\alpha}, \quad \frac{\cos\gamma}{\cos\alpha}$$

that appear here in this representation of the complex take the place of r, s, ρ, σ, resp., in equation (I) or p, q, π, κ, resp., in equation (III).

182. If we assume that the axes of all cylinders are parallel to a given plane, which we would like to take to be the *XZ* plane, then y' will vanish, as opposed to x' and z', or $\cos\alpha$ will equal zero. The foregoing general equation (24) will then go to the following one:

$$\begin{aligned}&[Fx'^2 - 2Lx'z' + Dz'^2]\,y^2 - 2[Kx' - Mz']\,x' \cdot yz + Ex'^2 \cdot z^2 \\ &+ 2[Qx'^2 - Nx'z' - Tz'^2]\,y - 2[Px' + Ux']\,x' \cdot z + [Ax'^2 + 2Hx'z' + Cz'^2] = 0.\end{aligned} \tag{26}$$

For the sake of agreement with the developments of the second paragraph, we would like to switch the *OX* and *OY* axes with each other, with consideration given to the permutation rules of the first paragraph. We would then find:

$$\begin{aligned}&[Fy'^2 - 2Ky'z' + Ez'^2]\,x^2 - 2[Ly' - Mz']\,y' \cdot xz + Dy'^2 \cdot z^2 \\ &- 2[Ry'^2 - Oy'z' - Uz'^2]\,x + 2[Sy' + Tx']\,y' \cdot z + [By'^2 + 2Gy'z' + Cz'^2] = 0.\end{aligned} \tag{27}$$

This equation represents the totality of cylinders whose axes are parallel to the *YZ*-plane, or – what amounts to the same thing – the *equatorial surface* that is enveloped by these cylinders.

When we divide by z^2, and after the division set:

$$\frac{y'}{z'} = \frac{\cos\alpha}{\cos\gamma} = \tan\gamma,$$

the last equation will go to the following one:

$$\begin{aligned}&[F \tan^2\gamma - 2K \tan\gamma + E]\,x^2 - 2[L \tan\gamma - M] \tan\gamma \cdot xz + D \tan^2\gamma \cdot z^2 \\ &- 2[R \tan^2\gamma - O \tan\gamma - U]\,x + 2[S \tan\gamma + J] \tan\gamma \cdot z \\ &+ [B \tan^2\gamma + 2G \tan\gamma + C] = 0.\end{aligned} \tag{28}$$

Finally, we would like to determine the intersection curve of the cylinder with those planes that are perpendicular to the axis of the cylinder, instead of the intersection curve with *XZ* that we have considered up to now. To that end, we switch z with $z \cdot \cos\gamma$ in the foregoing equation, while x remain unchanged. When we multiply by $\cos^2\gamma$ that will give:

$$\begin{aligned}&[F \sin^2\gamma - 2K \sin\gamma\cos\gamma + E \cos^2\gamma]\,x^2 - 2[L \sin\gamma - M \cos\gamma] \sin\gamma \cdot xz + D \sin^2\gamma \cdot z^2 \\ &- 2[R \sin^2\gamma - O \sin\gamma\cos\gamma - U \cos^2\gamma]\,x + 2[S \sin\gamma + T \cos\gamma] \sin\gamma \cdot z \\ &+ [B \sin^2\gamma + 2G \sin\gamma\cos\gamma + C \cos^2\gamma] = 0.\end{aligned} \tag{29}$$

183. In order to obtain the equation of the equatorial surface in plane coordinates, we next introduce into equation (28) the quotients of the coordinates v / w and u / w of a tangential plane to the cylinder that is also a tangential plane of the equatorial surface, which we shall do by means of the equation:

$$\tan\gamma = -\frac{v}{u}.$$

Equation (28) will then be converted into the following one when we first multiply by u^2:

$$\begin{aligned}&[Fv^2+2Kuv+Eu^2]\,x^2-2[Lv+Mu]v\cdot xz+Dv^2\cdot z^2\\&-2[Rv^2+Ouv-Uu^2]\,x+2(Sv-Tu)v\cdot z+[Bv^2-2Guv+Cu^2]=0.\end{aligned}\tag{30}$$

For a given value of γ, equation (28) represents the intersection curve of the relevant cylinder with the *XZ* coordinate plane in terms of the point coordinates *x* and *z*. We would like to introduce the coordinates of the tangents to that curve in place of these coordinates and take them to be t / u and w / u. However, these two coordinates of the tangent to the intersection curve are likewise two coordinates of the tangential plane of the cylinder and the equatorial surface whose third coordinate is v / u. In that way, after dividing by v^2, we will find *the equation of the equatorial surface in plane coordinates:*

$$\begin{aligned}&[(Lv+Mu)^2-D(Fv^2+2Kuv+Eu^2)]\,w^2\\&+2[(Sv-Tu)(Fv^2+2Kuv+Eu^2)-(Lv+Mu)(Rv^2+Ouv-Uu^2)\,w\\&+[(Rv^2+Ouv-Uu^2)^2-(Fv^2+2Kuv+Eu^2)(Bv^2-2Guv+Cu^2)]\\&-2[D(Rv^2+Ouv-Uu^2)-(Lv+Mu)(Sv-Tu)]tw\\&-2[(Lv+Mu)(Bv^2-2Guv+Cu^2)-(Sv-Tu)(Rv^2+Ouv-Uu^2)]t\\&+[(Sv-Tu)^2-D(Bv^2-2Guv+Cu^2)]t^2=0.\end{aligned}\tag{31}$$

The equatorial surfaces, like the meridian surfaces, are likewise of order four and class four. The *double axis* of the surface at infinity in *YZ* is distinguished in the foregoing equation by the fact that *u* and *v* are not present in any powers lower than two. From the second paragraphs, the double axis at infinity of the equatorial surface is likewise a double ray of it. We can then say that the equatorial surfaces have an *double line at infinity*.

184. The polar plane of the double line at infinity in *YZ* relative to an arbitrary complex cylinder that cuts the *XZ* plane in the curve (28) goes through that diameter of the intersection curve that is associated with the direction of the *OZ* coordinate axis. When we differentiate equation (28) with respect to z, we will obtain:

$$-(L\tan\gamma-M)\,x+D\tan\gamma\cdot z+(S\tan\gamma+T)=0$$

for the equation of that diameter, and from that:

$$-(L\tan\gamma-M)\,x-Dy+D\tan\gamma\cdot z+(S\tan\gamma+T)=0$$

for the equation of the polar plane. This equation will be satisfied independently of γ, in particular, when one simultaneously has:

$$\begin{aligned}Dz-Lx+S&=0,\\Dy-Mx-T&=0.\end{aligned}$$

The polar planes to the straight line at infinity in the *YZ* relative to all complex cylinders whose axes are parallel to that plane will then intersect in a fixed straight line that will be represented by the two foregoing equations.

These two equations are, however, the ones that we obtained earlier (no. **164**) for the determination of the diameter of the equatorial surface.

The diameter of an equatorial surface then has the double relationship to that surface that, in the one case, it is the geometric locus of the centers of the latitude curves that generate the surface, and on the other hand, it will be enveloped by the polar planes of those straight lines that lie at infinity in the planes of the latitude curves in relation to the enveloping complex cylinder.

185. The following three equations represent the bases in *YZ*, *XZ*, *XY* of those three complex cylinders whose axes are parallel to the three *OX*, *OY*, *OZ* coordinate axes, respectively:

$$\left.\begin{aligned} Fy^2 - 2Kyz + Ez^2 + 2Qy - 2Pz + A &= 0, \\ Fx^2 - 2Lxz + Dz^2 + 2Sz - 2Rx + B &= 0, \\ Ey^2 - 2Mxy + Dy^2 + 2Ux - 2Ty + C &= 0. \end{aligned}\right\} \tag{32}$$

The second of these equations is deduced immediately from equation (30) when we set *U* equal to zero in that equation, and the remaining two are deduced from the permutation rules of number **155**.

§ 6.

Analytical determination of the double points and double planes of complex surfaces.

186. Let:

$$a\,\alpha^2 + 2b\,\alpha\beta + c\,\beta^2 + 2d\,\alpha\gamma + 2e\,\beta\gamma + f\,\gamma^2 = 0$$

be a homogeneous equation of degree two in the variables α, β, γ. We then get the following algebraic decomposition:

$$\begin{aligned} & a(a\,\alpha^2 + 2b\,\alpha\beta + c\,\beta^2 + 2d\,\alpha\gamma + 2e\,\beta\gamma + f\,\gamma^2) \\ \equiv\ & [a\alpha + (b + \sqrt{b^2 - ac})\beta + (d + \sqrt{d^2 - af}\,)\gamma]\cdot[a\alpha + (b - \sqrt{b^2 - ac})\beta + (d - \sqrt{d^2 - af}\,)\gamma] \\ & - 2\,[(bd - ae) - \sqrt{(b^2 - ac)}\sqrt{(d^2 - af)}\,]\,\beta\gamma \\ \equiv\ & [a\alpha + (b + \sqrt{b^2 - ac})\beta + (d - \sqrt{d^2 - af}\,)\gamma]\cdot[a\alpha + (b - \sqrt{b^2 - ac})\beta + (d + \sqrt{d^2 - af}\,)\gamma] \\ & - 2\,[(bd - ae) + \sqrt{(b^2 - ac)}\sqrt{(d^2 - af)}\,]\,\beta\gamma \end{aligned}$$

Thus, if:

$$(bd - ae) - \sqrt{(b^2 - ac)}\sqrt{(d^2 - af)} = 0 \tag{33}$$

then the given second-degree equation will resolve into the following two first-degree equations:

$$\left.\begin{aligned} a\alpha + (b + \sqrt{b^2 - ac})\beta + (d + \sqrt{d^2 - af})\gamma = 0, \\ a\alpha + (b - \sqrt{b^2 - ac})\beta + (d - \sqrt{d^2 - af})\gamma = 0, \end{aligned}\right\} \tag{34}$$

and if:

$$(bd - ae) + \sqrt{(b^2 - ac)}\sqrt{(d^2 - af)} = 0 \tag{35}$$

then it will resolve into the following two:

$$\left.\begin{aligned} a\alpha + (b + \sqrt{b^2 - ac})\beta + (d - \sqrt{d^2 - af})\gamma = 0, \\ a\alpha + (b - \sqrt{b^2 - ac})\beta + (d + \sqrt{d^2 - af})\gamma = 0. \end{aligned}\right\} \tag{36}$$

We can combine the condition equations (33) and (35) into the following one:

$$(bd - ae)^2 - (b^2 - ac)(d^2 - af) = 0. \tag{37}$$

If this condition equation is satisfied then the given second-degree equation will resolve into two first-degree equations.

Two of the variables – viz., β and γ – enter into the equation forms (34) and (36) in the same way, while the third one – viz., α – enters in a distinguished way. We then obtain two entirely analogous decompositions along with the foregoing ones, and in fact, by a mere change of notation. Corresponding to them, we can also write the condition equation (37) in the following form:

$$\left.\begin{aligned} (be - cd)^2 - (b^2 - ac)(e^2 - cf) = 0, \\ (de - fb)^2 - (d^2 - af)(e^2 - cf) = 0. \end{aligned}\right\} \tag{38}$$

Finally, the foregoing three will go to the following one:

$$acf - ae^2 - cd^2 - fb^2 + 2bde = 0 \tag{39}$$

under the same equations when we develop them. The three equation forms (37) and (38) show that in the event that such a decomposition exists, the three expressions:

$$(b^2 - ae), \qquad (d^2 - af), \qquad (e^2 - cf)$$

will have the same sign. If these signs are positive then the decomposition will be a real one, while if they are negative then it will be an imaginary one. If two of the three expressions vanish simultaneously, which will imply the vanishing of the third one as a result of the condition equations (37) and (38), then the two equations into which the given one resolves will be identical. One likewise has:

$$(bd - ae) = 0, \qquad (be - cd) = 0, \qquad (de - fb) = 0.$$

The given second-degree, homogeneous equation resolves into the two first-degree equations (34) or the two equations (36), respectively, according to whether the condition equation (33) or the condition equation (35) is satisfied, respectively. That corresponds to the expression $(bd - ae)$ being positive in one case and negative in the other, in the case of a real decomposition, and conversely, the same expression is negative in one case and positive in the other in the case of an imaginary decomposition. This comes down to the same thing as saying that the decomposition (34) or (36) exists, resp., according to whether the expression $(bd - ae)$ *does or does not agree in sign*, respectively, with one of the three expressions:

$$(b^2 - ae), \qquad (d^2 - af), \qquad (e^2 - cf),$$

and thus, with all of them.

Some transformations are coupled with the foregoing equations (37) and (38) that will find immediate applications in the sequel.

Equation (37) gives:

$$\frac{bd - ae}{d^2 - af} = \frac{b^2 - ac}{bd - ae} = \pm\frac{\sqrt{b^2 - ac}}{\sqrt{d^2 - af}}. \tag{40}$$

The upper or lower sign in this is taken according to whether the decomposition (34) or (36) exists, respectively.

Furthermore, equations (38) give:

$$\frac{be - cd}{de - fb} = \pm\frac{\sqrt{b^2 - ac}}{\sqrt{d^2 - af}}, \tag{41}$$

in which the signs of the expressions $(be - cd)$ and $(de - fb)$ determine the double sign immediately. When no decomposition of the given second-degree function into two linear factors is possible, which would be expressed by the condition equation (39), we will get:

$$(bd - ac)(be - cd)(de - fb) = -(b^2 - ac)(d^2 - af)(e^2 - cf).$$

It follows from this that we have to take the upper or lower sign in equation (41) according to whether the decomposition (36) or (34) exists, respectively.

187. Complex surfaces, in their most general form, which we have called "meridian surfaces," are surfaces that are, on the one hand, generated by a variable complex curve whose plane rotates around a fixed line that lies in it, and are, on the other hand, enveloped by complex cones whose vertices advance along the same straight line. In connection with the first generation of the surface, we have obtained equation (15) as the analytic determination of the surface. For the sake of brevity, we set:

$$\left.\begin{aligned} (F\sin^2\varphi - 2K\sin\varphi\cos\varphi + E\cos^2\varphi) &\equiv a,\\ (R\sin^2\varphi - \quad O\sin\varphi\cos\varphi - U\cos^2\varphi) &\equiv b,\\ (B\sin^2\varphi + 2G\sin\varphi\cos\varphi + C\cos^2\varphi) &\equiv c,\\ -(Q\sin\varphi - P\cos\varphi) &\equiv d,\\ -(J\sin\varphi + H\cos\varphi) &\equiv e,\\ A &\equiv f, \end{aligned}\right\} \tag{42}$$

so we can write the equation in the following way:

$$aw^2 + 2btw + ct^2 + 2dvw + 2etv + fv^2 = 0. \tag{42}$$

In this, OX is taken to be the fixed straight line that will be a double line of the surface, and φ is the angle that the instantaneous meridian plane makes with a fixed plane – viz., the XZ coordinate plane. If we take the intersection of it with YZ in the instantaneous meridian plane to be the OZ axis, and denote it by OZ', while preserving the double line of the surface as the OX axis, then the latter equation will represent the relevant complex curve in its own plane in ordinary line coordinates.

Since the constants in the last equation are functions of φ, the complex curve that lies in the meridian plane will vary with φ – i.e., with the position of that plane. If we establish any condition equation between these constants and thus specialize the complex curve in it then this equation will give the meridian plane in which the curve, thus specialized, lies.

In particular, the complex curve degenerates into a system of two points when the condition equation (39) for the constants in its equation (43), which we can also write as:

$$f(b^2 - ae) + ae^2 + cd^2 - 2bde = 0, \tag{44}$$

is fulfilled.

When we revert to the constants of the complex and likewise divide by $\cos^2\varphi$, the foregoing equation will become:

$$\begin{gathered} A[(R\tan^2\varphi - O\tan\varphi - U)^2 - (F\tan^2\varphi - 2K\tan\varphi + E)(B\tan^2\varphi + 2G\tan\varphi + C)] \\ +(J\tan\varphi + H)^2(F\tan^2\varphi - 2K\tan\varphi + E) \\ +(Q\tan\varphi - P)^2(B\tan^2\varphi + 2G\tan\varphi + C) \\ -2(J\tan\varphi + H)(Q\tan\varphi - P)(R\tan^2\varphi - O\tan\varphi - U) = 0. \end{gathered} \tag{45}$$

This equation is of degree four with respect to tan φ. There are then four meridian planes, in general, in which the complex curves degenerate into systems of two points. Since these four planes go through the fixed coordinate axis OX, the four point-pairs will lie in the four planes on four straight lines that cut this axis. The point-pairs into which the four complex curves degenerate will be double points of the surface. We would like to call the four straight lines on which these point-pairs lie *singular rays of the complex surface.*

A complex surface has eight double points, in general, and four singular rays that cut the double lines of the surface, which will contain the double points, when taken pair-wise.

188. The four values of tan φ correspond to four groups of values for the constants in equation (43). For any group of values, this equation will then give the equations of the two points in its meridian plane. We can combine these equations into the following one:

$$aw + (b \pm \sqrt{b^2 - ac}\,)\, t + (d \pm \sqrt{d^2 - af}\,)\, v' = 0, \tag{46}$$

in which we must take square roots to have equal sign in one case and unequal signs in the other, according to whether the decomposition (34) or (36) exists, respectively. The two coordinates of the two points in the respective meridian planes are:

$$x = \frac{b \pm \sqrt{b^2 - ac}}{a}, \qquad z' = \frac{d \pm \sqrt{d^2 - af}}{a}, \tag{47}$$

in which we will get:

$$z = \frac{d \pm \sqrt{d^2 - af}}{a} \cdot \cos\varphi, \qquad y = \frac{d \pm \sqrt{d^2 - af}}{a} \cdot \sin\varphi, \tag{48}$$

instead of the value of z' above when we return to the original coordinates system. The singular ray that connects the two double points lies in the meridian plane that is determined by φ. We obtain:

$$x = \pm\frac{\sqrt{b^2 - ac}}{\sqrt{d^2 - af}} \cdot z' + \frac{b\sqrt{d^2 - af} \mp \sqrt{b^2 - ac}}{a\sqrt{d^2 - af}} \tag{49}$$

for its equation in that plane, or, with consideration to equation (40):

$$x = \frac{bd - ae}{d^2 - af} \cdot z' + \frac{de - fb}{d^2 - af} = \frac{b^2 - ac}{bd - ae} \cdot z' + \frac{e^2 - cf}{de - fb}. \tag{50}$$

We can replace z' with $\dfrac{y}{\sin\varphi}$ and $\dfrac{z}{\cos\varphi}$ in this equation, in succession, and then obtain the equations of the projections of that ray onto XY and XZ, resp.

The singular ray cuts out a segment:

$$x_0 = \frac{de - fb}{d^2 - af} = \frac{e^2 - cf}{de - fb} \tag{51}$$

from the double line OX and determines an angle δ with it that is determined by:

$$\tan\delta = \frac{d^2 - af}{bd - ae} = \frac{bd - ae}{b^2 - ac}. \tag{52}$$

The singular ray that corresponds to each value of φ that one finds is always real, although the expressions:

$$\sqrt{b^2 - ac} \qquad \text{and} \qquad \sqrt{d^2 - af}$$

can be real or imaginary. By contrast, the two double points on the singular ray are likewise real or imaginary according to these two expressions.

If an arbitrary line in space is taken to be the double line of the surface of a given second-degree complex then the determination of the four meridian planes that contain the double points of the surface will depend upon the solution of a fourth-degree equation. Thus, the singular line that connects the two double points in this meridian plane is given in a linear way. The determination of the two double points on the singular ray then depends ultimately upon the solution of a quadratic equation. The four meridian planes in which the singular rays of the surface lie can be pair-wise imaginary; the same will then be also true for the singular rays and the two double points. However, when the singular rays are real, the two double points that lie upon them can also be imaginary, as well as real.

189. In the following paragraph, we shall represent the same general complex surface by means of equation (20) that we determined in the third paragraph by means of equation (15) by starting with the second way of determining a complex. If we set:

$$\left.\begin{aligned} (Fx^2 - 2Rx + B) &\equiv a, \\ -(Kx^2 - Ox - G) &\equiv b, \\ (Ex^2 + 2Ux + C) &\equiv c, \\ (Qx - J) &\equiv d, \\ -(Px + H) &\equiv e, \\ A &\equiv f, \end{aligned}\right\} \tag{53}$$

while dropping the prime on x', then we can write this equation in the following way:

$$ay^2 + 2byz + cz^2 + 2dy + 2c + f = 0. \tag{54}$$

Once x has been chosen in YZ, it will represent the basis for the conic surface whose vertex lies on the double line of the surface and through which the choice of x on this double line will be determined.

The coefficients of the foregoing equation are functions of x. In particular, if we set:

$$f(b^2 - ae) + ae^2 + cd^2 - 2bde = 0 \tag{44}$$

then the base for the conic surface will no be longer a second-order curve, but that curve will degenerate into a system of two straight lines, so the corresponding conic surface will degenerate into a *system of two planes* whose line of intersection will meet the double line of the surface at the point that is determined by x. If we reintroduce the original constants of the complex into the foregoing equation then after dropping the common factor of x^2 we will get:

$$\begin{aligned} A\,[(Kx^2 - Ox - G)^2 - (Fx^2 - 2Rx + B)(Ex^2 + 2Ux + C)] \\ + (Px + H)^2\,(Fx^2 - 2Rx + B) + (Qx - J)^2\,(Ex^2 + 2Ux + C) \\ + 2(Px + H)(Qx - J)(Kx^2 - Ox - G) = 0. \end{aligned} \tag{55}$$

This equation is of degree four with respect to x. There are then four points on the double line, in general, that are no longer the centers of the circumscribed complex cones. These complex cones degenerate into systems of two planes whose line of intersection goes through the four points. The planes are *double planes* of the surface. The double planes of the surface arrange themselves into four pairs; the two double planes of any pair will intersect in four straight lines that meet the double lines of surface in the four points that are determined by the values of x. We call these four straight lines the *singular axes* of the meridian surface.

A complex surface has eight double planes, in general, which will intersect in the four singular axes of the surface, when taken pair-wise. Like the four singular rays, the four singular axes will cut the double lines of the surfaces.

190. The four values of x correspond to four groups of values for the constants in equation (51). For any group of values, this equation will represent a system of two straight lines in which the YZ coordinate plane will be cut by two associated double planes. These two lines will intersect in those points at which the singular axis (along which the two double planes intersect) meets the YZ-plane.

From the developments in number **185**, we immediately obtain the following equation for the equation of the two straight lines in YZ:

$$ay + (b \pm \sqrt{b^2 - ac})z + (d \pm \sqrt{d^2 - af}) = 0, \tag{56}$$

in which we have to take the square roots to have equal signs for both lines in one case and unequal signs in the other, according to the whether the decomposition (34) or (36) exists, respectively. The coordinates of the two straight lines in YZ are:

$$\frac{v}{u} = \frac{b \pm \sqrt{b^2 - ac}}{a}, \qquad \frac{w}{u} = \frac{d \pm \sqrt{d^2 - af}}{a}, \tag{57}$$

and we then get:

$$v = \pm\frac{\sqrt{b^2 - ac}}{\sqrt{d^2 - af}}\cdot w + \frac{b\sqrt{d^2 - af} \mp d\sqrt{b^2 - ac}}{a\sqrt{d^2 - af}}\cdot u \tag{58}$$

for the equation of their intersection point, or, with consideration given to equation (40):

$$v = \frac{bd - ae}{d^2 - af}\cdot w + \frac{de - fb}{d^2 - af}\cdot u = \frac{b^2 - ac}{bd - ae}\cdot w + \frac{e^2 - cf}{de - fb}\cdot u\,. \tag{59}$$

The coordinates of this point are then:

$$y = \frac{de - fb}{bd - ae} = -\frac{be - cd}{b^2 - ac} = -\frac{e^2 - cf}{be - cd}\,, \qquad z = -\frac{d^2 - af}{bd - ae} = -\frac{bd - ae}{b^2 - ac}\,. \tag{60}$$

The singular axis is determined analytically by means of equation (58), combined with the following one:

$$tx + w = 0. \tag{61}$$

The angle φ_0, which XZ defines with the meridian plane, which includes that axis, is given by the following equation:

$$\tan\varphi_0 = \frac{be - cd}{bd - ae} = -\frac{de - fb}{d^2 - af} = -\frac{e^2 - cf}{de - fb}\,. \tag{62}$$

We finally get:

$$x\tan\varepsilon = \sqrt{\frac{(bd - ae)^2 + (be - cd)^2}{(b^2 - ac)^2}} \tag{63}$$

for the determination of the angle ε that the singular axis makes with OX, which is the double line of the surface.

The determination of four singular axes of the meridian surface is linear, since the four points at which it intersects the double line will be determined by solving a fourth-degree equation. The determination of the two double planes of the surface, which intersect along one of the singular axes, depends upon the solution of a second-degree equation. The four points at which the singular axes cut the double line can be pair-wise imaginary; the same will also be true for the singular axes then. However, the double planes that intersect in the singular axes can also be imaginary, as well as real, when the singular axes are real.

191. Meridian surfaces of a special kind have lines of the complex for their double lines. In that case, the double lines of the surface will contact the generating curves in the various meridian planes. They will likewise be common lines of the complex cones that envelop the surface.

If we, in turn, take the OX axis to be the double line of the meridian surface then we will obtain the condition that A must vanish in the equation of the complex in order to express the idea that this line must belong to the complex. As a consequence of that, f will also vanish in equation (43), as well as in equation (54). Equation (45), by which the position of the meridian plane in which the singular rays lie is determined, reduces to the following one:

$$\begin{gathered}(J\tan\varphi + H)^2\,(F\tan^2\varphi - 2K\tan\varphi + E)\\ +\,(Q\tan\varphi - P)^2\,(B\tan^2\varphi + 2G\tan\varphi + C)\\ -\,2(J\tan\varphi + H)\,(Q\tan\varphi - P)(R\tan^2\varphi - O\tan\varphi - U) = 0.\end{gathered}\tag{64}$$

The equation remains of degree four with respect to $\tan\varphi$. The meridian surface then preserves its four singular rays. From (47), the two double points on it have the following coordinates:

$$x = \frac{b \pm \sqrt{b^2 - ac}}{a}, \qquad z' = \frac{2d}{a}, \qquad 0. \tag{65}$$

One of the two points falls upon the double line of the surface. Since this determination is independent of the instantaneous value of φ, one of the two double points will then fall upon each of the four singular rays in the double line of the surface.

The value of x_0 for that point on the double line at which the singular ray cuts the double line will be determined reduces to:

$$x_0 = \frac{e}{d} = \frac{J\tan\varphi + H}{Q\tan\varphi - P} \tag{66}$$

when we let f vanish in (51).

192. As a result of the assumption that the double line of the meridian surface is itself a line of the complex, equation (55), by means of which, the point at which the singular axes cut the double line is determined, will reduce to:

$$\begin{gathered}(Px+H)^2(Fx^2-2Rx+B)+(Qx-J)^2(Ex^2+2Ux+C)\\ -2(Px+H)(Qx-J)(Kx^2-Ox-G)=0\end{gathered}\tag{67}$$

by the vanishing of A. Since this equation remains of degree four with respect to x, the meridian surface will preserve its four singular axes. We obtain the following coordinates from equation (57):

$$u = a, \quad v = b \pm \sqrt{b^2 - ac}, \qquad w = 2d, \qquad 0 \tag{68}$$

for the two double planes that go through one of the four singular axes, and whose intersection with the double line will be determined by the foregoing equation. One of the two double planes of the surface that intersect in one of the four singular axes of the surface will then go through its double line.

When we let f vanish in equation (62), we will have:

$$\tan \varphi_0 = -\frac{e}{d} = \frac{Px+H}{Qx-J} \tag{69}$$

for the angle φ_0 that the meridian plane that goes through the singular axis defines with XZ.

193. We can write the two equations:

$$x_0 = \frac{J \tan\varphi + H}{Q\tan\varphi - P}, \tag{66}$$

$$\tan \varphi_0 = \frac{Px+H}{Qx-J} \tag{69}$$

in the following way:

$$\Phi(x_0, \tan\varphi) = 0, \qquad \Phi(x, \tan\varphi_0) = 0, \tag{70}$$

in which we denote the same function by Φ both times. If we introduce the foregoing value of x_0 into equation (64) and the value of $\tan\varphi_0$ into equation (67) then we will get:

$$\left.\begin{array}{c} x_0^2(F\tan^2\varphi - 2K\tan\varphi + E) + (B\tan^2\varphi + 2G\tan\varphi + C) \\ -2x_0(R\tan^2\varphi - O\tan\varphi - U) \equiv \\ \tan^2\varphi(Fx_0^2 - 2Rx_0 + B) + (Ex_0^2 + 2Ux_0 + C) \\ -2\tan\varphi(Kx_0^2 - Ox_0 - G) = 0, \\ \\ \tan^2\varphi_0(Fx^2 - 2Rx + B) + (Ex^2 + 2Ux + C) \\ -2\tan\varphi_0(Kx^2 - Ox - G) \equiv \\ x^2(F\tan^2\varphi_0 - 2K\tan\varphi_0 + E) + (B\tan^2\varphi_0 + 2G\tan\varphi_0 + C) \\ -2x(R\tan^2\varphi_0 - O\tan\varphi_0 - U) = 0. \end{array}\right\} \tag{71}$$

When we, in turn, denote the same function by Ψ, we can write the foregoing equations as:

$$\Psi(x_0, \tan\varphi) = 0, \qquad \Psi(x, \tan\varphi_0) = 0. \tag{72}$$

If we then eliminate x_0 from the first two equations in (70) and (72) and x from the second two equations in (70) and (72) then we will obtain the same fourth-degree equation for the determination of φ and φ_0 . If we eliminate $\tan\varphi$ from the same two equation pairs, in the one case, and $\tan\varphi_0$, in the other, then we will obtain the same fourth-degree equation for the determination of x_0 and x.

The four singular rays and the four singular axes intersect in the same points of the double lines of the planes that go through those double lines and lie in them, respectively.

We then obtain the determination of these points and planes when we combine these two equations:

$$\left.\begin{array}{l}\Phi(x, \tan\varphi) = 0, \\ \Psi(x.\tan\varphi) = 0,\end{array}\right\} \tag{73}$$

in which we consider x and $\tan\varphi$ to be variable (*).

(*) When x and $\tan\varphi \left(\equiv -\frac{v}{u}\right)$ are considered to be variable quantities, each of the two equations (73), when taken individually, will express a relation between the position of a point that moves along the OX coordinate axis and a plane that rotates around that axis: i.e., it will represent a geometric locus. The first equation, to which we would like to restrict ourselves here, determines, *in a general way, how any position of the point corresponds to a single position of the plane*, and conversely. That is the case, for example, when the point moves along a generator of a ruled surface, while the corresponding tangential plane rotates around that generator. For the analytic statement of that, let:

$$qy = pz$$

be the equation of such a ruled surface that has the OX coordinate axis for one of its generators when we denote any two linear functions by p and q. The equation of the tangential plane of the surface at any point on its generator, which corresponds to the function values p' and q', will then be the following one:

$$q'y = p'z.$$

This yields:

$$\tan\varphi = \frac{p'}{q'} = \frac{gx+h}{g'x+h'},$$

when x refers to the contact point, and g, h, g', h' mean the requisite constants to be determined. This equation has the form in question.

For the geometric interpretation of the dependency between a plane and a point that lies in it that is expressed by such an equation, we can, from the outset, assume that two straight lines are given arbitrarily, and when we let the plane rotate around one of these two lines, we can determine its various positions by $\tan\varphi$, while the position of the moving intersection point with the rotating plane on the second straight line will be determined by x. If, for example, we take *any two associated polars of a linear complex* to be the two straight lines then if a point moves on one of the two polars then the plane that corresponds to that point in the complex will rotate around the other one. The equation form above will give the law of rotation of the plane for a given motion of the point.

The same law of rotation is true for a plane that goes through a point that moves along a generator of a ruled surface, and likewise rotates around a second line of its generator. Finally, the same law is true for the rotation of the meridian plane around the double line of a complex surface when the plane is drawn through a point that moves on the polar of the complex surface. We immediately deduce the analytic statement of this latter geometric relation from number **170**, in which, the equation that was cited there:

$$\tan\varphi = \frac{Px+H}{Qx-J},$$

which we can also write in the following form:

$$x = \frac{J\tan\varphi+H}{Q\tan\varphi-P},$$

194. In the event that the double line of the complex surface lies at infinity, we have called such a surface an *equatorial surface.* If we take the YZ-plane to be the one in which the double line goes to infinity then we will obtain the following equation for the equation of the equatorial surface:

$$\begin{aligned} Dw^2 + 2(Lx - S)\,vw + (Fx^2 - 2Rx + B)v^2 \\ + 2(Mx + T)\,uw + 2(Kx^2 - Ox - G)\,uv + (Ex^2 + 2Ux + C) = 0. \end{aligned} \tag{3}$$

We thus think of the surface as being generated by a variable complex curve whose plane is parallel to YZ and moves parallel to that plane. The instantaneous plane of this curve is determined by x. In special cases, as is true for the meridian surfaces, the curve can degenerate into a system of points. The straight lines that connect two such points are *singular rays* of the equatorial surface, while the points themselves are *double points* of it. The singular rays of the equatorial surface are parallel to the YZ coordinate plane; in other words, they intersect the infinitely-distant double lines of that surface.

If, for the sake of brevity, we set:

$$\left.\begin{aligned} D &\equiv a, \\ (Lx - S) &\equiv b, \\ (Fx^2 - 2Rx + B) &\equiv c, \\ (Mx + T) &\equiv d, \\ (Kx^2 - Ox - G) &\equiv e, \\ (Ex^2 + 2Ux + C) &\equiv f \end{aligned}\right\} \tag{74}$$

then the foregoing equation (3) will go to the following one:

$$aw^2 + 2bvw + cv^2 + 2duw + 2euv + fu^2 = 0, \tag{75}$$

and in order to express the idea that this equation represents a system of two points, the development of (41) will give:

$$\begin{aligned} D\,[(Kx^2 - Ox - G)^2 - (Fx^2 - 2Rx + B)\,(Ex^2 + 2Ux + C)] \\ + (Mx + T)^2\,(Fx^2 - 2Rx + B) + (Lx - S)^2\,(Ex^2 + 2Ux + C) \\ + (Lx - S)\,(Mx + T)\,(Kx^2 - Ox - G) = 0. \end{aligned} \tag{76}$$

Since the degree of this equation with respect to x is four, an equatorial surface, like a meridian surface, will also have four *singular rays*, in general.

195. The center of the complex curve that generates the surface describes a diameter of the complex during that generation that we have referred to as the diameter of the

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gives the pole of the double line, relative to the complex curve in the meridian plane that is determined by φ, for a given value of φ on the polars of the complex surface by way of the corresponding value of x.

equatorial surface (no. **164**). If we take this diameter to be the remaining up-to-now undetermined axis OX then those terms in equation (3) that contain w in the first power will vanish, and in order for this to be true for every value of x, the four complex constants L, M, S, and T must vanish. The foregoing equation will then reduce to:

$$(Kx^2 - Ox - G)^2 - (Fx^2 - 2Rx + B)\,(Ex^2 + 2Ux + C) = 0. \tag{77}$$

Once we have determined the planes that contain the four singular rays by means of this equation, we will get:

$$y = \pm\sqrt{-\frac{c}{a}}, \qquad z = \pm\sqrt{-\frac{f}{a}} \tag{78}$$

for the determination of the two double points, when we set b and d equal to zero on each of these rays, in accordance with the coordinate system. According to whether the decomposition (34) or (36) exists – that is, according to whether e and f do or do not agree in sign, respectively – we must take the foregoing expressions for y and z for each of the two points to have equal or opposite signs, respectively. The singular line will be intersected by the diameter of the surface, and indeed in such a way that the two double points on it lie on both sides of the diameter at an equal distance from each other. The angle δ that the instantaneous singular ray makes with the XZ-plane will be determined by the equation:

$$\tan\delta = \pm\sqrt{\frac{c}{f}} = \frac{e}{f} = \frac{c}{e}, \tag{79}$$

in which the upper or lower sign is to be taken in the first or second of the two distinct cases above, respectively.

196. If we determine the same equatorial surface that have determined above by its latitude curves by enveloping cylinders whose axes are parallel to the YZ-plane then equation (28) will appear in place of equation (3). For the direction of the cylinder axis that is determined by γ, the new equation represents the intersection of that cylinder with the XZ-plane. If, for the sake of brevity, we set:

$$\left.\begin{aligned} (F\tan^2\gamma - 2K\tan\gamma + E) &\equiv a, \\ -(L\tan\gamma - M)\tan\gamma &\equiv b, \\ D\tan^2\gamma &\equiv c, \\ -(R\tan^2\gamma - O\tan\gamma - U) &\equiv d, \\ (S\tan\gamma + T)\tan\gamma &\equiv e, \\ (B\tan^2\gamma + 2G\tan\gamma + C) &\equiv f \end{aligned}\right\} \tag{80}$$

then the equation of the intersection curve will become:

$$ax^2 + 2bxz + cx^2 + 2dx + 2ez + f = 0. \tag{81}$$

In order to express the idea that this equation represents a system of two straight lines, and thus that the enveloping cylinder degenerates into a system of two planes that intersect the XZ-plane in these two straight lines, the development of equation (39) gives:

$$\begin{gathered} D\,[(R\tan^2\gamma - O\tan\gamma - U)^2 - (F\tan^2\gamma - 2K\tan\gamma + E)(B\tan^2\gamma + 2G\tan\gamma + C)] \\ + (S\tan\gamma + T)^2\,(F\tan^2\gamma - 2K\tan\gamma + E) + (L\tan\gamma - M)^2\,(B\tan^2\gamma + 2G\tan\gamma + C) \\ + 2\,(L\tan\gamma - M)\,(S\tan\gamma + T)\,(R\tan^2\gamma - O\tan\gamma - U) = 0. \end{gathered} \tag{82}$$

Corresponding to the four values of $\tan\gamma$ that the solution to this equation gives, there will then be *four pairs of double planes* of the equatorial surface into which four of the circumscribed cylinders will resolve; the two planes of each pair will intersect in one of the four singular axes of the surfaces. In each direction that is parallel to the YZ-plane, the equatorial surface will project onto second-order curves; the projections will be systems of two straight lines in the directions of the four singular axes.

197. If we take the diameter of the complex that is associated with the YZ-plane to be the OX axis then the foregoing equation will reduce to:

$$(R\tan^2\gamma - O\tan\gamma - U)^2 - (F\tan^2\gamma - 2K\tan\gamma + E)\,(B\tan^2\gamma + 2G\tan\gamma + C) = 0, \tag{83}$$

and the equation of the intersection curve of the relevant enveloping cylinder with the XZ-plane will reduce to:

$$ax^2 + cz^2 + 2ez + f = 0. \tag{84}$$

The equation resolves into the following two when the foregoing condition (83) is fulfilled:

$$ax \pm \sqrt{-ac}\cdot z \pm \sqrt{-af} = 0, \tag{85}$$

in which we must give the square roots the same or opposite signs according to whether condition (33) or condition (35) is fulfilled, resp., for each of the two lines that the equation represents.

The straight lines that are represented by the double equation (85) intersect in the same point. We get:

$$x = \mp\sqrt{-\frac{f}{a}} \tag{86}$$

for this intersection point. Thus, the singular axis of the equatorial surface, along which, two of its double planes will intersect, will also go through that point. Thus, the four singular axes, like the four singular rays of the surface, will, on the one hand, intersect the infinitely-distant double line because they are parallel to the YZ-plane, and on the other hand, will intersect the diameter of the surface, which we can consider to be its polar.

We get from (85):

$$\frac{z}{x \pm \sqrt{-\frac{f}{a}}} = \mp\sqrt{-\frac{a}{c}} \tag{87}$$

for the determination of the angle that the intersecting lines of the two double planes, which intersect along a singular axis, make with the OX with the XZ-plane in this plane. The two double planes, along with the two planes that intersect along the singular axis, one of which goes through the diameter of the surface and the other of which is associated with it, then define four harmonic planes, and are thus equally inclined with respect to them when the diameter is perpendicular to its associated planes.

198. We encounter a special kind of equatorial surface when we take an infinitely-distant line that belongs to the complex to be the double line of the surface. This comes down to saying that all latitude curves of the surface are parabolas.

As before, if we take the double line in the YZ-plane to be infinitely distant then the constant D will vanish in the equation of the complex, under the assumption that was made. Equation (76), by which, we have determined the distance between the singular rays, which are parallel to the coordinate plane, will then go to the following one:

$$\begin{gathered}(Mx+T)^2(Kx^2 - Ox - G) + (Lx - S)^2(Ex^2 + 2Ux + C) \\ + (Lx - S)(Mx + T)(Fx^2 - 2Rx + B) = 0.\end{gathered} \tag{88}$$

The surface has lost its diameter, which has gone to infinity.

If we set:

$$\left.\begin{aligned} (Ex^2 + 2Ux + C) &\equiv a, \\ (Rx^2 - Ox - G) &\equiv b, \\ (Fx^2 - 2Rx + B) &\equiv c, \\ (Mx + T) &\equiv d, \\ (Lx - S) &\equiv c, \\ D - f &\equiv 0 \end{aligned}\right\} \tag{89}$$

then equation (3) will go to the following one:

$$au^2 + 2buv + cv^2 + 2duw + 2cvw = 0, \tag{90}$$

and when we take x to be one of the four roots of equation (88), this equation will resolve into the following two:

$$\left.\begin{aligned} au + (b \pm \sqrt{b^2 - ac})v + 2dw &= 0, \\ au + (b \mp \sqrt{b^2 - ac})v \qquad &= 0, \end{aligned}\right\} \tag{91}$$

in which we have to take the upper or lower sign according to whether the decomposition (34) or (36) exists, resp.

One double point of the surface will then lie on the singular ray at infinity, while the other one will have:

$$y = \frac{a}{2d}, \qquad z = \frac{b \pm \sqrt{b^2 - ac}}{2d} \tag{92}$$

for its coordinates in its plane. The angle γ_0 that the direction of the singular ray defines with *OZ* is determined by the equation:

$$\tan \gamma_0 = \frac{a}{b \pm \sqrt{b^2 - ac}} = \frac{b \mp \sqrt{b^2 - ac}}{c} = \frac{d}{e}. \tag{93}$$

If we again introduce the constants of the complex then that will give:

$$\tan \gamma_0 = \frac{Mx + T}{Lx - S}. \tag{94}$$

199. If we are to determine the equatorial surface by its enveloping cylinder then we must start with equation (28). Under the assumption that was made that the infinitely-distant line in *YZ* belongs to the complex, equation (76), which expresses the idea that the curve that is represented by (28) resolves into a pair of lines, will become the following one:

$$\begin{gathered}(S \tan \gamma + T)^2 (F \tan^2 \gamma - 2K \tan \gamma + E)(L \tan \gamma - M)^2 (B \tan^2 \gamma + 2G \tan \gamma - C) \\ +2(L \tan \gamma - M)(S \tan \gamma + T)(R \tan^2 \gamma - O \tan \gamma - U) = 0.\end{gathered} \tag{95}$$

When we again introduce the constant determination (80) and let c vanish, for the sake of brevity, equation (28) will go to the following one:

$$ax^2 + 2bxz + 2dx + 2ez + f = 0, \tag{96}$$

and if $\tan \gamma$ is taken to be one of the roots of the foregoing equation then this equation will resolve to the following two:

$$\left.\begin{aligned} ax + 2bz + (d \pm \sqrt{d^2 - af}) &= 0, \\ ax \qquad + (d \mp \sqrt{d^2 - af}) &= 0, \end{aligned}\right\} \tag{97}$$

where we have to take the upper or lower sign according to whether the decomposition (34) or (36) exists, respectively.

One of the two double planes into which the complex cylinder that envelops the surface resolves will then always go to the double line at infinity of the surface. It cuts out a piece of *OX*:

$$x_0 = -\frac{d \mp \sqrt{d^2 - af}}{a} = -\frac{f}{d \pm \sqrt{d^2 - af}} = -\frac{e}{b}, \tag{98}$$

or, when we reintroduce the constants of the complex:

$$x_0 = \frac{S \tan \gamma + T}{L \tan \gamma - M}. \tag{99}$$

200. When we introduce the value of tan γ_0 from equation (94) into equation (88) and the value of x_0 from equation (99) into equation (95), we will arrive at the following theorem for equatorial surfaces of the special kind, as we did in number **193** for meridian surfaces:

The four singular rays and the four singular axes lie in the same plane, respectively, which goes through the double line at infinity of the surface, and are parallel to each other in that plane, respectively.

§ 7.

General considerations on complex surfaces, their double lines, double points, and double planes.

210. If a straight line moves in space then it will generate a ruled surface. It is therefore irrelevant whether we consider it to be a ray or an axis. We can represent the ruled surface by three equations in either ray coordinates or axial coordinates, which come down to a single equation in point coordinates, in the first case, and a single equation in plane coordinates in the second case.

202. In particular, if the straight line in space moves in such a way that any two successive positions of it are contained in the same plane, or – what amounts to the same thing – goes through the same point then it will describe a developable surface when it is considered to be a ray; when it is considered to be an axis, it will envelop a spatial curve. According to the two-fold conception of a straight line, the ruled surface will then go to a curve or developable surface, resp. The various positions of the straight line will then be represented by two complex equations in ray or axial coordinates. If we take:

$$y = sz + \sigma,$$
$$x = rz + \rho$$

to be the equations of two projections of the straight line that is considered to be a ray, then differentiate with respect to r, s, ρ, σ, and eliminate z after the differentiation, then we will get:

$$\frac{d\sigma}{ds} = \frac{d\rho}{dr}, \tag{100}$$

corresponding to the assumption that was made. On the other hand, when consider the straight line to be an axis and take:

$$u = qv + \kappa;$$
$$t = pv + \pi$$

to be the equations of its intersection points with two of the three coordinate planes then, corresponding to the same assumption, we will get the condition equation:

$$\frac{dq}{d\kappa} = \frac{dp}{d\pi}. \tag{101}$$

A spatial curve is simultaneously determined by any developable surface, and reciprocally, a developable surface is determined by any spatial curve. Equation (100) is the differential equation of the developable surface, while equation (101) is the differential equation of the spatial curve.

203. We obtain a second determination of a developable surface when we think of it as being enveloped by a plane that goes through two of the successive positions of the generating lines, and is thus represented by two equations in plane coordinates. The planes that envelop the developable surface belong to two surfaces as enveloping planes.

We obtain a second determination of the spatial curve when we think of it as being described by a point that is common to the enveloping axes in two successive positions, and will correspondingly be represented by two equations in point coordinates. A spatial curve is the intersection of two surfaces that are determined by points.

Developable surfaces are represented by a single equation in point coordinates. They are to be considered as ruled surfaces, insofar as we think of them as generated by a ray. Spatial curves are represented by a single equation in plane coordinates. They are to be considered as ruled surfaces, insofar as we think of them as generated by an axis.

204. A developable surface can degenerate into a conic surface upon further restriction. All rays then go through a fixed point, namely, the vertex of the conic surface. In order to express this, if (x^0, y^0, z^0) is the vertex of the conic surface then we will obtain the three linear condition equations:

$$\left.\begin{aligned} y^0 &= sz^0 + \sigma, \\ x^0 &= rz^0 + \rho, \\ ry^0 - sx^0 &= \eta. \end{aligned}\right\} \tag{102}$$

Two of these equations will imply the third one, assuming that r and s take on finite values. Once the fixed point is determined, the conic surface will be represented by a single complex equation in ray coordinates. If we take the fixed point to be the coordinate origin, in particular, then the three coordinates ρ, σ, and η will vanish simultaneously for all rays, and we will then obtain an equation between the two remaining coordinate r and s for the determination of the conic surface.

