“Mémoire sur les surfaces élastiques,” Mémoires de l'Académie Royale des Sciences de l'Institut Nationale **9** (1814-1816), 167-225.

On elastic surfaces

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Read on 1 August 1814

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The work of the geometers of the last century has brought general mechanics to a degree of perfection that leads one to regard that science as having concluded and to think that all that is left to conquer are the difficulties in integral calculus in each particular problem. Nonetheless, that is not true, and mechanics further poses several important questions that have not been addressed by calculations: The theory of elastic surfaces, which I propose to consider in this paper, offers a remarkable example of that. The differential equations of those surfaces in equilibrium, and more to the point, their equations of motion, are not known, either, except in the particular case where one is dealing with a cylindrical surface, which then leads back to the class of ordinary elastic strips. As one knows, Jacob Bernoulli was the first to give the equation of equilibrium of the elastic strip by basing it upon the hypothesis that the elasticity at each point is a force that is normal to the curve whose moment is proportional to the contingency angle or inversely proportional to the curvature at that point. Since the time of that great geometer, several other ones, and mainly Euler and Daniel Bernoulli, have published a great number of papers on the equilibrium conditions for elastic lines and the laws of the vibrations. However, all that appeared were some fruitless attempts that focused on elastic surfaces that were folded in two different directions. Thus, Euler presented some research into the sound of bells in his St. Petersburg papers in which he confined himself to considering the vibrations of each of the circular rings that a bell is composed of in isolation. That reduces the problem to that of simple elastic lines and leads to some results are not at all in agreement with experiments, moreover. In the same collection (from the year 1788), one also finds a paper by another Jacob Bernoulli that was written on the occasion of Chladni’s experiments on the vibration of resonant plates. That geometer considered a rectangular plate to be composed of two systems of parallel strips that were parallel to the sides of the rectangle and vibrated as if they were glued to together without hindering each other. Starting from that assumption, which Euler had already made in regard to the vibrations of drums, Bernoulli formed the partial differential equation that would serve to determine the small oscillation of the resonant plate. He himself then remarked that the consequences that can be deduced from it are not entirely in agreement with Chladni’s experiments, and indeed, one will see in my article that this equation is not the true equation, and that it lacks a term that the author could not find from his hypothesis.

About five years ago, the Institute proposed that the topic of a prize should be a theory of vibrations of resonant plates that is verified by comparison with experiments.

However, since that time, only one submission was received that was worthy of the attention of the class. At the beginning of that paper, the anonymous author presented, without proof and without saying what led him to it, a fundamental equation that included the term that Bernoulli's equation was lacking. Following the example of what Euler had done in regard to the equation of vibrating strips, the author of the article satisfied the equation that he had posed by means of some particular integrals that were composed of exponentials, sines, and cosines. Each of those integrals determined a particular figure of the vibrating plate that presented a certain number and disposition of nodal lines. In general, the sound that the plate made depended upon the number of those lines, and the integral established a relationship between that number and the corresponding sound. The author calculated the tone that related to each figure from that relationship, and then compared the calculated tone to the one that experiments gave for a similar figure. He found a satisfactory agreement between those two results, in such a way that the fundamental equation that he had started from, and to which we will arrive directly in this article, can be regarded as sufficiently verified by experiments, up to now. That comparison is the part of his work that motivated the judges to give him honorable mention. It drew upon a great number of experiments by Chladni and many others that the author of the cited study performed. There is another type of comparison that is much more difficult to undertake that relates to the figure that is produced by a given manner of putting the plate into vibration. One might also desire that the results of the calculation can be deduced from the general integral, and not from some particular integrals of the equation of the vibrating plates. Unfortunately, that equation can be integrated in finite form only for definite integrals that refer to imaginaries, and if one makes them disappear, as Plana did in the case of simple strips, then one will arrive at a very complicated equation that seems impossible to use.

Those are the only works on elastic surfaces that have appeared up to now, to my knowledge. That theory is one of the ones that merit the most attention from geometers, since on the one hand, it is attached to general mechanics by the search for the differential equations of equilibrium and motion, and on the other hand, it includes one of the most vast and curious branches of acoustics as an application. It is solely upon the basis of the first of those two relationships that I communicate this article to the class today, and in which my main goal was to arrive, with no hypotheses, at the equilibrium equations of elastic surfaces whose points are all acted upon by given forces.

This paper is divided into two parts. The first one relates to flexible, inelastic surfaces, whose equilibrium equation Lagrange has already given in the second edition of his *Mécanique analytique*. I arrived at that equation along a different path that has the advantage of showing the particular restriction to which it is subordinate. Indeed, it supposes that each element of the surface is equally stressed in all directions, which is a condition that is not fulfilled in large number of cases, and which will be impossible to fulfill, for example, in the case of a ponderous surface of unequal thickness. In order to solve the question completely, one must pay attention to the difference in the tensions that the same element will experience in two different directions. One will then find equilibrium equations that include those of analytical mechanics, but which are more general and also much more complicated.

In the second part, I consider elastic surfaces and determine the expression for the forces that are due to elasticity at each of their points. That quality of matter can be attributed to a repulsive force that is exerted between the molecules of the body, and its action extends over only immeasurable distances. The function that represents the law must become zero or negligible as the variable that represents the distances ceases to be extremely small. Now, one knows that similar functions generally disappear in the calculations and leave only total integral or arbitrary constants that are given by observation in the definitive results. Indeed, that is what happens in the theory of refraction, and even more so in the theory of capillary action, which is one of the most beautiful applications of analysis to physics to come from the geometers. The same thing is true for the present question, and that is what permits one to express the forces that are due to the elasticity of the surface in terms of quantities that depend solely upon its figure, such as its radii of curvature and their partial derivatives. Once those forces have been determined, it will be easy for me to define the equilibrium equation of the elastic surface by means of the equations that were found in the first part of the article. The same analysis can be applied to some surfaces whose thickness varies according to an arbitrary law. However, in order to not complicate the question, I have considered only the case of constant thickness. The equation to which I will arrive supposes, in addition, that the surface in question is naturally planar. It will not apply to elastic surfaces whose natural form is curved, such as bells, for example. The theory that has guided me cannot be applied to those surfaces without modifications that I shall not go into.

I have deduced the equation of motion of the elastic surface from its equilibrium equation from a known principle of mechanics, and upon assuming all of the limitations that the geometers have adopted for the problems of vibrating strings and strips, I found a linear equation in four variables for vibrating plates that does not differ essentially from that of the anonymous paper that I cited above.

In another paper, I shall apply the same considerations to elastic lines with simple or double curvature and with a thickness that is constant or varies according to a given law. That will lead me, in a manner that is direct and free from hypotheses, to not only their equations of equilibrium, but also to the expression for the forces that must be applied at their extremities in order to fix them and balance out the effect of elasticity.

CHAPTER I

EQUATION OF EQUILIBRIUM OF THE FLEXIBLE, INELASTIC SURFACE

1. – I consider a closed surface of a perfectly-flexible material that is devoid of elasticity and whose points are all acted upon by given forces. I will also suppose that it is inextensible, or at least only slightly extensible, in such a manner that the extension that it can exhibit will not alter its thickness appreciably, and that thickness can be constant or vary from one point of the surface to another, moreover. If the forces that are applied to them are given then I propose to find the equilibrium equation of the surface.

In order to do that, let x, y, z be the coordinates of an arbitrary point m of that surface, when referred to three rectangular axes that are chosen arbitrarily. We decompose all of the forces that act upon the point m along those axes and let X, Y, Z denote the components that point along the coordinates x, y, z, respectively, and tend to increase them. If those components are provided by gravity or other forces of attraction or repulsion that act upon all of the points of the matter that the surface is composed of then they will be proportional to its thickness, and the values of the quantities X, Y, Z will include a factor that is equal to the thickness that pertains to the point m. On the contrary, they will be independent of each other when they are provided with an external force, such as the pressure of a fluid on the surface, for example.

Divide the surface into infinitely-small elements for some planes that are perpendicular to the xy-plane, so one of them is parallel to the xz-plane and the other one is parallel to the yz-plane. As one knows, the element that pertains to the point m will be expressed by $k\,dx\,dy$, when one sets:

$$\frac{dz}{dx} = p, \qquad \frac{dz}{dy} = q, \qquad \sqrt{1+p^2+q^2} = k,$$

to abbreviate, and since the quantities X, Y, Z are deemed to be constant over the extent of that element, that will imply that the motivating forces that are applied to it will be equal to X, Y, Z, multiplied by $k\,dx\,dy$, to to:

$$X\,k\,dx\,dy, \qquad Y\,k\,dx\,dy, \qquad Z\,k\,dx\,dy\,.$$

However, along with those forces, there also exist other ones that arise from the coupling of the element under consideration with the ones that are adjacent to it, and which will be necessary to take under consideration.

Indeed, in the equilibrium state, each of the elements that comprise the surface is *tensed* by unknown forces that are directed in the plane of that element and which act in the opposite sense to its opposed extremities. Hence, the four sides that bound the arbitrary element $k\,dx\,dy$ will be pulled from the inside to the outside by forces that are found in the tangent plane at the point m. Furthermore, we suppose that they are perpendicular to the sides, but in order to embrace all of the cases that might present themselves, we shall not establish any particular relationship between the forces that act upon two adjacent sides; i.e., we regard each element of the surface as experiencing two mutually-independent *tensions*, and one effectively agrees that the same element can be, for example, not under any tension in one direction, while experiencing a considerable tension in the perpendicular direction.

Having said that, let T represent the force that pulls each point of the edge parallel to the yz-plane and adjacent to the point m. The length of that side is $dy\,\sqrt{1+q^2}$. The total force that one pulls from the inside to the outside will then be $T\,dy\,\sqrt{1+q^2}$. Let α, β, γ be the angles that its direction makes with the x, y, z axes, resp. Its components parallel to those axes will be:

$$T\,dy\,\sqrt{1+q^2}\cos\alpha, \quad T\,dy\,\sqrt{1+q^2}\cos\beta, \quad T\,dy\,\sqrt{1+q^2}\cos\gamma,$$

and they will act in the opposite sense to the coordinates x, y, z ; i.e., the forces will tend to diminish the coordinates. Now, the variable y will stay the same when we pass from the side that we are considering to the one that is opposite to it in the same element, and the variable x will change to $x + dx$. Those quantities will then become:

$$T\,dy\,\sqrt{1+q^2}\cos\alpha + dx\,dy\cdot\frac{d\left(T\sqrt{1+q^2}\cdot\cos\alpha\right)}{dx},$$

$$T\,dy\,\sqrt{1+q^2}\cos\beta + dx\,dy\cdot\frac{d\left(T\sqrt{1+q^2}\cdot\cos\beta\right)}{dx},$$

$$T\,dy\,\sqrt{1+q^2}\cos\gamma + dx\,dy\cdot\frac{d\left(T\sqrt{1+q^2}\cdot\cos\gamma\right)}{dx}$$

relative to the second side, and since those three forces act in the opposite sense to the preceding ones, it will follow that the element $k\,dx\,dy$ will be pulled in the sense of the coordinates x, y, z by forces:

$$dx\,dy\cdot\frac{d\left(T\sqrt{1+q^2}\cdot\cos\alpha\right)}{dx},$$

$$dx\,dy\cdot\frac{d\left(T\sqrt{1+q^2}\cdot\cos\beta\right)}{dx},$$

$$dx\,dy\cdot\frac{d\left(T\sqrt{1+q^2}\cdot\cos\gamma\right)}{dx},$$

which must be added to the given forces:

$$X\,k\,dx\,dy, \qquad Y\,k\,dx\,dy, \qquad Z\,k\,dx\,dy,$$

respectively.

Similarly, let T' represent the force at each of its points that pulls the second adjacent side at the point *m*, which is parallel to the *xz*-plane and equal in length to $dx\sqrt{1+p^2}$. Also, let α', β', γ' denote the angles that its direction makes with the coordinates axes. We will find by an argument that is similar to the preceding one that the element *k dx dy* is also pulled by forces that act on the opposite sense to the coordinates *x*, *y*, *z* and equal to:

$$dx\,dy \cdot \frac{d\left(T'\sqrt{1+q^2}\cdot\cos\alpha'\right)}{dx},$$

$$dx\,dy \cdot \frac{d\left(T'\sqrt{1+q^2}\cdot\cos\beta'\right)}{dx},$$

$$dx\,dy \cdot \frac{d\left(T'\sqrt{1+q^2}\cdot\cos\gamma'\right)}{dx},$$

respectively.

