CHAPTER V

INTEGRAL INVARIANTS

20. The Jacobi equations. Invariant differential forms. The concept of an integral invariant. – The fact that the equations of motion of a system:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\rho}\right) - \frac{\partial L}{\partial q_\rho} = 0 \tag{247}$$

come about as the **Euler** equations of a variational problem:

$$\int L(\dot{q}_1, \ldots, \dot{q}_n, q_1, \ldots, q_n, t)\, dt = \text{extrem.} \tag{248}$$

can be exploited for the integration of those equations in a different way that was first pointed out by **H. Poincaré** ([235]). In order to do that, one must start from a certain extremal space-time line:

$$q_\rho = \bar{q}_\rho(t) \qquad (\rho = 1, \ldots, n), \tag{249}$$

which might be referred to as the *base extremal*, and then consider its infinitesimally-close extremals. If one goes from the base extremal $\bar{q}_\rho(t)$ to a certain neighboring extremal $\bar{q}_\rho + \delta q_\rho$ then the n functions:

$$\delta q_\rho = \kappa_\rho(t) \qquad (\rho = 1, \ldots, n) \tag{250}$$

that mediate that transition will determine an extremal of a different variational problem in their own right. Namely, every base extremal of the variational problem (248) will be associated with a *variational problem for the second variation*:

$$\int \Lambda(\dot{\kappa}_1, \ldots, \dot{\kappa}_n, \kappa_1, \ldots, \kappa_n, t)\, dt = \text{extrem.}, \tag{251}$$

whose integrand is the quadratic form:

$$\Lambda = \tfrac{1}{2} \sum_{\rho,\sigma=1}^{n} \left\{ \frac{\partial^2 L}{\partial \dot{q}_\rho\, \partial \dot{q}_\sigma} \dot{\kappa}_\rho\, \dot{\kappa}_\sigma + 2 \frac{\partial^2 L}{\partial \dot{q}_\rho\, \partial q_\sigma} \dot{\kappa}_\rho\, \kappa_\sigma + \frac{\partial^2 L}{\partial q_\rho\, \partial q_\sigma} \kappa_\rho\, \kappa_\sigma \right\}. \tag{251.a}$$

([235]) **H. Poincaré**, "Sur le problème des trois corps et les équations de la dynamique," Acta math. **13** (1890), pp. 1 (esp., Chap. II), as well as *Les méthodes nouvelles de la mécanique céleste*, t. I, 162, *et seq.*, and t. III, p. 1, *et seq.*

The functions $\bar{q}_\rho(t)$ for the base extremal replace the q_ρ in its coefficients, such that the second derivatives will become known functions of time. The functions (250) are now solutions of the **Euler** equations:

$$\frac{d}{dt}\left(\frac{\partial \Lambda}{\partial \dot{\kappa}_\rho}\right) - \frac{\partial \Lambda}{\partial \kappa_\rho} = 0 \qquad (\rho = 1, \ldots, n) \tag{252}$$

of that variational problem (251), which define a system of *n linear* second-order differential equations. Conversely, every extremal of the variational problem (251) will also mediate the transition to a neighboring extremal of the chosen base extremal, since the $\kappa_1(t), \ldots, \kappa_n(t)$ (up to a proportionality factor) equal the differences $\delta q_1, \ldots, \delta q_n$ in the coordinates of a neighboring extremal compared to the base extremal. It would seem historically justified to call equations (252) the *Jacobi equations for the variational problem* (247), since **Jacobi** first recognized their meaning ([236]) when he sought the conditions for the occurrence of a true extremum. In the terminology of **H. Poincaré**, they were called the "équations aux variations" of equations (247).

As **Euler** equations of the variational problem (251), one can also put them into the form of a canonical system corresponding to the transformation of no. **9**, when one replaces the $\dot{\kappa}_\rho$ with:

$$\pi_\rho = \frac{\partial \Lambda}{\partial \dot{\kappa}_\rho}, \tag{253}$$

and introduce the function:

$$\mathsf{H}(\pi_\rho, \kappa_\rho, t) = \sum \pi_\rho \dot{\kappa}_\rho - \Lambda \tag{253.a}$$

in place of Λ. The function H, like Λ, is a quadratic form in its variables, and indeed:

$$\mathsf{H} = \tfrac{1}{2}\sum_{\sigma,\tau=1}^{n}\left\{\frac{\partial^2 H}{\partial p_\sigma\,\partial p_\tau}\pi_\sigma\,\pi_\tau + 2\frac{\partial^2 H}{\partial p_\sigma\,\partial q_\tau}\pi_\sigma\,\kappa_\tau + \frac{\partial^2 H}{\partial q_\sigma\,\partial q_\tau}\kappa_\sigma\,\kappa_\tau\right\}, \tag{254}$$

in which the functions for the base extremal are introduced into the second derivatives of H for p_ρ, q_ρ, such that those derivatives will become known functions of the independent variable t. The canonical system that belongs to (252) then reads:

$$\begin{cases} \dfrac{d\kappa_\rho}{dt} = \dfrac{\partial \mathsf{H}}{\partial \pi_\rho} = \displaystyle\sum_{\sigma=1}^{n}\left(\frac{\partial^2 H}{\partial p_\rho\,\partial p_\sigma}\pi_\sigma + \frac{\partial^2 H}{\partial p_\rho\,\partial q_\sigma}\kappa_\sigma\right), \\ \dfrac{d\pi_\rho}{dt} = -\dfrac{\partial \mathsf{H}}{\partial \kappa_\rho} = -\displaystyle\sum_{\sigma=1}^{n}\left(\frac{\partial^2 H}{\partial q_\rho\,\partial p_\sigma}\pi_\sigma + \frac{\partial^2 H}{\partial q_\rho\,\partial q_\sigma}\kappa_\sigma\right). \end{cases} \tag{255}$$

([236]) **C. G. J. Jacobi**, "Zur Theorie der Variationsrechnung und der Differentialgleichungen," J. f. Math. **17** (1837), pp. 68 = *Werke IV*, pp. 39.

It then represents a system of $2n$ *linear* first-order differential equations that belongs to the canonical system:

(256) $$\frac{dq_\rho}{dt} = \frac{\partial H}{\partial p_\rho}, \qquad \frac{dp_\rho}{dt} = -\frac{\partial H}{\partial q_\rho},$$

just like the **Jacobi** system (*équations aux variations*), namely the system of equations (252), is associated with the system of equations (247).

Now, a system of second-order linear differential equations that arises as the **Euler** equations of a variational problem (the associated canonical system, resp.) is distinguished from other linear differential equations by the fact that it defines a *self-adjoint system* [cf., II A 4.b (**E. Vessiot**), no. **26**], i.e., for two systems of solutions:

$$\kappa_\rho^{(1)}(t) \quad \text{and} \quad \kappa_\rho^{(2)}(t)$$

to (252) [for two systems of solutions:

$$\pi_\rho^{(1)}(t),\ \kappa_\rho^{(1)}(t) \qquad \text{and} \qquad \pi_\rho^{(2)}(t),\ \kappa_\rho^{(2)}(t)$$

of (255, resp.], one has the relation:

(257) $$\frac{d}{dt}\left[\sum_{\rho=1}^{n}(\pi_\rho^{(2)}\,\kappa_\rho^{(2)} - \pi_\rho^{(1)}\,\kappa_\rho^{(1)})\right] = 0,$$

or

(258) $$\sum_{\rho=1}^{n}(\pi_\rho^{(2)}\,\kappa_\rho^{(2)} - \pi_\rho^{(1)}\,\kappa_\rho^{(1)}) = \text{const.},$$

resp.

The obvious question of what meaning it might have in terms of the equations of motion themselves (247) when a relation like (258) exists for the associated **Jacobi** equations was discussed by **H. Poincaré** ([237]). In order to do that, he first started from an arbitrary system of r first-order differential equations:

(259) $$\frac{dx_1}{dt} = X_1, \ldots, \frac{dx_r}{dt} = X_r, \qquad X_\lambda = X_\lambda(x_1, \ldots, x_r, t).$$

With the help of any solution, namely, the base solution:

(259.a) $$x_1 = \bar{x}_1(t), \ldots, x_r = \bar{x}_r(t),$$

he associated it with the system of linear **Jacobi** equations:

([237]) **H. Poincaré**, *Méthode. nouv. I*, pp. 162.

(260) $$\frac{d\xi_\lambda}{dt} = \frac{\partial X_\lambda}{\partial x_1}\xi_1 + \cdots + \frac{\partial X_\lambda}{\partial x_r}\xi_r \qquad (\lambda = 1, \ldots, r),$$

whose coefficients are functions of the independent variable t in the same way as before. Every solution to these linear **Jacobi** equations:

(260.a) $$\xi_1 = \xi_1(t), \ldots, \xi_r = \xi_r(t)$$

gives (up to an arbitrary constant factor) the system of coordinate differences $\delta x_1(t), \ldots, \delta x_r(t)$ that mediate the transition from the base integral curve (259.a) to a neighboring integral curve of the system (259). From an integral ([238]):

(261) $$F(\xi_1, \ldots, \xi_r) = \text{const.}$$

of the **Jacobi** equations (260), one can then infer a relation:

(261.a) $$F(\delta x_1, \ldots, \delta x_r) = \text{const.}$$

for the original equations (259) that is valid for every integral curve that is close to the base integral curve. If one does not fix the base integral curve from the outset then an integral of the **Jacobi** equations will have the form ([239]):

(262) $$F(x_1, \ldots, x_r, \xi_1, \ldots, \xi_r) = \text{const.}$$

That will imply the relation:

(262.a) $$F(x_1, \ldots, x_r, \delta x_1, \ldots, \delta x_r) = \text{const.},$$

which is a statement about *any two* infinitesimally-close integral curves of the differential equations (259) ([240]). Ultimately, as was shown before in (258), such relations exist for not just a single neighboring solution, but for several of them, say s. If one has an invariant relation of the form ([241]):

(263) $$F(\xi_1^{(1)}, \ldots, \xi_r^{(1)}, \xi_1^{(2)}, \ldots, \xi_r^{(2)}, \xi_1^{(s)}, \ldots, \xi_r^{(s)}) = \text{const.}$$

([238]) Naturally, since equations (259.a) are linear, it must be homogeneous in the $\xi_1, \ldots, \xi_r$, but the degree can be arbitrary.

([239]) That relation must also be homogeneous in the $\xi_1, \ldots, \xi_r$.

([240]) The invariant differential forms were considered systematically by **E. Cartan**, *Leçons sur les invariants intégraux*, Paris, 1922.

([241]) That expression must also be homogeneous in the $\xi_1^{(1)}, \ldots, \xi_r^{(1)}, \xi_1^{(2)}, \ldots, \xi_r^{(2)}, \xi_1^{(s)}, \ldots, \xi_r^{(s)}$. Naturally, $s \le r$ in that.

or the form:

(264) $$F(x_1, \ldots, x_r, \xi_1^{(1)}, \ldots, \xi_r^{(1)}, \xi_1^{(2)}, \ldots, \xi_r^{(2)}, \xi_1^{(s)}, \ldots, \xi_r^{(s)}) = \text{const.} \qquad (s \leq r),$$

resp., then it will correspond to the fact that for an arbitrary base integral curve of the system (259) and s neighboring integral curves, the transition to which is mediated by:

$$\delta^{(\lambda)} x_1, \ldots, \delta^{(\lambda)} x_r \qquad (\lambda = 1, \ldots, s),$$

one will have the relation:

(263.a) $$F(\delta^{(1)} x_1, \ldots, \delta^{(1)} x_r, \delta^{(2)} x_1, \ldots, \delta^{(2)} x_r, \ldots, \delta^{(s)} x_1, \ldots, \delta^{(s)} x_r) = \text{const.}$$

or

(264.b) $$F(x_1, \ldots, x_r, \delta^{(1)} x_1, \ldots, \delta^{(1)} x_r, \delta^{(2)} x_1, \ldots, \delta^{(2)} x_r, \ldots, \delta^{(s)} x_1, \ldots, \delta^{(s)} x_r) = \text{const.},$$

resp. For example, it follows from the relation (258), which is true for the **Jacobi** equations (255), that for the canonical system (256) itself, the relation:

(258.a) $$\sum_{\rho=1}^{n} (\delta^{(1)} p_\rho \, \delta^{(2)} q_\rho - \delta^{(1)} q_\rho \, \delta^{(2)} p_\rho) = \text{const.}$$

must exist between any two integral curves that are close to a base integral curve.

For that invariant differential form, the transition from a point to a neighboring point takes place entirely within the manifold t = const. ($\delta t = 0$). That is not necessary. One will get ([242]) a more general differential form from an invariant differential form like (263.a) [(264.a), resp.], by which one can go to an arbitrary neighboring point ($\delta t \neq 0$) when one replaces δx_ρ in it with $\delta x_\rho - X_\rho \, \delta t$ ([242.a]). Thus, e.g., the more general invariant differential form:

(265) $$\sum_{\rho=1}^{n} \left\{ \left(\delta^{(1)} p_\rho + \frac{\partial H}{\partial q_\rho} \delta^{(1)} t \right) \left(\delta^{(2)} q_\rho - \frac{\partial H}{\partial q_\rho} \delta^{(2)} t \right) - \left(\delta^{(1)} q_\rho - \frac{\partial H}{\partial p_\rho} \delta^{(1)} t \right) \left(\delta^{(2)} p_\rho + \frac{\partial H}{\partial q_\rho} \delta^{(2)} t \right) \right\}$$

$$= \sum_{\rho=1}^{n} (\delta^{(1)} p_\rho \, \delta^{(2)} q_\rho - \delta^{(1)} q_\rho \, \delta^{(2)} p_\rho) - (\delta^{(1)} H \, \delta^{(2)} q_\rho - \delta^{(1)} q_\rho \, \delta^{(2)} H) = \text{const.}$$

will enter in place of (258.a).

([242]) Cf., **E. Cartan**, *loc. cit.* ([240]), pp. 28.

([242.a]) Obviously, that is the same idea that allowed us to see in no. **16.c** that along with the linear differential form (174), at the same time, the differential form (172) represents a total differential in a field of extremals. In general, in order to do that, one must first go on to the second-order invariant differential form (265). Cf., also ([247]).

If the relation (261.a), which depends upon *one* row of differentials, is a differential form of degree p :

(266) $$\sum a_{ik\cdots\sigma}(x_1,\ldots,x_r,t)\,\delta x_i\,\delta x_k\cdots\delta x_\sigma = \text{const.}$$

then one can convert it into a differential form of degree 1:

(267) $$\sqrt[p]{\sum a_{ik\cdots\sigma}\,\delta x_i\,\delta x_k\cdots\delta x_\sigma}$$

by extracting the root and employing it in that form as the integrand of a curve integral that extends along an arbitrary curve in a manifold t = const. The associated integral:

(268) $$\int\sqrt[p]{\sum a_{ik\cdots\sigma}\,\delta x_i\,\delta x_k\cdots\delta x_\sigma} = \int\sqrt[p]{\sum a_{ik\cdots\sigma}\frac{dx_i}{du}\frac{dx_k}{du}\cdots\frac{dx_\sigma}{du}}\,du$$

is then an *integral invariant*, as **H. Poincaré** had introduced it. Namely, it has the following property: If one lays the associated integral curve of the system (259) through each point of the integration path of (268) then they will collectively generate an M_2 that cuts out a curve segment (M_1) from any arbitrary manifold t = const. For any manifold t = const., the integral (268) will then have the same value when it is extended over the curve segment that is determined by M_2, that is, it will remain *invariant*. If one generalizes the differential form in the way that was described above by eliminating δt then one can choose an arbitrary curve on the M_2 as an integration path that connects an arbitrary point of the integral curve through the lower limit of the integral (268) with an arbitrary point on the integral curve through the upper limit of the integral. The integral has the same limit for all of those integration paths. That is the *concept of a first-order integral invariant* ([243]). The simplest example is the integral of a linear differential form:

(269) $$\int\sum a_i(x_1,\ldots,x_r,t)\,\delta x_i = \text{const.}$$

that is associated with the general integral invariant:

(269.a) $$\int\left\{\sum a_i\,\delta x_i - \left(\sum a_i\,X_i\right)\delta t\right\} = \text{const.}$$

The differential forms that are defined by *several rows of differentials* likewise yield integral invariants. For example, the bilinear differential form (258.a) can be regarded as the element of a double integral, and that leads to the integral invariant:

([243]) Just as one derives an integral invariant from an integral of the **Jacobi** equation, one will also naturally get an integral of the **Jacobi** equation from an integral invariant. That relation can be used to find new integral invariants from ones that are known already. Cf., **H. Poincaré**, *Méthode. nouv. III*, pp. 19.

(270) $$\iint \left(\sum_{\rho=1}^{n} \begin{vmatrix} \frac{\partial p_\rho}{\partial u} & \frac{\partial q_\rho}{\partial u} \\ \frac{\partial p_\rho}{\partial v} & \frac{\partial q_\rho}{\partial v} \end{vmatrix} \right) du\, dv = \text{const.}$$

Correspondingly, an invariant differential form (264) that is defined by s rows of differentials, when it can be regarded as the element of an integral ([244]), will generally lead to the integral invariant of order s :

(272) $$\underbrace{\iint \cdots \int}_{(s)} F(x_1, \ldots, x_r, \delta^{(1)}x_1, \ldots, \delta^{(1)}x_r, \ldots, \delta^{(s)}x_1, \ldots, \delta^{(s)}x_r) = \text{const.}$$

that is invariant in the following sense: The integral is thought of as extended over an s-dimensional region in an arbitrary manifold t = const. If one maps that segment to a corresponding segment of another manifold t = const. in such a way that one can construct an integral curve of (259) and intersects the integral curve with the new manifold t = const. then when the integral is extended over the image region, it will have the same value that it had for the original integration region. Here, as well, one can free oneself of the condition that the integration region must belong to a manifold t = const. by replacing δx_λ with $\delta x_\lambda - X_\lambda\, \delta t$. For example, from (265), one can replace the integral (270) with ([245]):

([244]) **E. Cartan** called such differential forms "formes extérieures" in *Leçons sur les invariants intégraux*, pp. 50. If that is to be true for a bilinear form:

$$\sum a_{ik}\, \delta^{(i)}x_i\, \delta^{(2)}x_k$$

then it would be necessary and sufficient that, for example, it should be alternating:

$$a_{ik} = -\, a_{ki} ,$$

such that one can write:

$$\sum a_{ik} \begin{bmatrix} \delta^{(1)}x_i & \delta^{(2)}x_i \\ \delta^{(1)}x_k & \delta^{(2)}x_k \end{bmatrix} = \sum a_{ik}\, \delta\omega_{ik} .$$

In general, a *forme extérieure* has the form:

(271) $$\sum a_{\lambda_1 \lambda_2 \cdots \lambda_s} \begin{bmatrix} \delta^{(1)}x_{\lambda_1} & \delta^{(2)}x_{\lambda_1} & \cdots & \delta^{(s)}x_{\lambda_1} \\ \delta^{(1)}x_{\lambda_2} & \delta^{(2)}x_{\lambda_2} & \cdots & \delta^{(s)}x_{\lambda_2} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{(1)}x_{\lambda_s} & \delta^{(1)}x_{\lambda_s} & \cdots & \delta^{(s)}x_{\lambda_s} \end{bmatrix} = \sum a_{\lambda_1 \lambda_2 \cdots \lambda_s}\, \delta\omega_{\lambda_1 \lambda_2 \cdots \lambda_s} .$$

The name *forme extérieure* is explained by the fact that the determinants are the components of an exterior product of s vectors, in the sense of **H. Grassmann**. In the language of tensor calculus, one refers to the system of coefficients as a (covariant) *alternating tensor of rank s*.

([245]) Here, as well, it will once more become clear that at the moment when one regards the position coordinates as being on a par with time, the impulse components and the energy will likewise be on a par with each other.

$$\iint \left\{ \sum_{\rho=1}^{n} (\delta^{(1)} p_\rho \, \delta^{(2)} q_\rho - \delta^{(1)} q_\rho \, \delta^{(2)} p_\rho) - (\delta^{(1)} H \, \delta^{(2)} t - \delta^{(1)} t \, \delta^{(2)} H) \right\} = \text{const.}$$

Along with the *absolute integral invariants* for a system of differential equations that are explained in that way, **H. Poincaré** ([246]) also posed the so-called *relative integral invariants*. For them, the differential forms that serve as integrals are *not invariant* by themselves, but the integrals that they define will remain invariant when one chooses *the integration region to be a closed manifold.* For example, if the linear form:

(273) $$a_1(x_1, \ldots, x_r, t)\, \delta x_1 + \ldots + a_r(x_1, \ldots, x_r, t)\, \delta x_r + b(x_1, \ldots, x_r, t)\, \delta t$$

is not also an invariant differential form in its own right then the integral:

(273.a) $$\oint (a_1 \, \delta x_1 + \cdots + a_r \, \delta x_r + b \, \delta t)$$

can nonetheless be a relative integral invariant when it is extended over a closed curve. In order to do that, the following must be true: If one draws that integral curves of (259) through all points of the closed integration path such that a tube is generated, and one lays a second closed curve around that tube then the integral must have the same value for both integral paths. If the summand with δt is missing then the path of integration must lie in a manifold t = const. ([247]). Analogously, one speaks of a relative integral invariant of order s of the system of differential equations (259):

(274) $$\underbrace{\iint \cdots \int}_{s} \sum a_{\lambda_1 \lambda_2 \cdots \lambda_s} \, \delta\omega_{\lambda_1 \lambda_2 \cdots \lambda_s}$$

as long as that integral remains constant in the sense that was just explained if and only if one extends it over a *closed* M_s as the domain of integration ([248]). The integrand in such a relative

([246]) **H. Poincaré**, *Méthode. nouv. III*, pp. 9.

([247]) δt *cannot* be introduced into a relative integral invariant with $\delta t = 0$ in the same way that it can be introduced into an absolute integral invariant.

([248]) In so doing:

$$\delta\omega_{\lambda_1 \lambda_2 \cdots \lambda_s} = \begin{vmatrix} \frac{\partial x_{\lambda_1}}{\partial u_1} & \frac{\partial x_{\lambda_1}}{\partial u_2} & \cdots & \frac{\partial x_{\lambda_1}}{\partial u_s} \\ \frac{\partial x_{\lambda_2}}{\partial u_1} & \frac{\partial x_{\lambda_2}}{\partial u_2} & \cdots & \frac{\partial x_{\lambda_2}}{\partial u_s} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_{\lambda_s}}{\partial u_1} & \frac{\partial x_{\lambda_s}}{\partial u_2} & \cdots & \frac{\partial x_{\lambda_s}}{\partial u_s} \end{vmatrix} du_1 \cdots du_s \, .$$

integral invariant is the sum of an invariant differential form ([249]) and a total differential of the same order ([250]). In that, one understands a total differential of order p to mean a differential form:

$$\sum C_{\lambda_1 \lambda_2 \cdots \lambda_s}(x_1, \ldots, x_r)\, \delta\omega_{\lambda_1 \lambda_2 \cdots \lambda_s} \tag{275}$$

whose coefficients satisfy the conditions ([251]):

$$\frac{\partial C_{\lambda_2 \cdots \lambda_s \lambda_{s+1}}}{\partial x_{\lambda_1}} - \frac{\partial C_{\lambda_1 \lambda_3 \cdots \lambda_s \lambda_{s+1}}}{\partial x_{\lambda_2}} + \cdots - \cdots + (-1)^s \frac{\partial C_{\lambda_1 \cdots \lambda_{s-1} \lambda_s}}{\partial x_{\lambda_{s+1}}} = 0\,. \tag{275.a}$$

Upon applying the generalized **Stokes**'s theorem ([252]), one will obtain an absolute integral invariant of the next-highest order ([253]) from a relative integral invariant (274). In order to do that, one must make the closed M_s that serves as the domain of integration for the relative integral invariant of order s (274) pass through an M_{s+1}, which is arbitrary moreover. One will then have:

$$\underbrace{\iint \cdots \int}_{s} \sum a_{\lambda_1 \cdots \lambda_s}\, \delta\omega_{\lambda_1 \cdots \lambda_s}$$
$$= \underbrace{\iiint \cdots \int}_{s} \sum \left(\frac{\partial a_{\lambda_2 \cdots \lambda_s \lambda_{s+1}}}{\partial x_{\lambda_1}} - \frac{\partial a_{\lambda_1 \lambda_3 \cdots \lambda_{s+1}}}{\partial x_{\lambda_2}} + \cdots - \cdots + (-1)^s \frac{\partial a_{\lambda_1 \cdots \lambda_s}}{\partial x_{\lambda_{s+1}}} \right) \delta\omega_{\lambda_1 \cdots \lambda_s \lambda_{s+1}}\,, \tag{276}$$

and the right-hand side is an integral invariant of order (s + 1) whose integral is a total differential, moreover ([254]).

In regard to the relationship between integral invariants and the integrals of the system (259), we might mention only the following things: Naturally, from an integral:

([249]) Which can be the integral of an absolute integral invariant in its own right.

([250]) Cf., **H. Poincaré**, *Méthod. nouv. III*, nos. 239 and 240, pp. 11, *et seq.*

([251]) Cf., **R. Weitzenböck**, *Invariantentheorie*, Groningen, 1923, pp. 398. There is further literature in that. The sum in (275) is taken over all combinations of s indices from 1 to r. In so doing, one regards the system of $C_{\lambda_1 \cdots \lambda_s}$ as the so-called alternating tensor, i.e., $C_{\lambda_1 \cdots \lambda_s}$ will change sign when any two of its indices are switched and will then be zero when two of its indices are equal.

One can call the tensor (275.a) the *derivative of the tensor* (275). **Weitzenböck** himself (*Invariantentheorie*, pp. 381) called it the **Stokes** tensor of (275). **E. Cartan**, *Leçons sur les inv. intégr.*, pp. 66, called it the *dérivée exterérieure* of the tensor (275). **E. Goursat**, (cf., e,g., *Leçons sur le problène de Pfaff*, Paris, 1922, pp. 210) called that process the *D operation*.

([252]) Cf., e.g., **R. Weitzenböck**, *Invariantentheorie*, pp. 398.

([253]) Naturally, instead of a relative integral invariant, one can also start from an absolute integral invariant whose domain of integration is a closed manifold. One then goes from an integral invariant of order p to an integral invariant of order p + 1.

([254]) Correspondingly, it is not possible to repeat the process, i.e., to extend the integral in the right-hand side in (276) over a *closed* M_{s+1} and then convert it into an integral of order (s + 2) with the help of the extended **Stokes** theorem. That is because the coefficients of that integral would vanish identically.

$$F(x_1, x_2, \ldots, x_r) = \text{const.}$$

of the system of equations (259), one will always get a first-order integral invariant whose integrand is a complete differential:

$$\int \left(\frac{\partial F}{\partial x_1} \delta x_1 + \cdots + \frac{\partial F}{\partial x_r} \delta x_r \right).$$

However, it *does not* follow that, conversely, from a first-order integral invariant:

$$\int (a_1 \, \delta x_1 + \cdots + a_r \, \delta x_r) \tag{277}$$

whose integral is a complete differential, one will arrive at an integral of the equations in form of the associated function:

$$U(x_1, \ldots, x_r) = \int (a_1 \, \delta x_1 + \cdots + a_r \, \delta x_r). \tag{277.a}$$

Rather, one must conclude from that, in general, that the expression:

$$V(x_1, \ldots, x_r) = \frac{\partial U}{\partial x_1} X_1 + \frac{\partial U}{\partial x_2} X_2 + \cdots + \frac{\partial U}{\partial x_r} X_r \tag{278}$$

will be an integral ([255]), and it is only when that expression (278) vanishes identically that $U(x_1, \ldots, x_r)$ will itself be an integral ([256]) of the system (259) ([257]).

([255]) Cf., **H. Poincaré**, *Méthod. nouv. III*, pp. 28. If one knows an integral:

$$a_1(x_1, \ldots, x_r)\, \xi_1 + \ldots + a_r(x_1, \ldots, x_r)\, \xi_r = \text{const.}$$

to the **Jacobi** equations then since, under the assumption that the X_λ are independent of t, the Ansatz:

$$\xi_1 = X_1, \ldots, \xi_r = X_r$$

will be a solution to the **Jacobi** equations, one must also have that:

$$a_1 X_1 + \ldots + a_r X_r = \text{const.}$$

However, since the $\xi_1, \ldots, \xi_r$ no longer appear in that, it will be an integral of the original system (259). A generalization of that result is in *Méthod. nouv. III*, pp. 34.

One can use the same process to arrive at an integral invariant of order $(p - 1)$ from one of order p, cf., *Méthod. nouv. III*, pp. 33. **H. Poincaré** also showed how one can deduce an integral when one knows several integral invariants, more generally. *Méthod. nouv. III*, pp. 26.

([256]) In general:

$$U - t\, V = W$$

will also be an integral of the system.

([257]) A survey of the literature on integral invariants is in **E. Cartan**, *Leçons sur les invar. intégr.*, Paris, 1922.

21. The first-order relative integral invariant. The associated Pfaffian expression and its bilinear covariant. The *n* characteristic absolute integral invariants of the canonical system. – For the equations of motion (247) [the associated canonical system (259), resp.], the boundary formula of the calculus of variations will immediate yield a relative integral invariant. That is because the result of no. **16.c** [cf., (170)] can be expressed by saying that:

(279) $$\oint (p_1\,\delta q_1 + \cdots + p_n\,\delta q_n - H\,\delta t)$$

is a relative integral invariant of the canonical system ([258]):

(280) $$\frac{dq_\rho}{dt} = \frac{\partial H}{\partial p_\rho}, \qquad \frac{dp_\rho}{dt} = -\frac{\partial H}{\partial q_\rho}.$$

If the integration path is chosen to be a curve in a manifold t = const. then that integral invariant will take the simpler form:

(281) $$\oint (p_1\,\delta q_1 + \cdots + p_n\,\delta q_n).$$

By means of **Stokes**'s theorem, the first-order relative integral invariant (281) will imply the second-order absolute integral invariant ([259]):

(282) $$\iint (\delta p_1\,\delta q_1 + \delta p_2\,\delta q_2 + \cdots + \delta p_n\,\delta q_n),$$

or when written more concisely:

([258]) In order to connect up with the considerations of the last section, it is convenient to interpret the q_ρ, p_ρ, t as coordinates of an R_{2n+1} and to regard the path of integration of (279) as an M_1 in that space. Correspondingly, the integral (279) is understood to mean:

(279.a) $$\int (0\,\delta p_1 + \cdots + 0\,\delta p_n + p_1\,\delta q_1 + \cdots + p_n\,\delta q_n - H\delta t).$$

([259]) If one starts from the form (279) for the relative invariant, instead of (281), then the second-order absolute integral invariant will have the form:

(282.b) $$\iint \left\{ \sum_{\rho=1}^{n} (\delta p_\rho^{(1)}\,\delta q_\rho^{(2)} - \delta q_\rho^{(1)}\,\delta p_\rho^{(2)}) - (\delta^{(1)} H\,\delta^{(2)} t - \delta^{(1)} t\,\delta^{(2)} H) \right\}.$$

One can formally associate that with the form (282.a) when one identifies the time t with q_{n+1} and the energy $(-H)$ with p_{n+1} and then lets the sum in (282.a) run from 1 to $(n + 1)$, instead of 1 to n. Analogous statements are true for all further integral invariants that will be defined in this section.

The bilinear differential form that served as the integrand is the *dérivée extérieure* of the linear differential form that appears as the integrand in the relative integral invariant.

(282.a)
$$\iint \sum_{\rho=1}^{n} (\delta p_\rho^{(1)}\,\delta q_\rho^{(2)} - \delta q_\rho^{(1)}\,\delta p_\rho^{(2)}) \ .$$

Now, further absolute integral invariants can be easily derived from that absolute integral invariant. Namely, if one now considers four neighboring curves to the base curve, instead of two, which is a transition that be mediated by:

(283) $\delta p_\rho^{(1)}$, $\delta q_\rho^{(1)}$ goes to $\delta q_\rho^{(2)}$, $\delta p_\rho^{(2)}$, $\delta q_\rho^{(3)}$, $\delta p_\rho^{(3)}$, $\delta q_\rho^{(4)}$, $\delta p_\rho^{(4)}$, resp.,

then each of the six alternating bilinear differential forms:

(284)
$$\Omega\,(\lambda, \mu) = \sum (\delta p_\rho^{(\lambda)}\,\delta q_\rho^{(\mu)} - \delta q_\rho^{(\lambda)}\,\delta p_\rho^{(\mu)})$$

that can be defined by two of the four systems (283) will be an invariant differential form of the canonical system. It will then follow that the sum ([260]):

(285)
$$\tfrac{1}{2}\{\Omega(1,2)\,\Omega(3,4) + \Omega(1,3)\,\Omega(4,2) + \Omega(1,4)\,\Omega(2,3)\}$$

$$= \sum_{\rho\neq\lambda} \left\{ \begin{vmatrix} \delta p_\rho^{(1)} & \delta q_\rho^{(1)} \\ \delta p_\rho^{(2)} & \delta q_\rho^{(2)} \end{vmatrix} \begin{vmatrix} \delta p_\rho^{(3)} & \delta q_\rho^{(3)} \\ \delta p_\rho^{(4)} & \delta q_\rho^{(4)} \end{vmatrix} + \begin{vmatrix} \delta p_\rho^{(1)} & \delta q_\rho^{(1)} \\ \delta p_\rho^{(3)} & \delta q_\rho^{(3)} \end{vmatrix} \begin{vmatrix} \delta p_\rho^{(4)} & \delta q_\rho^{(4)} \\ \delta p_\rho^{(2)} & \delta q_\rho^{(2)} \end{vmatrix} + \begin{vmatrix} \delta p_\rho^{(1)} & \delta q_\rho^{(1)} \\ \delta p_\rho^{(4)} & \delta q_\rho^{(4)} \end{vmatrix} \begin{vmatrix} \delta p_\rho^{(2)} & \delta q_\rho^{(2)} \\ \delta p_\rho^{(3)} & \delta q_\rho^{(3)} \end{vmatrix} \right\}$$

$$= \sum_{\rho\neq\lambda} \begin{vmatrix} \delta p_\rho^{(1)} & \delta q_\rho^{(1)} & \delta p_\lambda^{(1)} & \delta q_\lambda^{(1)} \\ \delta p_\rho^{(2)} & \delta q_\rho^{(2)} & \delta p_\lambda^{(2)} & \delta q_\lambda^{(2)} \\ \delta p_\rho^{(3)} & \delta q_\rho^{(3)} & \delta p_\lambda^{(3)} & \delta q_\lambda^{(3)} \\ \delta p_\rho^{(4)} & \delta q_\rho^{(4)} & \delta p_\lambda^{(4)} & \delta q_\lambda^{(4)} \end{vmatrix} = \Omega\,(1, 2, 3, 4)$$

is also an invariant (and indeed quadrilinear) differential form of the canonical system ([261]).

Therefore, that four-fold integral is:

([260]) The product of two differential forms is defined in such a way that all of the products of two terms in the bilinear form are set equal to zero when their small determinants are indexed over the same variables.

([261]) **E. Cartan** referred to that construction (up to a factor of 1/2) as the product of the form Ω (1, 2) with itself (viz., the square of the form Ω) by suitably defining the *multiplication extérieure* of two alternating forms (cf., *Leçons sur les invar. intégr.*, pp. 51, 55, and 78).

E. Goursat, who referred to alternating differential forms as *formes symboliques*, spoke of a *produit symbolique* accordingly. (*Leçons sur le problème de Pfaff,* Chap. 3)

$$(286) \qquad = \iiiint \sum_{\rho,\lambda} \begin{vmatrix} \delta p_\rho^{(1)} & \delta q_\rho^{(1)} & \delta p_\lambda^{(1)} & \delta q_\lambda^{(1)} \\ \delta p_\rho^{(2)} & \delta q_\rho^{(2)} & \delta p_\lambda^{(2)} & \delta q_\lambda^{(2)} \\ \delta p_\rho^{(3)} & \delta q_\rho^{(3)} & \delta p_\lambda^{(3)} & \delta q_\lambda^{(3)} \\ \delta p_\rho^{(4)} & \delta q_\rho^{(4)} & \delta p_\lambda^{(4)} & \delta q_\lambda^{(4)} \end{vmatrix}$$

$$= \iiiint \left\{ \sum_{\rho,\lambda} \begin{vmatrix} \frac{\partial p_\rho}{\partial u_1} & \frac{\partial q_\rho}{\partial u_1} & \frac{\partial p_\lambda}{\partial u_1} & \frac{\partial q_\lambda}{\partial u_1} \\ \frac{\partial p_\rho}{\partial u_2} & \frac{\partial q_\rho}{\partial u_2} & \frac{\partial p_\lambda}{\partial u_2} & \frac{\partial q_\lambda}{\partial u_2} \\ \frac{\partial p_\rho}{\partial u_3} & \frac{\partial q_\rho}{\partial u_3} & \frac{\partial p_\lambda}{\partial u_3} & \frac{\partial q_\lambda}{\partial u_3} \\ \frac{\partial p_\rho}{\partial u_4} & \frac{\partial q_\rho}{\partial u_4} & \frac{\partial p_\lambda}{\partial u_4} & \frac{\partial q_\lambda}{\partial u_4} \end{vmatrix} \right\} du_1\, du_2\, du_3\, du_4$$

will be a fourth-order integral invariant of the canonical system.

If one multiplies the differential form $\Omega^{(2)} = \Omega\,(1, 2, 3, 4)$ by $\Omega\,(1, 2)$ in the same sense ([262]) then one will get a new invariant differential form with six rows of differentials, which will then couple the base integral curve to six neighboring integral curves, and indeed that will give:

$$(287) \qquad \Omega^{(3)} = \Omega\,(1, 2, 3, 4, 5, 6) = \tfrac{1}{3}\{\Omega(1,2)\,\Omega(3,4,5,6)$$

$$-\ \Omega(1,3)\,\Omega(2,4,5,6) + \Omega(1,4)\,\Omega(2,3,5,6) - \Omega(1,5)\,\Omega(2,3,4,6) + \Omega(1,6)\,\Omega(2,3,4,5)$$

$$= \sum_{\rho,\sigma,\tau} \begin{vmatrix} \delta p_\rho^{(1)} & \delta q_\rho^{(1)} & \delta p_\sigma^{(1)} & \delta q_\sigma^{(1)} & \delta p_\tau^{(1)} & \delta q_\tau^{(1)} \\ \delta p_\rho^{(2)} & \delta q_\rho^{(2)} & \delta p_\sigma^{(2)} & \delta q_\sigma^{(2)} & \delta p_\tau^{(2)} & \delta q_\tau^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta p_\rho^{(6)} & \delta q_\rho^{(6)} & \delta p_\sigma^{(6)} & \delta q_\sigma^{(6)} & \delta p_\tau^{(6)} & \delta q_\tau^{(6)} \end{vmatrix},$$

which is the sum of all triples of three different indices from the sequence 1 to *n*. That corresponds to the fact that the canonical system also possesses the absolute integral invariant of order six:

([262]) In the sense of **Cartan**'s *multiplication extérieure* [**Goursat**'s analogous concept, resp.], this is the third power of the bilinear differential form (284).