The spatial curve can degenerate into a plane curve upon further restriction. All of the axes that envelop the curve will then lie in a fixed plane, which will be expressed by three linear condition equations:

$$\left.\begin{aligned} u^0 &= qv^0 + \kappa w^0, \\ t^0 &= pv^0 + \pi w^0, \\ pu^0 - qt^0 &= \omega w^0 \end{aligned}\right\} \tag{103}$$

when we take $\left(\frac{t^0}{w^0}, \frac{u^0}{w^0}, \frac{v^0}{w^0}\right)$ to be that plane. Two of these equations will imply the third one if one assumes that q and p remain finite. When the plane is determined, the curve will be represented in that plane by a single complex equation in axial coordinates. This equation will reduce to one equation in two of the five axial coordinates when we take one of the three coordinate planes to be the plane of the curve, in particular. If that plane is YZ then the three coordinates p, π, ω will vanish for all of the axes that envelop the curve, and we will obtain an equation in the two remaining axial coordinates q and κ for the enveloped curve, and we can also construe these two coordinates as line coordinates in the YZ-plane.

205. However, we can also consider a conic surface as being enveloped by a plane and correspondingly represent its vertex by the equation:

$$x^0\, t + y^0\, u + z^0\, v + w = 0.$$

The conic surface will then determine a second equation in plane coordinates when it is combined with this one. If we take its vertex to be the origin, with which, the foregoing equation will reduce to:

$$w = 0,$$

then the second equation alone will succeed in representing the conic surface. In an analogous way, we can think of a plane curve as being described by a point and represent its planes by the equation:

$$t^0\, x + u^0\, y + v^0\, z + w^0 = 0.$$

The plane curve will then determine a second equation in point coordinates when it is combined with this one. If the curve lies in one of the three coordinate planes, which we would like to take to be YZ, in turn, then we will obtain:

$$x = 0$$

instead of the foregoing equation, and a single equation between the two remaining point coordinates, which we can construct in the *YZ*-plane, will suffice to represent the curve.

206. One can speak of the order of a conic surface only when we think of it as being described by a straight line, namely, a ray. That order is equal to the degree of the equation by which the conic surface will be represented in point coordinates. One can speak of the class of a plane curve only when we think of it as being enveloped by a straight line, namely, an axis. This class is equal to the degree of the equation by which the curve will be represented in plane coordinates.

If we introduce the straight line into geometry as a spatial element, and consider the straight line to be a ray, in one case, and an axis, in the other, then we must put ordinary *plane geometry*, as completely coordinated, alongside *point geometry*, along with curves that are enveloped by axes in the plane and conic surfaces that will be defined by rays that go through the point. The conic surfaces are of a given order and the curves are of a given class. The class of a conic surface and the order of a curve appear as secondary concepts. It is only when we think of a conic surfaces as being enveloped by planes that go through two successive generating rays that we can speak of its class. That class will likewise be the class of its curves of intersection, and will be equal to the number of tangential planes to a conic surface that can be drawn through a straight line that goes through the vertex of that conic surface. It is only when we think of the plane curve as being described by the intersection of successive axes that we can speak of its order. That order will then likewise be the order of the conic surface that can be drawn through it, and will be equal to the number of points at which a curve will be cut by a straight line that that lies in its plane.

207. The following remarks, which are connected with the foregoing ones, touch upon the theory of the representation of spatial structures by means of line coordinates in an essential way.

In order to represent a conic surface in ray coordinates, we must express the idea that the rays that define it go through a fixed point (x^0, y^0, z^0), namely, its vertex. All three of equations (102) are necessary in order to achieve that completely. If we take just two of these three equations – say, the first two:

$$\begin{aligned} y^0 &= sz^0 + \sigma, \\ x^0 &= rz^0 + \rho, \end{aligned} \tag{104}$$

then they will express the idea that the relevant ray (r, s, ρ, σ) cuts those two lines that project the point (x^0, y^0, z^0) onto the two coordinate planes *YZ* and *XZ*. This includes the double geometric condition that *either* the ray (r, s, ρ, σ) goes through the given point (x^0, y^0, z^0) *or* it lies in the plane that contains the two projecting lines, and thus goes through the point (x^0, y^0, z^0) and is parallel to the *XY*-plane. It is only when the third equation:

$$ry^0 - sx^0 = \eta$$

enters in that the second interpretation for equation (104) will go away, and then all that will be expressed is the idea that the ray goes through the given point.

In order to represent a plane curve in axial coordinates, we must express the idea that the axes that envelop it lie in a fixed plane $\left(\frac{t^0}{w^0}, \frac{u^0}{w^0}, \frac{v^0}{w^0}\right)$. All three of equations (103) are necessary for this. If we take just two of these three equations – say, the first two:

$$\begin{aligned} u^0 &= qv^0 + \kappa, \\ t^0 &= pv^0 + \pi, \end{aligned} \tag{105}$$

then they will express the idea that the relevant axis (p, q, π, κ) goes through the line of intersection of the given plane $\left(\frac{t^0}{w^0}, \frac{u^0}{w^0}, \frac{v^0}{w^0}\right)$ and the two YZ and XZ coordinate planes. That will correspond geometrically to two possibilities: *Either* the axis (p, q, π, κ) lies in the given plane, *or* it goes through the point at which the OZ coordinate axis intersects that plane. The third equation:

$$pu^0 - qt^0 = w^0$$

must be added in order to exclude the second geometric relationship.

If we have questions about the foregoing – at first glance, paradoxical – relations on analytical grounds then that will be due to the fact that when r and s (p and q, resp.) become infinitely large (*) the third of equations (102) and (103) will no longer be an algebraic consequence of the first two.

208. When, along with the equation:

$$\Omega_n = 0,$$

(*) Infinity will be avoided by the use of homogeneous line coordinates. For example, if we replace the first two of equations (102) with the following ones:

$$\begin{aligned} y^0 (z - z') &= z^0 (y - y') + (yz' - y'z), \\ x^0 (z - z') &= z^0 (x - x') + (x'z - xz') \end{aligned}$$

then both equations will be satisfied simultaneously when:

$$x = x^0, \quad y = y^0, \quad z = z^0 ;$$

that is, when the relevant ray goes through the given point.

The same two equations will also be satisfied when:

$$z = z' = z^0 ;$$

that is, when all rays lie inside of a plane that is parallel to XY whose distance from that plane is equal to z^0.

which represents a line complex of an arbitrary degree n in ray coordinates, the two equations:

$$y^0 = s\,z^0 + \sigma,$$
$$x^0 = r\,z^0 + \rho,$$

which we can regard as two linear complex equations, exist simultaneously, the coordinates of all rays that, on the one hand, define the complex cone of order n whose vertex is (x^0, y^0, z^0), and on the other hand, envelop the complex curve of class n whose plane, which is parallel to XY, goes through the vertex of the cone, will satisfy the three foregoing equations. These three equations *then simultaneously represent a complex curve, along with the complex cone.*

Likewise, the system that consists of the equation:

$$\Phi_n = 0$$

of a complex of degree n in axial coordinates and the two linear equations:

$$u^0 = q\,v^0 + \kappa\,w^0,$$
$$t^0 = p\,v^0 + \pi\,w^0,$$

which we can regard as the equations of two first-degree complexes, *simultaneously represents a complex curve* whose plane is $\left(\frac{t^0}{w^0}, \frac{u^0}{w^0}, \frac{v^0}{w^0}\right)$ and a *complex conic surface* whose vertex lies in that plane.

There exists a geometric relationship between the conic surface of order n and the curve of class n that is represented in the foregoing by the three complex equations that the n lines along which the conic surface is cut by the plane of the curve are, at the same time, those n tangents to the curve that go through the vertex of the conic surface.

Only those geometric structures that are reciprocal to themselves can be represented by one or more equations in line coordinates.

If we go to *point coordinates* in the case of *ray coordinates* then we will tacitly introduce the third of the three linear equations (102) into the foregoing developments, and any trace of the curve that is enveloped by the rays will vanish from the analytical representation.

If we go from *axial coordinates* to *plane coordinates* then we will tacitly introduce the third of the three linear equations (103), and any trace of the conic surface that is enveloped by the axes will vanish from the analytical representation.

209. We have already presented the following two characteristic properties of a complex of degree n (no. **19**), which are mutually reciprocal:

Infinitely many lines of a complex of degree *n* lie in any plane that is drawn through space, which will envelop a curve of class *n*. Infinitely many lines of the complex go through any point of space, which will define a conic surface of order *n*.

The double construction of the surfaces of a complex of order *n* is linked to that fact immediately. Once we have chosen any fixed straight line, we can, in one case, define them by those complex curves of class *n* whose planes go through the fixed line, and in the other case, envelop them by those complex cones whose vertices lie on the fixed line.

Once the existence of the complex of degree *n* has be established, at all, we can couple each of the above two characteristic properties – which are only one, in principle – with the definition of such a complex, and that definition, when it is allowed, at all, which involves the imaginary in the domain of geometry, is referred to as a *geometric* one in the usual sense.

The double determination of a complex of degree *n* would lose its meaning, and we would search in vain for an analytical expression for the complex if we were to switch the words "order" and "class" in the definition.

When we determine complex surfaces by means of the complex that they belong to, that determination will be coupled with the consideration of straight lines and their coordinates. The surfaces of a complex of degree *n* are of equal order and class, which we would like to denote by *p*. We consider surfaces of order *p* to consist of points that are cut from a curve of order *p* in a plane by a straight line at *p* points. We consider surfaces of class *p* to be enveloped by planes; *p* planes of the surface will go through a straight line, and the enveloping cone will be of class *p*. Complex surfaces have a multiple line, along which, a multiple ray and a multiple axis coincide; let the line be *m*-fold. If we consider it to be a ray then it will cut *m* sheets of the surface: The surface will have *m* tangential planes at each point of the *m*-fold line. The *m*-fold line is the geometric locus of the *m*-fold points of the surface and all curves, along which the surface is cut by planes, will have an *m*-fold point on that line. The *m*-fold line, when considered as an axis, is a locus that is enveloped by *m*-fold planes of the surface. Any plane that goes through the *m*-fold line will contact the surface at *m* points that lie on that line. Any point of such a plane is the vertex of an enveloping cone that has *m* sheets that will be contacted by the plane, which is also an *m*-fold plane of the cone, along *m* lines of the cone that go through the *m* contact points on the surface.

210. The surfaces of a second-degree complex have a *double line*. They will be intersected by planes in curves of order four and enveloped by cones of class four. When the intersection is a meridian plane, in particular, and thus goes through the double line, the fourth-order intersection curve will decompose into a second-order curve and two rays that coincide in the double line. If we consider the curve to be enveloped by axes and we appeal to its analytic representation by line coordinates in its plane then its class will reduce to two when any trace of two coincident rays, which are foreign to the complex, drop out: The curve in the meridian plane will be a complex curve. On the other hand, if we choose the center of the enveloping cone to be on the double line of the second-degree complex, in particular, then such a cone, which will be of class four, in general, will degenerate into a cone of class two and two enveloping axes that coincide in the double line. Any trace of these two axes will vanish when we think of the cone as being described by a ray. The enveloping cone will then enter in as a second-order cone, and thus, as a complex cone.

211. In the general case, the surfaces of a complex of degree n will detach from their intersection curves when the intersecting plane goes through the m-fold line of the surface, in particular, which will coincide in that line. When we overlook these m rays, the order of the curve will reduce to $(p - m)$. On the other hand, since the intersection curves whose planes go through the m-fold line are complex curves, and as such are the general ones of class n, we will obtain $n\,(n - 1)$ for the order of these curves. In that way, we will find:

$$p = n\,(n - 1) + m. \tag{106}$$

When we choose the center of the enveloping cone to be on the m-fold line of the complex surface, in particular, that will separate m axes from that cone that coincide in the m-fold line, and when omit these m axes, the class of the enveloping cone will drop by p to $(p - m)$. It will then become a complex cone, and will be, as such, the general cone of order n, and will thus have class $n\,(n - 1)$. In that way, we will arrive at the foregoing equation, which includes a relationship between n, which is the degree of the complex that the surface belongs to, p, which is the order and class of that surface, and m, which is the number that gives how many rays, on the one hand, and how many axes, on the other, coincide in the multiple lines of the surfaces.

212. In order for a complex surface to be described completely by a complex curve, the meridian plane that contains that variable curve must rotate around the arbitrarily chosen multiple line by 180 degrees. Under this rotation, the complex curve will go through any given point of the multiple line of the complex surface in a certain number of positions of the meridian plane. This number will likewise be the number of sheets of the surface that intersect on the multiple line, and will thus be equal to m.

Any point of the multiple line of the complex surface is the vertex of a complex cone of order n, at which, since it is the general cone of that order, $n\,(n - 1)$ meridian planes can be drawn through the multiple lines that contact the conic surface. The $n\,(n - 1)$ lines of the cone along which this contact takes place likewise contact each other, since they are lines of the complex, namely, the meridian curves that lie in the same meridian plane at the center of the enveloping cone on the multiple line. The number $n\,(n - 1)$ will thus determine the number of meridian curves that go through the arbitrarily-chosen center of the enveloping cone on the multiple line, and thus, the number of sheets of the complex surface that intersect on the multiple line.

The multiple line is an $n\,(n - 1)$-fold line.

Any point of the $n\,(n - 1)$-fold line of the surface of a complex of degree n is the vertex of a complex cone of order n, at which, $n(n - 1)$ planes can be drawn through the $n(n - 1)$-fold line. The $n(n - 1)$ lines along which the cone is contacted by these planes *will themselves contact the* $n(n-1)$ *complex curves that intersect at this point at the vertex of the cone.*

Along with the foregoing theorem, one likewise states the following one:

Any meridian plane of the surface of a complex of degree n contains a complex curve that cuts the $n(n-1)$*-fold line of the surface in this plane at* $n(n-1)$ *points. The tangents to the curve at these* $n(n-1)$ *points are lines of* $n(n-1)$ *complex cones that have those points for their vertices, and contact the meridian plane along these lines.*

We can also immediately link the two foregoing theorems, which follow reciprocally from each other as the statement of correlative properties of a complex, to the definition above of the complex of degree n, and then obtain the following theorem:

The number of straight lines (rays and axes) that define the multiple line of a complex surface is equal to the order of the complex curves that generate the surface and the class of the complex cones that envelop it.

We have:

$$m = n\,(n-1), \tag{107}$$

so:

$$p = 2n\,(n-1) = 2m. \tag{108}$$

The surfaces of a complex of order n have order and class $2n\,(n-1)$, *and have an* $n(n-1)$*-fold line.*

213. In place of the foregoing geometric considerations, we can just as simply pose analytic ones. We would thus lie to start with the surfaces of the second-degree complex. We have represented the projections of the individual meridian curves of such complex surfaces onto XZ by the following equation (no. **169**):

$$\begin{aligned}(F\tan^2\varphi - 2K\tan\varphi + E)\,w^2\\ +2(R\tan^2\varphi - O\tan\varphi - U)\,tw\\ +(B\tan^2\varphi + 2G\tan\varphi + C)\,t^2\\ -2(Q\tan\varphi - P)\,vw - 2(J\tan\varphi + H)\,tv + Av^2 = 0,\end{aligned} \tag{14}$$

and thereby made the assumptions that all meridian planes go through the OX coordinate axis and that OZ is perpendicular to OX. Any arbitrary point of this axis is to be chosen as the origin of the coordinates. The plane of the instantaneous meridian curve will be determined by the angle φ that it defines with a fixed meridian plane. When we set w equal to zero in the foregoing equation under these assumptions and divide by t, we will obtain the following equation for the determination of the directions of the projections of the two tangents to the instantaneous meridian curve that is determined by φ that are drawn through the origin:

$$A\left(\frac{v}{t}\right)^2 - 2(J \tan \varphi + H)\left(\frac{v}{t}\right) + (B \tan^2 \varphi - 2G \tan \varphi + C) = 0. \qquad (109)$$

When the meridian curve goes through the coordinate origin, the two tangents that go through that point will coalesce, which is expressed analytically by saying that the foregoing quadratic equation in $\left(\frac{v}{t}\right)$ has equal roots. This demands that:

$$A\,(B \tan^2 \varphi + 2G \tan \varphi + C) - (J \tan \varphi + H)^2 = 0. \qquad (110)$$

This condition equation has degree two in tan φ. Two of the infinitely many meridian curves of the complex surface will then go through any arbitrary point that is chosen on the *OX* coordinate axis. That axis will then be a double line of the complex surface.

When we set:

$$\frac{v}{t} = -\tan \psi$$

in the last equation, it will become:

$$A \tan^2 \psi + B \tan^2 \varphi + C + 2G \tan \varphi + 2H \tan \psi + 2J \tan \varphi \tan \psi = 0. \qquad (111)$$

This equation is to be regarded as the equation of a conic surface. ψ means the angle that one line of it defines with the *YZ* plane, and $\left(\frac{\pi}{2} - \psi\right)$ means the angle that it defines with *OX*. Once the plane in which two lines of the cone lie has been determined by an arbitrary choice of φ, that will give two values of ψ by which the directions of the two lines will be given in that plane. However, φ is likewise the angle that the projection of this line of the cone onto *YZ* makes with the *OZ* coordinate axis, and ψ is the angle that its projection onto *XZ* defines with that axis; one thus comes to:

$$\tan \psi = r, \qquad \tan \varphi = s,$$

and the equation of the conic surface goes to the following one:

$$Ar^2 + Bs^2 + C + 2Gs + 2Hr + 2Jrs = 0. \qquad (112)$$

We will obtain the same equation when set the line coordinates ρ, σ – and as a result of that, η – equal to zero in the general complex equation (I). It will then represent the complex cone that has its vertex at the origin.

214. By generalizing these considerations, we obtain the determination of the planes of the n (n – 1) meridian curves of class n for a complex of arbitrary degree n, which intersect an arbitrary point of their n (n – 1) lines at the origin, and the tangents to those curves at that point. The equation of the complex cone whose vertex falls upon the origin is deduced immediately when we set ρ, σ, and η equal to zero in the general equation of the n-degree complex, as above. Let the resulting equation of degree n in r and s be:

$$\Phi(r, s) \equiv \Xi_n = 0;$$

when we differentiate this, that will give:

$$\frac{d\Xi_n}{dr} = 0.$$

By eliminating r from the two foregoing equations, we obtain n $(n - 1)$ values of s for the determination of the planes of the n $(n - 1)$ meridian curves of the complex surface that intersection at the origin, and thus the directions of the tangents to the meridian curves at the origin from the corresponding n $(n - 1)$ equal roots r of the penultimate equation.

215. In paragraph 6 of this section, we proved analytically that the surface of a second-degree complex has *eight double points* that lie pair-wise in the *four singular rays, and* eight double planes that intersect pair-wise in the *four singular axes.* Just as the four singular rays intersect the double line, the four singular axes will lie in the same plane as the double line, and will therefore likewise intersect it. We would like to call the planes that can be drawn through the double line and the four singular rays the *four singular planes of the complex surface*, and denote the former by S_1, S_2, S_3, S_4 and the latter by E_1, E_2, E_3, E_4, respectively. In a corresponding way, we would like to call the intersection points of the singular axes with the double line the *four singular points of the complex surface*, and denote the former by A_1, A_2, A_3, A_4 and the latter by P_1, P_2, P_3, P_4, respectively.

Any ray that encounters the double line as a double ray of the complex surface will intersect the surface, since it is of order four at *two* more points, in addition. Any of the four singular rays will contain a pair of double points, in addition to the points at which it cuts the double lines, and thus, six pair-wise coincident points of the complex surface: *It lies on that surface in its entirety.*

If we draw a plane through the double line as a double ray of the complex surface, which we have referred to as the meridian plane, then the surface, since it is of order four, will be cut by that plane in yet another curve of second order, in addition to the two rays that coincide on the double line. The meridian plane that goes through a singular ray is a tangential plane of the surface, since the second-order curve in it degenerates into two rays that coincide on the singular ray. *The meridian planes that go through the four singular rays will be contacted by the surface along these rays.* The complete fourth-order intersection curve will degenerate into four rays in this case that pair-wise coincide in the double ray and the singular ray. By contrast, if we consider the fourth-order meridian curve to be a complex curve of class two that is enveloped by axes, in which the two rays that coincide in the double line remain completely beyond consideration, then it will degenerate in the present case into *a system of two points* with which the double points that lie on the instantaneous ray will coincide. The tangential plane to the surface at any point of a singular ray will be the singular plane that goes through that ray and the double line.

216. Since the complex surface is of class four, one can draw two more planes on the surface through any axis that lies in a plane with the double line (double axis) of the surface, in addition to the double plane that goes through the double line. Any of the four singular axes will be contained in a pair of double planes, in addition to the double plane that goes through the double line; they will thus be contained in six pair-wise coincident planes of the complex surface. As a consequence of this, *any plane that is drawn through it will be a plane of the complex surface.*

The enveloping cone of a complex surface of class four, which has a point of the double line for its center, resolves into the two axes that coincide with the double line and a cone of class two. The singular points P at which the singular axes A cut the double line are the vertices of cones of class two that degenerate into two axes that coincide in the singular axes and contact the surface at the singular points. The complete enveloping cone degenerates in this case into four axes that coincide pair-wise in the double line and the singular axis, respectively. By contrast, if we consider the enveloping cone to be a second-order cone that is described by rays then it will degenerate into a system of two planes that coincide with the two double planes that go through the instantaneous singular axis. The contact point of all planes with the surface that go through a singular axis is the singular point at which that axis will cut the double line.

217. An arbitrary plane cuts the complex surface in a fourth-order curve that has a double point at its intersection with the double line. Either *two real* or *two imaginary branches* of the curve will intersect at that double point; in the latter case, the double point will be an isolated point of the curve. By going from one case to the other, it will become a cuspidal point. That transition will correspond to the fact that the plane of the curve goes through one of the four singular points P_1, P_2, P_3, P_4 at which the double line of the complex surface will be intersected by the four singular axes A_1, A_2, A_3, A_4 . The double line will be divided into four segments P_1P_2, P_2P_3, P_3P_4, P_4P_1 by these four points, where we shall count the two external segments that meet at infinity as a single one. The double line lies completely in the complex surface, but in such a way that it will cut two real sheets of the surface in two segments that do not meet each other, while the remaining two segments, which likewise do not meet each other, will be the real intersections of two imaginary sheets of the surface. The two tangents to the curve at its double point will likewise be real or imaginary, along with the two tangential planes of the complex surface at that point. They will lie in these two tangential planes and rotate around the common double point in these two planes when the plane of the curve is rotated around that point arbitrarily. If the intersecting plane goes through one of the four singular points then that point will generally be a cuspidal point of the intersection curve. The two tangential planes to the surface at such a point will coincide in those planes that go through the double line and the singular axis, respectively. The tangents to all intersection curves at their common cuspidal point that coincides with the singular point will lie in this plane, whose directions might also be in the intersecting plane. We can describe the complex surface by a varying curve of order four with a cuspidal point that we can rotate around the tangent at that point. These tangents can have all possible directions in the tangential planes to the surface; in particular, when they coincide with the singular axis, the intersection curve, like its plane, might rotate around that axis into all positions of the same two branches that contact on the singular axis at the singular point. If the plane that rotates around the singular axis coincides, in particular, with one of the two planes into which the complex cone degenerates when its center falls upon the singular point then the fourth-order curve that lies in it will resolve into two curves of order two that coincide in those second-order curves along whose entire extent the surface will contact the planes. Finally, if the plane that rotates around the singular axis likewise goes through the double line then the fourth-order intersection curve will resolve into a second-order curve and two straight lines that coincide in the double line that represent a second curve of order two that contacts the former at singular points.

218. Any point of space is the center of a cone of class four that envelops the complex surface and has those meridian planes that that go through the point for its double planes. These double planes will either contact two real sheets of the conic surface in two real lines of it or those two sheets will be imaginary, and with them, the two lines of the cone. In the latter case, the double contact will be imaginary; viz., the double plane will be an isolated one. The two lines along which the enveloping cone contact the double plane will cut the double line of the complex surface in two points; that surface will contact the double plane at these two points. The four singular planes E_1, E_2, E_3, E_4, which contain the four singular rays S_1, S_2, S_3, S_4 of the complex surface, will belong to the meridian planes. They will divide the infinite space into four spatial components E_1E_2, E_2E_3, E_3E_4, E_4E_1, each of which will be bounded by two successive singular planes and will consist of two components that meet at infinity. If the vertex of the enveloping cone of one of the four spatial components is found on a singular plane in the adjacent spatial component then the cone in question will be contacted along two of its lines at one of the two positions of its vertex, while in the other position of its vertex the meridian plane that goes through it will be an isolated double plane. In the transitional case where the vertex of the enveloping cone lies in the singular plane itself, this plane will osculate the enveloping cone; it will then be an inflection plane of the enveloping cone that simultaneously contacts it and cuts it. If the vertex of the enveloping cone changes position in the same meridian plane then the two lines along which the cone is contacted by that plane will rotate in that plane around two fixed points of the double line in which the complex surface will be contacted by the meridian plane. When the meridian plane rotates around the double line, the two contact points on that line will change position. In particular, when the vertex of the enveloping cone is chosen to be in one of the four singular planes, they will coincide in those points at which the singular ray meets the double line, respectively. We can envelop a complex cone by a varying cone of order four that has a given plane for its inflection plane and whose vertex moves along a straight line in that plane. The given plane will then be the singular plane of the complex surface and the given line in it will then cut the double line at the point at which it will be cut by the ray, respectively. In particular, if the vertex of the enveloping cone lies in the singular line in the singular plane and moves along it then the cone in question will have two sheets at all positions of its vertex that will contact the singular plane along the singular ray. Only when the vertex is chosen to be at one of two double points on the ray that goes through those two of the eight double points will the enveloping cone of class four resolve into two cones of class two that coincide in the contact cone of the double point. That cone will have the singular ray as one of its lines and will contact it along the respective singular plane.

219. Any point of space is the center of an enveloping cone of the complex surface that has eight double lines that go through the eight double points of the surface. All curves along which the surface will be contacted by circumscribing cones will also have the eight double points of the surface for their double points. Therefore, this relationship will also exist when the vertex of the cone falls upon the double line of the surface. However, four pairs of planes will then separate from the conic surface, which, as a conic surface of class four with a double plane that goes through the double plane will generally be of order ten, and these pairs of planes will then coincide with the four singular planes of the surface, with which, only one second-order cone will remain. The contact curve of this cone will go through the eight double points of the surface and will be cut by each of the four singular planes at two of these points.

If we, in agreement with the foregoing, project the surface onto an arbitrary plane (in order to illustrate the silhouette, we can take it to be the surface that is illuminated from a point of its double line) from a point that lies upon its double line and can move arbitrarily to infinity in it then we will obtain a conic section that perpetually moves with the change of position of the point on the double line, and likewise four straight lines that keep their positions. They will be the projections of four singular rays, or also – what amounts to the same thing – the intersection lines of the image plane with the four singular planes of the surface. They will all go through the point at which the double line of the surface meets the image plane, and will cut the conic section in the projection of the eight double points. When the vertex of the circumscribing cone moves along the double line the system of the conic section and the four straight lines will transform into a curve with eight double points.

In particular, when the vertex of the second-order circumscribing cone falls upon one of the four singular points of the complex surface, and as a result, resolves into a system of two planes, the spatial fourth-order contact curve will decompose into two second-order plane curves. The eight double points of the surface will then distribute themselves along these two curves.

220. Any plane cuts a complex surface in a curve that has order four, and will likewise have class ten, since it has a double point on the double line of the surface. The tangential planes of the surface at points of the intersection curve determine a developable surface. All such developable surfaces have the eight double planes of the complex surface in addition to their own. The intersection curve will be enveloped by all axes along which their planes are cut by the enveloping planes of the developable surface; the intersection lines with the eight double planes will be double axes of the intersection curve. These relations also still continue to exist when the intersecting plane goes through the double line of the complex surface. Eight points will then separate from the intersection curve that coincide pair-wise at the four singular points that lie on the double line, and all that will remain is a curve of class two that belongs to both the complex surface and complex. That curve will be enveloped by the eight intersections with the eight double planes that intersect pair-wise in the four singular points on the double line. If the intersecting plane coincides with one of the four singular planes of the surface, in particular, then the curve of class two will resolve into two points that coincide with two double points of the complex surface, and the developable surface of class four will resolve into the two contact cones of class two at those two points. Each of the two associated double planes will then go through one of the two double points.

221. By the restricting condition that no double plane can contain two double points that lie upon the same double ray, and thus that two double planes cannot go through any double point that intersect in its singular axis, one is given immediately, on the one hand, the distribution of the eight double points into four plus four points that lie upon each of two double planes that intersect along its singular axis, as well as, on the other hand, the distribution of the eight double planes into four plus four planes that go through any two double points that lie on singular rays of them.

We would like to denote the four singular rays by the symbols:

$$(1, 2),\ \ (3, 4),\ \ (5, 6),\ \ (7, 8)$$

and the double points on them by:

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8.$$

We obtain the following eight groups of points:

$$\begin{array}{ll} (1,3,5,7), & (2,4,6,8), \\ (1,3,6,8), & (2,4,5,7), \\ (1,5,4,8), & (2,6,3,7), \\ (1,7,4,6), & (2,8,3,5). \end{array} \tag{113}$$

No two double points that lie upon the same line will appear in either of the groups. Any two adjacent groups will contain eight double points, in all. The four double points of one of the two groups will lie on one of two double planes, which intersect along a singular axis, while the four of the other group will lie on the other one. In the same sequence, we would like to take the following, simpler, notation for the eight double planes, instead of the foregoing one:

$$\begin{array}{ll} \text{I}, & \text{II}, \\ \text{III}, & \text{IV}, \\ \text{V}, & \text{VI}, \\ \text{VII}, & \text{VIII}. \end{array}$$

One will then have:

$$\text{(I, II)}, \quad \text{(III, IV)}, \quad \text{(V, VI)}, \quad \text{(VII, VIII)}$$

for the symbols for the four singular axes along which the eight double planes I and II, III and IV, V and VI, VII and VIII intersect. From the schema (113), we immediately obtain the following schema for the distribution of the eight double planes into groups of four planes that go through their double points:

$$\left.\begin{array}{ll} \text{(I, III, V, VII)}, & \text{(II, IV, VI, VIII)}, \\ \text{(I, III, VI, VIII)}, & \text{(II, IV, V, VII)}, \\ \text{(I, V, IV, VIII)}, & \text{(II, VI, III, VII)}, \\ \text{(I, VIII, IV, VI)}, & \text{(II, VIII, III, V)}. \end{array}\right\} \tag{114}$$

The four double planes of the foregoing eight groups intersect in the eight double points, respectively, which we previously denoted by the symbols:

$$\begin{array}{ll} 1, & 2, \\ 3, & 4, \\ 5, & 6, \\ 7, & 8. \end{array}$$

These eight double points lie pair-wise on the four singular rays of the complex surface whose symbols are (1, 2), (3, 4), (5, 6), (7, 8).

Therefore, when the eight double points of the surface are given, we will immediately obtain its eight double planes, and conversely, when the latter are given, we will obtain the former. A remarkable geometric structure that is polar reciprocal to itself is present in the eight points and eight planes.

222. If we draw an arbitrary plane through the double line of a complex surface and choose a point of it arbitrarily then a complex curve of class two will lie in that plane and the point will be the vertex of a second-order complex cone. Two lines of the cone will be two tangents of the curve. The polar plane of the double line relative to the cone will go through the pole of its double line relative to the curve. This relationship will continue to exist no matter how the plane of the curve might rotate around the double line or how the vertex of the cone might change position on that double line. It will then follow from this immediately in a geometric way, as we previously proved analytically, that the poles of the double line of a complex surface relative to all of its meridian curves will lie on a straight line, and that the polar planes of the double line relative to all circumscribing complex cones will intersect along that straight line. We have called this line the *polar of the complex surface.* In order to determine it, we need only to construct the two poles of the double line relative to any two meridian curves of the surface or the two polar planes of the double line relative to any two circumscribing complex cones of the surface.

On the one hand, if we take, instead of the meridian curves, those two points that lie on a singular ray into which the curve degenerates when its plane coincides with one of the four singular planes of the complex surface, in particular, and on the other hand, instead of the circumscribing complex cone, those two planes that intersect along a singular axis into which the cone degenerates when its vertex falls upon one of the four singular points of the complex surface then we will immediately obtain the following theorem:

The polar of a complex surface, like the double line itself, intersects its four singular rays and its four singular axes. Any singular ray is harmonically separated with the two double points of the surface that it connects and the two intersections with double lines and polars. The two double planes that intersect along any singular axis and the two planes that go through this axis and the double line and polar define a system of four harmonic planes.

223. All of the singularities of a complex surface are determined in a linear way when we know the double line, the polar, and three double points 1, 3, 5, or in place of them, three double planes I, III, V of the surface. In this, we are assuming only that no two double points lie upon the same singular ray and no two double planes intersect along the same singular axis.

We can draw three straight lines through the three given points that intersect the double line and the polar. These three straight lines, which are three singular rays of the surface, go through the three associated double points 2, 4, 5, which we obtain immediately from the previous number. All that then remain unknown are two of the eight double points, whose symbols we would like to bracket, in order to distinguish them. The known eight double points will suffice to determine all eight double planes (no. **221**):

$$\begin{aligned} &(1,\ 3,\ 5,\ (7)) \equiv \text{I}, && (2,\ 4,\ 6,\ (8)) \equiv \text{II}, \\ &(1,\ 3,\ 6,\ (8)) \equiv \text{III}, && (2,\ 4,\ 5,\ (7)) \equiv \text{IV}, \\ &(1,\ 5,\ 4,\ (8)) \equiv \text{V}, && (2,\ 6,\ 3,\ (7)) \equiv \text{VI}, \\ &(1,\ 4,\ 6,\ (7)) \equiv \text{VII}, && (2,\ 3,\ 5,\ (8)) \equiv \text{VIII}, \end{aligned}$$

which intersect pair-wise in four singular axes. In each of the eight double planes, we obtain immediately, and in a linear way, the contact curve that goes through three known double points and contacts the respective singular axis at its intersection with the double line, moreover. Four of the eight double planes I, IV, VI, VII intersect at one of the two previously-unknown double points in (7), while the remaining four II, III, V, VIII intersect at the others (8). With that, the fourth singular ray is also determined.

If we start with the three double planes I, III, V then those three straight lines that connect the points at which three planes intersect the double line and the polar will be three singular axes of the surface, and from the previous number, we will likewise obtain the three new double planes II, IV, VI, which intersect the three given ones along these three singular axes. From number **221**, the three pairs of double planes will suffice to determine the eight double points and the eight contact cones at the eight double points.

The two still-unknown double planes VII and VIII are determined by the fact that they contain the eight double points, four on one and four on the other; their intersection is the fourth singular axis.

The remarkable geometric structure that was already referred to at the end of number **221** can thus be constructed by means of the double line and the polar of the surface – both lines have a completely equivalent relationship to it – and three points or planes of it. This structure then depends upon:

$$2 \cdot 4 + 3 \cdot 3 \equiv 17$$

constants. However, the general complex surface itself depends upon just as many constants. That surface will be determined when the geometric structure that depends upon it is determined.

224. Some remarkable linear constructions of the general complex surface are linked with the foregoing when the double line, the polar, and either three of its double points or three of its double planes are given.

Determination of the complex curve in an arbitrary meridian plane:

First construction. One constructs the eight double planes. A meridian plane cuts these eight double planes along eight straight lines, which will be contacted by the complex curves in them. Five of these straight lines will be sufficient for the linear determination of the curve.

Second construction. A meridian plane cuts the eight contact curves, except for the eight points that coincide pair-wise in the four singular points, at eight additional points. These eight points lie on the complex in the meridian plane. Five of them will suffice for the linear determination of the curve. From the first construction, we obtain the eight tangents in each meridian plane that can be drawn from the four singular points to the curve, and from the second construction, the contact points to these tangents.

Determination of the complex cone whose vertex is chosen arbitrarily upon the double line:

First. construction. One constructs the eight double points of the surface. The eight straight lines that connect the chosen vertex with these eight double points are eight lines of the complex cone, which is determined by five of these lines in a linear way.

Second construction. One constructs the eight contact cones at the eight double points. Two tangential planes to each of the eight contact cones can be drawn from their centers, which are chosen arbitrarily on the double line. Of the sixteen tangential planes, four times two of them will coincide in the four singular planes. The remaining eight tangential planes to the eight contact cones that do not go through the double line will contact the complex cone, which is determined in a linear way by five of these planes. In the first construction, the complex cone will determined by eight lines that lie pair-wise in four singular planes, and in the second construction, it will be determined by the planes that contact it along these lines.

In order to then describe the surface itself, we merely need to let the curve of order and class two that is determined by any position of the meridian plane rotate around the double line. In order to envelop this surface with a complex cone of order class two, which is determined for any position of its vertex, we merely need to let this vertex move along the double line.

225. The foregoing discussion of the singularities of complex surfaces of the general kind can likewise be carried over to the special case in which any line that belongs to the complex is taken to be the double line of the surface. Then, on the one hand, the double line will contact all meridian curves of the surface, and on the other hand, a common line to all circumscribed complex cones will fall on the double line. *The double line and the polar to the surface will coincide in a straight line.*

In the general case, there is no direct route from meridian curves that cut the double line to ones that do not cut it. If there is a single such curve that contacts the double line then that line will belong to the complex, and will then contact all meridian curves. Nonetheless, there is a direct route from circumscribing complex cones with the property that the double line lies outside them to complex cones with the property that the double line lies inside them. For the surfaces of the special kind, all meridian curves will be real, and none of the circumscribing complex cones will reduce to a point.

226. Whereas the double line will be enveloped by a meridian plane that rotates around it, it will likewise be described by the points at which it is contacted by the complex curve that lies in the meridian plane. Any line that goes through the contact point in an arbitrary position of the meridian plane will cut the surface in four points, three of which will coincide on the double line. Any arbitrary plane that goes through such a line of the meridian plane will cut the surface in a curve of order four that has a cuspidal point at the contact point and the line in question for its tangent. The meridian plane is the geometric locus of the cuspidal tangents to all intersection curves whose planes go through the contact point of the complex curve on the double line; the two tangential planes to the surface will coincide at that point. When that point moves along the double line, the tangential plane to the surface at that point will rotate around that line. The double line will be a *cuspidal ray of the complex surface.* It will no longer consist of segments that are the (always real) intersections of rotating real and imaginary sheets of the surface; two real sheets of the surface will coalesce on the cuspidal edge.

227. A complex cone whose vertex is chosen arbitrarily on the double line has a plane that goes through that double line for its tangent plane. If we draw an arbitrary straight line through the vertex of the cone in that tangential plane and take an arbitrary point of it to be the vertex of an enveloping cone of class four then the tangential plane to the complex cone will be an inflection plane for that cone, which will be osculated by it along the chosen straight line (viz., a line of inflection of the cone). It follows from this that an arbitrary meridian plane will be the common inflection plane for all circumscribing cones of class four whose vertices lie in it. If the meridian plane rotates around the double line then the vertex of the complex surface that it contacts will move on the double line. If we intersect a circumscribing cone of class four whose center lies in an arbitrary meridian plane with a second meridian plane then the intersection curve will be of class four, will have the double line for its inflection line, and will have those points on it for inflection points that are vertices of those complex cones that contact the former meridian plane. When this meridian plane of the vertex of the cone of class four rotates around the double line, the double line will continually remain the line of inflection of the intersection curve, while the inflection point will move along it. The double line that entered in the previous numbers as a *cuspidal ray* of the surface will now enter in this number as an *inflection axis.*

228. Exactly the same relationship exists between the advance of the contact point of the complex curves of a complex surface of the special kind along the double line and the rotation of its plane around that line as the one that exists between the advance of the vertex of the complex cone of the surface along the double line and the rotation of its tangential plane around that line. In number **170**, when we started with the general complex equation and chose those complex surfaces that had OX for their double line, under the assumption of rectangular coordinates, we arrived at the following equation:

$$\tan \varphi = \frac{Px + H}{QX - J},$$

where φ, in the general case that was already considered in the footnote in number **193**, meant the angle that an arbitrary meridian plane made with the XZ coordinate plane, and x corresponded to the pole of the double line relative to the complex curve that lay in the meridian plane. In the case of complex surfaces of the special kind, where the double line is a line of the complex, and thus the constant A vanishes in its equation, the position of the contact point of the complex curve with the double line will be given by x. When the complex surface of the special kind is, in the one case, described by a complex curve, and in the other case, enveloped by a complex cone, the foregoing equation will then express the relationship in question that exists between the motion of the contact point on the complex curve (the vertex of the complex cone, resp.) along the double line and the rotation of the plane of the complex curve (the tangential plane of the complex cone, resp.) around that double line.

When we ignore the origins of the surface, we can refer x in the foregoing equation to an arbitrary point of the surface that lies upon the double line and φ to its tangential plane at that point. If the contact point moves along the double line (viz., the cuspidal ray of the surface) then the tangential plane to the surface at that point will rotate around the double line (viz., the inflection axis of the surface) in the same way that the tangential plane rotates around a generator of a ruled surface of degree two when the contact point moves along it. The law by which the contact point and tangential plane are reciprocally determined will be the same in both cases (see the footnote in number **193**).

229. For the complex surface of the special kind that we consider here (for which, all complex curves contact the double line and that double line is simultaneously a common line of all complex cones, on the one hand, when the complex curve in any of the four singular planes degenerates into two points, one of the two points will coincide with the intersection of the respective ray and the double line, whereas on the other hand, when the vertex of the complex cone is one of the four singular points, and therefore the cone degenerates into a system of two planes), one of these two planes will go through the double line and the singular axis.

Complex curves and complex cones in the complex surface in question arrange themselves together pair-wise in such a way that points at which the curves contact the double line will be the vertices of cones and the planes of the curves will contact the cones along the double line. Those complex cones that are associated with a complex curve that degenerates into two points will then degenerate into two planes, in their own right. Under the assumption that was made, the complex cone must then contact the singular plane along the double line. Furthermore, it must contain the singular ray that lies in that plane, since it will belong to the surface completely and go through its vertex. These two conditions can exist simultaneously only when the cone degenerates into a system of two planes, one of which is the singular plane. It is then proved that *the four singular axes of the surface lie in the four singular planes, and the four singular rays go through the four singular points.*

229. In order to distribute the eight double points on four of the eight double planes according to the schema (113) in number **221** in the case of the complex surfaces in question whose double lines are lines of the complex, we take the points that were referred to as:

$$1, \quad 3, \quad 5, \quad 8$$

to be those four of those points that coincide on the double line with the four singular points:

$$P_1, \quad P_2, \quad P_3, \quad P_4 .$$

We would now like to represent the four remaining double points:

$$2, \quad 4, \quad 6, \quad 7,$$

which are the four vertices of a tetrahedron, by:

$$Q_1, \quad Q_2, \quad Q_3, \quad Q_4 ,$$

in such a way that:

$$P_1Q_1, \quad P_2Q_2, \quad P_3Q_3, \quad P_4Q_4$$

are the four singular rays. The cited schema then gives the eight double planes:

$$\begin{array}{ll} \mathrm{I}, & \mathrm{II}, \\ \mathrm{III}, & \mathrm{IV}, \end{array}$$

	V,	VI,
	VIII,	VII
the following symbols:		
	$P_1P_2P_3Q_4$,	$Q_1Q_2Q_3P_4$,
	$P_1P_2P_4Q_3$,	$Q_1Q_2Q_4P_3$,
	$P_1P_3P_4Q_2$,	$Q_1Q_3Q_4P_2$,
	$P_2P_3P_4Q_1$,	$Q_2Q_3Q_4P_1$.

The four planes I, III, V, VIII go through the four vertices of the tetrahedron $Q_1Q_2Q_3Q_4$, and all of them intersect along the double line $P_1\, P_2\, P_3\, P_4$. The four singular planes are then:

$$E_4, \quad E_3, \quad E_2, \quad E_1 .$$

The four planes II, IV, VI, VIII coincide with the four faces of the tetrahedron and cut the double line, moreover, at the four singular points $P_4\, P_3\, P_2\, P_1$, respectively. The four singular axes are:

$$\text{(I, II)}, \quad \text{(III, IV)}, \quad \text{(V, VI)}, \quad \text{(VIII, VII)}.$$

We deduce the following relations from the foregoing:

Any vertex of the tetrahedron is a double point of the surface, and its opposite side is a double plane. The latter intersects the double line at one of the four singular points, through which goes one of the four singular planes. The straight line that connects the vertex of the tetrahedron with the singular point is one of the four singular rays and the line of intersection of the opposite face of the tetrahedron with the singular plane is one of the four singular axes of the surface.

A singular ray and a singular axis lie in each of four singular planes that are meridian planes. They both intersect in that plane at the corresponding singular point along the double line.

230. Complex surfaces depend upon seventeen mutually-independent constants, in general, and complex surfaces that have a line of the complex for their double line will depend upon one less constant. These complex surfaces are determined completely when one is given their double line and those tetrahedra that have the four double points for their vertices and the four double planes for their faces. Double lines and tetrahedra can thus be assumed to be arbitrary from here on.

The foregoing yields the following simple constructions:

An arbitrary face of the tetrahedron (Q_2, Q_3, Q_4) is a double plane of the surface, the point at which it cuts the double line is a singular point P_1, and the straight line P_1Q_1 that connects that point with the opposite vertex Q_1 of the tetrahedron is a singular ray of the surface S_1 . If we project that singular ray onto the double plane (Q_2, Q_3, Q_4) parallel to the double line then the projection will likewise be the singular axis A_1 that goes through the singular point P_1 and the projecting plane will be the singular plane E_1 of the surface. The contact curve in the double plane will be determined by the fact that it goes through the three vertices Q_2, Q_3, Q_4, and the singular point P_1 in that plane and contact the singular axis A_1 at the latter point. The contact cone at Q_1 will be determined by the fact that it will be contacted by the three faces of the tetrahedron that intersect at that point, as well as the singular plane E_1, and in fact along the singular ray S_1 . This cone, with (Q_2, Q_3, Q_4) as its base, has a conic section that contacts the three tetrahedral edges Q_2Q_3, Q_3Q_4, Q_4Q_2 in that plane, along with the singular axis A_1, and in fact at its intersection point P_1 with the double line. The singular axis A_1 is then a common tangent to that base and the contact curve at the singular point P_1 at which both double lines intersect. Whereas the contact curve goes through the three vertices of the triangle $Q_2Q_3Q_4$, the base of the contact cone will contact its three sides. In the name of reciprocity, we obtain two cones – viz., the contact cone at the double point Q_1 and a cone with the same vertex that envelops the contact curve in the opposite face of the tetrahedron. Both cones have the singular ray S_1 for their common line and contact it in the singular plane E_1 that goes through the double line. Whereas the contact cone contact the faces of the tetrahedron that intersect at Q_1, the cone that envelops the contact curve contain the three edges of the tetrahedron that meet that point.

We can repeat the same constructions three more times, and then obtain all of the singularities of the complex surface.

We can thus determine the complex surface itself in two ways: In one case, by its meridian curves, and in the other, by its enveloping cones whose vertices lie upon the double line. In regard to the former manner of determination, to which we will restrict ourselves here, we draw any meridian plane through the double line that cuts the contact curves in the four double planes in any four points, in addition to the four singular points on the double line. The curve along which the surface will be cut by the meridian plane will go through these four points and contact the double line, moreover. There are two such meridian curves, and consequently, there are also two complex surfaces of the special kind that have all of their singularities in common – viz., the double line, the four singular points on them, the four singular planes that go through them, and finally, the four double points with their contact cones, as well as the four double planes with their contact curves (*).