Now, if one adds the set of forces that pull the element *k dx dy* parallel to that axis and in the same sense then in order to have equilibrium of that element, it will be necessary that the sums should be equal to zero. Furthermore, if one suppresses the common factor *dx dy* then one will have the three equations:

$$X\,k + \frac{d\left(T\sqrt{1+q^2}\cdot\cos\alpha\right)}{dx} + \frac{d\left(T'\sqrt{1+q^2}\cdot\cos\alpha'\right)}{dx} = 0,$$

$$Y\,k + \frac{d\left(T\sqrt{1+q^2}\cdot\cos\beta\right)}{dx} + \frac{d\left(T'\sqrt{1+q^2}\cdot\cos\beta'\right)}{dx} = 0,$$

$$Z\,k + \frac{d\left(T\sqrt{1+q^2}\cdot\cos\gamma\right)}{dx} + \frac{d\left(T'\sqrt{1+q^2}\cdot\cos\gamma'\right)}{dx} = 0,$$

which must be true over the entire extent of the surface, and which will be the equations of equilibrium.

2. – In order to develop them, one must replace $\cos\alpha$, $\cos\beta$, $\cos\gamma$ with their values, and that determination is only a question of simple geometry.

Therefore, let:

$$x' - x = a\,(z' - z), \qquad y' - y = b\,(z' - z)$$

be the two equations of the line that passes through the point m and along which, the force T is directed. x', y', z' are the variable coordinates of the points on that line, and a and b are two unknown constants that must be determined. The direction of the force T lies in the tangent plane to the point m, whose equation is:

$$z' - z = p\,(x' - x) + q\,(y' - y)\,.$$

It will then be necessary that the two preceding equations must satisfy that equation, which will give the first conditional equation:

$$1 - a\,p - b\,q = 0.$$

Furthermore, the line that we consider is assumed to be perpendicular to the side of the element $k\,dx\,dy$, which is parallel to the yz-plane, and the indefinite line that is the prolongation of that side will have the equations:

$$z' - z = q\,(y' - y), \qquad x' - x = 0.$$

Now, in order for those two sides to be perpendicular to each other, one must have:

$$1 + \frac{b}{q} = 0.$$

One will infer from that second condition equation, when it is combined with the preceding one, that:

$$b = -\,q, \qquad a = \frac{1+q^2}{p}\,;$$

consequently, the equations of the direction of the force T will become:

$$x' - x = \frac{1+q^2}{p}\,(z' - z), \qquad y' - y = -\,q\,(z' - z)\,.$$

From the known formulas, the cosines of the angles α, β, γ that the line makes with the coordinate axes will be:

$$\cos\alpha = \frac{a}{\sqrt{1+a^2+b^2}}\,,$$

$$\cos\beta = \frac{b}{\sqrt{1+a^2+b^2}}\,,$$

$$\cos\gamma = \frac{1}{\sqrt{1+a^2+b^2}}.$$

Upon replacing a and b with their values and always denoting the radical $\sqrt{1+p^2+q^2}$ by k, we will then have:

$$\cos\alpha = \frac{\sqrt{1+q^2}}{k}, \qquad \cos\beta = -\frac{q\,p}{k\sqrt{1+q^2}}, \quad \cos\gamma = \frac{p}{k\sqrt{1+q^2}}.$$

One will likewise find the angles α', β', γ' that refer to the direction of the force T from:

$$\cos\alpha' = -\frac{p\,q}{k\sqrt{1+p^2}}, \ \cos\beta' = \frac{\sqrt{1+p^2}}{k}, \qquad \cos\gamma' = \frac{q}{k\sqrt{1+p^2}}.$$

I shall substitute these values into the equations of the preceding section; they will then become:

$$X\,k + \frac{d\,[T(1+q^2)/k]}{dx} - \frac{d\,[T'p\,q/k]}{dy} = 0,$$

$$Y\,k + \frac{d\,[T(p\,q/k]}{dx} + \frac{d\,[T'(1+p^2)\cdot/k]}{dy} = 0,$$

$$Z\,k + \frac{d\,(T\;p/k)}{dx} + \frac{d\,(T'q/k)}{dy} = 0\,.$$

If one eliminates the two unknowns T and T' from those three equations, what will remain will be an equation that will include only the given quantities X, Y, Z, and the partial derivatives of z, and it will be the general equation of a flexible surface in equilibrium. In addition, those equations will serve to determine the two tensions T and T' that an arbitrary element of that surface will experience as functions of the coordinates that it corresponds to. The results of those calculations are very complicated in the general case. However, when one supposes that the two tensions are equal, one will find an equation that is quite simple and deserves special attention.

3. – Hence, let $T = T'$. If one performs the indicated differentiations and adds the three equilibrium equations, after multiplying the first one by $-p/k$, the second one by $-q/k$, and the third one by $1/k$, then one will get:

$$Z - p\,X - q\,Y + \frac{T}{k^2}\left[(1+q^2)\frac{d^2z}{dx^2} - 2\,pq\frac{d^2z}{dx\,dy} + (1+p^2)\frac{d^2z}{dy^2}\right] = 0, \tag{1}$$

when one observes that:

$$\frac{dp}{dx} = \frac{d^2z}{dx^2}, \qquad \frac{dq}{dx} = \frac{dp}{dy} = \frac{d^2z}{dx\,dy}, \qquad \frac{dq}{dy} = \frac{d^2z}{dy^2}.$$

If one successively combines the first and the third of these equations, and then the second and the fifth one, then one will also find that:

$$X + Z\,p + \frac{dT}{dx} = 0, \qquad Y + Z\,q + \frac{dT}{dy} = 0,$$

and since one has $dz = p\,dx + q\,dy$, at will give:

$$X\,dx + Y\,dy + Z\,dz + dT = 0, \tag{2}$$

which is an equation that will replace the preceding two identically, due to the independence of the two variables x and y. Now, that equation will present two distinct cases to examine.

1. If the formula $X\,dx + Y\,dy + Z\,dz$ is the exact differential of a function of the three variables x, y, z, which are regarded as independent, in such a way that one will have:

$$X\,dx + Y\,dy + Z\,dz = d \cdot f(x, y, z)$$

identically, then equation (2) will give:

$$T = f(x, y, z) + c,$$

in which c is the arbitrary constant. Upon substituting that value for T in equation (1), it will be the equation of the equilibrium surface, up to second-order partial derivatives.

2. If that formula is not a differential in three variables then one must determine z in such a way that it will become a differential in two variables in order for one to satisfy equation (2). The value of z must then fulfill the condition that is expressed by the equation:

$$\frac{d(X + Z\,p)}{dy} = \frac{d(Y + Z\,q)}{dx}.$$

Consequently, it is necessary that it must agree with equation (1), which will be true only in very special cases. Therefore, in general, it will be impossible to satisfy equation (2) with any value of the unknown T in the second case. However, one must not conclude that equilibrium will then be impossible for the flexible surface. All that has been proved is that the hypothesis $T = T'$ is not permissible, because one can always satisfy the equilibrium equations of the preceding section by means of two different tensions.

4. – The integrability condition for the formula:

$$X\,dx + Y\,dy + Z\,dz$$

is not the only necessary condition for the assumption $T = T'$ to be permissible. It is also necessary for that hypothesis to agree with the given forces that act at the free boundaries of the surface, and which determine the boundary values of the tensions T and T'. One must then examine whether that agreement is effectively true in each particular case. However, the result will be a condition that relates to the direction of the given forces that one can state in a general manner.

Indeed, suppose that the arbitrary point m belongs to the free contour of the surface. Let ds be an element of the contour that surrounds that point. Draw a plane through that point that is parallel to the yz-plane and draw a second plane through the other extremity of the element ds that is parallel to the xz-plane. In that way, we will define an infinitely-small triangle in the tangent plane to the surface that has the element ds for one of its sides, and the other two will be $dy\sqrt{1+q^2}$ and $dx\sqrt{1+p^2}$, as before. Now, the boundary tensions must bring equilibrium to the given force, which will also pull the element ds from the inside outward, and consequently, its components must be equal and opposite. Hence, represent that external force by $P\,ds$. In the equilibrium state, its direction will be necessarily included in the tangent plane to the point m, but it might be perpendicular to the side ds or oblique to it. If it is perpendicular and one decomposes it into two forces that are also perpendicular to the other two sides of our triangle then we will know, from the elements of statics, that the components will be proportional to the sides and represented by:

$$P\,dy\sqrt{1+q^2} \quad \text{and} \quad P\,dx\sqrt{1+p^2}\,.$$

Consequently, one will have $T = T' = P$ in this case. However, if the force $P\,ds$ is oblique to the side ds then its components, which are perpendicular to the other two sides, will no longer be proportional to those sides, and the boundary tensions T and T' will no longer be equal to each other. Therefore, we can conclude that the hypothesis that $T = T'$ over the entire extent of the surface will demand that the forces that are applied to its contour must be perpendicular to the direction of that contour at each point.

There is one particular case that we shall soon give an example of (no. **6**) in which the force $P\,ds$ is perpendicular to the side ds, while the two tensions T and T' are not equal. That case is the one in which the side ds is found to be parallel to one of the planes of xz or yz. The composition of the given force will no longer be true, as we had supposed. That force will then be equal to that of the two tensions that are directly opposite to it, and the other one, which is perpendicular, will be entirely independent of it.

One must also point out that the parts of the contour of the surface that define lines that are fixed and capable of taking on an indefinite resistance will be pulled at each of their points by the resultant of the two tensions *T* and *T′* that pertain to them, and that force will be found to be cancelled by those fixed lines without resulting in any particular condition that relates to the ratio or absolute magnitude of the boundary tensions.

5. – Equation (1) coincides with the one that Lagrange found in a different way in the new edition of *la Mécanique analytique* (*). However, from our analysis, one will see that it is subordinate to some special hypotheses that prevent it from being the general equation of the flexible surface in equilibrium. We shall nonetheless apply it to the most remarkable special cases.

1. Suppose that one has $X = 0$, $Y = 0$, $Z = 0$, in such a way that the points of the surface, are not subjected to any given force except for the ones on its contour. Equation (2) will reduce to $dT = 0$ in that case. It will then be necessary that the applied force at each point of its free contour should not vary from one point to another. That being the case, the tension *T* will be equal to that constant force, and equation (1) of the equilibrium surface will become:

$$(1+q^2)\frac{d^2z}{dx^2} - 2pq\frac{d^2z}{dx\,dy} + (1+p^2)\frac{d^2z}{dy^2} = 0.$$

As one knows, that is equation is the equation of a surface whose area is a *minimum* for a given contour.

2. Consider a flexible surface that covers a solid body of arbitrary form that supports the surface at all of its points. The force *X*, *Y*, *Z* will then be equal to the components of the unknown pressure that the surface exerts on the body at the point whose coordinates are *x*, *y*, *z*. Therefore, let *N* be that pressure; its direction will be normal to the surface. Consequently, one will have:

$$X = -\frac{pN}{k}, \qquad Y = -\frac{qN}{k}, \qquad Z = \frac{N}{k}$$

for its components along the coordinate axes. If one substitutes these values in equation (2) then it will reduce to $dT = 0$. The tension will then be constant, and as in the preceding case, it will be necessary that all of the points of the free contour should be pulled by equal forces that are tangent to the surface and perpendicular to that contour.

At the same time, equation (1) will become:

$$N + \frac{T}{k^2}\left[(1+q^2)\frac{d^2z}{dx^2} - 2pq\frac{d^2z}{dx\,dy} + (1+p^2)\frac{d^2z}{dy^2}\right] = 0.$$

Hence, the surface of the solid body will then be given by its equation, which will tell one the pressure that exists at each point, or rather, its relationship to the tension *T*. Up to

(*) Tome. I, pp. 149.

sign, the coefficient of T is nothing but the sum $\frac{1}{\rho}+\frac{1}{\rho'}$, in which ρ and ρ' denote the two principal radii of curvature of the surface at the point that one considers. It will then follow that this sum expresses the relationship of the force N to the force T. Hence, for example, a flexible surface that extends over a sphere will exert a pressure at each point that is equal to the tension that it experiences, divided by twice the radius of that sphere.

