"Sur les équations canoniques des systèmes non holonomes," C. R. Acad. Sci. Paris **156** (1913), 1829-1831.

On the canonical equations of non-holonomic systems

Note by **THEODOR POSCHL**

Presented by Appell.

Translated by D. H. Delphenich

Let $q_1, q_2, \ldots, q_n$ be the n true coordinates of a system, and let:

(1) $$\omega_i = \alpha_{i1}\,\dot{q}_1 + \alpha_{i2}\,\dot{q}_2 + \cdots + \alpha_{in}\,\dot{q}_n \qquad (i = 1, 2, \ldots, n)$$

be linear, non-integrable combinations of their derivatives with respect to time. Solving (1) for the $\dot{q}_i$ will give:

(2) $$\dot{q}_i = \beta_{1i}\,\omega_1 + \beta_{2i}\,\omega_2 + \ldots + \beta_{ni}\,\omega_n \qquad (i = 1, 2, \ldots, n).$$

We formally set $\omega_i = d\vartheta_i / dt$ and regard the ϑ_i, which do not exist in reality, as quasi-coordinates.

The equations of motion of that system, which are expressed in terms of the q_i and ω_i, result from Hamilton's principle:

(3) $$I = \int_{t_1}^{t_2} (T + U)\, dt = \text{min.},$$

when one defines the variations of the ϑ according to equations (1). We will then get:

(4) $$\delta I = \int \sum_i \left\{ \left[\frac{\partial (T+U)}{\partial \vartheta_i} \right] \delta\vartheta_i + \frac{\partial T}{\partial \omega_i}\, \delta \frac{d\vartheta_i}{dt} \right\} dt = 0 ,$$

in which $\left(\dfrac{\partial f}{\partial \vartheta_i}\right)$ denotes the symbolic derivative, i.e., $\sum_\nu \dfrac{\partial f}{\partial q_\nu}\beta_{i\nu}$. However, it will follow from (1) and (2) that:

$$\delta \frac{d\vartheta_i}{dt} = \frac{d}{dt}\,\delta\vartheta_i - \sum_{\mu,\nu} g_{\mu\nu i}\,\delta\vartheta_\mu \frac{d\vartheta_\nu}{dt} ,$$

in which:

$$g_{\mu\nu i} = \sum_{r,s} \left(\frac{\partial \alpha_{is}}{\partial q_r} - \frac{\partial \alpha_{ir}}{\partial q_s} \right) \beta_{\mu s}\, \beta_{\nu r} \ .$$

When the integration by parts is applied to the term $\delta \frac{d\vartheta_i}{dt}$ in (4), that will obviously produce the equations for the non-holonomic system in the Lagrange-Euler form (according to G. Hamel):

$$(5) \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \omega_i} \right) + \sum_{\mu,\nu} g_{i\mu\nu}\, \omega_\mu \frac{\partial T}{\partial \omega_\nu} - \left[\frac{\partial (T+U)}{\partial \vartheta_i} \right] = 0 \ .$$

We introduce the new variables:

$$p_i = \frac{\partial T}{\partial \omega_i}$$

here, express T in terms of q_i and p_i, and set:

$$K = \sum p_i\, q_i' - T \ .$$

We will then obtain:

$$-\left(\frac{\partial T}{\partial \vartheta_i} \right) = \left(\frac{\partial K}{\partial \vartheta_i} \right) , \qquad \omega_i = \frac{\partial K}{\partial p_i} \ .$$

Equations (5) get:

$$\frac{dp_i}{dt} + \sum_{\mu,\nu} g_{i\mu\nu} \frac{\partial K}{\partial p_\mu} \frac{\partial T}{\partial \omega_\nu} = \left[\frac{\partial (K-U)}{\partial \vartheta_i} \right] .$$

Finally, will let:

$$H = K - U = \sum p_i\, q_i' - T - U = 2T - T - U = T - U \ .$$

The following form for the equations will result, which corresponds to the canonical equations:

$$(6) \qquad \begin{cases} \omega_i = \dfrac{\partial H}{\partial p_i} , \\[2ex] \dfrac{dp_i}{dt} = -\displaystyle\sum_{\mu,\nu} g_{i\mu\nu} \frac{\partial H}{\partial p_\mu} \frac{\partial H}{\partial \omega_\nu} - \left(\frac{\partial H}{\partial \vartheta_i} \right) . \end{cases}$$

If the constraints do not contain time then those equations will admit the *vis viva* integral:

$$H = h \ ,$$

in which h is an integration constant, because it will follow immediately that since $g_{i\mu\nu} = -\, g_{\mu\nu i}$, one has:

$$\frac{dH}{dt} = \sum_i \left[\left(\frac{\partial H}{\partial \vartheta_i} \right) \omega_i + \frac{\partial H}{\partial p_i} \frac{dp_i}{dt} \right] = 0 \, .$$

A development of this with some extensions and applications of equations (6) to various examples will be given in a more detailed article.