231. The discussion of the singularities of the general complex surfaces carries over immediately to the special case of *equatorial surfaces* when we let the double line of the surface go to infinity. The planes of all complex curves (latitude planes of the surface) are mutually parallel, and their centers lie on the diameter of the surface, which enters here in place of the polar. The circumscribing complex cones will be complex cylinders

(*) If just the singularities of a complex surface are given then it will remain to be decided which of the two straight lines that cut all four singular rays and all four singular axes will be the double line of the surface and which one will be its polar. By this indeterminacy, the same singularities will correspond to two different complex surfaces that belong to two different second-degree complexes. In the general case, the determination of the double line and polar of the surface will depend upon the solution of a quadratic equation. In the special case where the double line and polar of the surface coincide in a line of the complex and cannot be separated from each other, the construction of the surface from it singularities will necessarily be based upon the solution of a quadratic equation, while in the general case the construction will depend upon a linear one, as long as we assume that of the two straight lines whose determination depends upon a quadratic equation, one of them is the double line, and therefore the other one will be the polar of the surface (224).

whose axes lie in latitude planes. There are four latitude planes – viz., the four singular planes of the surface – in which the complex curve, when considered as a curve of class two, degenerates into two coincident straight lines at two points, when considered as a curve of order two. The four lines that connect the four pairs of points are the four singular rays that lie in the surface entirely. The four singular planes contact the surface along the entire extent of their four singular rays. The diameter of the surface cuts these rays at the midpoint of the two double points that lie in them. Four of the circumscribed complex cylinders degenerate into systems of two planes, which are double planes of the surface. The lines of intersection of the four pairs of planes that meet the axes of the cylinder are the four singular axes of the surface; they lie in latitude planes and cut the diameter of the surface. The intersection curve of a complex cylinder that circumscribes the surface with a given plane can be considered to be the projection of the surface onto that plane along the direction of the cylinder axis. If we let the axes of the projecting cylinders rotate around the double line in the latitude planes then they will coincide with the singular axes of the surface in four special positions. The projections then go through two intersecting straight lines, namely, the intersections of the image plane with the two respective double planes. This corresponds to a transition from a hyperbola to a hyperbola whose real imaginary axes, which go through zero, have been switched ([**]). The contact curve in the two double planes that intersect along the same singular axis have that axis for their common asymptote, and are thus determined by the fact that they contain eight double points – four on one plane and four on the other – moreover. The contact cones at each of the two double points that lie upon the same singular ray will be contacted by that ray in the singular plane that goes through it, and will thus be determined by the fact that they contact eight double planes, four on one and four on the other.

232. Finally, if we specialize further and consider the case in which a line of the complex that lies at infinity is taken to be the double line of the complex surface then of the eight double points on the double line, in this case, as well, four double planes will be at infinity and four of them will coincide with the four singular planes. One of the four singular rays will lie in each of the latter planes, and parallel it, one of the four singular axes. The complex curves in all latitude planes will be parabolas, since they contact the double line at infinity. When their planes move parallel to themselves, the parabolas will change in a singular plane under the transition, in which they degenerate into two points, one of which is at infinity, in the sense of its extent. The circumscribing complex cylinders will have the double line at infinity for their common line and will contact it

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([**]) We may not draw the conclusions from this that the complex cylinders are all hyperbolic and the projections are all hyperbolas in the case of eight real double points and eight real double planes for the surface (this assumption is adapted to our terminology). Two parabolic cylinders can also be given (no. **182**), and that would then refer to the transition from hyperbolic to elliptic complex cylinders. Two projections would then be parabolas (corresponding to projection directions that both lie between the directions of two successive singular axes), with which hyperbolas would go to ellipses, and these into hyperbolas.

Under the assumptions that were made, the meridian curves between any two successive singular planes are either all ellipses or all hyperbolas. Ellipses and hyperbolas go to each other under the transitions between each of the four singular planes.

along a latitude plane. It is the hyperbolic cylinders that will have latitude planes for one of their asymptotic planes. The hyperbolas along which they will be cut by an arbitrary plane are to be regarded as the projections of the surface itself. If we give the axis of the projecting complex cylinder all possible directions then one of the two asymptotes of the hyperbola will move parallel to itself. In particular, if we project along the direction of a singular axis (which is parallel to a singular ray) then the hyperbola will degenerate into a system of two straight lines that are the intersections of the image plane with the singular plane and the double plane, which goes through the instantaneous singular axis (*).

The four double points that do not lie at infinity are the vertices of a tetrahedron whose faces are the double planes that do not go through the double line at infinity. A face of the tetrahedron and the singular latitude plane, which goes through the opposite vertex, intersect along a singular axis, and the respective singular ray goes parallel to it in the singular plane through the vertex of the tetrahedron. The contact curve in the double plane has the singular axis for its asymptote and goes through the three double points in that plane. The contact cone at the opposite double point cuts that double plane along a hyperbola that likewise has the singular axis that lies in it for its asymptote and the three edges of the tetrahedron that lie in it for its tangents.

233. The complex surfaces to whose general discussion the present first section is chiefly dedicated define a remarkable family of surfaces of order and class four, which we can also define independently of the consideration of the complex in their own right as those surfaces of that order and class that have eight double points and eight double planes (which are mutually implicit), along with a double line. The discussion of these surfaces thus takes on a surprising simplicity and symmetry due to the fact that we link their existence to the consideration of the complex, irrespective of the infinite variety of their forms and the great number of their constants. On the other hand, these surfaces serve as an invaluable tool for the analytic discussion and geometric visualization of the complex. In the next section, we will go on to the discussion of the complex itself, in order to come back to the discussion of its surface later on.

However, there is an even newer viewpoint from which complex surfaces can be considered that I shall not refrain from mentioning here. The complex surfaces that we consider will be enveloped by lines that belong to a congruence, and indeed to one that consists of the coincident lines of two complexes, one of which is a general one of degree two, and the other of which is a first-degree complex of the special kind, such that all of its lines cut a fixed straight line. Complex curves and complex cones will be enveloped and described, respectively, by successive lines of the congruence that intersect.

In an analogous way, any congruence has a reciprocal relationship with a certain surface. Two successive intersecting straight lines of a congruence determine the intersection point of the two lines and a plane that contains both of them. The point will be a point of the surface and the plane will be a plane of it.

The general expression, viz.:

(*) Whereas two parabolic complex cylinders appear for the equatorial surfaces, which refer to the bounding elliptic complex cylinders when they are real, here, the two parabolic cylinders will coincide, and the elliptic cylinders will not exist. In special cases – such as all equatorial surfaces and especially all parabolic ones – all complex cylinders can be parabolic.

$$2n\,(n-1),$$

that we obtained for the order and class of the surfaces of the complex of an arbitrary degree n (no. **212**) reduces to zero for a first-degree complex. In this case, there are no lines in the surface that is enveloped by the congruence. This surface will be met by two straight lines, and we can represent two such straight lines by a single equation in either point or plane coordinates. The two straight lines will be sufficient for the determination of the congruence, and conversely, when the congruence is given, we will obtain the two straight lines in question from its two directrices.

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Part II.

Discussion of the general equation of a second-degree complex.

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§ 1.

Diameter of a complex. System of three associated diameters. The complex cylinder that is associated with a three-axis system. Central parallelepiped. Center of the complex.

234. For any given plane (t', u', v'), if we consider t', u', v' to be constant and t, u, v to be variable then equation (IV) will immediately give the complex curve that this plane contains when it is represented in space by plane coordinates. If we introduce $\frac{t'}{w'}, \frac{u'}{w'}, \frac{v'}{w'}$, instead of t', u', v', resp., and $\frac{t}{w}, \frac{u}{w}, \frac{v}{w}$, instead of t, u, v, resp., then we can write the condensed equation in the following form:

$$\begin{aligned}
&(Dt'^2 + Eu'^2 + Fv'^2 + 2Ku'v' + 2Lt'v' + 2Mt'u')\, w^2 \\
&- 2\,(Dt'w' + Lv'w' + Mu'w' - O\,u'v' - Rv'^2 - Sr'v' + T\,t'u' + Uu'^2)\, tw \\
&- 2\,(Eu'w' + Kv'w' + M\,t'w' + N\,t'v' + Pu'v' + Qv'^2 - T\,t'u' + Uu'^2)\, uw \\
&- 2\,(Fv'w' + Ku'w' + L\,t'w' - (N - O)\,t'u' - Pu'^2 - Qu'v' + R\,t'v' + St'^2)\, vw \\
&- 2\,(Au'w' - Kw'^2 + G\,t'^2 - H\,t'v' - J\,t'v' - O\,t'w' + P\,u'w' - Qv'w')\, uv \\
&- 2\,(B\,t'v' - Lw'^2 - G\,t'u' + H\,u'^2 - J\,u'v' + N\,u'w' + R\,v'w' - S\,t'w')\, tv \\
&- 2\,(C\,t'u' - Mw'^2 - G\,t'v' - H\,u'v' + J\,v'^2 - (N - O)\,v'w' + T\,t'w' - U\,u'w')\, tu \\
&+ (Dw'^2 + Bv'^2 + Cu'^2 - 2\,G\,u'\,v' - 2\,S\,v'\,w' + 2T\,u'\,w')\, t^2 \\
&+ (Ew'^2 + Av'^2 + Cu'^2 - 2\,H\,t'\,v' - 2\,P\,v'\,w' - 2U\,t'\,w')\, u^2 \\
&+ (Fw'^2 + Av'^2 + Bu'^2 - 2\,J\,t'\,u' - 2\,Q\,u'\,w' + 2R\,t'\,w')\, v^2 = 0. \qquad \text{(X)}
\end{aligned}$$

When we differentiate the equation of the curve with respect to w, we will obtain the following equation for the center of the curve (*):

$$\begin{aligned}
&(Dt'^2 + Eu'^2 + Fv'^2 + 2Ku'v' + 2L\,t'v' + 2Mt'u')\, w \\
&- 2\,(Dt'w' + Lv'w' + Mu'w' - O\,u'v' - Rv'^2 - Sr'v' + T\,t'u' + Uu'^2)\, t \\
&- 2\,(Eu'w' + Kv'w' + M\,t'w' + N\,t'v' + Pu'v' + Qv'^2 - T\,t'u' + Uu'^2)\, u \\
&- 2\,(Fv'w' + Ku'w' + L\,t'w' - (N - O)\,t'u' - Pu'^2 - Qu'v' + R\,t'v' + St'^2)\, v = 0. \qquad (1)
\end{aligned}$$

If we next set:

(*) The complex curve appears as a surface of class two in the manner of representation in the text, and its center will be determined like the center of such a surface. *Geometrie des Raumes.*, pp. 192.

$$Dt'^2 + Eu'^2 + Fv'^2 + 2Ku'v' + 2L\,t'v' + 2Mt'u' \equiv \Xi',$$

for the sake of brevity, then the three coordinates of the center of the curve will be:

$$\left.\begin{aligned}
x &= -\frac{Dt' + Lv' + Mu'}{\Xi'}\cdot w' + \frac{Ou'v' + Rv'^2 + St'v' - Tt'u' - Uu'^2}{\Xi'},\\
y &= -\frac{Eu' + Kv' + Mt'}{\Xi'}\cdot w' + \frac{-Nt'v' - Pu'v' - Qv'^2 + Tt'^2 + Ut'u'}{\Xi'},\\
z &= -\frac{Fv' + Ku' + Lt'}{\Xi'}\cdot w' + \frac{(N-O)t'u' + Pu'^2 + Qu'v' - Rt'v' - St'^2}{\Xi'}.
\end{aligned}\right\} \tag{2}$$

The equation of the plane $\left(\frac{t'}{w'}, \frac{u'}{w'}, \frac{v'}{w'}\right)$ is:

$$t'\,x + u'\,y + v'\,z + w' = 0, \tag{3}$$

and this will be satisfied by the foregoing coordinate values.

235. If we consider t', u', v' to be constant and w' to be variable then the plane (3) will move parallel to itself, while the complex curve in it will change continually. If we let w' vanish, in particular, then we will obtain the following coordinate values for the center of the curve in the respective plane that goes through the origin and has the given direction, and whose equation is:

$$t'\,x + u'\,y + v'\,z = 0,$$

namely:

$$\left.\begin{aligned}
x' &= \frac{Ou'v' + Rv'^2 + St'v' - Tt'u' - Uu'^2}{\Xi'},\\
y' &= \frac{-Nt'v' - Pu'v' - Qv'^2 + Tt'^2 + Ut'u'}{\Xi'},\\
z' &= \frac{(N-O)t'u' + u'^2 + Qu'v' - Rt'v' - St'^2}{\Xi'}.
\end{aligned}\right\} \tag{4}$$

We can thus write the previous general coordinate values (2) in the following way:

$$\left.\begin{aligned}
x - x' &= \frac{Dt' + Lv' + Mu'}{\Xi'},\\
y - y &= \frac{Eu' + Kv' + Mt'}{\Xi'},\\
z - z' &= \frac{Fv' + Ku' + Lt'}{\Xi'}.
\end{aligned}\right\} \tag{5}$$

This will then yield the double equation below:

$$\frac{x-x'}{Dt'+Lv'+Mu'} = \frac{y-y'}{Eu'+Kv'+Mt'} = \frac{z-z'}{Fv'+Ku'+Lt'}, \tag{6}$$

which we can also give the following form:

$$\frac{x-x'}{\dfrac{d\Xi'}{dt}} = \frac{y-y'}{\dfrac{d\Xi'}{du}} = \frac{z-z'}{\dfrac{d\Xi'}{dv}}. \tag{7}$$

If we consider x, y, z to be variable in them then the foregoing double equations will represent a straight line; w' is eliminated from them. The straight line that is represented will then be the geometric locus of the centers of the complex curves in parallel planes that are represented by equation (3) for an arbitrary choice of w'. We call this line a *diameter* of the complex and say that it is *associated* with the system of parallel planes in the complex, and in particular, with each of those planes.

Any system of parallel planes in a second-degree complex is, in general, associated with a single diameter that contains the centers of all curves of class two that lie in the parallel planes.

The complex curves in parallel planes define an *equatorial surface:* The diameter of the surface is a diameter of the complex.

236. If the diameter of the complex that is represented by (6) is to be perpendicular to the plane (3) to which it is conjugate then we will obtain the following condition equations:

$$\left.\begin{aligned} \frac{Dt'+Lv'+Mu'}{Fv'+Ku'+Lt'} &= \frac{t'}{v'}, \\ \frac{Eu'+Kv'+Mt'}{Fv'+Ku'+Lt'} &= \frac{u'}{v'}, \end{aligned}\right\} \tag{8}$$

which we can combine into the double equation:

$$\frac{t'}{\dfrac{d\Xi'}{dt'}} = \frac{u'}{\dfrac{d\Xi'}{du'}} = \frac{v'}{\dfrac{d\Xi'}{dv'}}. \tag{9}$$

The diameter is an *axis* of the complex in this case. The last double equation is identical with the one that is obtained for the determination of the direction of the three principal sections of a surface of class two when one considers $\dfrac{t'}{w'}$, $\dfrac{u'}{w'}$, $\dfrac{v'}{w'}$ to be plane coordinates that are variable and lets k denote an arbitrary constant, which is represented by the equation (*):

$$\Xi' + k\,w'^2 = 0.$$

237. The latter surface depends upon the six complex constants D, E, F, K, L, M, for the time being. Since these constants will remain the same when we change the position of the origin of the coordinates arbitrarily, we can displace the surface parallel to itself without changing its relationship to the complex. Corresponding to the arbitrary choice of k, its dimensions can be changed by any arbitrary ratio. If we give other directions to the coordinates axes then the six complex constants above will assume other values and the same values will correspond to the six constants of the surface when we also refer them to the new coordinate axes.

We would like to call the surface thus defined, whose center and dimensions can be chosen arbitrarily, the *characteristic* of the complex. We would like to once more write the equation of the complex in the following way:

$$\begin{gathered} Ar^2 + Bs + C + D\sigma^2 + E\rho^2 + F\eta^2 \\ + 2Gs + 2Hr + 2Jrs + 2Ks\eta - 2L\,\sigma\eta - 2M\,\rho\sigma - 2N\,r\sigma + 2O\,s\rho \\ + 2Pr\rho + 2Qr\eta + 2Rs\eta - 2Ss\sigma - 2T\sigma + 2U\rho = 0. \end{gathered} \tag{I}$$

When we take the origin to be the center of this surface, and after suppressing the primes, we will obtain the following equation for the characteristic of the complex:

$$Dt^2 + Eu^2 + Fv^2 + 2Kuv + 2Ltv + 2Mtu + w^2 = \Xi + w^2 = 0. \tag{10}$$

We have set k equal to unity in this equation, with no loss of generality.

The characteristic of a complex relieves us of any necessity for analytically discussing the direction of its diameter. A system of parallel planes is associated with a diameter of the characteristic, and that diameter will be parallel to the one that is associated with those planes in the complex. Three associated diameters of the characteristic will be parallel to three diameters of the complex, which we would like to refer to as *three associated diameters of the complex*, in their own right. We can take any given diameter of the complex to be one of three associated diameters, so the other two planes that are associated with the given ones will be parallel. Each of three associated diameters will be associated with those planes that are parallel to both of the other ones each time.

A complex has a single system of three axes that are perpendicular to each other, in general. We would like to refer to the planes that these axes are parallel to when taken pair-wise as the *principal sections* of the complex. The axes will be associated with the principal sections.

For the sake of determining the associated diameter of a complex, we can replace its characteristic with its asymptotic cone, and displace that cone parallel to itself arbitrarily.

(*) See *Geometrie des Raumes*, nos. **103** and **152**.

If we take the origin of the coordinates to be its vertex then that cone will be represented by the following two equations:

$$\Xi = 0, \qquad w = 0$$

in plane coordinates and the single equation:

$$\begin{aligned}(K^2 - EF)\, x^2 + (L^2 - DF)\, y^2 + (M^2 - DE)\, z^2 \\ + (DK - LM)\, yz + 2\,(EL - KM)\, xz + 2\,(FM - KL)\, xy = 0.\end{aligned} \tag{11}$$

in point coordinates.

The associated diameters of a complex have essentially different directions with respect to each other, according to whether the characteristic of the complex is a (one or two-sheeted) hyperboloid with a real asymptotic cone or a (real or imaginary) ellipsoid whose asymptotic cone reduces to an ellipsoidal point. The latter case is indicated by the agreement in sign of the three expressions:

$$K^2 - EF, \qquad L^2 - DF, \qquad M^2 - DE, \tag{12}$$

while this agreement was not present in the former case.

238. If the characteristic is a surface of revolution, in particular, then the complex, like that surface, will have a principal axis and infinitely many axes along with it that are all directed perpendicular to the principal axis, when taken pair-wise, as well as to each other. Under the assumption of rectangular coordinate axes, this special case will be characterized by the fact that:

$$D - \frac{LM}{K} = E - \frac{KM}{L} = F - \frac{KL}{M}, \tag{13}$$

and therefore the following double equation:

$$Kx = Ly = Mz \tag{14}$$

will determine the direction of the principal axis (*).

A more subordinate case is the one in which the characteristic goes to a cone, which corresponds to solving the double condition equations (13) into the following equations:

$$\begin{gathered} K = 0, \;\; L = 0, \;\; M = 0, \\ D = E = F. \end{gathered}$$

(*) *Geometrie des Raumes*, no. **154**.

All of the planes that go through space will then be principal sections of the complex to which the associated diameters will be perpendicular. Any diameter of the complex will be one of its axes.

239. If we take the coordinate axes to which the general equation (I) of the second-degree complex is referred to be parallel to any three associated diameters of the complex then three constants in that general equation will vanish, as well as in the equation of the characteristic. Namely, one will have:

$$K = 0, \quad L = 0, \quad M = 0.$$

This will happen, in particular, when rectangular axes are taken to be parallel to the axes of the complex. This can happen infinitely often when the characteristic has an axis of rotation and the complex has a principal axis. One of the coordinate axes is then taken to be parallel to the principal axis, while any two straight lines that are perpendicular to each other and the principal axis can be taken to be the other two coordinate axes. When *OX*, *OY*, *OZ* are taken to be parallel to the principal axis in succession, the coefficients E and F, D and F, D and E will then become equal to each other, in turn. K, L, M will vanish, and the three coefficients D, E, F will be equal to each other in the equation if a complex that has only rectangular associated diameters and is referred to an arbitrary system of rectangular coordinate axes.

In this paragraph, we would like to restrict ourselves to the general case in which the characteristic is a surface of class two with a center. The cases in which the vanishing of K, L, M has the simultaneous vanishing of one of the three constants D, E, F as a consequence will thus still be excluded from the discussion, for the time being.

240. For those diameters that are associated with planes that are parallel to a given plane:

$$t' x + u' y + v' z = 0,$$

we have obtained the following double equation:

$$\frac{x - x'}{Dt' + Lv' + Mu'} = \frac{y - y'}{Eu' + Kv' + Mt'} = \frac{z - z'}{Fv' + Ku' + Mt'}. \tag{6}$$

Corresponding to the successive assumptions that:

$$\begin{aligned} u' &= 0 \quad \text{and } v' = 0, \\ t' &= 0 \quad \text{“} \quad v' = 0, \\ t' &= 0 \quad \text{“} \quad u' = 0, \end{aligned}$$

from no. **234**, one will have:

$$\Xi' = Dt'^2, \qquad x' = 0, \qquad y' = \frac{T}{D}, \qquad z' = -\frac{S}{D},$$

$$\Xi' = Eu'^2, \qquad x' = -\frac{U}{E}, \qquad y' = 0, \qquad z' = \frac{P}{E}, \tag{15}$$

$$\Xi' = Fv'^2, \qquad x' = \frac{R}{F}, \qquad y' = -\frac{Q}{F}, \qquad z' = 0,$$

respectively. The foregoing double equation then resolves into the following three pairs of equations:

$$\begin{aligned} &Mx - Dy + T = 0, && Lx - Dz = S = 0, \\ &My - Ex - U = 0, && Ky - Ez + P = 0, \\ &Lz - Fx + R = 0, && Kz - Fy - Q = 0, \end{aligned} \tag{16}$$

which represent those diameters of the complex that are associated with planes that are parallel to *YZ*, *XZ*, *XY*, respectively.

If we choose the three coordinate axes to be such that they are parallel to any three associated diameters of the complex, in particular, then the three constants *K*, *L*, *M* will vanish, and we will obtain the following three pairs of equations for the determination of the absolute positions of these three diameters that are parallel to the *OX*, *OY*, *OZ* coordinate axes:

$$\begin{aligned} &y = +\frac{T}{D}, && z = -\frac{S}{D}, \\ &x = -\frac{U}{E}, && z = +\frac{P}{E}, \\ &x = +\frac{R}{F}, && y = -\frac{Q}{F}. \end{aligned} \tag{17}$$

The associated diameters, when taken pair-wise, will thus not intersect, in general. However, like any three straight lines that do not intersect at all, they will determine a parallelepiped, which we will consider more closely here, since it is indicative of the complex, and we would like to call it a *central parallelepiped of the complex*.

The foregoing six equations (17), when taken individually, represent the six face planes of a central parallelepiped. Any of two opposite face planes will go

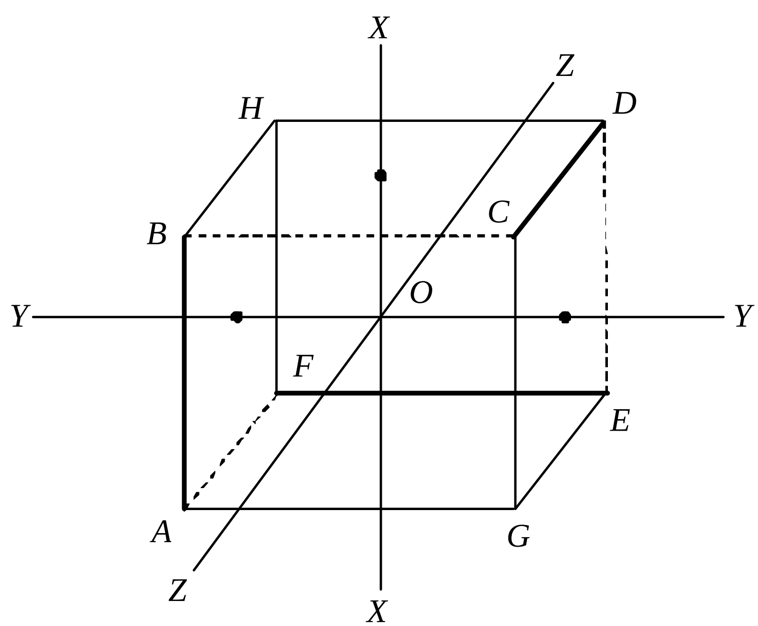

Figure 12.

through one of two of the three associated diameters and will be parallel to the other of the two. Three non-intersecting edges of the parallelepiped will be the three associated diameters, which we would like to take to be *AB*, *CD*, *EF* in the Figure 12. We can also arrange the same six equations (17) that represent the three associated diameters of the complex, when taken pair-wise, in the following way:

$$\begin{aligned} y &= -\frac{Q}{F}, & z &= +\frac{P}{E}, \\ x &= +\frac{R}{F}, & z &= -\frac{S}{D}, \\ x &= -\frac{U}{E}, & y &= +\frac{T}{D}. \end{aligned} \tag{18}$$

These three pairs of equations will then represent the equations of those three edges of the parallelepiped that are opposite to the three associated diameters. Those three edges *DE*, *FA*, *BC*, which also do not intersect, in their own right, will define a spatial hexangle *ABCDEF* with three edges that fall upon associated diameters. The vertices of the hexangle will be six of the eight vertices of the parallelepiped. Three diagonals of the parallelepiped will be the three diagonals of the hexangle, and the two points *G*, *H* that are linked with the fourth diagonal will have the coordinates:

$$\begin{aligned} x &= +\frac{R}{F}, & y &= +\frac{T}{D}, & z &= +\frac{P}{E}, \\ x &= -\frac{U}{E}, & y &= -\frac{Q}{F}, & z &= -\frac{S}{D}. \end{aligned} \tag{19}$$

One gets:

$$\frac{ER+FU}{EF}, \qquad \frac{DQ+FT}{DF}, \qquad \frac{DP+ES}{DE} \tag{20}$$

for the lengths of the edges that are parallel to the three coordinate axes *OX*, *OY*, *OZ*, respectively, and:

$$x^0 = \frac{ER-FU}{EF}, \qquad y^0 = \frac{DQ-FT}{DF}, \qquad z^0 = \frac{DP-ES}{DE} \tag{21}$$

for the center of the parallelepiped whose coordinates we would like to denote by x^0, y^0, z^0, to distinguish them.

241. The edges of the central parallelepiped that are represented by the pairs of equations (18) have a simple geometric relationship with the complex that we will obtain immediately when we revert to the equations of the three complex cylinders whose faces are parallel to the coordinate axes. The equations of this cylinder will be the following ones (Chapter I, § 5, eq. 32):

$$\left.\begin{aligned} Fy^2 + Ez^2 + 2Qy - 2Pz &= 0, \\ Fx^2 + Dz^2 - 2Rx + 2Sz &= 0, \\ Ex^2 + Dy^2 + 2Ux - 2Ty &= 0, \end{aligned}\right\} \tag{22}$$

when we take the coordinate axes OX, OY, OZ to be parallel to any three associated diameters of the complex, as in the previous number, and then set K, L, M equal to zero. The three axes of this cylinder will be represented by the three pairs of equations (18). Whereas three edges of the central parallelepiped AB, CD, EF will fall upon three associated diameters of the complex, the three opposite edges to them DE, FA, BC will fall upon *the axes of the three cylinders whose sides are parallel to the three associated diameters.*

242. If a second-degree complex is given, and we choose the direction of a plane arbitrarily then any line direction that is parallel to that plane direction will be associated with a second such line direction. Any given plane direction (any family of parallel planes) is associated with a single line direction, and conversely, each given line direction is associated with a single plane direction. Any given line direction is associated with infinitely many pairs of line directions, which will be parallel to the plane direction that is associated with the given line direction. There are then infinitely many systems of three associated line directions, in such a way that, on the one hand, every given line direction corresponds to infinitely many pairs of associated line directions that are parallel to the associate plane direction, and on the other hand, the plane direction that is parallel to any two of three associated line directions will be associated with the third of these directions. There are infinitely many systems of three associated plane directions: They are parallel to two of three associated line directions.

On the one hand, there are three associated diameters of a complex that have the direction of three associated line direction, and on the other hand, there are three axes of complex cylinders that have the same direction, and which we can refer to as *three conjugate cylinder axes*, in their own right. The three associated diameters and the three associated cylinder axes will define a spatial hexangle whose opposite sides are parallel. Its sides are, alternately, diameters and cylinder axes. Any diameter will be cut by two cylinder axes that are parallel to the plane direction that is associated with the direction of the diameter. Any cylinder axis will be cut by two diameters that are parallel to the plane direction that is associated with the direction of the cylinder axis.

A given plane is parallel to infinitely many diameters of the complex and the axes of infinitely many complex cylinders. On the one hand, any diameter will define a ruled surface, and on the other hand, so will any cylinder axis. The given plane is a associated with a diameter of the complex, just as it is associated with the axis of a complex cylinder. Any diameter is parallel to that cylinder axis. The axes of all complex cylinders that are parallel to the given plane cut the associated diameter, and all diameters of the complex that are parallel to the plane cut the associated cylinder axis.

243. It seems advisable to state and complete the foregoing geometric considerations with some analytical refinements.

The totality of all curves that lie in planes that are parallel to the *YZ* plane, and thus define an equatorial surface, is represented (Chap. I, § 2, no. **163**) by the following equation:

$$\begin{aligned} &Dw^2 + 2(Lx - S)\, vw + (Fx^2 - 2Rx + B)\, v^2 \\ &+ 2(Mx + T)\, uw + 2\,(Kx^2 - Ox - G)\, uv + (Ex^2 + 2Mx + C)\, u^2 = 0. \end{aligned} \tag{23}$$

The plane of the curve is determined by x, and the curve in that plane will then be determined by the line coordinates u / w and v / w. Should the axis *OX* have the direction that is associated with the *YZ* plane then L and M would have to vanish. Should it coincide with the diameter of the complex that is associated with that plane then the centers of all curves would have to lie upon it. This would demand that, along with:

$$L = 0, \quad M = 0,$$

one would also need to have:

$$S = 0, \quad T = 0.$$

The foregoing equation will then simplify into the following form:

$$Dw^2 + (Fx^2 - 2Rx + B)\, v^2 + 2\,(Kx^2 - X - G)\, uv + (Ex^2 + 2Ux + C)\, u^2 = 0. \tag{24}$$

The same equatorial surface that is represented by the foregoing equation by means of its breadth curves (*Breitencurven*) will be represented [Chap. I, § 5, eq. (30)], when one considers that L, M, S, and T vanish, by the following equation:

$$(FV^2 + 2Kuv + Eu^2)\, x^2 + Dv^2\, z^2 + 2\,(Rv^2 + Ouv - Uu^2)\, x + (Bv^2 - 2Guv + Cu^2) = 0 \tag{25}$$

by means of its circumscribing complex cylinder whose axes are parallel to the *YZ* coordinate plane. Once we have determined one of these circumscribing complex cylinders by an arbitrary choice of v / u for the axis direction, the last equation will represent the second-order curve in *XZ* along which the relevant cylinder cuts that coordinate plane. The axis of the cylinder that is parallel to *YZ* goes through the center of that curve of intersection that lies in the *OX* coordinate axis and is determined by the coordinate value:

$$x = \frac{Rv^2 + Ouv - Uu^2}{Fv^2 + 2Kuv + Eu^2} \tag{26}$$

on that axis. If we refer the coordinates y and z on any point of any cylinder axis that is parallel to *YZ* then we will have:

$$\frac{v}{u} = -\,\frac{y}{z},$$

and we will obtain:

$$x = \frac{Ry^2 - Oyz - Uz^2}{Fy^2 - 2Kyz + Ez^2}, \tag{27}$$

as the equation of the *geometric locus of the axes of complex cylinders that are parallel to the YZ plane.*

The last equation expresses the idea that a single cylinder axis will lie in any plane that is laid through a given diameter and is parallel to the plane direction that is associated with the diameter, while two cylinder axes that intersect along the diameter will lie in any plane that has that direction.

244. There is another way to determine the two cylinder axes that are contained in a given plane, which we have taken to be parallel to the *YZ* coordinate plane, here. Namely, if we differentiate equation (24) with respect to x then that will give:

$$(Fx - R)\, v^2 + (2Kx - O)\, uv + (Ex + U)\, u^2 = 0.$$

This equation immediately gives the value of x that was just found in terms of v and u (26). The direction of the two cylinder axes in the *YZ* itself is given by the roots of the following equation:

$$Rv^2 + Ouv - Uu^2 = 0.$$

A complex cylinder whose axis lies in a given plane has two tangents to the complex curve of class two that lies in that plane that are parallel to two of its sides. The axis of the cylinder will then go through the center of the complex curve. If we project the complex curve in the parallel plane that is close to the given plane onto that given plane along the direction that is conjugate to these planes then that projection will also be contacted by the two cylinder sides. In other words, the two parallel planes among them that contact the cylinder along these sides will simultaneously contact the equatorial surface that has *OX* for its diameter. One is then dealing with the determination of those points of the complex curve in the given plane at which the equatorial surface is contacted by planes that are parallel to the diameter of that surface. The cylinder that circumscribes the *equatorial surface* whose sides are parallel to its diameters contacts the surface along a spatial curve that is cut by a plane in four points. In particular, it will be cut by the given plane, which is the breadth plane of the surface, in four points that are the end points of two diameters of the complex curve in the given plane. The two diameters of the complex curve that are associated with these diameters will be the two axes of the cylinder to be constructed that lie in the given plane.

245. We would now like to displace the *OX* axis, which, from our assumption up to now, coincides with a diameter of the complex in such a way that it coincides with the axis of complex cylinder that is parallel to that diameter. The equation of the cylinder whose axis is parallel to the *OX* coordinate axis has the equation (no. **249**):

$$Fy^2 - 2Kyz + Ez^2 + 2Qy - 2Pz + A = 0.$$

We get the following two conditions for the axis of the cylinder to coincide with *OX*:

$$P = 0, \qquad Q = 0.$$

The general equation of the complex curves in plane coordinates (X), which we have placed at the apex of the developments in this paragraph, will then represent, in particular, the complex curve that is contained in an arbitrary plane:

$$u' y + v' z = 0$$

that is laid through the cylinder axis when we set t' and w' equal to zero in it. If we consider that P and Q vanish then we will obtain the following equation for that curve:

$$\begin{aligned}(Eu'^2 + 2Ku'v' + Fv'^2)\, w^2 - 2\,(Uu'^2 - Ou'v' - Rv'^2)\, tw \\ - 2(Hu'^2 - Ju'v')\, tv + 2\,(Hu'v' - Jv'^2)\, tu \\ + (Cu'^2 - 2Gu'v' + Bv'^2)\, t^2 \\ + A\,(u'v - v'u)^2 = 0.\end{aligned} \tag{28}$$

The center of this curve lies in the OX coordinate axis, and is determined on that axis by the coordinate value:

$$x = \frac{Rv'^2 + Ou'v' - Uu'^2}{Fv'^2 + 2Ku'v' + Eu'^2}. \tag{29}$$

When we consider v' / u' to be variable in it, the foregoing equation (28) will represent a meridian surface that has the axis of a complex cylinder for its double line. It is characterized by the fact that the centers of all of its meridian curves lie on the double line.

246. After exchanging v' / u' and v / u, the two equations (27) and (29) will be identical. If we then let v' / u' determine the direction of a cylinder axis that is parallel to the YZ plane and thus cuts the diameter of the complex that is parallel to OX then it will lie in a plane that cuts the cylinder axis OX at the point that is determined by (29). The straight line that lies in that plane goes through that point, and whose direction is conjugate to the direction of the plane of the complex curve that is determined by v' / u', and thus also to the direction of the cylinder axis that is determined by v / u, will be the desired diameter of the complex.

In order to then construct the diameter of the complex in question that goes through the center of the curve (28), we appeal to the characteristic of the surface. For the sake of simplicity, we would like to let the previously-undetermined direction of the two coordinate axes OY and OZ coincide with any two associated diameters of the intersection curve of the characteristic with the YZ coordinate plane. K will then vanish from the equation of the complex, and the equation of that curve of intersection will be:

$$Fv^2 + Eu^2 + kw^2 = 0.$$

We obtain the equation:

$$\frac{v'}{u'} \cdot \frac{v}{u} + \frac{E}{F} = 0 \tag{30}$$

for the determination of the direction that is associated with the direction v' / u', which we would like to denote by v / u. If we introduce v/ u into (29), in place of v' / u', by means of this equation then that will give:

$$x = -\frac{F^2Uv^2 + EFOuv - E^2Ru^2}{EF(Fv^2 + Eu^2)}. \tag{31}$$

Finally, if we refer y and z to any point of the diameter of the complex that is parallel to YZ then we will get:

$$\frac{v}{u} = -\frac{y}{z},$$

and thus:

$$x = \frac{-F^2Uy^2 + EFOyz + E^2Rz^2}{EF(Fy^2 + Ez^2)}. \tag{32}$$

When we consider x, y, z to be variable, this equation will represent *the geometric locus of the diameters of the given complex that are parallel to the YZ plane*. It says that a single diameter of the complex lies in any plane that is laid through the axis of a given complex cylinder, and it is parallel to the plane direction that is associated with the cylinder axis, while two diameters will lie in any plane with that direction that will intersect on the axis of the given cylinder.

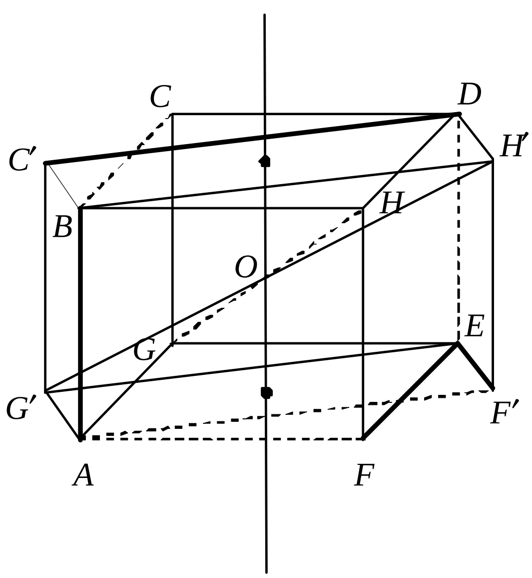

Figure 13.

An arbitrary plane $AFF'EGG'A$ cuts the diameter AB of the complex and the axis DE of the complex cylinder whose direction is associated with it in two points A and E. Two cylinder axes AF and AF' lie in this plane that cut the diameter AB at A and two complex diameters EF and EF' that cut the cylinder axis DE at E. The directions of the two diameters in this plane are conjugate to the directions of the two cylinder axes in it, respectively. They simultaneously belong to two central parallelepipeds that have two opposite edges in common that fall upon the diameter AB and the cylinder axis DE that is parallel to it. The opposite face-planes of the parallelepiped fall in the same plane $BC'CDH'HB$. The two diameters of the complex, CD and CD', that lie in these two planes are the opposite edges of the two parallelepipeds to the two cylinder axes in the first plane, as well as the two cylinder axes in the second, while BC and BC' are those of the two diameters in the first plane. The common center of the two central parallelepipeds lies in a plane that is parallel with the two opposite face-planes (which we would like to take to be parallel to the *YZ* coordinate plane, as before) and goes halfway between the two planes. It will then bisect the distance between a diameter and a cylinder axis of the complex that are parallel to each other and to *YZ*. Due to the fact that we have chosen the common direction of both of them to be parallel to *YZ* from the outset, the two opposite face-planes of a parallelepiped will be determined in a linear way.

If we refer equation (27), like equation (32), to the *OY* and *OZ* coordinate axes, which are parallel to two associated diameters of the complex, or – what amounts to the same thing – two associated cylinder axes of it, then *K* will also vanish on it, and we will get:

$$x = \frac{Ry^2 - Oyz - Uz^2}{Fy^2 + Ez^2}. \tag{33}$$

Under the assumption that y / z was assigned the same value in the foregoing equation (33) and equation (32), in both equations, x will mean the distance between a cylinder axis of the complex and one of its diameters, whose direction is the same and given by y / z, from the *YZ* coordinate plane. One-half the sum of that distance, which we would like to denote by x^0, will then give the distance from the middle plane of the relevant central parallelepiped to the same coordinate plane. If we add the equations in question, (32) and (33), then we will get:

$$x^0 = \tfrac{1}{2}\left\{\frac{R}{F} - \frac{U}{E}\right\}, \tag{34}$$

in agreement with (21). The value of x^0 is independent of the arbitrarily-chosen value of y / z. Moreover, the midpoints of all central parallelepipeds whose opposite edges fall on the diameter that is associated with *YZ* and the cylinder axis that is associated with that plane will lie on the midline between that cylinder axis and that diameter. We draw the conclusion from this that all central parallelepipeds with one edge that falls on a given diameter of the complex, so the opposite edge then falls on the cylinder axis that is parallel to the diameter, *will have a common center.*

The foregoing theorem immediately gives us a whole new series of central parallelepipeds that have the same point for center amongst themselves and with the parallelepipeds of the first series. To that end, we merely need to replace the given diameter with any new one that is associated with it and then proceed in such a way that each time new ones are replaced with ones that are associated with them. A given diameter of the characteristic of the complex is, however, associated with any diameter that lies in the given associated diametral plane. Two given diameters will then have *two* associated diameters along which the two diametral planes that are associated with the two given planes will intersect. We can then also go from a given diameter of a complex to any two given diameters of the same kind in such a way that we replace the first given diameter with a third diameter that is associated with them, and then replace that third one with the second given one that is associated with it, in its own right. We will then arrive at the following theorem:

All central parallelepipeds of a given complex have the same point for their centers.

We would like to call the common center of all central parallelepipeds the *center of the complex*, any plane that goes through it a *central plane*, and any straight line that goes through a *central line*.

A second-degree complex has one center, in general.

A plane that goes parallel to any two associated diameters or to any two associated cylinder axes of a complex and lies halfway between them is a central plane of the complex.

Any diameter of a complex is the axis of a cylinder that is parallel to it; the middle line between them is a central line of the complex.

247. If we take *YZ* to be a central plane of the complex and take the *OX* axis to be, first, its associated diameter and then, the cylinder axis that it is associated with it then the two ruled surfaces, one of which goes through all of the diameters that are associated with cylinder axes that are parallel to *YZ*, while the other one contains all of the diameters of the complex that are associated with the cylinder axes and parallel to *YZ*, will be represented by the following two equations:

$$x = \frac{Ry^2 - Oyz - Uz^2}{Fy^2 + Ez^2},$$

$$x = -\frac{Ry^2 - Oyz - Uz^2}{Fy^2 + Ez^2}.$$

If we displace the two ruled surfaces – and with them, at the same time, the relevant *OX* coordinate axis – parallel to themselves and to the central plane then their equations will not change. If, after the displacement, the conjugate diameter coincides with the conjugate cylinder axes then the foregoing equations will represent the two surfaces when they are referred to the same coordinate system. The geometric relationship between the two surfaces will then be the same as the one that we just described.

In this, we can always assume that the coordinate axes *OY* and *OZ* in *YZ*, which are parallel to any two associated diameters of the complex, are perpendicular to each other. In particular, if we take the given central plane to be one of the three principal sections of the complex that goes through its center then *OX* will also be perpendicular to *OY* and *OZ*. If we consider the central plane to be a reflecting plane then one of the two ruled surfaces will be the mirror image of the other one after a suitable reciprocal displacement of it.

238. When we take the center of the complex to be the origin of the coordinates and lay the three coordinate axes through it and parallel with any three associated diameters and cylinder axes then when we set K, L, M equal to zero, the equation of the complex will become:

$$\begin{gathered} Ar^2 + Bs^2 + C + D\sigma^2 + E\rho^2 + F\eta^2 \\ + 2Gs + 2Hr + 2Jrs \\ - 2Nr\sigma + 2Os\rho \\ + 2Pr\rho + 2Qr\eta + 2Rs\eta - 2Ss\sigma - 2T\sigma + 2U\rho = 0, \end{gathered} \tag{35}$$

by which, the following three condition equations (no. 240) will be fulfilled:

$$\frac{R}{F} = \frac{U}{E}, \qquad \frac{Q}{F} = \frac{T}{D}, \qquad \frac{P}{E} = \frac{S}{D}, \tag{36}$$

from which, the following one can be derived:

$$PRT = QSU. \tag{36a}$$

The three pairs of coordinates:

$$\left.\begin{array}{ll} y = \dfrac{T}{D} = \dfrac{Q}{F}, & z = -\dfrac{S}{D} = -\dfrac{P}{E}, \\ x = -\dfrac{U}{E} = -\dfrac{R}{F}, & z = \dfrac{P}{E} = \dfrac{S}{D}, \\ x = \dfrac{R}{F} = \dfrac{U}{E}, & y = -\dfrac{Q}{F} = -\dfrac{T}{D} \end{array}\right\} \tag{37}$$

will determine the position of the three associated diameters, and the same three coordinate pairs with the opposite signs will determine the position of the three associated cylinder axes.

The coordinate axes will be rectangular when we take them to be parallel to the three axes of the complex. The central parallelepiped that is determined by it will also be rectangular. The square of the length of one-half of its four diagonals will be:

$$\left(\frac{R}{F}\right)^2 + \left(\frac{P}{E}\right)^2 + \left(\frac{T}{D}\right)^2 = \left(\frac{Q}{F}\right)^2 + \left(\frac{U}{E}\right)^2 + \left(\frac{S}{D}\right)^2. \tag{38}$$

One of these four diagonals is distinguished by the fact that it cuts none of the three axes of the complex and none of the three cylinder axes that are parallel to then. If we denote the angles that they define with the three coordinate axes OX, OY, OZ by α, β, γ, respectively, then:

$$\cos\alpha : \cos\beta : \cos\gamma = \frac{U}{E} : \frac{Q}{F} : \frac{S}{D} = \frac{R}{F} : \frac{T}{D} : \frac{P}{E}. \tag{39}$$

The eighth part of the volume of the central parallelepiped is:

$$\frac{PRT}{DEF} \equiv \frac{QSU}{DEF}. \tag{40}$$

249. Once we have accounted for the six constants of the position, the number of constants of the complex will still amount to just *thirteen*, which will be recovered in equation (35) when we consider the condition equations (36). The single condition that must be satisfied if we would like to give the equation of the complex the foregoing form will consist of demanding that none of the three constants D, E, F vanish at the same time as K, L, M. Under the assumption of *rectangular coordinate axes*, we can then represent the complex by equation (35) in a single way, in general.

We will treat the special cases in which one or more of the three constants D, E, F vanishes at the same time as K, L, M later (§ 3).

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§ 2.

Specialization of the complexes that have a center. Complexes whose lines envelop a second-degree surface.

250. There are twenty constants in the general complex equation (I):

$$\begin{gathered} Ar^2 + Bs^2 + C + D\sigma^2 + E\rho^2 + F\eta^2 \\ + 2Gs + 2Hr + 2Jrs + 2K\rho\eta - 2L\sigma\eta - 2M\rho\sigma \\ - 2Nr\sigma + 2Os\rho \\ + 2Pr\rho + 2Qr\eta + 2Rs\eta - 2Ss\sigma - 2T\sigma + 2U\rho = 0, \end{gathered}$$

and when we divide by any of the remaining ones that will give *nineteen* constants that are necessary for the determination of the complex and its position; one can arrange them into the following six groups:

$$\begin{gathered} A, B, C \quad \text{and} \quad G, H, J, \\ D, E, F \quad \text{“} \quad K, L, M, \\ N, O, \\ P, Q, R, S, T, U. \end{gathered}$$

The six constants of the last group can be arranged in various ways, in their own right; e.g., two sets of three pairs:

P and Q,	R and S,	T and U,
P “ U,	R “ Q,	T “ S,

and one set of two groups of three:

$$P, R, T \qquad \text{and} \qquad Q, S, U.$$

251. In the first paragraph, we verified that when the three constants K, L, M vanish the three coordinate axes will be parallel to three associated diameters of the complex. We can then let three more constants drop out of the equation of the complex, moreover, by a suitable placement of the origin. If x_0, y_0, z_0 are the coordinates of the new origin then the six constants of the last group will take on the following new values, which we would like to distinguish by P_0, Q_0, R_0, S_0, T_0, U_0 (no. **157**):

$$\begin{aligned} P_0 &= P + E\,z_0\,, & Q_0 &= Q - F\,y_0\,, \\ R_0 &= R + F\,x_0, & S_0 &= S - D\,z_0\,, \\ T_0 &= T + D\,y_0\,, & U_0 &= U - E\,x_0\,. \end{aligned} \tag{41}$$

If we take one of the eight vertices of the relevant central parallelepiped to be the origin then three of the new constants will vanish. According to whether this vertex (Fig. 12) is one of the six at which a diameter and a cylinder axis intersect, or one of the two remaining vertices through which will go either one of the three conjugate diameters or one of the three conjugate cylinder axes, one will have the vanishing of:

$$\begin{array}{lll} S_0, T_0, U_0, & R_0, S_0, T_0, & Q_0, R_0, S_0 \\ P_0, Q_0, R_0, & U_0, P_0, Q_0, & T_0, U_0, P_0 \end{array}$$

and

$$S_0, Q_0, U_0, \qquad P_0, R_0, T_0,$$

respectively.