3. If the surface that is not defined on a solid body, but compressed at all of its points by a ponderous fluid then that case will be the same as the preceding one, with the difference that the pressure N, rather than being unknown, will be given and will depend upon the density and height of the fluid. Its value will then have the form:

$$N = a + b\,z,$$

if one supposes that the z ordinate is vertical and lets a and b denote two constant coefficients. The equation for the flexible surface in equilibrium will then become:

$$a + bz + \frac{T}{k^2}\left[(1+q^2)\frac{d^2z}{dx^2} - 2pq\frac{d^2z}{dx\,dy} + (1+p^2)\frac{d^2z}{dy^2}\right] = 0,$$

and if one observes that T is a constant quantity then one will see that this equation coincides with the one that Laplace found for the concave or convex *capillary* surface (*). Thus, it will result that when a ponderous liquid rises or falls in a capillary tube, it will take the form of a flexible surface that is compressed by a ponderous fluid at all of its points.

4. Finally, consider the ponderous surface and take the z-axis to be vertical and directed in the sense of gravity. We will then have $X = 0$, $Y = 0$, $Z = g\,\varepsilon$, if we denote gravity by g, and let ε be the thickness of the surface. Equation (2) will then become:

$$g\,\varepsilon\,dz + dT = 0.$$

Now, if ε is variable then that equation will be impossible to satisfy unless ε is a function of just the variable z; i.e., unless the thickness is not constant over the entire extent of each horizontal section of the surface. It will then result that in the case of ponderous surface of unequal thickness, the hypothesis of two equal tensions that we made above is not generally permissible. The equilibrium equation of such a surface must be deduced from the formulas in section **2**. However, if ε is constant then upon integrating the preceding equation and letting c denote the arbitrary constant, we will have:

$$T = c - g\,\varepsilon z,$$

and equation (1) will become:

(*) *Théorie de l'action capillaire,* pp. 19.

$$g\,\varepsilon + \frac{c - g\varepsilon\, z}{k^2}\left[(1+q^2)\frac{d^2z}{dx^2} - 2pq\frac{d^2z}{dx\,dy} + (1+p^2)\frac{d^2z}{dy^2}\right] = 0$$

in this case.

That equilibrium equation for the ponderous surface of equal thickness must include the usual equation of the *catenary*, which one indeed deduces by supposing that z is independent of one of the two variables x or y, for example. One will then have $q = 0$, and upon once more replacing p with dz / dx, moreover, that equation will become:

$$g\,\varepsilon + (c - g\,\varepsilon z)\,\frac{d^2z}{dx^2 + dz^2} = 0.$$

If one multiplies this by dz and divides by $c - g\,\varepsilon\, z$ then one will have:

$$\frac{g\,\varepsilon\, dz}{c - g\varepsilon z} + \frac{d^2z}{dx^2 + dz^2}\,dz = 0.$$

If c' is the arbitrary constant then integration will give:

$$c'\sqrt{dx^2 + dz^2} = (c - g\,\varepsilon z)\,dz\,,$$

and that equation is the equation of the catenary, as one can find directly (*).

6. – It is good to point out that the equation of the catenary is also included in the equilibrium equations of section **2** without one being obliged to suppose that the two tensions T and T' are equal to each other in order to deduce it. Indeed, if one has a rectangle that is composed of flexible cloth of constant thickness, and one suspends it by attaching two of its opposite sides to two fixed, horizontal, parallel lines then it will be obvious that the cloth will define a portion of a horizontal cylinder whose perpendicular section at its edges will be an ordinary catenary. Furthermore, it will also be obvious that the surface will experience no other tension in the direction of the horizontal edges and that its elements will experience only one tension in the direction of its sections perpendicular to the edges, which will vary from one point to another of the same section, but will be the same for all points of the same edge. If the cloth is thus suspended then one will change nothing in its figure when one applies equal and opposite forces to the extremities of each edge whose intensities vary as one pleases. The surface will then be stressed by its new forces in the direction of its edges in such a way that each of its elements will experience a second tension that will be the same along the length of each edge and will vary arbitrarily from one edge to another. From that, if one takes the *zx*-plane to be vertical and the *y*-axis to be parallel to the lines of suspension of the surface then one might satisfy the general equations of section **2** by supposing that the *z*-ordinate and the tension T, which is exerted parallel to the *xz*-plane independently of the variable

(*) See my *Traité mécanique*, t. I, pp. 201.

y, and taking the tension *T*, which is parallel to the *yz*-plane, to be an arbitrary function of *x*. Those assumptions must then give the equation of the ordinary catenary between *z* and *x*, and the known expression for the tension that the curve experiences at each of its various points for *T*.

Therefore, let:

$$X = 0, \qquad Y = 0, \qquad Z = g\,\varepsilon, \qquad q = \frac{dz}{dy} = 0, \quad \frac{dT}{dy} = 0, \qquad \frac{dT}{dy} = 0.$$

The equations of section **2** will then reduce to two, namely:

$$\frac{d\,(T\,/\,k)}{dx} = 0, \quad g\,\varepsilon\,k + \frac{d\,(T\;p\,/\,k)}{dx} = 0.$$

One will first have:

$$T = c'k = c'\sqrt{1 + \frac{dz^2}{dx^2}}$$

then, in which c' is a quantity that is independent of both *x* and *y*. If one substitutes that value for *T* in the second equation then it will become:

$$g\,\varepsilon + \frac{c'\,d^2z}{dx\sqrt{dx^2 + dz^2}} = 0.$$

If one multiplies this by *dz*, integrates, and lets *c* denote a second arbitrary constant then one will have:

$$c'\sqrt{dx^2 + dz^2} = (c - g\,\varepsilon\,z)\;dz,$$

which is an equation that is the same as the one that was found in the preceding section. The value of *T* will also become $T = c - g\ \varepsilon\, z$, as above. However, the tension T' will remain an arbitrary function of *x* that depends upon the forces that pull the cylinder under consideration at the extremities of its edges and will be zero when those forces do not exist.

7. – When one knows the equilibrium equations of a system of material points that are acted upon by arbitrary forces, one will know from the principles of mechanics how to deduce the equations of motion of that system immediately. In the present case, where the points of the surface are pulled by forces *X*, *Y*, *Z* that are parallel to the coordinate axes, if one is to get the general equations of its motion then it will then suffice to replace the forces in the equations of section **2** with:

$$X - \varepsilon\frac{du}{dt}, \qquad Y - \varepsilon\frac{dv}{dt}, \qquad Z - \varepsilon\frac{dw}{dt},$$

respectively, in which u, v, w denote the velocities that are parallel to those axes at the arbitrary point m, ε is the thickness of the surface that point, and t is the variable that represents time.

The only use that one makes of those equations is to employ them to the determination of small oscillations of surface that deviate only slightly from a plane, and that will suggest a problem that is analogous to that of a *vibrating string*, which is far from having been solved completely, moreover. For example, suppose that the surface deviates slightly from the xy-plane. The z-ordinate and its partial derivatives will then be very small, so one will neglect their squares and products in the calculations. Furthermore, one ignores gravity, in such a manner that the points of the surface will not be acted upon by any given force. Finally, suppose that the velocities u and v, which are parallel to the xy-plane, are zero or negligible, which amounts to saying that each point of the surface will constantly remain in the same line perpendicular to that plane. All of those restrictions are similar to the ones that one assumes in the theory of vibrating strings. Upon adopting them, the equation of motion of the surface will be obtained by setting $X = 0$, $Y = 0$ in those of section **2** and replacing Z with $-\varepsilon\, dw / dt$ in them. If one neglects the squares and products of the p and q, in addition, then those equations will reduce to:

$$\frac{dT}{dx} = 0, \qquad \frac{dT}{dy} = 0, \qquad -\varepsilon\frac{dw}{dt} + \frac{d\cdot Tp}{dx} + \frac{d\cdot T'q}{dy} = 0.$$

The first two show that T is a function of y and T' is a function of x. Furthermore, due to the fact that x and y are regarded as constants, one will have $dw / dt = d^2 z / dt^2$. If one then replaces p and q with dz / dx and dz / dy once more then the third equation will become:

$$\varepsilon\frac{d^2z}{dt^2} = T\frac{d^2z}{dx^2} + T'\frac{d^2z}{dy^2}.$$

If one desires that the two tensions T and T' should be equal in this case then it would be necessary that they should be independent of both x and y. If one then sets $T = T' = a^2$ then one will have this equation:

$$\varepsilon\frac{d^2z}{dt^2} = a^2\left(\frac{d^2z}{dx^2} + \frac{d^2z}{dy^2}\right),$$

which coincides with the one for the propagation of sound in a plane. It is the equation that Biot and Brisson appealed to in order to determine the different properties of vibrating surfaces (*). As one sees, they supposed that the surface was stretched the same in all directions and at all of its points. That is the case, for example, with *drums*, in such a way that the theory of their vibration will be contained in the preceding equation.

One will get a more general equation by supposing that the two tensions T and T' are constant, but unequal; i.e., upon setting $T = a^2$, $T' = b^2$. One will then have:

(*) Reports of the first Class of the Institute, tome IV, page 91.

$$\varepsilon \frac{d^2z}{dt^2} = a^2 \frac{d^2z}{dx^2} + b^2 \frac{d^2z}{dy^2},$$

which is an equation that Euler gave in his St. Petersburg papers (*) in order to determine the vibrations of a rectangular surface that was unequally stretched in the two directions of its length and its width. The x and y axes are parallel to the sides of that rectangle. a^2 is the tension parallel to the x-axis, and b^2 is the tension along the y-axis.

Furthermore, those cases are the simplest ones that one can consider, and meanwhile the equations that they refer to are not integrable in finite form. If one acts upon a surface whose limits consist of a fixed part and a moving part that is entirely free then one must keep the tensions T and T' variable and unequal, one of them being a function of x and the other one, of y, and determine those functions in such a manner that at all of the free limits, the tension in the direction that is perpendicular to the contour will be equal to zero. The values of T and T' will then depend upon the form of those curves, and the equation that one will have to treat will be even more complicated.

(*) *Novi commentarii*, t. X, pp. 247.

CHAPTER II

EQUATION OF THE ELASTIC SURFACE IN EQUILIBRIUM

8. – Having found the equations of equilibrium of a flexible, inelastic surface whose points are all acted upon by arbitrary forces, it is clear that one will deduce the equations of the elastic surface by including the forces that result from elasticity along with those forces. Now, no matter what the cause of that quality of matter, it consists of a tendency of the molecules in a body to mutually repel each other, and one can attribute that to a repulsive force that is exerted between those points according to a certain function of the distances between them. It is natural to think that this force, as well as all other molecular actions, is appreciable only for imperceptible distances. We then assume that hypothesis and consequently, we will assume that the function of distance that represents the elastic force is valid only for extremely small values of the variable that expresses the distances, and that it will become zero as soon as that variable becomes appreciable. Furthermore, that repulsive action is assumed to affect all points of the matter and to act upon all points that comprise the surface, so it will follow that for all equal distances, the repulsive force between two points will become proportional to the product of the thicknesses of the surface to which they correspond. However, for reasons that will soon become clear, we confine ourselves to considering surfaces of equal thickness over their entire extent, so the intensity of the repulsive force between two arbitrary points will then be expressed by the square of the constant thickness of the surface, multiplied by a function of the distance between those points that is subject to the condition that we just supposed.

As before, let m denote the point of the surface that pertains to the arbitrary coordinates x, y, z. Consider a second point m' that is located in the sphere of activity of the first one and whose coordinates are x', y', z'. Let r denote the distance from m' to m, and let $f\,r$ denote the function that expresses the law of the repulsive force with respect to the distances. Finally, let ε be the thickness of the surface: The intensity of the mutual repulsion between those two points m and m' will be equal to $\varepsilon^2 f\,r$. Hence, the point m will be repelled by an infinitude of forces that are similar to that one, and will originate at all points, such as m', that are found inside of its sphere of activity.