(287.a)
$$\iiint\iiint \sum_{\rho,\sigma,\tau} \begin{vmatrix} \delta p_\rho^{(1)} & \cdots & \delta q_\tau^{(1)} \\ \delta p_\rho^{(2)} & \cdots & \delta q_\tau^{(2)} \\ \vdots & \vdots & \vdots \\ \delta p_\rho^{(6)} & \cdots & \delta q_\tau^{(6)} \end{vmatrix} = \text{const.}$$

One will get a corresponding integral invariant of order eight, ten, etc., in an analogous way, and ultimately one will arrive at an invariant differential form with $2n$ rows of variables ([263]):

$$\Omega^{(n)} = \Omega\,(1, 2, 3, \ldots, 2n) = \begin{vmatrix} \delta p_1^{(1)} & \delta q_1^{(1)} & \delta p_2^{(1)} & \cdots & \delta q_n^{(1)} \\ \delta p_1^{(2)} & \delta q_1^{(2)} & \delta p_2^{(2)} & \cdots & \delta q_n^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta p_1^{(2n)} & \delta q_1^{(2n)} & \delta p_2^{(2n)} & \cdots & \delta q_n^{(2n)} \end{vmatrix}$$

or to the integral invariant of order $2n$:

(288)
$$\underbrace{\iint\cdots\int}_{2n} \begin{vmatrix} \delta p_1^{(1)} & \delta q_1^{(1)} & \delta p_2^{(1)} & \cdots & \delta q_n^{(1)} \\ \delta p_1^{(2)} & \delta q_1^{(2)} & \delta p_2^{(2)} & \cdots & \delta q_n^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta p_1^{(2n)} & \delta q_1^{(2n)} & \delta p_2^{(2n)} & \cdots & \delta q_n^{(2n)} \end{vmatrix} = \text{const.},$$

resp., which says that the volume of a chosen $2n$-dimensional structure in a manifold t = const. will remain unchanged ([264]) when the structure is carried to another manifold t = const. by means of the integral curves of the canonical system ([265]).

Moreover, one can also define a series of relative integral invariants analogously when one derives a new differential form from each of the integrands of the relative integral invariant (281) and the invariant differential forms Ω, $\Omega^{(2)}$, …, $\Omega^{(n-1)}$ that appear as integrals of the absolute

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([263]) Which one interprets as the n^{th} power of (284), in the spirit of **Cartan**.

([264]) When one writes:

(288.a)
$$\iint\cdots\int \delta p_1 \cdots \delta p_n \, \delta q_1 \cdots \delta q_n = \text{const.},$$

in the spirit of the usual notation of integral calculus, one must observe that every differential denotes a direction of advance here, so perhaps:

$$\begin{array}{lllll} \delta p_1^{(1)} = \delta p_1, & \delta q_1^{(1)} = 0, & \delta p_2^{(1)} = 0, & \ldots, & \delta q_n^{(1)} = 0, \\ \delta p_1^{(2)} = 0, & \delta q_1^{(2)} = \delta q_1, & \delta p_2^{(2)} = 0, & \ldots, & \delta q_n^{(2)} = 0, \\ \delta p_1^{(3)} = 0, & \delta q_1^{(3)} = 0, & \delta p_2^{(3)} = \delta p_2, & \ldots, & \delta q_n^{(3)} = 0, \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \delta p_1^{(2n)} = 0, & \delta q_1^{(2n)} = 0, & \delta p_2^{(2n)} = 0, & \ldots, & \delta p_n^{(2n)} = \delta q_n. \end{array}$$

([265]) Those n integral invariants of the canonical system have played a role in the development of quantum theory, cf., e.g., **M. Born**, *Vorlesungen über Atommechanik I*, Berlin, 1925, pp. 39.

integral invariants ([266]). The first of them, which is constructed from Ω, is the third-order relative integral invariant:

(289)
$$\iiint \sum_{\rho,\sigma} p_\rho \begin{vmatrix} \delta q_\rho^{(1)} & \delta p_\lambda^{(1)} & \delta q_\lambda^{(1)} \\ \delta q_\rho^{(2)} & \delta p_\lambda^{(2)} & \delta q_\lambda^{(2)} \\ \delta p_\rho^{(3)} & \delta p_\lambda^{(3)} & \delta q_\lambda^{(3)} \end{vmatrix} .$$

Meanwhile, they have no special meaning since the absolute invariant of one degree higher of the row that was just considered will emerge from each of them by using the generalized **Stokes**'s theorem.

That system of n integral invariants does not belong to the canonical system (259), but the converse is true. The integrand of the relative integral invariant (279) from which the absolute integral invariants arise is a **Pfaffian** expression:

(290)
$$p_1\,\delta q_1 + \ldots + p_n\,\delta q_n - H\,dt$$

in the $(2n + 1)$ variables $p_1, \ldots, p_n, q_1, \ldots, q_n, t$ that is already in normal form [cf., II A 5 (**E. von Weber**), Section III]. The integrand of the second-order absolute integral invariant (282.b):

(290.a)
$$(\delta^{(1)} p_1\,\delta^{(2)} q_1 - \delta^{(1)} q_1\,\delta^{(2)} p_1) + \cdots + (\delta^{(1)} p_n\,\delta^{(2)} q_n - \delta^{(1)} q_n\,\delta^{(2)} p_n) - (\delta^{(1)} H\,\delta^{(2)} t - \delta^{(1)} H\,\delta^{(2)} t)$$

is the associated *bilinear covariant* ([267]).

([266]) According to **E. Cartan**, in order to do that, one must form the *produit extérieure* of the two differential forms, cf., **E. Cartan**, *loc. cit.* ([240]), pp. 78.

([267]) In general, a **Pfaffian** expression:

(291)
$$a_1\,dx_1 + a_2\,dx_2 + \ldots + a_r\,dx_r$$

will, as a result of:

(292)
$$x_\rho = \varphi_\rho\,(y_1, \ldots, y_r) ,$$

take on the new form:

(291.a)
$$b_1\,dy_1 + b_2\,dy_2 + \ldots + b_r\,dy_r ,$$

in which the contravariant vector $(dx_1, \ldots, dx_r)$ and the covariant vector $(a_1, \ldots, a_r)$ will be substituted contragrediently. One can regard each of those two expressions as integrands of a relative integral invariant for a system of r first-order differential equations in each case. The relative integral invariant will then belong to the second-order absolute integral invariants:

(293)
$$\sum_{\lambda,\tau} \left(\frac{\partial a_\lambda}{\partial x_\tau} - \frac{\partial a_\tau}{\partial x_\lambda} \right) (\delta^{(1)} x_\lambda\,\delta^{(2)} x_\tau - \delta^{(1)} x_\tau\,\delta^{(2)} x_\lambda)$$

or

(293.a)
$$\sum_{\lambda,\tau} \left(\frac{\partial b_\lambda}{\partial x_\tau} - \frac{\partial b_\tau}{\partial x_\lambda} \right) (\delta^{(1)} y_\lambda\,\delta^{(2)} y_\tau - \delta^{(1)} y_\tau\,\delta^{(2)} y_\lambda) ,$$

Now, the canonical system is characterized by the fact that it represents the *characteristic system of the Pfaffian expression* (291) ([268]). Along with **G. D. Birkhoff** ([268.a]), one can relate the deduction of the characteristic system of the **Pfaffian** to the calculus of variations. If:

(294) $$a_1 (x_1, \ldots, x_r, t)\, \delta x_1 + \ldots + a_r (x_1, \ldots, x_r, t)\, \delta x_r + a_{r+1} (x_1, \ldots, x_r, t)\, \delta x_{r+1}$$

is the given **Pfaffian** expression then **Birkhoff** defined the variational problem:

(294.a) $$\int_{t_1}^{t_2} \left(\sum_{\rho=1}^{r} a_\rho (x_1, \ldots, x_r, t)\, \dot{x}_\rho + a_{r+1}(x_1, \ldots, x_r, t) \right) dt = \text{extrem.},$$

which he referred to as the ***Pfaffian** variational problem*. He obtained the characteristic system of the **Pfaffian** from that when be formally posed the **Euler** equations:

$$\frac{d}{dt}(a_\lambda) - \sum_{\rho=1}^{r} \frac{\partial a_\rho}{\partial x_\lambda}\, \dot{x}_\rho - \frac{\partial a_{r+1}}{\partial x_\lambda} = 0 ,$$

or

(295.a) $$\sum_{\rho=1}^{r} \left(\frac{\partial a_\lambda}{\partial x_\rho} - \frac{\partial a_\rho}{\partial x_\lambda} \right) \dot{x}_\rho - \left(\frac{\partial a_\lambda}{\partial t} - \frac{\partial a_{r+1}}{\partial x_\lambda} \right) = 0 \qquad (\lambda = 1, \ldots, r),$$

resp. Since the variation ([268.b]) of the integral (294.a):

$$\sum_{\rho=1}^{r} a_\rho\, \delta x_\rho \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left\{ \sum_{\rho=1}^{r} \left(\frac{da_\rho}{dt}\, \delta x_\rho - \delta a_\rho\, \dot{x}_\rho \right) - \delta a_{r+1} \right\} dt$$

$$= \sum_{\rho=1}^{r} a_\rho\, \delta x_\rho \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left\{ \sum_{\rho=1}^{r} \left(\sum_{\lambda=1}^{r} \left(\frac{\partial a_\rho}{\partial x_\lambda} - \frac{\partial a_\lambda}{\partial x_\rho} \right) \dot{x}_\rho + \left(\frac{\partial a_\rho}{\partial t} - \frac{\partial a_{r+1}}{\partial x_\rho} \right) \right) \delta x_\rho \right\} dt$$

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resp., whose integrands are therefore invariant differential forms for the associated characteristic system. Those differential forms will then go to each other under the transformation (292), like the **Pfaffian** expressions (291) [(291.a), resp.]. That is the origin of the term *covariant*.

([268]) The characteristic system of a **Pfaffian** expression is identical to the characteristic system of the bilinear covariant, which is regarded as a second-order differential form that one arrives at in the following way: For a given differential form, one asks what all systems of ordinary differential equations might be for which the differential form would be an invariant differential form. All of those systems have a certain number of common integrals, and those integrals are the integrals of a completely integrable **Pfaffian** system in their own right. That **Pfaffian** system will be the characteristic system of the differential form. Its meaning consists of the fact that it transforms *covariantly with the Pfaffian expression* under the introduction of new variables.

([268.a]) **G. D. Birkhoff**, *Dynamical Systems*, pp. 55.

([268.b]) Naturally, since the integral in (294.a) is a linear function of the $\dot{x}_\rho$, the values of the x_ρ are not prescribed at the limits t_1 and t_2 .

includes precisely the bilinear covariant of the **Pfaffian** expression (294) under the integral, the formal Ansatz of **Lagrange**'s equations (295) [(295.a), resp.] is equivalent to saying that one sets the derivatives of the bilinear covariant with respect to the δx_ρ equal to zero. If one multiplies equations (295) by $\dot{x}_\lambda$ and sums over λ then that will give:

$$\sum_{\lambda=1}^{r}\left(\frac{\partial a_\rho}{\partial x_\lambda}-\frac{\partial a_\lambda}{\partial x_\rho}\right)\dot{x}_\rho = 0$$

or

(295.b) $$da_{r+1}-\frac{\partial a_{r+1}}{\partial t}dt-\sum_{\lambda=1}^{r}\frac{\partial a_\lambda}{\partial t}dx_\lambda = 0 ,$$

resp., in which the left-hand side is precisely the derivative of the bilinear covariant with respect to δt. Applying the process to the variational problem of the linear differential form in (279) [(279.a), resp.] will then mean that one sets the derivatives of the associated bilinear covariant with respect to the δp_ρ, δq_ρ, δt equal to zero, as one prescribes in the theory of the **Pfaffian** problem ([269]). In fact, that gives the equations:

(296) $$\begin{cases} dq_\rho-\dfrac{\partial H}{\partial p_\rho}dt=0, \\ dp_\rho+\dfrac{\partial H}{\partial q_\rho}dt=0, \\ dH-\dfrac{\partial H}{\partial t}dt=0, \end{cases}$$

the first $2n$ of which define the canonical system, while the last one follows from it.

Obviously, the second-order absolute integral invariant from which the ones of higher order are all derived or the associated invariant differential form:

(297) $$(\delta^{(1)}p_1\,\delta^{(2)}q_1-\delta^{(1)}q_1\,\delta^{(2)}p_1)+\cdots+(\delta^{(1)}p_n\,\delta^{(2)}q_n-\delta^{(1)}q_n\,\delta^{(2)}p_n)-(\delta^{(1)}H\,\delta^{(2)}t-\delta^{(1)}t\,\delta^{(2)}H)$$
$$= \text{const.},$$

resp., is what has the fundamental meaning for the analytical treatment of the canonical system. The abbreviated form:

([269]) According to **E. Cartan**, *Leç. sur les invar. intégr.*, pp. 74, one will get the characteristic system of a *forme extérieure* when it defines its *dérivée extérieure* Ω' and then sets the derivatives of both forms with respect to a series of differentials equal to zero. The *dérivée extérieure* of a bilinear covariant that is itself the *dérivée extérieure* of a **Pfaffian** expression will be equal to zero, such that one must take only the derivatives of the bilinear covariant itself with respect to a series of differentials. For this, cf., the discussion in no. **15**.

(297.a) $$(\delta^{(1)} p_1 \delta^{(2)} q_1 - \delta^{(1)} q_1 \delta^{(2)} p_1) + \cdots + (\delta^{(1)} p_n \delta^{(2)} q_n - \delta^{(1)} q_n \delta^{(2)} p_n)$$

is essentially identical to the **Lagrange** bracket, moreover, which **Lagrange** had introduced into perturbation theory (cf., *supra*, no. **12**). That is because the derivatives of the q_ρ, p_ρ with respect to one of the constants c_λ will give a certain direction of advance in a manifold t = const., up to the factor δc_λ. Therefore:

(298) $$\left(\frac{\partial p_1}{\partial c_\lambda}\frac{\partial q_1}{\partial c_\rho} - \frac{\partial q_1}{\partial c_\lambda}\frac{\partial p_1}{\partial c_\rho}\right) + \cdots + \left(\frac{\partial p_n}{\partial c_\lambda}\frac{\partial q_n}{\partial c_\rho} - \frac{\partial q_n}{\partial c_\lambda}\frac{\partial p_n}{\partial c_\rho}\right) = [c_\lambda , c_\rho]$$

is identical to the form (297.a), up to the factor $\delta c_\lambda\, \delta c_\rho$, which is defined for the two directions that are determined by δc_λ [δc_ρ, resp.]. Therefore, the structure (298) must remain invariant when one advances along an integral curve of the canonical system, i.e., it must not change when t changes. The **Lagrangian** bracket must not include time t explicitly then, as **Lagrange** himself had proved by laborious calculations ([270]).

Among those second-order differential forms, the last of the series, namely, the differential form of order $2n$ (the associated integral invariant, resp.) plays an important role. It leads to the *theory of multipliers* that **Jacobi** addressed. Moreover, the existence of the integral invariant of order $2n$:

$$\iint \cdots \int \delta p_1 \cdots \delta p_n \, \delta q_1 \cdots \delta q_n = \text{const.}$$

for the canonical system can already be inferred from a remark by **J. Liouville** ([271]) such that one ordinarily refers to its existence in statistical mechanics as **Liouville**'s theorem [cf., IV 32 (**P. and T. Ehrenfest**), no. **8.c**]. If one interprets the integral curve of the canonical system as the trajectory of a fluid flow in the R_{2n} of $p_1, \ldots, p_n, q_1, \ldots, q_n$ (phase space of statistical mechanics) then the

([270]) It follows in a similarly-simple way that the **Poisson** brackets do not include time t explicitly, cf., the next section.

([271]) **J. Liouville**, "Sur la variation des constantes arbitraires," J. de math. **3** (1838), pp. 342. There, **Liouville** did not generally consider a canonical system, but a more general system of differential equations of the form (299). He showed that for the general solution:

$$x_1 = x_1 (t, c_1, \ldots, c_r) , \ldots, x_r = x_r (t, c_1, \ldots, c_r) ,$$

the functional determinant:

$$\left| \frac{\partial x_\rho}{\partial c_\sigma} \right|$$

will be independent of t in the event that the condition:

$$\frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_r}{\partial x_r} = 0$$

is fulfilled, which will go to (301) when M = const. That condition is fulfilled identically for a canonical system.

existence of the integral invariant will say that one is dealing with the flow of an incompressible fluid (i.e., a volume-preserving flow).

22. Integral invariants with the same order as the system. The Jacobi multiplier. – If one knows an integral invariant for a system of *r* first-order differential equations:

$$(299) \qquad \frac{dx_1}{dt} = X_1 (x_1, \ldots, x_r, t) , \qquad \ldots, \qquad \frac{dx_r}{dt} = X_r (x_1, \ldots, x_r, t)$$

that has the same order as the system:

$$(300) \qquad \iint \cdots \int M (t, x_1, \ldots, x_r)\, \delta x_1 \cdots \delta x_r$$

then the function *M* that appears in it will satisfy the differential equation:

$$(301) \qquad \frac{\partial M}{\partial t} + \frac{\partial (M\, X_1)}{\partial x_1} + \frac{\partial (M\, X_2)}{\partial x_2} + \cdots + \frac{\partial (M\, X_r)}{\partial x_r} = 0 .$$

Thus, that function will be a multiplier ([272]) of the system (299), with **C. G. J. Jacobi**'s definition. [Cf., II A 4.b (**E. Vessiot**), no.**12** and II A 5 (**E. Weber**), no. **12**]. For the canonical system, one can then deduce from the existence of the integral invariant of order 2*n* :

$$\iint \cdots \int \delta p_1 \cdots \delta p_n \;\; \delta x_1 \cdots \;\; \delta x_r = \text{const.}$$

that $M = 1$ is *a multiplier of the canonical system.*

C. G. J. Jacobi defined the word *multiplier* as a generalization of the concept of the **Euler** multiplier ([273]) $\mu (x, y)$ [II A 4.b (**E. Vessiot**), no. **5**], which is known to reduce the integration of a first-order differential equation:

$$(302) \qquad dy : dx = Y (x, y) : X (x, y)$$

([272]) That definition of the multiplier by **C. G. J. Jacobi** was in "Theoria novi multiplicatoris systemati aequationum differentialium vulgarium applicandi," J. f. Math. **27** (1844), pp. 199; *ibid.*, **29** (1845), pp. 213 and 333 = *Werke IV*, pp. 317. See also the presentation in **C. G. J. Jacobi**, *Vorlesungen*, *Werke Supplementband*, lectures 10 – 18, pp. 71, *et seq.* The connection between multipliers and integral invariants with the same order as the system was explained by **H. Poincaré** in "Sur le problème des trois corps et les équations de la dynamique," Acta math. **13** (1890), pp. and *Méthode. nouv. III*, pp. 41.

([273]) It is a solution of the partial differential equation:

$$\frac{\partial (\mu X)}{\partial x} + \frac{\partial (\mu Y)}{\partial y} = 0 ,$$

which is analogous to (301).

to the quadrature:

(302.a) $$\int \mu (X\,dy - Y\,dx) = \text{const.}$$

The analogy will become clear when one starts with the system of ([274]) $(r-1)^{\text{th}}$-order differential equations:

(303) $$dx_1 : dx_2 : \ldots : dx_r = X_1(x_1, \ldots, x_r, t) : X_2(x_1, \ldots, x_r, t) : \ldots : X_r(x_1, \ldots, x_r, t),$$

instead of (299), and as a generalization of the expression:

(304) $$X\,dy - Y\,dx,$$

defines the form of order $(n-1)$ in the differentials ([275]):

(305) $$\left\{X_1 \frac{\partial(x_2, x_3, \ldots, x_r)}{\partial(\sigma_1, \sigma_2, \ldots, \sigma_{r-1})} - X_2 \frac{\partial(x_1, x_3, \ldots, x_r)}{\partial(\sigma_1, \sigma_2, \ldots, \sigma_{r-1})} + - \cdots + (-1)^{r-1} X_r \frac{\partial(x_1, \ldots, x_{r-1})}{\partial(\sigma_1, \sigma_2, \ldots, \sigma_{r-1})}\right\} d\sigma_1\, d\sigma_2 \cdots d\sigma_{r-1}.$$

One can refer to that expression as an *exact differential* ([276]), in the generalized sense, when the $(r-1)$-fold integral:

([274]) The system (299) was correspondingly regarded as a system of $(r+1)$ variables:

$$dx_1 : \ldots : dx_r : dt = X_1(x_1, \ldots, x_r, t) : \ldots : X_r(x_1, \ldots, x_r, t) : 1.$$

The integral invariants with the same order as the system were accordingly first written out as integral invariants of order $(r+1)$:

$$\underbrace{\iint \cdots \int}_{r+1} M(x_1, \ldots, x_r, t)\, \delta x_1 \cdots \delta x_r\, \delta t.$$

If one now takes the $(r+1)$-dimensional domain of integration in the $(r+1)$-dimensional manifold to be "disk-shaped," i.e., one can bound it by two manifolds $t = c$ and $t = c + \delta c$ that are separated by δt, and chooses the two r-dimensional "base surfaces" in the two M_r to be congruent, and indeed such that they will go to each other under parallel translation in the t-direction, then one will see that the r-fold integral:

$$\underbrace{\iint \cdots \int}_{r} M(x_1, \ldots, x_r)\, \delta x_1 \cdots \delta x_r$$

is an integral invariant (of order r).

([275]) One understands:

$$\frac{\partial(x_2, x_3, \ldots, x_r)}{\partial(\sigma_1, \sigma_2, \ldots, \sigma_{r-1})}$$

to mean the functional determinant of the variables in the numerator with respect to the parameters in the denominator in the known manner.

([276]) On this, cf., **E. Cartan**, *Leç. sur les inv. intégr.*, pp. 71.

(305.a) $$\underbrace{\iint\cdots\int}_{r-1}\left\{X_1\frac{\partial(x_2,x_3,\ldots,x_r)}{\partial\,(\sigma_1,\sigma_2,\ldots,\sigma_{r-1})}-X_2\frac{\partial(x_1,x_3,\ldots,x_r)}{\partial\,(\sigma_1,\sigma_2,\ldots,\sigma_{r-1})}+\cdots-\cdots\right.$$
$$\left.+(-1)^{r-1}X_r\frac{\partial(x_1,\ldots,x_{r-1})}{\partial\,(\sigma_1,\sigma_2,\ldots,\sigma_{r-1})}\right\}d\sigma_1\,d\sigma_2\cdots d\sigma_{r-1}$$

is also equal to zero when one extends it over a (two-sided) *closed* M_{r-1}, i.e., when the r-fold integral over the region that is bounded by the closed M_{r-1} that emerges from (305.a) by the generalized **Stokes**'s theorem ([277]) is:

(306) $$\underbrace{\iint\cdots\int}_{r}\left(\frac{\partial X_1}{\partial x_1}+\cdots+\frac{\partial X_r}{\partial x_r}\right)\frac{\partial(x_1,x_2,\ldots,x_r)}{\partial\,(\tau_1,\tau_2,\ldots,\tau_r)}d\tau_1\cdots d\tau_r=0\,.$$

In order for that integral to vanish identically for *every* domain of integration, one must have:

$$\frac{\partial X_1}{\partial x_1}+\frac{\partial X_2}{\partial x_2}+\cdots+\frac{\partial X_r}{\partial x_r}=0\,,$$

which is a condition ([278]) that is naturally not fulfilled in general for the system (303). On the other hand, if $M(x_1, \ldots, x_r)$ is a **Jacobi** multiplier of the system (303) then from (301), one will have the relation:

(307) $$\frac{\partial(M\,X_1)}{\partial x_1}+\frac{\partial(M\,X_2)}{\partial x_2}+\cdots+\frac{\partial(M\,X_r)}{\partial x_r}=0\,,$$

i.e., multiplying by a **Jacobi** multiplier will convert the expression (305) into a complete differential in the generalized sense. When the integral:

(308) $$\underbrace{\iint\cdots\int}_{r-1}\left\{M\,X_1\frac{\partial(x_2,x_3,\ldots,x_r)}{\partial\,(\sigma_1,\sigma_2,\ldots,\sigma_{r-1})}-M\,X_2\frac{\partial(x_1,x_3,\ldots,x_r)}{\partial\,(\sigma_1,\sigma_2,\ldots,\sigma_{r-1})}+\cdots-\cdots\right.$$
$$\left.+(-1)^{r-1}M\,X_r\frac{\partial(x_1,x_2,\ldots,x_{r-1})}{\partial\,(\sigma_1,\ldots,\sigma_{r-1})}\right\}d\sigma_1\,d\sigma_2\cdots d\sigma_{r-1}$$

([277]) Cf., e.g., **R. Weitzenböck**, *Invariantentheorie*, pp. 398.
([278]) That is the analogue of the relation:

$$\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}=0$$

that must be fulfilled if the expression (304) is to be an exact differential in the ordinary sense.

is extended over a closed M_{r-1}, it will always be equal to zero. If one imagines that a fixed closed M_{r-2} is given and an arbitrary M_{r-1} is laid through it then the value of the integral (308), when extended over the piece of one such M_{r-1} that is bounded by M_{r-2}, will be independent of the special choice of that M_{r-1}, and will be determined only the bounding closed M_{r-2} ([279]).

C. G. J. Jacobi himself had not studied expressions such as (308). He exploited the knowledge of a multiplier for the integration of the system of differential equations in such a way that he showed how to likewise obtain a multiplier for the reduced system from a multiplier for the system, from which (303) can be reduced to the knowledge of an integral. If one, in fact, knows an integral of (303):

$$f(x_1, x_2, \ldots, x_r) = \text{const.} \tag{309}$$

then the integral curves will be associated with the simple infinitude of M_{r-1} that is represented by (307) when the numerical value of the constant varies, such that one will then have:

$$X_1 \frac{\partial(x_2, x_3, \ldots, x_r)}{\partial(\sigma_1, \sigma_2, \ldots, \sigma_{r-1})} - X_2 \frac{\partial(x_1, x_3, \ldots, x_r)}{\partial(\sigma_1, \sigma_2, \ldots, \sigma_{r-1})} + \cdots - \cdots + (-1)^{r-1} X_r \frac{\partial(x_1, \ldots, x_{r-1})}{\partial(\sigma_1, \ldots, \sigma_{r-1})} \tag{310}$$

$$= \begin{vmatrix} X_1 & X_2 & \cdots & X_r \\ \dfrac{\partial x_1}{\partial \sigma_1} & \dfrac{\partial x_2}{\partial \sigma_1} & \cdots & \dfrac{\partial x_r}{\partial \sigma_1} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial x_1}{\partial \sigma_{r-1}} & \dfrac{\partial x_2}{\partial \sigma_{r-1}} & \cdots & \dfrac{\partial x_r}{\partial \sigma_{r-1}} \end{vmatrix} = 0$$

when the $\sigma_1, \ldots, \sigma_{r-1}$ mean general coordinates on an integral M_{r-1} (309) ([280]).

Now, new coordinates $y_1, \ldots, y_r$ might be introduced into the r-dimensional space so the integral M_{r-1} (309) can be represented by:

$$y_r = \text{const.} \tag{311}$$

In order to do that, one must choose the $y_1, \ldots, y_r$ such that:

$$y_r = f(x_1, x_2, \ldots, x_r)\,. \tag{312}$$

([279]) From this standpoint, the *hydrodynamical interpretation* of the multiplier will become understandable, as it was given by **L. Boltzmann**, Math. Ann. **42** (1893), pp. 374 = *Ges. Abhandl. III*, pp. 497 and **J. Larmor**, Brit. Assoc. Rep. 1897) (Toronto), pp. 562 = *Papers II*, pp. 704. Namely, if one interprets the system (303) as the differential equations of a stationary fluid flow in an r-dimensional space then the **Jacobi** multiplier M will represent the *density* of that fluid flow. The integral (308) is the amount of fluid that flows through the M_{r-1} per unit time, so it is independent of the special form of the M_{r-1} and depends upon only the closed bounding-M_{r-2} that spans the M_{r-1}.

([280]) From the standpoint of the hydrodynamical interpretation, that means: No fluid will flow through an M_{r-1} that is defined streamlines of the stationary flow.

If (312) can be solved for, say x_r :

(312.a) $$x_r = \varphi(x_1, x_2, \dots, x_{r-1}, y_r)$$

then one can introduce:

$$y_1 = x_1 , \dots, y_{r-1} = x_{r-1} , y_r ,$$

in particular, as the new coordinates. When expressed in those coordinates, the system of equations (303) will take the form:

(313) $$dx_1 : dx_2 : \dots : dx_{r-1} : dy_r = \bar{X}_1 : \bar{X}_2 : \dots : \bar{X}_{r-1} : 0 ,$$

in which:

(313.a) $$\bar{X}_\lambda(x_1, \dots, x_{r-1}, y_r) = X_\lambda(x_1, x_2, \dots, x_{r-1}, \varphi(x_1, x_2, \dots, x_{r-1}, y_r)) .$$

If one recalculates the integral invariant of order r in the new coordinates:

$$\underbrace{\iint\cdots\int}_{r} M(x_1, \dots, x_r)\,\delta x_1 \cdots \delta x_r = \underbrace{\iint\cdots\int}_{r} \bar{M}(x_1, \dots, x_{r-1}, y_r)\frac{\partial\varphi}{\partial y_r}\delta x_1\,\delta x_2 \cdots \delta x_{r-1}\,\delta y_r$$

then that will give ([281]):

(314) $$\underbrace{\iint\cdots\int}_{r} \frac{\bar{M}}{\dfrac{\partial f}{\partial x_r}}\delta x_1\,\delta x_2 \cdots \delta x_{r-1}\,\delta y_r$$

as the integral invariant of the new system. Now, one needs only to choose the domain of integration to be a disc between the two infinitely-close M_{r-1} :

$$f(x_1, \dots, x_r) = y_r = c \quad \text{and} \quad f(x_1, \dots, x_r) = y_r = c + \delta c$$

then (314) will become:

$$\delta c \underbrace{\iint\cdots\int}_{r-1} \frac{\bar{M}}{\dfrac{\partial f}{\partial x_r}}\delta x_1\,\delta x_2 \cdots \delta x_{r-1} = \text{const.}$$

([281]) That is because one has:

$$y_r \equiv f(x_1, \dots, x_{r-1}, \varphi(x_1, \dots, x_{r-1}, y_r))$$

so:

$$1 = \frac{\partial f}{\partial x_r}\frac{\partial\varphi}{\partial y_r} .$$

One concludes from this that (r – 1)-fold integral, when extended over an arbitrary domain of integration on the M_{r-1} (309), will be an integral invariant for the system of equations:

$$(315) \qquad dx_1 : dx_2 : \ldots : dx_{r-1} = \bar{X}_1 : \bar{X}_2 : \ldots : \bar{X}_{r-1},$$

in which one imagines that one has introduced $y_r = c$ into the right-hand side. That system of equations determines the integral curves on each of the M_{r-1} on which they lie, according to (309). From the fact that this reduced system possesses the integral invariant:

$$(315.a) \qquad \underbrace{\iint\cdots\int}_{r-1} \frac{\bar{M}}{\frac{\partial f}{\partial x_r}} \delta x_1\, \delta x_2 \cdots \delta x_{r-1} = \text{const.},$$

it will follow immediately that the function:

$$(315.b) \qquad \frac{\bar{M}\,(x_1, \ldots, x_{r-1}, c)}{\frac{\partial f}{\partial x_r}}$$

is a multiplier for the reduced system (315) ([281.a]), such that the (r – 2)-fold integral:

$$(316) \quad \underbrace{\iint\cdots\int}_{r-2} \frac{\bar{M}}{\frac{\partial f}{\partial x_r}} \left\{ \bar{X}_1 \frac{\partial(x_2, x_3, \ldots, x_{r-1})}{\partial(\sigma_1, \ldots, \sigma_{r-2})} - \bar{X}_2 \frac{\partial(x_1, x_3, \ldots, x_{r-1})}{\partial(\sigma_1, \ldots, \sigma_{r-2})} + - \cdots + (-1)^{r-2} \bar{X}_{r-1} \frac{\partial(x_1, \ldots, x_{r-2})}{\partial(\sigma_1, \ldots, \sigma_{r-2})} \right\} d\sigma_1 \cdots d\sigma_{r-2}$$

on the M_{r-1} will be independent of the choice of the integration-M_{r-2} , i.e., it assumes the same value for all of the M_{r-3} with that same boundary that span the integration-M_{r-2} .

When one knows another integral, one can further reduce the system (313) and once more convert the multiplier (315.b) into a multiplier for the further-reduced system. One can also lower the order of the system (303) by two units all at once by means of two integrals:

$$(317) \qquad f_1\,(x_1, \ldots, x_r) = c_1 , \qquad f_2\,(x_1, \ldots, x_r) = c_2 .$$

One will then get a multiplier for the reduced system from a multiplier of (303) in the form of:

$$(317.a) \qquad M^* = \frac{\bar{M}}{\frac{\partial(f_1, f_2)}{\partial(x_{r-1}, x_r)}} .$$

([281.a]) In $\partial f / \partial x_r$, x_r is thought of as having been replaced with the function (312.a), in which one has set $y_r = c$. That shall be suggested by the overbar.

If one knows $(r-2)$ integrals of the system (303), so all of them but one:

(318) $$f_1(x_1, \dots, x_r) = c_1, \quad f_2(x_1, \dots, x_r) = c_2, \quad \dots, \quad f_{r-2}(x_1, \dots, x_r) = c_{r-2},$$

then the system will reduce to a differential equation:

(319) $$dx_1 : dx_2 = X_1^*(x_1, x_2, c_1, \dots, c_{r-2}) : X_2^*(x_1, x_2, c_1, \dots, c_{r-2}),$$

and one will get a multiplier of (319) from a multiplier $M(x_1, \dots, x_r)$ of the system (303) in the form of:

(318.a) $$M^*(x_1, x_2, c_1, \dots, c_{r-2}) = \frac{\bar{M}(x_1, \dots, x_r)}{\dfrac{\partial(f_1, \dots, f_{r-2})}{\partial(x_3, \dots, x_r)}}.$$

In place of the integral (308), one will then have the integral:

$$\int M^*(X_1^* dx_2 - X_2^* dx_1),$$

which is independent of the path of integration and will then produce the last integral that is still missing when one sets it equal to a constant. One will then finally get an **Euler** multiplier from the **Jacobi** multiplier. **Jacobi** called that the *principle of the last multiplier*. That should say that: If one knows a multiplier of the system (303), and one has found $(r-2)$ of the $(r-1)$ integrals then that will imply the *last* integral, since the multiplier will become an **Euler** multiplier by a mere *quadrature* ([282]).

([282]) **Jacobi** then found that for the motion of a point in a plane, besides the energy integral:

(a) $$H(p_1, p_2, q_1, q_2) = k$$

for the canonical system:

(b) $$dq_1 : dq_2 : dp_1 : dp_2 = \frac{\partial H}{\partial p_1} : \frac{\partial H}{\partial p_2} : -\frac{\partial H}{\partial q_1} : -\frac{\partial H}{\partial q_2},$$

one needs to know only one further integral:

(c) $$F(p_1, p_2, q_1, q_2) = c$$

if one is to complete the integration by quadratures. Namely, if one solves (a) and (c) for p_1, p_2:

$$p_1 = f_1(q_1, q_2, c, k), \qquad p_2 = f_2(q_1, q_2, c, k)$$

then the multiplier 1 for the canonical system will imply that the remaining differential equation:

$$\frac{\partial H}{\partial p_2} dq_1 - \frac{\partial H}{\partial p_1} dq_2 = 0$$

has the **Euler** multiplier:

The **Euler** multiplier has the property in common with the **Jacobi** multiplier that when *two* multipliers are known, their *quotient* will give *an integral* of the system of differential equations. That is because one will find an integral of the associated **Jacobi** equations from a multiplier in the expression:

$$M_1(x_1,\dots,x_r,t)\begin{vmatrix} \xi_1^{(1)} & \cdots & \xi_r^{(1)} \\ \vdots & \ddots & \vdots \\ \xi_1^{(r)} & \cdots & \xi_r^{(r)} \end{vmatrix} = \text{const.},$$

and analogously, the expression:

$$\frac{1}{\begin{vmatrix} \frac{\partial H}{\partial p_1} & \frac{\partial H}{\partial p_2} \\ \frac{\partial F}{\partial p_1} & \frac{\partial F}{\partial p_2} \end{vmatrix}},$$

such that the trajectory will be given by a quadrature. In particular, **Jacobi** found that one has:

$$\frac{\frac{\partial H}{\partial p_2}dq_1 - \frac{\partial H}{\partial p_1}dq_2}{\begin{vmatrix} \frac{\partial H}{\partial p_1} & \frac{\partial H}{\partial p_2} \\ \frac{\partial F}{\partial p_1} & \frac{\partial F}{\partial p_2} \end{vmatrix}} = \frac{\partial f_1}{\partial c}dq_1 + \frac{\partial f_2}{\partial c}dq_2 .$$

Therefore (on this subject, cf., the generalization arguments of no. **24**):

$$f_1\,dq_1 + f_2\,dq_2$$

will also be an exact differential $d\,\Theta\,(q_1, q_2, c, k)$, and the equation of the trajectory will read:

$$\frac{\partial\Theta}{\partial c} = \gamma .$$

In order to do that, one likewise calculates:

$$dt = \frac{\partial f_1}{\partial k}dq_1 + \frac{\partial f_2}{\partial k}dq_2 = \frac{\partial}{\partial k}d\Theta$$

then, so:

$$t - \tau = \frac{\partial\Theta}{\partial k} .$$

Cf., **C. G. J. Jacobi**, "Sur le mouvement d'un point et sur un cas particulier di problème des trois corps," C. R. Acad. Sci. Paris **3** (1836), pp. 59 = *Werke IV*, pp. 35.

$$M_2(x_1,\ldots,x_r,t)\begin{vmatrix} \xi_1^{(1)} & \cdots & \xi_r^{(1)} \\ \vdots & \ddots & \vdots \\ \xi_1^{(r)} & \cdots & \xi_r^{(r)} \end{vmatrix} = \text{const.}$$

will be an integral of the **Jacobi** equations for the second multiplier. Therefore, the quotient:

$$\frac{M_1(x_1,\ldots,x_r,t)}{M_2(x_1,\ldots,x_r,t)} = \text{const.}$$

must also be an integral of **Jacobi** equations. However, since it does not depend upon the $\xi_1, \ldots, \xi_e$ at all, it will also represent an integral of the given system of equations (299) in its own right.

In particular, knowing the multiplier $M = 1$ for the integration of the canonical system will also imply the statement that when one knows $(2n - 1)$ integrals, the last one will be obtained immediately by a quadrature.

23. Poincaré's recurrence theorem. Adiabatic invariants of a mechanical system. – The existence of the absolute integral invariant:

(320) $$V = \int\cdots\int \delta p_1 \cdots \delta p_n \, \delta q_1 \cdots \delta q_n$$

for the canonical system:

(321) $$\frac{dq_\rho}{dt} = \frac{\partial H}{\partial p_\rho}, \qquad \frac{dp_\rho}{dt} = -\frac{\partial H}{\partial q_\rho}$$

leads to an important theorem when one assumes that the time t does not appear explicitly in H and the trajectories remain entirely within a finite region of the $2n$-dimensional "phase space of the $p_1, \ldots, p_n, q_1, \ldots, q_n$." That theorem, which has found to be especially interesting in statistical mechanics, moreover [cf., IV 32 (**P. and T. Ehrenfest**), no. **7.b**], goes back to **H. Poincaré** ([283]).