The six new constants can vanish simultaneously only when the following three relations exist between the original ones:

$$\frac{R}{F}+\frac{U}{E} = 0, \qquad \frac{T}{D}+\frac{Q}{F} = 0, \qquad \frac{P}{E}+\frac{S}{D} = 0. \tag{42}$$

The result of the vanishing of the new constants is that *the three new coordinate axes coincide with three associated diameters of the complex.* The new origin will be the *center of the complex.* The complex curves in the three coordinate planes will also have that point for their common center, and at the same time, the three coordinate axes for the axes of three complex cylinders. Since the coordinate system still depends upon three arbitrary constants, there will generally be a system of three associated diameters in any complex that will intersect at its center. If we refer the complex to the three intersecting diameters as coordinate axes then its equation will become:

$$\begin{aligned} Ar^2 + Bs^2 + C + D\sigma^2 + E\,\rho^2 + F\,\eta^2 & \\ + 2Gs + 2Hr + 2Jrs & \\ - 2Nr\sigma + 2Os\rho = 0. & \end{aligned} \tag{43}$$

This equation contains ten mutually-independent constants, since the coordinate system is specified by nine conditions.

The coordinates of the centers of the complex in an arbitrary plane that goes through the center of the complex are:

$$\left.\begin{aligned} x &= \frac{Ou'v'}{Dt'^2 + Eu'^2 + Fv'^2}, \\ y &= \frac{-Nt'v'}{Dt'^2 + Eu'^2 + Fv'^2}, \\ z &= \frac{(N-O)t'u'}{Dt'^2 + Eu'^2 + Fv'^2}. \end{aligned}\right\} \tag{44}$$

Since the values of the three coordinates x, y, z vanish simultaneously only when two of the three coordinates of the plane t', v', r' vanish simultaneously, there will generally be no other diameters of the complex that go through its center besides the three associated diameters, which were taken to be coordinate axes.

When we eliminate t', u', v' from them, the three foregoing equations will give:

$$DO^2 y^2 z^2 + EN^2 x^2 z^2 + F(N-O)^2 x^2 y^2 - NO\,(N-O)\,xyz = 0. \tag{45}$$

This equation represents the geometric locus of the centers of the complex curve in the planes that go through the three associated diameters and are rotated arbitrarily around that point (*).

252. A specialization of the complex will come about when we let one of the three constants:

$$N, \quad O, \quad N-O$$

vanish, along with the six constants of the last group. If O is the vanishing constant then the three equations (44) will give:

$$x = 0, \qquad u'y + v'z = 0.$$

In any plane:

$$t'x + u'y + v'z = 0$$

that goes through the origin, the center of the complex curve will lie upon the straight line along which the YZ coordinate plane intersects that plane, and will advance upon that line when that plane is rotated around that line. When that plane goes through the OX coordinate axis, in particular, t' will vanish, and as a result of that, y and z will be equal to zero at the same time as x: The center of the curve will then coincide with the origin, or in other words, *all of the diameters that are associated with the OX coordinate axis will go through the origin and lie in the YZ plane.* Any line in this plane that goes through the origin will be a diameter of the complex, just as it will be the axis of a complex cylinder.

(*) The surface that is represented by equation (45) is a complex surface that has been specified in such a way that it will possess three double lines that intersect at a point: viz., the three coordinate axes OX, OY, OZ. Corresponding to that, they can be generated in three ways by rotating a variable conic section around a fixed axis.

If the two constants N and O of the foregoing group vanish at the same time as the six constants of the latter group then the values of x, y, z will vanish in equation (44). *All diameters of the complex will then go through its center.* They will likewise be the axes of the complex cylinder. Any complex curve whose plane goes through the center of the complex will also have that point for its center.

In this case, the general equation (I) will become:

$$\begin{aligned} &Ar^2 + Bs^2 + C + D\sigma^2 + E\rho^2 + F\eta^2 \\ &\quad + 2Gs + 2Hr + 2Jrs = 0. \end{aligned} \tag{45}$$

It represents a complex whose diameters all intersect at its center, so five of its constants have disappeared, and along with the six constants of position, it will depend upon *eight* constants that are again found in its equation. It will be referred to any three associated diameters as coordinate axes, which we can, despite the generality, also choose to be its three axes.

253. When all diameters of the complex intersect at its center and any three of these diameters that are associated with each other are taken to be coordinate axes, equations (3), (30), (12), (21) of the previous section will go to the following ones:

$$Dw^2 + (Fx^2 + B)\, v^2 - 2Guv + (Ex^2 + C)\, u^2 = 0, \tag{46}$$

$$\left(E\frac{u^2}{v^2} + F\right)x^2 + Dz^2 + \left(C\frac{u^2}{v^2} - 2G\frac{u}{v} + B\right) = 0, \tag{47}$$

$$\left(F\frac{y^2}{z^2} + E\right)w^2 + \left(B\frac{y^2}{z^2} + 2G\frac{y}{z} + C\right)t^2 - 2\left(J\frac{y}{z} + H\right)tv + Av^2 = 0, \tag{48}$$

$$\left(F\frac{y^2}{z^2} + E\right)w^2 + \left(B\frac{t^2}{w^2} + F\right)y^2 + 2\left(C\frac{t^2}{w^2} + E\right)z^2 + 2J\frac{t}{w}y + 2H\frac{t}{w}z + A = 0. \tag{49}$$

The first two of the foregoing equations (46) and (47) will represent the equatorial surface that has OX for its diameter in mixed coordinates, in one case, by its breadth curve, whose instantaneous plane is determined by x, and in the other case, by means of its enveloping complex cylinder whose axes define an angle with XZ whose trigonometric tangent is equal to $(-\,u\,/\,v)$. It will follow from equation (47) that the axes of all enveloping complex cylinders will lie in YZ and intersect the OY coordinate axis at the origin.

The last two of the foregoing equations – viz., (48) and (49) – represent (in mixed coordinates) the meridian surface that has the OX coordinate axis for its double line, in one case, by its meridian curves whose instantaneous plane is determined by $y\,/\,z$, the trigonometric tangent of the angle that it defines with XZ. In the other case, that surface will be represented by means of its enveloping complex cone whose instantaneous vertex lies in OX at a distance of $(-w/t)$ from the origin of the coordinates. As equation (48) shows, all meridian curves have a center that coincides with the center of the complex and should be regarded as *a center of the surface itself*.

254. If the constant G in the group:

$$G, H, I$$

vanishes along with the previous eleven constants then equation (46) will show that all breadth curves of the respective equatorial surface whose diameter is OX will have two associated diameters that are parallel to the two associated diameters of the complex. One specifies the equatorial surface whose diameters are OY and OZ by the vanishing of H and I in the same way that the equatorial surface whose diameter is OX was specified by the vanishing of G.

If H and I vanish at the same time then all complex cones whose midpoints lie on the double line of the surface will intersect the diametral plane that is conjugate to it along curve whose midpoints coincide with the midpoint of the complex.

If the three constants G, H, I vanish simultaneously then one can choose three associated diameters of the complex to be coordinate axes in such a way that all cones of the complex whose midpoints lie in one of these three associated diameters will intersect the plane of the other two instantaneous second-order curves whose centers all coincide with the center of the complex.

255. The six constants:

$$G, H, I, K, L, M$$

will vanish simultaneously when the coordinate axes are taken to be three diameters of the complex cone whose vertices fall upon the origin and are parallel to three associated diameters of the complex. This condition can be fulfilled for a given complex in a single way, in general. Any two concentric second-order surfaces – in particular, two cones with the same vertex – will have a single system of three associated diameters in common (*). We take the two cones to be the cone of the complex:

$$Ax^2 + By^2 + Cz^2 + 2Gyz + 2Hxz + 2Ixy = 0, \tag{50}$$

whose vertex falls upon the origin and the asymptotic cone of the characteristic, whose vertex we likewise place at the origin (11):

$$\begin{aligned}(K^2 - EF)\, x^2 + (L^2 - DF)\, y^2 + (M2 - DE)\, z^2 \\ + 2\,(DK - LM)\, yz + 2\,(EL - KM)\, xz + 2\,(FM - KL)\, xy = 0.\end{aligned} \tag{51}$$

The system of the two common three conjugate diameters will then be the coordinate system that was to be determined.

(*) See *Geometrie des Raumes*, no. 262.

256. In the case in which all diameters of the complex intersect at its vertex (and we will take its three diameters, which are associated with each other relative to the complex, as well as with respect to the complex cone that has the center of the complex for its vertex, to be coordinate axes) the equation of the complex will become:

$$Ar^2 + Bs^2 + C + D\sigma^2 + E\rho^2 + F\eta^2 = 0. \tag{52}$$

This equation will contain *five* mutually-independent constants, and together with the *nine* constants of position, that will give the *fourteen* constants upon which the complex still depends.

257. With the vanishing of G, H, I, the equations of the equatorial surface in mixed coordinates, (46) and (47), will go to:

$$w^2 + \frac{Fx^2 + B}{D} \cdot v^2 + \frac{Ex^2 + C}{D} \cdot u^2 = 0, \tag{53}$$

$$\frac{E\frac{u^2}{v^2} + F}{C\frac{u^2}{v^2} + B} \cdot x^2 + \frac{D}{C\frac{u^2}{v^2} + B} \cdot z^2 + 1 = 0, \tag{54}$$

and can be converted immediately into the following ones, which represent that equatorial surface in point and plane coordinates, respectively:

$$\frac{Dz^2}{Fx^2 + B} + \frac{Dy^2}{Ex^2 + C} + 1 = 0, \tag{55}$$

$$\frac{C\frac{u^2}{v^2} + B}{E\frac{u^2}{v^2} + F} \cdot t^2 + \frac{C}{D}u^2 + \frac{B}{D}v^2 + w^2 = 0. \tag{56}$$

The meridian surface whose double line is OX will be represented by the following equations in mixed coordinates in the case in question:

$$w^2 + \frac{B\frac{y^2}{z^2} + C}{F\frac{y^2}{z^2} + E} \cdot t^2 + \frac{A}{F\frac{y^2}{z^2} + E} \cdot v^2 = 0, \tag{57}$$

$$\frac{B\dfrac{t^2}{w^2}+F}{A}\cdot y^2+\frac{C\dfrac{t^2}{w^2}+E}{A}\cdot z^2+1=0, \tag{58}$$

respectively. We obtain the equations of that meridian surface in point and plane coordinates immediately from these equations:

$$\frac{F\dfrac{y^2}{z^2}+E}{B\dfrac{y^2}{z^2}+C}\cdot x^2+\frac{F}{A}\cdot y^2+\frac{E}{A}\cdot z^2+1=0, \tag{59}$$

$$\frac{A}{B\dfrac{t^2}{w^2}+F}\cdot u^2+\frac{A}{C\dfrac{t^2}{w^2}+E}\cdot v^2+w^2=0, \tag{60}$$

respectively.

The equatorial surface that has OX for its diameter and the meridian surface that has OX for its double line will also remain of order and class four after the specialization.

258. If the new condition equation:

$$BE = CF \tag{61}$$

is satisfied, from which:

$$\frac{Fx^2+B}{Ex^2+C}=\frac{F}{E}=\frac{B}{C},$$

then all breadth curves of the equatorial surface (55) will be second-degree curves that are similar and lie similarly. Their equation:

$$D\,(Fx^2+B)\,y^2+D\,(Ex^2+C)\,z^2+(Fx^2+B)\,(Ex^2+C)=0,$$

when we neglect the common factor:

$$DE\,(Fx^2+B)\equiv DF\,(Ex^2+C),$$

will be converted into the following one:

$$\frac{x^2}{D}+\frac{y^2}{E}+\frac{z^2}{F}+\frac{C}{DE}=0. \tag{62}$$

If we ignore the two planes:

$$E\,(Fx^2+B)\equiv F\,(Ex^2+C)=0 \tag{63}$$

that intersect in the double line at infinity of the surface and contact the surface along the *OX* axis then the equatorial surface will reduce to *a second-degree surface and lose its double ray that lies at infinity in YZ.*

The two planes that are represented by equation (63) are two planes in which the curve of class two that is enveloped by lines of the complex resolves to two points that coincide in one point.

In a similar way, when we multiply equation (56) by *DE* / *C* and consider that it will follow from the condition equation (61) that:

$$\frac{C\dfrac{u^2}{v^2}+B}{E\dfrac{u^2}{v^2}+F}=\frac{C}{E}=\frac{B}{F},$$

then equation (56) will be converted into the following one:

$$Dt^2 + Eu^2 + Fv^2 + \frac{DE}{C}\cdot w^2 = 0, \tag{64}$$

which is the equation, in plane coordinates, of the second-degree surface that we just represented by its equation (62) in point coordinates.

In this, we neglect two points:

$$Eu^2 + Fv^2 = 0, \tag{65}$$

which lie in the double axis at infinity, which will therefore likewise vanish. These two points will be ones for which the second-order cone that is defined by the complex lines will resolve into two planes that coincide.

When we multiply the equation of the meridian surface in point coordinates (59) by *A* / *EF*, it will reduce to:

$$\frac{A}{CF}x^2+\frac{y^2}{E}+\frac{z^2}{F}+\frac{A}{EF} = 0 \tag{66}$$

as a result of the condition equation (61). When we multiply the equation (60) of that surface in plane coordinates by:

$$\frac{F}{A}(Ct^2 + Ew^2) \equiv \frac{E}{A}(Bt^2 + Fw^2),$$

it will go to the following one:

$$\frac{CF}{A}t^2+Eu^2+Fv^2+\frac{EF}{A}w^2 = 0. \tag{67}$$

As a result of the condition equation (61), the meridian surface will reduce *to one of degree two and lose its double line.* If we consider it to be the geometric locus of points and accordingly represent it by equation (66) after the reduction then the basis for this reduction will lie in the fact that we are ignoring the two planes:

$$B\,(Fy^2 + Ez^2) \equiv F\,(By^2 + Cz^2) = 0, \tag{68}$$

which correspond to the neglected factor. These two planes intersect along *OX* and are the two tangential planes to the surface that go through *OX*. The complex curve in each of them has resolved into a system of two points that coincide. If we consider the meridian surface as being enveloped by planes and represented by equation (67) after the reduction then that reduction will be the result of the fact that we have ignored the two points:

$$E\,(Bt^2 + Fw^2) \equiv F\,(Ct^2 + Ew^2) = 0, \tag{69}$$

which correspond to the neglected factor. These two points will be the ones at which the surface will be cut by the *OX* axis. The complex cone that has either of these two points for its vertex will degenerate into a system of two planes that coincide in a point.

259. When we consider a second-degree surface to be an equatorial surface, a diameter of it will be likewise determined that is associated with a given plane direction. When we consider it to be a meridian surface, a diameter of it will be given immediately that corresponds to the previous double line.

260. As a result of the condition equation (61):

$$BE = CF,$$

the equatorial and meridian surface, which have *OX* for their diameter and double line, respectively, will both go to second-degree surfaces. When the double condition equation:

$$AD = BE = CF \tag{70}$$

is satisfied, *these two surfaces will be identical.* We can take the following equation to be their common equation in point coordinates:

$$\frac{x^2}{D} + \frac{y^2}{E} + \frac{z^2}{F} + \frac{A}{EF} = 0, \tag{71}$$

and also switch A / EF with B / DF and C / DE.

The double condition equation (70), in conjunction with the fact that G, H, I, K, L, M vanish, says that the complex cone and the asymptotic cone of the characteristic, which have the center of the complex for their common center, are identical. More generally, when the six constants above do not vanish, we will obtain the following *five-fold* condition equation from both equation (50) and (51) in order to express this identity:

$$\begin{array}{cccccc} K^2 - EF : & L^2 - DF : & M^2 - DE : & DK - LM : & EL - KM : & FM - KL \\ = \quad A : & B : & C : & G : & H : & I. \end{array} \tag{72}$$

However, when the two cones above are identical, we can take each system of its associated diameters to be coordinate axes and each arbitrary diameter to be the *OX* axis. The equatorial surface and the meridian surface, which has an arbitrary diameter of the complex for its diameter or double line, respectively, will be identical second-degree surfaces.

If we consider the two meridian surfaces that have *OX* and *OY* for their respective double lines in the chosen coordinate system then the intersection of these two surfaces will be identical with the three coordinate planes. Next, both surfaces have the complex curve that lies in *XY* in common. However, the intersection curves in *XZ* and *YZ* coincide, insofar as the complex curve that lies in each of the two coordinate planes is, on the one hand, the meridian curve of the one meridian surface, but it is also the breadth curve of the equatorial surface that is identical with the other meridian surface. As a result of this, *all* meridian surfaces and equatorial surfaces that have an arbitrary diameter of the complex for their double line or diameter, respectively, will coincide in the same second-degree surface.

All lines of a second-degree complex that has been specialized in that way, which now depends upon only nine constants, envelop a second-degree surface. We can say that this surface is represented by the equation of the complex.

It is only due to the fact that the general complex is subjected to a *ten-fold* restriction that it will go to one whose lines envelop a second-degree surface. We can summarize these restrictions by saying that *first* all diameters of the complex intersect in the same point and *second* the complex cone and the asymptotic cone of the characteristic of the complex that have that point for their common center are identical. The first assumption corresponds to *five* condition equations that we get in their most general form when we eliminate the three coordinates x^0, y^0, z^0 from the eight equations that we obtain by annihilating the last eight coefficients of the complex equation (VI) that refers to the new origin (x^0, y^0, z^0). The second assumption corresponds to the *five* condition equations (72).

If we satisfy the same condition equations in a different sequence then we will arrive at the same result by a different specialization.

261. We would like to turn back to equation (52) and denote the radii of the curves of the complex in the three coordinate planes *YZ*, *ZX*, *XY*, which fall upon *OY* and *OZ*, *OX* and *OZ*, *OZ* and *OY*, respectively, by b_1 and a_1, a_2 and c_2, a_3 and b_3 , resp. One will then have:

$$\left.\begin{array}{ll} b_1^2 = -\dfrac{C}{D}, & c_1^2 = -\dfrac{B}{D}, \\[2ex] a_1^2 = -\dfrac{C}{E}, & c_1^2 = -\dfrac{A}{E}, \\[2ex] a_3^2 = -\dfrac{B}{F}, & b_3^2 = -\dfrac{A}{F}. \end{array}\right\} \qquad (73)$$

The same six quantities, when combined in the following way:

$$b_3 \text{ and } c_2\,,\quad a_3 \text{ and } c_1\,,\; a_2 \text{ and } b_1\,,$$

will be, at the same time, the radii that fall along *OY* and *OZ*, *OX* and *OZ*, *OX* and *OY*, resp., of the bases in *YZ*, *XZ*, *XY*, resp., of the complex cylinders whose sides are parallel to *OX*, *OY*, *OZ*, resp. We get:

$$a_3^2\, b_1^2\, c_2^2 \;=\; a_2^2\, b_3^2\, c_1^2\,. \tag{74}$$

If the double condition equation:

$$AD = BE = CF$$

is satisfied by the six constants of equation (69) then the three complex curves will intersect in the three coordinate planes of the three coordinate axes at the same point. These three complex curves coincide with the bases of the three complex cylinders. If we suppress the symbols a, b, c then we will get:

$$\left.\begin{aligned} \frac{C}{E} &= \frac{B}{F} = -a^2, \\ \frac{C}{D} &= \frac{A}{F} = -b^2, \\ \frac{B}{D} &= \frac{A}{E} = -c^2. \end{aligned}\right\} \tag{75}$$

We can choose one of the six constants of the complex equation (52) arbitrarily. If we set:

$$C = a^2\, b^2$$

then the last equations will give:

$$A = b^2\, c^2, \qquad B = a^2\, c^2,$$
$$D = -\,a^2, \qquad E = -\,b^2, \qquad F = -\,c^2.$$

The equation in question will then go to the following one:

$$b^2\, c^2\, r^2 + a^2\, c^2\, s^2 + a^2\, b^2 = a^2\sigma^2 + b^2\rho^2 + c^2\eta^2. \tag{76}$$

It will represent a complex whose lines envelop a second-degree surface with a midpoint; it will represent the surface itself.

The equation of *the same surface* in point coordinates is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{77}$$

and in plane coordinates, it is:

$$a^2\, t^2 + b^2\, u^2 + c^2\, v^2 = w^2. \tag{78}$$

262. In order to represent a given second-degree surface, for whose general equation in point coordinates we would like to take the following one:

$$ax^2 + a'y^2 + a''z^2 + 2b''xy + 2b'xz + 2byz + 2cx + 2c'y + 2c''y + d = 0, \tag{79}$$

we need merely to determine the equation of the cone that circumscribes the surface, which has any given point (x', y', z') for its vertex. As is known (*), we will get:

$$\begin{gathered}(ax^2 + a'y^2 + a''z^2 + 2b''xy + 2b'xz + 2byz + 2cx + 2c'y + 2c''z + d)\\ (ax'^2 + a'y'^2 + a''z'^2 + 2b''x'y' + 2b'x'z' + 2by'z' + 2cx' + 2c'y' + 2c''z' + d)\\ = [(ax + b''y + b'z + c)\,x' + (b''x + a'y + bz + c')\,y + (b'x + by + a''z + c)\,z'\\ + (cx + c'y + c''z + d)]^2\end{gathered} \tag{80}$$

for this equation. If we consider x', y', z' to be variable, instead of x, y, z, then the equation of this cone will be the complex equation of the surface. We can actually write it in the general form:

$$\begin{gathered}A\,(x - x')^2 + B\,(y - y')^2 + C\,(z - z')^2\\ + D\,(yz' - y'z)^2 + E\,(x'z - xz')^2 + F\,(xy' - x'y)^2\\ + 2G\,(y - y')(z - z') + 2H\,(x - x')(z - z') + 2I(x - x')(y - y')\\ + 2K\,(xy' - x'y)\,(x'z - xz') + 2L\,(xy' - x'y)\,(yz' - y'z) + 2M\,(x'z - xz')\,(yz' - y'z)\\ + 2N'(x - x')(yz' - y'z) + 2O'(y - y')(x'z - xz') + 2V'(z - z')(xy' - x'y)\\ + 2P\,(x - x')(x'z - xz') + 2Q\,(x - x')\,(xy' - x'y)\\ + 2R\,(y - y')(xy' - x'y) + 2S\,(y - y')\,(yz' - y'z)\\ + 2T\,(z - z')\,(yz' - y'z) + 2U\,(z - z')(x'z - xz') = 0,\end{gathered}$$

in which, we have set:

(*) The equation in the text can be derived in the following way:

The equation of any second-degree surface that contacts a given second-degree surface:

$$\Omega = 0$$

along the intersection curve with a plane:

$$p = 0$$

takes the form:

$$\lambda\,\Omega - p^2 = 0,$$

in which λ denotes an arbitrary constant. Here, p is taken to be the polar plane of the point (x', y', z') relative to the given surface (Ω), and λ is determined such that the new surface will go through the point (x', y', z').

$$\left.\begin{array}{lll} ad-c^2=A, & a'd-c'^2=B, & a''d-c''^2=C, \\ a'a''-b^2=D, & aa''-b'^2=E, & aa'-b''^2=F, \\ bd-c'c''=G, & b'd-cc''=H, & b''d-cc'=I, \\ b'b''-ab=K, & bb''-a'b'=L, & bb'-a''b''=M, \\ b''c''-b'c'=N', & bc-b''c''=O', & b'c'-bc=V', \\ b'c-ac''=P, & ac'-b''c=Q, & \\ b''c'-a'c=R, & a'c''-bc'=S, & \\ bc''-a''c'=T, & a''c-b'c''=U. & \end{array}\right\} \tag{81}$$

Thus:

$$N'+O'+V'=0, \tag{82}$$

and

$$\left.\begin{array}{l} N=N'-V'=b''c''-2b'c'+bc, \\ O=O'-V'=-b''c''+2bc-b'c'. \end{array}\right\} \tag{83}$$

263. In order to determine the second-degree surface when its complex equation is given, we immediately obtain a series of relations from the foregoing equations (81) in which the constants enter into the equations of the complex and the surface linearly. For example, the six equations:

$$\begin{array}{lll} a'a''-b^2=D, & a\,a''-b'^2=E, & a\,a'-b''^2=F, \\ b'b''-ab=K, & bb''-a'b'=L, & bb'-a''b''=M \end{array}$$

will yield the following six for the determination of the ratios of a, a', a'', b, b', b'':

$$\left.\begin{array}{r} aL+b'F+b''K=0, \\ aM+b'K+b''E=0, \\ a'M+b''D+bL=0, \\ a'K+b''L+bF=0, \\ a''K+bE+b'M=0, \\ a''L+bM+b'D=0. \end{array}\right\} \tag{84}$$

We shall refrain from writing down these relations completely, from which, the elimination of the quantities a, a', etc., will yield immediately the conditions that the constants of general complex equation must fulfill in order for the complex lines to envelop a second-degree surface.

264. Just as when we appeal to the point coordinates x, y, z, the equation of the conic surface that circumscribes a given second-degree surface will be the complex equation of the surface in ray coordinates when we consider the coordinates x', y', z' of its vertex to be variable, as well, so will the equation of the intersection curve of a second-degree surface with an arbitrary intersecting plane (t', u', v') be the complex equation of that surface in axial coordinates when we appeal to plane coordinates t, u, v and consider its coordinates to be likewise variable. In a completely analogous way to the way that we went from the complex equation of a given second-degree surface in ray coordinates to its usual equation in point coordinates, we can likewise go from the complex equation of that surface in axial coordinates to the equation of the surface in plane coordinates. Since both of them will always exist at the same time, when one of the two equations of a complex is given in ray and axial coordinates, the foregoing will show the simplest way to go from one of the two equations of a second-degree surface in point and plane coordinates to the other one.

265. The basis for the representability of a second-degree surface by a complex equation lies in the property of these surfaces that any plane will cut it along a curve of *class two* and any point of it will be the vertex of an enveloping cone of *order two*.

The surface can, on the one hand, degenerate into a conic surface and, on the other, into a plane curve. In both cases, it can be represented by an equation in line coordinates.

In the first case, all of the complex cones will degenerate into a system of two planes, which will contact the conic surface that is being represented. All straight lines that go through the vertex of the surface will belong to the complex.

In the second case, the complex curve in an arbitrary plane will degenerate into a system of two points at which the curve being represented will be cut by the given plane. All of the straight lines that lie in the plane of the curve will be lines of the complex.

Whereas a plane curve cannot be represented by a single equation in point coordinates and a conic surface cannot be represented by a single equation in plane coordinates, both geometric structures will find a representation in line coordinates. However, whereas a conic surface is of *order two* and is determined by a second-degree equation in point coordinates, an a plane curve is of *class two* and is given by an equation in plane coordinates, a second-degree complex can represent only a cone of *class two* and a curve of *order two*.

A cone of class two can resolve into two axes that intersect at its vertex; a curve of order two can resolve into two rays that lie in its plane. With this specialization, the cone and the curve will be identical, and as before they will find their representation in an equation in line coordinates.

We come to the same specialization of the second-degree complex from yet another direction. Its equation can be resolved into linear factors, and these factors, in turn, can satisfy the condition that they represent first-degree complexes of the special kind whose lines all intersect a fixed straight line. When the two straight lines that are represented in this way go through the same point, or – what amounts to the same thing – lie in the same plane, we will have, in one case, the specialized cone of class two, and in the other case, the specialized curve of order two.

§ 3.

The lines at infinity of the complex. Classification of complexes by those lines.

266. If we choose a straight line in a given plane arbitrarily and move it parallel to itself ever further then it will lose every trace of its original direction in the plane at infinity. We can also replace the given plane, which contains the line shifted to infinity, with any other plane that is parallel to it. All straight lines at infinity in parallel planes will coincide in a single line at infinity. The straight line that was shifted to infinity will be the intersection of infinitely many parallel planes. It will have no relationship to finite points at infinity, except that it is parallel to a given plane direction of a given plane.

When a given plane is shifted ever further parallel to itself it will lose its direction, in its own right. The plane at infinity must be regarded as parallel to any given plane. The straight lines that lie in it have lost any relationship to finite points, and thus, any meaning in the usual sense.

These geometric insights find an immediate analytical expression. In order for a straight line:

$$x = r\,z + \rho, \qquad y = s\,z + \sigma,$$

to be contained in a plane:

$$tx + uy + vz + w = 0,$$

one must have the following three relations:

$$tr + us + v = 0,$$
$$t\rho + u\sigma + w = 0,$$
$$t\eta + v\sigma - ws = 0.$$

If the straight line lies at infinity in the given plane then ρ and σ – and as a result, also $\eta \equiv r\sigma - s\rho$ – will be infinitely large. The last two equations will then given the foregoing equations:

$$t : u : v = -\sigma : \rho : \eta,$$

while the first equation merely expresses the fact that the straight line that was shifted to infinity is parallel to the given plane.

If the given plane is shifted to infinity then w will become infinitely large, or – what amounts to the same thing – t, u, and v will vanish. Its equation will no longer express its direction, and the foregoing three relations will lose their meaning.

267. If we take the general equation of the second-degree complex to be the following one:

$$\begin{aligned} &Ar^2 + Bs^2 + C + D\sigma^2 + E\rho^2 + F\eta^2 \\ &+ 2Gs + 2Hr + 2Irs + 2K\rho\eta - 2L\sigma\eta - 2M\rho\sigma \\ &- 2Nr\sigma + 2Os\rho \\ &+ 2Pr\rho + 2Qr\eta + 2Rs\eta - 2Ss\sigma - 2T\sigma + 2U\rho = 0, \end{aligned} \tag{I}$$

and let ρ, σ, η become infinitely large in it, and thus neglect the two remaining variables, r and s, as well as constant quantities, in comparison to these three variables, and finally, neglect first powers of the first-mentioned three variables in comparison to their second powers then that will give:

$$D\sigma^2 + E\rho^2 + F\eta^2 + 2K\rho\eta - 2L\sigma\eta - 2M\rho\sigma = 0 \tag{85}$$

for the lines of the complex that lie at infinity.

This equation, like any equation in line coordinates, represents a complex. We would like to call it the *asymptotic complex of the given complex.* From the discussion in the previous paragraph, this complex will be subsumed by the one that represents a cone of class two. The vertex of this conic surface will coincide with the coordinate origin, and its intersection with the plane at infinity will be curve of class two that is enveloped by the lines of the complex that lie in that plane.

Any second-degree complex in whose equation the terms of second order in ρ, σ, η are multiplied by the same constants D, E, F, K, L, M that are in the equation of the given complex will represent the lines of the given complex that lie at infinity with the same precision as the complex whose equation is the foregoing one (85). It is the asymptotic complex, which has the same relationship to all of those complexes as the given one, in its own right, due to the simplicity of its equation, and corresponding to that, by the obvious grouping of its lines, by singling out a special position for the coordinate system, as well.

The degree of the approximation by which the asymptotic complex represents the lines of the given complex that lie at infinity is only the first degree, insofar as its equation agrees with the given one only in the terms of order two in the variables that come under consideration, but not with those of first order.

268. If we replace $-\sigma$, ρ, η in equation (85) with the values of t, u, v above that these coordinates will assume for straight lines at infinity then we will get the following equation:

$$Dt^2 + Eu^2 + Fv^2 + 2Kuv + 2Ltv + 2Mtu = 0$$

for the determination of those plane directions along which lines of the complex will lie at infinity. If we draw planes through the coordinate origin that have these directions then they will envelop a conic surface of class two, which is the conic surface was represented by equation (85) in line coordinates. We can displace the conic surface, and with it, the asymptotic complex, parallel to themselves arbitrarily without changing their relationships to the given complex. From the coordinate transformation formulas of number **157**, the coefficients D, E, F, K, L, M, which are the only ones that appear here, will remain unchanged under such a displacement. The tangential planes of the conic surface will move parallel to themselves under the displacement. All mutually parallel tangential planes will intersect along a line of the given complex that lies at infinity.

In the first paragraph of this section, we have used the term "the *characteristic* of a complex" to refer to a surface of class two whose center and absolute dimensions can be chosen arbitrarily, and which will be represented by the following equation:

$$Dt^2 + Eu^2 + Fv^2 + 2Kuv + 2Ltv + 2Mtu + kw^2 = 0$$

when we place its center at the origin of the coordinates, and let k denote an arbitrary constant. From the foregoing, the lines of the complex that lie at infinity will lie in the tangential planes to the asymptotic cone of the characteristic, and this asymptotic cone will be represented by equation (85) in line coordinates. A plane, which we can shift to infinity, but in such a way that it does not lose its original direction, will cut this asymptotic cone, and thus, the characteristics itself, as well, along a curve that will be enveloped by the lines of the complex that lie at infinity. One is therefore not dealing with a finite characteristic and its asymptotic cone.

269. We can *approximate the plane at infinity* (for which we hardly have a geometric representation) *in infinitely many ways*, when we start with a plane with a given direction and shift it ever further while preserving that direction. Such a plane will contain, on the one hand, a complex curve of class two, and on the other hand, a second such curve as its intersection with the characteristic. The two curves will coincide when their planes are shifted to infinity. In other words, the curves of all equatorial surfaces of a given complex that lie in breadth planes that are shifted to infinity will lie on the characteristic.

When a plane of a given direction is shifted, the complex curve in it will shift continually, and that will describe the equatorial surface. The directions of the two axes of the curve and their ratio will get closer to a certain limit when the plane is shifted ever further, corresponding to the direction of the plane. This limit is given by the constant direction and the constant ratio of the axes of the curve of intersection of the moving plane with the characteristic. Since complex curves and intersection curves with the characteristic that are contained in parallel planes will coincide at infinity, the diameter of the relevant equatorial surface of the given complex must be parallel to the diameter of the characteristic, which is associated with the plane that moves parallel to itself, as was confirmed by the analytical developments of the first paragraph.

The foregoing geometric insights point to the relations between the given complex and its characteristic. In agreement with that, we will get the following equations:

$$\left.\begin{aligned} Dw^2 + 2Lxvw + Fx^2v^2 + 2Mxuw + 2Kx^2uv + Ex^2u^2 &= 0, \\ Ew^2 + 2Mytw + Dy^2t^2 + 2Kyuw + 2Ly^2tv + Fy^2v^2 &= 0, \\ Fw^2 + 2Kzuw + Ez^2u^2 + 2Lztw + 2Mz^2tu + Dz^2t^2 &= 0 \end{aligned}\right\} \qquad (86)$$

from equations (7) in number **166**, which will give the three complex curves in planes that are shifted to infinity parallel to the arbitrarily-chosen coordinate planes YZ, XZ, XY, when we neglect the first powers of x, y, z, and constants of the second power, and those equations will coincide with the equations of the intersection curves of the three planes in question with the asymptotic cone of the characteristic.

270. The conic surfaces of class two that are enveloped by the lines of the asymptotic complex can be real or imaginary, and accordingly, the given second-degree complex might or might not include real lines that lie at infinity. Therefore, the general second-degree complexes will split into *two coordinated types.* We would like to call complexes of the first kind *hyperboloidal*, while complexes of the second will be *ellipsoidal*. In this classification, we first ignore complexes whose asymptotic complex has be specialized in some way.

Hyperboloidal complexes will have a characteristic with a real asymptotic cone, and will thus be defined analytically by the fact that only two of the three expressions:

$$D - \frac{LM}{K}, \qquad E - \frac{MK}{L}, \qquad F - \frac{KL}{M}$$

will have values with equal signs.

Ellipsoidal complexes have a characteristic whose asymptotic cone reduces to an ellipsoidal point; the three expressions above will have values that *all* agree in sign for such complexes.

271. In *hyperboloidal* complexes, the tangential planes of the asymptotic cone of the characteristic determine the directions of the planes along which lines of the complex will lie at infinity. The complex curves in such planes will be parabolas that contact the lines at infinity. If one moves such a plane parallel to itself then the parabola that lies in it that is enveloped by lines of the complex will describe a parabolic equatorial surface (no. **232**). The side along which the asymptotic cone of the characteristic is contacted by a breadth plane of the surface will determine the direction that the direction of the axis of the parabola will approach when its plane moves ever further, which can happen in two ways.

Any other plane direction, along which no line of the complex lies at infinity, determines an equatorial surface whose breadth curves possess a center. Here, we first emphasize that with increasing distance, when a plane that moves parallel to itself the complex curve in it will become a hyperbola or an ellipse, according to whether the plane cuts the asymptotic cone in a hyperbola or an ellipse, resp.

Two planes in which a line of a hyperboloidal complex lies at infinity will go through a given straight line, in general. If we take any point of the given straight line to be the vertex of the asymptotic cone of the characteristic then the two tangential planes to this cone that can be drawn through the given line will be the two planes in question. They will be real or imaginary according to whether the line lies outside or inside the cone, respectively, and will coincide in a tangential plane to the cone when the line is a side of the cone. Corresponding to them, two parabolas can appear among the meridian curves of the meridian surface of a hyperboloidal complex; they can also coincide. That will depend upon the direction of the double line of the meridian surface relative to the asymptotic cone of the characteristic of the complex.

The lines of the complex that are parallel to the double line of a meridian surface define a complex cylinder that circumscribes the meridian surface. This cylinder will be *hyperbolic* or *elliptic* (*), according to whether the two meridian planes in which

(*) Here, and in what follows, we understand hyperbolic and elliptic cylinders to mean ones that intersect the plane at infinity in two real or two imaginary lines, resp.; thus, the imaginary cylinder will also parabolic curves lie are real or imaginary, respectively. If the two planes coincide then the complex cylinder will be called *parabolic*.

From the foregoing, the cylinders that are defined by the lines of a hyperbolic complex will be elliptic or hyperbolic according to whether the direction of the complex lines that generate it do or do not lie in the asymptotic cone of the characteristic, resp. All complex cylinders whose generators are parallel to a side of the asymptotic cone are parabolic.

272. There are no parabolic curves whatsoever in *ellipsoidal* complexes. All equatorial surfaces are included between two planes that are found at a finite distance from each other. These planes refer to the transition from planes in which a real complex curve lies to ones in which an imaginary curve is enveloped by lines of the complex.

One finds no parabolas among the meridian curves of an arbitrary meridian surface that belongs to a complex. The two meridian planes in which parabolas are enveloped by lines of the complex in the case of hyperboloid complex will be imaginary in the case of ellipsoidal complexes that are independent of the direction of the double line. As a result of this, all cylinders that are defined by lines of an ellipsoidal complex will be elliptic cylinders.

273. In number 163, we obtained the following equation in mixed point and line coordinates x, u, v, w:

$$\begin{aligned} &Dw^2 + 2\,(Lx - S)\,vw + (Fx^2 - 2Rx + B)\,v^2 \\ &+ 2\,(Mx + T)\,uw + 2\,(Kx^2 - Ox - G)\,uv + (Ex^2 + 2Ux + C)\,u^2 = 0 \end{aligned} \tag{87}$$

for an *equatorial surface* whose breadth curves are parallel to the YZ plane. This equation contains *thirteen* constants, which gives the *fifteen* constants upon which the equatorial surface depends when one includes the *two* constants by which the coordinate system is specialized.

If we determine the OX axis in such a way that it runs parallel to the diameter of the complex that is associated with the arbitrary plane that is taken to be YZ then the constants L and M will vanish.; if it coincides with that diameter then S and T will vanish simultaneously. K will vanish when we give the two axes OY and OZ directions in YZ such that the three coordinate axes are parallel to the three associated diameters of the complex. The general equation of the equatorial surface will lose five more constants by this coordinate determination, and it will go to the following one:

$$Dw^2 + (Fx^2 - 2Rx + B)\,v^2 - 2\,(Ox + G)\,uv + (Ex^2 + 2Ux + C)\,u^2 = 0. \tag{88}$$

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be referred to as elliptic. In particular, one can resolve hyperbolic and elliptic cylinders into systems of two intersecting planes that will be real or imaginary, respectively.

If the two lines of intersection coincide with the plane at infinity in a straight line then the cylinder will be called parabolic, even if it has been specialized into a system of two parallel real or imaginary planes.

If we displace the coordinate plane *YZ* parallel to itself until it goes through the center of the complex then that will reduce the number of constants by a sixth unit, in accordance with the condition equation (36):

$$ER = FU.$$

274. All equatorial surfaces for ellipsoidal complexes are representable by an equation with the latter form (88), since we might also choose the direction of the *YZ* plane. However, if we take the breadth planes of the equatorial surface to be parallel to a tangential plane to the asymptotic cone of the characteristic for hyperboloidal complexes, in particular, then the associated diameter of the characteristic will be parallel to these planes, and as a result, the foregoing coordinate system will no longer be possible. The general equation of the equatorial surface (87) will then lose the constant D, such that the surface will depend upon only fourteen constants. We have called such equatorial surfaces *parabolic.*

The vanishing of D corresponds to the fact that *YZ* is a tangential plane to the asymptotic cone of the characteristic whose midpoint we have chosen to be the coordinate origin. We would like to let the *OZ* axis coincide with the side of the asymptotic cone along which it will contact the *YZ* plane. The constant M will then vanish in the equation for the equatorial surface. In the general case, the coordinates of the center of an arbitrary breadth curve will be:

$$y = -\frac{Mx+T}{D}, \qquad z = -\frac{Lx-S}{D}.$$

When D vanishes, the center in the plane of the curve will go to infinity, and the direction along which it lies at infinity will be determined by the equation:

$$\tan\alpha = \frac{Mx+T}{Lx-S},$$

in which α denotes the angle that this direction – viz., the direction of the axis of the parabola – makes with *OZ*. For the parabola at infinity, one will get:

$$\tan\alpha = \frac{M}{L}.$$

This axis direction will be parallel to the *OZ* axis when M vanishes.

The direction of the *OY* axis still remains undetermined, as of now. We can take it to be the one in *YZ* that is perpendicular to *OZ*. If we then draw a second tangential plane to the asymptotic cone through *OY* and take it to be the *XY* plane and the side along which it contacts the asymptotic cone to be the *OX* axis then the two constants F and K will vanish from the equation of the equatorial surface. One then writes the equation of the surface in the following form:

$$2\,(Lx - S)\,vw - (2Rx - B)\,v2 + 2Tuw - 2\,(Ox + G)\,uv + (Ex^2 + 2Ux + C)\,u^2 = 0. \qquad (89)$$

We can drop three more constants from this equation by a proper choice of origin.

275. When the expression:

$$Dt^2 + Eu^2 + Fv^2 + 2Kuv + 2Ltv + 2Mtu,$$

which corresponds to the condition equation:

$$DEF - DK^2 - EL^2 - FM^2 + 2KLM = 0, \tag{90}$$

resolves into two first-degree factors, that will specialize the complex by eliminating one of its nineteen constants.

The foregoing condition equation then comes down to saying that when we let K, L, M vanish by a suitable choice of directions for the three coordinate axes, as before, one of the three constants D, E, F will likewise vanish as a result. If D is the vanishing constant then the equation of the complex will become:

$$\begin{gathered} Ar^2 + Bs^2 + C + E\rho^2 + F\eta^2 \\ + 2Gs + 2Hr + 2Irs \\ - 2Nr\sigma + 2Os\rho \\ + 2Pr\rho + 2Qr\eta + 2Rs\eta - 2Ss\sigma - 2T\sigma + 2U\rho = 0. \end{gathered} \tag{91}$$

We can eliminate three more constants from this equation, in which we would like to choose the coordinate system to be rectangular, despite its generality, by determining the origin of the coordinates. It is essential in the following considerations that none of the other constants D, E, F vanish by the choice of directions for the coordinate axes, except for D.

276. We have represented the characteristic of the equation:

$$Dt^2 + Eu^2 + Fv^2 + 2Kuv + 2Ltv + 2Mtu + kw^2 = 0.$$

This characteristic is a second-degree surface with a center in the general case of hyperboloidal and ellipsoidal complexes. The center of that surface and its absolute dimensions, which are independent of the arbitrary constant k, can be chosen arbitrarily. In the case of complexes of the special kind that we are now considering, and which we have represented by equation (91), the characteristic will reduce to a second-degree curve with a center. We would like to call this curve *the characteristic curve* of the complex of the special kind.

A distinguished plane direction for the complex is given by the plane of the characteristic curve. If we take it to be parallel to the YZ coordinate plane then D, L, M will vanish, and the equation of the curve will go to the following one:

$$Eu^2 + Fv^2 + 2Kuv + kw^2 = 0.$$

If we take two associated diameters of the curve – in particular, its two axes – to be the *OY* and *OZ* coordinate axes then *K* will vanish, and then its coordinate axes will be the ones to which the complex in equation (91) is referred.

277. We have reduced the determination of the direction of the associated diameter of a complex of the general kind to the consideration of the diameter of its characteristic surface. We can regard the characteristic curve of a complex of the special kind as the limit of characteristic surfaces, and as a result, say that two associated diameters of the curve are simultaneously associated with all planes that can be drawn through other ones in an arbitrary direction each time. We can also say that each straight line that goes through the center of the curve that does not lie in the plane of that curve will be associated with that plane, and finally the fact that any such straight line and two diameters of the curve will define a system of three associated diameters of the curve.

These relations carry over immediately to complexes of the special kind. A given plane corresponds to a diameter of the complex that will be parallel to the plane of the characteristic curve and will remain parallel to this plane, even if the direction of the given plane might change. In other words, *the diameters of all equatorial surfaces of the complex are parallel to the plane of its characteristic curve.*

If the given plane rotates around its line of intersection with the plane of the characteristic curve then the associated diameter of the complex will move parallel to itself. There will then be infinitely many mutually-parallel diameters of the complex. Finally, if the rotating plane coincides with the plane of the characteristic curve then the diameter will be indeterminate. It will lose its direction when it goes to infinity.

In the general case of hyperboloidal and ellipsoidal complexes, we have shown that any two conjugate diameters will be cut by the axis of the complex cylinder whose sides are parallel to the third conjugate diameter. In the case of the special complex that we are considering here, the third conjugate diameter will be shifted to infinity every time. However, as before, any two arbitrary conjugate diameters that are parallel to the central plane will determine the directions of the sides of a complex cylinder whose axes cut the two diameters by the intersection of their associated planes. We say that this cylinder – and in particular, its axis – is associated with the system of two diameters.

278. In order to confirm and extend this result, we would like to return to equations (5), which represent the diameter that is associated with a given plane:

$$tx + uy + vz + w = 0$$

in the general case of complexes. When we use the equation of the complex of the special kind (91) as a basis, and let x, y, z keep their previous meaning, these equations will reduce to:

$$\left.\begin{aligned} x - x' &= 0, \\ y - y' &= \frac{Eu}{Eu^2 + Fv^2}, \\ z - z' &= \frac{Fv}{Eu^2 + Fv^2}. \end{aligned}\right\} \tag{92}$$

Thus:

$$(y - y')\, Fv = (z - z')\, Eu. \tag{93}$$

In accordance with the first of the three equations (92), the diameter is parallel to the *YZ* plane. Equation (93), when written in the following way:

$$\frac{y-y'}{z-z'} \cdot \frac{v}{u} = \frac{E}{F}, \tag{94}$$

immediately expresses the idea that the intersection of the given *YZ* plane and the diameter of the complex that is associated with that plane will have the direction of two associated diameters of the characteristic curve, which, from the vanishing of *K* will be represented by the equation:

$$Eu^2 + Fv^2 + kw^2 = 0.$$

The values of $x' : y' : z'$ that we have used as a basis for equations (92) are the following ones:

$$\left.\begin{aligned} x' &= \frac{Ouv + Rv^2 + Stu - Ttu - Uu^2}{Eu^2 + Fv^2}, \\ y' &= \frac{-Ntv - Puv - Qv^2 + Tt^2 + Utu}{Eu^2 + Fv^2}, \\ z' &= \frac{(N-O)tu + Pu^2 + Quv - Rtv - St^2}{Eu^2 + Fv^2}. \end{aligned}\right\} \tag{4}$$

The distance x' of the diameter from the *YZ* plane then remains the same for all planes whose coordinates satisfy the following equation:

$$(Eu^2 + Fv^2)\, x' = Ouv + Rv^2 + Stv - Ttu - Uu^2. \tag{95}$$

All such planes envelop a curve at infinity of class two. When the term in t^2 is missing from the foregoing equation, that curve will contact the straight line at infinity in *YZ*, independently of the choice of x'. We get:

$$Sv - Tu = 0 \tag{96}$$

for the contact point; x' no longer enters into this equation. It determines a *distinguished direction* for the complex.