In order to get the resultant of all those forces, one must decompose each of them along three fixed axes and then form the sum of the components in each direction using integral calculus. Now, the components of the force $\varepsilon^2 f\,r$, which are directed along the coordinates x, y, z at the point m and tend to increase them, are equal to:

$$\frac{x-x'}{r}\varepsilon^2 f\,r, \quad \frac{y-y'}{r}\varepsilon^2 f\,r, \quad \frac{z-z'}{r}\varepsilon^2 f\,r,$$

respectively. If one then lets X', Y', Z' denote the total components along those directions, and one lets w denote the surface element that pertains to the point m then, from the principles of integral calculus, one will have:

$$X' = \varepsilon^2 \cdot \iint \frac{x - x'}{r} \cdot f\, r \cdot \omega,$$

$$Y' = \varepsilon^2 \cdot \iint \frac{y - y'}{r} \cdot f\, r \cdot \omega,$$

$$Z' = \varepsilon^2 \cdot \iint \frac{z - z'}{r} \cdot f\, r \cdot \omega.$$

Those double integrals must be extended over all points of the surface that are situated around the point *m* and included in its sphere of activity.

It is then the forces X', Y', Z' that one must add to the other given forces *X*, *Y*, *Z* that act at all points of the surface (no. **1**). One substitutes those forces, thus-augmented, in the equations of section **2** in order to get the equations of equilibrium of the elastic surface. If one then eliminates the two unknowns *T* and T' from those three equations then one will get the equation of the surface itself. However, since the values of X', Y', Z' contain fourth-order partial derivatives, as one will see, the result of that elimination will lead to a very complicated equation that does not seem to have any utility. That is why we shall consider only the case in which the two tensions are equal to each other, except to prove that the forces that come from elasticity will satisfy the condition that this equality must exist, which consists of saying that the formula $X'dx + Y'dy + Z'dz$ must be the exact differential of a function of *x*, *y*, *z*.

9. – In the case of $T = T'$, the general equations of equilibrium reduce to equations (1) and (2) of section **3**. The elastic forces X', Y', Z' increase the left-hand side of equation (1) by the quantity $Z' - p\,X' - q\,Y'$, which we denote by *U*, to abbreviate, in such a way that upon replacing X', Y', Z' with the preceding integrals, we will have:

$$U = \varepsilon' \iint \frac{(z - z') - p\,(x - x') - q(y - y')}{r} \cdot f\, r \cdot \omega,$$

and equation (1) will become:

$$Z - p\,X - q\,Y + U + \frac{T}{k^2} \cdot \left[(1 + q^2)\frac{d^2 z}{dx^2} - 2pq\frac{d^2 z}{dx\,dy} + (1 + p^2)\frac{d^2 z}{dy^2} \right] = 0. \qquad (1)$$

At the same time, one must add the formula:

$$X'dx + Y'dy + Z'dz$$

to the left-hand side of equation (2). Now, upon observing that:

$$r^2 = (x - x')^2 + (y - y')^2 + (z - z')^2,$$

so it will follow that:

$$\frac{(x-x')\,dx+(y-y')\,dy+(z-z')\,dz}{r} = dr,$$

and we will have:

$$X'dx + Y'dy + Z'dz = \varepsilon' \cdot \iint f\,r \cdot \omega.$$

Everything will then come down to showing that this quantity is the differential of a function of *x*, *y*, *z*.

In order to do that, if one represents the integral of $f\,r \cdot dr$ by $F\,r$, or one makes $f\,r \cdot dr = d \cdot F\,r$ then one will first have:

$$X'dx + Y'dy + Z'dz = \varepsilon' \cdot \iint d \cdot F\,r \cdot \omega.$$

Now, by hypothesis, $f\,r$ is zero for any value of *r* is not extremely small. $F\,r$, or $\iint f\,r \cdot dr$, will then reduce to an arbitrary constant for any similar value of *r*. Consequently, one supposes that this integral is taken in such a way that it will vanish like $f\,r$ for the values of *r* that relate to the limits of the sphere of activity of the point *m*, which are also those of the double integral:

$$\iint d \cdot F\,r \cdot \omega.$$

Those limits depend implicitly upon the coordinates *x*, *y*, *z* of the point *m*. However, since the value of $F\,r$ that they refer to is equal to zero, it is further permissible to move the characteristic *d*, which indicates a differential relative to *x*, *y*, *z*, outside of the definite integral, in such a way that one will have:

$$\iint d \cdot F\,r \cdot \omega = d \cdot \iint F\,r \cdot \omega,$$

and due to the fact that the factor ε^2 is constant, it will follow that:

$$X'dx + Y'dy + Z'dz = d \cdot \varepsilon^2 \iint F\,r \cdot \omega.$$

In that way, equation (2) of section **3** will become:

$$X\,dx + Y\,dy + Z\,dz + d \cdot \varepsilon^2 \iint F\,r \cdot \omega + dT = 0.$$

That can be true only if $X\,dx + Y\,dy + Z\,dz$ is also an exact differential. Hence, let Π denote its integral, and one will have:

$$T = -\,\varepsilon^2 \iint F\,r \cdot \omega - \Pi,$$

in which one does not add an arbitrary constant, because it is assumed to be included in Π. This value of T is the one that must be substituted in equation (1).

10. – It is good to observe that the formula $X'dx + Y'dy + Z'dz$ will not generally satisfy the integrability condition when the thickness of the surface is not assumed to be constant. Indeed, upon letting ε denote the thickness at the point m and letting ε' denote the one that pertains to one of the surrounding points, one will find that:

$$X'dx + Y'dy + Z'dz = \varepsilon \cdot d \cdot \iint F\,r \cdot \varepsilon'\,\omega.$$

As always, $F\,r$ represents the integral $\int f\,r \cdot dr$, which is taken in such a manner that it will vanish at the limits of the sphere of activity of the first point. ε will now be a function of x, y, z, as well as the quantity $\iint F\,r \cdot \varepsilon'\,\omega$. Hence, except for the very special case in which those two quantities are functions of each other, the value of $X'dx + Y'dy + Z'dz$ will not be an exact differential. Consequently, the supposition that $T = T'$ is not generally permissible in the case of an elastic surface of unequal thickness.

It is solely for that reason that we shall confine ourselves to the case of a constant thickness, because the analysis that we just appealed to in order to determine the forces that are due to elasticity can likewise be applied to the case of a thickness that varies according to an arbitrary law.

11. – From what we just saw, we must determine the values of the two double integrals, namely: the integral $\iint F\,r \cdot \omega$, which enters into the value for the tension T, and the one that expresses the quantity U. In order to get the latter, it is necessary to give particular directions to the coordinate axes. Hence, draw three rectangular axes through the point m, one of which is normal to the surface, and the other two of which will be directed in the tangent plane at that point. Let u, s, s' be the coordinates of the point m, when referred to those axes, where u is the one that is parallel to the normal axis. The variables u, s, s' will be coupled with the other coordinates x', y', z' of the same point m', and from some known formulas, one will have:

$$\begin{aligned} x' &= x + \lambda\,u + \mu\,s + \nu\,s', \\ y' &= y + \lambda' u + \mu' s + \nu' s', \\ z' &= z + \lambda'' u + \mu'' s + \nu'' s'. \end{aligned}$$

In those equations, the nine coefficients λ, μ, etc., are the cosines of the angles that are subtended by the u, s, s' axes and the x, y, z axes or the x', y', z' axes.

The three cosines λ, λ', λ'' are those of the angles that the u-axis, or the normal to the point m, makes with the x, y, z axes, in such a way that one will have:

$$\lambda = -\frac{p}{k}, \qquad \lambda' = -\frac{q}{k}, \qquad \lambda'' = \frac{1}{k},$$

in which one sets:

$$\frac{dz}{dx} = p, \qquad \frac{dz}{dy} = q, \qquad \sqrt{1 + p^2 + q^2} = k,$$

to abbreviate.

As for the other six, their values will depend upon the directions of the s and s' axes in the tangent plane. However, some known relations will exist between them and the first three, but we shall dispense from using their values. The relations that we will need in what follows are:

$$\lambda^2 + \mu^2 + \nu^2 = 1,$$

$$\lambda'^2 + \mu'^2 + \nu'^2 = 1,$$

$$\lambda\lambda' + \mu\mu' + \nu\nu' = 0,$$

$$\lambda\mu + \lambda'\mu' + \lambda''\mu'' = 0,$$

$$\lambda\nu + \lambda'\mu' + \lambda''\nu'' = 0.$$

Upon replacing λ, λ', λ'' with their previous values, they will become:

$$\mu^2 + \nu^2 = \frac{1 + q^2}{k^2},$$

$$\mu'^2 + \nu'^2 = \frac{1 + p^2}{k^2},$$

$$\mu\mu' + \nu\nu' = \frac{-pq}{k^2},$$

$$p\,\mu + q\,\mu' - \mu'' = 0,$$

$$p\,\nu + q\,\nu' - \nu'' = 0.$$

We shall further cite another relation, which is likewise known, and which we will also use, namely:

$$(\mu\nu' - \mu'\nu)^2 = \lambda''^2 = \frac{1}{k^2}.$$

12. – If one replaces x', y', z' with their values in terms of u, s, s' in the expression for U then one will find, by virtue of the preceding relations, that it will reduce to:

$$U = -\varepsilon^2 k \cdot \iint f\, r \cdot \frac{u\,\omega}{r},$$

in which one might observe that when that quantity is divided by k, it will be, up to sign, the normal component of the force of repulsion that acts at the point m.

In order to perform that integration, we develop the quantities that are included in the double sign $\iint$ in powers of s and s'. We then change those variables into other ones that are more appropriate to the limits of the double integral.

The ordinate u is a function of s and s' that is determined by the unknown equation of the surface, and since the plane of s and s' is tangent to the point m, it will then follow that this function and its partial derivatives du / ds and du / ds' will be zero when one sets $s = 0$ and $s' = 0$. Consequently, the first three terms of the development in u in powers and products of s and s' will become:

$$u = A\, s^2 + A'\, s'^{\,2} + A''\, s\, s'.$$

However, upon conveniently determining the directions of the s and s' in the tangent plane, one can make the term $A''s\, s'$ disappear. The entire development will then be a series of the form:

$$u = A\, s^2 + A'\, s'^{\,2} + B\, s^3 + B'\, s^2\, s' + B''\, s\, s'^2 + B'''s'^{\,3}$$

$$+ C\, s^4 + C'\, s^3\, s' + C''\, s^2\, s'^2 + C'''\, s\, s'^3 + C^{\text{iv}}\, s'^4 + \text{etc.},$$

in which the coefficients A, A', B, etc. depend upon the position of the point m and are consequently functions of its coordinates x, y, z. The first two of them can be expressed immediately by means of two radii of principal curvature of the surface that pertain to that point.

Indeed, the directions that we gave to the s and s' axes amount to supposing that they were tangent to the two lines of principal curvature that intersect at the point m. If one then lets ρ denote the radius of curvature of the line that is tangent to the s-axis and one demands to know the value of that radius that relates to the point m then from the usual formulas, one will have:

$$\frac{1}{\rho} = \frac{d^2u}{ds^2}\left(1 + \frac{d^2u}{ds^2}\right)^{-3/2},$$

provided that one sets $s = 0$, $s' = 0$, after differentiation. That will give:

$$\frac{1}{\rho} = \frac{d^2u}{ds^2} = 2\,A.$$

Hence, one will infer that:

$$A = \frac{1}{2\rho},$$

and similarly, upon letting ρ' denote the second radius of principal curvature that relates to *m*, one will have:

$$A' = \frac{1}{2\rho'}.$$

The other coefficients *B*, *B′*, etc., in the development of *u* are expressed by means of the partial derivatives of the first two with respect to *x* and *y*, and we will give them values to the extent that we need them.

13. – The variables that I just substituted for the coordinates *s* and *s′* of the point *m* are the projection of the radius vector at that point (i.e., the one with radius *r*) onto the plane of *s* and *s′* and the angle that the projection makes with the *s*-axis. I denote that angle by φ and let α denote the projected radius that corresponds to it. I will then have:

$$s = \alpha \cdot \cos \varphi, \quad s' = \alpha \cdot \sin \varphi,$$

by means of which, the value of *u* will become:

$$u = P\,\alpha^2 + Q\,\alpha^3 + R\,\alpha^4 + \text{etc.},$$

in which one sets:

$$P = A \cdot \cos^2 \varphi + A' \cdot \sin^2 \varphi,$$

$$Q = B \cdot \cos^3 \varphi + B' \cdot \cos^2 \varphi \cdot \sin \varphi + B'' \cdot \cos \varphi \cdot \sin^2 \varphi + B''' \cdot \sin^3 \varphi,$$

$$R = C \cdot \cos^4 \varphi + C' \cdot \cos^3 \varphi \cdot \sin \varphi + C'' \cdot \cos^2 \varphi \cdot \sin^2 \varphi + C''' \cdot \cos \varphi \cdot \sin^3 \varphi + C^{\text{iv}} \cdot \sin^4 \varphi,$$

etc.,

to abbreviate.