It says, roughly, that a trajectory that starts from an arbitrary point in a region will generally get increasingly close to that point during the course of its motion and is therefore referred to as the ***Poincaré** recurrence theorem* ([284]). More precisely, **H. Poincaré** introduced suitable concepts from probability and made the statement of the theorem more precise by saying that it was "infinitely improbable" that a mass-point would not increasingly return to an arbitrary neighborhood of a starting point ([285]). However, since the proof that **H. Poincaré** gave is subject

([283]) Cf., **H. Poincaré**, "Sur les équations de la dynamique et le problème des trois corps," Acta math. **13** (1890), pp. 67, as well as the thorough presentation in **H. Poincaré**, *Méthod. nouv. III*, Chap. 26, pp. 140, *et seq.*

([284]) **H. Poincaré** himself referred to the behavior of the mechanical system that is described in the recurrence theorem as *stabilité à la Poisson* in connection with certain investigations of **S. D. Poisson** into the behavior of the semi-major axes of orbital ellipses in planetary systems.

([285]) For this formulation, one can also cf., the presentation by **P. Hertz** in the article "Statistische Mechanik" in the *Repertorium der Physik* by **R. H. Weber** and **R. Gans**, Bd. I^2, Leipzig and Berlin 1916, pp. 461, *et seq.*

to some objections, the question arose of what more precise conditions for the validity of the theorem might be ([286]). An exact formulation, and at the same time, a rigorous proof, was then given by **C. Carathéodory** ([287]) by appealing to the concept of the **Lebesgue** measure for point-sets [cf., II C 9 (**E. Borel-A.Rosenthal**), no. **20**]. He stated the theorem in the following way: The steady flow in phase space that is defined by the canonical system (321), which is spatially stable due to the integral invariant (320), takes place in a region G of phase-R_{2n} (with finite volume) that consists completely of finite points. Now, if a particle is found at a point P_0 of that region at the time $t = 0$, and one then determines the set of points $P_1, P_2, P_3, \ldots$ at which the particle is found at times $\tau, 2\tau, 3\tau, \ldots$ (where τ is understood to mean an arbitrary positive number) then that will establish the rule that P_0 is an accumulation point of the point-set $P_1, P_2, P_3, \ldots$ If there were a point P_0 in the region for which that statement were not correct then it would define at most a set that possessed **Lebesgue** measure zero.

The crux of the proof of that theorem, as well as the one by **Poincaré**, is the argument that the set of non-recurrent points must have the property that the regions in G that they occupy at the times $\tau, 2\tau, 3\tau, \ldots$ are all separate from each other. Otherwise, the phase points that fill up a sub-region Γ of G (with non-zero **Lebesgue** measure) at time $t = 0$ would fill up sub-regions of G, say, $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ at times $\tau, 2\tau, 3\tau, \ldots$ that would need to all have the same measure as Γ, due to the spatial stability of the phase flow (321) that is expressed in (320), and therefore not all of them could be separate in G. Therefore, if no two of the infinitude of regions $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ are to overlap then their common measure must have the value zero ([288]).

The volume of phase space also plays a role in a somewhat-different invariance property that was first recognized in statistical mechanics ([289]). In that way, one does not consider a mechanical system to be isolated, but one assumes that it is subject to external influences. Analytically, that is expressed by saying that the function H in (321) depends upon not only p_ρ, q_ρ, but also that a certain number of parameters a_ν (say r) will appear that one represents as given functions of time:

([286]) **L. Boltzmann**, "Über einen mechanischen Satz von Poincaré," Wien Sitzungsber. **106** II^a (1897), pp. 12 = *Ges. Abhandl.*, Bd. III, pp. 587.

([287]) **C. Carathéodory**, "Über den Wiederkehrsatz von Poincaré," Berlin Sitzungsberichte der Preuß. Akad. (1919), 2. Halbbd., pp. 580.

([288]) Moreover, it follows immediately from these arguments that the theorem can be generalized. It will remain valid when the canonical system is replaced with a system of differential equations:

$$\frac{dx_1}{dt} = X_1(x_1, \ldots, x_n), \qquad \ldots, \qquad \frac{dx_n}{dt} = X_n(x_1, \ldots, x_n),$$

and in place of the integral invariant (320), it will possess an integral invariant with the same order as the system:

$$\iint \cdots \int M\, \delta x_1\, \delta x_2 \cdots \delta x_n$$

whose integrand is:

$$M(x_1, x_2, \ldots, x_n) \geq 0,$$

whereby the set of points at which $M = 0$ must be a set of measure at most zero. Cf., **C. Carathéodory**, *loc. cit.* ([287]). pp. 583. The theorem was already expressed in this general context by **Poincaré** himself without appealing to the **Lebesgue** measure, cf., *Méthod. nouv. III*, pp. 155.

([289]) Cf., **P. Hertz**, *loc. cit.* ([285]), pp. 534.

(322) $$H = H(p_1, \ldots, p_n, q_1, \ldots, q_n, a_1, \ldots, a_r) .$$

In particular, if those parameters change very slowly in time then they will influence the motion ([290]) of the mechanical system in a way that is analogous to that of adiabatic changes in thermodynamics, such that one cases to refer to the variation of the motion that belongs to the variation of the a_ν as an *adiabatic process* ([291]). Statistical mechanics and quantum theory, which was initially developed from it, then ask what those quantities might be that remain invariant under adiabatic processes.

The energy integral:

(323) $$H = H(p_1, \ldots, p_n, q_1, \ldots, q_n, a_1, \ldots, a_r) = k$$

that the canonical system possesses for fixed values of the parameters a_ν is interpreted in phase R_{2n} of the p_ρ, q_ρ as an M_{2n-1} on which the integral curves lie. Now, if that M_{2n-1} is a *closed* manifold, in particular, then it will bound a region in phase space with a well-defined phase volume:

(324) $$V = \int \cdots \int \delta p_1 \cdots \delta p_n \, \delta q_1 \cdots \delta q_n .$$

Now, if the parameters a_ν vary *slowly* in time then the energy integral (323) will be valid at every moment during the motion, in the sense that on the one hand, the left-hand side of (323) is an analytical expression with varying values of the parameters a_ν, and that on the other hand, the value of the energy constant k also varies in time. Since the region over which the integral (324) is extended varies in time, the phase volume (324) can be a (likewise slowly-varying) function of time under the slow variation of the parameters. Meanwhile, that shows that under certain assumptions, the phase volume (324) will remain invariant under the adiabatic process and will represent a so-called *adiabatic invariant* ([292]) of the canonical system [cf., V 28 (**A. Smekal**), no. **3**]. The assumptions for the adiabatic invariance of the phase volume (324), which **T. Levi-Civita** ([293]) made more precise, consist of saying that, first of all, the energy integral is the only integral of the canonical system (321) that is not infinitely multivalued, or also that the system is (in **Levi-Civita**'s terminology) *simply imprimitive* ([294]), and that secondly, for fixed values of the parameters a_ν, *almost all* trajectories fill up the M_{2n-1} (323) densely everywhere [the so-called *quasi-ergodic hypothesis*, cf., IV 32 (**P. and T. Ehrenfest**), no. **10.a** and V 28 (**A. Smekal**), no.

([290]) Which can be calculated by the methods of perturbation theory, moreover.

([291]) Cf., **P. Hertz**, *loc. cit.* ([235]), pp. 533.

([292]) The term goes back to **P. Ehrenfest**, "Adiabatic Invarianten und Quantentheorie," Ann. Phys. (Leipzig) (4) **51** (1916), pp. 327, also appeared in Amsterdam Versl. van Akad. Wet. **25** (1916), pp. 412, as well as London Phil. mag. (6) **33** (1917), pp. 500.

([293]) **T. Levi-Civita**, "Drei Vorlesungen über adiabatische Invarianten," Hamburg Abh. aus dem math. Sem. d. U. **6** (1928), pp. 323. Cf., also **T. Levi-Civita**, "A general survey on the theory of adiabatic invariants," J. of math. and phys. **13** (1934), pp. 18.

([294]) **T. Levi-Civita** [*loc. cit.* ([293])] referred to mechanical systems that possess only integrals that are infinitely-multivalued as *primitive*. The number of integrals that are not infinitely multivalued determines the *order of imprimitivity* of the system.

1], i.e., that one such trajectory comes arbitrarily close to *every* point in the M_{2n-1} ([295]). With those assumptions, it is possible to replace a temporal mean along a trajectory with a spatial mean over the M_{2n-1} (323), and indeed in that way, as can be shown, the density of that distribution is equal to the reciprocal value of the magnitude of the gradient of the family of M_{2n-1} that is defined by $H = k$ as k varies. One can easily conclude from this the change in the phase volume (324) that will occur when one fixes the a_ν in the left-hand side of (323), but gives a new value to k that is equal and opposite to the change that one will obtain when one gives new values to the a_ν on the left-hand side, but fixes the value of k. Therefore, if the total change in the phase volume V for an adiabatic change in all of the parameters is equal to zero then V will be an adiabatic invariant.

Such adiabatic invariants must have special significance in the development of quantum theory. That is because the first preliminary attempts to explain the radiation phenomena, etc., in terms of classical mechanics came out of the Ansatz of establishing quantization conditions [cf., V 28 (**A. Smekal**), no. **14**], so there must be quantities that remain individually constant during the motion of a mechanical system whose values could be arbitrary real numbers according to classical mechanics, but could assume only certain distinguished values, i.e., they should not vary continuously, but can change only in jumps (i.e., quantum jumps). Now, if external influences act upon a mechanical system that change very slowly ([296]) then the quantities that would remain constant in the absence of external influences will (slowly) vary continuously in time. However, should such a quantity be used as a quantum condition, then it can change only in jumps, so it must remain completely constant under the slow change in the parameters ([297]), i.e., it must be an adiabatic invariant. One must then look for the quantities to be quantized among the adiabatic invariants.

If the mechanical system has only one degree of freedom and the energy integral:

$$H(p, q) = k \tag{325}$$

determines a closed curve in phase ([298]) then the area of that energy curve (325) that surrounds a surface patch in the phase plane must prove to be the adiabatic invariant to be quantized:

$$V = \iint \delta p\, \delta q = \oint p\, \delta q\,.$$

Of the systems with several degrees of freedom, the constrained periodic systems offer the simplest examples for the introduction of quantization conditions, i.e., the systems whose **Hamilton-Jacobi** equations can be integrated by separation of variables (cf., no. **19**) [cf., V 28 (**A. Smekal**), no. **15**]. The essential basis for that preferred status for the constrained periodic systems is that when one integrates the **Hamilton-Jacobi** by separation of variables (cf., no. **19**), along with the imprimitive energy integral, $(n-1)$ further imprimitive integrals will appear, and each of them will be quadratic in the impulse components p_ρ, moreover. That will raise the inevitable question of whether

([295]) From the recurrence theorem, it must come arbitrarily close to it arbitrarily often then.
([296]) I.e., slowly enough that it cannot produce any quantum jumps.
([297]) On this, cf., e.g., **M. Born**, *Vorlesungen über Atommechanik I*, Berlin 1925, pp. 58 and 109.
([298]) Such that the motion is perioidic.

adiabatic invariants can also be given for systems whose canonical equations exhibit a number of imprimitive integrals.

The simplest case of such imprimitive integrals is the one in which a number of cyclic coordinates appear, such that:

$$p_1 = c_1 , \ldots, p_n = c_n \qquad (m < n) \tag{326}$$

are the imprimitive integrals that appear. The canonical system will then be immediately reduced to a canonical system for the unknowns $p_{m+1} , \ldots, p_n , q_{m+1} , \ldots, q_n$, such that the associated phase space will have dimension 2 ($n - m$). It will once more be assumed that the energy integral:

$$H (c_1, \ldots, c_r, p_{m+1}, \ldots, p_n, q_{m+1}, \ldots, q_n) = k \tag{326.a}$$

represents a closed $M_{2(n-m)-1}$ in that phase-$M_{2(n-m)}$. Now, if slowly-varying parameters a_ν appear once more in H then the cyclic impulses will be independent of them. One can then regard those (constant) cyclic impulses as parameters that are added to the a_ν . The motion will then be described by the canonical system:

$$\frac{dq_\rho}{dt} = \frac{\partial H}{\partial p_\rho}, \qquad \frac{dp_\rho}{dt} = -\frac{\partial H}{\partial q_\rho} \qquad (\rho = m + 1, \ldots, n), \tag{326.b}$$

and from the results above, the volume of the region that is bounded by the closed energy-$M_{2(n-m)-1}$ in phase-$M_{2(n-m)}$:

$$V = \int \cdots \int \delta p_{m+1} \cdots \delta p_n \, \delta q_{m+1} \cdots \delta q_n \tag{326.c}$$

will be an adiabatic invariant of the reduced canonical system (326.b), but in that way, it also be an adiabatic invariant of the original system.

That can be easily generalized by saying that the canonical system (321) possesses a number of more general imprimitive integrals:

$$F_1 (p_1, \ldots, p_n, q_1, \ldots, q_n) = c_1 , \ldots, F_m (p_1, \ldots, p_n, q_1, \ldots, q_n) = c_m , \tag{327}$$

assuming that they lie *in involution* (cf., *infra*, no. **26**), so all **Poisson** brackets will be:

$$(F_\rho , F_\sigma) = 0 . \tag{327.a}$$

Namely, if one solves the m integrals (327) for $p_1, \ldots, p_m$:

$$p_\rho = f_\rho (p_{m+1}, \ldots, p_n, q_1, \ldots, q_n, c_1 , \ldots, c_m) \qquad (\rho = 1, \ldots, m), \tag{327.b}$$

and in that way, takes $H (p_1, \ldots, p_n, q_1, \ldots, q_n, c_1 , \ldots, c_m)$ to a function:

(328) $$\bar{H}\,(p_{m+1}, \ldots, p_n, q_1, \ldots, q_n, c_1, \ldots, c_m, a_1, \ldots, a_r)\,,$$

then

(329) $$\bar{H} = k$$

might likewise represent a closed manifold in the phase space of $p_{m+1}, \ldots, p_n, q_{m+1}, \ldots, q_n$. The position coordinates $q_1, \ldots, q_m$ that still appear in $\bar{H}$ can take on values that are fixed, in any case, but intrinsically arbitrary, such that this closed $M_{2(n-m)-1}$ will also depend upon the m parameters $q_1, \ldots, q_m$, in addition to the other parameters a_ν, c_ρ . The volume that is enclosed by it:

(330) $$W = \int\cdots\int \delta p_{m+1}\cdots\delta p_n\,\delta q_{m+1}\cdots\delta q_n$$

is therefore an adiabatic invariant ([299]) of the canonical system:

(331) $$\frac{dq_\rho}{dt} = \frac{\partial\bar{H}}{\partial p_\rho}\,,\qquad \frac{dp_\rho}{dt} = -\frac{\partial\bar{H}}{\partial q_\rho}\qquad (\rho = m + 1, \ldots, n),$$

so one must also regard the $q_1, \ldots, q_m$ as adiabatic parameters, in addition to the a_ν, c_ρ . However, that property also remains preserved when one subsequently thinks of the $q_1, \ldots, q_m$ as arbitrarily variable, such that ultimately the phase volume (330) also proves to be an adiabatic invariant of the original canonical system (321) ([300]).

Now, however, every imprimitive integral (327) is on a par with the energy integral H, in the following sense: If one chooses any of them – say, F_μ – and forms the canonical system from it:

(332) $$\frac{dq_\rho}{dt} = \frac{\partial F_\mu}{\partial p_\rho}\,,\qquad \frac{dp_\rho}{dt} = -\frac{\partial F_\mu}{\partial q_\rho}\qquad (\rho = m + 1, \ldots, n)$$

then that system, which possesses the m imprimitive integrals:

(332.a) $$H = k\,,\qquad F_1 = c_1\,, \ldots,\quad F_{\mu-1} = c_{\mu-1}\,,\quad F_{\mu+1} = c_{\mu+1}\,,\quad F_m = c_m\,,$$

will have the same trajectories as the system (331) in phase-$M_{2(n-m)-1}$. Therefore, if $F_\mu = c_\mu$ is a closed $M_{2(n-m)-1}$ in phase space then the volume that it encloses, which is an adiabatic invariant of the system (332), will be likewise an adiabatic invariant of the system (332), and therefore of the

([299]) Which does not depend upon the choice of impulse components for which the system (327) is solved, moreover.

([300]) Cf., **T. Levi-Civita**, "Drei Vorles. über adiab. Inv.," Hamburg Abhandl. aus d. math. Sem. **6** (1928), pp. 323, esp. pp. 361.

system (321), such that a system with (m + 1) imprimitive integrals will yield precisely (m + 1) adiabatic invariants.

Now, if the canonical system can be solved by separation of variables, in particular, then along with $H = k$, there will be (n – 1) further imprimitive integrals that are quadratic in the p_ρ that are also in involution with each other. One will then have n adiabatic invariants in that case, for which one can make the quantization Ansätze. If one solves the (n – 1) integrals that are added to $H = k$ for $p_1, \ldots, p_{n-1}$ then one will get the canonical system:

$$\frac{dq_n}{dt} = \frac{\partial \bar{H}}{\partial p_n}, \qquad \frac{dp_n}{dt} = -\frac{\partial \bar{H}}{\partial q_n}, \tag{333}$$

which will make the $p_1, \ldots, p_{n-1}$ drop out of $\bar{H}$ completely, along with the $q_1, \ldots, q_{n-1}$. Since the phase space will then reduce to a plane in which $H = k$ represents a closed curve for constrained periodic motion, the area that is enclosed by that curve:

$$W = \iint \delta p_n \, \delta q_n = \oint p_n \, \delta q_n \tag{334}$$

will be an adiabatic invariant. If one successively replaces the integral H with one of the other quadratic integrals then that will correspondingly yield the (n – 1) adiabatic invariants:

$$W_\sigma = \iint \delta p_\sigma \, \delta q_\sigma = \oint p_\sigma \, \delta q_\sigma \qquad (\sigma = 1, \ldots, n-1). \tag{335}$$

The n expressions (334) and (335) will yield precisely n quantum conditions for constrained periodic systems when one sets them equal to whole-number multiples of the **Planck** quantum of action [cf., V 28 (**A. Smekal**), no. **15**].

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CHAPTER VI

THE SYSTEMATIC INTEGRATION OF THE CANONICAL SYSTEM.

24. The $2n$ **integrals of the equations of motion and their geometric interpretation.** – For the systematic integration of the equations of motion in the spirit of the **Jacobi** school (cf., no. **15**), one prefers to not start from the equations of motion in the form of the **Euler** equations, but to convert them into the associated canonical form (cf., no. **19**):

$$\frac{dq_\rho}{dt} = \frac{\partial H}{\partial p_\rho}, \qquad \frac{dp_\rho}{dt} = -\frac{\partial H}{\partial q_\rho} \qquad (\rho = 1, \ldots, n), \tag{336}$$

$$H = H(p_1, \ldots, p_n, q_1, \ldots, q_n, t) .$$

One can then interpret the individual solution:

$$q_\rho = q_\rho(t, c_1, \ldots, c_{2n}), \qquad p_\rho = p_\rho(t, c_1, \ldots, c_{2n}) \tag{336.a}$$

as a curve (M_1) in the $(2n + 1)$-dimensional phase space ($p_1, \ldots, p_n, q_1, \ldots, q_n, t$) ([301]). If t does not appear explicitly in H then one ordinarily restricts oneself to the system of "trajectories" in the M_{2n} of ($p_1, \ldots, p_n, q_1, \ldots, q_n$) that one likes to refer to as a system of streamlines of a fluid flow in that manifold, and indeed a spatially stable fluid, since one indeed has:

$$\frac{\partial}{\partial q_\rho}\left(\frac{\partial H}{\partial p_\rho}\right) + \frac{\partial}{\partial p_\rho}\left(-\frac{\partial H}{\partial q_\rho}\right) = 0 ,$$

and therefore, the sum over ρ is also equal to zero, while its vanishing represents the condition for spatial stability (cf., no. **23**).

An integral ([302]) of the canonical system:

$$F(p_1, \ldots, p_n, q_1, \ldots, q_n, t) = \text{const.} \tag{337}$$

([301]) Cf. ([258]). *One* curve goes through each point of phase space (except for singularities).

([302]) In what follows, we will always consider the general case in which t enters into H explicitly, since we can easily reduce the results to the case in which t does not enter into H explicitly.

represents a (one-parameter family of, resp.) M_{2n} in the $(2n + 1)$-dimensional phase space ([303]), on which the integral curves of the canonical systems are arranged. Therefore, the function F ([304]) must be a solution of the *linear* first-order partial differential equation that is associated with the canonical system (336):

(337.a) $$\frac{\partial F}{\partial t}+\sum_{\rho=1}^{n}\left(\frac{\partial F}{\partial q_\rho}\frac{\partial F}{\partial p_\rho}-\frac{\partial F}{\partial p_\rho}\frac{\partial F}{\partial q_\rho}\right)=0\,,$$

which one can also write:

(337.b) $$\frac{\partial F}{\partial t}+(H,F)=0\,,$$

since the sum represents the ***Poisson*** *bracket* of F and H that was introduced in no. **12**. In total, $2n$ such integrals:

(338) $$\begin{cases} F_1(p_1,\ldots,p_n,q_1,\ldots,q_n,t)=c_1\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ F_{2n}(p_1,\ldots,p_n,q_1,\ldots,q_n,t)=c_{2n} \end{cases}$$

will be required for the complete integration of the canonical system ([305]) that will give an analytical representation of the set of ∞^{2n} integral curves. In the **Jacobi** school, one then refers to the integration of the canonical system as an integration problem of order $2n$.

([303]) Since the function F for such an integral is generally an infinitely-multivalued function of its arguments, that integral will basically play a role in only integration in the small. By contrast, knowing an integral curve lies on an M_{2n} will say nothing at all about the way that one might approach integration in the large. [On this, cf., no. **23**, esp. ([294])]

([304]) There can be *no integral* of the canonical system that *is free of the impulse components*. That is because an integral:

$$f(q_1,\ldots,q_n,t)=\text{const.}$$

will imply the relation:

$$\frac{\partial f}{\partial q_1}\dot q_1+\cdots+\frac{\partial f}{\partial q_n}\dot q_n+\frac{\partial f}{\partial t}=0$$

between the velocity components (an analogous relation between the impulse components, resp.). However, such a relation is impossible since the impulse components at a space-time point can be prescribed *arbitrarily*.

([305]) If t does not enter into H explicitly then there will be $(2n - 1)$ integrals that are free of t :

(338.a) $$\begin{cases} F_1(p_1,\ldots,p_n,q_1,\ldots,q_n)=c_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ F_{2n-1}(p_1,\ldots,p_n,q_1,\ldots,q_n)=c_{2n-1}, \end{cases}$$

[one of which is the energy integral $H(p_1,\ldots,p_n,q_1,\ldots,q_n)$, moreover]. They determine the trajectories in the M_{2n} of (p_ρ, q_ρ) . A relation of the form:

(338.b) $$t-\tau=G(p_1,\ldots,p_n,q_1,\ldots,q_n)$$

Now, **Hamilton-Jacobi** theory has shown that one can represent the $2n$ integrals with help of the principal function [a complete solution to the **Hamilton-Jacobi** partial differential equation $S(q_1, \dots, q_n, t, c_1, \dots, c_n)$, resp.]. On the other hand, since:

$$\frac{\partial S}{\partial q_1} = p_1 , \qquad \dots, \qquad \frac{\partial S}{\partial q_n} = p_n , \qquad \frac{\partial S}{\partial t} = - H ,$$

one can obtain such a complete solution by a quadrature when one knows $p_1, \dots, p_n$, so as functions of the $q_1, \dots, q_n, t$ (and n arbitrary constants), such that:

(339) $$p_1\ \delta q_1 + \dots + p_n\ \delta q_n$$

is a complete differential ([306]). Instead of the $2n$ integrals (338) of the canonical system, one therefore needs to know only n such integrals:

(340) $$\begin{cases} F_1(p_1, \dots, p_n, q_1, \dots, q_n, t) = c_1 , \\ \dots\dots\dots\dots\dots\dots \\ F_n(p_1, \dots, p_n, q_1, \dots, q_n, t) = c_n , \end{cases}$$

but they must be of a special type. The $p_1, \dots, p_n$ must be computable from them as functions of the $q_1, \dots, q_n, t, c_1, \dots, c_n$ such that they will make the expression (339) into an exact differential. One would then get a complete solution of the **Hamilton-Jacobi** differential equation by a quadrature, and the missing n integral can be obtained from it by mere differentiation and eliminations. **C. G. J. Jacobi** correspondingly posed the problem of determining *n integrals* (340) with the desired property ([307]).

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must then be added as a $2n^{\text{th}}$ integral. That integral assigns the time duration to the individual trajectories that are given by (338.a), and indeed each curve is associated with ∞^1 types of temporal evolution for the motion. Moreover, according to (337.b), the integrals (338.a) will satisfy the equation:

(337.c) $$(H, F_\rho) = 0 ,$$

while one will have:

(337.d) $$(H, G) = 1$$

for the integral (338.b). The relation (338.b) has been called the "clock reading" of the mechanical process, cf., **Ph. Frank**, "Die Grundbegriffe the analytischen Mechanik als Grundlage der Quanten- und Wellenmechanik," Phys. Zeit. **30** (1929), pp. 209. In the terminology of no. **28**, the energy integral and the clock reading are conjugate integrals.

([306]) As was stated in no. **16.c**:

$$p_1\ \delta q_1 + \dots + p_n\ \delta q_n - H\ \delta t$$

will indeed be a complete differential as well then.

([307]) **C. G. J. Jacobi**, "Nova methodus, aequationes differentiales partiales primi ordinis inter numerum variabilium quemcunque propositas integrandi," J. f. Math. **60** (1862), pp. 1 = *Werke V*, pp. 1. The treatise was published by **A. Clebsch** with the permission of **Jacobi**'s estate.

The intrinsic meaning of that requirement can be seen in the following way: A system of integrals (340) of the desired type will determine an M_{n+1} for each choice of numerical values for the $c_1, \ldots, c_n$ in the phase space of R_{2n+1} that will carry a system of ∞^n integral curves of the canonical system (336). If one now returns from the canonical system to the associated **Euler** equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\rho}\right) - \frac{\partial L}{\partial q_\rho} = 0 \tag{341}$$

then the ∞^n chosen integral curves of the canonical system will correspond to a system of ∞^n extremals of that variational problem in the R_{n+1} of the $(q_1, \ldots, q_n, t)$, such that in general one extremal will go through each point of the R_{n+1}. In this conception of things, equations (340) yield the impulse components that are assigned to the point $(q_1, \ldots, q_n, t)$ by the extremal. If the integral (340) satisfies the demand that was imposed then the p_ρ will be the derivatives of a function $S(q_1, \ldots, q_n, t, c_1, \ldots, c_n)$:

$$p_\rho = \frac{\partial S}{\partial q_\rho},$$

i.e., the family of ∞^n extremals defines a field, and the function S is the value of the extremal integral for the field, such that the M_n that are defined by:

$$S(q_1, \ldots, q_n, t, c_1, \ldots, c_n) = \text{const.}$$

will each be the ∞^1 transversal M_n of a field for fixed numerical values of the $c_1, \ldots, c_n$.

From what was explained in no. **21**, the condition for the **Pfaffian** expression (339) to be a total differential is identical to the condition that the associated bilinear covariant:

$$\sum_{\rho=1}^{n} (\delta^{(1)} p_\rho \, \delta^{(2)} q_\rho - \delta^{(1)} q_\rho \, \delta^{(2)} p_\rho) = 0 \tag{342}$$

for any two arbitrary directions of advance $\delta^{(1)} q_\rho$, $\delta^{(1)} p_\rho$ and $\delta^{(2)} q_\rho$, $\delta^{(2)} p_\rho$ that belong to an M_{n+1} (340) in the phase-R_{2n+1} ([308]). Now, one can effortlessly succeed in converting the condition

([308]) That is because if one is to have:

$$\oint (p_1 \, \delta q_1 + \cdots + p_n \, \delta q_n) = 0$$

for *every* closed curve then the integrand in the integral (282.a):

$$\iint \sum_\rho (\delta^{(1)} p_\rho \, \delta^{(2)} q_\rho - \delta^{(1)} q_\rho \, \delta^{(2)} p_\rho) = 0$$

would have to vanish.

(342) into a condition between the partial derivatives of the functions (340) ([309]), and indeed that its precisely the same problem as in no. **11** of the transition from the **Lagrange** brackets to the **Poisson** brackets ([310]). According to no. **20**, the argument for that is conveniently linked with the system of linear differential equations that is associated with the canonical system. In that way, one will likewise arrive, in a completely natural way, at the connection between an integral of the canonical system and a one-parameter group of transformations that take the integral curves of the canonical system into each other that was developed systematically by **S. Lie** (cf., also no. **18.b**).

25. Connection between an integral and an infinitesimal transformation. – From no. **20**, the **Jacobi** system of linear differential equations that belongs to the canonical system (336) for which one must make that connection has the form:

$$(348)\qquad \begin{cases} \dfrac{d\kappa_\rho}{dt} = \displaystyle\sum_\lambda \left(\frac{\partial^2 H}{\partial p_\rho\,\partial p_\lambda}\pi_\lambda + \frac{\partial^2 H}{\partial p_\rho\,\partial q_\lambda}\kappa_\lambda \right), \\ \dfrac{d\pi_\rho}{dt} = -\displaystyle\sum_\lambda \left(\frac{\partial^2 H}{\partial q_\rho\,\partial p_\lambda}\pi_\lambda + \frac{\partial^2 H}{\partial q_\rho\,\partial q_\lambda}\kappa_\lambda \right). \end{cases}$$

A solution of (348) will mediate (cf., no. **20**) the transition of an integral curve that is to be performed at constant *t*, and indeed, the integral curve of the canonical system (336) that is introduced in the coefficients of (348), to a neighboring integral curve. If one writes the solution of the **Jacobi** equations in the form:

$$(349)\qquad \begin{cases} \pi_\rho = \varphi_\rho(p_1(t),\dots,p_n(t),q_1(t),\dots,q_1(t),t), \\ \kappa_\rho = \psi_\rho(p_1(t),\dots,p_n(t),q_1(t),\dots,q_1(t),t) \end{cases}$$

in order to emphasize the fact that the **Jacobi** equations themselves, and therefore their solutions as well, are meaningful only when an integral curve of the canonical system (336) is given then that will likewise express the idea that such a solution will mediate the *transition to an infinitesimally-close integral curve* for *any* integral curve, so in the spirit of **S. Lie**, it will then represent an *infinitesimal transformation:*

$$(350)\qquad \begin{cases} \delta p_\rho = \varphi_\rho(p_1(t),\dots,p_n(t),q_1(t),\dots,q_1(t),t)\,\delta\alpha, \\ \delta q_\rho = \psi_\rho(p_1(t),\dots,p_n(t),q_1(t),\dots,q_1(t),t)\,\delta\alpha, \\ \delta t = 0 \end{cases}$$

([309]) Cf., **C. G. J. Jacobi**, "Nova methodus…," J. f. Math. **60** (1862), pp. 1 – *Werke V*, pp. 1.
([310]) It was already suggested in no. **21** that the **Lagrange** brackets and the bilinear covariants are essentially identical.

that takes every integral curve of the canonical system into an infinitesimally-close one (cf., also no. **18.b**). One will get a one-parameter group of transformations that take the integral curves of the canonical system to each other by integrating (350) in a known way. [Cf., II A 6 (**L. Maurer** and **H. Burkhardt**), no. **4**].

Now, a first integral of the **Jacobi** equations (348), which must be linear and homogeneous in the π_ρ, κ_ρ :

(351) $$\sum_\rho (A_\rho \pi_\rho + B_\rho \kappa_\rho) = \text{const.}$$

might be found, which will make the A_ρ, B_ρ into known functions of time t for every integral curve of the canonical system (336). In order to suggest that, one might write:

(351.a) $$\begin{cases} A_\rho = A_\rho(p_1(t), \dots, p_n(t), q_1(t), \dots, q_1(t), t), \\ B_\rho = B_\rho(p_1(t), \dots, p_n(t), q_1(t), \dots, q_1(t), t), \end{cases}$$

just as in (349). The relation ([311]):

$$\sum_\rho \left\{ \pi_\rho \left[\frac{dA_\rho}{dt} - \sum_\lambda \left(\frac{\partial^2 H}{\partial p_\rho \, \partial q_\lambda} A_\lambda - \frac{\partial^2 H}{\partial p_\rho \, \partial p_\lambda} B_\lambda \right) \right] + \pi_\rho \left[\frac{dB_\rho}{dt} - \sum_\lambda \left(\frac{\partial^2 H}{\partial q_\rho \, \partial q_\lambda} A_\lambda - \frac{\partial^2 H}{\partial q_\rho \, \partial p_\lambda} B_\lambda \right) \right] \right\} = 0$$

must obviously be true for *any* solution π_ρ, κ_ρ to the **Jacobi** equations (348). Due to the arbitrariness in π_ρ and κ_ρ, it must follow that the factors of π_ρ and κ_ρ must vanish by themselves ([312]). Therefore:

(351.b) $$\pi_\rho = A_\rho(t), \qquad \kappa_\rho = -B_\rho(t)$$

is likewise a solution of the **Jacobi** equations (348), and the formulas:

(351.c) $$\delta q_\rho = A_\rho(t)\,\delta\alpha, \qquad \delta p_\rho = -B_\rho(t)\,\delta\alpha \qquad (\delta t = 0)$$

will likewise mediate the transition from the integral curve of the canonical system (336) in question to an infinitesimally-close one ([313]).

([311]) In which the second derivatives of H are assumed to be known functions of time t. One must then imagine substituting a well-defined integral curve of the canonical system.

([312]) Cf., **H. Poincaré**, *Méthod. nouv. I*, pp. 168. One can always interpret such a linear integral as the relationship between two solutions then, and from no. **20**, that expresses the fact that the linear **Jacobi** equations (348) define a self-adjoint system.

([313]) That means that the solution A_ρ, $-B_\rho$ must be a linear combination of the different systems of displacements $\delta^{(\lambda)} q_\rho$, $\delta^{(\lambda)} p_\rho$ that mediate the transition from the integral curve in question to a neighboring one. For one such system of displacements δp_ρ, δq_ρ, one must indeed have:

Now, if one has an integral:

(352) $$F(p_1, \ldots, p_n, q_1, \ldots, q_n, t) = \text{const.}$$

of the canonical system (336) then it will follow immediately that:

(352.a) $$\frac{\partial F}{\partial p_1}\pi_1 + \cdots + \frac{\partial F}{\partial p_n}\pi_n + \frac{\partial F}{\partial q_1}\kappa_1 + \cdots + \frac{\partial F}{\partial q_n}\kappa_n = \text{const.}$$

is an integral of the associated **Jacobi** system (348), in the sense of (351), (351.a). Thus:

(336.b) $$\pi_\rho = -\frac{\partial F}{\partial q_\rho}, \qquad \kappa_\rho = \frac{\partial F}{\partial p_\rho}$$

represents a solution of the **Jacobi** equations (348), in the sense of (349), and one will have in:

(353) $$\delta q_\rho = \frac{\partial F}{\partial p_\rho}\delta\alpha, \qquad \delta p_\rho = -\frac{\partial F}{\partial q_\rho}\delta\alpha$$

an infinitesimal transformation that will take every individual extremal to an (infinitesimally-close) extremal. Since that is likewise true of the associated one-parameter group, one will have the theorem:

An integral (352) *of the canonical system* (336) *belongs to a one-parameter group with the infinitesimal transformation* (353) *that takes the integral curves of the canonical system to each other.*

The integral curves of the system (353), which is likewise a canonical system (as one might expect), are the "orbits" ([314]) of the one-parameter group. Since an orbit runs through every point in the phase-R_{2n+1}, the orbits that run through the points of an individual integral curve of the canonical system (336) will generate an M_2 . All of the ∞^1 integral curves of the canonical system (336) that emerge from the original integral curve by the transformations of the one-parameter group will then lie on one such M_2 , and indeed one will get it when one measures out segments on all orbits that belong to the same increase $\delta\alpha$. Correspondingly, the M_2 will carry nets that are defined by ∞^1 integral curves of the canonical system (336) and ∞^1 orbits, i.e., integral curves of (353).

$$\sum_\rho (\delta^{(\lambda)} q_\rho\, \delta^{(\lambda)} p_\rho - \delta^{(\lambda)} p_\rho\, \delta^{(\lambda)} q_\rho) = \text{const.}$$

since the **Jacobi** equations are self-adjoint.

([314]) That word might enter in place of the usual term "trajectories of the group" here, as it did before in no. **18.b**, since that might easily lead to confusion in the applications to mechanics.

Now, since obviously the integral (352) of the canonical system is likewise an integral of the system (353), all integral curves of the canonical system that emerge from one of them by the transformations of the one-parameter group will have the same numerical values for the constants in the relation (352), or in other words: If an integral curve belongs to the M_{2n} (351), so the entire M_2 that arises from it by the one-parameter group will, as well. Basically, this argument only repeats what was done in no. **18.c**, moreover, which was achieved by generalizing the results on cyclic coordinates there ([315]). The individual transformation of the group is now regarded as a point transformation in the phase space of (p_ρ, q_ρ, t), while at the time, it was interpreted as a transformation of the field elements in the R_{n+1} of the $(q_1, \ldots, q_n, t)$ that will become a point transformation in the space of the $(q_1, \ldots, q_n, t)$ only in special cases, such as cyclic coordinates ([316]). Such a degeneracy will occur if and only if the function $F(p_1, \ldots, p_n, q_1, \ldots, q_n, t)$ is a *linear function that impulse components* ([317]):

$$(354) \quad F(p_1, \ldots, p_n, q_1, \ldots, q_n, t)$$

$$= A_1(q_1, \ldots, q_n, t)\, p_1 + \ldots + A_n(q_1, \ldots, q_n, t)\, p_n + A_{n+1}(q_1, \ldots, q_n, t)\,,$$

and therefore, the infinitesimal transformation (351) will assume the form ([318]):

([315]) If q_n is a cyclic coordinate then the following integral of the canonical system will be known:

$$p_n = \text{const.}$$

The infinitesimal transformation of the associated one-parameter group of transformation will then read simply:

$$\delta q_1 = 0\,, \ldots, \qquad \delta q_{n-1} = 0\,, \qquad \delta q_n = \delta\alpha\,, \qquad \delta p_1 = 0\,, \ldots, \qquad \delta p_n = 0\,,$$

which yields the "parallel displacement" in the q_n-direction. The M_2 here are the structures that one will obtain when one lays the curve:

$$p_1 = \text{const.}, \quad \ldots, \quad p_n = \text{const.}, \quad q_1 = \text{const.}, \quad \ldots, \quad q_{n-1} = \text{const.}, \quad t = \text{const.},$$

along which only q_n is variable, through each point of an integral curve

([316]) In the case where, e.g., p_n is a cyclic coordinate, one will have simply the parallel displacement in the p_n -direction.