The coordinates of the point x', y', z', which determine the position of the diameter, will become infinitely large when u and v vanish at the same time. As equations (92) show, the diameter will then lose its direction at infinity. However, the quotient y' / z' will keep a finite and well-defined value. When we let u and v vanish, we will get from (4) that:

$$\frac{y'}{z'} = -\frac{T}{S}. \tag{97}$$

The diameter will then be shifted to infinity in the direction that is indicated by the foregoing equation. This direction will coincide with the one that we have determined by equation (96). We can say that the infinitude of diameters that are associated with the plane of the characteristic curve in the complex intersect that plane in the same point at infinity. That point will be the center of curve that is enveloped by lines of the complex in the plane of the characteristic curve, and will remain unchanged when the plane moves parallel to itself. We will obtain the analytic confirmation of this geometric consequence in the following number.

279. We get:

$$\left.\begin{array}{ll} x = -\dfrac{U}{E}, & x = \dfrac{P}{E}, \\[2ex] x = \dfrac{R}{F}, & y = -\dfrac{Q}{F} \end{array}\right\} \tag{98}$$

for the intersection of the two diameters that are associated with the XZ and XY coordinate planes and parallel to OY and OZ with these two coordinate planes. If we set:

$$P = 0, \qquad Q = 0$$

then we will displace the XZ and XY planes in such a way that after the displacement the two diameters that are associated with these two planes will cut the OX axis.

Of the pairs of equations (18) in number 240, by which the axes of three complex cylinders whose sides are parallel to the OX, OY, OZ coordinate axes were represented, the first one:

$$y = -\frac{Q}{F}, \qquad z = \frac{P}{E}, \tag{99}$$

showed that one of the cylinder axes coincided with OX. The other two pairs of equations gave:

$$\left.\begin{array}{ll} x = \dfrac{R}{F}, & z = \infty, \\[2ex] x = -\dfrac{U}{E}, & y = \infty. \end{array}\right\} \tag{100}$$

The other two cylinder axes in the same planes that are parallel to *YZ*, and in which the two associated diameters lie, will be shifted to infinity.

Of the three coordinates of the center of the central parallelepiped whose edges are parallel to *OX*, *OY*, *OZ*, respectively, for which we have obtained:

$$x^0 = \frac{ER - FU}{2EF}, \qquad y^0 = -\frac{DQ - FT}{2DF}, \qquad z^0 = \frac{DP - ES}{2DE} \tag{21}$$

in the general case, only x^0 remains finite and determined completely, while y^0 and z^0 will become infinitely large; the ratio of y^0 and z^0 will remain determinate. We will get:

$$\frac{y^0}{z^0} = -\frac{T}{S} \tag{101}$$

for it.

The same thing will be determined by this equation that we obtained in the previous number (97).

The center of the central parallelepiped that we have chosen lies in a plane that is determined, not only in direction, but also in position, in which the sense that is determined by equation (101) is at infinity. If we keep the *OX* axis as a side of the central parallelepiped and take *OY* and *OZ* arbitrarily to be two conjugate diameters of the characteristic curve then we will obtain a series of central parallelepipeds. The same considerations that we posed in number **246** in the case of hyperboloidal and ellipsoidal complexes, show us here that the center of all of these central parallelepipeds will be shifted to infinity in the same direction and in the same plane that is parallel to the plane of the characteristic curve.

If we choose another cylinder axis of the complex in place of the *OX* axis then we will obtain a new series of central parallelepipeds. The centers of all these parallelepipeds will be shifted to infinity in the same direction, as before, parallel to the plane of the characteristic curve, since the determination of that direction was independent of the choice of the *OX* coordinate axis. By contrast, the plane in which the center of the parallelepiped is shifted to infinity will generally be different. If we then choose any two conjugate diameters of the complex and replace the one with another one that is parallel to it then that will change the central plane that goes halfway between the two conjugate diameters.

We have then come to the following theorems:

In the complexes of the special kind that we are considering, the center is shifted to infinity parallel to the plane of the characteristic curve in the given direction:

$$\frac{y}{z} = \frac{y_0}{z_0} = -\frac{T}{S}.$$

All central parallelepipeds that have the same finite cylinder axis for one of their edges will possess the same central plane parallel to the plane of the characteristic curve.

280. We will get the following equation for the complex of the special kind:

$$\begin{aligned} &Ar^2 + Bs^2 + C + E\rho^2 + F\eta^2 \\ &+ 2Gs + 2Hr + 2Irs \\ &- 2Nr\sigma + 2Os\rho \\ &+ 2Rs\eta - 2S\sigma - 2T\sigma + 2U\rho = 0 \end{aligned} \tag{102}$$

when we let the axis of any of its cylinders coincide with the *OX* coordinate axis and choose *OY* and *OZ* to be any two diameters of the characteristic curve. We can add to that the condition equation:

$$ER = FU \tag{103}$$

and then determine that the central plane that belongs to the *OX* axis will coincide with the *YZ* coordinate plane. Finally, we can let *S* or *T* vanish at will when we take one of the two axes *OY*, *OZ* to be parallel to the direction that is determined by equation (101).

When one considers that simplification, equation (102) will contain *eleven* mutually-independent constants. When we add to them the *seven* constants by which the coordinate system was specialized, we will obtain the *eighteen* constants of the complex of the special kind.

281. The asymptotic cone of the characteristic surface of a complex of the general kind will be represented by the two asymptotes of the characteristic curve for the complexes of the special kind that we consider here.

In the case of the general complex, the curve along which a given plane cuts the asymptotic cone will determine the nature of the complex curve in the plane that is shifted to infinity parallel to the given one. In complexes of the special kind, this curve will resolve into the two intersection points of the given plane with the asymptotes. The complex curve will then degenerate into *a system of two points* that lie at infinity in the direction of the two asymptotes in the plane that has been shifted to infinity.

All equatorial surfaces whose breadth planes are parallel to one of the two asymptotes are parabolic. We will also obtain a parabolic equatorial surface when we take its breadth planes to be parallel to the plane of the characteristic curve. The equation of this surface is:

$$\begin{aligned} &- 2Svw + (Fx^2 - 2Rx + B)\, v^2 + 2Tuw \\ &- 2(Ox + G)\, uv + (Ex^2 + 2Ux + C)\, u^2 = 0, \end{aligned} \tag{104}$$

and the surface will be specialized in such a way that the axes of the parabola are directed the same in all breadth planes. This direction is, in agreement with number **278**, the one along which the center of the complex is shifted to infinity.

If we determine that equatorial surface by its enveloping cylindrical surfaces, instead of by its breadth curves, then we will get the following equation from the developments of number **182**:

$$\begin{aligned} &(Fy'^2 + Ez'^2)\, x^2 - 2\,(Ry'^2 - Oy'z' - Uz'^2)\, x \\ &+ 2\,(Sy' + Tz')\, y' \cdot z + (By'^2 + 2Gy'z' + Cz'^2) = 0. \end{aligned} \tag{105}$$

This represents the intersection of XZ with the complex cylinder whose sides are parallel to the direction that is determined by the ratio y' / z'.

The term in z^2 is missing from the foregoing equation. All complex cylinders whose sides are parallel to the plane of the characteristic curve are parabolic cylinders. Their diametral planes are parallel to the stated plane. In particular, those two cylinders whose sides are parallel to one of the two asymptotes of the characteristic curve will resolve into a system of two planes, one of which is shifted to infinity. As a result of this, the equation of the cylinder will reduce to one of first degree. If we finally give the sides of the cylinder the direction in which the center of the complex is shifted to infinity then we will get:

$$Sy' + Tz' = 0,$$

and the cylinder will decompose into two planes that are both parallel to the plane of the characteristic curve.

282. We would like to call a complex of the special kind *hyperbolic* or *elliptic* according to whether the two asymptotes of the characteristic curve are real or imaginary, respectively.

In both kinds of complexes, a line of the complex will lie at infinity in planes that are parallel to the plane of the characteristic curve. There no other planes that contain lines of the complex at infinity in elliptic complexes. In hyperbolic complexes, two real planes can be drawn through any line in space, which are parallel to the two asymptotes of the characteristic curve, respectively. The complex cures in these planes will be parabolas. With the exception of the complex cylinders whose sides run parallel to the plane of the characteristic curve, all cylinder surfaces that belong to a hyperbolic complex will be hyperbolic, and the cylinders that belong to an elliptic complex will be elliptic.

We can say that the curves of the complex that lie in the plane at infinity resolve into a system of *two real points* in the case of hyperbolic complexes and a system of *two imaginary points* in the case of elliptic complexes.

283. If we consider only the terms of degree two in ρ, σ, η in order to represent the totality of lines of the complex that lie at infinity, as we did in the general case (no. **267**), then we will get:

$$E\rho^2 + F\eta^2 = 0 \tag{106}$$

from equation (102).

This equation represents the two asymptotes of the characteristic curve in line coordinates.

However, with a greater approximation than one gets by using the characteristic curve, we can determine the lines of the given complex at infinity when we neglect, as before, first powers of ρ and η in comparison to the second powers, as well as the variables r, s, and constants, *while we keep first powers of* σ. In this way (*), we will get the following equation:

$$E\rho^2 + F\eta^2 - 2\,(Ss + T)\,\sigma = 0. \tag{107}$$

A term with N or O does not enter in. Namely, one has:

$$-N\,r\sigma + Os\rho = -N\eta + (O - N)\,s\rho\,;$$

that is, $r\sigma$ will always have the same order as the terms with η and $s\rho$, so it will not come under consideration.

The foregoing equation represents a new complex that we would like to call the *asymptotic complex of the given one.*

As in the general case, the approximation of the asymptotic complex is of first degree to the given one, while it would be only of degree 1/2 by neglecting the terms of first order in σ.

If we displace the origin of the coordinates arbitrarily then the two constants S and T will remain unchanged in equation (91), which does not include D, K, L, M. Since we might then displace the given complex and its asymptotic complex parallel to themselves with respect to each other, *their reciprocal relationship will remain the same.*

The equation of the asymptotic complex will be satisfied when we simultaneously have:

$$\rho = 0, \quad \sigma = 0, \quad \eta = 0.$$

All of the straight lines that go through the coordinate origin will belong to the asymptotic complex. The complex further encompasses all straight lines that obey the two equations:

$$E\rho^2 + F\eta^2 = 0, \qquad \sigma = 0,$$

or the following two:

$$E\rho^2 + F\eta^2 = 0, \qquad Ss + T = 0.$$

Any straight line that cuts the OX axis and the two asymptotes of the characteristic curve that lie in YZ will then be a line of the asymptotic complex. Moreover, it will also contain any straight line that cuts one of the two asymptotes and is parallel to the plane through the origin:

$$Sy + Tz = 0,$$

which refers to the direction in which the center of the given complex is shifted to infinity in the plane of the characteristic curve. As a result of this, the complex curve will degenerate into the system of two points in the YZ plane, one of which will coincide with the coordinate origin and the other of which will be shifted to infinity in the direction that

(*) Analogously, a curve branch with a parabolic asymptote has only one point that lies at infinity, when taken absolutely, namely, the one in which it is intersected by the diameter of the parabolic asymptote. We will get a more precise insight into the position of the infinitely-close points by the parabolic asymptote itself whose points will lie at infinity in the direction of the axis, as well as perpendicular to it when they are shifted to infinity. However, this happens in such a way that when the magnitude of the distance to the axis is of first order, the order of the magnitude of the distance from the axis will only be 1/2.

was specified by the foregoing equation. The equatorial surface of the asymptotic complex whose breadth planes are parallel to *YZ* consists of parabolas, like that of the given complex. All of these parabolas will contact the two planes that can be drawn through the *OX* axis and the two asymptotes of the characteristic curve. If the breadth plane is shifted to infinity parallel to *YZ* then the parabola in it will degenerate into a system of two points at infinity. We have imagined the transition from a parabola to two points at infinity in such a way that the contact points will be shifted to infinity along two fixed tangents to the curve.

284. If the plane in which the center of a given complex is shifted to infinity contains one of the two asymptotes or is undetermined then we will obtain a corresponding specialization of the complex relative to the position of its diameter and the arrangement of its lines at infinity. In general, such complexes will depend upon *seventeen* or *sixteen* constants, respectively.

Here, we would like to consider only the latter case, in which *S* and *T* vanish in the general complex equation, along with *K*, *L*, *M*. *The variable* σ *will then drop out of the equation of the complex, thus-specialized, completely.*

The most general form of equation in which these variables are missing is:

$$\begin{aligned} Ar^2 + Bs^2 + C + 2Gs + 2Hr + 2Irs \\ + E\rho^2 + F\eta^2 + 2K\rho\eta \\ + 2\,(O - N)\,s\rho - 2N\eta \\ + 2Pr\rho + 2Qr\eta + 2Rs\eta + 2U\rho = 0. \end{aligned} \tag{108}$$

K will vanish in this equation due to the fact that we take the *OY* and *OZ* coordinate axes to be parallel to two associated diameters of the characteristic curve. *P* and *Q* will vanish when we let the *OX* axis (which was assumed to be arbitrary, up to now) coincide with the axis of a complex cylinder. Finally, by displacing the *YZ* plane parallel to itself, we will obtain the relation:

$$ER = FU.$$

The equation:

$$\begin{aligned} Ar^2 + Bs^2 + C + 2Gs + 2Hr + 2Irs \\ + E\rho^2 + F\eta^2 \\ + 2\,(O - N)\,s\rho - 2N\eta \\ + 2Rs\eta + 2U\rho = 0, \end{aligned} \tag{109}$$

in which:

$$ER = FU,$$

is then to be regarded as the *general equation* of the complex that has been specialized in the manner in question. It will include *ten* mutually-independent constants, to which, one must add the *six* constants of position, which arise from the fact that the *YZ* plane is determined by the complex, the fact that the two axes, *OY* and *OZ*, have associated directions relative to the characteristic curve, and finally, the fact that *OX* is a cylinder axis of the complex.

The condition that a second-degree can then be represented by a second-degree equation in only *four* of the five variables:

$$r,\ s,\ \sigma,\ \rho,\ (r\sigma - s\rho \equiv \eta),$$

corresponds to a *three-fold specialization* of the complex (*).

285. It is interesting to examine the complex thus-specialized more closely.

For the distance from the diameter of the complex that is associated with a given plane:

$$tx + uy + vz + w = 0$$

to the *YZ* coordinate plane that is parallel to it, we find:

$$x = \frac{Rv^2 + Ouv - Uu^2}{Eu^2 + Fv^2} \tag{110}$$

from the formulas of number **278**, when we set *S* and *T* equal to zero. If we then rotate the given plane arbitrarily around its intersection with the plane of the characteristic curve then the diameter that is associated with it will always remain in the same plane that is determined by the foregoing value of *x*, while the distance between the diameter that is associated with it and the *YZ* plane will change under rotation of the given plane in the general case of hyperbolic and elliptic complexes.

With that, the previously-obtained result will go to the following one:

The diameters of the complex that are parallel to any two associated diameters of the characteristic plane lie in two parallel planes that have the same distance from a *fixed plane*. We would like to call this plane the *central plane* of the given complex.

The coordinates of the center of the complex in the central plane are no longer infinitely large; their values take the form 0 / 0. The center no longer lies at infinity. *Any point of the central axis can be regarded as the center of the complex.*

286. For the complexes of the special kind that we consider, as in the general case of hyperbolic and elliptic complexes, lines at infinity will lie in all planes that are parallel to one of the two asymptotes of the characteristic curve, and the complex curves in them will be parabolas. However, in planes that are parallel to the central plane – and thus, both asymptotes – the complex curves will be represented by the equation:

(*) Instead of letting σ drop out, as we did in the text, we can also choose η by taking the *XY* coordinate plane to be the plane of the characteristic curve. The equation of the complex is then written immediately as the general second-degree equation in the four variables r, s, σ, ρ that we encounter when we determine the straight line by its projections onto *XZ* and *YZ*. Instead of the previous constants *K*, *P*, *Q*, we can let *M*, *T*, *U* vanish here, and obtain the relation:

$$DP = ES$$

by a suitable displacement of the *XY* coordinate plane.

$$(Fx^2 - 2Rx + B)\, v^2 - 2\,(Ox + G)\, uv + (Ex^2 + 2Ux + C)\, v^2 = 0, \tag{111}$$

due to the vanishing of S and T. They will cease to be parabolas and will degenerate into systems of two points that lie at infinity in directions that change from one plane to another.

The lines of the complex on one of the planes that are parallel to the central plane will then consist of all lines in the plane that are parallel to two given ones. These lines can be real or imaginary, and they can lie at infinity. If the plane moves even further from the central plane then the directions of the two line systems will always approach the directions of the two asymptotes of the characteristic curve more closely.

To summarize, the complex is then specialized by the fact that any point of a straight line in the plane at infinity is the center of a complex cone that resolves into the system of two planes that intersect in the line in question, or – what amounts to the same thing – that any plane that can be drawn through a distinguished straight line in the plane at infinity will contain a complex curve that resolves into the system of two points that lie upon the straight line in question.

287. In the foregoing, we have discussed the case in which the complex that is represented by the general second-degree equation is specialized in relation to its lines at infinity as a result of the fact that the expression:

$$D\sigma^2 + E\rho^2 + F\eta^2 + 2K\rho\eta - 2L\sigma\eta - 2M\rho\sigma$$

resolves into two linear factors. We would now like to consider a new specialization of the complex, by which, the same expression will be *the square of a linear function*, which would correspond to the fact that one simultaneously has:

$$K^2 - EF = 0, \qquad L^2 - DF = 0, \qquad M^2 - DE = 0. \tag{112}$$

It will then come down to the fact that in the associated determination of the directions of the coordinate axes, two of the three constants D, E, F will vanish along with K, L, M. If E and F are the two vanishing constants then the equation of the complex will be the following one:

$$\begin{aligned} Ar^2 + Bs^2 + C + 2Gs + 2Hr + 2Irs \\ + D\sigma^2 \\ - 2Nrs + 2Os\rho \\ + 2Pr\rho + 2Qr\eta + 2Rs\eta - 2Ss\sigma - 2T\sigma + 2U\rho = 0. \end{aligned} \tag{113}$$

288. Equations (2) of number **234** will give the following equations for the determination of the diameter of the complex that is associated with the given plane:

$$tx + uy + vz + w = 0,$$

namely:

$$\left.\begin{aligned} x &= -\frac{w}{t} + x' = -\frac{w}{t} + \frac{Ouv + Rv^2 + Stv - Ttu - Uu^2}{Dt^2}, \\ y &= y' = \frac{-Ntv - Puv - Qv^2 + Tt^2 + Utu}{Dt^2}, \\ z &= z' = \frac{(N-O)tu + Pu^2 + Quv - Rtv - St^2}{Dt^2}. \end{aligned}\right\} \tag{114}$$

All diameters of the complex are parallel to the OX axis. The direction of the *OX* axis is then given by the complex. The *sixteen* constants of the complex that is specialized by the three conditions (112) are found in the *fourteen* constants of its equation (113) and the *two* constants by which we have determined the direction of the aforementioned axis. With no loss of generality, we can then take the coordinate system to which the complex is referred in equation (113) to be a rectangular one.

For the determination of the three cylinder axes that are parallel to the three coordinate axes *OX*, *OY*, *OZ*, respectively, we will get:

$$\left.\begin{aligned} &y = \infty, \qquad z = \infty, \\ &x = \infty, \qquad z = -\frac{S}{D}, \\ &x = \infty, \qquad y = \frac{T}{D} \end{aligned}\right\} \tag{115}$$

from equations (18). *All cylinder axes of the complex are shifted to infinity.*

We can eliminate the two terms in equation (113) that are endowed with $s\sigma$ and σ by a parallel displacement of the *OX* axis. We then choose the center of the complex curve that lies in the *YZ* plane to be the coordinate origin, which will be represented by the following two equations in the case of equation (113):

$$y = \frac{T}{D}, \qquad z = -\frac{S}{D}.$$

The *OX* axis will then become the diameter of the complex that is cut by the two axes of the cylinders that are parallel to *OY* and *OZ*, and are shifted to infinity along *OX*. Of the edges of the central parallelepiped that is determined by the directions of the three coordinate axes in the complex, only one of them will remain at infinity. Corresponding to that, the coordinates of the center of the complex, as we have determined then using equations (21), will all be infinitely large. The quotient of any two of them will take the form 0 / 0. *The center of the central parallelepiped is shifted to infinity in an undetermined direction.*

289. For an arbitrary plane that contains the *OX* axis, and is therefore parallel to all of the diameters of the complex that lie at infinity, the coordinates x', y', z' (114) will take on infinitely large values when t vanishes; however, their ratios will remain finite. The complex curve in any such plane will be a parabola, and the direction of the diameter of that parabola will be indicated by that finite ratio. From (114), we find this direction to be:

$$x' : y' : z' = (Ouv + Rv^2 - Uu^2) : -v\,(Pu + Qv) : u\,(Pu + Qv),$$

and when we set:

$$\frac{u}{v} = -\frac{z'}{y'}$$

and drop the prime, that will give us:

$$x\,(Pz - Qy) = Ry^2 - Oyz - Uz^2. \tag{116}$$

This equation will represent a second-order conic surface whose vertex falls upon the coordinate origin, and which will contain the *OX* axis as one side. Those two sides along which the conic surface is cut by an arbitrary plane that is drawn through the *OX* axis will give the direction in which the vertex is shifted to infinity in the chosen plane. This direction will remain unchanged when the chosen plane is displaced parallel to itself. From the transformation formulas of number **158**, the coefficients O, P, Q, R, U that enter into the foregoing equation will remain unchanged under a displacement of the coordinate system as long as the constants E, F, K, L, M vanish, as in the special case that we are considering. The equatorial surfaces of the complex whose breadth curves are parallel to the *OX* axis will then be specialized in such a way that their breath curves (which will be parabolas) will possess the same direction for their diameters. The common direction of the diameters of all parabolas will be given by equation (116).

In the case of the elliptic and hyperbolic complexes, there is an equatorial surface that was specialized in that way; viz., the one whose breadth planes were parallel to the plane of the characteristic curve. The axis direction that is common to all breadth curves in that parabolic equatorial surface is indicated by the center of the complex that lies in the plane at infinity. Corresponding to that, we will get infinitely many directions along which the center of the complex is shifted to infinity for the complexes of the special kind that we are considering, and this infinitude of directions will be indicated by equation (116).

The center of the complexes of the special kind that we are considering will be undetermined. The geometric locus of them is one of the second-order curves that lie in the plane at infinity.

290. The complex curves in all planes that are parallel to *OX* are parabolas in the case that we are considering. Consistent with that, from equations (115), all complex cylinders will be parabolic cylinders whose diametral planes are parallel to the *OX* axis. All lines that lie in the plane at infinity and cut the *OX* axis will belong to the complex. We can say that the curve that is enveloped by the lines of the complex in the plane at infinity *has resolved into a system of two points that coincide at infinity along the OX axis.* We would like to call such a degenerate complex that corresponds to the previous relationship a *parabolic* complex.

The cylinder whose side is parallel to the common direction of all diameters of the complex resolves into a system of two planes, one of which is at infinity, and as in the case of hyperbolic and elliptic complexes, if the cylinder whose sides indicate the direction in which the center of the complex was shifted to infinity decomposes into two planes that are parallel to the plane of the characteristic curve then a cylinder in a parabolic complex whose sides possess any direction in which the midpoint of the complex is shifted to infinity will resolve into a system of two planes that are parallel to *OX*. We will find the analytical confirmation of this assertion in equation (27) of number **182**, which determines those cylinders whose sides are parallel to the *YZ* plane – which is an arbitrarily-chosen plane that has no distinguished relationship to the complex whatsoever – by its intersection with *XZ*. This equation is the following one:

$$Dy'^2 \cdot z^2 - 2\,(Ry'^2 - Oy'z' - Uz'^2)\,x + (By'^2 + 2Gy'z' + Cz'^2) = 0.$$

The assumption corresponds to the fact that:

$$Ry'^2 - Oy'z' - Uz'^2 = 0;$$

that is, that the sides of the complex cylinder have the direction of one of the two straight lines along which the conic surface (116) is cut by the *YZ* plane, so it will decompose into two linear factors in which x no longer occurs, and will thus represent two planes that are parallel to *OX*.

291. If we neglect first powers of the variables ρ, σ, η, as well as r, s, and constants, in the complex equation (113) when compared with second powers of ρ, σ, η then we will find that the lines of the complex at infinity can be represented by:

$$D\sigma^2 = 0. \tag{117}$$

All lines that cut the *OX* coordinate axis will belong to the previous complex. The two asymptotes of the characteristic curve for hyperbolic and elliptic complexes will then coincide in a single straight line for parabolic complexes.

However, we can represent lines of the complex at infinity to a higher degree of approximation than is possible in the foregoing equation when we keep the first powers of ρ and η, along with the second power of σ. The resultant equation:

$$D\sigma^2 - 2N\eta + 2\,(O - N)\,s\rho + 2\,(Pr + U)\,\rho + 2\,(Qr + Rs)\,\eta = 0 \tag{118}$$

will represent a new complex that we would like to call the *asymptotic complex of the given one.* Since we might also displace the given complex and its asymptotic complex with respect to each other, *their reciprocal relationship to each other will remain the same.* According to the rules in number **157**, the coefficients D, N, O, P, Q, R, U will then keep the same values under a displacement of the coordinate system parallel to itself in the case that we are considering.

The asymptotic complex that is represented by equation (118) will subsume all lines that go through the coordinate origin. When we let the constants B, C, E, F, G, K, L, M, S, T vanish, we will obtain the following equation from number **165** for the equation of its equatorial surface whose breadth planes are parallel to the YZ coordinate plane by solving for the factor x:

$$2Dry^2 - 2DOyz + 2DUz^2 + O2x - 4RUx = 0. \tag{119}$$

This equation has degree two and represents a paraboloid that contacts the YZ plane at the coordinate origin, and whose diameter is parallel to OX. The reduction of the fourth-degree equation for the general equatorial surface to degree two comes about here, in agreement with the developments of number **258**, as a result of the fact that the equatorial surface splits into two planes, in which two points that coalesce into one will be enveloped by the lines of the complex. In the present case, they will be the YZ coordinate plane and the plane at infinity.

From number **169**, we get the following equation in mixed coordinates for the meridian surface that has OX for the double lines:

$$(R \tan^2 \varphi - O \tan \varphi - U)\, tw - (Q \tan \varphi - P)\, vw = 0. \tag{120}$$

In an arbitrary meridian plane, the curve will then resolve into a system of two points, one of which coincides with the coordinate origin, and the other of which is shifted to infinity in the direction that is indicated by the equation:

$$(R \tan^2 \varphi - O \tan \varphi - U)\, t - (Q \tan \varphi - P)\, v = 0. \tag{121}$$

The cylinders of the complex whose sides possess that direction will resolve into a system of two planes that are parallel to the plane that is determined by the value of $\tan \varphi$.

292. We obtain one last specialization of the complex when we let six constants from the group:

$$D, E, F, K, L, M$$

vanish at the same time. The general equation of the complex will then contain only *thirteen* mutually-independent constants.

In order to represent lines of the complex – thus-specialized – that belong to the (absolute) plane at infinity, we obtain the identity:

$$0 = 0.$$

In complexes of the special kind that we consider, *any straight line that lies in the plane at infinity will belong to the complex.* The complex curve in an arbitrary plane is a parabola. All of the complex cylinders decompose into systems of two planes and reduce to first degree when one shifts one of them to infinity. We say nothing further about central parallelepipeds of complexes. *The complex has lost its center.*

We refer to the complex whose equation is derived from the given complex by neglecting the variables r, s, and constants in comparison to the first powers of ρ, σ, η as the *asymptotic complex*. We thus obtain:

$$-2Nr\sigma + 2Os\rho + 2Pr\rho + 2Qr\eta + 2Rs\eta - 2Ss\sigma - 2T\sigma + 2U\rho = 0. \tag{122}$$

As a result of the form of this equation, the relationship of the asymptotic complex to the given one will not change when one displaces it parallel to itself through a finite segment.

We might next remark that the asymptotic complex in whose equation the constants:

$$A, B, C, G, H, I,$$

as well as the constants:

$$D, E, F, K, L, M,$$

are missing is specialized with respect to the origin in a manner that is analogous to the way that it is specialized with respect to the plane at infinity. All lines that lie at infinity, as well as all lines that go through the coordinate origin, belong to the asymptotic complex.

As in the case of the given complex, all of the cylinder surfaces that are defined by lines of the asymptotic complex will degenerate into systems of planes, one of which is shifted to infinity. However, a new specialization appears, in that the other plane will go through the coordinate origin in every case. Whereas a parabola in an arbitrary plane in space will be enveloped by lines of the complex, the complex curve in any plane that goes through the coordinate origin will split into a system of two points, one of which will coincide with the coordinate origin, while the other of which will be shifted to infinity. As a result of this, any equatorial surface of the complex will degenerate into a cone of order two whose vertex will fall upon the coordinate origin and that will be cut by the associated breadth planes in parabolas. In particular, any breadth plane that goes through the coordinate origin will contact the conic surface along a side that points in the direction in which one of the points into which the complex curve has resolved in the plane in question has been shifted to infinity.

293. In the foregoing, we have discussed the position of the straight lines at infinity and the behavior of the corresponding diameters for second-degree complexes, and illustrated this, in particular, by means of a simpler second-degree complex that we referred to as the "asymptotic complex." In summary, we have thus arrived at a *sixteen-fold* distinction between second-degree complexes.

In *hyperbolic* complexes, the lines of the complex at infinity will envelop a real curve of class two, and in *elliptic* ones, the curve will be imaginary. In the case of *hyperbolic* complexes, this curve will resolve into a system of two real points, and in the case of *elliptic* complexes, the points will be imaginary. If these two points coincide then the complex will be *parabolic*. Finally, the case can come about, in which *all* of the straight lines that belong to the plane at infinity are lines of the complex.

§ 4.

Tangential and polar complexes of degree one.

294. The results that were obtained in the foregoing can be generalized with no further assumptions when we carry over all of the considerations that we previously posed for the plane at infinity to *an arbitrary plane* and to *an arbitrary point*, according to the rules of the principle of reciprocity. However, we can propose a series of other arguments that are intended to extend the theorems of the foregoing paragraphs and bring them under a more general viewpoint.

Let Ω_n be a homogeneous function of degree n in arbitrarily many variables $p, q, r, \ldots$ In accordance with the known theorems of homogeneous functions, we will then get:

$$\frac{\delta\Omega_n}{\delta p}\cdot p+\frac{\delta\Omega_n}{\delta q}\cdot q+\frac{\delta\Omega_n}{\delta r}\cdot r+\ldots \equiv n\cdot\Omega_n . \tag{123}$$

We can thus also write the equation:

$$\Omega_n = 0 \tag{124}$$

in the following way:

$$\frac{\delta\Omega_n}{\delta p}\cdot p+\frac{\delta\Omega_n}{\delta q}\cdot q+\frac{\delta\Omega_n}{\delta r}\cdot r+\ldots \equiv 0. \tag{125}$$

Thus, if $p', q', r', \ldots$ are given values that satisfy equation (124) then these values will satisfy equation (125). The partial differential quotients that enter into this equation and are generally homogeneous functions of degree $n - 1$ will then take on constant values that we would like to enclose in parentheses below, in order to distinguish them. If we go from the given values $p', q', r', \ldots$ to neighboring ones then we will find from (124) that:

$$\left(\frac{\delta\Omega_n}{\delta p}\right)dp+\left(\frac{\delta\Omega_n}{\delta q}\right)dq+\left(\frac{\delta\Omega_n}{\delta r}\right)dr+\ldots = 0. \tag{126}$$

The following equation:

$$\left(\frac{\delta\Omega_n}{\delta p}\right)p+\left(\frac{\delta\Omega_n}{\delta q}\right)q+\left(\frac{\delta\Omega_n}{\delta r}\right)r+\ldots \equiv \Pi = 0, \tag{127}$$

in which the bracketed differential quotients have the meaning that was just given to them, is an equation of degree one in the variables $p, q, r, \ldots$ The given values $p', q', r', \ldots$ satisfy the foregoing equation, just as they satisfy equation (124), which has degree n. If we then write the latter equation in the form (125) then we will get, in agreement with both equations:

$$\left(\frac{\delta\Omega_n}{\delta p}\right)p'+\left(\frac{\delta\Omega_n}{\delta q}\right)q'+\left(\frac{\delta\Omega_n}{\delta r}\right)r'+\ldots = 0.$$

However, when we go from the given values p', q', r', … to neighboring ones, equation (127) will give us the same equation (126) that gave us the n^{th} degree equation (124) above. *Corresponding to that, we would like to call* Π *a linear tangential function of the given homogeneous function* Ω_n *of degree n.*

If we assume that the constant values of p', q', r', … are completely arbitrary, instead of assuming that they satisfy the given function Ω_n, then the form of the function Π will not be changed in any way. In this general case, we would like to call Π *a linear polar function of the given function* Ω_n. A polar function will go to a tangential function by the assumption above.

In particular, when $n = 2$, the differential quotients of Ω_n will be functions of degree one in the variables. We can then exchange the variable quantities p, q, r, … with their constant values p', q', r', … in the polar function Π without changing anything about in the function. Consistent with that, we can write equation (127) in the following two ways:

$$\left(\frac{\delta\Omega_2}{\delta p}\right) p + \left(\frac{\delta\Omega_2}{\delta q}\right) q + \left(\frac{\delta\Omega_2}{\delta r}\right) r + \ldots = 0, \tag{128}$$

$$p'\frac{\delta\Omega_2}{\delta p} + q'\frac{\delta\Omega_2}{\delta q} + r'\frac{\delta\Omega_2}{\delta r} + \ldots = 0. \tag{129}$$

The foregoing carries over immediately to the more general case of *inhomogeneous* functions. To that end, we can make the inhomogeneous function homogeneous by the introduction of new variables, derive the polar function for the function that has been made homogeneous, which will be a homogeneous function of degree one, and set the variables that have been introduced into it, along with their constant values, equal to unity.

If the given variables p, q, r, … are not mutually independent, but have to satisfy arbitrarily many (m) condition equations:

$$\Phi = 0, \;\; \Phi' = 0, \;\ldots \tag{130}$$

(whose degree we would like to make the same as that of Ω_n, for the sake of simplicity) then the foregoing considerations will be modified. The same values of the variables p, q¸ r, …that must satisfy the equation:

$$\Omega_n = 0$$

will each satisfy an equation of the following form:

$$\Omega_n + \lambda\Phi + \lambda'\,\Phi' + \ldots = 0, \tag{131}$$

where λ, λ', … mean undetermined constants. We will obtain *a polar function that is linear with respect to any equation of that form* that corresponds to a given system of values for the variables.

These polar functions will represent linear equations when they are set equal to zero. They will also be satisfied by the values of the variables *p*, *q*, *r*, … that satisfy the following $m + 1$ equations:

$$\left.\begin{array}{l}\left(\frac{\delta\Omega_n}{\delta p}\right)p+\left(\frac{\delta\Omega_n}{\delta q}\right)q+\left(\frac{\delta\Omega_n}{\delta r}\right)r+\cdots=0,\\ \left(\frac{\delta\Phi}{\delta p}\right)p+\left(\frac{\delta\Phi}{\delta q}\right)q+\left(\frac{\delta\Phi}{\delta r}\right)r+\cdots=0,\\ \left(\frac{\delta\Phi'}{\delta p}\right)p+\left(\frac{\delta\Phi'}{\delta q}\right)q+\left(\frac{\delta\Phi'}{\delta r}\right)r+\cdots=0,\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{array}\right\} \tag{132}$$

The infinitude (viz., ∞^n) of linear polar functions that correspond to a given system of values of the variables *p*, *q*, *r*, … define *an* ($m + 1$)*-parameter group* (*).

We can choose any linear polar function from the *m*-fold infinitude of them, corresponding to an arbitrary choice of λ, λ', … In particular, if $n = 2$ then the variables in it can be exchanged with the corresponding differential quotients without changing the form of the polar function, as in the case of independent variables. However, whereas in the case of independent variables the *one* linear polar function that it gave has an exclusive relationship to the system of given values for the variables and to the given equation, now, any arbitrarily-chosen linear polar function will be as good as any other one. We can say that the given constant values p', q', r', … are not associated with any individual polar function as they are with the *m*-fold infinite family of all polar functions.

295. If we restrict ourselves to three variables then we will have:

$$\Omega_n = f(p, q, r),$$

and we will get:

$$\Pi = \left(\frac{\delta\Omega_n}{\delta p}\right)p+\left(\frac{\delta\Omega_n}{\delta q}\right)q+\left(\frac{\delta\Omega_n}{\delta r}\right)r\,.$$

If we give the variables the meaning of point coordinates in the plane then p', q', r' will determine a point, and the homogeneous equation:

$$\Omega_n = 0 \tag{124}$$

will represent a curve of order *n*, while:

(*) We thus ignore the case in which one finds linear Φ among the condition equations. The corresponding equation (132) will be satisfied with this assumption, anyway, since the condition equations themselves will not differ.

$$\Pi \equiv \left(\frac{\delta\Omega_n}{\delta p}\right) p + \left(\frac{\delta\Omega_n}{\delta q}\right) q + \left(\frac{\delta\Omega_n}{\delta r}\right) r = 0 \tag{133}$$

will represent the equation of the polars of the given point relative to the curve, and in particular, when the point lies upon the curve, it will represent the equation of the tangent to the curve at that point.

The principle of reciprocity that relates to second-order curves rests upon the two-fold form that the latter equation will assume in the case of $n = 2$.

If we give the three variables the meaning of line coordinates in the plane then a straight line will be determined by three constant values of them, and equation (124) will represent a curve of class n, while equation (133) will represent the pole of that straight line relative to that curve; in particular, when a straight line is a tangent to the curve, it will represent its contact point.

The remarks that were made in relation to curves of order two will be true for curves of class two.

296. In the case of four variables, let:

$$\Omega_n = f(p, q, r, s)$$

and

$$\Pi = \left(\frac{\delta\Omega_n}{\delta p}\right) p + \left(\frac{\delta\Omega_n}{\delta q}\right) q + \left(\frac{\delta\Omega_n}{\delta r}\right) r + \left(\frac{\delta\Omega_n}{\delta s}\right) s .$$

If we give the four variables the meaning of point coordinates in space then the equation:

$$\Omega_n = 0 \tag{124}$$

will represent a surface of order n, and:

$$\Pi = 0 \tag{134}$$

will be the equation of the polar plane to the point (p', q', r', s') relative to the surface; in particular, when the point lies upon the surface, it will represent the tangential plane to the surface at that point.

If p, q, r, s means plane coordinates then equation (124) will represent a surface of class n and (p', q', r', s') will refer to a given plane. (134) will then be the equation of the pole of that plane relative to the surface; in particular, when the plane contacts the surface, it will be the equation of the contact point.

The double form of equation (134) in the case of $n = 2$ includes the principle of reciprocity for surfaces of order two and surfaces of class two, which was first developed by *Gergonne* in an elegant way for curves and surfaces of order two.

We can also consider the four variables to be point or line coordinates in the plane, but a linear condition equation must exist between them in this case, and thus between their constant values, as well. Equation (124) will then, in turn, represent a curve of order n or class n, and equation (134) will represent the polar of the point (p', q', r', s') or the pole of the straight line (p', q', r', s'), respectively, with respect to the curve. Polars and poles will go to tangents and contact points when the given point lies upon the curve or the given straight line contacts the curve, respectively. We can add to the given equation of degree n, the linear condition equation that the variables p, q, r, s must satisfy, when it is multiplied by an arbitrary (homogeneous) function of degree $n - 1$. However, equation (134) for the polars (poles, resp.) will not be changed, insofar as the variables p, q, r, s, as well as their fixed values p', q', r', s', must satisfy the linear condition equation in question.

297. Finally, if:

$$\Omega_n = f(p, q, r, s, t, u)$$

then we will obtain:

$$\Pi = \left(\frac{\delta\Omega_n}{\delta p}\right)p + \left(\frac{\delta\Omega_n}{\delta q}\right)q + \left(\frac{\delta\Omega_n}{\delta r}\right)r + \left(\frac{\delta\Omega_n}{\delta s}\right)s + \left(\frac{\delta\Omega_n}{\delta t}\right)t + \left(\frac{\delta\Omega_n}{\delta u}\right)u .$$

We would like to give the variables the meaning of line coordinates, and indeed, we will first take them to be line coordinates:

$$(x - x'),\ (y - y'),\ (z - z'),\ (yz' - y'z),\ (x'z - xz'),\ (xy' - x'y)$$

and then axial coordinates:

$$(uv' - u'v),\ (t'v - tv'),\ (tu' - t'u),\ (t - t'),\ (u - u'),\ (v - v') .$$

The homogeneous equation:

$$\Omega_n = 0 \tag{124}$$

will represent the same complex of degree n with either choice, and when we refer the constant values p', q', r', s', t', u' that the partial differential quotients include to a straight line (whether a ray or axis), the equation:

$$\Pi = 0 \tag{135}$$

will represent a linear complex that we would like to call the *polar complex* of the given straight line (p', q', r', s', t', u') relative to the given complex of degree n. In particular, if the given straight line belongs to the complex itself then the polar complex will go to a *tangential complex*; that is, in a complex of degree one that contains the given straight line and all of the lines of the given complex that lie infinitely close to it.

298. The six coordinates of the straight line are not independent of each other, but must satisfy a second-degree equation that finds its expression in the identity:

$$(x - x')(yz' - y'z) + (y - y')(x'z - xz') + (z - z')(xy' - x'y) = 0.$$

Corresponding to that, from the discussion in number **294**, we will obtain a *two-parameter group of linear polar complexes* that all have the same relationship to the given straight line and the given complex. The two-parameter group of linear polar complexes that is associated with a given straight line will determine a *linear congruence*, about which, we can say, in particular, that it is associated with the given straight line relative to the complex of degree n.

In the sequel, as before, we would like to denote the six line coordinates in the foregoing sequence by:

$$r,\ s,\ h,\ -\sigma,\ \rho,\ \eta\ .$$

The condition equation that the line coordinate must satisfy will then be written in the following form:

$$-r\sigma + s\rho + h\eta = 0. \tag{136}$$

We assign the coordinates r', s', h', $-\sigma'$, ρ', η' to the given straight line.

Without changing the given complex of degree n:

$$\Omega_n = 0,$$

we can add that equation to equation (136), when it has been multiplied by a homogeneous function of degree $n - 2$. We can then add a term $2V\eta$ to the general equation (I) of the second-degree complex at will. With no loss of generality, we would like to denote the arbitrary function of degree $n - 2$ by λ and consider it to be constant in the definition of the polar function. The terms in the polar function that we therefore neglect will then appear to be multiplied by the factor $(-r'\sigma' + s'\rho' + h'\eta')$, and that factor will be equal to zero, since the coordinates r', s', h', $-\sigma'$, ρ', η' of the chosen straight line must satisfy equation (136).

We can thus take the equation of the given complex to be the following one:

$$\Omega_n + \lambda(-r\sigma + s\rho + h\eta) = 0. \tag{137}$$

The equation of the polar complex will then become:

$$\Pi + \lambda(-r\sigma' + s\rho' + h\eta' - r'\sigma + s'\rho + h'\eta) = 0, \tag{138}$$

where Π denotes the function:

$$\Pi = \left(\frac{\delta\Omega_n}{\delta r}\right)r + \left(\frac{\delta\Omega_n}{\delta s}\right)s + \left(\frac{\delta\Omega_n}{\delta h}\right)h + \left(-\frac{\delta\Omega_n}{\delta\sigma}\right)\sigma + \left(\frac{\delta\Omega_n}{\delta\rho}\right)\rho + \left(\frac{\delta\Omega_n}{\delta\eta}\right)\eta\ .$$

Each value of λ will correspond to a different polar complex.

299. Of the two directrices of the congruence that is determined by the two-parameter group of the polar complex, one of them will coincide with the given straight line. When we take λ to be infinitely large, equation (138) will become:

$$-r\sigma' + s\rho' + h\eta' - r'\sigma + s'\rho + h'\eta = 0, \tag{139}$$

and, from the discussion of number **45**, this equation will represent a linear complex that subsumes all lines that cut the given straight line. In connection with the considerations of number **71**, we can expression this theorem as follows:

A given straight line corresponds to the same straight line as its conjugate polar relative to the two-parameter group of its associated linear polar complexes.

This latter straight line is the second directrix of the congruence that is determined by the polar complex. We say that this straight line is *associated* with the given one relative to the complex of degree *n*, and call it *the polar of the given straight line relative to the complex of degree n* (*).

We can choose the undetermined constant λ in equation (138) in such a way that the equation represents a complex of degree one whose lines all cut a fixed straight line. To that end, we would like to write equation (138) in the following way:

$$\begin{aligned}&\left[\left(\frac{\delta\Omega_n}{\delta r}\right)-\lambda\sigma'\right]r+\left[\left(\frac{\delta\Omega_n}{\delta s}\right)-\lambda\rho'\right]r+\left[\left(\frac{\delta\Omega_n}{\delta h}\right)-\lambda\eta'\right]h\\ -&\left[\left(-\frac{\delta\Omega_n}{\delta\sigma}\right)+\lambda r'\right]\sigma+\left[\left(\frac{\delta\Omega_n}{\delta\rho}\right)+\lambda s'\right]\rho+\left[\left(\frac{\delta\Omega_n}{\delta\eta}\right)+\lambda h'\right]\eta = 0.\end{aligned} \tag{140}$$

From number **45**, we would then obtain:

$$\begin{aligned}&\left[\left(\frac{\delta\Omega_n}{\delta r}\right)-\lambda\sigma'\right]\cdot\left[\left(-\frac{\delta\Omega_n}{\delta\sigma}\right)+\lambda r'\right]+\left[\left(\frac{\delta\Omega_n}{\delta s}\right)+\lambda\rho'\right]\cdot\left[\left(\frac{\delta\Omega_n}{\delta\rho}\right)+\lambda s'\right]\\ &+\left[\left(\frac{\delta\Omega_n}{\delta h}\right)+\lambda\eta'\right]\cdot\left[\left(\frac{\delta\Omega_n}{\delta\eta}\right)+\lambda h'\right] = 0\end{aligned} \tag{141}$$

for the determination of λ. As a result of equation (136), a root of the foregoing equation will be infinitely large, which corresponds to the fact that one directrix of the congruence that is determined by the two-parameter group (138) will coincide with the given straight line. Equation (141) will then reduce to degree one, and if we set:

(*) Here, we might just as well remark that a straight line and its polar do not have the same reciprocal relationship to each other. The polar of the given straight line corresponds to a new straight line as the polar that is associated with it, etc. There are only a finite number of straight lines that are the polars of their own polars.

$$-\frac{\delta\Omega_n}{\delta r}\cdot\frac{\delta\Omega_n}{\delta\sigma}+\frac{\delta\Omega_n}{\delta s}\cdot\frac{\delta\Omega_n}{\delta\rho}+\frac{\delta\Omega_n}{\delta h}\cdot\frac{\delta\Omega_n}{\delta\eta}=\Phi,$$

for the sake of brevity, that will give:

$$\lambda = -\frac{1}{2}\left(\frac{\Phi}{\Omega_n}\right) \tag{142}$$

for the determination of the second directrix, which we have referred to as the polar of the given straight line. In this last expression, the values of the coordinates of the given straight line are substituted in Φ and Ω_n, and we have employed the parentheses for that reason.

300. If the given straight line belongs to the given complex Ω_n, in particular, then we will get a two-parameter group of *tangential complexes*, in place of the two-parameter group of polar complexes.

The two directrices of the congruence that is determined by them coincide with the given straight line. Since Ω_n will vanish for the coordinates of the given straight line, the value of λ, as we have determined it by means of equation (142), will then become infinitely large. The congruence has been specialized in such a way that it will subsume all of the lines of a linear complex that cut a fixed straight line that itself belongs to the complex (cf., no. **68**). The fixed straight line will be the given one (r', s', h', $-\sigma'$, ρ', η').