The values of α are always very small, since they are limited by the extent of the sphere of activity at the point *m*, and that will imply that this series is very convergent. In the following calculation, we will never need to consider terms above the third, and generally we can neglect the fifth power of the quantity α.

The projection of the element ω that pertains to the point *m* onto the plane of *s*, *s′* will be equal to $\alpha\, d\alpha\, d\varphi$, and that element will have the expression:

$$\omega = \alpha\, d\alpha\, d\varphi \cdot \sqrt{1 + \frac{du^2}{ds^2} + \frac{du^2}{ds'^2}}.$$

Now, one has:

$$\frac{du}{ds} = 2\,A\,s + \text{etc.} = 2\,A\,\alpha \cdot \cos \varphi + \text{etc.},$$

$$\frac{du}{ds'} = 2\,A'\,s + \text{etc.} = 2\,A'\,\alpha \cdot \cos\varphi + \text{etc.}$$

Hence, one will conclude that:

$$\omega = \alpha\, d\alpha\, d\varphi\,(1 + 2\,P'\,\alpha^3 + \text{etc.}),$$

upon setting:

$$P' = A^2 \cdot \cos^2\varphi + A'^2 \cdot \sin^2\varphi\,.$$

Finally, the radius r will be equal to $\sqrt{\alpha^2 + u^2}$. Upon setting u equal to its value and developing it in powers of α, one will get:

$$r = \alpha + \tfrac{1}{2} P^2\,\alpha^3 + P\,Q\,\alpha^4 + \text{etc.}\,,$$

and upon developing $f\,r$ similarly, one will have:

$$f\,r = f\,\alpha + \tfrac{1}{2} P^2 \alpha^3 \cdot \frac{d\,f\alpha}{d\alpha} + P\,Q\,\alpha^4 \cdot \frac{d\,f\alpha}{d\alpha} + \text{etc.}$$

I shall now substitute those various series for the value of U. I shall arrange the quantity that is found under the $\iint$ sign in powers of α. I neglect the fifth power, while preserving only the terms that are multiplied by $\alpha^3 \cdot \frac{d\,f\alpha}{d\alpha}$, which will have the same order after integrating over α as the terms that are multiplied by $\alpha^4\,f\,\alpha$. Having completed all calculations, I will then find that:

$$U = -\,\varepsilon^2\,k \iint \left[P\,f\alpha + Q\,\alpha\,f\alpha + \left(R + 2PP' - \tfrac{1}{2}P^2\right)\alpha^3\,f\alpha + \tfrac{1}{2}P^2\alpha^3 \frac{d\,f\alpha}{d\alpha} \right] \alpha^2\,d\alpha\,d\varphi\,.$$

14. – The limits of that integral will differ according to whether the point m is very close to the contour of the surface or at a much larger distance than the radius of the sphere of activity of the repulsive forces. In the second case, in order to extend the double integral to all points that act on the one that one considered, one must obviously integrate over α from $\alpha = 0$ up to α equal to that radius, and over φ from $\varphi = 0$ to $\varphi = 2\pi$, where π denotes the ratio of the circumference to the diameter, as usual. In truth, the radius of the sphere of activity is not a well-defined quantity, but since $f\,\alpha$ is zero or negligible for any value of α that is not less than that radius, it will follow that one can extend the integral over α up to a value of that variable that as large as one might desire, and even infinity, and not introduce any error.

On the contrary, if the point m is located on the contour of the surface, or it is only slightly separated from it, then the sphere of activity of the repulsive force around that point will no longer be complete; i.e., a portion of its extent will not include any points of the surface. The integrals over α and φ will then be taken between other limits, but in order to find the differential equation of the elastic surface in equilibrium, it will suffice to consider the interior points that are located at an arbitrary distance from its contour, and one needs to examine what happens at the boundary points only in order to determine the particular forces that one must apply to the limits of the surface in order to put it into equilibrium. That determination is very delicate, and I would like to return to it later, but I shall not address it in this article.

Therefore, upon considering only the points that do not belong to the limits of the surface, the integrations over α and φ will become mutually independent, and since each of the terms that are found under the $\iint$ sign is the product of two factors in which those variables are separate, nothing will be easier now than to complete that double integration.

15. – In regard to the limits $\varphi = 0$ and $\varphi = 2\pi$, one will have:

$$\int \cos^n \varphi \cdot \sin^{n'} \varphi \cdot d\varphi = 0,$$

except when the exponents n and n' are two numbers that are even or zero. That consideration will make the second term in U disappear, because from the form of the quantity Q, it will follow that one will have $\int Q\, d\varphi = 0$. Moreover, let a^2 and b^2 denote the values of the integrals $\int \alpha^2 f\alpha \cdot d\alpha$, $\int \alpha^4 f\alpha \cdot d\alpha$, when they are taken between convenient limits, in such a way that one will have:

$$\int \alpha^2 f\alpha \cdot d\alpha = a^2, \qquad \int \alpha^4 f\alpha \cdot d\alpha = b^2,$$

where a^2 and b^2 represent some essentially-positive constants. Upon integrating by parts, one will get:

$$\int \alpha^5 \cdot d \cdot f\alpha = \alpha^5 f\alpha - 5\int \alpha^4 f\alpha \cdot d\alpha,$$

but the term $\alpha^2 f\alpha$ will vanish at the two limits. We will then have simply:

$$\int \alpha^5 \cdot d \cdot f\alpha = -5\int \alpha^4 f\alpha \cdot d\alpha = -5b^2.$$

Having done that, the value of U in section **13** will reduce to:

$$U = -\varepsilon^2 k a^2 \cdot \int P\, d\varphi - \varepsilon^2 k b^2 \cdot \int R\, d\varphi + \varepsilon^2 k b^2 \cdot \int (3P^2 - 2PP')\, d\varphi,$$

in which all that remains is to perform the integrations over the variable φ.

Upon resetting P, R, and P' equal to their values and keeping only even powers of $\sin \varphi$ and $\cos \varphi$ under the $\int$ sign, one will have:

$$\int P\,d\varphi = A\int \cos^2\cdot\varphi\cdot d\varphi + A'\int \sin^2\cdot\varphi\cdot d\varphi,$$

$$\int R\,d\varphi = C\int \cos^4\cdot\varphi\cdot d\varphi + C''\int \cos^2\cdot\varphi\cdot\sin^2\varphi\cdot d\varphi + C^{\text{iv}}\int \sin^4\cdot\varphi\cdot d\varphi,$$

$$\begin{aligned}\int (3P^3 - 2PP')\,d\varphi \\ &= A^3\left(3\int\cos^6\varphi\cdot d\varphi - 2\int\cos^4\varphi\cdot d\varphi\right) + A'^3\left(3\int\sin^6\varphi\cdot d\varphi - 2\int\sin^4\varphi\cdot d\varphi\right) \\ &+ A^2A'\left(9\int\cos^4\varphi\cdot\sin^2\varphi\cdot d\varphi - 2\int\cos^2\varphi\cdot\sin^2\varphi\cdot d\varphi\right) \\ &+ A\,A'^2\left(9\int\sin^4\varphi\cdot\cos^2\varphi\cdot d\varphi - 2\int\cos^2\varphi\cdot\sin^2\varphi\cdot d\varphi\right).\end{aligned}$$

If one integrates from $\varphi = 0$ up to $\varphi = 2\pi$ then one will find that:

$$\int\cos^2\varphi\cdot d\varphi = \int\sin^2\varphi\cdot d\varphi = \pi,$$

$$\int\cos^2\varphi\cdot\sin^2\varphi\cdot d\varphi = \frac{\pi}{4},$$

$$\int\cos^4\varphi\cdot\sin^2\varphi\cdot d\varphi = \int\sin^4\varphi\cdot\cos^2\varphi\cdot d\varphi = \frac{\pi}{8},$$

$$\int\cos^6\varphi\cdot d\varphi = \int\sin^6\varphi\cdot d\varphi = \frac{5\pi}{8},$$

from which, it will result that:

$$\int P\,d\varphi = \pi(A + A'),$$

$$\int R\,d\varphi = \frac{\pi}{4}(3C + C' + 3C''),$$

$$\int (3P^3 - 2PP')\,d\varphi = \frac{\pi}{8}(3A^3 + 3A'^3 + 5A\,A' + 5A\,A'^2),$$

and consequently:

$$\begin{aligned}U = -\,\varepsilon^2\,k\,a^2\,\pi(A + A') - \tfrac{1}{4}\varepsilon^2\,k\,a^2\,\pi(3C + C'' + 3C^{\text{iv}}) \\ + \tfrac{1}{8}\varepsilon^2\,k\,b^2\,\pi(3A^3 + 3A'^3 + 5A\,A' + 5A\,A'^2).\end{aligned}$$

16. – We can calculate the value of the integral $\iint Fr\cdot\omega$ that enters into the expression for T (no. **9**) similarly. As in no. **13**, we will first have:

$$\omega = \alpha\, d\alpha\, d\varphi\,(1 + 2P'\alpha^2 + \text{etc.}),$$

$$F\,r = F\,\alpha + \tfrac{1}{2}P^2\alpha^2\cdot\frac{d\cdot F\alpha}{d\alpha} + \text{etc.}$$

Upon observing that $d\cdot F\,\alpha = f\,\alpha\cdot d\,\alpha$, we will conclude that:

$$\iint F\,r\cdot\omega = \iint\left[F\alpha + 2P'\alpha^2\cdot F\alpha + \tfrac{1}{2}P^2\alpha^3\cdot f\,\alpha + \text{etc.}\right]\alpha\, d\,\alpha\, d\varphi\,.$$

If we consider the points of the surface that are not very close to its contour then, as we saw above, the integrals must be taken from $\varphi = 0$ up to $\varphi = 2\pi$, and then from $\alpha = 0$ up to a reasonable value of α. Now, upon integrating by parts, we will have:

$$\int F\alpha\cdot\alpha\cdot d\alpha = \tfrac{1}{2}\alpha^2\cdot F\alpha - \tfrac{1}{2}\cdot\int\alpha^2 f\,\alpha\cdot d\alpha\,,$$

$$\int F\alpha\cdot\alpha^3\cdot d\alpha = \tfrac{1}{4}\alpha^4\cdot F\alpha - \tfrac{1}{4}\cdot\int\alpha^4 f\,\alpha\cdot d\alpha\,,$$

but the products $\alpha^2\cdot F\,\alpha$ and $\alpha^4\cdot F\,\alpha$ will be zero at the two limits. In the former case, it is due to the factor α, and in the latter, it is because $F\,\alpha$ will vanish for any value of α is not extremely small, by hypothesis (no. **9**). We will then have simply:

$$\int F\alpha\cdot\alpha\cdot d\alpha = -\tfrac{1}{2}\cdot\int\alpha^2 f\,\alpha\cdot d\alpha = -\tfrac{1}{2}a^2\,,$$

$$\int F\alpha\cdot\alpha^3\cdot d\alpha = -\tfrac{1}{4}\cdot\int\alpha^4 f\,\alpha\cdot d\alpha = -\tfrac{1}{4}b^2\,.$$

From that, upon neglecting the fifth power of α, the value of $\iint F\,r\cdot\omega$ will become:

$$\iint F\,r\cdot\omega = -\tfrac{1}{2}a^2\int d\varphi + \tfrac{1}{2}b^2\int(P^2 - P')\,d\varphi\,.$$

Upon integrating from $\varphi = 0$ to $\varphi = 2\pi$, one will first have $\int d\varphi = 2\pi$; moreover:

$$\int(P^2 - P')\,d\varphi = A^2\left(\int\cos^4\varphi\cdot d\varphi - \int\cos^2\varphi\cdot d\varphi\right) + A'^2\left(\int\sin^4\varphi\cdot d\varphi - \int\sin^2\varphi\cdot d\varphi\right)$$

$$+\ 2AA'\int\cos^2\varphi\cdot\sin^2\varphi\cdot d\varphi\,,$$

and that equation is the same thing as:

$$\int (P^2 - P')\,d\varphi = (2AA' - A^2 - A'^2)\cdot \int \cos^2\varphi \cdot \sin^2\varphi \cdot d\varphi$$
$$= -\frac{\pi}{4}(A - A')^2,$$

and consequently:

$$\iint F\,r\cdot\omega = -a^2\,\pi - \frac{b^2\pi}{8}\cdot(A - A')^2 .$$

That will give:

$$T = \varepsilon^2\,a^2\,\pi + \frac{\varepsilon^2 b^2 \pi}{8}\cdot(A - A')^2 - \Pi .$$

17. – We will soon see that we will need to know the value of T that refers to the points on the contour of the surface. That value will differ from the one that we just calculated for the interior points, but fortunately, it is, like the latter, independent of law of the repulsive force, and can be determined easily. The same thing will not be true if we consider a point that is not located on that contour, but is separated from it by a distance that is less than the radius of activity of the repulsion: The value of T will then depend upon the law of that force, in such a way that it can be determined only by making some hypothesis in regard to the form of the function $f\,r$.