([317]) The simplest of those cases is just the case of cyclic coordinates.

([318]) If one puts that transformation into the form of a *parallel translation* in the q_n-direction by introducing new variables then the integral (354) will go to:

$$p_n = \text{const.},$$

i.e., q_n will become a cyclic coordinate. Thus, the case of an integral that is linear in the impulse components seems to be closely related to the case of cyclic coordinates. Cf., *infra*, no. **29**, as well as **E. T. Whittaker**, *Dynamics*, pp. 328.

(354.a)
$$\begin{cases} \delta q_\rho = A_\rho(q_1,\dots,q_n,t)\,\delta\alpha\,, \\ \delta p_\rho = -\left(\dfrac{\partial A_1}{\partial q_\rho}\,p_1 + \dfrac{\partial A_2}{\partial q_\rho}\,p_2 + \cdots + \dfrac{\partial A_n}{\partial q_\rho}\,p_n + \dfrac{\partial A_{n+1}}{\partial q_\rho}\right)\delta\alpha\,, \\ \delta t = 0\,, \end{cases}$$

in which the n differential equations in the first row define an infinitesimal transformation in only the $q_1, \dots, q_n$. The n equations in the second row then give the infinitesimal transformation of the impulse components p_ρ that it is coupled with (the velocity components $\dot{q}_\rho$, resp.), such that the entire transformation (354.a) will represent an extended *point transformation*, in **Lie**'s terminology [cf., II A 6 (**L. Maurer** and **H. Burkardt**), no. **13**].

Special emphasis should be placed on the case in which the function H in the canonical system (336) is free of the independent variable t, and therefore:

(355)
$$H(p_1, \dots, p_n, q_1, \dots, q_n, t) = k$$

will be an integral of the canonical system (energy integral). From (355), that integral is associated with the infinitesimal transformation:

(355.a)
$$\delta q_\rho = \frac{\partial H}{\partial p_\rho}\,\delta\alpha\,, \qquad \delta p_\rho = -\frac{\partial H}{\partial q_\rho}\,\delta\alpha\,,$$

whose equations will then coincide with the canonical system itself, up to the independent variables. The projection of the integral curves in phase-R_{2n+1} onto the M_{2n} of the (p_ρ, q_ρ), i.e., the trajectories of the motion will then be transformed into other ones by the one-parameter group of transformations that arises from the energy integral (cf., nos. **10** and **18.a**), and the transformation will generate a different time ordering of the individual points along the trajectory. Now, since a comparison of (336) and (355.a) will further show that dt and $\delta\alpha$ are proportional, so the difference between the old and new time values will have the same magnitude for all points of a trajectory, the transformation in the R_{2n+1} of $p_1, \dots, p_n, q_1, \dots, q_n, t$ will also be generated by a parallel translation in t-direction ([319]).

The following relation, which comes close to the argument in no. **18**, is important for the relationship between the integrals of the canonical system and the fields of extremals of the associated variational problem in ($q_1, \dots, q_n, t$)-space, on which the systematic integral of the equations of motion is based.

If one has a field of extremals for the variational problem that will all belong to the same M_{2n}:

$$F(p_1, \dots, p_n, q_1, \dots, q_n, t) = \text{const.}$$

([319]) Or in other words: The independent variable t is the analogue of a cyclic coordinate.

when they are converted into integral curves of the canonical system then an integral curve will also simultaneously belong to the entire one-parameter family of integral curves of the field that it generates by way of the one-parameter group of transformations with the infinitesimal transformation (353). That is because since the p_ρ are given as functions of the $q_1, \dots, q_n, t$ in the field, when one substitutes those functions in the right-hand side of:

$$\delta q_\rho = \frac{\partial F}{\partial p_\rho}\,\delta\alpha\,, \tag{356}$$

one will get an infinitesimal point transformation of the $(q_1, \dots, q_n, t)$-manifold, in which an integral curve of the field is associated with a neighboring integral curve of the **Euler** equations. However, they must also belong to the field, because from (356), they possess the components ([320]):

$$p_\rho + \left(\sum_\lambda \frac{\partial p_\rho}{\partial q_\lambda}\frac{\partial F}{\partial p_\lambda}\right)\delta\alpha = p_\rho + \left(\sum_\lambda \frac{\partial F}{\partial p_\lambda}\frac{\partial p_\lambda}{\partial q_\rho}\right)\delta\alpha = p_\rho - \frac{\partial F}{\partial q_\rho}\,\delta\alpha\,.$$

However, those are precisely the changes that the impulse components of the associated integral curves of the canonical system will experience under the infinitesimal transformation (353). If an integral curve of the manifold F = const. belongs to the field then all ∞^1 integral curves of the M_2 that is spanned by the orbits of the group that run through the points of the original extremal will belong to the field.

26. The involution relation between two integrals and Poisson's theorem. – If one has *two integrals* of the canonical system (336):

$$F_1(p_1, \dots, p_n, q_1, \dots, q_n, t) = c_1\,, \qquad F_2(p_1, \dots, p_n, q_1, \dots, q_n, t) = c_2 \tag{357}$$

then from no. **25**, each of them will belong to the infinitesimal transformations:

$$\delta q_\rho = \frac{\partial F_1}{\partial p_\rho}\,\delta\alpha\,, \qquad \delta q_\rho = -\frac{\partial F_1}{\partial p_\rho}\,\delta\alpha\,, \tag{357.a}$$

([320]) One should observe that one has $\dfrac{\partial p_\rho}{\partial q_\lambda} = \dfrac{\partial p_\lambda}{\partial q_\rho} = \dfrac{\partial^2 S}{\partial q_\lambda\,\partial q_\rho}$. Since:

$$F(p_1, \dots, p_n, q_1, \dots, q_n, t) = \text{const.}$$

will further become an identity when one replaces the $p_1, \dots, p_n$ with functions of the $p_1, \dots, p_n, t$, one will have:

$$\frac{\partial F}{\partial q_\rho} + \sum_\lambda \frac{\partial F}{\partial p_\lambda}\frac{\partial p_\lambda}{\partial q_\rho} = 0.$$

or

(357.b) $$\delta q_\rho = \frac{\partial F_2}{\partial p_\rho}\,\delta\alpha\,, \qquad \delta q_\rho = -\frac{\partial F_2}{\partial p_\rho}\,\delta\alpha\,,$$

resp., of the integral curves to other ones, i.e., the right-hand sides of (357.a) and (357.b) are two solutions of the **Jacobi** equations (348). Now, since (258) in no. **20** says that two solutions:

$$\delta^{(1)}q_\rho\,,\ \delta^{(1)}p_\rho \quad \text{and} \qquad \delta^{(2)}q_\rho\,,\ \delta^{(2)}p_\rho$$

of those equations will satisfy the relation:

$$\sum_{\rho=1}^{n}(\delta^{(1)}p_\rho\,\delta^{(2)}q_\rho - \delta^{(1)}q_\rho\,\delta^{(2)}p_\rho) = \text{const.},$$

one will also have:

(358) $$\sum_{\rho=1}^{n}\left(\frac{\partial F_1}{\partial p_\rho}\frac{\partial F_2}{\partial q_\rho} - \frac{\partial F_1}{\partial q_\rho}\frac{\partial F_2}{\partial p_\rho}\right) = \text{const.}$$

Since the left-hand side of (358) is the ***Poisson** bracket* (cf., no. **12**) that is constructed from the functions F_1 and F_2 ([321]):

(359) $$(F_1\,, F_2) = \left(\frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial q_1} - \frac{\partial F_1}{\partial q_1}\frac{\partial F_2}{\partial p_1}\right) + \cdots + \left(\frac{\partial F_1}{\partial p_n}\frac{\partial F_2}{\partial q_n} - \frac{\partial F_1}{\partial q_n}\frac{\partial F_2}{\partial p_n}\right),$$

and (358) says that this Poisson bracket that is defined by two integrals will be constant along every integral curve, so along with F_1 = const. and F_2 = const., at the same time:

(358.a) $$(F_1\,, F_2) = \text{const.}$$

will also represent an integral of the canonical system when the **Poisson** bracket is a new function of q_ρ, p_ρ, t that is independent of F_1 and F_2 ([322]).

([321]) If follows from the definition of the **Poisson** bracket (cf., no. **12**) that:

(359.a) $$(F_1, F_2) = -(F_1, F_2)\,, \qquad (F, F) = 0\,,$$

as well as:

(359.b) $$\begin{cases} (G_1 + G_2, F) = (G_1, F) + (G_2, F), \\ (G_1 \cdot G_2, F) = G_1 \cdot (G_2, F) + G_2 \cdot (G_1, F)\,. \end{cases}$$

Moreover, one should observe that the constancy of the **Poisson** brackets along the integral curves is inferred from precisely the same argument that gives the constancy of the **Lagrange** brackets in no. **21**.

([322]) In the terminology of **Lie**'s theory of groups are the symbols of the two infinitesimal transformations (357.a) and (357.b), resp.:

As will be worked out in no. **12** [cf., exp. ([119])], **Poisson** had defined those **Poisson** brackets based upon the Ansatz of his perturbation equations and verified by laborious calculations that they would be free of t when $p_1, \dots, p_n, q_1, \dots, q_n$ are replaced with a solution of the canonical system. **C. G. J. Jacobi** was the person who first recognized that the following theorem would emerge from that fact, which **Poisson** only regarded as remarkable: A new integral of the canonical equations can be obtained from two known integrals by mere differentiation using the **Poisson** bracket.

In general, **Jacobi** seemed to have initially overestimated the significance of that theorem. He probably believed that all integrals of a mechanical problem could be defined by repeatedly forming the **Poisson** brackets of two known integrals ([323]). It seemed to him to be an exception when forming the **Poisson** bracket did not yield a new integral, but one of the cases:

(357.c) $$X_1 f = (F_1, f) , \qquad X_1 f = (F_1, f) .$$

[Cf., II A 6 (**L. Maurer** and **H. Burkhardt**), no. **4**] and the associated bracket expression (cf., II A 6, no. **5**) is:

$$(X_1, X_2) f = ((F_1, F_2), f) .$$

Since every integral of the canonical system yields a one-parameter group of transformation that transforms the set of integral curves into itself, in **Lie**'s theorem of transformation groups, **Poisson**'s theorem means that the associated bracket expression that is defined by two infinitesimal transformations (357.c) will also produce an infinitesimal transformation of the integral curves.

([323]) **C. G. J. Jacobi**, "Sur un théorème de Poisson," C. R. Acad. Sci. Paris **11** (1841), pp. 529 – *Werke IV*, pp. 143, where he called that remark *la plus profonde découverte de M. Poisson* ("Poisson's most profound discovery").

Jacobi proved this theorem by starting from the so-called ***Jacobi** identity* [cf. II A 5 (**E. von Weber**), as well as II A 6 (**L. Maurer** and **H. Burkhardt**), no. **5**]. Namely, for three functions F_1, F_2, F_3, one has:

$$((F_1, F_2), F_3) + ((F_2, F_3), F_1) + ((F_3, F_1), F_2) = 0$$

identically.

Jacobi had considered the case in which time t did not appear explicitly in H, such that:

$$H (p_1, \dots, p_n, q_1, \dots, q_n) = k$$

would be an integral of the canonical system. Therefore, in order for a function:

$$F (p_1, \dots, p_n, q_1, \dots, q_n) = c$$

to be an integral of the canonical system, it is necessary and sufficient that one must have:

$$(H, F) = 0 .$$

If one has two integrals of the canonical system:

$$F_1 (p_1, \dots, p_n, q_1, \dots, q_n) = c_1 , \qquad F_2 (p_1, \dots, p_n, q_1, \dots, q_n) = c_2 ,$$

and thus has:

$$(H, F_1) = 0 , \qquad (H, F_2) = 0 ,$$

then it will follow from the **Jacobi** identity that when one introduces $F_3 = H$:

$$(H, (F_1, F_2)) = 0 ,$$

i.e.:

$$(F_1, F_2) = \text{const.}$$

1. The **Poisson** bracket is identically zero: $(F_1, F_2) \equiv 0$.

2. The **Poisson** bracket is identically constant:

$$(F_1, F_2) \equiv C .$$

3. The **Poisson** bracket is a function of both of them.

The construction of related functions F_1, F_2 :

$$(F_1, F_2) = \varphi\,(F_1, F_2)$$

can occur ([324]). On the other hand, it was **Jacobi** who recognized the meaning of the first of those special cases, namely, the identical vanishing of the **Poisson** bracket of two integrals, for the systematic integration of the canonical system and utilized it ([325]). A deeper insight into the

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is an integral of the canonical system. (Cf., **C. G. J. Jacobi**, "Nova Methodus…," *Werke V*, esp. pp. 46, as well as in *Probleme der Mechanik, Werke V*, pp. 217, esp. pp. 348).

([324]) **C. G. J. Jacobi**, "Nova Methodus…," *Werke V*, pp. 1, esp. pp. 48, as well as *Vorlesungen*, *Werke Suppl.-Bd*, pp. 270.

He explained the observation that, e.g., the **Poisson** bracket of two area integrals will always yield precisely the third area integral, so one will not leave the domain of the area integrals by forming the **Poisson** brackets (cf., "Nova Methodus…," *Werke V*, esp. pp. 112), by saying that the area integrals are common to a large class of mechanical problems and therefore cannot succeed in integrating a particular mechanical problem. A complete integration of a mechanical problem can be achieved by forming the **Poisson** brackets of two integrals only when those integrals are peculiar to the problem being solved. **S. Lie** was the first to discover the intrinsic basis for the behavior of area integrals. The one-parameter group that arises from an area integral is a group of point transformations of (x, y, z)-space, namely, the group of rotations around a coordinate axis. It is included as a subgroup in the three-parameter group of rotations around the coordinate origin, which is already determined by two of the one-parameter groups of rotations around each of two coordinate axes.

Corresponding statements are true for the center of mass integrals, each of which arises from a one-parameter group of parallel displacements in the direction of a coordinate axis. However, since the parallel displacements in a plane once more define a group, the **Poisson** bracket of two center of mass integrals will not give a new integral, but rather it is identically zero.

The motions in three-dimensional (x, y, z)-space, which define a six-parameter group, correspondingly belong to the first three center of mass integrals and the three area integrals. One will not leave the domain of those six integrals by forming the **Poisson** brackets. Rather, the **Poisson** brackets will always once more give one of the six integrals, as long as they do not vanish.

It was the theory of relativity that first gave rise to the extension of the group of motions in **Euclidian** space to the so-called **Galilei** group, and therefore to also classify the energy integral and the second center of mass integral (which are, however, interpreted as *first* integrals) within that sphere of ideas. Since those ten integrals corresponding to the ten-parameter **Galilei** group, one can once more not leave the realm of the ten integrals by forming the **Poisson** brackets. Cf., on this, the papers by **F. Klein** that were concerned with that: **F. Engel**, "Über die zehn allgemeinen Integrale der klassischen Mechanik," Gött. Nachr. (1916), pp. 270 and **F. Engel**, "Nochmals die allgemeinen Integrale der klassischen Mechanik," Gött. Nachr. (1917), pp. 189, as well as the presentation in **F. Engel**, *Die Liesche Theorie der partiellen Differentialgleichungen erster Ordnung*, (ed., by **K. Faber**), Leipzig and Berlin, 1932, Chap. 10, pp. 348.

([325]) The meaning of the vanishing of the **Poisson** bracket was explained in Lecture 32 of his *Vorlesungen* (*Werke, Suppl.-Bd.*), while the formation of new integrals by means of the **Poisson** bracket was treated in Lecture 34. Analogous things were done in "Nova Methodus…," *Werke V*, pps. 22 and 47.

relationships was first achieved with the preparatory work of **J. Bertrand** ([326]) and **E. Bour** ([327]) by **S. Lie** with his introduction of the concept of *function groups* ([328]) [cf., also II A 4 (**E. Von Weber**), nos. **40** and **41**]. **Lie** defined a function group to be the set of functions:

$$\Phi_1 (p_1, \dots, p_n, q_1, \dots, q_n), \dots, \Phi_r (p_1, \dots, p_n, q_1, \dots, q_n), \tag{360}$$

with the property that the **Poisson** bracket of any two of those functions will again be expressed as a function of the r functions:

$$(\Phi_\lambda , \Phi_\mu) = g_{\lambda\mu} (\Phi_1, \dots, \Phi_r) . \tag{360.a}$$

Now, if one has two integrals F_1 and F_2 of the canonical system then one can define their **Poisson** bracket. In the three special cases, the two functions F_1 and F_2 define a function group by themselves. By contrast, in the general case, one can append the **Poisson** bracket:

$$(F_1, F_2) = F_3$$

of the two integrals and once more define the **Poisson** brackets (F_1, F_3), (F_2, F_3). If one of them (or both of them) produces a new integral then one appends it (both of them, resp.) to F_1, F_2, F_3 . If one proceeds in the same way then the two integrals F_1 and F_2 will produce a certain number of integrals:

$$F_1 = c_1, \quad F_2 = c_2, \quad \dots, \quad F_k = c_k \tag{361}$$

that represent a system of functions that can no longer be extended by defining the **Poisson** brackets. The k functions $F_1, F_2, \dots, F_k$, which are mutually independent, then define a k-parameter *function group*, and indeed that will be the smallest function group that includes the functions F_1 and F_2 of the initial integrals.

Naturally, the set of $2n$ integrals of the canonical system:

$$F_1 = c_1, \quad F_2 = c_2, \quad \dots, \quad F_{2n} = c_{2n} , \tag{362}$$

([326]) **J. Bertrand**, "Sur la théorème de Poisson," Note VII to t. I of Lagranges's *Mécanique analytique*. Cf., **J. L. Lagrange**, *Œuvres XI*, pp. 484.

([327]) **E. Bour**, "Sur l'intégration des équations de la mécanique analytique," J. de math. **20** (1855), pp. 185.

([328]) Cf., **S. Lie**, "Begründing einer Invariantentheorie der Berührungstransformationen," Math. Ann. **8** (1875), pp. 215 = *Werke IV*, pp. 1, cf., esp., the second section, *Werke IV*, pp. 36. In that article, **Lie** referred to the function groups more briefly as "groups." Only later was the term "function group" introduced in order to distinguish between the various transformation groups. Cf., **S. Lie**, *Theorie der Transformationsgruppen II*, Arch. f. Math. og Naturw. **1** (1876), pp. 152 = *Werke V*, pp. 42, esp., pp. 68.

The original ideas that led **Lie** to the concept of function group were probably expressed most clearly in the treatise: **S. Lie**, "Zur Theorie der Transformationsgruppen," Chritiania Forhandl. (1888), pp. 3= *Werke V*, pp. 553, esp., Section II, *Werke V*, pp. 554.

along with the $2n$ functions $F_1, \ldots, F_{2n}$, also determine a function group, and indeed a $2n$-parameter function group. That is because since $2n$ independent integrals are no longer present, one must necessarily have:

$$(F_\lambda, F_\mu) = g_{\lambda\mu}(F_1, \ldots, F_{2n}) \tag{362.a}$$

for any two of those functions. The k-parameter function group (361) is included in that $2n$-parameter functions group as a subgroup.

At this point, one must next investigate the special case in which one of the three exceptional cases that were given above:

$$(F_1, F_2) \equiv 0\,, \quad (F_1, F_2) \equiv C\,, \quad (F_1, F_2) \equiv \varphi(F_1, F_2)\,, \text{ resp.},$$

will appear when one starts from two integrals F_1 = const., F_2 = const., such that the two functions F_1 and F_2 determine a two-parameter function group. The first and second of those two exceptional cases are different, as well be shown. If:

$$(F_1, F_2) \equiv 0$$

then **S. Lie** said that the two functions F_1 and F_2 are *in involution* ([329]),

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([329]) **S. Lie**, "Kurzes Résumé mehrerer neuer Theorien," Christiania Forhandlinger i Vidensk.-Sels. (1872), pp. 24 = *Werke III*, pp. 1. The term "involution," which is borrowed from geometry, shall then reproduce the following state of affairs. On the one hand, every integral is associated with an infinitesimal transformation on the manifold t = const.:

$$\delta q_\rho : \delta p_\rho = \frac{\partial F_1}{\partial p_\rho} : -\frac{\partial F_1}{\partial q_\rho}\,, \tag{a}$$

or

$$\delta q_\rho : \delta p_\rho = \frac{\partial F_1}{\partial p_\rho} : -\frac{\partial F_1}{\partial q_\rho}\,. \tag{b}$$

On the other hand, each of the two integrals:

$$F_1 = \text{const.}, \qquad F_2 = \text{const.}, \qquad \text{resp.},$$

determines a tangent M_{2n-1} in the manifold t = const.:

$$\sum_{\rho=1}^{n}\left(\frac{\partial F_1}{\partial q_\rho} dq_\rho + \frac{\partial F_1}{\partial p_\rho} dp_\rho\right) = 0\,, \tag{c}$$

or

$$\sum_{\rho=1}^{n}\left(\frac{\partial F_2}{\partial q_\rho} dq_\rho + \frac{\partial F_2}{\partial p_\rho} dp_\rho\right) = 0\,, \tag{d}$$

resp. If one now chooses the differentials in dq_ρ, dp_ρ in (c) to be the displacement components δq_ρ, δp_ρ then equation (c) will be satisfied since:

$$(F_1, F_2) = 0\,,$$

whereas in the second case ([330]), $F_1 = c_1$ and $F_2 = c_2$ will be two so-called *canonically conjugate* integrals, with **Lie**'s terminology. The third exceptional case:

$$(F_1, F_2) \equiv \varphi\,(F_1, F_2)$$

can be reduced to the second one, as **J. Bertrand** pointed out before ([331]). Here, one can immediately determine a function $G\,(F_1, F_2)$ for which one has:

$$(F_1, G) = 1\,. \tag{363}$$

Namely, one has:

$$(F_1, G) = \frac{\partial G}{\partial F_2} \cdot (F_1, F_2) = \frac{\partial G}{\partial F_2} \cdot \varphi\,(F_1, F_2)\,,$$

in general, so one needs only to calculate G from:

$$\frac{\partial G}{\partial F_2} = \frac{1}{\varphi(F_1, F_2)} \tag{364}$$

by a quadrature. Conversely, since:

$$F_2 = \psi\,(F_1, G)\,, \tag{365}$$

one can also say that in the third exceptional case, F_2 is an integral that is itself a function of the integral F_1 and the integral G that is canonically conjugate to it.

The case of involution:

$$(F_1, F_2) = 0 \tag{366}$$

has special meaning in terms of the systematic integration. The two one-parameter groups of transformations that belong to F_1 and F_2 collectively define a two-parameter group of transformations here ([332]). The ∞^2 transformations of that group associate each individual integral curve of the canonical system with a set of ∞^2 integral curves that fill up a characteristic M_3 that belongs completely to the M_{2n-1} in which the starting integral curve lies:

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i.e., the displacement components of the transformation that arises from F_2 will lie in the tangent M_{2n-1} that belongs to F_1, and likewise, the displacement components of the transformation (a) that arises from F_1 will lie in the tangent manifold (d) to the function F_2. That reciprocity of the relation corresponds completely to the involution relations of geometry. (Cf., the remarks of **F. Engel** in Bd. III on **S. Lie**'s *Werke*, esp. pp. 602).

([330]) One can assign the numerical value of 1 to the constants with no loss of generality.

([331]) **J. Bertrand**, *loc. cit.* ([326]), cf., **J. L. Lagrange**, *Œuvres XI*, pp. 488.

([332]) **S. Lie** initially spoke of commuting (i.e., permutable) transformations; cf., **S. Lie**, "Kurzes Résumé …," ([329]) = *Werke III*, pp. 1.

The *composition constants* of the two-parameter group [cf., II A 6 (**L. Maurer** and **H. Burkhardt**), no. **5**] are equal to zero here.

(367) $$F_1 = c_1 , \quad F_2 = c_2 .$$

Since one can also imagine that this M_3 is constructed from the characteristic M_2 that belong to the two one-parameter groups that are generated by F_1 (F_2, resp.), when an integral curve belongs to a field, that must mean that the entire M_3 should, as well: In other words, all of the integral curves that comprise it must belong to the field. Conversely, if one is to be able to select a field from the integral curves of the M_{2n-1} (367) then the two integrals (367) will be in involution. That is because if (F_1, F_2) is non-zero then the M_2, which is generated by an integral curve of one of the groups (say, the one that belongs to F_1 = const.), will indeed go to another M_2 that is generated by the other group (that arises from F_2) and no longer belongs to the M_{2n-1} (367). One can select a *field* from the ∞^{2n-2} integral curves of a M_{2n-1} :

$$F_1 = c_1 , \qquad F_2 = c_2$$

if and only if the **Poisson** bracket that is formed from F_1 and F_2 :

$$(F_1, F_2) = 0 ,$$

so the two integrals are in involution.

An immediate generalization of that is the theorem:

If one has n integrals of the canonical system:

(368) $$F_1 = c_1 , \qquad F_2 = c_2 , \qquad \ldots, \qquad F_\nu = c_n$$

that determine an M_{n+1} with ∞^n integral curves then the ∞^n integral curves of such an M_{n+1} (368) *will always define a field of extremals of the **Euler** equations if and only if the **Poisson** bracket of any two of those integrals vanishes identically:*

(368.a) $$(F_\lambda , F_\mu) = 0 ,$$

so the n integrals will be pairwise in involution.

The set of all one-parameter groups that arise from the individual integrals will then define an n-parameter group, and the transformations of that group will be generated by any integral curve from the ∞^n integral curves of the field (a characteristic M_{n+1} that includes the extremals of the field, resp.).

The general infinitesimal transformation of that n-parameter group finds its analytical expression in:

(369) $$\begin{cases} \delta q_\rho = \left(\mu_1 \dfrac{\partial F_1}{\partial p_\rho} + \mu_2 \dfrac{\partial F_2}{\partial p_\rho} + \cdots + \mu_n \dfrac{\partial F_n}{\partial p_\rho}\right)\delta\alpha , \\ \delta p_\rho = -\left(\mu_1 \dfrac{\partial F_1}{\partial q_\rho} + \mu_2 \dfrac{\partial F_2}{\partial q_\rho} + \cdots + \mu_n \dfrac{\partial F_n}{\partial q_\rho}\right)\delta\alpha \end{cases} \qquad (\delta t = 0).$$

Here, one also convinces oneself that the integral curves of the M_{n+1} (368) define a field, because for any two directions of advance $\delta^{(1)}q_\rho, \delta^{(1)}p_\rho$ and $\delta^{(2)}q_\rho, \delta^{(2)}p_\rho$ that belong to the M_n that arises from (368) by adding t = const., one has from (368) and (368.a) that:

$$\sum_\rho (\delta^{(1)}q_\rho\, \delta^{(2)}p_\rho - \delta^{(1)}p_\rho\, \delta^{(2)}q_\rho) = \left[\sum_{\lambda,\sigma} \mu_\lambda^{(1)} \mu_\sigma^{(2)} (F_\lambda, F_\sigma)\right] \delta\alpha\, \delta\beta = 0 ,$$

i.e., the bilinear covariant of two arbitrary directions of advance is always zero. If one therefore determines the $p_1, \ldots, p_n$ as functions of the $q_1, \ldots, q_n, t$ from the n relations (368) then:

$$p_1\, dq_1 + \ldots + p_n\, dq_n$$

will be an exact differential, and indeed it will be the differential of the function S of a field in ($q_1, \ldots, q_n, t$)-space for a manifold t = const. Correspondingly, one has the differential of the field function S ($q_1, \ldots, q_n, t$) in:

$$\delta S = p_1\, \delta q_1 + \ldots + p_n\, \delta q_n - H\, \delta t ,$$

in which one has likewise replaced the $p_1, \ldots, p_n$ in H with the calculated functions. The systematic integration of the canonical system then comes down to finding n integrals that are pairwise in involution. If one has determined n such integrals, i.e., one half of the integrals that are required in order to complete the integration, then when one calculates the $p_1, \ldots, p_n$ as functions of the $q_1, \ldots, q_n, t$, and the constants $c_1, \ldots, c_n$, one will get an n-parameter family of fields, and therefore, according to no. **17**, one will get a *complete solution* S ($q_1, \ldots, q_n, t, c_1, \ldots, c_n$) *to the Hamilton-Jacobi* partial differential equation ([333]) from the quadrature:

(370) $$\int (p_1\, \delta q_1 + \cdots + p_n\, \delta q_n - H\, \delta t) = S\,(q_1, \ldots, q_n, t, c_1, \ldots, c_n) .$$

With that, one then has the n other integrals immediately, because from no. **17**, one will find them from one such complete solution by means of the relations:

(371) $$\frac{\partial S}{\partial c_1} = \gamma_1 , \quad \ldots, \qquad \frac{\partial S}{\partial c_n} = \gamma_n .$$

([333]) The theorem of **C. G. J. Jacobi** ["Nova methodus…," J. f. Math. **60** (1862), pp. 1 = *Werke V*, pp. 1, cf., esp., pp. 22] is expressed in that formalism. **Jacobi** referred to the theorem as the *theorema gravissimum* there.

In order to give them the form:

$$\text{(372)}\qquad \begin{cases} G_1\,(p_1,\ldots p_n, q_1,\ldots q_n, t) = \gamma_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ G_n\,(p_1,\ldots p_n, q_1,\ldots q_n, t) = \gamma_n, \end{cases}$$

one must only introduce the functions (369) for the $c_1, \ldots, c_n$. On the other hand, the integrals (369) are naturally equivalent to the relations:

$$\text{(371.a)}\qquad \frac{\partial S}{\partial q_1} = p_1\,, \qquad \frac{\partial S}{\partial q_2} = p_2\,, \qquad \ldots, \qquad \frac{\partial S}{\partial q_n} = p_n\,,$$

which are indeed nothing by the relations (369), when solved for the $p_1, \ldots, p_n$.

From (371), the n infinitesimal transformations that belong to the integrals (372):

$$\text{(373)}\qquad \delta^{(\lambda)} q_\rho = \frac{\partial G_\lambda}{\partial p_\rho}\,\delta\alpha\,, \qquad \delta^{(\lambda)} p_\rho = -\frac{\partial G_\lambda}{\partial q_\rho}\,\delta\alpha$$

have the form:

$$\text{(373.a)}\qquad \begin{cases} \delta^{(\lambda)} q_\rho = \left(\sum_{\sigma=1}^{n} \frac{\partial^2 S}{\partial c_\lambda\,\partial c_\sigma}\frac{\partial F_\sigma}{\partial p_\sigma}\right)\delta\alpha\,, \\ \delta^{(\lambda)} p_\rho = -\left(\frac{\partial p_\sigma}{\partial c_\lambda} + \sum_{\sigma=1}^{n} \frac{\partial^2 S}{\partial c_\lambda\,\partial c_\sigma}\frac{\partial F_\sigma}{\partial q_\sigma}\right)\delta\alpha\,. \end{cases}$$

Each of them is a superposition of the infinitesimal transformation ([334]):

$$\text{(373.b)}\qquad \delta^{(\lambda)} q_\rho = 0\,, \qquad \delta^{(\lambda)} p_\rho = -\frac{\partial p_\rho}{\partial c_\lambda}\,\delta\alpha\,,$$

with the n infinitesimal transformations that arise from the integrals $F_\sigma = c_\sigma$.

Therefore ([335]):

([334]) The infinitesimal transformation (373.b) obviously means the superposition of a field that belongs to certain values of the constants $c_1, \ldots, c_n$ with a neighboring field with altered values of the constants c_λ. Therefore, (373) is also an infinitesimal transformation that takes the integral curves of the first field to those of that neighboring field, except that the integral curves of the field have been permuted with each other with respect to the infinitesimal transformation (373.b)

([335]) Because F_τ still remains unchanged under all of the transformations that arise from $F_1, \ldots, F_n$. Therefore, the change in F_τ under the infinitesimal transformation (373) is equal to the change that the infinitesimal transformation (373.b) generates, so:

$$(G_\lambda, F_\tau) = \sum_\rho \frac{\partial F_\tau}{\partial p_\rho}\frac{\partial p_\rho}{\partial c_\lambda} = \frac{\partial F_\tau}{\partial c_\lambda}\,.$$

(374) $$(G_\lambda, F_\tau) = \frac{\partial F_\tau}{\partial c_\lambda} = \begin{cases} 0 & \lambda \neq \tau, \\ 1 & \lambda = \tau, \end{cases}$$

and furthermore ([336]):

(375) $$(G_\lambda, G_\mu) = 0 ,$$

i.e., the integrals of the canonical system that arise from a complete solution of the **Hamilton-Jacobi** equation and can, from no. **19**, be divided into two subsets:

(376) $$F_1 = c_1 , \qquad \ldots, \qquad F_n = c_n ,$$

and

(376.a) $$G_1 = \gamma_1 , \qquad \ldots, \qquad G_n = \gamma_n ,$$

which have the property: Any two integrals from one and the same subset are in involution. Moreover, an arbitrary integral from one subset is also in involution with all integrals of the other subset, with the exception of the integrals from the other subset that carry the same number. The **Poisson** bracket of will be equal to 1 for those two. Two such associated integrals were already referred to above as *conjugate integrals* of the canonical system ([337]). The $2n$ integrals (376) and (376.a) define a canonical basis ([338]) for the function group of the $2n$ integrals ([339]).

One can then recognize the meaning of the exceptional cases of **Poisson**'s theorem, in which the **Poisson** bracket of two integrals does not give a new integral. If one starts from the transformation groups that are coupled with the integrals then one can characterize them as

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([336]) Because from (373.a) and (374), the change in G_μ under the transformation (373) is:

$$\delta G_\mu = \left(- \sum_\rho \frac{\partial G_\mu}{\partial p_\rho} \frac{\partial p_\rho}{\partial c_\lambda} + \frac{\partial^2 S}{\partial c_\lambda\, \partial c_\mu} \right) \delta\alpha = \left[- \frac{\partial}{\partial c_\lambda} \left(\frac{\partial S}{\partial c_\mu} \right) + \frac{\partial^2 S}{\partial c_\lambda\, \partial c_\mu} \right] \delta\alpha .$$

([337]) All of the functions of the same subset will then remain invariant under an infinitesimal transformation that arises from one of those integrals, and likewise all functions of the other subset, with the exception of conjugate functions. The manifold $F_\lambda = c_\lambda$ will go to the manifold $F_\lambda = c_\lambda + \delta\alpha$ under the transformation that arises from the G_λ. That corresponds to the fact that the manifold $G_\lambda = \gamma_\lambda$ will go to the manifold $G_\lambda = \gamma_\lambda - \delta\alpha$ under the transformation that arises from the F_λ.

([338]) Cf., **S. Lie**, "Über partielle Differentialgleichungen erster Ordnung," Christiania Forhandlingar i Vidensk. Sels. (1874), pp. 16 = *Werke III*, pp. 32, in particular, pp. 45.

([339]) One can, correspondingly, go from every basis for the function group of the $2n$ integrals (i.e., from every system of $2n$ independent integrals) of the canonical system to such a canonical basis. On that subject, also cf., the arguments of **J. Bertrand** in the note: "Sur la théorème de Poisson," in **Lagrange**'s *Mécanique analytique* (**J. L. Lagrange**, *Œuvres XI*, pp. 484). Just as one does with the $2n$-parameter complete function group of the $2n$ integrals, one can also put any k-parameter subgroup that it contains into canonical form (cf., no. **28**, in which the meaning of a k-parameter function group of k integrals that is obtained with the help of **Poisson**'s theorem in the context of the systematic integration of the canonical system will be treated.)

follows: In the three exceptional cases, the integral curves of the canonical system that belong to the M_{2n-1} :

$$F_1 = c_1\,, \qquad F_2 = c_2 \tag{377}$$

will go to integral curves under a transformation of the two one-parameter groups that arise from F_1 (F_2, resp.), all of which again belong to one and the same M_{2n-1} :

$$F_1 = \text{const.}, \quad F_2 = \text{const.} \tag{378}$$

However, whereas the **Poisson** bracket (F_1, F_2) will yield a new integral from F_1 and F_2 in the general case, that is no longer the case. Under a transformation of one of the group groups, only those integrals of the M_{2n-1} (377) will again go to integral curves of the same M_{2n-1} (378) for which the constant in:

$$(F_1, F_2) = \text{const.} \tag{379}$$

have a fixed numerical value, such that a further decomposition of the integrals in the M_{2n-1} (377) into ∞^1 subsets will be achieved in that way.

That explains the expression that **Jacobi** gave to **Poisson**'s theorem, in essence. For **Jacobi** himself and his immediate followers, it had the character of something wondrous. Therefore, it would seem explainable that many have attempted to generalize **Poisson**'s theorem ([340]), whether by formal calculations or by more intuitive arguments.

All of those *extensions of the Poisson's theorem* will become directly understandable when one starts from the fact that **Poisson**'s theorem represents only the combination of the two facts:

1. Each integral of the canonical system (336);

$$F(p_1, \ldots, p_n, q_1, \ldots, q_n, t) = \text{const.} \tag{380}$$

belongs to a solution of the **Jacobi** equations (348):

([340]) For example, **E. Schering**, "Verallgemeinerung der Poisson-Jacobischen Störungsformalen," Gött. Abh. **19** (1874), pp. 3 = *Werke I*, pp. 249 (cf., esp., pp. 271) gave the following theorem: If the function H of the canonical system is free of the independent variables t and one then knows an integral of the problem that depends upon t:

$$F(p_1, \ldots, p_n, q_1, \ldots, q_n, t) = \text{const.}$$

then $\partial F / \partial t$ = const. will also be an integral of the problem. That is because one will indeed once more obtain an integral curve by displacing it in the t-direction under the given assumption. Therefore, the two equations:

$$F(p_1, \ldots, p_n, q_1, \ldots, q_n, t) = \text{const.}$$

and

$$F(p_1, \ldots, p_n, q_1, \ldots, q_n, t + \Delta t) = \text{const.}$$

must be simultaneously true, and the theorem will follow from that immediately.