Only in the special case in which the given straight line belongs to the following complex:

$$\Phi \equiv -\frac{\delta\Omega_n}{\delta r}\cdot\frac{\delta\Omega_n}{\delta\sigma}+\frac{\delta\Omega_n}{\delta s}\cdot\frac{\delta\Omega_n}{\delta\rho}+\frac{\delta\Omega_n}{\delta h}\cdot\frac{\delta\Omega_n}{\delta\eta}, \tag{143}$$

along with the given complex Ω_n, will the value of λ that is given by (142) be indeterminate. Since Ω, as well as Φ, vanishes, λ will take the form 0 / 0. For each arbitrary choice of λ, we will obtain a tangential complex whose lines all cut a fixed straight line. If we choose λ to be infinitely large then that straight line will coincide with the given one. The given straight remains, as before, one of the directices of the congruence that is determined by the two-parameter group of tangential planes. From the discussion in number **68**, this congruence will have been specialized in such a way that it will possess infinitely many directions that lie in a plane and go through a point in it. All lines that lie in the plane that is determined by the directices or go through their point of intersection will belong to the congruence.

We would like to refer to those straight lines that belong to the given complex of degree n:

$$\Omega_n = 0,$$

as well as the complex of degree 2 (n – 1) that is derived from them:

$$\Phi \equiv -\frac{\delta\Omega_n}{\delta r}\cdot\frac{\delta\Omega_n}{\delta\sigma}+\frac{\delta\Omega_n}{\delta s}\cdot\frac{\delta\Omega_n}{\delta\rho}+\frac{\delta\Omega_n}{\delta h}\cdot\frac{\delta\Omega_n}{\delta\eta} = 0, \tag{143}$$

as the *singular lines of the given complex*.

The singular lines of a complex of degree n define a congruence of order and class $2n \cdot (n-1)$.

From the foregoing discussion, each singular line corresponds to a plane and a point in a distinguished way. We would like to call that plane *a singular plane* and the point, *a singular point* of the complex, and refer to them as being associated with or corresponding to the chosen singular line.

One last case still remains to be considered. If:

$$r' : s' : h' : -\sigma' : \rho' : \eta'$$

$$= \left(-\frac{\delta\Omega_n}{\delta\sigma}\right):\left(\frac{\delta\Omega_n}{\delta\rho}\right):\left(\frac{\delta\Omega_n}{\delta\eta}\right):\left(\frac{\delta\Omega_n}{\delta r}\right):\left(\frac{\delta\Omega_n}{\delta s}\right):\left(\frac{\delta\Omega_n}{\delta h}\right)$$

then the polar complex of the given straight line will represent the totality of all of those straight lines that cut the given one independently of the special values that we might assign to λ. The polar complexes will be identical to each other and will no longer determine a linear congruence. Each arbitrary straight line will be regarded as the polar of the given straight line.

We would like to call the given straight line a *double line* of the complex.

Whereas two conditions must be fulfilled in order for a given straight line to be a singular line of the complex, and there will then be a congruence of singular lines in a given complex, there are *five* conditions that must be satisfied in order for a given straight line to be a double line of the complex. Since a straight line depends upon four constants, a given complex will contain no double lines, in general. *One* specialization of it will then be necessary.

301. We restrict ourselves to the complexes of degree two in what follows.

Let the general equation of the complex in ray coordinates be:

$$\begin{gathered} Ar^2 + Bs^2 + C + Ds^2 + Er^2 + Fh^2 \\ + 2Gsh + 2Hrh + 2Irs + 2K\rho\eta - 2L\sigma\eta - 2M\rho\sigma \\ - 2Nr\sigma + 2Os\rho + 2Vh\eta \\ + 2Pr\rho + 2Qr\eta + 2Rs\eta - 2Ss\sigma - 2Th\sigma + 2Uh\rho = 0. \end{gathered} \tag{V}$$

We then obtain the equation:

$$\begin{gathered} (Ar' + Hh' + Is' - N\sigma' + P\rho' + Q\eta')\, r \\ + (Bs' + Gh' + Ir' + O\rho' + R\eta' - S\sigma')\, s \\ + (Ch' + Gs' + Hr' + V'\eta - T\,\sigma' + U\rho')\, h \\ - (-D\sigma' + L\eta' + M\rho' + Nr' + Ss' + Th')\, \sigma \end{gathered}$$

$$
\begin{aligned}
&+ (E\rho' + K\eta' - M\sigma' + Os' + Pr' + Uh')\,\rho \\
&+ (F\eta' + K\rho' - L\sigma' + Vh' + Qr' + Rs')\,\eta = 0
\end{aligned}
\tag{144}
$$

for the equation of the polar complex of a given straight line (r', s', h', $-\sigma'$, ρ', η') in ray coordinates. We can set h and h' equal to unity in the foregoing equation arbitrarily.

If we start with the equation of the complex in axial coordinates (III) and determine the given straight line by its axial coordinates (p', q', l', $-\kappa'$, π', ω') then we will get:

$$
\begin{aligned}
&(Dp' + Ll' + Mq' - N\kappa' + S\pi' + T\omega')\,p \\
&+ (Eq' + Kl' + Mp' + O\pi' - P\kappa' + U\omega')\,q \\
&+ (Fl' + Kq + Lp' + V\omega' - Q\pi' + R\pi')\,l \\
&- (-Ak' + Hw' + Ip' + Np' + Pq' + Ql')\,\kappa \\
&+ (B\pi' + G\omega' - I\kappa' + Oq' + Rl' + Sp')\,\pi \\
&+ (C\omega' + G\pi' - H\kappa' + Vl' + Tp' + Uq')\,\omega = 0
\end{aligned}
\tag{145}
$$

for the equation of that complex. Thus, one has:

$$r' : s' : h' : -\sigma' : \rho' : \eta' = -\kappa' : \pi' : \omega' : p' : q' : l'.$$

302. In particular, if we set:

$$
\begin{gathered}
s', h', \rho', \sigma', \eta', \\
r', h', \rho', \sigma', \eta', \\
r', s', \rho', \sigma', \eta'
\end{gathered}
$$

equal to zero, respectively, in the general equation of the polar complex (144) then the three resulting equations:

$$
\left.
\begin{aligned}
Ar + Hh + Is - N\sigma + P\rho + Q\eta = 0, \\
Bs + Gh + Ir + O\rho + R\eta - S\sigma = 0, \\
Ch + Gs + Hr + V\eta - T\sigma + U\rho = 0
\end{aligned}
\right\}
\tag{146}
$$

will represent the polar complexes of the three coordinate axes OX, OY, OZ, resp. We can write these equations in the following forms:

$$\frac{\delta\Omega_2}{\delta r} = 0, \qquad \frac{\delta\Omega_2}{\delta s} = 0, \qquad \frac{\delta\Omega_2}{\delta h} = 0, \tag{146b}$$

resp., which will be deduced immediately when we go back to equation (144).

If we take one of the three straight lines that lie at infinity in the YZ, XZ, XY planes to be the given one then:

$$
\begin{gathered}
r', s', h', \rho', \eta', \\
r', s', h', \sigma', \eta', \\
r', s', h', \sigma', \rho'
\end{gathered}
$$

will vanish, respectively. We then get:

$$\left.\begin{array}{r} -D\sigma + L\eta + M\rho + Nr + Ss + Th = 0, \\ E\rho + K\eta - M\sigma + Os + Pr + Uh = 0, \\ F\eta + K\rho - L\sigma + Vh + Qr + Rs = 0 \end{array}\right\} \tag{147}$$

for the polar complexes of these three lines, or, when written in another form:

$$\frac{\delta\Omega_2}{\delta\sigma} = 0, \qquad \frac{\delta\Omega_2}{\delta\rho} = 0, \qquad \frac{\delta\Omega_2}{\delta\eta} = 0, \tag{148}$$

resp.

303. If we set r, ρ, and as a result, η, as well, equal to zero in the general equation (V) for the second-degree complex, which we would like to write in the following way:

$$\Omega_2 = 0$$

then we will find:

$$Bs^2 + Ch^2 + D\sigma^2 + 2Gsh - 2s\sigma - 2Th\sigma \equiv \Omega_2^0 = 0 \tag{149}$$

for the determination of the complex curve in YZ. We will find:

$$\frac{1}{2}\frac{d\Omega_2^0}{dh} \equiv Ch + Gs - T\sigma = 0 \tag{150}$$

for the equation of the pole of this complex curve relative to OZ, in a known way. On the other hand, the equation of the polar complex to the OZ axis will be:

$$\frac{1}{2}\frac{d\Omega_2}{dh} \equiv Ch + Gs - T\sigma + U\rho + V\eta = 0.$$

r, ρ, and η, are, in turn, equal to zero for all lines of the polar complex that lie in YZ, so the following equation:

$$Ch + Gs - T\sigma = 0 \tag{150}$$

will represent the point at which these lines intersect.

The pole of the OZ axis relative to the complex curve in YZ coincides with the points at which all lines of the polar complex that lie in YZ intersect. This point of intersection describes a straight line when the YZ plane rotates around OZ. This line is then, at the same time, the geometric locus of the pole of OZ relative to the complex curves whose planes go through OZ. We then obtain the following theorem:

An arbitrary straight line corresponds to a meridian surface in the complex. The polar of this meridian surface coincides with the straight line that we have referred to as the polar of the given line relative to the complex (*).

In particular, a diameter of the complex will be the polar of the straight line at infinity in the parallel planes that are associated with it.

If we reduce the proof of the foregoing theorem to its simplest form then it will rest upon the fact that it is all the same whether we first set r, ρ, and η equal to zero in the function Ω_2 and then differentiate with respect to h or we first differentiate with respect to h and then set r, ρ, and η equal to zero after the differentiation. However, that is obvious.

304. The foregoing gives a geometric definition for the congruence of the polar complex that is associated with a given straight line.

A complex curve of class two lies in an arbitrary plane that is drawn through the given straight line. It will be cut by the given straight line in two points, in general. The tangents to the complex curve at these points will belong to the congruence in question.

An arbitrary point of the given straight line is the midpoint of a complex cone of order two. Two tangential planes to it can be drawn through the given straight line, in general. The two sides along which it will be contacted by those two planes are likewise lines of the congruence.

The congruence that is determined by the polar complex of a given straight line belongs to the lines of the given complex that cut a next line of that complex that lies with it in the plane that is drawn through the given straight line at a point of the given line.

When the given straight line is itself a line of the complex, it will be contacted by all complex curves that lie in the planes that are drawn through it, and will be a common side of all complex cones whose vertices are chosen to lie along it. The polar will then coincide with the given straight line. All lines that lie in a plane that is drawn through the given straight line and go through the contact point of the relevant complex curve with the given straight line, or – what amounts to the same thing – all lines that go through a point of the given straight line and are contained in the plane by which the relevant complex cone is contacted by the given straight line, *will belong to the congruence that is determined by the tangential complex to the given straight line.*

The congruence in question has all lines *that are the next line to the given one and cut it* in common with the given second-degree complex.

305. When the given straight line is a *singular line* of the complex, the congruence that is determined by the tangential complex will subsume all lines that lie in a certain plane that is drawn through the given straight line, as well as all such lines that go

(*) This theorem can be carried over immediately from complexes of second degree to complexes of arbitrary degree.

through a well-defined point of it. We have called the plane and the point the associated singular plane and the associated singular point, respectively.

A singular line of the complex will thus be contacted at a *fixed* point by all complex curves that lie in the planes that go through it, and all complex cones whose midpoints were chosen to lie along a singular line of the complex will contact a *fixed* plane that goes through it.

We can confirm this result analytically. If we demand that the OX coordinate axis must be a singular line of the given complex then if we set the variables:

$$s, h, \sigma, \rho, \eta$$

equal to zero in the two equations:

$$\Omega_2 = 0, \qquad \Phi = 0,$$

we will obtain the following two relations between the constants of the given complex equation:

$$A = 0, \qquad PI + HQ = 0. \tag{151}$$

We have determined the contact point of OX with the complex curve in an arbitrary plane that goes through it by means of the following equation (no. **191**):

$$x_0 = \frac{I \tan \varphi + H}{Q \tan \varphi - P}, \tag{152}$$

under the assumption that OX is a line of the given complex, so the constant A would have the value zero. φ denotes the angle between the arbitrarily-chosen plane and the XY coordinate plane, and x_0 denotes the distance from the contact point to the origin of the coordinates. That distance will be constant, as long as the second of the conditions equations (151) is fulfilled.

Moreover, we have found that:

$$\tan \varphi_0 = \frac{Px + H}{Qx - I}, \tag{153}$$

under the same assumption that was used in number **192** for the determination of the tangential plane to an arbitrary complex cone that is laid through OX and has its vertex on OX, and also that the latter expression will take on a constant value when the second of equations (151) is fulfilled.

306. When the second of equations (151):

$$PI + HQ = 0$$

is fulfilled, equation (64) of number **191**, by which, we determined the planes that go through *OX* for which the complex curve will resolve into a system of two points, will possess the double root:

$$\tan\varphi = -\frac{H}{I} = \frac{P}{Q}. \tag{154}$$

This value of tan φ will correspondingly resolve the complex curve into a system of two points that both lie along the *OX* axis. The value of x_0 (152) that determined the contact point of the complex curve with *OX* will then take the form of 0 / 0 for the value (154) of tan φ.

Under the same assumption, equation (67) in number **192**, by which we determined the point on the *OX* axis for which the complex cone would resolve into a system of two planes, will have the double root:

$$x = -\frac{H}{P} = \frac{I}{Q}. \tag{155}$$

This value of x will correspondingly resolve the complex cone into a system of two planes that intersect along *OX*. The associated value of tan φ_0 (153) will then take the form 0 / 0.

This gives the following geometric definition of the singular lines, points, and planes of a second-degree complex.

The connecting line of two such points into which a complex curve will resolve for certain position of its plane, or – what amounts to the same thing – the *line of intersection* of two such planes into which a complex cone decomposes for a special choice of its midpoint, is a *singular line* of the complex. The planes and points for which the complex curves and complex cones, respectively, are specialized in the manner in question are *singular planes* and *singular points* of the complex, respectively.

In particular, the eight lines of a complex surface that we have referred to as its singular rays and singular axes (no. **187**, **189**), and the four singular planes and four singular points of a complex surface (no. **215**), are singular lines, planes, and points of the complex, respectively.

In the previous paragraphs (no. **275-283**), we have taken the plane at infinity to be a singular plane of the complex and chosen the singular line that corresponds to it to be parallel to *YZ*. In agreement with the foregoing, we found that the complex curves are parabolas in all planes that are parallel to *YZ* whose diameter direction is the same (no. **281**). The common direction of the diameters of all parabolas points to the singular point that is associated with the singular line at infinity in *YZ*.

307. If:

$$A, H, I, P, Q$$

vanish simultaneously then that relation will specialize the singular line that coincides with *OX* to the complex. The values of x_0 (152) and tan φ_0 (153) then take the form 0 / 0, independently of the choice of variables tan φ and x. Corresponding to that, the complex curve an arbitrary plane that goes through *OX* will resolve into a system of two points that lie along *OX*, and the complex cone whose vertex is an arbitrary point of *OX* will decompose into two planes that intersect along *OX*.

With this constant determination, we will obtain the following equation:

$$\sigma = 0$$

for the equation of the polar complex of the *OX* axis, which represents all lines that cut the *OX* axis. The *OX* is a *double line* of the given complex (cf., no. **300**). A double line is then a singular line whose relationship to the complex has been specialized in such a way that any point that is chosen along it will be a singular point and any plane that goes through it will be a singular plane of the complex.

In numbers **284-286**, we chose the line at infinity in *YZ* to be a double line of the complex, and correspondingly found that all complex cylinders whose sides are parallel to the *YZ* plane will decompose into systems of two planes that are parallel to *YZ*.

In general, a given second-degree complex will contain no double line. That would require a *simple* specialization of it.

308. We would, in turn, like to write the equation of the given second-degree complex in the following form:

$$\Omega = 0. \tag{156}$$

We can add the identity:

$$-r\sigma + s\rho + h\eta = 0, \tag{157}$$

when it is multiplied by an arbitrary factor, to that equation without changing the complex itself. We will then obtain:

$$\Omega + \lambda(-r\sigma + s\rho + h\eta) = 0. \tag{158}$$

Corresponding to that, the equation of the polar complex that is associated with a given straight line (r', s', h', $-\sigma'$, ρ', η') will become the following one:

$$\left[\left(\frac{\delta\Omega}{\delta r}\right) - \lambda\sigma'\right] r + \left[\left(\frac{\delta\Omega}{\delta s}\right) + \lambda\rho'\right] s + \left[\left(\frac{\delta\Omega}{\delta h}\right) + \lambda\eta'\right] h - \left[\left(-\frac{\delta\Omega}{\delta\sigma}\right) + \lambda r'\right]\sigma$$

$$+ \left[\left(\frac{\delta\Omega}{\delta\rho}\right) + \lambda s'\right]\rho + \left[\left(\frac{\delta\Omega}{\delta\eta}\right) + \lambda h'\right]\eta = 0, \tag{159}$$

which can also be written in the other form:

$$\left(\frac{\delta\Omega}{\delta r} - \lambda\sigma\right) r' + \left(\frac{\delta\Omega}{\delta s} + \lambda\rho\right) s' + \left(\frac{\delta\Omega}{\delta h} + \lambda\eta\right) h'$$

$$-\left(\frac{\delta\Omega}{\delta\sigma}+\lambda r\right)\sigma'+\left(\frac{\delta\Omega}{\delta\rho}+\lambda s\right)\rho'+\left(\frac{\delta\Omega}{\delta\eta}+\lambda h\right)\eta' = 0. \tag{160}$$

If we assign a fixed value to λ then this double form of the same equation will be linked, in the same sense, with a *theory of reciprocity* that *Gergonne* first developed for plane curves and surfaces of second order (*). We can summarize it in the following words:

Any straight line that belongs to the polar complex of a given straight line will correspond to a polar complex that, conversely, the given straight line belongs to.

The totality of all straight lines in space, along with their polar complexes, defines a *polar system*. We can regard the foregoing two equations (159) and (160) – which are identical to each other – as its equation, when we consider r', s', h', $-\sigma'$, ρ', η' to be variable, along with r, s, h, $-\sigma$, ρ, η, but independently of them. Corresponding to another choice of the undetermined constant λ, we would obtain another polar system from the given second-degree complex that would have the same relationship to the complex as the originally-chosen one. Whereas a second-degree complex depends upon *nineteen* constants, each of the polar systems that are associated with it will be determined by *twenty* constants.

309. In order to express the idea that the given straight line itself belongs to the polar complex that is associated with it, independently of the special values that we would like to assign to the constant λ, if we recall equation (157) then we will get the following condition:

$$\left[\frac{\delta\Omega}{\delta r}\right]r'+\left[\frac{\delta\Omega}{\delta s}\right]s'+\ldots = 0. \tag{161}$$

The lines that belong to their own associated polar system are the same in all polar systems and coincide with the lines of the given second-degree complex.

If the polar complex of a given straight line in a polar system, for whose equation we would like to consider (159), should be a complex of the special kind whose lines all cut a fixed straight line then, from the discussion in number **45**, when we set:

$$(\Phi) \equiv -\left(\frac{\delta\Omega}{\delta r}\right)\cdot\left(\frac{\delta\Omega}{\delta\sigma}\right)+\left(\frac{\delta\Omega}{\delta s}\right)\left(\frac{\delta\Omega}{\delta\rho}\right)+\left(\frac{\delta\Omega}{\delta h}\right)\left(\frac{\delta\Omega}{\delta\eta}\right),$$

as in number **299**, if we recall equation (157) then we will get:

$$(\Phi) + \lambda\,(\Omega) = 0. \tag{162}$$

(*) *Geometrie des Raumes*, no. **258**.

This equation will be fulfilled independently of the special value that we gave to λ as long as the two equations:

$$(\Phi) = 0, \qquad (\Omega) = 0$$

are satisfied. These are the same equations by which we determined the singular lines of the given complex in number **300**. In agreement with what we said before, we will then get the theorem that the polar complexes of the singular lines of the given complexes in all associated polar systems *are complexes of the special kind in which all lines will cut a fixed straight line.*

310. In what follows, we would like to set λ equal to zero, with no loss of generality, and subject the polar system that is determined by that value of λ to a closer consideration. The equation of the polar system will then be written in the double form:

$$\left(\frac{\delta\Omega}{\delta r}\right) r + \left(\frac{\delta\Omega}{\delta s}\right) s + \left(\frac{\delta\Omega}{\delta h}\right) h - \left(-\frac{\delta\Omega}{\delta\sigma}\right) \sigma + \left(\frac{\delta\Omega}{\delta\rho}\right) \rho + \left(\frac{\delta\Omega}{\delta\eta}\right) \eta = 0 \tag{163}$$

and

$$\frac{\delta\Omega}{\delta r}\cdot r' + \frac{\delta\Omega}{\delta s}\cdot s' + \frac{\delta\Omega}{\delta h}\cdot h' + \frac{\delta\Omega}{\delta\sigma}\cdot \sigma' + \frac{\delta\Omega}{\delta\rho}\cdot \rho' + \frac{\delta\Omega}{\delta\eta}\cdot \eta' = 0. \tag{164}$$

Let a straight line be given. It will correspond to a polar complex in the polar system. Any line of the latter will belong to a polar complex that contains the given straight line. However, in general, the polar complexes that correspond to the lines of an arbitrarily-chosen line complex will have no straight lines in common with each other. That would require a special position of it with respect to the given polar system.

The polar complexes that correspond to *two* given straight lines determine a linear congruence. The two given straight lines will belong to each of the polar complexes that correspond to the lines of the congruence. Conversely, the polar complexes of all lines of an arbitrarily-chosen linear congruence will have two fixed lines in common. Four lines of the congruence will then determine two straight lines by their polar complexes. The polar complexes of these two straight lines will have four lines of the given congruence in common, and thus all of them.

If three straight lines are given then the polar complex will determine a hyperboloid by way of the lines of one of its generators. An arbitrary line of that generator – which we would like to refer to as the *first* one – will possess a polar complex that the three given straight lines will belong to. As lines of the first generator, the three given lines will determine a second hyperboloid. The two hyperboloids mutually correspond to each other. The lines of the first generator of the second hyperboloid will belong to the polar complex of the lines of the first generator of the first hyperboloid, and likewise the lines of the first generator of the first hyperboloid will belong to the polar complex of the lines of the first generator of the second hyperboloid. The second generator of each of the two hyperboloids will thus not come under consideration any longer. Correspondingly, each generator of a given hyperboloid will be associated with a second one.

Since we chose the three given straight lines especially so that they would intersect at a point or lie in a plane, they will determine all lines that go through a fixed point or are contained in a fixed plane. Thus, in the polar system, every point and every plane will then correspond to one generator of a hyperboloid. The polar complex that belongs to an arbitrary line of that generator will be of the special kind that all of its lines cut a fixed straight line. That fixed straight line will go through the given point or lie in the given plane, respectively. The lines of the one generator of a hyperboloid will belong to the complex that is determined by equation (162). The hyperboloid itself is not specialized, only its position in respect to the polar system.

If we take the three intersecting straight lines to be the three coordinate axes *OX*, *OY*, *OZ*, in particular, or the three straight lines at infinity in the coordinate planes *YZ*, *XZ*, *XY* then we will get the following three equations for the determination of the hyperboloid that is associated with the coordinate origin or the plane at infinity, respectively:

$$\frac{\delta\Omega}{\delta r} = 0, \qquad \frac{\delta\Omega}{\delta s} = 0, \qquad \frac{\delta\Omega}{\delta h} = 0, \tag{165}$$

or

$$\frac{\delta\Omega}{\delta \sigma} = 0, \qquad \frac{\delta\Omega}{\delta \rho} = 0, \qquad \frac{\delta\Omega}{\delta \eta} = 0. \tag{165}$$

Equation (162):

$$\Phi \equiv - \frac{\delta\Omega}{\delta r}\cdot\frac{\delta\Omega}{\delta \sigma} + \frac{\delta\Omega}{\delta s}\cdot\frac{\delta\Omega}{\delta \rho} + \frac{\delta\Omega}{\delta h}\cdot\frac{\delta\Omega}{\delta \eta} = 0$$

will be fulfilled with both assumptions, and it represents the straight lines that correspond to polar complexes whose lines all intersect a fixed straight line in the given polar system ($\lambda = 0$).

§ 5.

Surfaces of order and class four that are defined by the singular points of complexes and enveloped by their singular planes.

311. We have a point whose complex cone resolves into a system of two planes – viz., a *singular point* – and a plane whose complex curve degenerates into a system of two points – viz., a *singular plane* – of the complex.

The line of intersection of the two planes into which the complex cone, whose vertex is a singular point, has resolved, as well as the connecting line of the two points into which the complex curve, whose plane is a singular plane, decomposes, are *singular lines* of the complex. In that sense, any singular line will correspond to a singular point and a singular plane. All complex curves that lie in planes that are drawn through a singular line will contact that line at the corresponding singular point, and all complex cones whose vertices are assumed to be on a singular line will contact the corresponding singular plane along it. We would like to refer to the singular point and singular plane that correspond to a singular line as *associated* with each other.

The singular lines of a given complex define a congruence of degree four. It is determined by the two-parameter group of two complexes of degree two, one of which is given, and the other of which will be obtained when one replaces the line coordinates in the second-degree condition equation that the line coordinates must satisfy with the partial differential quotients of the equation of the given complex with respect to them. In general, four of the tangents to a given complex curve and four of the lines (*Seiten*) of a given complex cone will be singular lines. In particular, if the complex curve resolves into a system of two points or the complex cone resolves into a system of two planes then two of the four singular lines will coalesce into the connecting line of the two points or the line of intersection of the two planes.

312. When the complex curve in a given plane is specialized in such a way that it resolves into two points that coalesce into one, we would like to call the plane a *double plane* of the complex. We refer to a point that is the center of a complex cone that degenerates into a system of two coincident planes as a *double point.* No point that is enveloped by the lines of the complex in a double plane can be, in turn, a double point, any more than the planes that are defined by the lines of the complex that go through a double point can be double planes.

Any line of the given complex that goes through a double plane or a double point is a singular line of it. The double planes and double points are those planes and points that contain infinitely many lines of the congruence of singular lines.

Any singular line that lies in a double plane corresponds to it as a singular plane. Any singular point that corresponds to each singular line, in turn, still does not coincide with the singular point at which they all intersect. Moreover, any singular line corresponds to a second singular point that is generally different from the first one. If the singular line in the double plane rotates around the fixed point that is enveloped in it by the lines of the complex then the corresponding singular point will describe a *second-order curve* that goes through the fixed point. Whereas a singular point is generally associated with a singular point, a double plane will correspond to *infinitely many* associated singular points that lie on a second-order curve.

Any singular line that goes through a double point will correspond to it as a singular point. However, any of these singular lines will, in general, correspond to a singular plane that does not coincide with the fixed plane that is defined by the lines of the complex that go through the double point. All of these planes envelop a *conic surface of class two* that contacts the fixed plane, in particular. Whereas a singular point is, in general associated with one singular plane, a double point will correspond to *infinitely many* associated singular planes that envelop a conic surface of class two.

We derive the analytical statements of the foregoing geometric results from numbers **289** and **290**, in which the plane infinity is taken to be a double plane of the complex.

313. A singular line can be specialized in such a way that any plane that goes through it is a singular plane and that any point on it can be assumed to be a singular point. We have called such a line a "double line" of the complex (no. **307**). However, one arrives at a specialization of the given complex when one requires that it should contain a double line. We shall exclude the possibility that the given complex is specialized in the manner that is necessary for this to be true from further consideration.

There can be distinguished points or planes in a given complex with the property that all of the straight lines that go through them (all of the ones that lie in them, respectively) are lines of the complex. Any plane that goes through the plane will then be a singular plane and any point that is taken from the plane will be a singular point. In number **292**, we let such a plane coincide with the plane at infinity. From the discussion in that number, one requires a six-fold specialization in order for the plane at infinity for a given complex to be of this special kind. In general, then, there will be no planes and points of that kind. In the sequel, we shall omit the possibility that the given complex has been correspondingly specialized.

314. In order for a given cone of order two to resolve into a system of two planes or for a given curve of class two to resolve into a system of two points, a condition equation must be fulfilled. A surface will then be defined by the singular points of a complex, and a surface will be enveloped by its singular planes. By contrast, three conditions will be fulfilled when the two planes into which a complex cone (the two points into which a complex curve, resp.) resolves must coincide. There are then an *infinite number* of double points and double planes.

In the sixth and seventh paragraphs of the previous chapter, we proved that *four* singular points will lie on the *OX* coordinate axis, which we took to be the double line of a complex surface and was assumed to be completely arbitrary, and that *four* singular planes will go through it (cf., no. **215**). We thus immediately obtain the following theorem:

The surface that is defined by the singular points of a complex of degree two has order four.

The surface that is enveloped by the singular planes of a complex has class four.

315. In order to obtain the equation for the surface of singular points in point coordinates, we start with equation (II) of the second-degree complex in ray coordinates. We have to express the idea that the cone that equation of the complex represents will resolve into a system of two planes as soon as we assign fixed values to the values x, y, z. Of the six ray coordinates:

$$(x - x'), \quad (y - y'), \quad (z - z'), \quad (yz' - y'z), \quad (x'z - xz'), \quad (xy' - x'y),$$

we can write the last three in the following way:

$$((y - y')\, z - y\, (z - z')), \qquad (x\, (z - z') - (x - x')\, z), \qquad ((x - x')\, y - x\, (y - y')).$$

In that way, equation (II) of the complex will assume the following form:

$$a(x-x')^2+2b(x-x')(y-y')+c(y-y')^2 \qquad (167)$$
$$+2d(x-x')(z-z')+2e(y-y')(z-z')+f(z-z')^2=0,$$

where a, b, c, d, e, f are functions of degree two in x, y, z. In particular, we find that:

$$(168)\qquad \begin{cases} a=A+Ez^2+Fy^2-2Kyz-2Pz+2Qy, \\ b=I-Fxy+Kxz+Lyz-Mz^2+(N-O)z-Qx+Ry, \\ c=B+Dz^2+Fx^2-2Lxz-2Rx+2Sz, \\ d=H-Exz+Kxy-Ly^2+Myz-Ny+Px-Uz, \\ e=G-Dyz-Kx^2+Lxy+Mxz+Ox-Sy+Tz, \\ f=C+Dy^2+Ex^2-2Mxy-2Ty+2Ux. \end{cases}$$

In order to express the idea that the cone that is represented by equation (167) resolves into a system of two planes, we get the following condition from number **186**:

$$(169)\qquad acf+2bde-ae^2-cd^2-fb^2=0.$$

We will obtain the desired equation for the surface when we substitute the values of a, b, c, d, e, f from equation (168) into this equation; it will obviously be of degree six. When one actually carries out the suggested multiplications in (169), the terms of order five and six will then drop out.

316. We obtain the equation of the surface that is enveloped by the singular planes of the complex in plane coordinates when we exchange:

$$x, y, z \quad \text{with} \quad t, u, v$$

in the present equations (168), using the exchange rules of number **153**, and reciprocally exchange:

$$A, B, C, \qquad G, H, I, \qquad P, Q, R$$

with

$$D, E, F, \qquad K, L, M, \qquad S, T, U,$$

respectively. The constants:

$$N, O$$

will remain unchanged by this. If we write a' in place of a, b' in place of b, etc., after the exchange then we will get:

$$
(170)\qquad \begin{cases}
a' = D + Bv^2 + Cu^2 - 2Guv - 2Sv + 2Tu, \\
b' = M - Ctu + Gtv + Huv - Iv^2 + (N - O)v - Tt + Uu, \\
c' = E + Av^2 + Ct^2 - 2Htv - 2Ut + 2Pv, \\
d' = L - Btv + Gtu - Hu^2 + Iuv - Nu + St - Rv, \\
e' = K - Auv - Gt^2 + Htu + Itv + Ot - Pu + Qv, \\
f' = F + Au^2 + Bt^2 - 2Itu - 2Qu + 2Rt,
\end{cases}
$$

and we will get the equation of the surface in the following form:

$$
(171)\qquad a'\, c'\, f' + 2b'\, d'\, e' - a'\, e'^2 - c'\, d'^2 - f\, b'^2 = 0.
$$

The analogous statement from the previous number regarding the reduction of the present equation, which is clearly of degree six in t, u, v, to degree four in those variables is still valid.

If we make the expressions (170) for a', b', c', d', e', f' homogeneous by the introduction of a fourth variable w, and then substitute into equation (171) then it will generally be of degree six, and will reduce to degree four only when a factor of w^2 has been separated from it. When interpreted geometrically, equation (171) says not so much that the complex curve in a given plane t, u, v, w resolves into a system of two points as much as that those cones of class two that can be drawn through the complex curve in question with the coordinate origin as their centers will decompose into a system of two enveloped axes. That will then be true for any plane that goes through the coordinate origin, and therefore, when the factor w^2 represents the coordinate origin when it is set to zero.

317. An arbitrarily-chosen straight line cuts the surface of the singular points at four points, in general, and one can, in general, draw four tangential planes to the surface of singular planes. If the chosen straight line is a singular line of the complex, in particular, then two of the four singular points will coalesce into the corresponding point and two of the four singular planes will coalesce into the corresponding plane (cf., no. **306**). A singular line will then contact the surface of singular points, as well as the surface of singular planes. The contact point with the former surface will be the corresponding singular point, while the contact plane with the latter surface will be the corresponding singular plane.

For singular lines that lie in a double plane of the complex, two of the four intersection points with the surface of singular points will coincide pair-wise. Such lines will then be *double tangents of the surface of singular points,* in the sense that they contact two distinct points of this surface.

Likewise, two of the four tangential planes that can generally be drawn through a given straight line on the surface of singular planes will coalesce pair-wise into one, as long as the given straight line is one of the singular lines that goes through a double point of the complex. These lines will then be *double tangents to the surface of singular planes,* in the sense that they will contact that surface at two distinct planes.

318. The four singular lines of the complex that lie in an arbitrary plane contact the curve in that plane that is enveloped by lines of the complex in the singular points that correspond to them. The same straight lines are contained in the fourth-order surface of singular points at the same points. The fourth-order intersection curve of this surface with an arbitrary plane then contacts the complex curve that lies in this plane at four points. Of the eight intersection points that the two curves must have, in any event, two of them will coalesce into one contact point.

Likewise, those complex cones that have an arbitrary point of space for their vertex will contact the cone of class four that can be drawn from the arbitrarily-chosen point to the surface of class four that is enveloped by the singular planes at four straight lines, which are the four singular lines that go through them. The common tangential planes to the two cones along these four straight lines will be the corresponding singular planes.

We would like to choose a singular plane to be an arbitrarily-chosen plane, in particular. The locus in it that is enveloped by lines of the complex will be, as before, contacted by the intersection curve of the singular planes with the surface of singular points at four points; the contact points of the singular lines will be the corresponding singular lines, respectively. For a singular plane, two of the four singular lines that will generally lie in a given plane will coalesce into the singular line that corresponds to it. The other two will each go through one of the two points with which the complex curve resolved in arbitrary directions. Two of the four contact points of the intersection curve of the surface of singular points with the locus that is enveloped by the lines of the complex will then coalesce into the two points into which the complex curve has resolved in the singular planes, while the other two will coincide with the singular points that are associated with the given singular planes.

The fourth-order intersection curve of the surface of singular points with an arbitrary singular plane has a double point at the singular points that are associated with that plane.

In the same way, we can prove the theorem:

The conic surface of class four that can be drawn from an arbitrary singular point to the surface of singular planes has the singular plane that is associated with that point for its double plane.

319. We derive the analytic statement of these geometric results from equations (169) and (171), which represent the surface of singular points and the surface of singular planes, respectively, in point and plane coordinates, respectively. If we assume that the XZ-plane is a singular plane and that the corresponding singular line coincides with OX, while the associated singular point coincides with O, then we will get the following determination of the constants from numbers **305** and **306**:

$$A = 0, \quad H = 0, \quad I = 0, \quad P = 0.$$

In that way, when we let all of the y' in them vanish, the expressions a, b, c, d, e, f in (168) will take on the following values:

$$(172)\qquad \begin{cases} a = Ez^2, \\ b = Kxz - Mz^2 + (N - O)z - Qx, \\ c = B + Dz^2 + Fx^2 - 2Lxz - 2Rx + 2Sz, \\ d = -Exz - Uz, \\ e = G - Kx^2 + Mxz + Ox + Tz, \\ f = C + Ex^2 + 2Ux. \end{cases}$$

If we neglect x and z in them compared to constants, as well as the second powers of x and z compared to the first, then we will obtain:

$$(173)\qquad \begin{cases} a = Ez^2, \\ b = (N - O)z - Qx, \\ c = B, \\ d = -Uz, \\ e = G, \\ f = C. \end{cases}$$

When we substitute these values into equation (169):

$$acf + 2bde - ae^2 - cd^2 - fb^2 = 0,$$

we will find the following equation:

$$(174)\qquad BCEz^2 - 2GUz((N - O)z - Qx) - EG^2z^2 - BU^2z^2 - C((N - O)z - Qx)^2 = 0$$

for the representation of singular points that lie in the singular plane XZ in the vicinity of the associated singular point O. This equation includes only terms of second degree in x and z. The intersection curve of the surface of singular points with the XZ-plane then possesses a double point at the coordinate origin, in agreement with the concluding remarks of the previous number.

We may further remark that this double point will be a point of regression when then constant Q vanishes, in addition to the constants A, H, I, P. From the discussions in number **307**, the OX-axis will be a double line of the complex.

In the same way, we can prove that the cone of class four that can be drawn from an arbitrary singular point to the surface of singular planes has the singular plane that is associated with the chosen singular point for a double plane.

320. From the foregoing, any singular plane is a tangential plane of the fourth-order surface that is defined by the singular points; its contact point is the associated singular point. Conversely, any singular point is a point of the surface of class four that is enveloped by the singular planes. The tangential plane to it is the associated singular plane.

The fourth-order surface that is defined by the singular points of the complex and the surface of class four that is enveloped by its singular planes are identical.

Any singular line of the complex contacts the fourth-order surface of singular points and the surface of class four of singular planes. The contact point with the surface is the corresponding singular point, and the contact plane to it is the corresponding singular plane. The two remaining intersection points of the singular line with the surface are those two points with which the complex curve resolves in the corresponding singular planes. Likewise, the two remaining tangential planes that can be drawn through the singular lines to the surface are those two planes with which the complex cone whose vertex is the associated point decomposes. The direction of the one singular plane and the singular line that corresponds to its associated singular point are still not given for the surfaces of singular points and singular planes, resp. The surface depends upon fewer arbitrary constants then the second-degree complex that determines it.

321. The surface that is defined by the singular points of the complex and the one that is enveloped by its singular planes are of order four and class four, respectively. It will then have sixteen double points and sixteen double planes, in general. The possibility that the surface possesses further singularities – in particular, a double ray and a double axis that coincides with it – by which, the number of double points and double planes will be reduced, remains excluded as long as the given complex is not itself specialized.

The tangential planes to the surface at a double point on it envelop a cone of class two and the contact points of it with its double planes will define a curve of order two. In number **312**, we proved that the singular planes that go through a double point of the complex also envelop a cone of class two and that the singular points that lie in a double plane of the complex define a curve of order two.

The double points and double planes of the complex coincide with the double points and double planes of the surfaces that are defined by the singular points and enveloped by the singular planes, respectively.

From this:

In any second-degree complex there are, in general, sixteen double points and sixteen double planes.

Which point in a double plane is enveloped by the lines of the complex and which plane is defined at a double point by the lines of the complex are still not determined for the surface of singular points and the surface of singular planes, respectively. The point can be an arbitrary point of the contact curve, while the plane can be an arbitrary plane of the contact cone.

322. We return to the equation for the surface of singular points and planes in plane coordinates (171):

$$a' c' f' + 2b' d' e' - a' e'^2 - c' d'^2 - f' b'^2 = 0.$$

We would like make this homogeneous through the introduction of a fourth variable w. It will then come to pass that we will make the expressions (170) that were found for a', b', c', d', e', f' homogeneous by the introduction of these variables, and neglect the factor w^2 that will arise after replacing these expressions in equation (171).

We would like to write the equation for the surface in the following way:

$$f = 0. \tag{175}$$

According to number **296**, we will then obtain the following equation for the pole of a given plane (t', u', v', w') relative to this surface:

$$\left(\frac{\delta f}{\delta t}\right) t + \left(\frac{\delta f}{\delta u}\right) u + \left(\frac{\delta f}{\delta v}\right) v + \left(\frac{\delta f}{\delta w}\right) w = 0.$$

The rectangular coordinates of the poles are then:

$$x' = \frac{\left(\frac{\delta f}{\delta t}\right)}{\left(\frac{\delta f}{\delta w}\right)}, \quad y' = \frac{\left(\frac{\delta f}{\delta u}\right)}{\left(\frac{\delta f}{\delta w}\right)}, \quad z' = \frac{\left(\frac{\delta f}{\delta v}\right)}{\left(\frac{\delta f}{\delta w}\right)}. \tag{176}$$

If we substitute the expressions a', b', c', d', e', f', which have been made homogeneous, in equation (171) then we will obtain the equation:

$$F = 0, \tag{177}$$

and from the foregoing, one will have:

$$F = w^2 f. \tag{178}$$

It is then permissible to replace the differential quotients of the function f with respect to t, u, v, w in formulas (176) with the following functions:

$$\left(\frac{\delta F}{\delta t}\right), \qquad \left(\frac{\delta F}{\delta u}\right), \qquad \left(\frac{\delta F}{\delta v}\right), \qquad \left(\frac{\delta F}{\delta w} - \frac{2F}{w}\right),$$

respectively.

323. We would like to choose the plane at infinity, in particular. For the sake of simplicity, we set – as is always permissible – the constants K, L, M equal to zero in the equation of the given complex. We then obtain the following values for the expressions a', b', c', d', e', f', which have been made homogeneous:

$$(179)\qquad \left\{\begin{aligned} a' &= Dw^2 + Bv^2 + Cu^2 - 2Guv - 2Svw + 2Tuw, \\ b' &= -Ctu + Gtv + Huv - Iv^2 + (N - O)vw - Ttw - Uuw, \\ c' &= Ew^2 + Av^2 + Ct^2 - 2Htv - 2Utw + 2Pvw, \\ d' &= -Btv + Gtu - Hu^2 + Iuv - Nuw + Stw - Rvw, \\ e' &= -Auv - Gt^2 + Htu + Itv + Otw - Puw + Qvw, \\ f' &= Fw^2 + Au^2 + Bt^2 - 2Itu - 2Quw + 2Rtw. \end{aligned}\right.$$

If we substitute the coordinates of the plane at infinity:

$$t' = 0,\quad u' = 0,\quad v' = 0,\quad w' = w'$$

in these expressions and their differential quotients with respect to t, u, v, w then we will get:

$$(180)\qquad a' = D\,w'^2,\quad b' = 0,\quad c' = E\,w'^2,\quad d' = 0,\quad e' = 0,\quad f' = F\,w'^2,$$

and

$$(181)\qquad \left\{\begin{aligned} &\left(\frac{\delta a'}{\delta t}\right) = 0, && \left(\frac{\delta c'}{\delta t}\right) = -2Uw', && \left(\frac{\delta f'}{\delta t}\right) = 2Rw', \\ &\left(\frac{\delta a'}{\delta u}\right) = 2Tw', && \left(\frac{\delta c'}{\delta u}\right) = 0, && \left(\frac{\delta f'}{\delta u}\right) = -2Qw', \\ &\left(\frac{\delta a'}{\delta v}\right) = -2Sw', && \left(\frac{\delta c'}{\delta v}\right) = 2Pw', && \left(\frac{\delta f'}{\delta v}\right) = 0, \\ &\left(\frac{\delta a'}{\delta w}\right) = 2Dw', && \left(\frac{\delta c'}{\delta w}\right) = 2Ew', && \left(\frac{\delta f'}{\delta w}\right) = 2Fw'. \end{aligned}\right.$$

It is unnecessary for what follows to write down the differential quotients of b', d', e'.

From the foregoing equations, the four expressions:

$$\left(\frac{\delta F}{\delta t}\right),\qquad \left(\frac{\delta F}{\delta u}\right),\qquad \left(\frac{\delta F}{\delta v}\right),\qquad \left(\frac{\delta F}{\delta w} - 2\frac{F}{w}\right),$$

where:

$$F \equiv a'\,c'\,f' + 2\,b'\,d'\,e' - a'\,e'^{\,2} - c'\,d'^2 - f'\,b'^{\,2}$$

in which, just the one term:

$$a' \, c' \, f'$$

comes under consideration for the plane at infinity, will take on the following values:

$$\text{(182)} \qquad \begin{cases} \left(\dfrac{\delta F}{\delta t}\right) = 2Dw'^5(ER - FU), \\ \left(\dfrac{\delta F}{\delta u}\right) = 2Ew'^5(FT - DQ), \\ \left(\dfrac{\delta F}{\delta v}\right) = 2Fw'^5(DP - ES), \\ \left(\dfrac{\delta F}{\delta w} - 2\dfrac{F}{w}\right) = 6DEFw'^5 - 2DEFw'^5 = 4DEFw'^5. \end{cases}$$

The coordinates of the pole of the plane at infinity with respect to the surface – or, as we can say, the coordinates of the center of the surface – will then become:

$$\text{(183)} \qquad x' = \frac{ER - FU}{2EF}, \qquad y' = -\frac{DQ - FT}{2DF}, \qquad z' = \frac{DP - ES}{2DE}.$$

These are the same expressions that we found in number **240** for the coordinates of the center of the complex. With that, we have the theorem:

The center of a second-degree complex coincides with the centers of the surfaces of its singular points and planes.

In agreement with that, the center of the complex goes to infinity when the plane at infinity is a singular plane, in particular, and those points at which it contacts the surfaces of singular points and planes (cf., no. **279**) coincide with its associated singular points at infinity.

If the plane at infinity is a double plane of the complex then its center will become undetermined. Its geometric locus will be a second-order curve that lies in the plane at infinity. That curve will be the contact curve of the double plane with the surface of singular points and planes (cf., no. **289**).

§ 6.

Pole of a given plane and polar plane of a given point that are associated relative to the complex.

324. We now return to the considerations of the first three – and especially the third – sections of this Part of this Chapter. In them, we investigated the relationship between the given second-degree complex and the plane at infinity. We next concerned ourselves with the totality of all *diameters* of the complex – viz., straight lines that are associated with the straight lines in the plane at infinity as polars relative to the complex – then the totality of all *cylinders* of the complex – viz., complex cones whose midpoint lies on the plane at infinity – and the axes of the these cylinders – viz., their polar lines relative to the plane at infinity that goes through their midpoint. We then considered the curves that were enveloped by lines of the complex in the plane at infinity and in planes that were infinitely close to it and represented them by a complex of exceptional simplicity and a characteristic with respect to the coordinate system, namely, the *asymptotic complex* of the given one.

We can carry over all of these considerations, and in turn, all of the results that we found, from the plane at infinity to an *arbitrary* plane in space using known rules that already find their expression in the foregoing. The basis for this convertibility lies in the identity of the analytic operations that correspond to the geometric considerations in the one case, as in the other.

In particular, we would like to let an arbitrarily-chosen plane coincide with one of the three coordinate planes. The exchange of the plane at infinity with one of the corresponding coordinate planes gives us an exchange of the line coordinates among themselves, and therefore a reciprocal exchange of constants in the equation of the given complex. In what follows, we will pose the rules for these exchanges, and then we will be spared any further analytic development in any coordinate plane, since it will suffice to switch the variables, as well as the constants, in all previous formulas according to these rules.

With the conversion of the theorems that were posed for the plane at infinity to an arbitrary plane, we will extend the previously-obtained results, to the extent that we are allowed by the foregoing two paragraphs to introduce the elements of complexes – viz., their singular points, lines, and planes – into the geometric considerations in a way that is more intuitive that was previously possible.