Therefore, suppose that m is one of the points on the curve that bounds the surface. Draw a tangent to the curve through that point and a tangent plane to the surface, and to fix ideas, suppose that the s-axis, starting from which, one will measure the angle φ over that plane, points along the same side in which one finds the surface. Let θ be the acute angle that is subtended by that axis and the tangent to the curve. It is obvious that in order to extend the integral $\iint Fr\cdot\omega$ over all points of the surface that act upon the point m, one must take it from $\alpha = 0$ up to α equal to the radius of activity of the repulsive force, and then from $\varphi = -\theta$ up to $\varphi = \pi - \theta$, which are the values of that angle that the two extreme positions of the radius α subtend with the tangent plane at m.

If the limits that relate to α are the same as in the preceding section then we will further have:

$$\iint Fr\cdot\omega = -\tfrac{1}{2}a^2\int d\varphi + \tfrac{1}{2}b^2\int (P^2 - P')\,d\varphi .$$

Furthermore, from the values of P and P' (no. **13**), one will always have:

$$\iint (P^2 - P')\,d\varphi = -(A - A')^2\cdot\int \cos^2\varphi\cdot\sin^2\varphi\cdot d\varphi .$$

If one integrates from $\varphi = -\theta$ up to $\varphi = \pi - \theta$ then one will find that:

$$\int d\varphi = \pi, \qquad \int \cos^2\varphi\cdot\sin^2\varphi\cdot d\varphi = \frac{\pi}{8},$$

from which, one will conclude that:

$$\iint Fr \cdot \omega = -\tfrac{1}{2}a^2\pi - \tfrac{1}{16}b^2\pi \cdot (A - A')^2,$$

and for the value of the boundary tension:

$$T = \tfrac{1}{2}\varepsilon^2 a^2\pi + \tfrac{1}{16}\varepsilon^3 b^2\pi \cdot (A - A')^2 - \Pi.$$

On might point out that the part of this expression that is due to elastic forces is precisely one-half the part of the value of the internal tension that we determined before.

18. – It is now necessary for us to express the values of the quantities A, A', and $3C + C'' + 3C^{\text{iv}}$ that enter into the expressions for U and T that we have found as functions of the x, y, z coordinates of the point m. Now, in regard to the first two, we already saw (no. **12**) that:

$$A = \frac{1}{2\rho}, \qquad A' = \frac{1}{2\rho'}.$$

All that remains to be done then is to replace ρ and ρ' with the known expressions for the two radii of principal curvature. However, we prefer to determine the values of A and A' directly by the same analysis that we appealed to in order to find the value of the quantity $3C + C'' + 3C^{\text{iv}}$.

In order to do that, consider a second point m' on the surface that corresponds to the coordinates x', y', z' and recall the formulas for the coordinate transformation that were cited in section **11**, namely:

$$x' = x + \mu s + \nu s' - \frac{pu}{k},$$

$$y' = y + \mu' s + \nu' s' - \frac{qu}{k},$$

$$z' = z + \mu'' s + \nu'' s' + \frac{u}{k},$$

in which one has replaced λ, λ', λ'' with their values.

The variable z is a function of x and y, and z' is likewise a function of x' and y'. Hence, if one develops the latter using Taylor's theorem and observes that $dz / dx = p$, $dz / dy = q$ then one will have:

$$z' = z + p\left(\mu s + \nu s' - \frac{pu}{k}\right) + q\left(\mu' s + \nu' s' - \frac{qu}{k}\right)$$

$$+ \frac{1}{2}\frac{d^2 z}{dx^2}\left(\mu s + \nu s' - \frac{pu}{k}\right)^2 + \frac{1}{2}\frac{d^2 z}{dy^2}\left(\mu' s + \nu' s' - \frac{qu}{k}\right)^2$$

$$+\ \frac{d^2z}{dx\,dy}\left(\mu s+\nu s'-\frac{pu}{k}\right)\left(\mu' s+\nu' s'-\frac{qu}{k}\right)+\text{etc.}$$

If we equate that value of z' to the preceding one and move all terms of the first power in u, s, and s' that result to the left-hand side of the equation then we will have:

$$(1 + p^2 + q^2)\ \frac{u}{k} + (\mu'' - \mu\,p - \mu'q)\ s + (\nu'' - \nu\,p - \nu'q)\ s'$$

$$= \frac{1}{2}\frac{d^2z}{dx^2}\left(\mu s+\nu s'-\frac{pu}{k}\right)^2+\frac{1}{2}\frac{d^2z}{dy^2}\left(\mu' s+\nu' s'-\frac{qu}{k}\right)^2$$

$$+\frac{d^2z}{dx\,dy}\left(\mu s+\nu s'-\frac{pu}{k}\right)\left(\mu' s+\nu' s'-\frac{qu}{k}\right)+\text{etc.}$$

The coefficients of s and s' are zero, by virtue of the condition equations of section **11**. The left-hand side will then reduce to $k\,u$, with $1 + p^2 + q^2 = k^2$. If one also replaces u with its development in powers of s and s' and arranges the left-hand side of that equation similarly then it will become:

$$k\,A\,s^2 + k\,A'\,s'^2 + \text{etc.} = \left(\frac{d^2z}{dx^2}\cdot\frac{\mu^2}{2}+\frac{d^2z}{dx\,dy}\cdot\mu\,\mu'+\frac{d^2z}{dy^2}\cdot\frac{\mu'^2}{2}\right)s^2$$

$$+\left(\frac{d^2z}{dx^2}\cdot\frac{\nu^2}{2}+\frac{d^2z}{dx\,dy}\cdot\nu\,\nu'+\frac{d^2z}{dy^2}\cdot\frac{\nu'^2}{2}\right)s'^2$$

$$+\left(\frac{d^2z}{dx^2}\cdot\mu\nu+\frac{d^2z}{dx\,dy}(\mu\nu'+\mu'\nu)+\frac{d^2z}{dy^2}\cdot\mu'\nu'\right)s\,s'+\text{etc.}$$

This must be an identity in s and s'. Hence, if one equates the coefficients of the terms of second power in the two sides of the equation then that will lead to these three equations:

$$\frac{d^2z}{dx^2}\cdot\frac{\mu^2}{2}+\frac{d^2z}{dx\,dy}\cdot\mu\,\mu'+\frac{d^2z}{dy^2}\cdot\frac{\mu'^2}{2}=k\,A,$$

$$\frac{d^2z}{dx^2}\cdot\frac{\nu^2}{2}+\frac{d^2z}{dx\,dy}\cdot\nu\,\nu'+\frac{d^2z}{dy^2}\cdot\frac{\nu'^2}{2}=k\,A,$$

$$\frac{d^2z}{dx^2}\cdot\mu\nu+\frac{d^2z}{dx\,dy}(\mu\nu'+\mu'\nu)+\frac{d^2z}{dy^2}\cdot\mu'\nu'=0.$$

The first two will yield the values of A and A', and the third one will determine the directions of the s and s' axes, which were supposed to be tangent to the two lines of principal curvature that intersect at the point m (no. **12**). It is true that the quantities μ, μ', ν, ν' enter into those values of A and A', but one can make them disappear in the following manner:

1. I add the first two equations. I let H denote the sum $A + A'$, and upon considering the condition equations of section **11**, I will find that:

$$H = \frac{1+q^2}{2k^3}\cdot\frac{d^2z}{dx^2} - \frac{pq}{k^3}\cdot\frac{d^2z}{dx\,dy} + \frac{1+p^2}{2k^3}\cdot\frac{d^2z}{dy^2}.$$

2. I multiply the first two equations with each other. I then subtract the square of the third one from the product; that gives:

$$k^2 A A' = \tfrac{1}{4}(\mu\nu' - \mu'\nu)^2\left[\frac{d^2z}{dx^2}\cdot\frac{d^2z}{dy^2} - \left(\frac{d^2z}{dx\,dy}\right)^2\right],$$

but one will have (no. **11**):

$$(\mu\,\nu' - \mu'\,\nu)^2 = \frac{1}{k^2}.$$

If one then lets G lets denote the product $A\,A'$ then one will have:

$$G = \frac{1}{4k^4}\cdot\left[\frac{d^2z}{dx^2}\cdot\frac{d^2z}{dy^2} - \left(\frac{d^2z}{dx\,dy}\right)^2\right].$$

The sum and the product of A and A' are then known, so one can form a second-degree equation whose roots will be those two quantities. It is easy to verify that these results coincide with known formulas that serve to determine the two radii of principal curvature of a given surface.

Since one has:

$$(A - A')^2 = (A + A')^2 - 4AA' = H^2 - 4G,$$

it will then follow that the internal tension that was determined in section **16** will become:

$$T = \varepsilon^2 a^2 \pi + \tfrac{1}{8}\varepsilon^2 b^2 \pi \cdot H^2 - \tfrac{1}{2}\varepsilon^2 b^2 \pi \cdot G - \Pi,$$

and the boundary tension (no. **17**) will become:

$$T = \tfrac{1}{2}\varepsilon^2 a^2 \pi + \tfrac{1}{16}\varepsilon^2 b^2 \pi \cdot H^2 - \tfrac{1}{4}\varepsilon^2 b^2 \pi \cdot G - \Pi.$$

Hence, those two quantities are now expressed as functions of the coordinates of the point *m* that they describe.

19. – In order to likewise express the quantity $3C + C'' + 3C^{\text{iv}}$ that enters into the value of *U*, I let H' denote what *H* will become at the point m', in such a way that H' will be the same function of x' and y' that *H* is of *x* and *y*. Upon developing that using Taylor's theorem, we will have:

$$H' = H + \frac{dH}{dx}\left(\mu s + \nu s' - \frac{pu}{k}\right) + \frac{dH}{dy}\left(\mu' s + \nu' s' - \frac{qu}{k}\right)$$

$$+ \frac{1}{2}\frac{d^2H}{dx^2}\left(\mu s + \nu s' - \frac{pu}{k}\right)^2 + \frac{1}{2}\frac{d^2H}{dy^2}\left(\mu' s + \nu' s' - \frac{qu}{k}\right)^2$$

$$+ \frac{d^2H}{dx\,dy}\left(\mu s + \nu s' - \frac{pu}{k}\right)\left(\mu' s + \nu' s' - \frac{qu}{k}\right) + \text{etc.}$$