(381) $$\delta q_\rho = \frac{\partial F}{\partial p_\rho}\,\delta\alpha\,, \qquad \delta p_\rho = -\frac{\partial F}{\partial q_\rho}\,\delta\alpha\,.$$

2. Any two solutions of the **Jacobi** equations will give a constant value to the bilinear covariant:

$$\sum_{\rho=1}^{n}(\delta^{(1)}p_\rho\,\delta^{(1)}q_\rho - \delta^{(1)}q_\rho\,\delta^{(1)}p_\rho)\,.$$

Now, that bilinear covariant is the element of the second-order characteristic integral invariant of the canonical system (cf., no. **21**). However, along with that second-order integral invariant, there will also be characteristic integral invariants of order four, six, …, $2n$ of the canonical system, whose integrands likewise remain constant along an integral curve. If one then replaces the δq_ρ, δp_ρ in those integrands with systems of solutions to the **Jacobi** equations that are derived from integrals of the canonical system in the manner of (381) then one will get expressions that remain constant along the integral curves of the canonical system and can then possibly produce new integrals of the canonical system. For example, the fourth-order integral invariant (286) of the canonical system implies that if one uses the four integrals of the canonical system:

(382) $$F_1 = c_1\,, \qquad F_2 = c_2\,, \qquad F_3 = c_3\,, \qquad F_4 = c_4$$

in order to form the expression:

(382.a) $$\sum_{\rho,\sigma=1}^{n}\begin{vmatrix} \frac{\partial F_1}{\partial p_\rho} & \frac{\partial F_2}{\partial p_\rho} & \frac{\partial F_3}{\partial p_\rho} & \frac{\partial F_4}{\partial p_\rho} \\ \frac{\partial F_1}{\partial q_\rho} & \frac{\partial F_2}{\partial q_\rho} & \frac{\partial F_3}{\partial q_\rho} & \frac{\partial F_4}{\partial q_\rho} \\ \frac{\partial F_1}{\partial p_\sigma} & \frac{\partial F_2}{\partial p_\sigma} & \frac{\partial F_3}{\partial p_\sigma} & \frac{\partial F_4}{\partial p_\sigma} \\ \frac{\partial F_1}{\partial q_\sigma} & \frac{\partial F_2}{\partial q_\sigma} & \frac{\partial F_3}{\partial q_\sigma} & \frac{\partial F_4}{\partial q_\sigma} \end{vmatrix} = \text{const.}$$

then that will be true along every integral curve of the system ([341]). A new integral can be derived analogously from six, eight, etc., integrals, and that sequence conclude with the fact that when one uses all $2n$ integrals of the canonical system:

(383) $$F_1 = c_1\,, \qquad F_2 = c_2\,, \qquad \ldots, \qquad F_{2n} = c_{2n}$$

to form the determinant of order $2n$:

([341]) That fact was expressed by **H. Laurent**, "Sur un théorème de Poisson," J. de math. (3) **17** (1872), pp. 422, and should probably be referred to as ***Laurent****'s theorem*.

$$(383.a)\qquad \begin{vmatrix} \frac{\partial F_1}{\partial p_1} & \frac{\partial F_2}{\partial p_1} & \cdots & \frac{\partial F_{2n}}{\partial p_1} \\ \frac{\partial F_1}{\partial q_1} & \frac{\partial F_2}{\partial q_1} & \cdots & \frac{\partial F_{2n}}{\partial q_1} \\ \frac{\partial F_1}{\partial p_2} & \frac{\partial F_2}{\partial p_2} & \cdots & \frac{\partial F_{2n}}{\partial p_2} \\ \frac{\partial F_1}{\partial q_2} & \frac{\partial F_2}{\partial q_2} & \cdots & \frac{\partial F_{2n}}{\partial q_2} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_1}{\partial p_n} & \frac{\partial F_2}{\partial p_n} & \cdots & \frac{\partial F_{2n}}{\partial p_n} \\ \frac{\partial F_1}{\partial q_n} & \frac{\partial F_2}{\partial q_n} & \cdots & \frac{\partial F_{2n}}{\partial q_n} \end{vmatrix} = \text{const.}$$

then that will remain true along the individual integral curves. In this last case, it is obvious that the right-hand side of (383.a) must be a function of the $2n$ integrals (383), so it must belong to the $2n$-parameter function group of all integrals. However, it is also generally true that one will not arrive at the function group that is determined by the integrals that are used in the construction of expressions like (382.a), etc., so those expressions cannot produce anything essentially new in comparison to the simple **Poisson** brackets ([342]). One can also easily derive the expressions (382.a) directly ([343]).

([342]) That was shown by **S. Lie**, "Begründung einer Invariantentheorie der Berührungstransformationen," Math. Ann. **8** (1875), pp. 215 = *Werke IV*, pp. 1; cf., esp., **§ 26**, pp. 300 (pp. 93, resp.).

([343]) In fact, one has [cf., also (285)]:

$$\sum_{\rho,\sigma} \begin{vmatrix} \delta^1 q_\rho & \delta^2 q_\rho & \delta^3 q_\rho & \delta^4 q_\rho \\ \delta^1 p_\rho & \delta^2 p_\rho & \delta^3 p_\rho & \delta^4 p_\rho \\ \delta^1 q_\sigma & \delta^2 q_\sigma & \delta^3 q_\sigma & \delta^4 q_\sigma \\ \delta^1 p_\sigma & \delta^2 p_\sigma & \delta^3 p_\sigma & \delta^4 p_\sigma \end{vmatrix}$$

$$= \sum_{\rho,\sigma} \left\{ \begin{vmatrix} \delta^1 q_\rho & \delta^2 q_\rho \\ \delta^1 p_\rho & \delta^2 p_\rho \end{vmatrix} \cdot \begin{vmatrix} \delta^3 q_\sigma & \delta^4 q_\sigma \\ \delta^3 p_\sigma & \delta^4 p_\sigma \end{vmatrix} - \begin{vmatrix} \delta^1 q_\rho & \delta^3 q_\rho \\ \delta^1 p_\rho & \delta^3 p_\rho \end{vmatrix} \cdot \begin{vmatrix} \delta^2 q_\sigma & \delta^4 q_\sigma \\ \delta^2 p_\sigma & \delta^4 p_\sigma \end{vmatrix} \right.$$

$$+ \begin{vmatrix} \delta^1 q_\rho & \delta^4 q_\rho \\ \delta^1 p_\rho & \delta^4 p_\rho \end{vmatrix} \cdot \begin{vmatrix} \delta^2 q_\sigma & \delta^3 q_\sigma \\ \delta^2 p_\sigma & \delta^3 p_\sigma \end{vmatrix} + \begin{vmatrix} \delta^2 q_\rho & \delta^3 q_\rho \\ \delta^2 p_\rho & \delta^3 p_\rho \end{vmatrix} \cdot \begin{vmatrix} \delta^1 q_\sigma & \delta^4 q_\sigma \\ \delta^1 p_\sigma & \delta^4 p_\sigma \end{vmatrix}$$

$$\left. - \begin{vmatrix} \delta^2 q_\rho & \delta^4 q_\rho \\ \delta^2 p_\rho & \delta^4 p_\rho \end{vmatrix} \cdot \begin{vmatrix} \delta^1 q_\sigma & \delta^3 q_\sigma \\ \delta^1 p_\sigma & \delta^3 p_\sigma \end{vmatrix} + \begin{vmatrix} \delta^3 q_\rho & \delta^4 q_\rho \\ \delta^3 p_\rho & \delta^4 p_\rho \end{vmatrix} \cdot \begin{vmatrix} \delta^1 q_\sigma & \delta^2 q_\sigma \\ \delta^1 p_\sigma & \delta^2 p_\sigma \end{vmatrix} \right\}$$

$$= 2\sum_{\rho} \begin{vmatrix} \delta^1 q_\rho & \delta^2 q_\rho \\ \delta^1 p_\rho & \delta^2 p_\rho \end{vmatrix} \sum_{\rho} \begin{vmatrix} \delta^3 q_\sigma & \delta^4 q_\sigma \\ \delta^3 p_\sigma & \delta^4 p_\sigma \end{vmatrix} - 2\sum_{\rho} \begin{vmatrix} \delta^1 q_\rho & \delta^3 q_\rho \\ \delta^1 p_\rho & \delta^3 p_\rho \end{vmatrix} \sum_{\sigma} \begin{vmatrix} \delta^2 q_\sigma & \delta^4 q_\sigma \\ \delta^2 p_\sigma & \delta^4 p_\sigma \end{vmatrix}$$

27. Simplifying the canonical system when one knows an integral. – From the results of the previous section, in order to perform the integration of the canonical system, one must determine n integrals that are pairwise in involution. For a systematic search for those integrals, one can appeal to the idea that the existence of a cyclic coordinate will make it possible to reduce the canonical system of order $2n$ to a canonical system of order $(2n - 2)$, since the *impulse component* that belongs to the cyclic coordinate is constant. Namely, since the constancy of the impulse component means nothing but the fact that if one knows a first integral of the original canonical system (which be just p_n = const. when q_n is the cyclic coordinate) then that will suggest the question of whether knowing an arbitrary first integral of the canonical system:

$$\frac{dq_\rho}{dt} = \frac{\partial H}{\partial p_\rho}, \quad \frac{dp_\rho}{dt} = -\frac{\partial H}{\partial q_\rho} \qquad (\rho = 1, \ldots, n) \tag{384}$$

will imply a corresponding simplification (on this, cf., esp., no. **18.b**). Now, from no. **9**, when a cyclic coordinate q_n appears, the reduced canonical system:

$$\frac{dq_\rho}{dt} = \frac{\partial H}{\partial p_\rho}, \quad \frac{dp_\rho}{dt} = -\frac{\partial H}{\partial q_\rho} \qquad (\rho = 1, \ldots, n-1) \tag{385}$$

will determine the projections of the space-time curves in the M_{n+1} of the $(q_1, \ldots, q_n, t)$ onto the M_n of the $(q_1, \ldots, q_{n-1}, t)$, resp., which means the same thing as the "cylindrical" M_2 that is defined by the space-time curves and the generators that are parallel to the q_n-direction. However, since those parallel generators are nothing but the orbits of the one-parameter group of parallel displacements in the q_n-direction whose infinitesimal transformation:

$$+ 2\sum_\rho \begin{vmatrix} \delta^1 q_\rho & \delta^4 q_\rho \\ \delta^1 p_\rho & \delta^4 p_\rho \end{vmatrix} \sum_\sigma \begin{vmatrix} \delta^2 q_\sigma & \delta^3 q_\sigma \\ \delta^2 p_\sigma & \delta^3 p_\sigma \end{vmatrix} .$$

The sum of the fourth-order determinants is then composed of the sums of second-order determinants in the simplest way. Analogous statements are also true for the sums of determinants of order six, eight, etc. [cf., (287)]. That can then be adapted to determinants of the same form as (382.a), etc., to (382.a), which can be constructed from **Poisson** brackets in the same way. Cf., **H. Poincaré**, *Méthod. nouv. III*, pp. 23.

For example, when one denotes the determinant (382.a) by D, that will give:

$$D^2 = \begin{vmatrix} (F_1, F_{n+1}) & \cdots & (F_1, F_{2n}) & (F_1, F_1) & \cdots & (F_1, F_n) \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ (F_n, F_{n+1}) & \cdots & (F_n, F_{2n}) & (F_n, F_1) & \cdots & (F_n, F_n) \\ (F_{n+1}, F_{n+1}) & \cdots & (F_{2n}, F_{n+1}) & (F_1, F_{n+1}) & \cdots & (F_n, F_{n+1}) \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ (F_{n+1}, F_{2n}) & \cdots & (F_{2n}, F_{2n}) & (F_1, F_{2n}) & \cdots & (F_n, F_{2n}) \end{vmatrix} .$$

(386) $$\begin{cases} \delta p_\rho = 0, & (\rho = 1, \ldots, n), \\ \delta q_1 = \cdots = \delta q_{n-1} = 0, \quad \delta q_n = \delta\alpha & (\delta t = 0) \end{cases}$$

belongs to the integral:

(387) $$p_n = c_n ,$$

one can also say that the reduced system (385) determines the characteristic M_2 that arises from the integral (387) that belongs to the cyclic coordinate.

Now, that can be adapted to an arbitrary first integral of the canonical system (384) ([344]):

(388) $$G(p_1, \ldots, p_n, q_1, \ldots, q_n, t) = \text{const.} = c .$$

That is because, here as well, for each integral curve of the canonical system in the R_{2n+1} of (p_ρ, q_ρ, t), the orbits of the one-parameter group that arises from the integral (388), i.e., the integral curves of the system:

(388.a) $$\delta q_\rho = \frac{\partial G}{\partial p_\rho}\delta\alpha , \qquad \delta p_\rho = -\frac{\partial G}{\partial q_\rho}\delta\alpha \qquad (\delta t = 0),$$

will determine a characteristic M_2 that is spanned by a net that is formed from ∞^1 integral curves of the canonical system (384) and ∞^1 orbits of the group (388.a). Thus, it is natural to use the given canonical system (384) in order to develop a system of equations for the determination of that characteristic M_2 if one wishes to generalize the ideas that led from the original canonical system (384) to the simplified canonical system (385) in the case of cyclic coordinates.

In order to do that, one solves the integral (388) for one of the impulse components (say, p_n) and writes out:

(389) $$p_n + h(p_1, \ldots, p_{n-1}, q_1, \ldots, q_n, t, c) = 0 ,$$

in place of (388). It is obvious then how one would introduce the position coordinate q_n on the characteristic M_2 as the independent variable, along with t, such if one is to obtain the M_2 then one would have to determine $p_1, \ldots, p_{n-1}, q_1, \ldots, q_{n-1}$ as functions of q_n and t :

(390) $$\begin{cases} p_1 = p_1(q_n, t), & \cdots \quad p_{n-1} = p_{n-1}(q_n, t), \\ q_1 = q_1(q_n, t), & \cdots \quad q_{n-1} = q_{n-1}(q_n, t), \end{cases}$$

while (389) would then imply p_n as a function of q_n and t.

For that choice of coordinates on M_2, one must employ q_n as the independent variable on the orbits, since the curves t = const. are indeed on M_2 . Their differential equations then read:

([344]) The symbol G will be chosen to be the function symbol in order to coincide with the notation in no. **18.b**.

(389.a) $$\frac{\delta q_\rho}{\delta q_n} = \frac{\partial h}{\partial p_\rho}, \qquad \frac{\delta p_\rho}{\delta q_n} = -\frac{\partial h}{\partial q_\rho}, \qquad \delta t = 0 \qquad (\rho = 1, \ldots, n-1),$$

which corresponds to the form (389) that the first integral took ([345]).

On the other hand, one will a new representation for the integral curves of the given canonical system (384) that lie on an M_2 when one replaces the impulse coordinate p_n in H ($p_1, \ldots, p_n, q_1, \ldots, q_n, t$) with the function (389), and thus takes H to:

(391) $$\begin{aligned} &\bar{H}(p_1, \ldots, p_{n-1}, q_1, \ldots, q_{n-1}, q_n, t, c) \\ = \ &\bar{H}(p_1, \ldots, p_{n-1}, -h(p_1, \ldots, p_{n-1}, q_1, \ldots, q_n, t, c), q_1, \ldots, q_n, t). \end{aligned}$$

The canonical system (384) will then go to:

(392) $$\left\{ \begin{aligned} \frac{dq_\rho}{dt} &= \frac{\partial H}{\partial p_\rho} = \frac{\partial \bar{H}}{\partial p_\rho} + \frac{\partial H}{\partial p_n}\frac{\partial h}{\partial p_\rho}, \\ \frac{dp_\rho}{dt} &= -\frac{\partial H}{\partial q_\rho} = -\left(\frac{\partial \bar{H}}{\partial q_\rho} + \frac{\partial H}{\partial p_n}\frac{\partial h}{\partial q_\rho}\right), \end{aligned} \right. \qquad (\rho = 1, \ldots, n-1),$$

to which one must add:

(392.a) $$\frac{dq_n}{dt} = \frac{\partial H}{\partial p_n}.$$

Now, the 2 (n – 1) equations (392), when taken by themselves, represent a **Pfaff** system:

([345]) If one writes:

$$\frac{\partial S}{\partial q_n} + h\left(\frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_{n-1}}, q_1, \ldots, q_{n-1}, q_n, t, c\right) = 0,$$

instead of (389), then one will see that one can go from the **Hamilton-Jacobi** equation of the parametric problem:

$$G\left(\frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_{n-1}}, q_1, \ldots, q_n, t\right) - c = 0$$

to the associated function problem with q_n as the independent variable, which has the form:

$$\int g\left(\frac{dq_1}{dq_n}, \ldots, \frac{dq_{n-1}}{dq_n}, q_1, \ldots, q_{n-1}, q_n, t, c\right) dq_n = \text{extrem.}$$

(Cf., no. **18.b**)

(393)
$$\begin{cases} dq_\rho = \dfrac{\partial \bar{H}}{\partial p_\rho}\,dt + \dfrac{\partial h}{\partial p_\rho}\,dq_n\,, \\ dp_\rho = -\left(\dfrac{\partial \bar{H}}{\partial q_\rho}\,dt + \dfrac{\partial h}{\partial q_\rho}\,dq_n\right), \end{cases} \qquad (\rho = 1, \ldots, n-1),$$

and that will show, in particular, that it defines a completely-integrable system ([346]) of $(2n - 2)$ total differential equations. Namely, every integral:

(394)
$$F(p_1, \ldots, p_{n-1}, q_1, \ldots, q_{n-1}, q_n, t) = \text{const.}$$

of that system must simultaneously satisfy the two linear partial differential equations ([347]):

(394.a)
$$\begin{cases} \dfrac{\partial F}{\partial t} + \{\bar{H}, F\} = 0, \\ \dfrac{\partial F}{\partial q_n} + \{h, F\} = 0, \end{cases}$$

in which one has set:

(395)
$$\{\Phi, \Psi\} = \sum_{\rho=1}^{n-1}\left(\frac{\partial \Phi}{\partial p_\rho}\frac{\partial \Psi}{\partial q_\rho} - \frac{\partial \Phi}{\partial q_\rho}\frac{\partial \Psi}{\partial p_\rho}\right),$$

to abbreviate. Those two linear partial differential equations are then equivalent to the **Pfaff** system (393), and one can easily see that those two equations (394.a) define a *complete system* ([348]) in the

([346]) That was emphasized by **G. Morera**, "I sistemi canonici d'equazioni ai differentiali totali nella teoria di gruppi di transformazioni," Turin Atti dell'acc. di sc. **38** (1902-03), pp. 940. Cf., also the presentation in **T. Levi-Civita**, "Drei Vorlesungen über adiabatische Invarianten," Hamburg Abhandl. aus. math. Sem. **6** (1928), pp. 323, esp., pp. 352-358.

([347]) This system of two linear partial differential equations is also the basis for the investigations of **E. Bour**, who first succeeded in lowering the number of unknown functions in the canonical system by two when one knows an integral. Cf., **E. Bour**, "Sur l'intégration des équations diff. de la mécanique analytique," J. de math. **20** (1855), pp. 185.

([348]) In particular, it is a **Jacobi system.** If one denotes the left-hand side of (394.a) by $X_1 f$ ($X_2 f$, resp.) then the bracket expression will be:

$$(X_1, X_2)\,f = \frac{\partial}{\partial t}\{h, F\} - \frac{\partial}{\partial q_n}\{\bar{H}, F\}$$
$$= \left\{\frac{\partial h}{\partial t} - \frac{\partial H}{\partial q_n}, F\right\} + \{h, \{\bar{H}, F\}\} - \{H, \{F, h\}\}\,.$$

Upon appealing to the **Jacobi** identity, that will then imply that:

$$(X_1, X_2)\,f = \left\{\frac{\partial h}{\partial t} - \frac{\partial \bar{H}}{\partial q_n} + \{H, h\}, F\right\}.$$

sense of the theory of linear partial differential equations [cf., II A 5 (**E. von Weber**), no. **13**], from which the complete integrability of (393) will follow immediately.

However, the M_2 that are obtained as solutions to the system (393) are precisely the desired characteristic M_2 . That is because one finds on them, on the one hand, the orbits of the one-parameter group, since one will indeed get the equations (389.a) when one sets $dt = 0$ in (393), and on the other hand, the integral curves of the canonical system that one gets when one adds the differential equation (392.a) to (393). However, the integration of the completely-integrable canonical system (393) comes down to the integration of an ordinary canonical system with $(2n - 2)$ independent functions $p_1, \ldots, p_{n-1}, q_1, \ldots, q_{n-1}$. Namely, if one know the values $p_1^{(0)}, \ldots, p_{n-1}^{(0)}$, $q_1^{(0)}, \ldots, q_{n-1}^{(0)}$ [and $p_n^{(0)}$ from (389)] at a point $P_1(q_1^{(0)}, t_0)$ on an M_2 then one will get the values $p_1, \ldots, p_{n-1}, q_1, \ldots, q_{n-1}$ at an arbitrary point P (q_n, t) on M_2 when one connects the two points by a curve $q_n(\lambda)$, $t(\lambda)$ and determines the changes $p_1, \ldots, p_{n-1}, q_1, \ldots, q_{n-1}$ along that curve. That is because due to the complete integrability of (392), the final values that one achieves are independent of the curve along which one arrives at P from P_0 . Now, if one sets:

$$\begin{aligned}(396)\qquad &\bar{H}(p_1,\ldots,p_{n-1},q_1,\ldots,q_{n-1},q_n(\lambda),t(\lambda))\frac{dt}{d\lambda} \\ &\quad + h(p_1,\ldots,p_{n-1},q_1,\ldots,q_{n-1},q_n(\lambda),t(\lambda))\frac{dq_n}{d\lambda} \\ &\quad = K(p_1,\ldots,p_{n-1},q_1,\ldots,q_{n-1},\lambda)\end{aligned}$$

along the chosen curve $q_n(\lambda)$, $t(\lambda)$ then equations (393) will read:

$$(397)\qquad \frac{dq_\rho}{dt} = \frac{\partial K}{\partial p_\rho}, \qquad \frac{dp_\rho}{dt} = -\frac{\partial K}{\partial q_\rho} \qquad (\rho = 1, \ldots, n-1)$$

along the curve. Conversely, the integration of that canonical system with $2(n - 1)$ unknown functions will yield the values of $q_1, \ldots, q_{n-1}, p_1, \ldots, p_{n-1}$ at an arbitrary point (q_n, t), and thus the characteristic M_2 ([349]). Hence, the given canonical system has been reduced to a canonical system

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On the other hand, if one replaces the impulse component p_n by (389) in the equation:

$$\frac{\partial G}{\partial t} + (H, G) = 0,$$

which says that G is an integral of the canonical system, then that will imply that:

$$\frac{\partial h}{\partial t} - \frac{\partial \bar{H}}{\partial q_n} + \{H, h\} = 0,$$

such that the bracket expression $(X_1, X_2)\,f$ will be, in fact, identically zero.

([349]) **S. Lie** proceeded in such a way that he intersected M_2 with the bundle of M_{2n} :

that includes two less unknown functions when one known an arbitrary first integral, exactly as in the case of a cyclic coordinate.

The canonical system (397) belongs to the **Hamilton-Jacobi** partial differential equation:

$$(400) \qquad \frac{\partial V}{\partial \lambda} + K\left(\frac{\partial V}{\partial q_1}, \ldots, \frac{\partial V}{\partial q_{n-1}}, q_1, \ldots, q_{n-1}, \lambda\right) = 0 .$$

A solution $V(q_1, \ldots, q_{n-1}, \lambda)$ to that equation determines a field of the system (397), i.e., a family of ∞^{n-1} extremals of the associated variational problem that cut the family of ∞^1 manifolds V = const. However, one can derive the associated M_2 that is a solution to the system (393) from any extremal, such that one will get a family of ∞^{n-1} M_2, and since precisely ∞^1 integral curves of the original canonical system (384) lie on each M_2, one will then also have a field of the original system. One will get the field function $S(q_1, \ldots, q_n, t)$ of that field in the same way that one does with V by going from the integral curves of the canonical system (397) to the M_2 . However, the field function must (cf., no. **18.b**) simultaneously satisfy the two partial differential equations ([350]):

$$(398) \qquad q_n - q_n^{(0)} = \mu\,(t - t_0) \qquad \left(\mu = \frac{q_n^{(1)} - q_n^{(0)}}{t_1 - t_0}\right).$$

He could then arrive at any point P of M_2 from P_0 along the line of intersection that the M_{2n} cut out with the correct value of μ . Since he used t as the independent variable along that line of intersection, from (396), he had:

$$(398.a) \qquad K(p_1, \ldots, p_{n-1}, q_1, \ldots, q_{n-1}, t) = \bar{H} + \mu\, h ,$$

and the system (397) read:

$$(398.b) \qquad \frac{dq_\rho}{dt} = \frac{\partial K}{\partial p_\rho}, \qquad \frac{dp_\rho}{dt} = -\frac{\partial K}{\partial q_\rho} \qquad (\rho = 1, \ldots, n-1).$$

It will be symmetric one sets:

$$(399) \qquad q_n - q_n^{(0)} = (q_n^{(1)} - q_n^{(0)})\,\lambda , \qquad t - t_0 = (t_1 - t_0)\,\lambda ,$$

in which λ is the independent variable, as in the text [**G. Morera**, *loc. cit.*, ([346])], such that from (396), one will have:

$$(399.a) \qquad K(p_1, \ldots, p_{n-1}, q_1, \ldots, q_{n-1}, \lambda)$$

$$= (t_1 - t_0)\,\bar{H}(p_1, \ldots, p_{n-1}, q_1, \ldots, q_{n-1}, q_n^{(0)} + (q_n^{(1)} - q_n^{(0)})\lambda, t + (t_1 - t_0)\lambda)$$
$$+ (q_n^{(1)} - q_n^{(0)})\,h(p_1, \ldots, p_{n-1}, q_1, \ldots, q_{n-1}, q_n^{(0)} + (q_n^{(1)} - q_n^{(0)})\lambda, t + (t_1 - t_0)\lambda) .$$

One will then get the associated M_2 that is the solution to (396) from an integral curve of (397) when one simply takes $\lambda = 1$ and simultaneously replaces $q_n^{(1)}$ with the variables q_n, t_1 .

One will get an integral of the system (396) from an integral of (397) in that way.

([350]) From this standpoint, the condition:

$$\frac{\partial G}{\partial t} + (H, G) = 0 ,$$

(401)
$$\begin{cases} \dfrac{\partial S}{\partial t} + \bar{H}\left(\dfrac{\partial S}{\partial q_1}, \dots, \dfrac{\partial S}{\partial q_{n-1}}, q_1, \dots, q_n, t\right) = 0, \\ \dfrac{\partial S}{\partial q_n} + h\left(\dfrac{\partial S}{\partial q_1}, \dots, \dfrac{\partial S}{\partial q_{n-1}}, q_1, \dots, q_n, t\right) = 0, \end{cases}$$

or also

(401.a)
$$\begin{cases} \dfrac{\partial S}{\partial t} + H\left(\dfrac{\partial S}{\partial q_1}, \dots, \dfrac{\partial S}{\partial q_{n-1}}, q_1, \dots, q_n, t\right) = 0, \\ G\left(\dfrac{\partial S}{\partial q_1}, \dots, \dfrac{\partial S}{\partial q_{n-1}}, q_1, \dots, q_n, t\right) - c = 0. \end{cases}$$

Hence, one also finds, conversely, a function $S(q_1, \dots, q_n, t)$ that is a simultaneous solution to the two partial differential equations (401) in such a way that one starts from a solution to the partial differential equation (400) and redefines it in the given way.

If the characteristic M_2 are determined by integrating (393) then one will get the integral curves of the canonical system that lie on an individual M_2 by a quadrature. That is because since the integral curves of an M_2 all belong to a field, the expression:

$$p_1\,dq_1 + \dots + p_n\,dq_n - H\,dt$$

must be an exact differential on that M_2. When one introduces the functions (390) that represent the M_2 for $p_1, \dots, p_{n-1}, q_1, \dots, q_{n-1}$ and introduces the p_n from (389), one will then have that:

(402)
$$S^*(q_n, t, c) = \int\left(p_1\frac{\partial q_1}{\partial q_n} + \cdots + p_{n-1}\frac{\partial q_{n-1}}{\partial q_n} + p_n\right)dq_n + \int\left(p_1\frac{\partial q_1}{\partial t} + \cdots + p_{n-1}\frac{\partial q_{n-1}}{\partial t} - H^*(q_n, t)\right)dt$$

is the field function for the field that is formed by the integral curves on the M_2. Therefore, the integral curves of the canonical system on the M_2 will be determined by the equation:

(403)
$$\frac{\partial S^*}{\partial c} = \gamma.$$

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which says that $G = c$ is an integral of the given canonical system, means that the two partial differential equations (401.a) possess a common solution. Cf., **Bour**, "Sur l'intégration des équations differ. part. du premier et du second ordre," Paris J. Éc. Polyt. **22**, cah. 39 (1862), pp. 149, cf., esp., pp. 164.

That relation corresponds to an integral of the given canonical system that is canonical conjugate to the starting integral (388) (in the canonical basis for the function group of all $2n$ integrals) that is used for the reduction.

The simplified canonical system (397) has $(2n - 2)$ integrals that are naturally simultaneously integrals of the completely-integrable system (393). Those integrals likewise produce integrals of the given canonical system (384), and indeed they are integrals of the system that is in involution with the integral $G = c$ that mediates the transition to the simplified canonical system (397) ([351]). Namely, if:

(404) $$f(p_1, \ldots, p_n, q_1, \ldots, q_n, t, c) = \text{const.}$$

is an integral of the completely-integrable system (393) then f will be a solution to the two partial differential equations (394.a):

(405) $$\begin{cases} \dfrac{\partial f}{\partial t} + \{\bar{H}, f\} = 0, \\[2mm] \dfrac{\partial f}{\partial q_n} + \{h, f\} = 0. \end{cases}$$

If one now goes over to a function $F(p_1, \ldots, p_n, q_1, \ldots, q_n, t)$ in such a manner that one replaces c with the function:

$$G(p_1, \ldots, p_n, q_1, \ldots, q_n, t)$$

then:

(406) $$F(p_1, \ldots, p_n, q_1, \ldots, q_n, t) = \text{const.}$$

will be, on the one hand, an integral of the original canonical system, since one has:

$$\frac{\partial F}{\partial t} + (H, F) = \frac{\partial f}{\partial t} + \{\bar{H}, f\} = 0$$

from the first of equations (405). On the other hand, one has:

$$(G, F) = -\frac{\partial G}{\partial p_n}\left(\frac{\partial f}{\partial q_n} + \{h, f\}\right)$$

for the **Poisson** bracket, i.e., from the second of the equations (405), one has:

(407) $$(G, F) = 0\,.$$

([351]) **E. Bour**, *loc. cit.* ([347]). With **Lie**'s terminology, the integrals define a $(2n - 2)$-parameter function group that is reciprocal to the two-parameter function group for which the integral $G = c$ and integral that is canonically conjugate to it represent a basis.

In that way, one ultimately gets the following path to the determination of the n integrals that are in involution for a canonical system with $2n$ unknown functions, which are required for one to perform the integration completely. One next determines an integral of the given canonical system itself and goes over to a canonical system with $(2n - 2)$ unknown functions with the help of that integral. One once more determines an integral of that system, exhibits a new canonical system with $(2n - 4)$ unknown functions ([352]), and proceeds in that way to canonical systems with $(2n - 6)$, …, 2 unknown functions. In total, one will then have to perform an operation of order $2n$, $2n - 2$, …, 4, 2, with **Lie**'s terminology, in order to completely integrate the canonical system with $2n$ unknowns, and one only needs to add a quadrature to that. The individual canonical system that one exhibits in that way are auxiliary systems for that integration. They represent the completely-integrable systems with 2, 3, …, n independent variables that determine the characteristic M_2, M_3, …, M_n on which the ∞^1, ∞^2, …, ∞^{n-1}, resp., integral curves of a field are arranged.

If several (say r) integrals:

$$F_1(p_1, \dots, p_n, q_1, \dots, q_n, t) = c_1\,, \qquad \dots, \qquad F_r(p_1, \dots, p_n, q_1, \dots, q_n, t) = c_r \tag{408}$$

of the given canonical system have been found, instead of a single integral, and they are pairwise in involution:

$$(F_\lambda, F_\mu) = 0 \tag{408.a}$$

then one can exhibit a completely-integrable system with r independent variables directly [the associated canonical system with $2(n - r)$ unknown functions, resp.]. That is because, from **no. 26**, the integrals (408) generate an r-parameter group of transformations that is assigned to the individual integral curves of a characteristic M_{r+1}. One can then next determine that M_{r+1} and thus obtain a completely-integrable system with $2(n - r)$ unknown functions whose integration is equivalent to that of the auxiliary canonical system with $2(n - r)$ unknown functions in a manner that corresponds entirely to procedure for the special case of an individual integral.

In order to find that completely-integrable system, one solves the r integrals (408) for r of the impulse components p_ρ (say, for p_{n-r+1}, …, p_n):

$$\left\{\begin{array}{l} p_{n-r+1} + f_1(p_1,\dots,p_{n-r},q_1,\dots,q_{n-r},q_{n-r+1},\dots,q_n,t,c_1,\dots,c_r) = 0\,, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ p_n \quad + f_r(p_1,\dots,p_{n-r},q_1,\dots,q_{n-r},q_{n-r+1},\dots,q_n,t,c_1,\dots,c_r) = 0 \end{array}\right. \tag{408.b}$$

([352]) The integrals of that system simultaneously produce integrals of the canonical system with $(2n - 2)$ unknown functions that are all in involution with the integral of that system that served to simplify things. Hence, that will imply integrals of the given canonical system that are in involution with *both* known integrals of that system. With each further step, along with the integral that is used to simplify things, at the same time, the integral that is conjugate to it will single out all $2n$ integrals from the basis for the function group.

and imagines that the associated position coordinates $q_{n-r+1}, \ldots, q_n$ have been introduced, along with t, as the independent variables on the characteristic M_{r+1} ([353]). The involution relations (408.a) imply the corresponding relations for the functions (408.b) ([354]):

$$(p_{n-r+\sigma} + f_\sigma, p_{n-r+\tau} + f_\tau) = 0 , \tag{408.c}$$

which one can also write in the form:

$$\frac{\partial f_\tau}{\partial q_{n-r+\sigma}} - \frac{\partial f_\sigma}{\partial q_{n-r+\tau}} + \{f_\sigma, f_\tau\} = 0 \qquad (\sigma, \tau = 1, \ldots, r), \tag{409}$$

when one introduces:

$$\{f_\sigma, f_\tau\} = \sum_{\rho=1}^{n-r} \left(\frac{\partial f_\sigma}{\partial p_\rho}\frac{\partial f_\tau}{\partial q_\rho} - \frac{\partial f_\sigma}{\partial q_\rho}\frac{\partial f_\tau}{\partial p_\rho} \right), \tag{409.a}$$

to abbreviate.

If one defines the function:

$$\begin{aligned} \bar{H}\,(p_1, \ldots, p_{n-r}, q_1, \ldots, q_{n-r}, q_{n-r+1}, \ldots, q_n, t) \\ = (p_1, \ldots, p_{n-r}, - f_1, q_1, \ldots, - f_1, q_1, \ldots, q_n, t) , \end{aligned} \tag{410}$$

which corresponds precisely to (391), then the relations:

$$\frac{\partial F_\lambda}{\partial t} + (H, F_\lambda) = 0 \qquad (\lambda = 1, \ldots, r), \tag{411}$$

which express the idea that the $F_\lambda = c_\lambda$ are integrals of the given canonical system, next imply the following relations for the functions (408.b) ([355]):

([353]) On this topic, cf., **T. Levi-Civita**, "Drei Vorlesungen über adiabatische Invarianten," Hamburg Abhandl. aus d. math. Sem. **6** (1928), esp., pp. 352.

([354]) That is because, it will next follow from (408.a) that:

$$\sum_{\sigma,\tau=1}^{r} \frac{\partial F_\lambda}{\partial p_{n-r+\sigma}} \frac{\partial F_\mu}{\partial p_{n-r+\tau}} \,(p_{n-r+\sigma} + f_\sigma, p_{n-r+\tau} + f_\tau) = 0 ,$$

and that will imply (408.c).

([355]) That is because (411) can be rewritten as:

$$\sum_{\rho=1}^{r} \frac{\partial F_\lambda}{\partial p_{n-r+\rho}} \left(\frac{\partial f_\rho}{\partial t} + (\bar{H}, p_{n-r+\rho} + f_\rho) \right) = 0 ,$$

which implies that:

$$\frac{\partial f_\rho}{\partial t} + (\bar{H}, p_{n-r+\rho} + f_\rho) = 0 ,$$

as well as (412).

(412) $$\frac{\partial f_\rho}{\partial t} - \frac{\partial H}{\partial q_{n-r+\rho}} + \{\bar{H}, f_\rho\} = 0 \qquad (\rho = 1, \ldots, r),$$

in which the curly brackets mean the same abbreviations as in (409.a). In so doing, one employs the relations:

(413) $$\frac{\partial \bar{H}}{\partial p_\rho} = \frac{\partial H}{\partial p_\rho} - \sum_{\sigma=1}^{r} \frac{\partial H}{\partial p_{n-r+\sigma}} \frac{\partial f_\sigma}{\partial p_\rho}, \qquad \frac{\partial \bar{H}}{\partial q_\rho} = \frac{\partial H}{\partial q_\rho} - \sum_{\sigma=1}^{r} \frac{\partial H}{\partial p_{n-r+\sigma}} \frac{\partial f_\sigma}{\partial q_\rho}.$$

If one now introduces all of those relations into the given canonical system then that will give the relations:

(414) $$\begin{cases} dq_\rho = \dfrac{\partial H}{\partial p_\rho} dt + \sum_{\sigma=1}^{r} \dfrac{\partial f_\sigma}{\partial p_\rho} dq_{n-r+\sigma}, \\ dp_\rho = -\left(\dfrac{\partial H}{\partial q_\rho} dt + \sum_{\sigma=1}^{r} \dfrac{\partial f_\sigma}{\partial q_\rho} dq_{n-r+\sigma} \right) \end{cases} \qquad (\rho = 1, \ldots, n - r),$$

in place of the first $(n - r)$ pairs of equations, to which one must add the r differential equations:

(415) $$dq_{n-r+\sigma} = \frac{\partial H}{\partial p_{n-r+\sigma}} dt \qquad (\sigma = 1, \ldots, r),$$

along with the r equations (408.b). The expressions (408.b) have likewise been substituted for $p_{n-r+1}, \ldots, p_n$ in their right-hand sides.

Equations (414) also represent a completely-integrable system of total differential equations here, since the associated $(r + 1)$ partial differential equations:

(416) $$\begin{cases} \dfrac{\partial \Phi}{\partial t} + \{\bar{H}, \Phi\} = 0, \\ \dfrac{\partial \Phi}{\partial q_{n-r+\sigma}} + \{f_\sigma, \Phi\} = 0 \end{cases} \qquad (\sigma = 1, \ldots, r)$$

define a complete (and in fact **Jacobi**) system ([356]). The **Pfaff**ian equations (416) will then possess ∞^{2n-2r} M_{r+1} as solutions, and those M_{r+1} will be precisely the characteristic M_{r+1}. That is because,

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([356]) That is because, on the one hand, one has from (414) [which follows in the same way as in ([348])] that:

$$\frac{\partial}{\partial t}\{f_\sigma, F\} - \frac{\partial}{\partial q_{n-r+\sigma}}\{H, F\} = \left\{ \frac{\partial f_\sigma}{\partial t} - \frac{\partial \bar{H}}{\partial q_{n-r+\sigma}} + \{\bar{H}, f_\sigma\}, F \right\} = 0.$$

On the other hand:

on the one hand, the integral curves of the canonical system traverse it, since one will get them when one adds equations (415) to (414). On the other hand, the orbits of each of the r one-parameter groups will also lie on them. That is because when one starts from integrals in the form (408.b), each of which is an integral curve of one of the r systems:

$$\frac{dp_\rho}{dq_{n-r+\sigma}} = \frac{\partial f_\sigma}{\partial q_\rho}, \qquad \frac{dq_\rho}{dq_{n-r+\sigma}} = -\frac{\partial f_\sigma}{\partial p_\rho} \qquad (\rho = 1, \ldots, n-r),$$

they will traverse the $M_{2n+1-2r}$ that one obtains when one adds the equations:

$$t = \text{const.}, \quad q_{n-r+1} = \text{const.}, \ldots, q_{n-r+\sigma-1} = \text{const.}, \quad q_{n-r+\sigma+1} = \text{const.}, \ldots, q_n = \text{const.}$$

$$(\sigma = 1, \ldots, r)$$

to the relations (408.b).