325. We would like to use equation (V) for the second-degree complex as the basis for what follows, which we make homogeneous by the introduction of a sixth variable h, and make symmetric by the addition of a term $2Vh\eta$. The equation is then:

$$\begin{aligned} &Ar^2 + Bs^2 + Ch^2 + D\sigma^2 + E\rho^2 + F\eta^2 \\ &+ 2Gsh + 2Hrh + 2Irs + 2K\rho\eta - 2L\sigma\eta - 2M\rho\sigma \\ &- 2Nr\sigma + 2Os\rho + 2Vh\eta \\ &+ 2Pr\rho + 2Qr\eta + 2Rs\eta - 2Ss\sigma - 2Th\sigma + 2Uh\rho = 0. \end{aligned} \tag{V}$$

In number 10, we obtained the following six proportional expressions for the ray coordinates:

$$r,\ s,\ h,\ -\sigma,\ \rho,\ \eta,$$

namely:

$$(x\tau' - x'\tau),\ (y\tau' - y'\tau),\ (z\tau' - z'\tau),\ (yz' - y'z),\ (x'z - xz'),\ (xy' - x'y).$$

The ratios:

$$\frac{x}{\tau},\ \frac{y}{\tau},\ \frac{z}{\tau} \qquad \text{and} \qquad \frac{x'}{\tau'},\ \frac{y'}{\tau'},\ \frac{z'}{\tau'}$$

then denote the coordinates of two points that are chosen arbitrarily on the straight line.

The exchange of the plane at infinity with the coordinate plane *YZ* will correspond to the exchange of:

$$x \text{ with } \tau \quad \text{and} \quad x' \text{ with } \tau', \text{ resp.}$$

Corresponding to that, the six line coordinates:

$$r, s, h, -\sigma, \rho, \eta$$

will be replaced with the following ones:

$$-r, -\eta, r, -\sigma, h, -s,$$

resp. This exchange will not affect the coefficients:

$$A, D, R, U,$$

while

$$B, C, I, M$$

and

$$F, E, Q, T$$

respectively, will not change their signs, and:

$$G, H, L, O$$

and

$$K, P, S, V,$$

resp., will be reciprocally exchanged with a simultaneous change of sign, and *N* will change its sign.

Two of the plane coordinates:

$$t, u, v, w,$$

namely, *t* and *w*, will be exchanged reciprocally.

In particular, the equation for the curve that is enveloped by lines in the plane at infinity is:

$$Dt^2 + Eu^2 + Fv^2 + 2Kuv + 2Ltv + 2Mtu = 0,$$

and we will obtain the following equation from it for the complex curve in *YZ* by using the foregoing rules of exchange:

$$Dw^2 + Cu^2 + Bv^2 - 2Guv - 2Svw + 2Tuw = 0,$$

in agreement with number **166**.

We will get completely analogous rules of exchange that correspond to an exchange of the plane at infinity with one of the other two coordinate planes, *XZ* or *XY*. We shall not write them down here, since we shall refer back to the exchange rules of number **155**, which correspond to an exchange of the three planes *YZ*, *XZ*, *XY* among themselves.

326. Let an arbitrary plane P be given. A curve K will be enveloped by lines of the complex in it. The polar that corresponds to an arbitrarily-chosen straight line a in P with respect to the complex, and which we would like to denote by b, will cut the plane P in the pole of the line a relative to the curve K. The polar of a straight line relative to the complex will then be the geometric locus of its poles relative to the curves that are enveloped by lines of the complex in the plane that goes through it. Consistent with that, in number **236**, we have constructed the direction of the diameters of the complex that are associated with a given system of parallel planes by means of the complex curve that lies in the plane at infinity.

Let a, a', a'' be three straight lines in the plane P, which define a self-conjugate triangle with respect to the curve K; call the three associated polars b, b', b'', resp. b, b', b'' will then go through the intersections of a' and a'', a'' and a, a and a', respectively. We would like to call b, b', b'' three mutually-conjugate polars relative to the plane P, or also, more briefly, *three mutually-conjugate polars*, since the plane P remains fixed. In the case of an arbitrarily-chosen plane, the system of three conjugate polars substitutes for the system of three conjugate diameters in the case of the plane that is shifted to infinity.

The intersection points (a', a''), (a'', a), and (a, a') are the vertices of three complex cones A, A', A''. If we consider the plane P to be fixed, as before, then we will refer to them as the three complex cones that are associated with the straight lines a, a', a'', resp.

This plane is associated with a straight line relative to any complex cone whose vertex is chosen to be in P. This line is the intersection of the tangential planes that contact the complex cone along the two edges, along which, it is intersected by the plane P. If the plane P is shifted to infinity, in particular, then the complex cone will become a complex cylinder, and the straight line in question will become the cylinder axis. We would like to refer to this straight line as the "polar line of the complex cone relative to the plane P, or more briefly, as its *polar line*, in order to distinguish it from the "polar," which is the term that we used in order to refer to the straight line that is associated with the given one relative to complex.

Let the polar lines of the three complex cones A, A', A'' be c, c', c''. We call these three polar lines mutually-conjugate and the three straight lines a, a', a'', like their polars b, b', b'', resp., associated. Each polar line cuts the polar that it is associated with at a point of the plane P.

327. The polars of an arbitrary straight line will be enveloped by its polar planes with respect to all complex cones whose vertices lie along it. For example, b will then be the intersection of the two polar planes of the straight line a relative to the two complex cones A' and A''. However, one will also find the polar lines of the cones A' and A'', which we have previously denoted by c' and c'', in the same two planes. b will thus cut c' and c''.

Each of three mutually-conjugate polars cuts the polar lines that are associated with the other two.

Thus, each of three conjugate polar lines will also cut the polars that are associated with the other two.

If the three polars b, b', b'' are given then the three polar lines c, c', c'', resp., can be constructed in a linear way. Each of them will then go through the point of intersection of one of the three polars with the plane P and cut the other two. b, b', b'' will be determined in the same way when c, c', c'' are given.

Three arbitrary straight lines – in particular, the three polars b, b', b'' – determine a hyperboloid as lines of one generator. All of them that cut the three given straight lines will belong to the second generators as lines. The polar lines c, c', c'' will then be lines of the second generator of the hyperboloid that is determined by the polars b, b', b'' as lines of it. The six straight lines b, b', b'', c, c', c'' determine a hexangle $bc'b''cb'c''$ (cf., no. **109**) that is drawn in the hyperboloid. For an arbitrary choice of the plane P, this hexangle will substitute for the central parallelepiped that is determined by three conjugate diameters and the cylinder axes that are parallel to them in the case of the plane at infinity.

The three planes (b, c), (b', c'), (b'', c''), which are the tangential planes to the hyperboloid that we speak of at the three points (a', a''), (a'', a), (a, a') that lie in P, intersect at a point O, which is the pole of the plane P relative to the hyperboloid. We can determine this point in yet another way. The plane (b, c) cuts the plane P in a line d. The fourth harmonic to b, c, and d, which we would like to denote by e, will go through the desired point. The three diagonals to the hexangle $bc'b''cb'c''$ will intersect at the same point.

When the plane P is shifted to infinity, the point O will become the midpoint of the central parallelepiped. We can define the midpoint of such a parallelepiped to be either the point of intersection of the three planes that go through a diameter and the cylinder axis that is parallel to it or finally as the common intersection of the diagonals of the central parallelepiped.

328. Precisely the same calculations and considerations that allowed us to prove, in numbers **245** and **246**, that all central parallelepipeds of a given complex have the same center (which we referred to as the *center of the complex*) will show that the pole O of the plane P relative to the hyperboloid that is determined by b, b', b'' will be independent of the choice of those three conjugate polars.

The pole of the plane P relative to a hyperboloid that is determined by three conjugate polars is independent of the choice of these polars.

The point O is then associated with the plane P that goes through the given complex. We would like to call it the *pole of the plane P relative to the complex.*

For a second-degree complex, one point is associated with a given plane in a unique way, in general.

In number **323**, we proved that the center of the complex coincides with the center of the surface that is determined by its singular points and planes. We then have the theorem:

The pole of a given plane relative to a second-degree complex coincides with the pole of the same plane with respect to the surface that is defined by the singular points of the complex and enveloped by its singular planes.

329. Let an arbitrary line *a* in the plane *P* be given. Let the polar that is associated with it be *b* and let the polar line be *c*. We then construct the straight line *c* that connects the pole of the plane *P* with the point of intersection of the two straight lines *b* and *c* in such a way that we draw a plane through *b* and *c* and determine the fourth harmonic to *b*, *c*, and the line of intersection *d* of that plane with the plane *P*. We next examine the extent to which this construction will maintain its validity when the chosen straight line *a* belongs to the complex; in particular, when it is singular line of it.

Let *a* be a line of the given complex. The polar *b* then coincides with it. However, the polar line *c* is also no different from *a* and *b*. The pole of the straight line *a* relative to the complex curve that lies in *P* is its contact point with that curve, and the complex cone whose vertex is that point will contact the plane *P* along the tangent at that point; that is, along the chosen line *a*. Thus, *b* and *c*, and therefore, also *d*, will coincide with the line *a*. The geometric construction of the connecting lines of the pole of the straight line *a* relative to the complex curve that lies in *P* with the pole of the plane *P* relative to the complex will become illusory.

If the straight line *a*, in particular, coincides with one of the four singular lines that lie in *P* then its polar *b* will next be determined. It can be chosen arbitrarily from the straight lines that go through the associated singular point in the associated singular plane. This point is the contact point of the singular line *a* with the complex curve that lies in *P*. The complex cone that has it for its vertex will decompose into two planes that intersect along the singular line *a*. The polar line *c*, like the polar *b*, will then be undetermined, and will be subject to the single condition that the plane that is harmonic to the aforementioned two and the plane *P* must go through the contact point that lies along *a*. The desired line *e* will be contained in the fourth harmonic plane to the given plane *P* and the two planes in which *b* and *c* lie, respectively, but it will not be determined completely inside of them by the general construction.

330. When the straight line *a* does not belong to the given complex, the associated polar *b* and the associated polar line *c* will generally be different. That corresponds to the fact that three conjugate polars will not intersect, in general. From the discussion in number **251**, there is a system of three associated diameters that go through the center of the complex. They coincide with the cylinder axes that parallel to them. Correspondingly, there are three mutually-associated polars for any plane that go through the pole of the plane, and thus coincide with the polar lines that are associated with it.

If we refer the complex to the three diameters that intersect in its center as coordinate axes, as in number **251**, then its equation will be the following one:

$$\begin{aligned}Ar^2 + Bs^2 + Ch^2 + D\sigma^2 + E\rho^2 + F\eta^2 \\ + 2Gsh + 2Hrh + 2Irs \\ - 2Nr\sigma + 2Os\rho + 2Vh\eta \equiv \Omega = 0.\end{aligned} \tag{184}$$

We will then get:

$$Dt^2 + Eu^2 + Fv^2 = 0 \tag{185}$$

for the curve that is enveloped by its lines in the plane at infinity. For the curve of the complex in the same plane whose equation is the following one:

$$-\frac{\delta\Omega}{\delta r}\cdot\frac{\delta\Omega}{\delta\sigma}+\frac{\delta\Omega}{\delta s}\cdot\frac{\delta\Omega}{\delta\rho}+\frac{\delta\Omega}{\delta h}\cdot\frac{\delta\Omega}{\delta\eta} = 0,$$

we will find:

$$DNt^2 + EOu^2 + FVv^2 = 0. \tag{187}$$

The variables enter into the two equations (185) and (187), only as squares. The two curves that are represented by these equations will then be of class two when referred to a coordinate system that is self-conjugate with respect to them.

We have determined the singular lines of the given complex by means of equation (186), along with the equation of that complex. They are the four singular lines of the common tangents to the two conic sections that are represented by equations (185) and (187) in the plane at infinity. The three points at which the coordinate axes *OX*, *OY*, *OZ* cut the plane at infinity are then *the three points at which the diagonals of the complete tetragon that is defined by the four singular lines that lie in that plane.* Its three diagonals are the straight lines that lie in the plane at infinity, and whose associated polar and polar line coincide without themselves belonging to the complex.

The foregoing considerations carry over immediately from the plane at infinity to an arbitrarily-chosen one.

331. For a given plane, there is, in general only *one* system of three associated polars that intersect in the pole of the plane: It is the one that we constructed in the foregoing number. The construction will be undetermined when the four singular lines coincide pair-wise in the chosen plane *P*, which would require a two-fold specialization of the relationship between the given complex and it. A point of intersection *o* and a straight line *p*, which is the polar of *o* relative to the complex curve that lies in *P*, will be determined by the two straight lines in which the four singular lines coincide. The polars of *p* relative to the complex will go through the point *o* and the pole of the plane *P* relative to the complex. Conversely, the polar to any given line that can be drawn through *O* in *P* will go through a point of the straight line *p* and pole of the plane *P* relative to the complex. There are then *infinitely many polars* that intersect in the pole of the plane *P*. One of them will be distinguished, while the rest of them will all be conjugate to that one and will lie in a plane that goes through the pole.

If all polars of the straight lines that lie in *P* are to go through the pole of *P* then, from the geometric construction that we consider, all lines of the complex that lie in *P* must be singular lines of that complex. As long as the given plane is not a singular plane, this will require a five-fold specialization of the relationship of the given plane to the complex.

That will then demand that either the curve that is enveloped by lines of the complex (186) in the given curve P is no different from the curve of the given complex in that plane, or that every line in the plane P belongs to the complex. Whether the one or the other case is pertinent will depend upon the choice of the extra term in the equation of the given complex.

If the given second-degree complex is of the particular type for which its *lines envelop a second-degree surface* then the polars of all such straight lines that lie in an arbitrary plane will intersect at the pole of this plane relative to the complex that coincides with its pole relative to the surface, and generally speaking all lines of such a complex will be regarded as singular lines. The surface that is defined by the singular points of the general second-degree complex and is enveloped by its singular planes is no different from the latter in the case of the special complex that represents a second-degree surface.

332. If the given plane P is a *singular* plane then, from what was explained in numbers **279** and **323**, its pole relative to the complex will coincide with the singular point that is associated with it. That point will be the contact point of the given singular plane with the surface of the singular points and planes.

We easily convince ourselves of the validity of this result. Corresponding to the assumption, the complex curve in the given plane P has resolved into the system of two points K_1 and K_2 . Their connecting line (K_1, K_2) is the singular line that is associated with the given singular plane. The associated singular point O that is the pole of the plane P is similarly arranged.

Let an arbitrary straight line a be given in the plane P. Its polar b cuts the plane P at a point of the singular line (K_1, K_2). The complex cone whose vertex is that point of intersection contacts the given plane P along (K_1, K_2). The polar line c that is associated with the arbitrarily-chosen straight line a then coincides with (K_1, K_2). This is expressed by saying that we seek the pole of the plane P on the singular line (K_1, K_2). The plane that is drawn through b and c must then cut the plane P again along c, and the fourth harmonic to b, c, and this line of intersection must coincide with c, since b and c do not themselves coincide.

In order to determine the pole on the singular line (K_1, K_2), we let the arbitrarily-chosen straight line a coincide with (K_1, K_2). Infinitely many straight lines will then correspond to it as polars – namely, all of the ones that go through the associated singular point O in the given plane P. The proof is complete with that. The fourth harmonic to such a polar, the polar line to a complex cone whose vertex is chosen arbitrarily on it, and the line of intersection of the plane that is determined by the polar and the polar line with the given one P will coincide with the chosen polar itself.

333. If the given plane P is a *double plane* of the complex then the position of its pole will be undetermined. The geometric locus of them will be the second-order curve along which the double plane contacts the surface of singular points and planes. The complex curve in the double plane will resolve into a system of two points that coincide in one point of the second-order contact curve. The direction of the connecting line of the two points will be undetermined. Each line in *P* that goes through the point at which the two coincide will be a singular line. The singular point that corresponds to each of them can be regarded as the pole of the plane *P* relative to the complex. If the singular line in *P* rotates around the fixed point then the corresponding singular point will describe the second-order curve along which the double plane contacts the surface of singular points and planes.

We still have to mention the case in which *all* of the lines that lie in a given plane *P* belong to the complex. One can say nothing further about a well-defined pole in such a plane with respect to the complex. That corresponds to the fact that the plane is separated from the surface of singular points as an isolated plane, which will reduce to order three.

334. We have represented the lines at infinity of the given complex by means of an especially simple complex whose equation was then more transparent when we put it into a close relationship with the coordinate system, namely, the *asymptotic complex.* We obtained the equation of the asymptotic complex in the general case when we let the three variables r, s, h vanish in the equation of the given one. It then represented a conic surface of class two whose vertex fell upon the coordinate origin, and which cut out a curve that was enveloped by lines of the complex from the plane at infinity. If the relationship of the plane at infinity to the given complex is specialized in that way then further terms beyond those of second order in ρ, σ, η must be selected from the equation of the given complex for inclusion in the equation of the asymptotic complex in order for it to represent the lines of the complex at infinity with the same degree of approximation as in the general case. The degree of approximation of the asymptotic complex with respect to the given complex is the first in all cases; that is, the relationship of the asymptotic complex to the given complex will remain unchanged when we displace them with respect to each other parallel to themselves through a finite line segment.

Similar considerations can be posed for an arbitrary plane, in particular, for each of the three coordinate planes. We say "asymptotic complex of the given complex relative to a coordinate plane" to mean the complex that has in common with the given complex all lines that lie in this plane and, up to higher-order quantities, in all planes that differ infinitely little from the coordinate plane, and which is the simplest of the complexes that are endowed with that property, in and of themselves, as well as in relation to the coordinate system.

335. In order to exhibit the equation of the asymptotic complex that is associated with a coordinate plane, we proceed as before in the case of the plane at infinity. In particular, if we select the *YZ* plane then we will next get:

$$Bs^2 + Ch^2 + D\sigma^2 + 2Gsh - 2Ss\sigma - 2Th\sigma = 0, \tag{188}$$

since we let r, ρ, η vanish in the equation of the given complex, for which we would like to take equation (V). This equation represents a complex whose lines envelop a cylinder surface of class two whose sides are parallel to *OX*, and which will cut out the complex curve that lies in the *YZ* plane from that plane.

By a suitable choice of coordinate axes *OY* and *OZ* in the fixed *YZ* plane, we can, in general, bring the foregoing equation into the following form:

$$Bs^2 + Ch^2 + D\sigma^2 = 0. \tag{189}$$

If the complex curve in *YZ* resolves into a system of two points when *YZ* is a singular plane then one of the three constants *B*, *C*, and *D* will vanish. If *D* vanishes then we must add terms from the equation of the given complex that contain the variable σ in the first power to equation (189), which represents the asymptotic complex in the general case. In this way, the equation of the asymptotic complex will become:

$$Bs^2 + Ch^2 - 2\,(L\eta + M\rho)\,s = 0. \tag{190}$$

A term in $r\sigma$ does not enter into this. One then has:

$$-Nr\sigma + Os\rho + Vh\eta = (O - N)\,s\rho + (V - N)\,h\eta\,.$$

If the two points into which the complex curve in *YZ* has resolved, corresponding to the assumption that *YZ* is a double plane of the given complex, coincide in a point then two of the three constants *B*, *C*, *D* will vanish in equation (189). If *B* and *C* are the two vanishing constants then when we add the terms of first order in *s* and *h* to equation (189) the equation of the asymptotic complex will become:

$$Ds^2 + 2\,(Ir + Rh)\,s + 2\,(Hr + Ur)\,h + 2\,(O - N)\,s\rho + 2\,(V - N)\,h\eta = 0. \tag{191}$$

Finally, if the three constants *B*, *C*, *D* vanish together in equation (189), corresponding to the assumption that every straight line in the *YZ* plane belongs to the given complex, when we select the terms of first order in *s*, *h*, *s* from the equation of the given complex, we will get:

$$(Ir + R\eta)\,s + (Hr + U\rho)\,h - (L\eta + M\rho)\,\sigma - Nr\sigma + Os\rho + Vh\eta = 0 \tag{192}$$

for the equation of the asymptotic complex.

We will pursue these considerations no further here, but refer to the developments of the third paragraph, and in particular, we will go no further into a more detailed discussion of the complexes that are represented by equations (190), (191), (192).

336. A line complex represents a self-reciprocal structure, in the sense that its equation has a double interpretation, according to whether we consider the straight line to be a ray or an axis. An exchange of the two viewpoints will correspond to an exchange of the coordinates of the straight line among themselves. The form of the equation of the complex will then remain unchanged. In this fact, one finds the justification for carrying over all of the considerations and results that are contained in the foregoing from an arbitrary *plane* to an arbitrary *point* using the rules of the principle of reciprocity.

In particular, we would like to let the arbitrary point coincide with the coordinate origin. All analytic developments and relationships that we have posed for the plane at infinity will carry over to it, when we exchange point and plane coordinates, and ray and axial coordinates everywhere, corresponding to which, from the rules in number **153**, the following constants in the complex equation:

$$A, B, C, \qquad G, H, I, \qquad P, Q, R$$

will mutually switch with the constants:

$$D, E, F, \qquad K, L, M, \qquad S, T, U,$$

respectively.

In particular, we have obtained the equation:

$$Dt^2 + Eu^2 + Fv^2 + 2Kuv + 2Ltv + 2Mtu = 0$$

for the complex curve that lies in the plane at infinity. The equation that is derived from it according to the foregoing exchange rules:

$$Ax^2 + By^2 + Cz^2 + 2Gyz + 2Hxz + 2Ixy = 0$$

represents the complex cone whose vertex is the coordinate origin.

If we shift the arbitrarily-chosen point that we have made coincide with coordinate origin to infinity then we can choose it to be any of the three points at which the plane at infinity is cut by the coordinate axes OX, OY, OZ, respectively. The exchange rules that correspond to such an assumption are derived immediately from the foregoing ones when we next replace the plane at infinity with the coordinate planes XZ, YZ, XY, respectively, from the discussions of number **325**.

337. In what follows, we will restrict ourselves to expressing, with no further proof, the essential results that we previously derived for an arbitrary plane for an arbitrary point.

Let O be the chosen point, and let a, a', a'' be three arbitrary lines that go through it that are conjugate to each other relative to the cone K whose vertex falls upon O. The polars of these three straight lines relative to the complex – which we would like to denote by b, b', b'' – will lie in the three planes (a', a''), (a'', a), (a, a'), respectively. We call the three polars mutually conjugate. The polar lines of the point O relative to the complex curves that lie in the three planes (a', a''), (a'', a), (a, a') – which might be called c, c', c'', respectively – are said to be associated with the three polars and the three given lines a, a', a''. We refer to them as three mutually-conjugate polar lines. The following theorem is then true:

Any of three conjugate polars is cut by any of the polar lines that are associated with the other two.

Thus, any of three polar lines will also cut each of the polars that are associated with the other two. When the point *O* and three conjugate polars or polar lines are given, the associated polar lines (polars, respectively) can be constructed linearly using this theorem.

Three conjugate polars determine a hyperboloid, as lines of one generator, and the associated polar lines will determine it as lines of the other generator. The polar plane of the point *O* relative to that hyperboloid will be the plane *P* that contains the three points of intersection of any of the three conjugate lines with their associated polar line. This plane will not change when we replace the chosen three conjugate planes with any other three.

The polar plane of the point O relative to a hyperboloid that is determined by three conjugate polars is independent of the choice of these polars.

The plane *P* is then associated with the point *O* through the given complex. We would like to call it the *polar plane of the point O relative to the complex.*

In a second-degree complex, a given point will be in one-to-one correspondence with a plane, in general.

We construct this plane as the *polar plane of the given point relative to the surface that is defined by the singular points of the complex and enveloped by singular planes of it.*

338. If the chosen three straight lines a, a', a'' do not themselves belong to the complex then their associated polars will not intersect, in general. In general, there is *only one* system of associated polars that do intersect, and which will then coincide with their associated polar lines in the polar plane of the given point. The corresponding three straight lines a, a', a'' are easy to construct.

Four singular lines of the complex go through the given point *O*. The three lines of intersection of any two planes that collectively contain the four singular points will be the desired one.

Corresponding to a double specialization of the relationship between the second-degree complex and the given point, we can let the four singular lines that go through it coincide pair-wise. The polars of all straight lines that go through *O* in the plane that contains the two singular lines will then intersect in a point inside the polar plane *P* of the point *O*, namely, the point at which the polar plane *P* will be cut by the polar line of the aforementioned plane relative to the complex cone whose vertex falls upon *OP*, and the polar of this latter line will also fall in the plane *P*. It is the line of intersection with the plane that is drawn through the two singular lines.

A five-fold specialization is required when all polars are to be contained in the polar plane *P*. Each polar will then coincide with its associated polar line. This would demand that all of the complex lines that go through the point *O* must be singular lines of it. This condition is fulfilled, in particular, in the case of complex whose lines envelop *a second-degree surface.* All lines of such a complex are to be regarded as singular lines of it. The polar plane of an arbitrary point relative to such a complex will coincide with its polar plane relative to the surface that it envelops. The latter will make the surface be one of order and class four, which will be determined by the singular points and planes of the complex in the general case.

339. If the given point O is a *singular* point, in particular, then its polar plane will coincide with the associated singular plane. The same thing will be true for the tangential planes to the surface at the singular point and the planes at the given singular point.

If the given point O is a *double point* of the complex then its polar plane will be undetermined. It can be selected arbitrarily from the enveloping planes of a cone of class two that has the chosen point for its vertex. The point O will then be a double point of the surface of singular points and planes. The conic surface of class two that is enveloped by its polar planes will be the tangential cone of the surface at the double point.

Finally, all of the straight lines that go through the point O can belong to the complex. One could then no longer speak of a well-defined polar plane relative to the complex. That would correspond to the fact that the polar was separated from the surface of singular planes as an isolated point, which would reduce it to class three.

In conclusion, let it be remarked that in the case of the general second-degree complex, the correspondence between the polar plane and the given point is not reciprocal, as it is for second-degree surfaces. If the pole of the polar plane relative to the complex should once more coincide with the initially-given point then a three-fold specialization of the position of the plane in the complex would be necessary. There are then, in general, only a *finite number* of points and planes in a given complex that correspond reciprocally relative to the complex.

340. We have represented the lines of the complex that lie in a given plane or in the neighborhood of it by its *asymptotic complex* relative to the given plane. We can determine the straight lines in the complex that go through a given point and all of its neighboring points in a similar way.

Let the given point be the origin of the coordinates. We will then get:

$$Ar^2 + Bs^2 + Ch^2 + 2Gsh + 2Hrh + 2Irs = 0 \tag{193}$$

for the asymptotic complex of the given complex relative to that point when we neglect the variables ρ, σ, η, as well as first powers of r, s, h in the equation of the latter. This equation represents a *curve* of second order in the plane at infinity. The straight lines that go through the coordinate origin and cut that curve will belong to the given complex.

By an appropriate choice of directions for the coordinate axes, we can bring the foregoing equation (193) into the form:

$$Ar^2 + Bs^2 + Ch^2 = 0. \tag{194}$$

When the coordinate origin is a *singular* point of the complex, in particular, one of the three constants A, B, C will vanish; let A be the vanishing constant. In order to represent the lines of the given complex in the neighborhood of the coordinate origin to the same degree of approximation as before, we must then keep the terms of first order in r, along with the terms of second order in s and h, in the equation of the asymptotic complex. We then find:

$$Bs^2 + Ch^2 + 2\,(P\rho + Q\eta)\,r = 0. \tag{195}$$

No term in $r\sigma$ will enter in, since:

$$-Nr\sigma + Os\rho + Vh\eta = (O - N)\,s\rho + (V - N)\,h\eta.$$

The simultaneous vanishing of two of the three constants A, B, C – say, B and C – would correspond to the case in which the coordinate origin is a *double point* of the complex, and we would get the following equation for the asymptotic complex:

$$Ar^2 + 2\,(Rh - Ss)\,s + 2\,(-Ts + Ur)\,h + 2\,(O - N)\,s\rho + 2\,(V - N)\,h\eta = 0, \tag{196}$$

since we must consider first powers of s and h, along with second powers of r. Finally, when all of the straight lines that go through the origin belong to the complex, and correspondingly A, B, C vanish at the same time, the equation of the asymptotic complex will become:

$$(P\rho + Q\eta)\,r + (R\eta - S\sigma)\,s + (-T\sigma + U\rho)\,h - Nr\sigma + Os\rho + Vh\eta = 0 \tag{197}$$

This is the same equation that we found in number **292** in order to represent the lines at infinity of the given complex in the case in which terms of second order in the variables ρ, σ, η were missing from its equation.

We can impose considerations that are similar to the ones that we made for the coordinate origin for any of three points that are shifted to infinity along the three coordinate axes OX, OY, OZ.

341. Here, we suspend our foregoing developments, whose objective was the discussion of the general equation of the second-degree complex, in order to once more turn to the investigation of complex surfaces. In particular, we emphasize the great analogy that prevails between the theory of those complexes and the theory of second-degree surfaces, an analogy that that finds its explanation in the fact that the latter can be regarded as second-degree complexes of a special kind. The totality of all conditions that must be fulfilled in order for a given complex to represent a surface of that degree can be summarized in the statement that *all lines of such a complex are singular lines of it.*

Chapter Three.

Classification of the surfaces of a general second-degree complex. Construction and discussion of the equatorial surfaces.

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342. We have understood the term "*a complex surface*" to mean a surface that is the geometric locus of the curves that are determined by the lines of a given complex that lie in a plane that is drawn through a fixed straight line, or – what amounts to the same thing – a surface that will be enveloped by all cones of a given complex whose vertices lie along a fixed straight line. We can say that a complex surface represents the totality of all lines of a given complex that cut a fixed line. The consideration of these surfaces has the same meaning for the investigation of the complexes for which we have regarded the straight line as space element that the consideration of the plane curves of intersection or enveloping cones has for the investigation of surfaces.

In the case of second-degree complexes, a complex surface will be of order and class four, in general. The fixed straight line that determines the complex surface, along with the given complex, is a *double line* of the surface, in the two-fold sense that it appears a *double ray* and a *double axis* of it. For four distinguished positions of the plane that rotates around the double line, the curve that is enveloped by the lines of the complex in it, and which generates the surface, will resolve into a system of two points. These points will be double points of the surface. We have called the plane a *singular plane* of the surface and the connecting line of the two double points in it, a *singular ray*. The singular rays lie completely on the surface, in the sense that any point of the ray will be a point of the surface. The surface will be contacted by the singular planes when they are extended. Four of the points that lie along the double line are distinguished by the fact that they are the vertices of complex cones that have resolved into two planes. These planes will be double planes of the surface. We have called such a point a *singular point* and the line of intersection of the two double planes that it determines a *singular axis* of the surface. The singular axis will belong to the surface entirely, insofar as each of the planes that is drawn through it will be a plane of the surface. The common contact point of these planes will be the singular point.

343. These definitions take on an immediate clarity when we think of introducing the surfaces of order and class four that are defined by the singular points of the complex and enveloped by its singular planes. The four singular planes of a complex surface are the four tangential planes to that surface that can be drawn through its double line; the four singular points are the four points of intersection of the double line with that surface. The four singular rays and four singular axes are the singular lines of the complex that are associated with the four singular planes and four singular points, respectively.

In what follows, for the sake of brevity, we would like to denote the surface of order and class four that is determined by the singular points and planes of the given second-degree complex by the symbol Φ and denote the fixed straight line that generates the complex surface that we are considering, along with the given complex, by *d*.

We immediately obtain a classification of the complex surfaces, and in particular, those of the given complex, when we successively distribute the straight lines *d* over all different positions with respect to the surface Φ. The analysis in the fifth section of the previous chapter gives us enough material to proceed with such a discussion.

If we ignore the relationship of the complex surface to the given complex then the selected classification principle will emerge that allows us to decide how the singular elements of such a surface are grouped with respect to each other, and how many of them coincide, in particular. Here, we especially emphasize that the order and class of the complex surface are gradually reduced by the coincidence of singular elements until the surface is finally of order and class two.

For the sake of a further classification of the complex surfaces, we can decide whether the fixed straight line *d* does or does not belong to the given complex, and furthermore, whether the singularities that the complex surface possesses – viz., its singular planes and points, its double points and double planes – are real or imaginary. This is not the place to go into the classification of complex surfaces in greater detail. We will restrict ourselves to just the first, and most essential, classification principle: viz., to examine the relationship of the fixed line *d* to the surface Φ.

344. We thus obtain the classification below of the surfaces of a given second-degree complex in *seven ways* (*). These ways are not coordinated with each other. Moreover, each of the foregoing can be subsumed as a limiting case of the following one.

I.

The straight line *d* is chosen arbitrarily.

In section six of the last chapter, we subjected the case that we referred to as the *general* case to a thorough discussion and examined the mutual position of the singularities of the surface. There, under the assumption of real singularities, we found, in particular, a linear construction for the surface that must be replaced with a construction of second degree only when the given straight line *d* is itself a complex line. As the object of consideration, we first emphasize that the straight line *d* is a double ray and a double axis of the surface, and then that the four singular rays and four singular axes of the surface are simple rays, null axes and null rays, and simple axes of it. The order and class of the surface are both four. The number of arbitrary constants upon which such a surface depends is *seventeen*.

(*) It is easy to derive even more sub-classifications from the same principle, although we shall not go into that here.

II.

The straight line d contacts the surface Φ.

Two of the four singular points coincide with the contact point, and two of the four singular planes coincide with the respective tangential planes. The two associated singular rays and singular axes will coincide in the same straight line: viz., the singular line that is associated in the complex with the contact point of the surface Φ with the straight line d and the tangential plane to it. This line will be a double line of the surface, in the sense that it is a double ray, as well as a double axis. The complex surface possesses *two intersecting double lines.* The relationship between the two double lines and the surface is not the same. The surface will be cut by a plane that goes through d in a curve of class two, and a point that is chosen along d will be the vertex of an enveloping cone of order two. For the second double line, one switches the words "order" and "class."

The complex surfaces that we consider are of order and class four as in the general case. The number of arbitrary constants upon which they depend has been reduced to *sixteen*.

III.

The straight line d is a double tangent to the surface Φ.

The four singular planes and four singular points of the complex surface coincide pair-wise. Two double lines of the surface appear in place of the four singular rays and four singular axes. The surface contains *three double lines*, one of which (d) cuts the other two. If we lay a plane through one of the last two lines then it will cut the complex surface in a curve of order two that will have a double point on the other one, and which will have then resolved into a system of two straight lines. The complex surface has become a *ruled surface.* Its order and class remain four. Its constant count is *fifteen.*

IV.

The straight line d lies in a double plane of the surface Φ.

Of the four singular planes of the complex surface, two of them will coincide in the double plane, while the other two have an arbitrary direction. The four singular points coincide pair-wise in the two points of intersection of the straight line d with the conic section, along which, the surface Φ is contacted by the given double plane. Like the line d, they will be double lines of the surface. Since the complex surface is cut by the double plane in three double lines, the double line will belong to the surface when it is extended. Of the four singular rays of the complex surface, two of them will have an arbitrary position. The other two will be contained in the double plane and will be undetermined in it. They can be chosen arbitrarily among the complex lines that lie in that plane. We thus obtain the following result: If we regard the complex surface as being enveloped by planes then it will remain of class *four*. It will possess *three double axes that lie in a plane.* If we consider the surface as being defined by points then that will separate an isolated plane from it. In that way, the order of the surface will be reduced to *three.* The separated plane will be contacted triply, since it cuts the surface along three simple rays. After removing that plane, the surface *will have lost its double ray.*

V.

The straight line *d* goes through a double point of the surface Φ.

While one can derive no new kinds of complex surfaces from the first three using the principle of reciprocity, but one will only arrive at the same kinds all over again, this principle will lead from the aforementioned kind to a new one that is coordinated with it. We obtain it when we do not choose the straight line *d* to be in a double plane of the surface Φ, but to go through one of its double points. We will then find a surface *of order four and class three that has three double rays that intersect at a point and are simple axes* of the surface (*). The reduction of the class from four to three comes about because one point of the surface – viz., the point of intersection of the three double rays – is separated as a disjoint locus of class one. The complex surface then loses its double axes.

The surface, like the foregoing one, depends upon *fifteen* arbitrary constants.

VI.

The straight line *d* in a double plane of the surface Φ goes through one of its double points.

This assumes that a number of double points lie in any double plane of the surface Φ, which is easy to prove. Two contact curves of order two lie in two arbitrary double planes of the surface Φ. The line of intersection of the two double planes will be cut by these curves in the same two points. These two points are double points of the surface.

Corresponding to the assumption that the straight line *d* in a double plane of the surface goes through one of its double points, we will obtain a kind of complex surface that has the same relationship to the last two kinds that were posed and is again reciprocal to itself. The surface is of *order three and class three.* It will possess a *double ray* that cuts the straight line *d* and is a *simple axis*, along with a *double axis* that likewise cuts the straight line *d* and is a *simple ray.* The straight line *d* is then a simple line of the surface. As a surface of order three, or – what amounts to the same thing – a surface of class three with a double axis, the complex surface will be a *ruled surface* (**). The constant number has been reduced to *fourteen*.

(*) We encountered such a surface in number **251**. It was the geometric locus of the midpoints of all complex curves whose planes were drawn through the center of the complex.

(**) We have already encountered such surfaces several times from completely different viewpoints. The axes of the complexes of a linear congruence define such a surface whose double axis is shifted to infinity

VII.

The straight line *d* is the line of intersection of two double planes of the surface Φ and the connecting line of two of its double points.

From the remark that was made just now, the line of intersection of two double planes is also always the connecting line of two double points. The complex surface will then reduce to *order two and class two* (*), due to the fact that two isolated planes (viz., the two double planes of the surface Φ) or two isolated points (viz., the two double points of the surface) will be separated from it, according to whether we think of it as being determined in point or plane coordinates, resp. It has *lost all of its singularities*. In particular, the line *d* has become a null line of the surface.

Since there are 16 double planes (points, resp.) in a given complex, in general, the line *d* can assume a position in which its associated complex surface is of order and class two $16 \cdot 15 / 2 = 120$ times.

We find *thirteen* for the number of arbitrary constants upon which such a complex surface depends. *Nine* of them come from the second-degree surface, and *four* of them come from the latter straight line *d*, which is still in no way determined.

345. A reduction in the order or class of a complex surface will come about when one separates isolated planes (points, resp.) from the surface for special choices of the straight line *d*. We remark here incidentally that a reduction in order or class can come about in yet another way. Let the given complex be specialized in such a way that all straight lines that go through a fixed point that is chosen along the straight line *d* belong to it. The complex curve in an arbitrary plane that goes through *d* will then resolve into a system of two points, one of which will coincide with the given fixed point, and the other of which will have an arbitrary position. If we determine that curve in point coordinates then the doubly-counted connecting line of the two points will enter in place of the system of those two points. The complex surface will then be cut by a plane that goes through the straight line *d* in two coincident straight lines, along with *d* itself, that go through a fixed point that is given along *d*. The complex surface will have then degenerated into *a cone of order two* (**). The reduction of order from four to two will result when the surface resolves into a system of two surfaces of order two that coincide.

346. We have called complex surfaces that are defined by complex curves in parallel planes *equatorial surfaces.* We now go on to a discussion of these equatorial surfaces and first consider a restricted family of them. It will guide us to the double insight that, first of all, we will find a confirmation of the results up to now in simple and easily-

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and is perpendicular to the double ray (no. **86**). We found an entirely similar surface in the general theory of second-degree complexes as the geometric locus of cylinder axes that cut a diameter or the diameters that cut a cylinder axis (no. **243**, **246**).

(*) We have already considered this kind of reduction in order and class of a complex surface in number **258**.

(**) We represented a complex surface of this special kind in mixed coordinates in number **292**.

constructible surfaces. Then, however, we will arrive at, perhaps, an *intuition* into the multifaceted character of complex surfaces, to begin with, and with that, the distribution of straight lines in a second-degree complex using these surfaces. In what follows, we will thus observe not just the number and positions of the singularities of the surface, as in the foregoing, but especially the *form of the surface components*, which define a transition between the singularities, and the *structural character of that transition.* In these investigations, we will not consider equatorial surfaces in full generality, at first, but subject them to a number of simplifying conditions. The possibilities that we thus exclude are of minor significance for our purposes, and their consideration would only complicate the argument unnecessarily. We will also pass over the generation of equatorial surfaces by enveloping cylinders, and consider only their emergence from the advance of complex curves in parallel planes. This way of determining a surface will lie incomparably closer to our intuition that the other way by enveloping cylinders.

347. In number **273**, when we assumed that the straight lines at infinity in the breadth planes did not themselves belong to the complex, we obtained the following equation for the general equation of such a surface:

$$Dw^2 + (Fx^2 - 2Rx + B)\, v^2 - 2\,(Ox + G)\, uv + (Ex^2 + 2Ux + C)\, u^2 = 0. \tag{1}$$

The coordinate plane *YZ* is then chosen to be parallel to the breadth planes of the complex surface. The *OX* axis coincides with the diameter of the given complex, which is associated with the system of breadth plane, and which we have referred to as the *diameter of the equatorial surface.* Finally, *OY* and *OZ* have the directions in *YZ* of two diameters of the complex that are conjugate to each other and to *OX*.

Equation (1) contains *eight* mutually-independent constants, and along with the *seven* that specialize the coordinate system, that will give *fifteen* constants upon which an equatorial surface will depend. The coordinate origin on *OX* (*) and the angle between the *OY* and *OZ* coordinate axes can be chosen arbitrarily.

In what follows, for the sake of simplicity and intuitiveness, we would like to assume that the coordinate system that equation (1) is based upon is rectangular, which will require a *two-fold* specialization of the equatorial surface. The diameter of the surface will then be perpendicular to the direction of the parallel planes that are associated with it in the complex, and will thus coincide with one of the three principal axes of the complex.

(*) In the cited number, we determined the origin on *OX* by the following condition:

$$ER = FU,$$

which says that the plane *YZ* goes through the midpoint of the complex. We can drop this condition here as being inessential for the following considerations.

348. We next link our further developments with the assumption that the two constants G and O in equation (1) vanish. The equation of the surface, in which we would like to set $D = 1$, with no loss of generality (*), will then be the following one:

$$w^2 + (Fx^2 - 2Rx + B)\, v^2 + (Ex^2 + 2Ux + C)\, u^2 = 0. \tag{2}$$

The complex curve that lies in an arbitrary breadth plane is referred to the two coordinate axes OY and OZ, just as it is to its own axes. Corresponding to the vanishing of G and O, the equatorial surface (2) has been specialized in such a way that *the axes of its breadth curves point in the same direction.* Thus, these equatorial surfaces will approach the type of second-order surfaces and will come closer our intuition. Two of the four singular rays of the surface will be pair-wise parallel. We can say that the surfaces that we consider are distinguished by the fact that *their singular rays cut two of their double lines.*

Equation (2) contains *six* mutually-independent constants. The rectangular coordinate system to which it is referred is determined completely, up to the position of the origin, which can still be chosen arbitrarily along OX. Thus, equatorial surfaces that are represented by equation (2) will depend upon *eleven* constants, while this number will amount to *fifteen*, in general.

We immediately obtain the equation for the surface in point coordinates from equation (1):

$$\frac{y^2}{Ex^2 + 2Ux + C} + \frac{z^2}{Fx^2 - 2Rx + B} + 1 = 0. \tag{3}$$

The surface remains of order four. It will be cut by the coordinate planes XY, XZ in two second-order curves, since two singular rays of the surface will lie in that plane, along with the second-order intersection curve.

349. We would like to refer to the second-order curves along which the equatorial surface is cut by the two coordinate planes XY and XZ as the two *characteristics.* If we let z and y vanish in the foregoing equation, in succession, then we will get the following two equations for them:

$$\left.\begin{aligned} y^2 + Ex^2 + 2Ux + C = 0, \\ z^2 + Fx^2 - 2Rx + B = 0. \end{aligned}\right\} \tag{4}$$

Of the two axes of these two conic sections, one will fall along the OX coordinate axis, while the other will be parallel to OY and OZ, respectively.

In number **185**, we obtained the following two equations in order to represent the complex cylinder whose sides are parallel to the OZ and OY coordinate axes, respectively:

(*) The assumption $D = 0$ corresponds to the parabolic equatorial surfaces, which will remain excluded here.

$$\left.\begin{aligned} Fx^2 - 2L\,x\,z + Dz^2 + 2Sz - 2Rx + B = 0, \\ Ex^2 - 2Mxy + Dy^2 + 2Ux - 2Ty + C = 0. \end{aligned}\right\} \tag{5}$$

If we let L, M, S, and T vanish in these two equations, and set D equal to unity then they will coincide with the two equations (4). As is also geometrically clear, the equatorial surface that is represented by equation (4) will be contacted by the two complex cylinders whose sides are parallel to OZ and OY, respectively, along the two characteristics that lie in XY and XZ, resp.

If the two characteristics are given then the equatorial surface will be determined uniquely and its geometric construction will be given. The two equations of the characteristics will then collectively contain precisely the same number of independent constants as the equation of the surface itself. *The classification principle for equatorial surfaces of this kind is then borrowed from the differing natures and relative positions of the two characteristics.*

350. We would like to rotate the XZ plane, along with the characteristic that it contains, around OX in such a way that it coincides with XY. If we then draw a parallel to OY through an arbitrary point of the OX then the four pair-wise equal line segments that are cut out of these straight lines by the two characteristics will give the magnitudes of the two semi-axes of the complex curve that lies in the breadth plane that goes through this straight line.

If the breadth curve is a hyperbola or an imaginary ellipse, and we would like to construct its imaginary axes as a real straight line in the same way, then we must add *a second curve of order two* to each characteristic that has a midpoint and an axis that falls upon OX in common, while the second axis equal to the second axis of the characteristic in absolute magnitude, but will be imaginary or real according to whether the latter is real or imaginary, respectively. Once the two characteristics are extended in that way, the construction of the surface will be given in all cases.

We have brought the two characteristics into the same plane by rotating them around OX. Their four points of intersection in it will determine the two breadth planes by which the equatorial surface will be intersected in real circles. Imaginary circles, as curves of intersection with the surface, will lie in the two breadth planes that are given by the four points of intersection of the two second-order curves that we have affected the characteristics with. Finally, the four points of intersection of a characteristic with the other associated extension curve will each time determine two breadth planes that will cut the equatorial surface in equilateral hyperbolas.

351. In particular, if we give the breadth plane one of the four positions that are determined by the four points of intersection of the two characteristics with the OX axis then the magnitude of the one axis of the breadth curve will vanish, and as a result of that, it will reduce to two straight lines that coincide in one of them. These straight lines will be the singular rays of the equatorial surface.

Consistent with that, equation (77) of number **195**, by which the breadth planes of the four singular rays were determined in the general case of equatorial surfaces, such that *K*, *G*, *O* vanished, will decompose into the two equations:

$$\left.\begin{array}{l} Ex^2 + 2Ux + C = 0, \\ Fx^2 - 2Rx + B = 0, \end{array}\right\} \tag{6}$$

and these two equations will determine the points of the diameter of the equatorial surface at which it will be cut by the two characteristics (4).

The two singular rays that lie in the plane of one characteristic perpendicular to the diameter will go through the intersection of the other characteristic with the diameter.

If the diameter of the surface is not cut by the either of two characteristics at any real point then the four singular rays *will all be imaginary*, and the equatorial surface will define an undivided whole.

If the diameter is cut by one characteristic in real points, but not by the other one, then *two of the four singular rays will be real, and two of them will be imaginary*. When we consider the two external parts of the surface, which merge together at infinity into the complex curve at infinity, to be a single surface component, then that surface will decompose into two components, one of which is finite, while the other of which is at infinity, and between which the two singular rays will point to the transition.

If the diameter intersects both characteristics in real points then *the four singular rays will all be real*, and the surface will decompose into four parts that are separated by the singular rays when we once more consider the two external surface components to be a single one.

352. We can distinguish two kinds of singular rays (cf., no. **188**). *Singular rays of the first kind* are the connecting lines of two real double points of the surface, and define the transition between real complex ellipses and complex hyperbolas. *Singular rays of the second kind* are the connecting lines of two imaginary double points of the surface, and define the transition from complex hyperbolas to imaginary complex ellipses.