If one replaces *u* with its value as a series and arranges things in powers and products of *s* and *s*′ then that will give:

$$H' = H + \left(\frac{dH}{dx}\mu + \frac{dH}{dy}\mu'\right)s + \left(\frac{dH}{dx}\nu + \frac{dH}{dy}\nu'\right)s'$$

$$+ \left(\frac{d^2H}{dx^2}\cdot\frac{\mu^2}{2} + \frac{d^2H}{dx\,dy}\cdot\mu\mu' + \frac{d^2H}{dy^2}\cdot\frac{\mu'^2}{2} - \frac{Ap}{k}\cdot\frac{dH}{dx} - \frac{Aq}{k}\cdot\frac{dH}{dy}\right)s^2$$

$$+ \left(\frac{d^2H}{dx^2}\cdot\frac{\nu^2}{2} + \frac{d^2H}{dx\,dy}\cdot\nu\nu' + \frac{d^2H}{dy^2}\cdot\frac{\nu'^2}{2} - \frac{A'p}{k}\cdot\frac{dH}{dx} - \frac{A'q}{k}\cdot\frac{dH}{dy}\right)s'^2$$

$$+ \text{etc.}$$

However, one can get another value for H' by replacing the coordinates *x*, *y*, *z* of the point *m* with the coordinates *s*, s', *u* of the point m' in the *H* of the preceding section. That will give:

$$H' = \frac{1+q'^2}{2k'^3}\cdot\frac{d^2u}{ds^2} - \frac{p'q'}{k'^3}\cdot\frac{d^2u}{ds\,ds'} + \frac{1+p'^2}{2k'^3}\cdot\frac{d^2u}{ds'^2},$$

if one sets:

$$\frac{du}{ds} = p', \qquad \frac{du}{ds'} = q', \qquad \sqrt{1+p'^2+q'^2} = k,$$

to abbreviate. If one confines oneself to terms of the second power in s and s' then the value of u as a series will give:

$$p'^2 = 4A^2 s^2, \qquad q'^2 = 4A'^2 s'^2, \qquad p'q' = 4AA' s s',$$

$$\frac{1}{k'^2} = 1 - 6A^2 s^2 - 6A'^2 s'^2,$$

$$\frac{d^2u}{ds^2} = 2A + 6Bs + 2B' s' + 12Cs^2 + 6C' s s' + 2C'' s'^2,$$

$$\frac{d^2u}{ds'^2} = 2A' + 6B'' s + 2B'' s + 12C'' s'^2 + 6C'' s s' + 2C'' s^2 .$$

Using that, the second value of H', when arranged in powers and products of s and s' will become:

$$H' = A + A' + (3B + B)\, s + (3B + B')\, s'$$

$$+ (6C + C'' - 6A^3 - 2A^2 A')\, s^2 + (6C^{\text{iv}} + C'' - 6A'^3 - 2AA'^2)\, s'^2$$

$$+ \text{etc.}$$

If one compares that value to the first one and equates the coefficients of s^2 and s'^2 on one side and the other then one will have:

$$6C + C'' - 6A^3 - 2A^2 A' = \frac{\mu^2}{2}\cdot\frac{d^2H}{dx^2} + \mu\mu'\cdot\frac{d^2H}{dx\,dy} + \frac{\mu'^2}{2}\cdot\frac{d^2H}{dy^2} - \frac{A}{k}\left(p\frac{dH}{dx} + q\frac{dH}{dy}\right),$$

$$6C^{\text{iv}} + C'' - 6A'^3 - 2AA'^2 = \frac{\nu^2}{2}\cdot\frac{d^2H}{dx^2} + \nu\nu'\cdot\frac{d^2H}{dx\,dy} + \frac{\nu'^2}{2}\cdot\frac{d^2H}{dy^2} - \frac{A'}{k}\left(p\frac{dH}{dx} + q\frac{dH}{dy}\right) .$$

If one adds those two equations and reduces the result as in the preceding section then one will find that:

$$3C + C'' + 3C^{\text{iv}} = \frac{1+q^2}{4k^2}\cdot\frac{d^2H}{dx^2} - \frac{pq}{4k^2}\cdot\frac{d^2H}{dx\,dy} + \frac{1+p^2}{4k^2}\cdot\frac{d^2H}{dy^2} - \frac{H}{2k}\left(p\frac{dH}{dx} + q\frac{dH}{dy}\right)$$

$$+ 3A^3 + 3A'^3 + (A + A')\, AA' .$$

I will presently substitute that value for that of U in section **15**, and while always keeping H and G in place of $A + A'$ and AA', it will become:

$$U = -\,\varepsilon^2\, k\, a^2\, \pi H - \tfrac{1}{8}\varepsilon^2 b^2 \pi\left[\frac{1+q^2}{2k}\cdot\frac{d^2H}{dx^2} - \frac{p\,q}{k}\cdot\frac{d^2H}{dx\,dy} + \frac{1+p^2}{2k}\cdot\frac{d^2H}{dy^2}\right.$$
$$\left. -\,p\,H\,\frac{dH}{dx} - qH\,\frac{dH}{dy} + 3k\,H^3 - 12\,k\,G\,H\right].$$

20. – Now recall equation (1) of section **9**. Upon recalling the value of *H*, one can then write:

$$Z - p\,X - q\,Y + U + 2\,T\,k\,H = 0,$$

and if one replaces *U* with its value and replaces *T* with the first of those two expressions in section **18** then one will have:

$$Z - p\,X - q\,Y + (\varepsilon^2\, a^2\, \pi - 2\Pi)\, k\,H + \tfrac{1}{2}\varepsilon^2\, b^2\, \pi\;GH$$
$$(a)$$
$$-\tfrac{1}{8}\varepsilon^2\, b^2\pi\left[\frac{1+q^2}{2k}\cdot\frac{d^2H}{dx^2} - \frac{p\,q}{k}\cdot\frac{d^2H}{dx\,dy} + \frac{1+p^2}{2k}\cdot\frac{d^2H}{dy^2} - pH\,\frac{dH}{dx} - qH\,\frac{dH}{dy} + kH^3\right] = 0.$$

That equation, the study of which defines the main topic of this article, is that of the elastic surface in equilibrium. From the form of the quantity *H*, as one sees, it contains fourth-order partial derivatives and is linear with respect to the highest derivatives. Meanwhile, if one pays attentions to the nature of the constants that are denoted by a^2 and b^2 (no. **15**), which represent the definite integrals:

$$\int \alpha^2 f\alpha\cdot d\alpha,\quad \int \alpha^4 f\alpha\cdot d\alpha,$$

then it will be obvious that the second one is negligible, if not infinitesimal, with respect to the first one, because b^2 depends upon the fourth power of the radius of activity of the repulsive force, while a^2 depends upon only its square. It might then seem as if we must neglect the terms in our equation that are multiplied by b^2, which will lower its order to two. However, one must observe that the term that is multiplied by a^2 will be combined with the arbitrary constant that is contained in Π, and we shall prove that the former term will disappear along with that arbitrary constant once we have determined its value.

Indeed, that constant will depend upon the particular forces that pull the surface to its boundary and are equal and directly opposite to the boundary tensions, as one saw in section **4**. Therefore, suppose that *m* is one of the points on the contour. Let *V* denote the given force that acts upon that point tangentially to the surface and in a direction that is perpendicular to its contour. If one equates that force to the boundary tension that pertains to that point (i.e., the second value of *T* in section **18**) then one will have:

$$V = \tfrac{1}{2}\varepsilon^2 a^2 \pi + \tfrac{1}{16}\varepsilon^2 b^2 \pi\cdot H - \tfrac{1}{4}\varepsilon^2 b^2 \pi\cdot G\; - \Pi. \qquad (b)$$

Now, if one applies that equation to a well-defined point of the contour and infers the value of the arbitrary constant that is included in Π, and one then eliminates it from equation (*a*) then it will be obvious that the term that is multiplied by a^2 will simultaneously disappear, in such a way that all that will remain will be the given forces and some terms that come from the elastic forces that are multiplied by b^2.

21. – In the case where the forces *X*, *Y*, *Z* are zero and the elastic surface is a cylinder, one must find the usual equation of the *elastic strip* for the section that is perpendicular to the edges. In order to verify that, take the *y*-axis to be parallel to those edges, in which case, *z* will be independent of *y*, and one will have:

$$q = 0, \quad k = \sqrt{1+p^2}, \quad \frac{d^2z}{dx^2} = \frac{dp}{dx}, \quad H = (1+p)^{-3/2} \cdot \frac{1}{2}\frac{dp}{dx}, \qquad G = 0.$$

Moreover, the quantity Π, which represents the integral of $X\,dx + Y\,dy + Z\,dz$, will reduce to an arbitrary constant. I let *c* denote a similar constant, and I put:

$$2\Pi - \varepsilon^2 a^2 \pi = \tfrac{1}{16}\varepsilon^2 a^2 \pi \cdot c,$$

to simplify. Equation (*a*) will then become:

$$\frac{1}{\sqrt{1+p^2}} \frac{d\cdot\left[(1+p^2)^{-3/2}\cdot\frac{dp}{dx}\right]}{dx} d\cdot\left[(1+p^2)^{-3/2}\cdot\frac{dp}{dx}\right] - \frac{p}{(1+p^2)^{3/2}}\frac{dp}{dx}\cdot d\cdot\left[(1+p^2)^{-3/2}\cdot\frac{dp}{dx}\right]$$

$$+ \frac{1}{2(1+p^2)^4}\frac{dp^2}{dx^2}dp + \frac{c}{1+p^2}dp = 0.$$

One does not recognize the equation of the elastic strip in this. However, it is possible to convert it into the usual form by a sequence of transformations that I shall now indicate.

First, divide all terms by $\sqrt{1+p^2}$ and observe that:

$$\frac{1}{(1+p^2)}\frac{d}{dx}\left[(1+p^2)^{-3/2}\cdot\frac{dp}{dx}\right]\cdot d\cdot\left[(1+p^2)^{-3/2}\cdot\frac{dp}{dx}\right]$$

$$= \frac{d}{dx}\left[(1+p^2)^{-5/2}\cdot\frac{dp}{dx}\right]\cdot d\cdot\left[(1+p^2)^{-3/2}\cdot\frac{dp}{dx}\right]$$

$$+ \frac{6p}{(1+p^2)^2}\frac{dp}{dx}\cdot\left[(1+p^2)^{-3/2}\cdot\frac{dp}{dx}\right] + \frac{2}{(1+p^2)^{9/2}}\frac{dp}{dx}\cdot dp,$$

so our equation will become:

$$\frac{d}{dx}\left[(1+p^2)^{-5/2}\cdot\frac{dp}{dx}\right]\cdot d\cdot\left[(1+p^2)^{-5/2}\cdot\frac{dp}{dx}\right]+\frac{5p}{(1+p^2)^2}\frac{dp}{dx}\cdot d\cdot\left[(1+p^2)^{-3/2}\cdot\frac{dp}{dx}\right]$$

$$+\frac{5}{2(1+p^2)^{9/2}}\frac{dp^2}{dx^2}dp+c\cdot d\cdot\frac{p}{\sqrt{1+p^2}}=0.$$

and due to the fact that:

$$\frac{2p}{(1+p^2)^2}\frac{dp}{dx}\cdot d\cdot\left[(1+p^2)^{-3/2}\cdot\frac{dp}{dx}\right]+\frac{1}{(1+p^2)^{9/2}}\frac{dp^2}{dx^2}\cdot dp$$

$$= d\cdot\left[p(1+p^2)^{-7/2}\cdot\frac{dp^2}{dx^2}\right],$$

it will change into:

$$\frac{d}{dx}\left[(1+p^2)^{-5/2}\cdot\frac{dp}{dx}\right]\cdot d\cdot\left[(1+p^2)^{-5/2}\cdot\frac{dp}{dx}\right]+\frac{5}{2}\cdot d\cdot\left[p(1+p^2)^{-3/2}\cdot\frac{dp^2}{dx^2}\right]+c\cdot d\cdot\frac{p}{\sqrt{1+p^2}}=0.$$

If one integrates this and denotes the arbitrary constant by c then it will become:

$$\frac{d}{dx}\left[(1+p^2)^{-5/2}\cdot\frac{dp}{dx}\right]+\frac{5p}{2(1+p^2)^{7/2}}\frac{dp^2}{dx^2}+\frac{cp}{\sqrt{1+p^2}}=c'.$$

I multiply this by 2 dp and observe that:

$$2\frac{dp}{dx}\cdot d\cdot\left[(1+p^2)^{-5/2}\cdot\frac{dp}{dx}\right]+\frac{5p}{2(1+p^2)^{7/2}}\frac{dp^2}{dx^2}dp = d\cdot\left[(1+p^2)^{-5/2}\cdot\frac{dp^2}{dx^2}\right],$$

which will reduce the preceding equation to:

$$d\cdot\left[(1+p^2)^{-5/2}\cdot\frac{dp^2}{dx^2}\right]+\frac{2cp}{\sqrt{1+p^2}}dp = 2c'\,dp\,.$$

I integrate a second time, and let c'' denote the arbitrary constant; I will then have:

$$\frac{1}{(1+p^2)^{-5/2}}\frac{dp^2}{dx^2}+2c\sqrt{1+p^2}=2c'\,dp+c''.$$

If one solves that equation for *dx* then one will have:

$$dx = \frac{(1+p^2)^{-5/4}}{\sqrt{2c'p + c^2 - 2c\sqrt{1+p^2}}}\,dp\,.$$

If one multiplies this by *p* and replaces *p dx* with *dz* then it will become:

$$dz = \frac{(1+p^2)^{-5/4}}{\sqrt{2c'p + c^2 - 2c\sqrt{1+p^2}}}\,p\,dp\,.$$

The last two equations are effectively the ones for the elastic strip in the form that Euler found for it at the end of his treatise on the isoperimetric problem (*). It is easy to reduce them to just one between *z* and *x*, but we will not stop to perform that transformation.