The system (414) can also be converted into a canonical system with 2 $(n - r)$ unknown functions in precisely the same way as one did with the system (393). That is because due to the complete integrability of (414), one can find the values of $p_1, \ldots, p_{n-r}, q_1, \ldots, q_{n-r}$ at an arbitrary point P of the M_{r+1} from its values at a given point P in such a way that one connects P_0 and P with an arbitrary curve:

(417) $$q_{n-r+1} = q_{n-r+1}(\lambda), \quad \ldots, \quad q_n = q_n(\lambda), \quad t = t(\lambda)$$

and calculates the change in the $p_1, \ldots, p_{n-r}, q_1, \ldots, q_{n-r}$ along that curve. However, along that curve, the completely-integrable system (414) will go to the ordinary canonical system:

(418) $$\frac{dq_\rho}{d\lambda} = \frac{\partial K}{\partial p_\rho}, \quad \frac{dp_\rho}{d\lambda} = -\frac{\partial K}{\partial q_\rho} \qquad (\rho = 1, \ldots, n-r),$$

in which one has set:

(419) $K(p_1, \ldots, p_{n-r}, q_1, \ldots, q_{n-r}, \lambda)$

$$\frac{\partial}{\partial q_{n-r+\sigma}}\{f_\tau, F\} - \frac{\partial}{\partial q_{n-r+\tau}}\{f_\sigma, F\}$$
$$= \left\{\frac{\partial f_\tau}{\partial q_{n-r+\sigma}} - \frac{\partial f_\sigma}{\partial q_{n-r+\tau}}, F\right\} + \{f_\tau, \{F, f_\sigma\}\} - \{f_\sigma, \{F, f_\tau\}\}$$
$$= \left\{\frac{\partial f_\tau}{\partial q_{n-r+\sigma}} - \frac{\partial f_\sigma}{\partial q_{n-r+\tau}} + \{f_\tau, f_\sigma\}, F\right\} = 0,$$

since the first quantity in that **Poisson** bracket is zero, from (409).

$$= \bar{H}\,(p_1,\ldots,p_{n-r},q_1,\ldots,q_{n-r},q_{n-r+1}(\lambda),\ldots,q_n(\lambda),t\,(\lambda))\frac{dt}{d\lambda}+\sum_{\sigma=1}^{r} f_\sigma \frac{dq_{n-r+\sigma}}{d\lambda}\,,$$

corresponding to (396) ([357]).

The ($2n$ – $2r$) integrals of (418) produce immediate integrals of the completely-integrable system (414), as well as integrals of the original given canonical system that are in involution with the integrals (408), and the system will go to (418) with their help ([358]). That is because an integral of the canonical system (418) [the completely-integrable system (414), resp.] simultaneously satisfies the (r + 1) partial differential equations (416), the first of which says that it is also an integral of the original canonical system after the reformation, while the following r equations express the idea that it will be in involution with the r integrals (408) after the reformation ([359]).

([357]) That is the so-called fundamental theorem of **S. Lie**. The **Hamilton-Jacobi** equation that belongs to (418):

$$\frac{\partial V}{\partial \lambda}+K\left(\frac{\partial V}{\partial q_1},\ldots,\frac{\partial V}{\partial q_{n-r}},q_1,\ldots,q_{n-r},\lambda\right)=0$$

is equivalent to the system of (r + 1) partial differential equations:

$$\frac{\partial S}{\partial t}+H\left(\frac{\partial S}{\partial q_1},\ldots,\frac{\partial S}{\partial q_n},q_1,\ldots,q_n,t\right)=0\,,$$

$$F_\rho\left(\frac{\partial S}{\partial q_1},\ldots,\frac{\partial S}{\partial q_n},q_1,\ldots,q_n,t\right)-c_\rho=0 \qquad (\rho = 1, \ldots, r).$$

S will emerge from V when one goes from the integral curves of the system (418) to the characteristic M_{r+1} . Cf., **S. Lie**, "Allgemeine Theorie der partiellen Differentialgleichungen erster Ordnung," Math. Ann. **9** (1876), pp. 245 = *Werke IV*, pp. 97, esp., **§ 10**, *Werke IV*, pp. 136.

([358]) They will then define a subgroup of the function group of the $2n$ integrals of the original canonical system that no longer includes the r integrals that are canonically conjugate to the r integrals (408) in involution, nor does it include the latter.

([359]) If one has solved the canonical system (418), i.e., one has determined the characteristic M_{r+1} :

(420)
$$\begin{cases} p_1 = p_1(t,q_{n-r+1},\ldots,q_n), & \ldots, \quad p_{n-r} = p_{n-r}(t,q_{n-r+1},\ldots,q_n), \\ q_1 = q_1(t,q_{n-r+1},\ldots,q_n), & \ldots, \quad q_{n-r} = q_{n-r}(t,q_{n-r+1},\ldots,q_n) \end{cases}$$

that are solutions of the completely-integrable system (414), then the solution to the given canonical system will naturally come down to a quadrature here, as well. That is because since the integral curves of a characteristic M_{r+1} all belong to the same field, the expression:

$$p_1\,dq_1 + \ldots + p_n\,dq_n - H\,dt = \left(\sum_{\rho=1}^{n-r} p_\rho \frac{\partial q_\rho}{\partial q_{n-r+1}}+p_{n-r+1}\right)dq_{n-r+1}+\cdots+\left(\sum_{\rho=1}^{n-r} p_\rho \frac{\partial q_\rho}{\partial q_n}+p_n\right)dq_n+\left(\sum_{\rho=1}^{n-r} p_\rho \frac{\partial q_\rho}{\partial t}-H^*\right)dt$$

will be the total differential of a function S of the (r + 1) variables t, q_{n-r+1}, …, q_n on M_{r+1} , in which the r integrals (408) will appear, along with the constants c_1, …, c_r . The individual integral curves on the M_{r+1} will then be fixed by the relations:

(420.a)
$$\frac{\partial S}{\partial c_1} = \gamma_1\,,\ \ldots,\ \frac{\partial S}{\partial c_r} = \gamma_r\,.$$

28. Simplifying the integration when one knows a function group of integrals. – The special case that was treated in the previous number, viz., when one knows a number of integrals of the canonical system that are all in involution with each other, does not generally exist. Rather, from the arguments in no. **26**, one would generally expect that one will construct a system of integrals that forms a function group from two (or more) integrals by repeatedly forming the **Poisson** bracket. Let:

$$(421) \qquad \begin{cases} F_1(p_1, \ldots, p_n, q_1, \ldots, q_n, t) = c_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ F_k(p_1, \ldots, p_n, q_1, \ldots, q_n, t) = c_k \end{cases}$$

be the integrals of one such function group, such that for any two of them, one has:

$$(421.a) \qquad (F_\lambda, F_\mu) = \varphi_{\lambda\mu}(F_1, \ldots, F_k) \qquad (\varphi_{\lambda\mu} = -\varphi_{\mu\lambda}),$$

in which zero or a constant can also appear in place of the function on the right-hand side. Since the k functions (421) are mutually independent, they define a *basis* for the function group.

In order to decide what simplifications of the integration can arise when one knows such a function group, **S. Lie** proceeded in such a way that he sought a so-called *canonical basis* ([360]) for the function group. He next had to establish whether there were functions in the function group that were all in involution with all functions in the basis (421) (and therefore with all functions of the function group) ([361]). Since one such distinguished function:

$$(422) \qquad U(F_1, \ldots, F_k)$$

must obviously satisfy the k equations ([362]):

Those r relations are equivalent to the integrals of the original canonical system that are canonically conjugate to the r initial integrals (408).

([360]) **S. Lie**, "Über partielle Differentialgleichungen erster Ordnung," Christiania Forhandlingar i Vidensk.Selsk. (1872), pp. 16 = *Werke III*, pp. 32, esp., **§ 3**, pp. 42.

([361]) The functions that are in involution with all functions of the given function group likewise define a function group, namely, the so-called *polar group* of the given function group [cf., II A 5 (**E. von Weber**), no. **40**]. The distinguished functions of a function group therefore simultaneously belong to the function group itself and its polar group. Cf., **S. Lie**, *loc. cit.* ([360]).

([362]) Naturally, the **Poisson** brackets in (423) are thought of as being replaced with the functions (421.a). Obviously, one will then have to deal with only k independent variables $F_1, \ldots, F_k$, instead of the $2n$ independent variables p_ρ, q_ρ.

The idea of going from the $2n$ independent variables p_ρ, q_ρ to the $F_1, \ldots, F_k$ as independent variables defined the nucleus of the starting point for **S. Lie**'s theory of function groups. In that sense, the relations (421.a) *define* the **Poisson** brackets for the independent variables and thus exhibit the generalization of the relations:

$$(p_\rho, p_\rho) = 0, \qquad (q_\rho, q_\rho) = 0, \qquad (p_\rho, q_\rho) = \begin{cases} 0 & (\rho \neq \sigma), \\ 1 & (\rho = \sigma), \end{cases}$$

(422.a) $(F_1, U) = 0\,, \qquad (F_2, U) = 0\,, \quad \ldots, \qquad (F_k, U) = 0\,,$

i.e., the *k* linear partial differential equations:

$$
(423) \qquad \begin{cases}
(F_1,F_1)\dfrac{\partial U}{\partial F_1}+(F_1,F_2)\dfrac{\partial U}{\partial F_2}+\cdots+(F_1,F_k)\dfrac{\partial U}{\partial F_k}=0\,, \\
(F_2,F_1)\dfrac{\partial U}{\partial F_1}+(F_2,F_2)\dfrac{\partial U}{\partial F_2}+\cdots+(F_2,F_k)\dfrac{\partial U}{\partial F_k}=0\,, \\
\dotfill \\
(F_k,F_1)\dfrac{\partial U}{\partial F_1}+(F_k,F_2)\dfrac{\partial U}{\partial F_2}+\cdots+(F_k,F_k)\dfrac{\partial U}{\partial F_k}=0\,,
\end{cases}
$$

and conversely every solution of that system is a distinguished function, the existence of distinguished functions of the functions will depend upon whether the skew-symmetric determinant:

$$
(423.a) \qquad D = \begin{vmatrix} (F_1,F_1) & \cdots & (F_1,F_k) \\ \vdots & \ddots & \vdots \\ (F_k,F_1) & \cdots & (F_k,F_k) \end{vmatrix}
$$

is equal to zero or non-zero. If it is non-zero then the function group will possess *no* distinguished functions. By contrast, if $D = 0$ then *at least one* distinguished function will exist ([363]). Now, since the skew-symmetric certainly vanishes when *k* is an odd number, there will be at least one distinguished function in the *k*-parameter function group as long as *k* is an odd number. Moreover, since as long one minor of a skew-symmetric determinant is non-zero, there will always be at least one non-zero principal of equal order ([364]), as well, while on the other hand, the principal minors of a skew-symmetric determinant are again skew-symmetric determinants, it will further follow that ([365]): If *k* is an odd number then the function group with the basis (421) will have either 1, 3,

---

that were true for the original variables, in which one must imagine that the skew-symmetric system of $\varphi_{\lambda\mu}$ has been given. Obviously, those functions $\varphi_{\lambda\mu}$ cannot be chosen to be entirely arbitrary, rather they must satisfy a system of first-order partial differential equations, because one must also require the general validity of the ***Jacobi*** *identity* for those generalized **Poisson** brackets if one would like to construct the theory of function groups. That original thought of **Lie** [cf., **S. Lie**, "Zur Theorie der Transformationsgruppen," Christiania Forhandlingar (1888), no. **13** = *Werke V*, pp. 533, esp., **§ II**, pp. 554] was not established in the later presentations of the theory of function groups. It was first moved back to the focus of attention by **C. Carathéodory**, *Variationsrechung*, Leipzig and Berlin, 1935, Chap. 9.

([363]) **S. Lie**, *loc. cit.* ([360]), *Werke III*, pp. 41.

If time does not appear explicitly then one will have in the energy integral:

$$H = \text{const.}$$

as a first integral that will be a distinguished function in every group that it belongs to.

([364]) Cf., e.g., **G. Kowalewski**, *Einführung in die Determinantentheorie*, Leipzig, 1909, pp. 145.

([365]) **S. Lie**, "Begründung einer Invariantentheorie der Berührungstransformationen." Math. Ann. **8** (1875), pp. 215 = *Werke IV*, pp. 1, cf., esp., *Werke IV*, pp. 44.

or 5 distinguished functions. If k is an even number then the number of distinguished functions will amount to either 0, 2, or 4, etc. In order to fix the number of distinguished functions in the function group, one then needs only to examine the determinant (423.a) (its odd-order principal minors, resp.). If the determinant (423.a) has rank r, in which r must be *an even number*, then there will be $(k - r)$ distinguished functions in the function group. Naturally, those distinguished functions are also in involution with each other.

In order to find the $(k - r)$ distinguished functions of the function group, one must select r linearly-independent equations from the k equations of the system (423), so perhaps (with a suitable numbering) the r linear partial differential equations:

$$(424) \qquad \left\{ \begin{array}{l} (F_1, F_1)\dfrac{\partial U}{\partial F_1} + (F_1, F_2)\dfrac{\partial U}{\partial F_2} + \cdots + (F_1, F_k)\dfrac{\partial U}{\partial F_k} = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (F_r, F_1)\dfrac{\partial U}{\partial F_1} + (F_r, F_2)\dfrac{\partial U}{\partial F_2} + \cdots + (F_r, F_k)\dfrac{\partial U}{\partial F_k} = 0. \end{array} \right.$$

They define a complete system ([366]) [cf., II A 5 (**E. von Weber**), no. **13**], and the determination of the $(k - r)$ unknown functions will then require that one must determine an integral of a system of $(k - r)$ [$(k - r - 1)$, $(k - r - 2)$, …, 2, 1, resp.] first-order ordinary differential equations for each of them, so an operation of order $(k - r)$, $(k - r - 1)$, …, 2, 1, resp., in the usual terminology ([367]).

([366]) That is because if one denotes the left-hand sides by $X_1 U$, …, $X_r U$ then one will indeed have:

$$X_1 U = (F_1, U), \ldots, X_r U = (F_r, U),$$

and thus, from the **Jacobi** identity, one will have the bracket expression:

$$\begin{aligned} (X_\lambda, X_\mu)\, U &= (F_\lambda, (F_\mu, U)) - (F_\mu, (F_\lambda, U)) \\ &= (F_\lambda, (F_\mu, U)) = (\varphi_{\lambda\mu}(F_1, \ldots, F_k), U) \\ &= \frac{\partial \varphi_{\lambda\mu}}{\partial F_1}(F_1, U) + \cdots + \frac{\partial \varphi_{\lambda\mu}}{\partial F_k}(F_k, U), \end{aligned}$$

and it will follow from this that:

$$(X_\lambda, X_\mu)\, U = A_1 X_1 U + \ldots + A_r X_r U.$$

([367]) The general procedure employs the following ideas [cf., II A 5 (**E. von Weber**), nos. **13** and **15**]. One then gives the complete system (424), which one solves for r of the derivatives – say, $\dfrac{\partial U}{\partial F_1}, \ldots, \dfrac{\partial U}{\partial F_r}$ – the form of a **Jacobi** system:

The $(k - r)$ distinguished functions that are thus determined can then be taken from the basis for the function group (421), which will then take the form:

(425) $$U_1, \ldots, U_{k-2s}, \qquad F_1^*, \ldots, F_{2s}^*$$

when one sets the even number $r = 2s$. Therefore, one now has:

(425.a) $$\begin{cases} (U_\lambda, U_\mu) = 0, \quad (U_\lambda, F_\sigma^*) = 0, \\ (F, F_\tau^*) = \varphi_{\sigma\tau}(U_1, \ldots, U_{k-2}, F_1^*, \ldots, F_{2s}^*). \end{cases}$$

With the help of the integrals $U_1 = c_1, \ldots, U_{k-2s} = c_{k-2s}$ that arise from the $(k - 2s)$ distinguished functions, one can now reduce the given system (384) to a canonical system with $2(n - k + 2s)$ unknown functions, say:

(424.a) $$\begin{cases} \dfrac{\partial U}{\partial F_1} + a_{11}\dfrac{\partial U}{\partial F_{r+1}} + \cdots + a_{1.k-r}\dfrac{\partial U}{\partial F_k} = 0, \\ \dfrac{\partial U}{\partial F_2} + a_{21}\dfrac{\partial U}{\partial F_{r+1}} + \cdots + a_{2.k-r}\dfrac{\partial U}{\partial F_k} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{\partial U}{\partial F_r} + a_{r1}\dfrac{\partial U}{\partial F_{r+1}} + \cdots + a_{r.k-r}\dfrac{\partial U}{\partial F_k} = 0. \end{cases}$$

One will then introduce new independent variables $\tau_2, \ldots, \tau_r$ in [cf., II A 5 (**E. von Weber**), no. **17**] in place of $F_2, \ldots, F_r$ by setting:

$$F_2 - F_2^{(0)} = \tau_2 \left(F_1 - F_1^{(0)}\right), \ldots, F_r - F_r^{(0)} = \tau_r \left(F_1 - F_1^{(0)}\right).$$

(If one interprets $F_1, \ldots, F_r$ as ordinary rectangular coordinates of an R_r then $\tau_2, \ldots, \tau_r$ will represent the coordinates of the line bundle through the point $F_1^{(0)}, \ldots, F_r^{(0)}$.) If one then sets $F_1 - F_1^{(0)} = \tau_1$ then the system (424.a) will go to a corresponding **Jacobi** system:

(424.b) $$\begin{cases} \dfrac{\partial U}{\partial \tau_1} + b_{11}\dfrac{\partial U}{\partial F_{r+1}} + \cdots + b_{1.k-r}\dfrac{\partial U}{\partial F_k} = 0, \\ \dfrac{\partial U}{\partial \tau_2} + b_{21}\dfrac{\partial U}{\partial F_{r+1}} + \cdots + b_{2.k-r}\dfrac{\partial U}{\partial F_k} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{\partial U}{\partial \tau_r} + b_{r1}\dfrac{\partial U}{\partial F_{r+1}} + \cdots + b_{r.k-r}\dfrac{\partial U}{\partial F_k} = 0, \end{cases}$$

which has the property: Every solution of the first equation of the system is likewise a solution of the other $(r - 1)$ partial differential equations.

However, that *one* partial differential equation is equivalent to a system of $(k - r)$ first-order ordinary differential equations.

(426) $$\frac{dq_\rho}{d\lambda} = \frac{\partial K}{\partial p_\rho}, \qquad \frac{dp_\rho}{d\lambda} = -\frac{\partial K}{\partial q_\rho} \qquad (\rho = 1, 2, \ldots, n-k+2s),$$

as in no. **27**. In that way, since the integrals $F_1^* = \gamma_1, \ldots, F_{2s}^* = \gamma_{2s}$ are in involution with the integrals $U_1 = c_1, \ldots, U_{k-2s} = c_{k-2s}$ that are used for the reduction, they will go to $2s$ integrals:

(427) $$G_\sigma(p_1, \ldots, p_{n-k+2s}, q_1, \ldots, q_{n-k+2s}, \lambda) = \gamma_\sigma \qquad (\sigma = 1, \ldots, 2s)$$

of the canonical system (426). They represent a function group:

(426.a) $$(G_\sigma, G_\tau) = \psi_{\sigma\tau}(G_1, \ldots, G_{2s}),$$

in which *no* distinguished function exists now ([368]).

However, according to **S. Lie**, the basis for a function group that has no distinguished functions can be put into *canonical form*, i.e., one can introduce a basis:

(428) $$\begin{cases} \Phi_1, & \Phi_2, & \cdots & \Phi_s, \\ \Psi_1, & \Psi_2, & \cdots & \Psi_s \end{cases}$$

for the function group (427) such that the functions Φ_ρ and Ψ_ρ will represent canonically-conjugate integrals of (426) when they are set equal to constants, such that one will have:

(428.a) $$\begin{cases} (\Phi_\rho, \Phi_\sigma) = 0, \quad (\Psi_\rho, \Psi_\sigma) = 0 \qquad (\rho, \sigma = 1, \ldots, s), \\ (\Phi_\rho, \Psi_\sigma) = \begin{cases} 0 & (\rho \neq \sigma) \\ 1 & (\rho = \sigma) \end{cases}. \end{cases}$$

In order to achieve that, one must start with one of the functions G_ρ – say, G_1 – and then determine the function $K(G_1, \ldots, G_{2s})$ in such a way that one will have:

$$(G_1, K) = (G_1, G_1)\frac{\partial K}{\partial G_1} + (G_1, G_2)\frac{\partial K}{\partial G_2} + \cdots + (G_1, G_{2s})\frac{\partial K}{\partial G_{2s}} = 1.$$

The two functions:

$$G_1 = \Phi_1, \qquad K = \Psi_1$$

obviously define a two-parameter function group by themselves that is included in the $2s$-parameter function group (427) as a *subgroup*. If one now takes those two functions Φ_1 and Ψ_1 from the basis of the function group (427) then the remaining $(2s-2)$ functions in the basis can be chosen such that they will be in involution with Φ_1 and Ψ_1 and likewise define a subgroup of $2s$-

([368]) One can easily verify that by direct calculation.

parameter function group (427) ([369]). That $(2s - 2)$-parameter subgroup can be treated in the same way as the $(2s)$-parameter group (426), such that one will arrive at the canonical basis (428) for the group (427) in s steps.

There will then be a system in involution of precisely s functions in the function group with no distinguished functions $G_1, \ldots, G_{2s}$. Such a thing is defined by, e.g., the functions $\Phi_1, \ldots, \Phi_s$ of the canonical basis (428). On the other hand, a system in involution that is included in the function group (427) cannot consist of more than s functions. The original k-parameter function group (421) that was given will then contain a system in involution of $(r + s)$ parameters $(k = r + 2s)$, while no system in involution will have more than $(r + s)$ parameters.

From no. **27**, an $(r + s)$-parameter system in involution that is contained in the k-parameter function group will make it possible to convert the given canonical system into one that contains $2\,(r + s)$ unknown functions. The remaining s integrals that are included in the function group are, in a certain sense, worthless for the integration process ([369.a]). They belong to the $(r + s)$ integrals that lie on the individual characteristic M_{r+s+1} that are the solutions of the simplified canonical system [the corresponding completely-integrable system with $(r + s + 1)$ independent variables, resp.], which determine the ∞^{r+s} integral curves. When one recalls the equations for the M_{r+s+1}, they must then be included among the ∞^{r+s} integrals of the form (420.a) in ([359]).

Therefore, if a k-parameter function group of integrals is known for the given canonical system then if one is to integrate the system, one will generally have to look for one of the $(k - s)$-parameter systems in involution of greatest extent that is contained in the group, and one can reduce the given canonical system to a canonical system that includes $2\,(k - s)$ fewer unknown functions by using those $(k - s)$ integrals. One will then conclude the integration in the way that was given at the conclusion of no. **27**, whereby one must also finally recover the s unused integrals of the canonical basis for the function group. In order to find the system in involution of greatest extent, one must

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([369]) **S. Lie**, "Begründung einer Invariantentheorie der Berührungstr.," Math. Ann. **8** (1875), pp. 89 = *Werke IV*, pp. 1 (see esp., pp. 46) showed the following: He imagined determining the polar groups $H_1, \ldots, H_{2n-2s}$ to the $2s$-parameter function groups $G_1, \ldots, G_{2s}$, whose functions would satisfy:

$$(G_\sigma, H_\rho) = 0 ,$$

from the definition of the polar group. If one now adds the two functions Φ_1 and Ψ_1 to the function group of the H_ρ then another function group will arise, and indeed one whose number of parameters is equal to $(2n - 2s + 2)$, and for which the functions:

$$\Phi_1, \Psi_1, H_1, \ldots, H_{2n-2s}$$

will represent a basis. Its polar group will then be a $(2s - 2)$-parameter function group, and any basis $G_1^*, \ldots, G_{2s-2}^*$ for that function group will fulfill the conditions:

$$(G_\sigma^*, \Phi_1) = 0 , \qquad (G_\sigma^*, \Psi_1) = 0 , \qquad (G_\sigma^*, H_\rho) = 0 .$$

The functions $G_1^*, \ldots, G_{2s-2}^*$ must obviously be contained in the group $G_1, \ldots, G_{2s}$, since it indeed subsumes all of the functions for which one has $(G_\sigma, H_\rho) = 0$. They can then be taken from the basis for the given function group, along with Φ_1, Ψ_1, and will then define a subgroup with the required property in their own right.

([369.a]) On this topic, one can cf., the article by **G. D. Mattioli**, "Sulla riduzione di rango dei sistemi canonici mediante integrali generici," Roma Lin. Rend. (6) **15** (1932), pp. 437.

solve a series of complete systems of first-order partial differential equations with the independent variables $F_1, \ldots, F_k$ of the original basis (421) of the function group. Namely, if p functions:

$$V_1 (F_1, \ldots, F_k) , \qquad \ldots, \qquad V_p (F_1, \ldots, F_k)$$

have been found, every two of which are in involution, then one will get a further function $V (F_1, \ldots, F_k)$ that is in involution with all of them as a solution to the complete system:

$$(V_1, V) = 0 , \quad \ldots, \quad (V_p, V) = 0 ,$$

although those equations will not generally be mutually independent ([369.b]), such that one must single out individual equations from them.

If one would like to systematically enumerate the operations that are required for the integration, in the spirit of **Lie**, then it would be convenient for one to imagine that the integration proceeds in the following way: One first determines the $(k - 2s)$ distinguished functions that are included in the group, which will require that one must solve a system of $(k - 2s)$, $(k - 2s - 1)$, …, 3, 2, 1 ordinary differential equations in each case, so an operation of order of $(k - 2s)$, $(k - 2s - 1)$, …, 3, 2, 1, with **Lie**'s terminology. One then goes on to exhibit the canonical basis for the group, which will come down to an operation of order $(2s - 2)$, $(2s - 4)$, …, 4, 2 ([370]).

That general line of reasoning will determine the number of necessary operations in the most unfavorable case. However, some significant simplifications can enter in particular cases, namely, when one knows subgroups of the function group (421) ([371]). One can next determine whether such a subgroup possesses distinguished functions and how many of those distinguished functions are, at the same time, distinguished functions of the total k-parameter function group. If:

$$G_1, G_2, \ldots, G_i, G_{i+1} , \ldots, G_k \tag{429}$$

([369.b]) For example, if V_ρ is a distinguished function of the function group then the equation $(V_\rho , V) = 0$ will be fulfilled identically for any function $V (F_1, \ldots, F_k)$.

([370]) That is because when one starts from G_1, one must look for a function:

$$K_1 (G_1, \ldots, G_{2s})$$

that fulfills the condition:

$$(G_1, K) = (G_1, G_1)\frac{\partial K_1}{\partial G_1} + (G_1, G_2)\frac{\partial K_1}{\partial G_2} + \cdots + (G_1, G_{2s})\frac{\partial K_1}{\partial G_{2s}} = 0 ,$$

i.e., since $(G_1, G_1) = 0$, it will be a function that satisfies a linear partial differential equation with $(2s - 1)$ independent variables, such that the determination of that function will come down to determining an integral of a system of $(2s - 2)$ first-order ordinary differential equations. One then determines a function $K_2 (G_1, \ldots, G_{2s})$ that satisfies the two conditions:

$$(G_1, K_2) = 0 , \qquad (K_1, K_2) = 0 .$$

Those two linear partial differential equations with $2s$ independent variables, which define a complete system, possess the two known solutions $K_2 = G_1$ and $K_2 = K_1$. One will then have to solve a system of $(2s - 2)$ first-order ordinary differential equations, for which one already knows two integrals, which will come down to the solution of a system of $(2s - 4)$ first-order ordinary differential equations [cf., II A 5 (**E. von Weber**), nos. **13** and **11**].

([371]) On this subject, cf., **S. Lie**, *loc. cit.* ([369]), esp. *Werke IV*, pp. 51, *et seq.*

is a basis for the k-parameter function group (421), and:

(429.a) $$G_1, G_2, \ldots, G_i$$

is a basis for the i-parameter function subgroup then the common distinguished functions must be solutions of the $(2k - i)$ partial differential equations:

$$(G_1, U) = 0\,, \quad (G_2, U) = 0\,, \quad \ldots, \quad (G_k, U) = 0\,,$$

$$\frac{\partial U}{\partial G_{i+1}} = 0\,, \quad \ldots, \quad \frac{\partial U}{\partial G_k} = 0\,.$$

One can decide, with no further analysis, how large the number w of its common solutions is and then find the distinguished functions by operations of order w, $w - 1$, $w - 2$, …, 2, 1. Hence, one will then already know w of the distinguished functions of the function group (429), such that the problem of determining its distinguished functions will be reduced to lower-order operations. However, even for the further operations for determining the largest-possible system of involution in the function group (429), it will be preferable in situations to first ascertain all distinguished functions of the function subgroup (429.a) (to determine the largest-possible system in involutions in that subgroup, resp.). Naturally, that will also be a system in involution for the k-parameter function group (429) then, such that one will already know a number of solutions for the determinations of the largest-possible system in involution in that function group. Naturally, one can begin with the determinations of the still-unknown distinguished functions of the group (429) for the determination of that largest-possible system in involution that is contained in (428).

It is obvious how to diminish the order of the operations even further when a function subgroup is again contained in the subgroup in its own right ([372]).

29. Integrals of special form. In particular, ones that are rational in the impulses. – The general arguments in regard to the significance of knowing integrals for a given mechanical problem will be completed by investigations that bring integrals of a particular form under consideration. In so doing, one directs one's attention to the dependency of the integral on the impulse components. That is obvious, since indeed, of the ten general integrals of the **Galilei** group, which appear, in total or in part, in many mechanical problems, nine of them, namely, the center of mass integrals and the area integrals, are linear in the impulses, while the energy integral proves to be quadratic in the impulse components.

([372]) Obviously, the distinguished functions bring with them a complication of the integration problem, in a certain sense, when their number is large, and in that way, they will diminish the advantage in the integration that occurs when one knows the function group. Since there are only systems in involution with n parameters in the $2n$-parameter function group of *all* integrals, the number of distinguished functions in a function group must naturally always remain below n.

If one makes that standpoint one's own then the *integrals that are linear and homogeneous in the impulse components*, so the ones that generally possess the form ([373]):

$$a^1(q_1,\dots,q_n)\,p_1+\cdots+a^n(q_1,\dots,q_n)\,p_n = \text{const.}, \tag{430}$$

will prove to be the simplest class of integrals. According to no. **25**, the one-parameter group of transformations of the p_ρ, q_ρ that take of the space-time lines to other ones that belong to such an integral (430) has the infinitesimal transformation:

$$\delta q_1 = a^1(q_1,\dots,q_n)\,\delta\alpha,\ \dots,\ \delta q_n = a^n(q_1,\dots,q_n)\,\delta\alpha \qquad (\delta t = 0), \tag{431}$$

$$\left\{\begin{aligned} \delta p_1 &= -\left(\frac{\partial a^1}{\partial q_1}p_1+\cdots+\frac{\partial a^n}{\partial q_1}p_n\right)\delta\alpha, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \delta p_n &= -\left(\frac{\partial a^1}{\partial q_n}p_1+\cdots+\frac{\partial a^n}{\partial q_n}p_n\right)\delta\alpha. \end{aligned}\right. \tag{431.a}$$

However, equations (431) define a system of n first-order differential equations between the $q_1, \dots, q_n$ in their own right, and in which the impulse components $p_1, \dots, p_n$ do not enter at all. Thus, they likewise already represent the infinitesimal transformation of a one-parameter transformation group by themselves, and indeed a *transformation group for the position coordinates* $q_1, \dots, q_n$ *alone.* Since that one-parameter group also takes each space-time line of the motion in the R_{n+1} of the $(q_1, \dots, q_n, t)$ to another such space-time line, an integral that is linear in the impulse components will imply the existence of a one-parameter group of transformations of only the position coordinates that take all of the space-time line of the motion in the R_{n+1} of the $(q_1, \dots, q_n, t)$. Equations (431.a) represent the infinitesimal transformation of the impulse components that they produce ([374]).

Even when one has the more general linear integral:

([373]) In the ordinary case where the kinetic energy is a quadratic form $T = \frac{1}{2}\sum_{\lambda,\mu=1}^{n} g_{\lambda\mu}\,\dot q_\lambda\,\dot q_\mu$, so the M_n of the $q_1, \dots, q_n$ is a **Riemann**ian space with the arc-length element $ds^2 = \sum_{\lambda,\mu=1}^{n} g_{\lambda\mu}\,dq_\lambda\,dq_\mu$, the following is true: The left-hand side of (430) must be a scalar. Therefore, since the impulse components $p_1, \dots, p_n$ represent the components of a covariant vector, the $a^1, \dots, a^n$ will represent the components of a contravariant vector.

([374]) One can arrive at it in such a way that one first derives the extended point transformation that produces the infinitesimal transformation of the velocity components $\dot q_1, \dots, \dot q_n$ from the transformation of the $q_1, \dots, q_n$ and then goes over to the infinitesimal transformation of the impulse components by means of the relations between the velocity and impulse components.

In particular, if the **Lagrangian** function L is quadratic in the velocity components then the impulse components will be connected with the impulse components in such a way that an integral that is linear in the impulse components will, at the same time, represent an integral that is linear in the velocity components.

$$a^1(q_1,\ldots,q_n,t)\,p_1+\cdots+a^n(q_1,\ldots,q_n,t)\,p_n+a(q_1,\ldots,q_n,t)\;=0\,,$$

instead of (430), the relations:

$$\text{(431.b)}\qquad \delta q_1 = a^1(q_1,\ldots,q_n)\,\delta\alpha,\;\ldots,\;\delta q_n = a^n(q_1,\ldots,q_n)\,\delta\alpha\,,\qquad \delta t = 0\,,$$

which correspond to (431), will represent an infinitesimal point transformation in the R_{n+1} of the $q_1, \ldots, q_n, t$ that transforms the system of space-time lines of motion into itself.

One can then introduce new coordinates $q_1^*, \ldots, q_n^*$ in place of the position coordinates $q_1, \ldots, q_n$ such that the transformation group of the position coordinates will become the group of "parallel translations" in the direction of one of the new coordinates, say q_n^* ([375]). That is to say, however: With the new coordinates, q_n^* will be a cyclic coordinate of the system. Thus, if a mechanical problem possesses an integral that is linear in the impulse components then it will go to a problem with one cyclic coordinate when one introduces suitable new *position coordinates* (by contrast, cf., the more general conception of things in no. **9**).

If several integrals exist that are linear in the impulse components then one can indeed employ any one of them in order to introduce a cyclic coordinate. Meanwhile, for the canonical system that has been reduced by two units with the help of the cyclic coordinate, only the integrals that are in involution will again be integrals. If r linear integrals that are in involution exist then they will determine an r-parameter group of transformations of the position coordinates $q_1, \ldots, q_n$. If one puts into the normal form of parallel translations in r directions then the system of equations of motion will contain r cyclic coordinates after one introduces the new coordinates ([375.a]).

If one would like to employ linear integrals directly to simplify the equations of motion then that would suggest that one might appeal to the concept of *quasi-coordinates* (no. **2**). If one introduces them in such a way that the individual linear integral is equivalent to the constancy of a quasi-component of *impulse* then one will have a direct generalization of the concept of cyclic

([375]) In order to do that, one must form the $(n-1)$ first integrals of the differential equations (431.b) that are independent of α and introduce those integrals as the first $(n-1)$ of the new coordinates:

$$\varphi_1(q_1, \ldots, q_n, t) = q_1^*,\qquad \ldots,\qquad \varphi_{n-1}(q_1, \ldots, q_n, t) = q_{n-1}^*,$$

to which one then adds:

$$q_n^* = \int \frac{\delta q_n}{a^n(q_1,\ldots,q_n,t)}\,,$$

where the $q_1, \ldots, q_n$ in a^n are replaced with $q_1^*, \ldots, q_{n-1}^*$. In fact, one will then have:

$$\delta q_1^* = 0\,,\;\ldots,\;\delta q_{n-1}^* = 0\,,\quad \delta q_{n-1}^* = 1\cdot\delta\alpha\,.$$

M. Lévy has discussed that result for the special problem of determining the geodetic lines of an arc-length element in "Sur les conditions que doit remplir un espace, pour qu'on y puisse déplace un système invariable…," C. R. Acad. Sci. Paris **86** (1878), pp. 875.

([375.a]) Cf., on this, also **É. Delassus**, "Sur les integrales linéaires des équations de Lagrange," C. R. Acad. Sci. Paris **153** (1911), pp. 40.

coordinates ([375.b]). However, can also employ the left-hand sides of the integrals that are linear in the $\dot{q}_\rho$ directly to define the quasi-components of the *velocity*. For example, if k integrals that are linear in the $\dot{q}_\rho$ exist:

$$a_{11}\,\dot{q}_1 + \cdots + a_{1n}\,\dot{q}_n + a_{1,n+1} = c_1\,,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{k1}\,\dot{q}_1 + \cdots + a_{kn}\,\dot{q}_n + a_{k,n+1} = c_k$$

then one can, corresponding to (39) and (39.a), introduce quasi-components of the velocity ω_ρ , of which:

$$\omega_{n-k+1} = c_1\,,\ \ldots,\ \omega_n = c_k$$

are constant and give the equations of motion a form like (40.a).

P. Woronetz, has appealed to the process that he developed as an Ansatz for the equations of motion with non-holonomic constraints (cf., no. **4**) in order to simplify the equations of motion when linear integrals exist ([375.c]).

One can generally pose the problem of characterizing the mechanical problems whose equations of motion possess linear integrals or more general integrals of a prescribed form. Since the individual mechanical problem is constrained, on the one hand, by the form of the kinetic energy T (by the associated arc-length element $ds = \sqrt{2T}\,dt$, when one has a **Riemann**ian space, resp.), and on the other hand, by the force components $Q_1, \ldots, Q_n$, which possibly arise from a potential Φ, the conditions that characterize the appearance of integrals of a particular form for the equations of motion:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\rho}\right) - \frac{\partial T}{\partial q_\rho} = Q_\rho \qquad (\rho = 1, \ldots, n) \tag{432}$$

will split into two classes, one of which refers to the kinetic energy and the other of which refers to the force components. The conditions in the first class will then coincide with the one that the differential equations of the geodetic lines of the arc-length element must possess an integral of the prescribed form ([376]).

([375.b]) Cf., on this, **G. Hamel**, "Über die virtuellen Verschiebungen in der Mechanik," Math. Ann. **59** (1904), pp. 416, esp., pp. 430, in which the connection between the constancy of a quasi-impulse and an infinitesimal point transformation is treated. **Hamel**'s considerations must be inverted for the consideration of linear integrals.