The surface component between two successive singular rays will be defined by curves of the same kind. One can then find no parabolas among the breadth curves as transitions between curves of different kind, since otherwise the lines at infinity in the breadth planes would belong to the complex, which would contradict the assumption. According to whether the breadth curves between two successive singular rays are real ellipses, hyperbolas, or imaginary ellipses, we would like to refer to the surface component as *elliptic*, *hyperbolic*, or *imaginary*, respectively.

If an elliptic and a hyperbolic surface component follow in succession then they will merely merge into two points of the singular ray that separates them. These two points (viz., the two into which the complex curve degenerates in the relevant singular plane) will divide the ray into a middle, finite segment and two external, infinite segments, which are to be regarded as one. The middle segment will belong to the elliptic surface component, and should be regarded as an ellipse whose one axis vanishes. The two external segments will belong to the hyperbolic surface component and should be regarded as a hyperbola whose auxiliary axis vanishes, or – what amounts to the same thing – and whose asymptotic angle has become equal to zero.

If an imaginary surface component follows a hyperbolic one then the former will end up as an unbounded straight line along sides of the latter. This straight line is to be regarded as a hyperbola whose principal axis vanishes, or – what amounts to the same thing – whose asymptotic angle has become equal to π.

353. An elliptic surface component is bounded by two singular rays of the first kind (*). They can be parallel or perpendicular to each other.

The ratio of the two axes of the generating ellipse, which is equal to zero in the two limiting positions, will be a maximum in the former case. When this maximum is smaller than one, if one starts from one of the two limiting positions then the generating ellipse will approach a circle up to the maximum, and will then move away from it without having reached it, until it once more goes to a straight line at the other limit. The major axis of the ellipse will always remain in the same direction. When the maximum is greater than one, the generating ellipse will go through two circles between the two limiting positions. Under this transition, the directions will switch their major and minor axes. The major axis is directed perpendicular to the two limiting singular rays. Finally, when the maximum is equal to one, there is a circle among the generating ellipses that should be regarded as two coincident ones. There are then always two ellipses that are similar to the given one.

In the second case, the generating ellipse will go through a single circle on its way from one limiting position to the other. Under the transition through the circle, the major and minor axes of the generating ellipse will switch their directions reciprocally. There are two ellipses that are similar to a given ellipse for a crossed direction of their major axes.

We will get an equal classification into two different kinds for the imaginary surface components. They are bounded by either two parallel rays of the second kind or two mutually-perpendicular rays of the second kind.

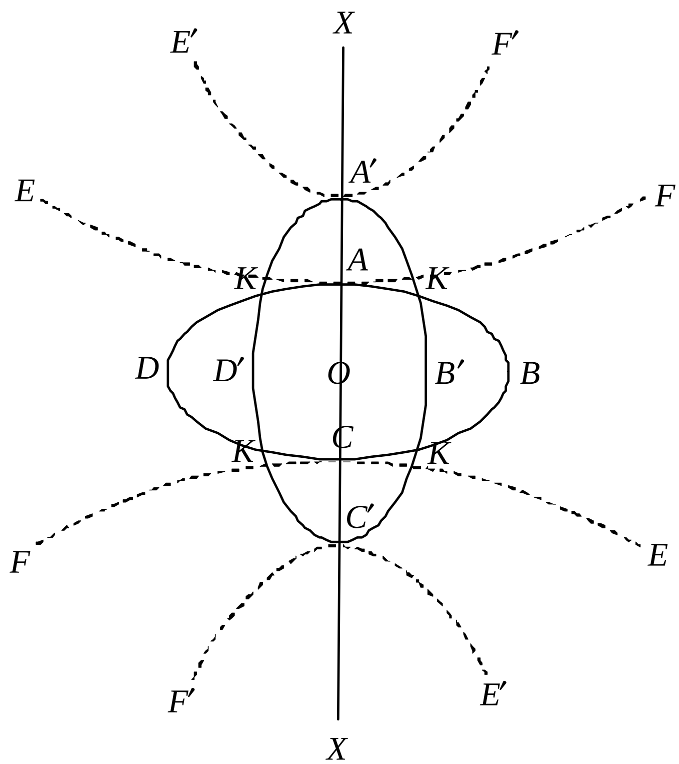
We must distinguish two cases for hyperbolic surface components, as well, according to whether the two limiting singular rays are parallel or cross. In the former case, the two rays will have the same kind, while in the latter, they will have different kinds. If the two rays are parallel then for all complex hyperbolas of the surface piece, either the principal axis or the auxiliary axis will have a finite magnitude. This will depend upon whether the two rays are of the first or second kind, resp. Under both assumptions, there will be two real or imaginary or coincident breadth planes, which will cut the surface component in equilateral hyperbolas. By contrast, of the two singular rays that bound the hyperbolic surface piece, there will then be one, but also only one, equilateral hyperbola among the generating complex hyperbolas. Whereas, in the first case, the asymptotic angle of the complex hyperbola will start from 0 (π, resp.) and will once more return to its initial value, in the second case, it will increase continuously from one of these values to the other one.

(*) In the text, we exclude the assumption that the equatorial surface consists of an undivided whole.

354. We would like to explain the foregoing with an example. Let the two ellipses *ABCD* and *A′B′C′D′* be the two characteristics in the figure, which we have brought into a plane by rotation around *OX*. We extend the two ellipses, in the sense that was established above (no. **350**), two hyperbolas; the two hyperbolas are *AECF* and *A′E′C′F′*. In order to suggest that the breadth curves will be determined, up to magnitude, by the imaginary axes, we have not removed them, but dotted them.

If we now draw a perpendicular to *OX* in the picture, corresponding to an arbitrary *x*, then the two sections of it, which are determined by the two characteristics and their extension curves, will represent the axes of the complex curve in the breadth plane that goes through *x*.

Corresponding to the vertex tangents at *A′*, *A*, *C*, *C′*, we will get the four singular rays of the equatorial surface. The rays that go through *C* (*A*, resp.) are of the first kind, while the other two are of the second kind.



Between *A* and *C*, the breadth curves will be ellipses. Among them, one finds two circles, corresponding to the points of intersection *K*. The elliptical surface piece is of the first kind. Two hyperbolic surface pieces of the second kind will close up on it that reach from *A* to *A′* (*C* to *C′*, resp.). The two breadth planes that contain equilateral hyperbolas are the ones that are determined by the intersection of the ellipse *A′B′C′D′* with the hyperbola *AECF*. Imaginary breadth curves will ensue from *A′* (*C′*, resp.) onward. The equatorial surface is included completely between the breadth planes that are given by *A′* and *C′*.

355. In what follows, we would like to denote an elliptical surface component by *E*, a hyperbolic one by *H*, and an imaginary one by *I*, and let the applied numerals 1, 2 distinguish whether a surface component is bounded by parallel or mutually-perpendicular singular rays, resp. The hyperbolic surface component of the first kind H_1 can be bounded by singular rays of either the first kind or the second; we correspondingly denote them by H_1' and H_1'', resp. In all cases in which surface components are no longer bounded on both sides by singular rays of a definite direction, we will use simply the symbols *E*, *H*, *I*.

We will get a *symbol* for any equatorial surface when combine the symbols that were introduced for the individual surface components in such a way that we begin with the surface component that extends to infinity and also conclude with it, as well. Thus:

$$I_1\, H_2\, E_1\, H_2\, I_1$$

represents the surface that was considered in the previous number. Such a symbol suggests, not only the type of the individual surface component, but also the type and position of the singular rays of the surface, such that they will succeed in characterizing an equatorial surface, as we have considered it here.

356. One deduces the present enumeration of *seventeen coordinates types* (*) at once from the equatorial surfaces that are represented by equation (3) when one decides whether the two characteristics are imaginary ellipses, real ellipses, or hyperbolas (**), and likewise directs one's attention to the relative positions of their points of intersection with the diameters of the surface. Each time, we shall give the type and position of the characteristics and the sequence of surface components that would require. The seventeen types are arranged into three groups according to the reality of their singular rays (***).

First group: The singular rays are all imaginary.

1. Two imaginary ellipses. *I*.
2. An imaginary ellipse and a hyperbola whose auxiliary axis falls upon the diameter. *H*.
3. Two hyperbolas whose auxiliary axes fall upon the diameter. *E*.

Second group: Two of the four singular rays are real, and two of them are imaginary.

4. An imaginary ellipse and a real ellipse. $I_1\, H_1''\, I_1$.
5. An imaginary ellipse and a hyperbola whose principal axes fall upon the diameter. $H_1''\; I_1\, H_1''$.
6. A real ellipse and a hyperbola whose auxiliary axis falls upon the diameter. $H_1'\, E_1\, H_1'$.
7. A hyperbola whose principal axis and a hyperbola whose auxiliary axis falls upon the diameter. $E_1\, H_1'\; E_1$.

(*) [Plücker has made models of most of the surfaces that are discussed in what follows, which eases the imagination of them considerably. F. K.]

(**) Here, we might emphasize incidentally that that the complex that determines the equatorial surface will necessarily be a *hyperboloidal* one, as long as one finds a hyperbola among the characteristics of the surface.

(***) Among the presently-enumerated surfaces, the ones whose characteristics possess the same midpoint are distinguished by their symmetry. Such surfaces correspond to the assumption that all of the diameters that are associated with the *OX* axis in the complex that determines the equatorial surface will intersect at the center of the complex (cf., no. **252**).

Third group: The singular rays are all real.

A. Two real ellipses.

8. The two points of intersection of the diameter with one ellipse follow from the two points of intersection of it with the other one. $I_2\, H_1''\, I_2\, H_1''\, I_2$.

9. The intersection with the one ellipse with the surface lies between the two intersections with the other ellipse on the diameter of the surface (*). $I_1\, H_2\, E_1\, H_2\, I_1$.

10. The points of intersection of one and the other ellipse with the surface lie alternately on its diameter. $I_2\, H_2\, E_1\, H_2\, I_2$.

B. A real ellipse and a hyperbola whose principal axis falls upon the diameter of the surface.

11. The diameter of the surface is cut by the ellipse in two points that lie between the two points of intersection with the hyperbola.

12. The intersection of the surface with the ellipse lies on the diameter of the surface inside of its branch of the hyperbola. $H_2\, I_1\, H_2\, E_1\, H_2$.

13. The vertex of the hyperbola lies between the intersections of the ellipse with the diameter of the surface. $H_1'\, E_2\, H_1'\, E_2\; H_1'$.

14. Of the two vertices of the hyperbola, one of them lies outside of the ellipse and the other one lies inside of it. $H_2\, I_2\, H_2\, E_2\, H_2$.

C. Two hyperbolas whose principal axes fall upon the diameter of the surface.

15. The two vertices of the one hyperbola lie on the diameter between the two vertices of the other. $E_1\, H_2\, I_1\, H_2\, E_1$.

16. The vertex of the one hyperbola follows the vertex of the other one along the diameter. $E_2\; H_1'\, E_2\; H_1'\, E_2$.

17. One vertex of each of the two hyperbolas lies between the two vertices of the other one along the diameter. $E_2\, H_2\, I_1\, H_2\, E_2$.

357. In our next discussion, we would like to emphasize the particular case in which two of the singular rays of the equatorial surface *fall in its breadth plane.* The corresponding surfaces should be regarded as *transitional forms* between two of the previously-enumerated seventeen types. They will depend upon one less constant than the surfaces that were considered up to now, and thus, upon *ten* constants. Due to the fact that two singular rays of the surface fall in the same breadth plane, the surface component that is included between them will vanish. The breadth plane will no longer refer to the transition between a hyperbolic and an elliptic or an imaginary surface component, as before.

The singular rays can fall pair-wise into the same breadth plane; three of them can lie in the same plane, etc. All such surfaces are again found among the various kinds of

(*) The surface that was considered in number **354**.

complex surfaces that we obtained in number **344** by a general classification when we assume that the relationship of the straight line *d* that determines the complex surface, along with the given complex, to the surface Φ of singular points and planes of the complex is specialized in some way.

We next obtain two specializations of the kind in question, since we can assume that either two parallel or two crossed singular rays fall upon the same breadth plane, so the equatorial surface will lose a surface component of the first or second kind, resp.

358. Two parallel singular rays of the equatorial surfaces will coincide when we assume that one of the two characteristics *has resolved into a system of two straight lines.* The two coincident singular rays will then appear as a *double ray* of the equatorial surface. When separated by the double line, two elliptic or two imaginary or two hyperbolic surface components that open in the same sense will connect with each other. In general, the double ray does not lie on a real component of the surface for its entire extent, but moves across it as an isolated straight line. If the two coincident singular rays are of the first kind then a bounded piece of the double ray will define the transition between two elliptic or two hyperbolic surface components. If they are of the second kind then the (always real) double ray will define the transition between two successive hyperbolic or imaginary surface components. In the latter case, the double ray will be an isolated straight line.

The surfaces that we consider here should be regarded as transition forms between ones that belong to either the first and second group of equatorial surfaces that were enumerated in number **356** or to the second and third group. In the general classification of the complex surfaces that we gave in number **344**, they will be found there among the type that was denoted by II.

When we decide whether the line-pair into which the one characteristic decomposes is real or imaginary, and furthermore, whether the second characteristic is an imaginary ellipse or a hyperbola or a real ellipse, and when we fix our attention upon the position of the (always real) intersection of the two straight lines into which the one characteristic has resolved with respect to the second characteristic, we will obtain the following classification of such surfaces into *twelve* types. We denote them by the numbers **18-29**, in turn, and give the sequence of surface components for each of them and those two of the seventeen types that were enumerated up to now that define the transition between them. We denote the double ray into which two parallel singular rays coincide by one or two vertical lines according to whether it is of the first or second kind, respectively. We will then get the following table:

18.	$I_1 \parallel I_1$.	1, 4.
19.	$H_1'' \parallel H_1''$.	2, 5.
20.	$H_1' \mid H_1'$,	2, 6.
21.	$E_1 \mid E_1$.	3, 7.
22.	$I_2 \parallel I_2\ H_1''\ I_2$.	4, 8.
23.	$I_1\ H_2\ H_2 \mid I_1$.	4, 9.
24.	$H_1''\ I_2 \parallel I_2\ H_1''$.	5, 11.

25. $H_2\, I_1\, H_2 \mid H_2$. 5, 11.
26. $H_2 \parallel H_2\, E_1\, H_2$. 6, 12.
27. $H_1'\, E_2 \mid E_2\, H_2$. 6, 13.
28. $E_1\, H_2 \parallel H_2\, E_1$. 7, 15.
29. $E_2\, H_1'\, E_2 \mid E_2$. 7, 16.

359. *Both* characteristics can degenerate into systems of two (real or imaginary) straight lines. Since the parallel singular rays coincide, the surface will then obtain two crossed double rays and will be a ruled surface. It will still depend upon *nine* constants. Such surfaces belong to the third of the types that were exhibited by the general classification of complex surfaces. They should be regarded as transitional cases between the twelve previously-enumerated cases. We distinguish *three* types of them, according to the reality of the two line-pairs into which the characteristics decompose:

30. $I_2 \parallel I_1 \parallel I_1$. 18, 22.
31. $H_2 \parallel H_2 \mid H_2$. 19, 25; 20, 26.
32. $E_2 \mid E_2 \mid E_2$. 21, 29.

360. Two mutually-perpendicular singular rays of the equatorial surface will fall in the same breadth plane when we assume that *the two characteristics of the surface intersect.* It emerges from this that the two characteristics contact at a point of the diameter after one brings them into the same plane by a rotation around the diameter. The lines of the complex will envelop a system of two points that coincide on the diameter of the surface in the breadth plane that is determined by that contact point. The breadth plane will then be a double plane of the complex. The equatorial surfaces that we consider here belong to the fourth of the types of complex surfaces that were exhibited in number **344**. They still depend upon *ten* constants. When the double plane is separated from the equatorial surface as an isolated plane, that surface will become one of order three and will lose its double ray at infinity. The double plane will cut the surface along three simple rays, one of which will lie at infinity, and the other two of which will intersect in the diameter.

We will obtain the analytical confirmation of this result immediately from equations (6), by which we have determined the four points of the diameter of the equatorial surface at which it is cut by the four singular rays of the surface. When they possess a common root x', we can write them in the following form:

$$\left.\begin{aligned} E(x-x')(x-x_1) &= 0, \\ F(x-x')(x-x_2) &= 0. \end{aligned}\right\} \tag{7}$$

Equation (9), by which we have represented the equatorial surface in point coordinates, will then go to the following one:

$$(x - x')\left(\frac{y^2}{E(x-x_1)} + \frac{z^2}{F(x-x_2)} + (x - x')\right) = 0. \tag{8}$$

The linear factor:

$$x - x' = 0 \tag{9}$$

corresponds to the plane that is separated from the equatorial surface, and the equation:

$$\frac{y^2}{E(x-x_1)} + \frac{z^2}{F(x-x_2)} + (x - x') = 0, \tag{10}$$

which represents the surface itself, will be of order three.

If we set x equal to x', in particular, then the foregoing equation will go to the following one:

$$\frac{y^2}{E(x-x_1)} + \frac{z^2}{F(x-x_2)} = 0, \tag{11}$$

which is an equation that represents the real or imaginary line-pair according to whether the surface is or is not cut by the plane that is determined by equation (9) in its plane at infinity, in addition, respectively. The plane (9) contacts the surface (10) at the three points of intersection of these three lines.

The equatorial surface has lost two of its singular rays by the separation of an isolated plane. This plane defines the boundary of two successive elliptic and imaginary surface components or between hyperbolic surface components whose hyperbolas open in different senses. The second-order surfaces will give an intuitive example of both kinds of transition, when we think of them as being generated by curves in the plane that is moved parallel to itself.

The two characteristics of an equatorial surface that we consider here can be only real ellipses or hyperbolas whose principal axis falls upon the diameter. We then obtain the following enumeration of *seven* coordinate types, which characterize, in the previous way, by the givens of their surface component and those of the first seventeen surfaces between which the transition is defined. We have thus indicated the separated breadth planes by a cross:

33.	$I_2\, H \times H\, I_2$.	8, 10.
34.	$I \times E\, H_2\, I$.	9, 10.
35.	$H\, I_2\, H \times H$.	11, 14.
36.	$H_2\, I \times E\, H_2$.	12, 14.
37.	$H \times H\, E_2\, H$.	13, 14.
38.	$E\, H_2\, I \times E$.	15, 17.
39.	$E_2\, H \times H\, E_2$.	16, 17.

361. If the characteristics intersect the surface at two points then the surface *will go to a second-order surface*, since it will have lost all of its singularities. Among the breadth planes, there will then be *two* of them that are double planes of the complex that determines the surface and that contact the second-order surface. These planes will themselves be given by the second-order surface, insofar as, by assumption, they are perpendicular to one of the three principal axes of that surface. The equatorial surface will then depend upon just as many constants as a general second-degree surface. In fact, we find that its number of constants will be *nine*, which is one less than in the case that was treated in the previous number. If we have found *thirteen* constants for a complex surface that degenerates into a second-degree surface by the general classification of complex surfaces then four of the thirteen constants will belong to the straight line *d*, which has no particularly distinguished relationship to the surface, at all.

Here and in what follows, we shall not go further into the equatorial surfaces that degenerate into second-order surfaces.

362. We will find further types of the equatorial surfaces that were considered here when we assume *that one of the two intersecting characteristics has resolved into a system of two straight lines.* Such equatorial surfaces depend upon *nine* constants. They do not correspond to any special kind that was described in the general classification of complex surface, although we shall not go into greater detail about that. We will obtain one when we assume that the straight line *d* that determines the complex surface, along with the given complex, is contained in a double plane of the complex and contacts the conic section that this plane has in common with the surface Φ of singular points and planes of the complex.

In particular, we emphasize the singularity that such equatorial planes possess in their breadth plane that goes through the point of intersection of the two characteristics. Three singular rays will fall in these breadth planes. The breadth plane will then separate from the surface as an isolated plane, by which the order of the surface will become three, and the surface will lose two of its singular rays. The equatorial surface will have then lost two of its surface components since three of its singular rays will have been shifted into the same breadth plane. One of the two remaining ones will necessarily be hyperbolic, while the other one will be elliptic or imaginary, according to whether the line-pair into which the one characteristic has resolved is real or imaginary, resp. In both cases, the hyperbolic part will be contacted along the entire extent of the remaining three singular rays by the separated breadth plane. The point at which this singular ray cuts the diameter of the surface will be a *double point* of it. The tangents to the surface at them will lie in two separate, real or imaginary, planes, namely, the planes that go through the singular ray and the two straight lines into which the characteristic has resolved. The surface will be contacted by the two planes after the extension of these two straight lines. Every plane that contains the singular ray that goes through the equatorial surface at the double point can be regarded as a tangential plane to that surface. Whereas the cone of order two that is defined by the tangents to a surface at a double point, in general, will resolve into a system of two planes, in our case, the cone of class two that is enveloped by the tangential planes of a surface at a double point, in general, will degenerate into the system of two enveloped axes that coincide in the singular ray, in our case.

Next, let the line-pair into which a characteristic has resolved be real. An elliptic part of the surface will then follow a hyperbolic one. When the moving breadth plane approaches the distinguished position from the side of the hyperbolic component, the real, as well as the imaginary, axis of the hyperbola that is contained in it will always decrease in such a way that the asymptotic angle will always become larger and take on the value of π in the limit. Once the moving breadth plane has exceeded the distinguished position, it will contain an infinitely-small ellipse whose axes are to be regarded as infinitely different. It is the larger of the axes whose direction coincides with that of the singular ray.

If the line-pair into which the one characteristic has resolved is imaginary then an imaginary surface component will follow a hyperbolic one. One has to think of the transition as being such that the principal and auxiliary axes of the hyperbola that is moving to the boundary both decrease, although the former will do so faster than the latter. The hyperbolic part will then conclude in a hyperbola whose asymptotic angle is equal to zero against the imaginary one.

If we consider whether the line-pair into which the one characteristic has resolved is imaginary or real and whether the second characteristic is a real ellipse or a hyperbola whose principal axis falls along the diameter of the surface then we will get the enumeration of *four* types below. In it, we denote the distinguished breadth plane by a horizontal line. The equatorial surfaces that we consider here can be regarded as transitional forms between two surfaces whose one characteristic is a line-pair that does not cut the second characteristic, as well as ones between surfaces whose characteristics intersect without one of them resolving into a line-pair. We then obtain the following table:

40.	$I_2 - H_2\, I_2$.	22, 23; 33, 24.
41.	$H_2 - I_2\, H_2$.	24, 25; 35, 36.
42.	$H_2 - E_2\, H_2$.	26, 27; 36, 37.
43.	$E_2 - H_2\, E_2$.	28, 29; 38, 39.

It might finally be remarked that when both of the intersecting characteristics become line-pairs, the equatorial surface will reduce to order two when it is *conic surface.*

363. It still remains for us to discuss the case in which one or more of the singular rays of the surface are shifted to *infinity* (*).

If we assume that one of the two characteristics is a *parabola* then one of the singular rays will be moved to infinity. When a singular ray is moved to infinity, the surface will be divided into two parts by the plane at infinity. As long as no further singularities occur, one of these parts will be hyperbolic, while the other one will be elliptic or imaginary. Such a surface should be regarded as a transitional form between two of the types that were enumerated up to now that have a common characteristic, while the other one is a real ellipse and a hyperbola whose principal axis fall along the diameter, respectively. It depends upon on less constant than each of the two surfaces between

(*) Such a surface will give one an intuition into the distribution of the lines in complexes for which the plane at infinity is a singular plane or a double plane; i.e., into *hyperbolic, elliptic,* and *parabolic* complexes.

which the transition is defined. We immediately obtain the following enumeration of the cases that are possible here, although we shall not enter into a deeper discussion of them:

44. $I_1 H_1''$. 4, 5.
45. $H_1' E_1$. 6, 7.
46. $I_2 H_1'' I_2 H_1''$. 8, 11.
47. $I_1 H_2 E_1 H_2$. 9, 12.
48. $I_2 H_2 E_2 H_2$. 10, 14.
49. $H_2 I_1 H_2 E_1$. 12, 15.
50. $H_1' E_2 H_1' E_2$. 13, 16.
51. $H_2 I_2 H_2 E_2$. 14, 17.
52. $I_1 H_2 \mid H_2$. 23, 24.
53. $I_1 H_2 \mid H_2$. 23, 25.
54. $H_2 \parallel H_2 E_1$. 26, 28.
55. $H_1' E_2 \mid E_2$. 27, 29.
56. $I_1 H \times H$. 33, 35.
57. $I \times E H_2$. 34, 36.
58. $H_2 I \times E$. 36, 38.
59. $H \times H E_2$. 37, 39.
60. $I_2 - H_2$. 40, 41.
61. $H_2 - E_2$. 42, 43.

In the foregoing, the parabolic characteristic can be replaced everywhere with a system of *two parallel, real or imaginary, straight lines*. The surface will then obtain a second double ray at infinity. Such surfaces can take the form of limiting cases of the previously-enumerated surfaces whose one characteristic was a line-pair. They will accordingly depend upon one less constant. We combine the different types that we have encountered here into the following table, in which we denote the double ray of the surface in the previous way, also once it is moved to infinity, and in which we cite the previously-named type of equatorial surface from which the new one is derived.

62. $\parallel I \parallel$. 18.
63. $\parallel H_1'' \parallel$. 19.
64. $\mid H_1' \mid$. 20.
65. $\mid E_1 \mid$. 21.
66. $\parallel I_2 H_1'' I_2 \parallel$. 22.
67. $\mid H_2 I_1 H_2 \mid$. 25.
68. $\parallel H_2 E_1 H_2 \parallel$. 26.
69. $\mid E_2 E_2 \mid$. 29.
70. $\parallel I_2 \parallel I_2 \parallel$. 30.
71. $\parallel H_2 \parallel H_2 \parallel$. 31.
72. $\mid H_2 \mid H_2 \mid$. 31.
73. $\mid E_2 \mid E_2 \mid$. 32.

Both characteristics of the equatorial surface can be parabolas. The plane at infinity will then be a double plane of the complex that determined the equatorial surface. A plane is separated from the surface as isolated, and in that way the surface will become one of order three. We obtain the following enumeration, which is understandable with no further explanation:

74.	$\times I\, H_2\, E \times$.	34, 38; 47, 48, 49, 51.
75.	$\times H\, I_2\, H \times$.	35; 46, 51.
76.	$\times H\, E_2\, H \times$.	37; 48, 50.

Finally, of the two characteristics, *one of them can be a parabola and the other one, a real or imaginary pair of parallel straight lines.* We will then obtain the following two surfaces:

77.	$- I_2\, H_2 -$.	40, 41; 66, 67; 74, 75.
78.	$- H_2\, E_2 -$.	42, 43; 68, 69; 74, 76.

Corresponding to the assumption that *both* characteristics decompose into pairs of real or imaginary, parallel straight lines, the equatorial surface will be of order two and will degenerate into a *cylinder surface.*

364. The various cases of equatorial surfaces that are represented by equation (3) will be exhausted by this classification into **78** types, provided that their order does not drop below two. All of these equatorial surfaces will belong to the *first four* of the types that were exhibited in number **344** for the classification of complex surfaces. Since they are distinguished from the general types of surfaces that belong to it by structural simplicity and clarity, they can certainly be used as representatives of them. For equatorial surfaces that belong to the fifth or sixth of the types that were exhibited in number **344**, we will have to include an addendum.

Here, our first problem is to examine what value the enumeration of the 78 types that we gave here will have in the general discussion of the equatorial surfaces. The single specializing condition that we subjected the equatorial surface to in the foregoing was that we assumed that the axes of its breadth curves were equally-directed. In the general case, the sequence of surface components, the type of singular rays, etc., remained the same as it was under that special assumption. *We will get an intuition for the general equatorial surface when we think of the breadth curves of one of the surfaces that were considered up to now as being rotated with respect to the other one in their planes.*

We can think of the equatorial surfaces as *twisted* when their breadth curves possess fixed axis directions that are generally *rotated* with respect to each other.

This determination of a general equatorial surface is obviously only an approximate one. When the breadth curves rotate in their planes, their dimensions must change accordingly if the surface that arises is to be an equatorial surface. Meanwhile, these changes are only of order two when the magnitude of the rotation is of order one. We would like to use equation (2) as a basis:

$$w^2 + (Fx^2 - 2Rx + B)\, v^2 + (Ex^2 + 2Ux + C)\, u^2 = 0.$$

If we rotate the breadth curves in their plane, which is determined by x, from XZ to XY through an angle α then the equation of the surface that is defined by that will be:

$$\begin{aligned} w^2 &+ (Fx^2 - 2Rx + B)\,(u \sin \alpha + v \cos \alpha)^2 \\ &+ (Ex^2 + 2Ux + C)\,(u \cos \alpha - v \sin \alpha)^2 = 0. \end{aligned} \tag{12}$$

We would like to determine the angle α by the equation:

$$\sin \alpha = \frac{ax+b}{\sqrt{(Fx^2 - 2Rx + B) - (Ex^2 + 2Ux + C)}}. \tag{13}$$

Equation (12) will then go to the following one:

$$\begin{aligned} w^2 &+ (Fx^2 - 2Rx + B - (ax+b)^2)\, v^2 \\ &+ 2\,(ax+b) \cdot \sqrt{1-(ax+b)^2} \cdot uv \\ &+ (Ex^2 + 2Ux + C - (ax+b)^2)\, u^2 = 0. \end{aligned} \tag{14}$$

Up to quantities that are of second order in $(ax + b)$, we can set the square root that occurs in the latter equation equal to unity. The equation of the surface will then become:

$$\begin{aligned} w^2 &+ (Fx^2 - 2Rx + B - (ax+b)^2)\, v^2 \\ &+ 2\,(ax+b)\, uv \\ &+ (Ex^2 + 2Ux + C - (ax+b)^2)\, u^2 = 0, \end{aligned} \tag{15}$$

and will agree in form completely with the general equation (1) for the equatorial surfaces. In this equation, it is obvious that we must assume that either the two recently-introduced constants a, b are infinitely-small or that the consideration is coupled to only those breadth curves of the equatorial surface whose planes are close to the plane that is determined by the equation:

$$ax + b = 0.$$

365. In general, an equatorial surface whose equation in mixed coordinates is, in turn, the following one:

$$\begin{aligned} w^2 + (Fx^2 - 2Rx + B)\, v^2 - 2\,(Ox + G)\, uv \\ + (Ex^2 + 2Ux + C)\, u^2 = 0, \end{aligned} \tag{1}$$

will be cut by the two coordinate planes XZ, XY in two curves of order four. These curves determine the four breadth planes in which the singular rays lie by their intersection with the diameter of the surface.

However, the two cylinders that project the surface along OY and OZ, respectively, will remain of degree two, as before. We think of them as being given by their bases in XZ and XY, respectively. If we direct our attention to their type and relative positions then we will obtain precisely the same enumeration of 78 different cases as we did under the assumption that was used up to now that the bases of both cylinders were characteristics of the surface. The two projection cylinders will no longer have the exclusive relationship to the surface that they did before. The planes of the singular rays will not be determined by their intersection with the diameter of the equatorial surface, but by breadth planes in which hyperbolas will be enveloped by lines of the complex that possess an asymptote that is parallel to *OY* or *OZ*, respectively. For an arbitrary breadth curve, the two cylinders give only four pair-wise parallel tangents (*). A new condition must be added in order to determined the breadth curve completely.

As such a condition, we can take *the directions of its axes or their magnitude ratio*.

366. If we let φ denote the angle that one of the axes of the breadth curve that is determined by x defines with the *OZ* coordinate axis then we will get, as is known (**):

$$\tan 2\varphi = \frac{-2(Ox+G)}{(Fx^2-2Rx+B)-(Ex^2+2Ux+C)}. \tag{16}$$

For any point on one of the two axes, this will give:

$$\tan\varphi = \frac{y}{z}, \qquad \tan 2\varphi = \frac{-2yz}{y^2-z^2}.$$

With that, one will get:

$$\frac{yz}{y^2-z^2} = \frac{Ox+G}{(Fx^2-2Rx+B)-(Ex^2+2Ux+C)}. \tag{17}$$

The fourth-order surface that is represented by this equation is the geometric locus of the axes of the breadth curves of the equatorial surface that is determined by equation (1). It is a ruled surface with two mutually-perpendicular double lines, one of which coincides with the diameter of the equatorial surface, while the other one lies at infinity in the breadth plane of the line.

Here, we must distinguish two essentially different cases according to whether the constant O in equation (16) does or does not vanish.

In the first case, the rotation of the axes will arrive at a maximum or minimum that corresponds immediately to the minimum or maximum of the denominator, respectively. Starting from $x = -\infty$, where, in general, the axes of the breadth curve are parallel to the coordinate axes *OY*, *OZ*, the system of two axes will be rotated to a certain limiting position, and from that position, for $x = +\infty$ the initial position will again be assumed. Whether the maximum of the rotation is larger or smaller than 45° will depend upon the reality of the roots of the following quadratic equation:

(*) One must especially emphasize the case in which the bases of the two projection cylinders have two points of intersection in common with the diameter of the surface. The four planes in which the singular rays lie will then be determined by equations of degree two. When we, in turn, let G and O vanish, the equatorial surface will generate into a surface of degree two.

(**) *Analytisch geometrische Entwicklungen II,* no. **501**.

$$(Fx^2 - 2Rx + B) - (Ex^2 + 2Ux + C) = 0. \tag{18}$$

The direction of the axes will be the same at equal distances from a breadth plane to the ones that correspond to the maximal rotation of the axes.

In the second case, there are *two* breadth planes for which the rotation of the axes is a maximum or a minimum. These breadth planes can be imaginary or real. In the latter case, they will lie at equal distances from both sides of the plane that is represented by the equation:

$$Ox + G = 0. \tag{19}$$

If the two maximal values are imaginary then the axes of the breadth curves will rotate through 180° when x increases from $-\infty$ to $+\infty$. In the plane that is given by equation (19), the rotation will amount to 90°.

If the two maximal values are real then the axes of the breadth curves will rotate up to a certain limiting position when x increases from $-\infty$, and then reverse in their path until they again arrive at their initial position in the plane (19), then continue their rotation until they reach another limiting position, then turn around, and once more assume their origin directions for $x = +\infty$. The magnitude of the rotation will be the same for two values of x that correspond to breadth planes that lie harmonically with the two limiting positions. When equation (18) has real roots in the case that we are considering, they will determine two breadth planes that lie harmonically to the limiting positions and contain complex curves whose axes are rotated by 45° around OY, OZ. The one will then exceed the maximum rotation of 45°, but not the other one.

The construction of a breadth curve whose axes are not given, as far as position is concerned, and that is inscribed in the right angle that is determined by the two cylinders that are projected along OY and OZ requires no further explanation here. We merely remark that in the case where the sides of the right angle are all real, a second tetrangle will be obtained at once that contacts the breadth curve when we describe a circle around the given right angle and connect the four points in which the two axes cut the circle with four new straight lines. If two or four of the sides of the circumscribed right angle are imaginary then we can extend the corresponding projection cylinder in a manner that is similar to the way that did in number **350** with the two characteristics.

367. For the determination of the two asymptotes of a breadth curve that is given by x, we get:

$$(Fx^2 - 2Rx + B)\, v^2 - 2\,(Ox + G)\, uv + (Ex^2 + 2Ux + C)\, u^2 = 0$$

from equation (1) when we let w vanish. If we set:

$$-\frac{v}{u} = \frac{y}{z}$$

then that will give:

$$(Fx^2 - 2Rx + B)\, y^2 + 2\,(Ox + G)\, yz + (Ex^2 + 2Ux + C)\, z^2 = 0. \tag{19.b}$$

This equation represents a ruled surface of order four that is the geometric locus of the asymptotes of the breadth curves.

If we denote the asymptotic angles by ψ and $\pi - \psi$, and the angles that are defined by the same associated diameters by ω and $\pi - \omega$ then when we consider a given (real or imaginary) ellipse to be a hyperbola or a given hyperbola to be an ellipse, we will get (*):

$$\tan^2 \psi = -\sin^2 \omega .$$

We then find (**):

$$\tan^2 \psi = -\sin^2 \omega =$$

$$= \frac{4[(Ox+G)^2 - (Fx^2 - 2Rx + B)(Ex^2 + 2Ux + C)]}{(Fx^2 - 2Rx + B) - (Ex^2 + 2Ux + C)} . \tag{20}$$

This equation shows that, in general, there are four of the breadth curves of an equatorial surface that are similar to a given conic section.

For the complete determination of the breadth curve, we get the square of its semi-axis (***):

$$r^2 = -\tfrac{1}{2}[(Fx^2 - 2Rx + B) + (Ex^2 + 2Ux + C)]$$

$$\pm \tfrac{1}{2}\sqrt{[(Fx^2 - 2Rx + B) - (Ex^2 + 2Ux + C)]^2 + 4(Ox+G)^2} \tag{21}$$

The foregoing expression will serve as the model of a rotated equatorial surface in the calculations.

368. We now turn to the consideration of those equatorial surfaces *whose breadth planes contain a double point of the complex at infinity;* that is, the equatorial surfaces that belong to the fifth and sixth types of complex surfaces that were presented in number **344**.

If the equatorial surface belongs to the fifth type, as we would like to first assume, then it will possess three double rays that intersect in a point that are simple axes. The other two will be parallel to each other and to the breadth curves. The order of the surface will be four and its class will be three. The number of independent constants that enter into the equation of the surface will be *thirteen*.

What distinguishes such equatorial surfaces is the fact that *their breadth curves are all hyperbolas whose one asymptote has a fixed direction.* It will likewise be the direction of the two mutually-parallel double rays of the surface. This direction will point to the double point at infinity of the complex in the breadth plane.

The general linear construction of such equatorial surfaces is given by the foregoing remark. Here, as in the general case, the two projection cylinders along *OY* and *OZ* determine four tangents to such a breadth curve. A fifth tangent is given by the fixed direction of an asymptote. Of the thirteen constants upon which the surface depends, six of them will enter this construction for the determination of the breadth plane and the

(*) *System der analyt. Geometrie*, no. **33**.
(**) *Analytisch geometrie Entwicklungen, II*, no. **490**.
(***) *Ibidem*, no. **512**.

diameter, while six more will determine the two projection cylinders that are parallel to *OY*, *OZ*, and finally, one will determine the fixed direction of the one asymptote.

We can put the equation of such a surface into a simpler form by letting the *XZ* plane coincide with the plane that refers to the fixed direction of the one asymptote. The term in u^2 will then vanish in the equation of the equatorial surface. It is only then, in general, that it is not permitted to assume that *OY* and *OZ* have the directions of two associated diameters of the complex, and thus, that the constant *K* vanishes in equation (3) of number **163**. We thus obtain the following equation for the equation of the surface:

$$w^2 + (Fx^2 - 2Rx + B)\, v^2 + 2\,(Kx^2 - Ox - G)\, uv = 0. \tag{22}$$

The two points at which the diameter is cut by the two double rays that are parallel to *OZ* are determined by the equation:

$$Kx^2 - Ox - G = 0. \tag{23}$$

369. In the general case of equatorial surfaces, the asymptotes of the breadth curve will define a ruled surface of order and class four for which the diameter of the surface and the line at infinity in its breadth plane will be double lines. When one of the two asymptotes of each breadth curve has a fixed direction, a plane that goes through the diameter will separate from this ruled surface, along with a point that lies on the line at infinity. When we ignore those elements, the ruled surface will be of order and class three. Thus, the diameter will remain a double axis of the surface, while it will be a simple ray of it. Each point of it will be cut by a real generator of the ruled surface. Every plane that goes through it will contain two generators of the surface that will be real an imaginary, resp., and can also coincide. In general, there will be two planes in which the two generators coincide; they can be real or imaginary (*). Correspondingly, there will or will not be not two maximum rotations for the asymptotes of the breadth curves, resp.

The two generators along which the ruled surface is cut by the totality of the asymptotes of a certain surface that point in the same direction are the two double rays of the equatorial surface. In the case where there is a maximum for the rotation of the second asymptote, it can be real or imaginary or coincide. If there is no maximum for the second asymptote then the double rays will always be real.

Thus, when we first exclude the special assumption that the two double rays coincide, we will have *three* essentially different forms to distinguish for the equatorial surface that they belong to.

If the two double rays are imaginary then the equatorial surface will consist of an undivided whole. Among the breadth curves, there will be a hyperbola whose asymptotic angle is a maximum, and another one whose asymptotic angle is a minimum.

If the two double rays are real then the equatorial surface will decompose into two parts, one of which will extend to infinity along both sides. Here, as in number **358**, we must next distinguish between double rays of the first and second kind. Double rays of the first kind should be regarded as hyperbolas whose imaginary axis is equal to zero.

(*) A coincidence of the two either assumes a decomposition of the ruled surface or demands that the diameter of the surface be moved to infinity. Both possibilities remain excluded here.

They are divided into two segments: an internal, finite one and an external, infinite one. Now, the latter lies on real shells of the surface. A double ray of the second kind is to be regarded as a hyperbola whose principal axis is equal to zero. Its entire extent will lie on a real component of the surface. Under the transition of the breadth plane through the plane of a double ray, the second asymptote of the hyperbola that is contained in it will go to the first, fixed asymptote on the other side. Thus, the asymptotic angle will arrive at the value of zero or 180^o, according to whether the double ray is of the first or second kind, resp.

The two parallel double rays of the surface can be of *the same* or *different* types. There is a maximum and a minimum of the rotation of the second asymptote in the first case, but not in the second case. The ruled surface that is defined by the second asymptote still does determine whether the two double rays are of the first or second kind in the first case and which of the two double rays belong to the first kind and which of them belong to the second kind, in the second case. That will leave us with one arbitrary assumption.

A surface component that is bounded by two double rays of the first kind consists of hyperbolas whose asymptotic angles increase from zero up to a certain maximum and then decrease until they vanish again.

A surface component that is bounded by two double rays of the second kind will consist of hyperbolas whose asymptotic angle will increase continually up to the limiting value π.

In all cases, one of the two double rays can shift to infinity. The surface will then decompose into two parts that come together, once at finite points and once at infinity.

We deduce the analytical confirmation of the foregoing geometric statements immediately from equation (22). In particular, the case in which one of the two parallel double rays shifts to infinity will be characterized by the vanishing of K.

370. We now turn to the consideration of the case *in which the two parallel double rays of the surface coincide.* Such a surface is reciprocally coordinated with the kind that was treated in number **362**. It is distinguished by the fact that the straight line that lies at infinity in its breadth plane will go through a double point of the complex and contact the cone of class two that is defined by the singular planes that are associated with double points of the complex.

If the two double rays coincide in a straight line then they will contact two shells of the surface when they are extended. The common tangential plane at all of its points is the plane that can be laid through it and the diameter. The tangents in that plane will envelop two points that lie along the straight line, which will be real or imaginary according to whether the two coincident rays are of the first or second kind, resp. The two points are contact points of all planes that can be laid through two such straight lines and connect the vertices of the hyperbolas that are contained in neighboring planes. An arbitrary plane that goes through the straight line in which the two double rays coincide will contact the equatorial surface at the double point of the complex that lies at infinity on it.

371. It remains for us to discuss one last case, in which the equatorial surface degenerates into *a ruled surface of order and class three.* It seems unnecessary to enter into a deeper discussion of that surface, which was already mentioned several times in the foregoing (nos. **344**, **369**). Here, we shall only stress that in this case the surface that is defined by the asymptotes of the breadth curves will be a hyperbolic paraboloid. It is derived from the surface of asymptotes that was considered in number **369** when one separates a plane that is parallel to the breadth plane from the latter as an isolated plane. Correspondingly, the equatorial surfaces in question will be characterized by the fact that when we represent them by an equation of the form (22), the two second-degree expressions:

$$Fx^2 - 2Rx + B, \qquad Kx^2 - Ox - G$$

will possess a common factor.

Among these surfaces, one can distinguish the ones for which:

$$Kx^2 - Ox - G$$

is the square of a linear expression, and thus the double ray that such a surface possesses will coincide with its double axis. The straight lines at infinity in the breadth planes of such an equatorial surface that go through a double point in a double plane of the complex will then contact the second-order curve that is defined by the singular points that are associated with the double plane, or – what amounts to the same thing – it is a side of the cone of class two that is enveloped by the singular planes that are associated with the double point. The ruled surface of order and class three whose double ray and double axis coincide can then be regarded as transitional forms between the types of equatorial surfaces that were exhibited in numbers **362** and **370**.

372. We have thus exhausted the different cases of equatorial surfaces whose breadth curve possess a midpoint. It remains for us to discuss the ones whose breadth curves are parabolas, and which we have correspondingly referred to as *parabolic*. Here, we must speak briefly and settle for few explanations. The general classification of complex surfaces that we gave in number **344** will also retain its validity here. When we link the generation of the surface to a given second-degree complex, the special character of the surface – and thus, the grouping of its singularities – will, in all cases, be determined by the fact that the straight line at infinity in the breadth planes is a line of the complex.

We single out only two forms that we have encountered already in the foregoing.

We have discussed how the singularities arrange themselves with respect to each other for the *general case* of the parabolic equatorial surface in the sixth and seventh paragraphs of the first chapter (nos. **198**, **199**; no. **231**). According to whether the four singular rays that such a surface possesses are or are not all imaginary, the surface will define an undivided whole or decompose into several parts. The singular rays define the transition between parabolas that open in different senses.

In particular, the straight line at infinity in the breadth plane can be a *singular line* of the complex. The equatorial surface is then distinguished by the fact that the axes of its breadth curves point in the same direction (*). Two of its four singular rays will coincide with the straight line at infinity in the breadth plane.

In conclusion, we will summarize the formulas that serve to determine a parabola from its equation in line coordinates (*). We then start with the general equation of the parabolic equatorial surface, as we would deduce from equation (3) in number **163** when we let the constant D vanish in it. It is the following one:

$$\begin{gathered} 2\,(Lx - S)\,vw + (Fx^2 - 2Rx + B)\,v^2 \\ + 2\,(Mx + T)\,uw + 2\,(Kx^2 - Ox - G)\,uv \\ + (Ex^2 + 2Ux + C)\,u^2 = 0, \end{gathered} \tag{24}$$

which we would like to write in the form below, for the sake of brevity:

$$2bvw + cv^2 + 2duw + 2euv + fu^2 = 0. \tag{25}$$

When we let α denote the angle that the axis of the parabola makes with the OZ coordinate axis, we will get:

$$\tan\alpha = \frac{d}{b} \tag{26}$$

for the direction of that axis. The coordinates of the focal point are (**):

$$\left.\begin{aligned} y &= \frac{2be - d(c - f)}{2(b^2 + d^2)}, \\ z &= \frac{2de + b(c - f)}{2(b^2 + d^2)}, \end{aligned}\right\} \tag{27}$$

and the parameter will become:

$$\Pi = \pm\frac{d^2c - 2bde + b^2 f}{(b^2 + d^2)^{3/2}}. \tag{28}$$

373. The foregoing numbers were dedicated to the consideration of equatorial surfaces. We can discuss the various kinds of *meridian surfaces* in exactly the same way. Here, let us emphasize just one point: Among the complex curves that generate such a curve, one will find two parabolas, in general (no. **251**), whose planes are real or imaginary and can also coincide. These parabolas will define the transition between real ellipses and hyperbolas. We will get an intuition for the type of such a transition, when we consider the succession of intersection curves of a given one-shelled hyperboloid with a plane that rotates around a fixed straight line that intersects the hyperboloid at two real

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(*) We have considered such an equatorial surface in number **281**.

(*) *Analytisch geometrische Entwicklungen, II*, no. **480**, **506**.

(**) In particular, we can assume:

$$e = 0, \quad c = f.$$

The focal point of the parabola will then move along the OX axis.

points. Whereas, for equatorial surfaces, a surface component that is bounded by two singular rays will necessarily be defined by complex curves of the same kind, amongst the components into which a meridian surface is decomposed by its singular rays, there can be two of them that are generated by the various kinds of complex curves. This is the basis for the fact that there is a larger manifold of forms for meridian surfaces than the one that is defined by equatorial surfaces. We ascend from the discussion of equatorial surfaces to a discussion of meridian surfaces when we invoke the two planes that contain the parabolas arbitrarily from among the breadth planes of the equatorial surface. We shall then pursue the viewpoint that is suggested by that no further. *If has sufficed for the purpose of showing us how easy it is to arrive at a geometric understanding of the variegated surfaces of the second-degree complexes.*

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