22. – If the elastic surface differs only slightly from a plane, which is the *xy*-plane, and one consequently neglects the squares and products of the partial derivatives of ordinate *z* in its equations, then one will have:

$$k = 1, \quad G = 0, \quad H = \frac{1}{2}\left(\frac{d^2z}{dx^2} + \frac{d^2z}{dy^2}\right),$$

which will reduce equation (*a*) to simply:

$$Z - p\,X - q\,Y + \left(\tfrac{1}{2}\varepsilon^2 a^2 \pi - \Pi\right)\left(\frac{d^2z}{dx^2} + \frac{d^2z}{dy^2}\right) - \tfrac{1}{32}\varepsilon^2 a^2 \pi \cdot \left[\frac{d^4z}{dx^4} + 2\frac{d^4z}{dx^2\,dy^2} + \frac{d^4z}{dy^4}\right] = 0,$$

and equation (*b*) will reduce to:

$$V = \tfrac{1}{2}\varepsilon^2 a^2 \pi - \Pi\,.$$

We can deduce the equations of motion of the elastic surface immediately from its equilibrium equations by using the principles of mechanics that we recalled in section **7**, and if we confine ourselves to considering the case in which the surface makes very small oscillations on one side and the other of a fixed plane then one must start with the reduced equations that we just wrote down. Suppose, in addition, that each point oscillates along a line that is perpendicular to that plane and ignore the weight of the surface. As in the cited section, we will then have:

(*) "Methodus inveniendi lineas curvas, …," page 249.

$$X = 0, \quad Y = 0, \quad Z = -\varepsilon \frac{d^2 z}{dt^2},$$

in which t is the variable that represents time.

The quantity Π denotes the integral of $X\,dx + Y\,dy + Z\,dz$, so one will then have:

$$d\,\Pi = -\varepsilon \frac{d^2 z}{dt^2} dz = -\frac{\varepsilon}{2} \cdot d \cdot \frac{d^2 z}{dt^2},$$

which is a value that one must regard as zero, since one neglects the terms of second order in z. Hence, Π will be an arbitrary constant, and from the preceding value of V, that force, which acts tangentially to the boundary of the surface, must also be constant or zero.

With those new restrictions, one will have the equation:

$$\varepsilon \frac{d^2 z}{dt^2} - V\left(\frac{d^2 z}{dx^2} + \frac{d^2 z}{dy^2}\right) + \tfrac{1}{32}\varepsilon^2 b^2 \pi \left(\frac{d^4 z}{dx^4} + \frac{d^4 z}{dx^2\,dy^2} + \frac{d^4 z}{dy^4}\right) = 0$$

for the determination of the vibrations of the surface.

The coefficient b^2 depends upon the natural elasticity of the surface. It will then vary with the matter that comprises it, and it is assumed to be given for each particular surface. If one suppose that it is zero then one will have the equation of an inelastic surface that is acted upon by a force V (no. **7**). On the contrary, if one sets $V = 0$ then the equation of motion will reduce to the simplest form that it can have in the case of elasticity, namely:

$$\frac{d^2 z}{dt^2} + n^2 \left(\frac{d^4 z}{dx^4} + \frac{d^4 z}{dx^2\,dy^2} + \frac{d^4 z}{dy^4}\right) = 0,$$

in which n^2 is an essentially-positive constant coefficient that is proportional to the thickness ε and the quantity b^2. It no longer contains the equation of the inelastic surface then, which can vibrate only when it is acted upon by a force that is applied to its boundary.

The latter equation is the one that one finds, without proof, in the anonymous piece that I spoke of at the beginning of this article. Now that it has been deduced from a rigorous theory, it can provide a basis for some of the studies that one might undertake regarding the laws of the vibrations of resonant plates.

23. – I shall conclude this paper by exhibiting a curious property of the elastic surface in equilibrium. The one that I shall consider is a plate of equal thickness that is bent by given forces that act upon its contour, and to simplify, I shall ignore its weight. Now, I say that in the equilibrium state, among all of the surfaces with the same area it will be the surface for which the integral:

$$\iint \left(\frac{1}{\rho} + \frac{1}{\rho'} \right)^2 k\,dx\,dy$$

is a *maximum* or a *minimum*. As before, ρ and ρ' will denote the two principal radii of curvatures that pertain to an arbitrary point. $k\,dx\,dy$ represents the element that relates to that point, and the double integral is extended over the entire surface. In order to verify that theorem, it will suffice to show that it will lead to the equation of the surface that was found before.

Indeed, the integral that represents the area of the surface will be $\iint k\,dx\,dy$. From the rules of the *calculus of variations*, the equation of the *maximum* or *minimum* that is presently at issue will then be:

$$\delta \cdot \iint \left(\frac{1}{\rho} + \frac{1}{\rho'} \right)^2 k\,dx\,dy + c\,\delta \cdot \iint k\,dx\,dy = 0,$$

in which c is an arbitrary constant. Moreover, since one only wants to know what the equation of the surface is without looking at what happens at its limits, one can regard dx and dy as constants, in such a way that when one moves the characteristic δ under the $\iint$ sign, combines the two integrals into one and sets:

$$\frac{1}{\rho} + \frac{1}{\rho'} = R$$

to abbreviate, the preceding equation will become:

$$\iint (\delta \cdot k\,R^2 + c\,\delta k)\; dx\; dy = 0.$$

However, upon setting $dz = dx = p$, $dz / dy = q$, as always, one will have:

$$k = \sqrt{1 + p^2 + q^2}\;,$$

$$R = \frac{1+q^2}{k^3} \cdot \frac{d^2 z}{dx^2} - \frac{2pq}{k^3} \cdot \frac{d^2 z}{dx\,dy} + \frac{1+p^2}{k^3} \cdot \frac{d^2 z}{dy^2}\,.$$

Hence, one infers that:

$$\delta k = \frac{p}{k} \cdot \frac{d \cdot \delta z}{dx} + \frac{q}{k} \cdot \frac{d \cdot \delta z}{dy}\,,$$

$$\delta \cdot k\,R^2 = 2R \left(\frac{1+q^2}{k^2} \cdot \frac{d^2 \cdot \delta z}{dx^2} - \frac{2pq}{k^2} \cdot \frac{d^2 \cdot \delta z}{dx\,dy} + \frac{1+p^2}{k^2} \cdot \frac{d^2 \cdot \delta z}{dy^2} \right)$$

$$+\,2k\,R\left(\frac{dR}{dp}\cdot\frac{d\cdot\delta z}{dx}+\frac{dR}{dq}\cdot\frac{d\cdot\delta z}{dy}\right)+R^2\left(\frac{p}{k}\cdot\frac{d\cdot\delta z}{dx}+\frac{q}{k}\cdot\frac{d\cdot\delta z}{dy}\right).$$

I substitute those values in the preceding equation. Upon integrating by parts, I will make the first and second derivatives of δz disappear. I neglect the terms that can go outside the double ∬ sign, which refer, as one knows, to the limits of the surface, which we shall not consider here. Finally, upon observing that:

$$\frac{dR}{dp}=-\frac{3p}{k}\cdot R+\frac{2p}{k^3}\frac{d^2z}{dy^2}-\frac{2q}{k^3}\cdot\frac{d^2z}{dx\,dy},$$

$$\frac{dR}{dq}=-\frac{3p}{k}\cdot R+\frac{2q}{k^3}\frac{d^2z}{dx^2}-\frac{2q}{k^3}\cdot\frac{d^2z}{dx\,dy},$$

$$\frac{d\cdot(p/k)}{dx}+\frac{d\cdot(q/k)}{dy}=R\,.$$

After some reductions, I will find that:

$$\iint\left[2\frac{d^2}{dx^2}\cdot\left(\frac{1+q^2}{k^2}\cdot R\right)-2\frac{d^2}{dx\,dy}\cdot\left(\frac{pq}{k^2}\cdot R\right)+2\frac{d^2}{dy^2}\cdot\left(\frac{1+p^2}{k^2}\cdot R\right)\right.$$

$$-\,4\frac{d}{dx}\left(\frac{pR}{k^2}\right)\cdot\frac{d^2z}{dy^2}-4\frac{d^2}{dx\,dy}\left(\frac{qR}{k^2}\right)\cdot\frac{d^2z}{dx^2}+4\frac{d}{dx}\left(\frac{qR}{k^2}\right)\cdot\frac{d^2z}{dx\,dy}+4\frac{d}{dy}\left(\frac{pR}{k^2}\right)\cdot\frac{d^2z}{dx\,dy}$$

$$\left.+\frac{10pR}{k}\cdot\frac{dR}{dx}+\frac{10qR}{k}\cdot\frac{dR}{dy}+5R^3-cR\right]\cdot\,\delta z\cdot dx\,dy=0.$$

Upon equating the coefficient of δz to zero (i.e., the quantity that is found inside the brackets), one will have the equation of the surface that was sought. It will not be presented in the same form as the equation of the elastic surface that we found in section **20**, but if one performs the differentiations of the products that are only indicated there, one will find, after reducing and dividing by two, that:

$$\frac{1+q^2}{k^2}\cdot\frac{d^2R}{dx^2}-\frac{2pq}{k^2}\cdot\frac{d^2R}{dx\,dy}+\frac{1+p^2}{k^2}\cdot\frac{d^2R}{dy^2}$$

$$-\,\frac{pR}{k}\cdot\frac{dR}{dx}-\frac{qR}{k}\cdot\frac{dR}{dy}+\frac{R}{2}(R^2-c)-\frac{R}{2k^4}\left[\frac{d^2z}{dx^2}\cdot\frac{d^2z}{dy^2}-\left(\frac{d^2z}{dx\,dy}\right)^2\right]=0.$$

It is easy to recognize the coincidence of that equation with the one in section **20** when one suppresses the forces X, Y, Z in it and one replaces the quantity Π by an arbitrary constant.

Furthermore, the integral $\iint R\,k\,dx\,dy$ is not the only one that enjoys the property of being a *maximum* or a *minimum*. It also belongs to the integrals:

$$\iint\left(\frac{1}{\rho}+\frac{1}{\rho'}\right)^2 k\,dx\,dy\,, \qquad \iint\left(\frac{1}{\rho^2}+\frac{1}{\rho'^2}\right)k\,dx\,dy\,,$$

and generally all of the integrals that can be deduced from the first one by adding the integral $\iint\frac{k\,dx\,dy}{\rho\,\rho'}$ to it, multiplied by a constant coefficient. That amounts to saying that one has:

$$\delta\cdot\iint\frac{k\,dx\,dy}{\rho\,\rho'} = \iint\delta\left(\frac{k}{\rho\,\rho'}\right)\cdot dx\,dy = 0$$

identically, when one considers only the terms that will remain under the double $\iint$ sign after integrating by parts and one ignores the ones that pass outside of it. One will effortlessly verify that assertion upon starting from the known value of the quantity $k/(\rho\,\rho')$, namely:

$$\frac{k}{\rho\,\rho'} = \frac{1}{k^3}\left[\frac{d^2z}{dx^2}\frac{d^2z}{dy^2}-\left(\frac{d^2z}{dx\,dy}\right)^2\right].$$

The property of the elastic surface that we just proved includes the property of the strip, which was first imagined by D. Bernoulli and which Euler later verified at the end of his treatise on the *isoperimetric problem* that was cited above (*). Indeed, in the case of the strip, one of the two principal radii of curvature – for example, the radius ρ' – will become infinite. Furthermore, the surface element will change into that of the elastic curve, which we will call *ds*, and the double integral:

$$\iint R\,k\,dx\,dy$$

will become the simple integral $\int ds/\rho^2$, which must be effectively a *minimum*, from the principle of D. Bernoulli.

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(*) On that topic, also see the note that follows the paper by Laplace on *double refraction.* Mémoires de l'Institut in the year 1809.