([375.c]) **P. Woronetz**, "Transformation der dynam. Gleich. vermittels linearen Integrale der Bewegung," Kiev (1906), in Russian, = Fortschr. d. Math. **37** (1906), pp. 728; cf., also **A. Bilimovitch**, "Der Bewegungsgleichungen konservativer Syst. mit linearen Bewegungenintegralen," Math. Ann. **69** (1910), pp. 586.

([376]) That connection was treated systematically for the linear integrals by **M. Lévy**. Cf., **M. Lévy**, "Sur les conditions pour qu'une forme quadratique de n différentielles puisse être transformées de façon que ses coefficients perdent une partie de la totalité des variables qu'ils renferment," C. R. Acad. Sci. Paris **86** (1878), pp. 463, in which he expressed the theorem in this book about the connection between linear integrals and cyclic coordinates for the geodetic lines in R^n . Moreover, one already finds the theorem that a two-dimensional surface can be bent into a surface of revolution (angle of rotation – cyclic coordinate) if and only if the equations of the geodetic lines possess an integral that is linear in the impulse (velocity, resp.) components in the *Thèse* of **F. Massieu**, "Sur les intégrales

Moving beyond the linear integrals, one must next further ask about the integrals of the equations of motion that are *quadratic* in the impulse components. The simplest of those integrals is the *energy integral:*

(434) $$H = T + \Phi = \tfrac{1}{2}\sum g^{\lambda\mu} p_\lambda\, p_\mu + \Phi = k$$

(its generalization for rheonomic constraints, resp.). Its existence depends upon the fact that the independent variable, namely, time t, plays the role of a cyclic coordinate (cf., no. **10**). Further quadratic integrals are known in the case when the **Hamilton-Jacobi** differential equation can be integrated by separating the variables (cf., no. **19**), and indeed one will then have a system of n quadratic integrals that includes the energy integral. That is because equations (232) are identical to:

(435) $$p_\rho^2 = k\,\psi_\rho(q_\rho) + \{c_1\,\varphi_\rho^{(1)}(q_\rho) + \cdots + c_{n-1}\,\varphi_\rho^{(n-1)}(q_\rho)\} - \chi_\rho(q_\rho) \qquad (\rho = 1, \ldots, n),$$

and when one solves this for the constants k, c_1, …, c_{n-1}, one will get n expressions that are quadratic in the p_ρ . Each of those quadratic integrals belongs to a one-parameter group of transformations, although there seems to have been no investigations of its meaning up to now.

One must pose the question of whether that theorem can be inverted, i.e., whether the existence of n quadratic integrals will imply that the equations of motion can be solved by separation of variables. That is easy to answer for two degrees of freedom. In that case, when a second quadratic integral exists in addition to the energy integral, one must always, in fact, integrate the **Hamilton-Jacobi** equation that belongs to the problem by separating the variables and the integration of the

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algébriques des problèmes de mécanique," Paris, 1861 [cf., also, C. R. Acad. Sci. Paris **49** (1859), pp. 352]. **M. Lévy** treated the same problem in C. R. Acad. Sci. Paris **86** (1878), pp. 947. **E. Bour**, "Théorie de la deformation des surfaces," J. Éc. Polyt. **22**, cah. 39 (1862), pp. 1 (esp., pp. 79) also referred to it.

V. Cerruti treated the problem in more a modern notation in "Sopra una proprietà degli integrali di un problema di meccanica che sono lineari rispetto alle component della velocità," Roma Linc. Rend. (5) **4**[1] (1895), pp. 283, in which he gave not only the conditions that the arc-length element must satisfy, but he also showed that in order for the contravariant force components Q^ρ to admit an integral that is linear in the velocity components:

$$a_1(q_1, \ldots, q_n)\,\dot{q}_1 + \cdots + a_n(q_1, \ldots, q_n)\,\dot{q}_n = \text{const.},$$

it must satisfy the condition that:

(433) $$\sum g_{\lambda\mu} Q^\lambda\, \delta q_\mu = \delta\alpha \sum g_{\lambda\mu} Q^\lambda a^\mu = 0, \qquad \text{i.e.,} \qquad \sum Q^\lambda a_\lambda = 0.$$

For the most important of the linear integrals, viz., the first center of mass integral and the area integrals, **G. Bisconcini**, "Di una classificazione dei problemi dinamici," Il nuovo Cimento (5) **1** (1901), pp. 253, followed the ideas of **T. Levi-Civita** and gave normal forms for the arc-length elements that belong to such center of mass and surface integrals and whose associated infinitesimal transformations will define the six-parameter group of motions in Euclidian R_3 (one of its subgroups, resp.).

equations of motion with the help of quadratures alone will be possible ([377]). However, that result has no special significance, since one indeed always comes down to quadratures in this case as soon as an *arbitrary* further integral is known, along with the energy integral. [Cf., ([282])]

G. di Pirro ([378]) treated the condition for further quadratic integrals to appear along with the energy integral for the general case of a system of n degrees of freedom, but generally only under the assumption that the kinetic energy T possessed an orthogonal form. More generally, **P. Stäckel** showed that ([379]): m quadratic integrals will appear when the n variables $q_1, \ldots, q_n$ can be arranged into m classes such that the **Hamilton-Jacobi** equation can be split into m equations that each include only the variables from one class. **P. Painlevé** also gave a similar treatment of the question ([380]). Moreover, will arrive at the existence of quadratic integrals when one asks a different question, namely, when one investigates when two mechanical problems (whose kinetic energy is a quadratic form in the velocity components and whose forces depend upon only the position coordinates) will possess the same trajectories ([381]). (Cf., *infra*, no. **36**)

If one proceeds systematically then one can pose the question of the integrals of the equations of motion that are whole rational functions or even more general fractional rational functions of the impulse (velocity, resp.) components. However, an examination of that question beyond the first principles has not materialized. For example, **T. Levi-Civita** ([382]) showed that for force-free motion, i.e., for the differential equations of the geodetic lines of an M_n with the arc-length element:

$$ds^2 = 2T\,dt^2 = \sum_{\lambda,\mu=1}^{n} g_{\lambda\mu}\, dq_\lambda\, dq_\mu\,,$$

a homogeneous whole rational function of the $\dot{q}_\rho$ will represent an integral:

$$\sum A_{r_1\cdots r_m}\, \dot{q}_{r_1}\cdots\dot{q}_{r_m} = \text{const.}$$

([377]) Cf., e.g., the presentation in **G. D. Birkhoff**, *Dynamical Systems*, pp. 48. That theorem was first given for the special case of the differential equations of the geodetic lines on ordinary surfaces by **M. Massieu**, *loc. cit.* ([376]) [cf., also III B 3 (**R. von Lilienthal**), no. **18**].

([378]) Cf., **G. di Pirro**, "Sugli integrali primi quadratici delle equazioni della meccanica," Ann. di mat. (2) **24** (1896), pp. 315. **G. Pennacchietti** had already posed the question of quadratic integrals in some special cases in Mailand Lomb. Ist. Rend. (2) **18** (1885), pp. 242 and pp. 269, as well as **G. Vivanti**, Mailand Lomb. Ist. Rend. (2) **25** (1892), pp. 689.

([379]) **P. Stäckel**, "Über quadratische Integrale der Differentialgleichungen der Dynamik," Ann. di mat. (2) **25** (1897), pp. 55.

([380]) **P. Painlevé**, "Sur les intégrales quadratiques des équations de la dynamique," C. R. Acad. Sci. Paris **124** (1897), pp. 221. Cf., also the note by **T. Levi-Civita**, *ibid.*, pp. 392.

([381]) That question comes from a generalization of the problem of mapping two surfaces to each other in such a way that the geodetic lines of one will go to the geodetic lines of the other. Cf., **G. Darboux**, *Théorie des surfaces*, III, Chap. 3. That is basically also the approach that **R. Liouville** took when he sought to treat the question of quadratic integrals with some specialized methods, cf., **R. Liouville**, "Sur les équations de la dynamique," Acta math. **19** (1895), pp. 251.

([382]) **T. Levi-Civita**, "Sugli integrali algebrici delle equazioni dinamiche," Turin Atti **31** (1895/96), pp. 816.

if and only if the covariant derivatives [cf., III D 10 (**R. Weitzenböck**), Part. 2, no. **19**] of the $A_{r_1 \cdots r_m}$ define a semi-symmetric system ([383]).

On the other hand, if the equations of motion have the form (432) with $T = \sum g_{\lambda\mu}\,\dot{q}_\lambda\,\dot{q}_\mu$, and an integral exists that is a fractional rational function of the velocity components then the quotient of the highest-order terms in the numerator and the denominator must be an integral of the associated differential equation of the geodetic lines ([384]). In general, the relationship between the integrals of the equations of motion (432) and the integrals of the associated problem of geodetic lines was investigated as a generalization of the ideas of **G. Darboux** ([385]) by **P. Painlevé** ([386]). Finally, one must still bring the integrals that depend upon the velocity (impulse, resp.) components algebraically under consideration. **G. Koenigs** ([387]), and the **T. Levi-Civita** ([387.a]) could show that the existence of an algebraic integral would necessarily imply that of a rational integral ([388]).

Up to now, it was assumed that the force components Q_ρ depended upon only the position coordinates $q_1, \ldots, q_n$ (and possibly time t). By contrast, the force components Q_ρ can also depend upon the velocity components $\dot{q}_1, \ldots, \dot{q}_n$, so one must generalize the results. Hence, should the system of equations of motion, which might be written in the form:

$$\ddot{q}_\rho + \sum_{\lambda,\mu} \begin{Bmatrix} \lambda\ \mu \\ \rho \end{Bmatrix} \dot{q}_\lambda\,\dot{q}_\mu = Q^\rho , \tag{436}$$

possess an integral:

$$F(\dot{q}_1, \ldots, \dot{q}_n, q_1, \ldots, q_n, t) = C , \tag{437}$$

in which F is a prescribed function, then one must have:

([383]) That is intended to mean: If $A_{r_1 \cdots r_m r_{m+1}}$ is the covariant derivative of $A_{r_1 \cdots r_m}$ with respect to $q_{r_{m+1}}$, and one adds those of the derivatives whose indices emerge from $r_1 \ldots r_m\ r_{m+1}$ by cyclic permutations then all of those sums must vanish.

([384]) It follows for the equations of geodetic lines themselves that for an integral that is fractional rational in the direction coefficients, the quotient of the highest-order terms in the numerator and the denominator must likewise already be an integral.

([385]) **G. Darboux**, C. R. Acad. Sci. Paris **108** (1889), pp. 449.

([386]) **P. Painlevé**, "Sur les intégrales de la dynamique," C. R. Acad. Sci. Paris **114** (1892), pp. 1168.

([387]) **G. Koenigs**, "Sur les intégrales algébriques des problèmes de la dynamique," C. R. Acad. Sci. Paris **103** (1886), pp. 460.

([387.a]) **T. Levi-Civita**, *loc. cit.* ([382]).

([388]) The question of the appearance of algebraic integrals plays a major role in the two main problems in analytical mechanics – viz., the *n-body problem*, as well as that of the *top*. **H. Bruns** has proved that for the three-body problem, there can be no integrals beyond the ten integrals that arise from the **Galilei** group that are algebraic functions of p_ρ, q_ρ, t [cf., VI_2 12 (**E. T. Whittaker**), no. **4**]. In particular, the known investigations of **S. Kowalewski** in the theory of the top were guided by the goal of arriving at a new algebraic integral [cf., IV **6** (**P. Stäckel**), no. **36**].

(438)
$$\frac{\partial F}{\partial t}+\sum_{\rho}\left[\frac{\partial F}{\partial q_\rho}\dot{q}_\rho+\frac{\partial F}{\partial \dot{q}_\rho}\left(Q^\rho-\sum_{\sigma,\lambda}\begin{Bmatrix}\sigma\ \lambda\\ \rho\end{Bmatrix}\dot{q}_\sigma\,\dot{q}_\lambda\right)\right] = 0$$

identically in the q_ρ, $\dot{q}_\rho$, t. If one, with **J. Bertrand** ([389]), imagines that the kinetic energy T has been given then one can seek to determine the force components Q_ρ (Q^ρ, resp.) in such a way that equation (438) will become an identity. Naturally, that is not possible in general. However, in some special case, the Q^ρ can be given uniquely from that demand ([390]), or several solutions can also be possible. In the latter case, the given integral would be common to several mechanical problems, such as would be true of, e.g., the center of mass and area integrals. Furthermore, **J. Bertrand** examined the conditions that a function would have to satisfy in order for its forces Q^ρ to be determined in the given way. In that way, he went into more detail regarding the integrals that are rational in the velocity components ([391]).

30. Stationary motions for cyclic coordinates and their generalization. – It is not rare for a problem to arise in which one does not perform the general integration of the equations of motion, but must determine a certain, more specifically characterized, class of solutions. **E. J. Routh** ([392]) treated an important example of that, namely, the *stationary motion* (*steady motion*) of a mechanical system with n degrees of freedom such that the last r of its general coordinates $q_1, \ldots, q_n$ (namely, $q_{n-r+1}, \ldots, q_n$) are cyclic (hidden) coordinates. One will get a distinguished case of the motion of such a system when the acyclic (observable) coordinates remain constant such that the system will ostensibly appear to be at rest, while in reality the apparent state of equilibrium will be maintained by the (unobservable) motion in the cyclic coordinates ([393]). The **Lagrangian** function of the system:

(439)
$$L(\dot{q}_1,\ldots,\dot{q}_n,q_1,\ldots,q_{n-r}) = T-\Phi\,,$$

([389]) **J. Bertrand**, "Sur les intégrales communes a plusieurs problèmes de mécanique," J. de math. **17** (1852), pp. 121.

([390]) In that case, one can differentiate the identity (438) with respect to the $\dot{q}_1, \ldots, \dot{q}_n$, which will give n linear equations for the unknowns $Q_1, \ldots, Q_n$, from which one can possibly calculate them.

([391]) **J. Bertrand**, "Mémoires sur quelqu'unes des forms les plus simples que puissant prendre les intégrales des équations différentielles du movement d'un point matériel," J. de math. (2) **2** (1857), pp, 113, in which he generally assumed that the kinetic energy had the form $T = \frac{1}{2}(\dot{x}^2+\dot{y}^2)$. For that form of T, he treated the cases in which the integral was a whole rational function of order one, two, or three, as well as a fractional rational function of order one in the velocity components. An extension to the Euclidian R_3 was given by **G. Vivanti**, Rend. circ. mat. Palermo **6** (1892), pp. 127.

([392]) **E. J. Routh**, *A treatise on the stability of a given state of motion*, London, 1877, referred to a trajectory whose linear "**Jacobi** equations" (cf. no. **20**) included coefficients that were independent of time as a *steady motion*.

([393]) The meaning of such a motion is based in the fact that its stability can be resolved with the help of the energy criterion.

which one might assume does not contain time t explicitly, in agreement with **Routh**, is such that one will have:

$$T + \Phi = H = k \tag{440}$$

for the energy integral of the motion. Since the cyclic impulse components are constant:

$$p_{n-r+1} = \frac{\partial T}{\partial \dot{q}_{n-r+1}} = c_1 , \qquad \ldots, \qquad p_n = \frac{\partial T}{\partial \dot{q}_n} = c_r , \tag{441}$$

by the **Routh** transformation (cf., no. **9**), one will then obtain, in place of the original **Lagrangian** function L, the modified function:

$$L^* (\dot{q}_1, \ldots, \dot{q}_n, q_1, \ldots, q_{n-r}, c_1, \ldots, c_r) = L - \sum_{\sigma=1}^{r} \dot{q}_{n-r+\sigma}\, c_\sigma , \tag{442}$$

which represents the **Lagrangian** function for the observable motion of the system. Hence, should rest prevail in observable coordinates, then the associated **Euler** equations:

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \dot{q}_\rho}\right) - \frac{\partial L^*}{\partial q_\rho} \qquad (r = 1, \ldots, n - r) \tag{443}$$

would have to be satisfied by:

$$q_1 = \gamma_1 , \qquad \ldots, \qquad q_{n-r} = \gamma_{n-r} , \tag{444}$$

and therefore:

$$\dot{q}_1 = 0 , \qquad \ldots, \qquad \dot{q}_{n-r} = 0 . \tag{444.a}$$

However, since L^* does not depend upon time explicitly, that means that one has to calculate the values (444) from the equations:

$$\frac{\partial L^*}{\partial q_1} = 0 , \qquad \ldots, \qquad \frac{\partial L^*}{\partial q_{n-r}} = 0 , \tag{445}$$

in which the velocity components are set equal to zero ([394]). In order to determine the motion in cyclic coordinates, one must then calculate the values of the derivatives $\dot{q}_{n-r+1}, \ldots, \dot{q}_n$ from (441),

([394]) Since one has:

which will then prove to be constant, such that the cyclic coordinates themselves will become linear functions of time ([395]):

$$q_{n-r+1} = a_1\, t + \alpha_1\,, \qquad \ldots, \qquad q_n = a_r\, t + \alpha_r\,. \tag{446}$$

For fixed numerical values of the cyclic impulses (441), equations (445) [(444), resp.] will determine a certain M_r in the R_n of the $q_1, \ldots, q_n$ on which the $q_{n-r+1}, \ldots, q_n$ vary arbitrarily. It represents a characteristic M_r that belongs the r integrals (441). That is because it is created from an orbit (444), (446) of the r-parameter group of parallel displacements in the directions of q_{n-r+1}, …, q_n that arises from the r integrals (441). ∞^r orbits lie on that M_r [corresponding to the r arbitrary constants α_1, …, α_r in (446)], which emerge from one of them by way of the ∞^r parallel displacements in the r-parameter group. Moreover, since one can prescribe the numerical values of the cyclic impulses arbitrarily, one does not have just a single M_r, but a family of ∞^r such M_r on which each of the ∞^r orbits lie. In total, one has then found ∞^{2r} special orbits of the system.

If one employs the associated canonical system for the equations of motion, in place of the **Euler** equations, then the **Euler** equations (443) will correspond to the canonical system:

$$\frac{dq_\rho}{dt} = \frac{\partial H}{\partial p_\rho}, \qquad \frac{dp_\rho}{dt} = -\frac{\partial H}{\partial q_\rho} \qquad (\rho = 1, \ldots, n-r), \tag{447}$$

in which one imagines that the cyclic impulses have been replaced with the constant values (441) in H. One will then get the stationary motion (444), (446) from the equations:

$$\frac{\partial H}{\partial p_\rho} = 0\,, \qquad \frac{\partial H}{\partial q_\rho} = 0 \qquad (\rho = 1, \ldots, n-r), \tag{448}$$

$$\frac{\partial L^*}{\partial q_\rho} = \frac{\partial L}{\partial q_\rho} \qquad (\rho = 1, \ldots, n-r),$$

from (442), one can also determine the values (444) from the $(n - r)$ equations:

$$\frac{\partial L}{\partial q_\rho} = 0 \qquad (\rho = 1, \ldots, n-r), \tag{445.a}$$

in which one has given the velocity components $\dot q_1, \ldots, \dot q_{n-r}$ the value zero, and the velocity components $\dot q_{n-r+1}, \ldots, \dot q_n$ are then replaced with the r values that follow from (441). (German translation, pp. 77)

([395]) **Watt**'s centrifugal governor when the prime mover has constant angular velocity will serve as a simple example of such a motion. **E. J. Routh**, *Advanced Rigid Dynamics*. (German translation, pp. 81) which will simultaneously yield the coordinates (444), as well as the constant values of the impulses $p_1, \ldots, p_{n-r}$ that belong to (444.a) ([396]). One must again append the values (446) of the $q_{n-r+1}, \ldots, q_n$ to them. The r integrals (441) determine an M_{2n-r} in the space of (p_ρ, q_ρ) on which the ∞^{2n-r} orbits lie. Among them, the orbits that are given by (448) and (446) are distinguished by the fact that the energy H assumes the value for all of them, and indeed, an extremal value.

T. Levi-Civita ([397]) generalized that **Routh** motion for cyclic coordinates by replacing the special r integrals (441) with r general integrals:

(449) $$F_1(p_1, \ldots, p_n, q_1, \ldots, q_n) = c_1, \qquad \ldots, \qquad F_r(p_1, \ldots, p_n, q_1, \ldots, q_n) = c_r$$

of the canonical system:

(450) $$\frac{dq_\rho}{dt} = \frac{\partial H}{\partial p_\rho}, \qquad \frac{dp_\rho}{dt} = -\frac{\partial H}{\partial q_\rho} \qquad (\rho = 1, \ldots, n),$$

which one assumes are in involution, such that one will have:

(449.a) $$(F_\sigma, F_\tau) = 0$$

for it. Here, as in no. **27**, if one solves the relations (449) for r of the impulse components, say for $p_{n-r+1}, \ldots, p_n$:

(451) $$p_{n-r+\sigma} + f_\sigma(p_1, \ldots, p_{n-r}, q_1, \ldots, q_n, c_1, \ldots, c_r) = 0 \qquad (\sigma = 1, \ldots, r),$$

then one can next determine (cf., no. **26**) the characteristic M_r that belongs to the integrals (449). In order to do that, one appeals to the completely-integrable **Pfaff**ian system (cf., no. **27**):

([396]) The impulses that belong to the observable coordinates are constant, but generally non-zero. For example, in the simplest case, one has:

$$p_\rho = \frac{\partial T}{\partial \dot{q}_\rho} = g_{\rho 1}\,\dot{q}_1 + \cdots + g_{\rho n}\,\dot{q}_n$$
$$= g_{\rho, n-r+1}\,\dot{q}_{n-r+1} + \cdots + g_{\rho n}\,\dot{q}_n \qquad (\rho = 1, \ldots, n-r),$$

in which the $\dot{q}_{n-r+1}, \ldots, \dot{q}_n$ are replaced with the values that are calculated from:

$$c_\rho = p_{n-r+\sigma} = g_{n-r+\sigma, n-r+1}\,\dot{q}_{n-r+1} + \cdots + g_{n-r+\sigma, n}\,\dot{q}_n .$$

([397]) **T. Levi-Civita**, "Sulla determinazione di soluzioni particolari di un sistema canonico, quando se ne conosce qualche integrale o relazione invariante," Roma Linc. Rend. (5) 10^1 (1901), pp. 3 and pp. 35. An extended and generalized presentation is found in **T. Levi-Civita**, "Sur la recherche des solutions particulières," Warschau Prace matematyczno-fisycne **17** (1906), pp. 1, cf., also **T. Levi-Civita** and **U. Amaldi**, *Lezioni II*, 2, pp. 339, *et seq.*

(452)
$$\begin{cases} dq_\rho = \dfrac{\partial \bar{H}}{\partial p_\rho} dt + \sum_{\sigma=1}^{r} \dfrac{\partial f_\sigma}{\partial p_\rho} dq_{n-r+\sigma}, \\ dp_\rho = -\left(\dfrac{\partial \bar{H}}{\partial q_\rho} dt + \sum_{\sigma=1}^{r} \dfrac{\partial f_\sigma}{\partial q_\rho} dq_{n-r+\sigma} \right), \end{cases}$$

in which $\bar{H}$ arises from H, in such a way that the functions (451) are substituted for the $p_{n-r+1}, \ldots, p_n$.

As a generalization of (448), **T. Levi-Civita** posed the 2 ($n - r$) equations:

(453)
$$\frac{\partial \bar{H}}{\partial p_\rho} = 0, \qquad \frac{\partial \bar{H}}{\partial q_\rho} = 0 \qquad (\rho = 1, \ldots, n - r),$$

from which one can calculate the $p_1, \ldots, p_{n-r}, q_1, \ldots, q_{n-r}$ as functions of the $q_{n-r+1}, \ldots, q_n$. However, those functions determine a characteristic M_r. That is because the relations (453) are $2(n-r)$ integrals of the likewise completely-integrable system ([398]):

(454)
$$\begin{cases} dq_\rho = -\sum_{\sigma=1}^{r} \dfrac{\partial f_\sigma}{\partial p_\rho} dq_{n-r+\sigma}, \\ dp_\rho = \sum_{\sigma=1}^{r} \dfrac{\partial f_\sigma}{\partial q_\rho} dq_{n-r+\sigma}, \end{cases}$$

since indeed one will have:

$$d\left(\frac{\partial \bar{H}}{\partial p_\rho}\right) = \sum_{\sigma=1}^{r} \left(\left\{ \frac{\partial \bar{H}}{\partial p_\rho}, f_\sigma \right\} + \frac{\partial^2 \bar{H}}{\partial p_\rho \, \partial q_{n-r+\sigma}} \right) dq_{n-r+\sigma}$$

from those equations, so from (453), one will have:

$$d\left(\frac{\partial \bar{H}}{\partial p_\rho}\right) = \frac{\partial}{\partial p_\rho} \sum_{\sigma=1}^{r} \left(\frac{\partial \bar{H}}{\partial q_{n-r+\sigma}} + \{\bar{H}, f_\sigma\} \right) dq_{n-r+\sigma} = 0,$$

and that will imply that:

([398]) This method of proof is in **P. Burgatti**, "Sopra un teorema di Levi-Civita riguardante la determinazione di soluzioni particulari di un sistema Hamiltoniano," Roma Linc. Rend. (5) **11**[1] (1902), pp. 309.

One should observe that the coefficients of the differentials on the right-hand side of (452) are independent t. The entire argument will proceed in an entirely-similar way when H and the $F_1, \ldots, F_r$ also depend upon t, but the coefficients on the right-hand side of (452) will not include one of the coordinates $q_{n-r+1}, \ldots, q_n$. In the event that the coordinate $q_{n-r+\sigma}$ does not appear, one must not go over to (454) then, but a system that arises from (452) when one sets the coefficient of $dq_{n-r+\sigma}$ equal to zero, and calculates $p_1, \ldots, p_{n-r}, q_1, \ldots, q_{n-r}$ as functions of t, $q_{n-r+1}, \ldots, q_{n-r+\sigma-1}$, $q_{n-r+\sigma+1}, \ldots, q_n$. from it. That would likewise determine a characteristic M_r.

$$d\left(\frac{\partial \bar{H}}{\partial p_\rho}\right) = 0 ,$$

analogously.

If one imagines that the constants $c_1, \ldots, c_r$ in (449) can be chosen arbitrarily then equations (453) will determine a family of ∞^r characteristic M_r . ∞^r integral curves of the canonical system lie on each of those characteristic M_r that one will obtain by a quadrature (cf., no. **27**).

However, according to **T. Levi-Civita**, it is not necessary for the relations (449) that facilitate the reduction should be integrals. Rather, one can make the same argument in the case where one knows only *r invariant relations* ([399]):

(455) $F_1(p_1, \ldots, p_n, q_1, \ldots, q_n) = 0 , \qquad \ldots, \qquad F_r(p_1, \ldots, p_n, q_1, \ldots, q_n) = 0$

that are involution. One can then arrive at equations (453) in the same way, but one will not get only a single M_r with ∞^r integral curves. One must determine those integral curves by means of the system:

$$\frac{dq_{n-r+1}}{dt} = \frac{\partial H}{\partial p_{n-r+1}}, \ldots, \frac{dq_n}{dt} = \frac{\partial H}{\partial p_n},$$

whose right-hand sides are functions of $q_{n-r+1}, \ldots, q_n$, which will then require at most an $(r - 1)$ operation (in **Lie**'s terminology) ([400]).

Just as in **Routh**'s special case, those ∞^r orbits are singled out from the ∞^{2n-r} orbits on the M_{2n-r} (455) by the fact that the energy H assume the same extreme value for all of them ([401]).

([399]) That is:

$$\frac{dF_\rho}{dt} = (H, F_\rho)$$

is not identically zero, but only when one recalls (455). **P. Painlevé**, *L'intégr. des équations diff. de méc.*, pp. 286, referred to such a relation as an *intégrale première particularisée*.

([400]) Cf., **T. Levi-Civita**, Roma Linc. Rend. (5) **10**[1] (1901), pp. 3.

([401]) The special position that is enjoyed by canonical systems emerges when one seeks to adapt those arguments to general differential systems with $(2n + 1)$ variables:

(a) $dx_0 : dx_1 : \ldots : dx_{2n} = X_0(x_0, x_1, \ldots, x_{2n}) : X_1 : \ldots : X_{2n}$,

cf., **T. Levi-Civita**, Warschau Prace mat. fis. **17** (1906), pp. 1. Just as the canonical system:

(α) $$\frac{dq_\rho}{dt} = \frac{\partial H}{\partial p_\rho}, \qquad \frac{dp_\rho}{dt} = -\frac{\partial H}{\partial q_\rho} \qquad (\rho = 1, \ldots, n)$$

has the integral:

(β) $H = k$,

which determines a cylindrical (in the t-direction) M_{2n} in the phase-R_{2n+1} of the (q_ρ, p_ρ, t), one must also assume that one knows a first integral:

(b) $$f(x_0, x_1, \ldots, x_{2n}) = c$$

that determines an M_{2n} (a family of M_{2n}, resp.) in the R_{2n+1} of the $(x_0, x_1, \ldots, x_{2n})$.

If one now ignores the appearance of cyclic coordinates in the canonical system (α) then, corresponding to (448), one will have to pose the relations:

(γ) $$\frac{\partial H}{\partial q_\rho} = 0 , \qquad \frac{\partial H}{\partial p_\rho} = 0 \qquad (\rho = 1, \ldots, n),$$

to which one would have to add the identity relation:

(γ') $$\frac{\partial H}{\partial t} \equiv 0 .$$

They determine a point $p_\rho^{(0)}, q_\rho^{(0)}$ in the M_{2n} of the (p_ρ, q_ρ) [a curve, resp., namely, a parallel to the t-direction in the R_{2n+1} of the (p_ρ, q_ρ, t)]. That is the *one* stationary solution to the canonical system that one will get in this case. For the general system of differential equations (a) with the first integral (b), when one sets:

(c) $$\frac{\partial f}{\partial x_0} = 0 , \qquad \frac{\partial f}{\partial x_1} = 0 , \qquad \ldots, \qquad \frac{\partial f}{\partial x_{2n}} = 0 ,$$

corresponding to (γ), that will yield a system of $2n$ independent equations that determine an M_1, in general, so *one* stationary integral curve. **T. Levi-Civita** (cf., *loc. cit.*, pp. 37) deduced the meaning that this stationary solution had in comparison to the other solutions.

Now, if the canonical system (α) has r cyclic coordinates then r of the equations (γ) will be fulfilled identically. Thus, equations (γ) will not determine an M_1 in this case, but an M_{r+1} with the property that every integral curve that has a point in common with the M_{r+1} will belong to it completely. Correspondingly, equations (c) can also establish an $M_{\tau+1}$ ($\tau \neq 0$) with the same property instead of an M_1 .

For a canonical system, the appearance of r cyclic coordinates is always coupled with the existence of r first integrals, namely, the r relations (441). One can correspondingly start from them (the more general relations that **T. Levi-Civita** introduced, resp.). If one assumes in the same way that for the general system (a), along with the integral (b), one knows r further integrals, or more generally, invariant relations:

(d) $$f_0(x_0, x_1, \ldots, x_{2n}) = 0 , \qquad \ldots, \qquad f_r(x_0, x_1, \ldots, x_{2n}) = 0 ,$$

that determine an M_{2n-r} in conjunction with (b), then one can set:

(e) $$F = f + \lambda_1 f_1 + \ldots + \lambda_n f_n$$

and prescribe the relations:

(f) $$\frac{\partial F}{\partial x_0} = 0 , \qquad \frac{\partial F}{\partial x_1} = 0 , \qquad \ldots, \qquad \frac{\partial F}{\partial x_{2n}} = 0 .$$

In that way, precisely *one* stationary integral curve will once more be determined on the M_{2n-r} then, while one will get an $M_{\tau+1}$ only in exceptional cases. In contrast to that, one will always get an M_{r+1} for the canonical system when one follows the procedure in the text in the event that the invariant relations satisfy the given assumptions.

The behavior of the canonical systems is precisely analogous to that of the **Pfaffian** systems, which arise from a **Pfaffian** expression (i.e., a linear differential form) in the way that was given in no. **21**. It was treated by **T. Levi-Civita**, "Sulle soluzione stazionarie dei sistemi pfaffiani," Roma Linc. Rend. (6) **19**[1] (1934), Nota I, pp. 261 and Nota II, pp. 369. Based upon the concept that **É. Cartan** introduced of the higher-order *dérivée extérieure* of a **Pfaff**ian form (cf., no. **20**), he could introduce relations for a **Pfaff**ian system that are the natural generalizations of the involution relations for canonical systems and subsume them as special cases. If one has not only a first integral:

(g) $$f(x_0, x_1, \ldots, x_{2n}) = 0$$

for a **Pfaff**ian system, but also r invariant relations:

(h) $$f(x_0, x_1, \ldots, x_{2n}) = 0\,, \quad \ldots, \quad f_r(x_0, x_1, \ldots, x_{2n}) = 0\,,$$

that are in involution (in the general sense) with each other and with (g) then one will always get an M_{r+1} on which ∞^r stationary integral curves lie from the Ansatz of (e) and (f), precisely as one does with the canonical system.

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CHAPTER VII

THE CANONICAL TRANSFORMATION

31. The canonical system as the characteristic Pfaffian system of a linear differential form. The bilinear covariant. Historical connection with the perturbation calculations. ([401.a]) – From what was explained in the previous chapter, the simplifications in the integration of the equations of motion will be implied by essentially their canonical form:

(456) $$\frac{dq_\rho}{dt} = \frac{\partial H}{\partial p_\rho}, \qquad \frac{dp_\rho}{dt} = -\frac{\partial H}{\partial q_\rho} \qquad (\rho = 1, \ldots, n).$$

If one is compelled to introduce new coordinates in place of the p_ρ, q_ρ for any reason, such as, e.g., in perturbation calculations, then the canonical form will not be preserved, in general, and one would sacrifice the advantage that it would give in integrating the transformed equations. Thus, one might try to choose the new coordinates in such a way that the transformed system will again possess the canonical form. A coordinate transformation with that property will then be referred to as a *canonical transformation*.

In order to determine those canonical transformations, one will most conveniently appeal to the *Pfaff problem* [cf., II A 5 (**E. von Weber**), no. **18**, *et seq.*], in which the results of no. **21** will take on new meaning. A linear differential form (a so-called **Pfaff**ian expression):

(457) $$\Phi_d = X_1(x_1, \ldots, x_m)\, dx_1 + \ldots + X_m(x_1, \ldots, x_m)\, dx_m$$

is associated with the so-called *characteristic Pfaffian system* ([402]). One arrives at it (cf., no. **21**) most simply when one forms the *bilinear covariant* ([403]):

(458) $$\delta\Phi_d - d\,\Phi_d = \sum_{\lambda,\mu=1}^{m} \left(\frac{\partial X_\lambda}{\partial x_\mu} - \frac{\partial X_\mu}{\partial x_\lambda}\right)(\delta x_\mu\, dx_\lambda - \delta x_\lambda\, dx_\mu)$$

and sets the factors of δx_μ equal to zero in it:

(459) $$\left(\frac{\partial X_1}{\partial x_\mu} - \frac{\partial X_\mu}{\partial x_1}\right) dx_1 + \cdots + \left(\frac{\partial X_m}{\partial x_\mu} - \frac{\partial X_\mu}{\partial x_m}\right) dx_m = 0$$

$$(\mu = 1, \ldots, m).$$

([401.a]) For the historical development, see also Chapter III.

([402]) It is also referred to as the "first **Pfaff** system" of the differential form.

([403]) Cf., no. **21**, pp. 671, *et seq.*

If one now introduces new coordinates into the linear differential form (457), such that it will go to, say:

(457.a) $$\Phi_d = Y_1 (y_1, \ldots, y_m)\, dy_1 + \ldots + Y_m (y_1, \ldots, y_m)\, dy_m ,$$

then the system of differential equations (459) will be converted into the characteristic **Pfaff**ian system for the linear differential form (457.a) under that coordinate transformation. One will then need only to form the bilinear covariant of the **Pfaff**ian expression (457.a):

(458.a) $$\delta \Phi_d - d\, \Phi_d = \sum_{\lambda,\mu=1}^{m} \left(\frac{\partial Y_\lambda}{\partial y_\mu} - \frac{\partial Y_\mu}{\partial y_\lambda} \right) (\delta y_\mu\, dy_\lambda - \delta y_\lambda\, dy_\mu)$$

and then set the factors of δy_μ in it equal to zero, in order for:

(459.a) $$\left(\frac{\partial Y_1}{\partial y_\mu} - \frac{\partial Y_\mu}{\partial y_1} \right) dy_1 + \cdots + \left(\frac{\partial Y_m}{\partial y_\mu} - \frac{\partial Y_\mu}{\partial y_m} \right) dy_m = 0$$

$$(\mu = 1, \ldots, m)$$

to be the system of differential equation to which the system (459) will go under the coordinate transformation.

Now [cf., no. **21**, eq. (296)], the canonical system (465) is the characteristic **Pfaff**ian system of the linear differential form ([404]):

(460) $$\Phi_d = p_1\, dq_1 + \ldots + p_n\, dq_n - H (p_1, \ldots, p_n, q_1, \ldots, q_n, t)\, dt ,$$

such that one will arrive at it when one introduces new coordinates into the canonical system (456). In that way, one will easily obtain, e.g., the form of the perturbation equations, as **Lagrange** gave them (cf., no. **12**). Namely, let, say, (460) be the linear differential form of the unperturbed motion. For the perturbed motion, let it be the corresponding form:

(461) $$\Phi_d^* = p_1\, dq_1 + \ldots + p_n\, dq_n - (H + \Omega)\, dt ,$$

in which the perturbing function Ω can now be thought of as depending upon, more generally than in no. **12**, not only the position coordinates q_ρ , but also the impulse components p_ρ . If one then introduces the constants $c_1, \ldots, c_{2n}$ of the unperturbed problem in place of the q_ρ, p_ρ as new variables in the perturbed problems by way of the transformation formulas:

([404]) The Ansatz for the equations of motion was studied systematically from this standpoint by **G. Morera**, "Sulle equazioni dinamiche di Hamilton." Turin Atti **39** (1904), pp. 364. In it, he investigated, in particular, how one would have to consider non-holonomic constraints on the mechanical system for the derivation of the canonical system from the bilinear covariant [cf., also the citation in ([78])].

(462) $q_\rho = \varphi_\rho(t, c_1, \dots, c_{2n})$, $p_\rho = \psi_\rho(t, c_1, \dots, c_{2n})$

then the linear differential form (461) will go to the differential form:

(463) $$\Phi_d^* = \left(\sum_{\rho=1}^{n} p_\rho \frac{\partial q_\rho}{\partial c_1}\right) dc_1 + \cdots + \left(\sum_{\rho=1}^{n} p_\rho \frac{\partial q_\rho}{\partial c_{2n}}\right) dc_{2n} + \left(\sum_{\rho=1}^{n} p_\rho \frac{\partial q_\rho}{\partial t} - H - \Omega\right) dt ,$$

whose bilinear covariant will assume the form ([405]):

(464) $$\delta\Phi_d^* - d\,\Phi_d^* = \sum_{\lambda,\mu=1}^{2n} [c_\lambda, c_\mu](\delta c_\lambda\, dc_\mu - \delta c_\mu\, dc_\lambda) + \sum_{\sigma=1}^{2n} \frac{\partial \Omega}{\partial c_\sigma}(\delta c_\sigma\, dt - \delta t\, dc_\sigma)$$

when one introduces the **Lagrange** brackets [cf., eq. (96)]. The equations of motion (456) will correspondingly go to:

(465) $$[c_1, c_\mu]\frac{dc_1}{dt} + \cdots + [c_{2n}, c_\mu]\frac{dc_{2n}}{dt} = \frac{\partial \Omega}{\partial c_\mu} \qquad (\mu = 1, \dots, 2n)$$

under the transformation (462) ([406]), and those are precisely **Lagrange**'s perturbation equations (95). In order for that system to again take the canonical form, it is necessary and sufficient that all of the **Lagrange** brackets vanish except for those of the $2n$ brackets for which the one index differs from the other by exactly n and half of them equal (+ 1), while the other half equal (– 1). Now, **Lagrange** remarked that that will occur when one selects the $2n$ constants $c_1, \dots, c_{2n}$ suitably (cf., the conclusion of no. **12**), e.g., when one chooses the initial values $q_\rho^{(0)}$, $p_\rho^{(0)}$ of the q_ρ, p_ρ at time $t = t_0$ to be varying constants ([407]).

Whereas **Lagrange** hit upon the idea of a canonical transformation largely at random, **W. R. Hamilton** ([408]) (cf., also no. **14**) found the systematic way that would lead to the canonical form for the perturbation equations. Namely, he started from the principal function:

([405]) In so doing, one should observe that (462) is the solution to the unperturbed problem, such that one will then have:

$$\frac{\partial}{\partial c_\sigma}\left(\sum_{\rho=1}^{n} p_\rho \frac{\partial q_\rho}{\partial c_1} - H\right) - \frac{\partial}{\partial t}\left(\sum_{\rho=1}^{n} p_\rho \frac{\partial q_\rho}{\partial c_\sigma}\right) = 0.$$

([406]) The relation:

$$\sum_{\sigma=1}^{2n} \frac{\partial \Omega}{\partial c_\sigma} dc_\sigma = 0$$

gets added to that, which expresses the idea that the q_ρ, p_ρ must have the same values for the perturbed and unperturbed problem, respectively.

([407]) That is because the **Lagrange** brackets will then have the correct values for $t = t_0$, and those values will be preserved for all t, since they are independent of time t (cf., nos. **12** and **21**).

([408]) **W. R. Hamilton**, "Second essay on a general method in dynamics," Trans. Phil. Soc. London **2** (1835), pp. 95.

$$S(q_1,\dots,q_n,t;q_1^{(0)},\dots,q_n^{(0)},t_0)$$

of the unperturbed problem (cf., no. **14**) and employed the equations:

(466) $$\frac{\partial S}{\partial q_\rho} = p_\rho, \qquad \frac{\partial S}{\partial q_\rho^{(0)}} = -\, p_\rho^{(0)} \qquad (\rho = 1, \dots, n)$$

in order to introduce the constants $q_\rho^{(0)}$, $p_\rho^{(0)}$ of the unperturbed problem as the new variables in the perturbed problem. Namely, since one has:

(467) $$dS = \sum_{\rho=1}^{n} p_\rho\, dq_\rho - H\, dt - \sum_{\rho=1}^{n} p_\rho^{(0)}\, dq_\rho^{(0)},$$

when one regards t_0 as a fixed parameter [cf., eq. (153)], the linear differential form (461) of the perturbed problem will go to the form:

(468) $$\Phi_d^* = (p_1^{(0)}\, dq_1^{(0)} + \cdots p_n^{(0)}\, dq_n^{(0)} - \Omega\, dt) + dS$$

under the transformation (466), whose bilinear covariant will be:

(468.a) $$\delta\Phi_d^* - d\,\Phi_d^* = \sum_{\rho=1}^{n}(\delta p_1^{(0)}\, dq_1^{(0)} + \cdots \delta p_n^{(0)}\, dq_n^{(0)}) - (\delta\Omega\, dt - d\,\Omega\,\delta t)$$

since the exact differential will make no contribution. The characteristic **Pfaff**ian system will then read:

(469) $$\frac{dq_\rho^{(0)}}{dt} = \frac{\partial\Omega}{\partial p_\rho^{(0)}}, \qquad \frac{dp_\rho^{(0)}}{dt} = -\frac{\partial\Omega}{\partial q_\rho^{(0)}},$$

i.e., the perturbation equations have the canonical form.

C. G. J. Jacobi ([409]) took up **Hamilton**'s ideas and likewise generalized them in such a way that he replaced **Hamilton**'s principal function S with an arbitrary complete solution S (q_1, …, q_n, t, c_1, …, c_n) of the **Hamilton-Jacobi** partial differential equation:

(470) $$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q_1},\dots,\frac{\partial S}{\partial q_n}, q_1,\dots,q_n,t\right) = 0$$

([409]) **C. G. J. Jacobi**, "Note sur l'intégration des équat. diff. de la dynamique," C. R. Acad. Sci. Paris **5** (1837), pp. 61 = *Werke IV*, pp. 131, as well as in *Probleme der Mechanik,* in the case where a force function exists, as well as the theory of perturbations, *Werke V*, esp., Theorem IX, pp. 355.

and defined the transformation by the equations:

(471) $$p_\rho = \frac{\partial S}{\partial q_\rho}, \qquad \frac{\partial S}{\partial c_\rho} = -\gamma_\rho ,$$

in which the relations $\partial S / \partial c_\rho = -\gamma_\rho$ represent the equations of the space-time lines of the unperturbed problem ([410]). Now, since:

$$dS = \sum_{\rho=1}^{n} p_\rho\, dq_\rho - H\, dt - \sum_{\rho=1}^{n} \gamma_\rho\, dc_\rho ,$$

the differential form (461.a) will go to:

(472) $$\Phi_d^* = \sum_{\rho=1}^{n} \gamma_\rho\, dc_\rho - \Omega\, dt + dS$$

under the transformation (471), from which the perturbation equations will once more emerge in canonical form:

(472.a) $$\frac{dc_\rho}{dt} = \frac{\partial \Omega}{\partial \gamma_\rho}, \qquad \frac{d\gamma_\rho}{dt} = -\frac{\partial \Omega}{\partial c_\rho}.$$

At the same time, **Jacobi** achieved the breakthrough that this argument was basically entirely independent of the perturbation calculations ([411]). If one introduces new coordinates P_ρ , Q_ρ into the canonical system in place of the p_ρ , q_ρ with the help of an arbitrary function U (q_1, …, q_n, Q_1, …, Q_n) by way of the formulas:

(473) $$p_\rho = \frac{\partial U}{\partial q_\rho}, \qquad -P_\rho = \frac{\partial U}{\partial Q_\rho} \qquad (\rho = 1, \ldots, n)$$

then the transformed system will once more be canonical. That is because the linear differential form (460) that belongs to (456) will go to:

(473.a) $$\Phi_d = \sum_{\rho=1}^{n} P_\rho\, dQ_\rho - H\, dt + dU$$

under the transformation (473), and its characteristic **Pfaff**ian system will have the form:

([410]) Cf., eq. (186), in which a different sign was chosen for the constants γ_ρ .

([411]) **C. G. J. Jacobi**, *Werke IV*, pp. 136, as well as *Probleme der Mech.*, *Werke V*, Theorem X, pp. 371. Moreover, it is indicative of the direct connection with the perturbation calculations that **Jacobi** seemed to feel that the general canonical transformation was only a transition from one system of canonical perturbation equations to another canonical system of perturbation equations.

(473.b) $$\frac{dQ_\rho}{dt} = \frac{\partial H}{\partial P_\rho}, \qquad \frac{dP_\rho}{dt} = -\frac{\partial H}{\partial Q_\rho}.$$

That expresses the fact that one will obtain a canonical transformation that corresponds to the formulas (473) from a *substitution function* ([412]) like $U(q_1, \ldots, q_n, Q_1, \ldots, Q_n)$ that can be chosen arbitrarily. **Jacobi** had also already given a generalization of the Ansatz (473) of the canonical transformation ([413]). Namely, one is occasionally given a number of relations between the old and new position coordinates for the system from the outset by the nature of the problem itself. In that spirit, if one prescribes r relations between the old position coordinates $q_1, \ldots, q_n$ and the new ones $Q_1, \ldots, Q_n$, say:

(474) $$\varphi_\sigma(q_1, \ldots, q_n, Q_1, \ldots, Q_n) = 0 \qquad (\sigma = 1, \ldots, r)$$

then one will need only to introduce r **Lagrang**ian factors and replace the arbitrary function U with the expression:

$$U + \lambda_1 \varphi_1 + \ldots + \lambda_r \varphi_r ,$$

and one will then get a canonical transformation from:

(475) $$\begin{cases} p_\rho = \dfrac{\partial U}{\partial q_\rho} + \lambda_1 \dfrac{\partial \varphi_1}{\partial q_\rho} + \cdots + \lambda_r \dfrac{\partial \varphi_r}{\partial q_\rho}, \\[2ex] P_\rho = -\left(\dfrac{\partial U}{\partial Q_\rho} + \lambda_1 \dfrac{\partial \varphi_1}{\partial Q_\rho} + \cdots + \lambda_r \dfrac{\partial \varphi_r}{\partial Q_\rho} \right) \end{cases}$$

when one calculates the old coordinates q_ρ, p_ρ as functions of the new ones Q_ρ, P_ρ from (474) and (475) by eliminating $\lambda_1, \ldots, \lambda_r$. In that way, r can assume the values $0, 1, \ldots, n$ ([414]). In fact, the differential form (460) will go to:

(476) $$\Phi_d = \sum_{\rho=1}^{n} P_\rho\, dQ_\rho - H\, dt + dU + \lambda_1\, d\varphi_1 + \cdots + \lambda_r\, d\varphi_r$$

under the transformation (475), such that the characteristic **Pfaff**ian system will again assume the canonical form (473.b).

([412]) The term was first used by **E. Schering**, "Hamilton-Jacobische Theorie für Kräfte, deren Maß von der Bewegung der Körper abhängt," Gött. Abh. **18** (1873), pp. 3 = *Werke I*, pp. 193. In that article, **Schering** was also the first to treat the canonical transformation of the bilinear covariant [cf., eq. (8) in that treatise (*Werke I*, pp. 212)], from which he then derived the canonical equations [cf., eq. (8*)] as the characteristic **Pfaff**ian system. He then used the bilinear covariant in the usual form from eq. (13) onwards (*Werke I*, pp. 230).

([413]) **C. G. J. Jacobi**, *Probleme der Mech.*, *Werke V*, esp., § 38, pp. 373, *et seq.*

([414]) In the case of $r = n$, one will have a transformation that allows one to represent the new position coordinates $Q_1, \ldots, Q_n$ as functions of the old position coordinates $q_1, \ldots, q_n$.

Finally, one can further generalize the transformation in such a way that one also replaces the independent variable t with a new variable T. The problem is then to introduce new variables P_ρ, Q_ρ, T in place of the original variables p_ρ, q_ρ, t by a transformation:

$$(477)\qquad \begin{cases} p_\rho = \varphi_\rho(P_1,\ldots,P_n,Q_1,\ldots,Q_n,T), \\ q_\rho = \psi_\rho(P_1,\ldots,P_n,Q_1,\ldots,Q_n,T), \\ t = \chi(P_1,\ldots,P_n,Q_1,\ldots,Q_n,T) \end{cases}$$

such that a canonical system will once more arise from every canonical system.

The transformations that are defined in that way will take *every* canonical system to another canonical system might also have the same function H. Those are the *canonical transformations in the proper sense*. By contrast, it seems that for an *individual* canonical system (456) with a fixed function H, one might also possibly ask the question of what the transformations would be that would only take that one system to another canonical system ([415]) (cf., no. **34**), in which one must

([415]) **S. Lie**, "Die Störungstheorie und die Berührungstransformationen," Arch. for Math. og Naturvodensk. **2** (1877), pp. 129 = *Werke III*, pp. 295. There, the problem is treated as Problem III, and the last of formulas (477) is assumed in the form $t = T$. – Whereas the proper canonical transformations must take the bilinear covariant:

$$(a)\qquad \sum(\delta p_\rho\, dq_\rho - \delta q_\rho\, dp_\rho) - (\delta H\, dt - dH\,\delta t)$$

for the original system to:

$$(b)\qquad \sum(\delta P_\rho\, dQ_\rho - \delta Q_\rho\, dP_\rho) - (\delta K\, dt - dK\,\delta t),$$

only the differential form (b) needs to exist for the desired transformation (**S. Lie** argued). However, it does not need to arise from (a) under the transformation, but it can arise from a different system of associated second-order differential forms under the transformation.

For the transformation of a bilinear differential form, cf., **S. Kantor**, "Über einen neuen Gesichtspunkt in der Theorie des Pfaffschen Problems, der Funktionengruppen und der Berührungstransformationen," Wien Sitzungsber. **110** (1901), II[a], pp. 1147. In that article, he juxtaposed the *normal form* of a (skew-symmetric) bilinear differential form in $2r$ variables:

$$(478)\qquad \sum_{\rho=1}^{r}(\delta x_\rho\, dx_{r+\rho} - dx_\rho\,\delta x_{r+\rho})$$

with the general (skew-symmetric) bilinear differential form in $2r$ variables:

$$(478.a)\qquad \sum_{\rho,\sigma=1}^{2r} c_{\rho\sigma}(u_1,\ldots,u_{2r})(du_\rho\,\delta u_\sigma - du_\rho\,\delta u_\sigma)$$

in precisely the same way that the theory of quadratic differential forms (i.e., the arc-length element for an M_r) juxtaposes the general form:

$$(479)\qquad \sum_{\rho,\sigma=1}^{r} g_{\rho\sigma}(u_1,\ldots,u_r)\, du_\rho\, du_\sigma$$

with the Euclidian normal form:

further demand, in particular (as would seem natural based upon the applications to the theory of perturbations), that the form of the function $K(P_1, \ldots, P_n, Q_1, \ldots, Q_n, t)$ can be prescribed from the outset for the newly-created canonical system:

$$(480) \qquad \frac{dQ_\rho}{dt} = \frac{\partial K}{\partial P_\rho}, \qquad \frac{dP_\rho}{dt} = -\frac{\partial K}{\partial Q_\rho}.$$

Meanwhile, in contrast to the opinion of **S. Lie**, **C. Carathéodory** drew attention to that problem statement without pointing out its intrinsic connection with the canonical form. That is because if one imagines, on the one hand, integrating the canonical system (456):

$$(481) \qquad p_\rho = p_\rho(t, c_1, \ldots, c_{2n}), \qquad q_\rho = q_\rho(t, c_1, \ldots, c_{2n}),$$

and on the other hand, the canonical system (480), as well:

$$(481.a) \qquad P_\rho = P_\rho(t, c_1, \ldots, c_{2n}), \qquad Q_\rho = Q_\rho(t, c_1, \ldots, c_{2n}),$$

then one will need only to set:

$$(482) \qquad c_\sigma = \varphi_\sigma(C_1, \ldots, C_{2n}) \qquad (\sigma = 1, \ldots, 2n)$$

in which the φ_σ are completely-arbitrary functions, in order to obtain a transformation of the desired kind. Namely, every integral curve of the one system will be associated with an integral curve of the other system by (482). Therefore, when one eliminates c_σ, C_σ from equations (481), (481.a), and (482), one must obtain the transformation that takes the canonical system (456) to the canonical system (480). However, since an arbitrary system of differential equations can go to a corresponding system that is it associated with it the same way, no problem will arise that is specifically linked with the canonical form of the differential equations.

32. The substitution function. – Should the transformation (477) take *any* canonical system:

$$(479.a) \qquad \sum_{\rho=1}^{r} dx_\rho^2 .$$

Just as an arbitrary quadratic differential form (479) cannot always be brought into the Euclidian form (479.a) by a coordinate transformation, similarly, a bilinear form (478.a) cannot always be brought into the normal form (478). For the quadratic differential form (479), in order for the transformation to (479.a) to be possible, it is necessary that the curvature tensor must vanish. For the differential form (478.a), one must correspondingly be able to give $2r$ functions $U_1(u_1, \ldots, u_{2r}), \ldots, U_{2r}(u_1, \ldots, u_{2r})$ with whose help one can put the $2r(2r-1)/2$ coefficients $c_{\rho\sigma}$ into the form:

$$c_{\rho\sigma} = \frac{\partial U_\rho}{\partial u_\sigma} - \frac{\partial U_\sigma}{\partial u_\rho}.$$

(483) $$\frac{dq_\rho}{dt} = \frac{\partial H}{\partial p_\rho}, \qquad \frac{dp_\rho}{dt} = -\frac{\partial H}{\partial q_\rho} \qquad (\rho = 1, \ldots, n)$$

to another canonical system:

(483.a) $$\frac{dQ_\rho}{dT} = \frac{\partial K}{\partial P_\rho}, \qquad \frac{dP_\rho}{dT} = -\frac{\partial K}{\partial Q_\rho} \qquad (\rho = 1, \ldots, n),$$

then the relation:

(484) $$\sum_{\rho=1}^{n} (\delta p_\rho \, dq_\rho - \delta q_\rho \, dp_\rho) - (\delta H \, dt - dH \, \delta t) = 0$$

would have to go to the relation:

(484.a) $$\sum_{\rho=1}^{n} (\delta P_\rho \, dQ_\rho - \delta Q_\rho \, dP_\rho) - (\delta K \, dT - dK \, \delta T) = 0$$

under that transformation, i.e., one would need to have ([416]):

(484.b) $$\sum_{\rho=1}^{n} (\delta p_\rho \, dq_\rho - \delta q_\rho \, dp_\rho) - (\delta H \, dt - dH \, \delta t) = \sum_{\rho=1}^{n} (\delta P_\rho \, dQ_\rho - \delta Q_\rho \, dP_\rho) - (\delta K \, dT - dK \, \delta T).$$

It will then follow from that relation (484.b) that the linear differential form that belongs to (483):

(485) $$\sum_{\rho=1}^{n} p_\rho \, dq_\rho - H \, dt$$

can differ from the linear differential form that belongs to (483.a):

(485.a) $$\sum_{\rho=1}^{n} P_\rho \, dQ_\rho - K \, dT$$

and to which (485) will go under the transformation (477), by only the total differential of a function of the P_ρ, Q_ρ, T ([416.a]). The relation:

(485.b) $$\sum_{\rho=1}^{n} p_\rho \, dq_\rho - H \, dt = \sum_{\rho=1}^{n} P_\rho \, dQ_\rho - K \, dT + dW,$$

in which:

([416]) The proportionality of the two expressions can be easily converted into an equality (cf., no. **34**).
([416.a]) Because the bilinear covariant of a total differential is identically zero.

(485.c) $$W = W(P_1, \ldots, P_n, Q_1, \ldots, Q_n, T)$$

is an arbitrary function of the given variables, must then become an identity as a result of the transformation (477) ([416.b]).

In order to capture the essence of the transformation (477) thus-characterized more precisely, one can follow **C. Carathéodory** ([417]) and imagine that the solutions of the canonical systems (483) and (483.a) have been determined as functions of the independent variables t (T, resp.) and the initial values $p_\rho^{(0)}$, $q_\rho^{(0)}$ ($P_\rho^{(0)}$, $Q_\rho^{(0)}$, resp.) that belong to t_0 (T_0, resp.):

(486) $$\begin{cases} p_\rho = f_\rho(t, p_1^{(0)}, \ldots, p_n^{(0)}, q_1^{(0)}, \ldots, q_n^{(0)}), \\ q_\rho = f_\rho(t, p_1^{(0)}, \ldots, p_n^{(0)}, q_1^{(0)}, \ldots, q_n^{(0)}) \end{cases}$$

or

(486.a) $$\begin{cases} P_\rho = F_\rho(T, P_1^{(0)}, \ldots, P_n^{(0)}, Q_1^{(0)}, \ldots, Q_n^{(0)}), \\ Q_\rho = G_\rho(T, P_1^{(0)}, \ldots, P_n^{(0)}, Q_1^{(0)}, \ldots, Q_n^{(0)}), \end{cases}$$

resp. If one defines the principal functions:

$$S_1(q_1, \ldots, q_n, t, q_1^{(0)}, \ldots, q_n^{(0)}) \qquad [S_2(Q_1, \ldots, Q_n, T, Q_1^{(0)}, \ldots, Q_n^{(0)}), \text{ resp.}]$$

for the variational problems that belong to the canonical systems (483) [(483.a), resp.] then one will have:

$$\sum_{\rho=1}^{n} p_\rho \, dq_\rho - H \, dt = \sum_{\rho=1}^{n} p_\rho^{(0)} \, dq_\rho^{(0)} + dS_1$$

and correspondingly ([417.a]):

$$\sum_{\rho=1}^{n} P_\rho \, dQ_\rho - K \, dt = \sum_{\rho=1}^{n} P_\rho^{(0)} \, dQ_\rho^{(0)} + dS_2 .$$

It will then follow from (485.b) that:

([416.b]) It follows immediately from the definition that the composition of two canonical transformations will again be a canonical transformation. The canonical transformations will then have the *group* property, and indeed the set of all canonical transformations will define an *infinite group*, due to the appearance of the arbitrary functions. Cf., II A 6 (**L. Maurer** and **H. Burkhardt**), no. **22**.

([417]) **C. Carathéodory**, "Les transformations canoniques de glissement et leur application à l'optique géometrique," Roma Linc. Rend. (6) **12** (1930), pp. 353.

([417.a]) Therefore, when one thinks of t as being fixed in advance, formulas (486) will mediate a canonical transformation between the $p_\rho^{(0)}$, $q_\rho^{(0)}$ and the p_ρ, q_ρ. If one regards t as a variable parameter then one will have a one-parameter family of canonical transformations. Similarly, formulas (486.a) represent a family of canonical transformations between the $P_\rho^{(0)}$, $Q_\rho^{(0)}$ and the P_ρ, Q_ρ for which the parameter of the family is T. **C. Carathéodory** referred to those special canonical transformations as *sliding transformations*, because an individual point will slide along the space-time line along with its impulse vector as t (T, resp.) varies (cf., no. **34**).

(487) $$\sum_{\rho=1}^{n} p_\rho^{(0)}\,dq_\rho^{(0)} - \sum_{\rho=1}^{n} P_\rho^{(0)}\,dQ_\rho^{(0)} = dW - dS_1 + dS_2 = dV\,,$$

in which, as **C. Carathéodory** showed, V can depend upon only the $P_\rho^{(0)}$, $Q_\rho^{(0)}$, while it is independent of T ([417.b]):

(487.a) $$V = V\,(P_1^{(0)},\ldots,P_n^{(0)},Q_1^{(0)},\ldots,Q_n^{(0)})\,,$$

as long as one imagines that W, S_1, and S_2 are represented as functions of $P_\rho^{(0)}$, $Q_\rho^{(0)}$ with the help of (477), (486), and (486.a) ([417.c]).

The transformation (477), which takes the canonical system (483) with the given function H to the canonical system (483.a) with the given function K, is then well-defined when one, on the one hand, prescribes the function V [the associated function V^*, resp., ([417.d])], which exhibits the connection between the integral curves of both systems, along with the relations:

([417.b]) That says: The function V mediates a transformation that associates every integral curve of the one canonical system with an integral curve of the other canonical system.

([417.c]) The functional determinant of a canonical transformation is always non-zero, cf., ([424.a]).

Since one further has that due to the fact that:

$$p_\rho\,dq_\rho = -\,q_\rho\,dp_\rho + d\,(p_\rho\,q_\rho)\,,$$

the transformation:

$$p_\rho^* = -\,q_\rho\,, \qquad q_\rho^* = p_\rho\,,$$

i.e., the permutation of *one* pair of variables p_ρ, q_ρ with the same index, will also be a canonical transformation, namely, it represents a so-called *elementary canonical transformation*, with **C. Carathéodory**'s terminology (cf., **C. Carathéodory**, *Variationsrechnung*, Leipzig and Berlin 1935, Chap. 6), one can then assume (once one has possibly performed a number of elementary canonical transformations) that the functional determinants:

$$\left|\frac{\partial P_\rho}{\partial q_\sigma}\right|, \text{ etc.,}$$

will be non-zero, such that all of the conversions that one has imagined performing in the text will actually be possible.

([417.d]) It is assumed in so doing that one can solve the transformation formulas:

$$p_\rho^{(0)} = p_\rho^{(0)}\,(P_1^{(0)},\ldots,P_n^{(0)},Q_1^{(0)},\ldots,Q_n^{(0)})\,,$$
$$q_\rho^{(0)} = q_\rho^{(0)}\,(P_1^{(0)},\ldots,P_n^{(0)},Q_1^{(0)},\ldots,Q_n^{(0)})\,,$$

for the $P_1^{(0)}$, …, $P_n^{(0)}$, and in that way, one can take V to a function V^* of the $q_1^{(0)}$, …, $q_n^{(0)}$, $Q_1^{(0)}$, … $Q_n^{(0)}$.

V and V^* are coupled by the relation:

$$V^* = V\left(-\frac{\partial V}{\partial Q_1^{(0)}},\ldots,-\frac{\partial V}{\partial Q_n^{(0)}},Q_1^{(0)},\ldots,Q_n^{(0)}\right),$$

(488) $$p_\rho^{(0)} = \frac{\partial V^*}{\partial q_\rho^{(0)}}, \quad P_\rho^{(0)} = -\frac{\partial V^*}{\partial Q_\rho^{(0)}},$$

and on the other hand, prescribes the last of the transformation formulas (477):

(488.a) $$t = \chi(P_1, \dots, P_n, Q_1, \dots, Q_n, T)$$

arbitrarily. In that way, from (487), the function W will then be given in the formula (485.b) by:

(489) $$W = S_1 - S_2 + V,$$

in which one must express the quantities q_ρ, $q_\rho^{(0)}$, $Q_\rho^{(0)}$, and t in the right-hand side in terms of P_ρ, Q_ρ, T by means of (486), (486.a), (488), and (488.a). Conversely, the transformation (477) will also be determined for given functions H and K when the function $W(P_1, \dots, P_n, Q_1, \dots, Q_n, T)$ is prescribed arbitrarily.

For the applications in mechanics, essentially the only special case that comes under consideration is the one in which the independent variables remain unchanged, so the relation (488.a) will have the form:

(490) $$t = T.$$

The relation (485.b) will then simplify to:

(491) $$\sum_{\rho=1}^{n} p_\rho \, dq_\rho - \sum_{\rho=1}^{n} P_\rho \, dQ_\rho - (H - K)\, dt = dW(P_1, \dots, P_n, Q_1, \dots, Q_n, t),$$

which must become an identity under the transformation:

(492) $$\begin{cases} p_\rho = \varphi_\rho(P_1, \dots, P_n, Q_1, \dots, Q_1, t), \\ q_\rho = \psi_\rho(P_1, \dots, P_n, Q_1, \dots, Q_1, t). \end{cases}$$

If it is possible to solve the second group of those equations for the $P_1, \dots, P_n$ and to give the transformation formulas (492) the form:

(492.a) $$\begin{cases} p_\rho = g_\rho(q_1, \dots, q_n, Q_1, \dots, Q_1, t), \\ P_\rho = h_\rho(q_1, \dots, q_n, Q_1, \dots, Q_1, t) \end{cases}$$

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which one can regard as a partial differential equation for the determination of V^* when one is given the function $V(P_1^{(0)}, \dots, P_n^{(0)}, Q_1^{(0)}, \dots Q_n^{(0)})$. V^* would then be determined as the complete solution to that differential equation with the constants $q_1^{(0)}, \dots, q_n^{(0)}$.

then the relation (491) will go to:

(493) $$\sum_{\rho=1}^{n} p_\rho \, dq_\rho - \sum_{\rho=1}^{n} P_\rho \, dQ_\rho - (H-K)\,dt = dW^*(q_1, \ldots, q_n, Q_1, \ldots, Q_n, t) .$$

Now, should that be true identically in the q_ρ, Q_ρ, t then one would need to have:

(494) $$p_\rho = \frac{\partial W^*}{\partial q_\rho}, \qquad P_\rho = -\frac{\partial W^*}{\partial Q_\rho}, \qquad K = H + \frac{\partial W^*}{\partial t},$$

such that one will necessarily be led to a generalization of the Ansatz by which **C. G. J. Jacobi** (cf., no. **31**) first arrived at a canonical transformation upon starting from **Hamilton**'s theory of perturbations. The canonical transformation in the form (492) then arises from a single function $W^*(q_1, \ldots, q_n, Q_1, \ldots, Q_n, t)$, and according to **E. Schering** ([418]), that is again referred to as a *substitution function* ([419]). The general transformation (477), under which time is also transformed,

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([418]) **E. Schering**, *loc. cit.* ([412]), *Werke I*, pp. 214.

([419]) One sees from these formulas (494) that the integration of the canonical system (483) can also be regarded as a problem in canonical transformation, because the integration of the system (483) will be complete when one takes it to a system of the form:

(495) $$\frac{dQ_\rho}{dt} = 0, \qquad \frac{dP_\rho}{dt} = 0$$

by a canonical transformation (492). Since K obviously cannot include the P_ρ, Q_ρ, it would be simplest for one to take $K \equiv 0$. From (494), one must then determine the substitution function W^* from the partial differential equation:

$$\frac{\partial W^*}{\partial t} + H\left(\frac{\partial W^*}{\partial q_1}, \ldots, \frac{\partial W^*}{\partial q_n}, q_1, \ldots, q_n, t\right) = 0,$$

i.e., from the **Hamilton-Jacobi** equation of the given canonical system, and indeed as a complete solution to the equation. If one denotes the n essential constants in one such solution by $Q_1, \ldots, Q_n$ then the equations:

(496) $$\frac{\partial W^*}{\partial Q_1} = -P_1, \ldots, \frac{\partial W^*}{\partial Q_n} = -P_n,$$

together with:

(496.a) $$\frac{\partial W^*}{\partial q_1} = p_1, \ldots, \frac{\partial W^*}{\partial q_n} = p_n,$$

will represent the canonical substitution. However, from the results of no. **17**, equations (496) are precisely the equations of the integral curves of the given canonical system (483). That remark is also found in **E. Schering**, *loc. cit.* ([412]), *Werke I*, pp. 218.

If one has the special case in which H is independent of t, and therefore, the energy integral exists:

$$H = k$$

then one will have:

$$W^* = -k\,t + V,$$

differs from the transformation (492) by the fact that the function K of the transformed canonical system can no longer be prescribed now when the substitution function W is prescribed arbitrarily, but must be determined by way of the substitution W corresponding to (494). Conversely, should K be a prescribed function (such as, e.g., in perturbation theory), then the substitution W could not be given arbitrarily ([419.a]).

One will get the special case that **Jacobi** was the first to treat when one further assumes that the transformation (492) does not include time t explicitly, so it possesses the form:

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in which V is a complete solution to the differential equation:

$$H\left(\frac{\partial V}{\partial q_1},\ldots,\frac{\partial V}{\partial q_n},q_1,\ldots,q_n\right) = k\,.$$

If one correspondingly takes $V(q_1, \ldots, q_n, k, c_1, \ldots, c_{n-1})$ to be one such complete solution and sets:

$$k = Q_1, \qquad c_1 = Q_2\,, \qquad \ldots, \qquad c_{n-1} = Q_n$$

then the relations:

$$-\,t + \frac{\partial V}{\partial Q_1} = -P_1\,, \qquad \ldots, \qquad \frac{\partial V}{\partial Q_n} = -P_n\,,$$

$$\frac{\partial V}{\partial q_1} = p_1\,, \qquad \ldots, \qquad \frac{\partial V}{\partial q_n} = p_n$$

will enter in place of (496) and (496.a). The transformation:

$$\frac{\partial V}{\partial Q_1} = -\,P_1^*, \qquad \ldots, \qquad \frac{\partial V}{\partial Q_n} = -\,P_n^*,$$

$$\frac{\partial V}{\partial q_1} = p_1\,, \qquad \ldots, \qquad \frac{\partial V}{\partial q_n} = p_n$$

will then convert the canonical system (483) into:

$$\frac{dQ_1}{dt} = 0\,, \qquad \frac{dP_1^*}{dt} = 1\,,$$

$$\frac{dQ_\rho}{dt} = 0\,, \qquad \frac{dP_\rho^*}{dt} = 0 \qquad\qquad (\rho = 2, \ldots, n),$$

according to (495).

([419.a]) From (499), W^* must satisfy the partial differential equation:

$$\frac{\partial W^*}{\partial t} + H\left(\frac{\partial W^*}{\partial q_1},\ldots,\frac{\partial W^*}{\partial q_n},q_1,\ldots,q_n,t\right) = K\left(-\frac{\partial W^*}{\partial Q_1},\ldots,-\frac{\partial W^*}{\partial Q_n},Q_1,\ldots,Q_n,t\right)\,.$$

For the determination of the transformation, cf., no. **34**.

(497) $$\begin{cases} p_\rho = p_\rho(P_1, \dots, P_n, Q_1, \dots, Q_1), \\ q_\rho = q_\rho(P_1, \dots, P_n, Q_1, \dots, Q_1). \end{cases}$$

In so doing, one must assume that the function W is independent of t, so W^* will naturally be independent to t, as well. Obviously, from (494), one will then have:

(497.a) $$K = H ,$$

i.e., the function K will arise from H simply by introducing the new variables by means of (497).

As **E. Schering** ([419.b]) had shown before, the transformation (492), in which t appears explicitly, can be reduced to the **Jacobi** special case in which t does not appear explicitly. In order to do that, one takes the temporal derivative of the function W^* that appears in (493):

(494.a) $$\frac{\partial W^*}{\partial t} = E ,$$

such that from (494):

(494.b) $$E = K - H ,$$

and imagine that E is expressed as a function of the P_ρ, Q_ρ, t. If one then writes down the canonical system:

(498) $$\frac{dQ_\rho}{dt} = \frac{\partial E}{\partial P_\rho}, \qquad \frac{dP_\rho}{dt} = -\frac{\partial E}{\partial Q_\rho} \qquad (\rho = 1, \dots, n)$$

then its solutions:

(498.a) $$\begin{cases} P_\rho = P_\rho(t, P_1^*, \dots, P_n^*, Q_1^*, \dots, Q_n^*), \\ Q_\rho = Q_\rho(t, P_1^*, \dots, P_n^*, Q_1^*, \dots, Q_n^*), \end{cases}$$

in which the P_ρ^*, Q_ρ^* might be the initial values of the P_ρ, Q_ρ for any value t^* of t, will produce a canonical transformation of the P_ρ, Q_ρ into the P_ρ^*, Q_ρ^*. That is because Ψ ($Q_1, \dots, Q_n, t, Q_1^*, \dots, Q_n^*$) is the principal function of the variational problem that the canonical system (498) belongs to, so one will have:

(498.b) $$d\,\Psi = \sum_{\rho=1}^{n} P_\rho \, dQ_\rho - E\,dt - \sum_{\rho=1}^{n} P_\rho^* \, dQ_\rho^* .$$

If one combines that relation with (491) then it will follow that:

([419.b]) **E. Schering**, "Verallgemeinerung der Poisson-Jacobischen Störungsformeln," Gött. Abh. **19** (1874), pp. 3 = *Werke I*, pp. 247, cf., esp. pp. 259.

(499) $$\sum_{\rho=1}^{n} p_\rho \, dq_\rho - \sum_{\rho=1}^{n} P_\rho^* \, dQ_\rho^* = d\,(W + \Psi)\,,$$

in which the time t can no longer appear in the function:

(499.a) $$U\,(P_1^*, \ldots, P_n^*, Q_1^*, \ldots, Q_n^*) = W + \Psi\,.$$

The relations between the p_ρ , q_ρ and the P_ρ^*, Q_ρ^* that one obtains when one substitutes (498.a) in (492) must then be free of t and represent a canonical transformation with the same form as **Jacobi**'s special case.

One can also attempt to classify the general canonical transformation (477), under which the independent variable t is also transformed, within **Jacobi**'s special case. That is because one can correspondingly set simply:

(500) $$\begin{cases} t = q_{n+1}, & -H = p_{n+1}, \\ T = Q_{n+1}, & -K = P_{n+1} \end{cases}$$

in the relation (485) and corresponding introduce a relation of the form:

(500.a) $$\sum_{\rho=1}^{n+1} p_\rho \, dq_\rho - \sum_{\rho=1}^{n+1} P_\rho \, dQ_\rho = dW\,(P_1, \ldots, P_{n+1}, Q_1, \ldots, Q_{n+1})$$

in place of (485.b) ([420]), in which one should observe that here the function W depends upon one more independent variable than the similarly-denoted function in (485.c). If:

(501) $$\begin{cases} p_\rho = \varphi_\rho(P_1, \ldots, P_{n+1}, Q_1, \ldots, Q_{n+1}), \\ q_\rho = \psi_\rho(P_1, \ldots, P_{n+1}, Q_1, \ldots, Q_{n+1}) \end{cases} \qquad (\rho = 1, \ldots, n + 1)$$

is the most general transformation that fulfills that condition then one can get from it to the desired transformation of the form (477) in the following way: The $p_1, \ldots, p_n, p_{n+1}, q_1, \ldots, q_n, q_{n+1}$ are not independent, but are coupled by the relation:

(502) $$p_{n+1} + H\,(p_1, \ldots, p_n, q_1, \ldots, q_n, q_{n+1}) = 0\,,$$

which is combined with (501). If that relation (502) goes to the equation:

(503) $$F\,(P_1, \ldots, P_n, P_{n+1}, Q_1, \ldots, Q_n, Q_{n+1}) = 0$$

([420]) Cf., **G. Morera**, "Sulla trasformazione delle equazioni differenziali di Hamilton, Nota I," Roma Acc. Linc. Rend. (5) **12**[1] (1903), pp. 113 (cf., esp., no. 5, pp. 119).

under the transformation and one solves it for P_{n+1} ([421]) then one will get:

$$P_{n+1} + K(P_1, \ldots, P_n, Q_1, \ldots, Q_n, Q_{n+1}) = 0 . \tag{503.a}$$

Thus, the function cannot be chosen arbitrarily here, unlike before, but is determined by the transformation. That was to be expected, since the function W in (500.a) indeed no longer includes an independent variable, so it will have a more general character than the likewise-denoted function (485.c). Now, if the function K is established in that way then one will get the desired transformation of the form (477) when one substitutes that function for P_{n+1} in the transformation formulas (501). Therefore, one can now drop the formula:

$$p_{n+1} = \varphi_{n+1}(P_1, \ldots, P_n, -K, Q_1, \ldots, Q_{n+1})$$

from (501), since it must be identical to (502) as long as one introduces the newly-obtained expressions for the $p_1, \ldots, p_n, q_1, \ldots, q_n, q_{n+1}$ in the function H. Finally, if one again writes t in place of q_{n+1} and T in place of Q_{n+1} then one will get the desired general transformation of the form (477).

For the connection between the canonical transformation that is defined by (491) and its substitution function, one should observe the following: If the second group of transformation formulas (492) [(497), resp.] *cannot* be solved for the $P_1, \ldots, P_n$ then when one eliminates the $P_1, \ldots, P_n$ from them, one will get a number – say, k – of relations:

$$\left\{\begin{array}{l} \Omega_1(q_1, \ldots, q_n, Q_1, \ldots, Q_n, t) = 0, \\ \Omega_2(q_1, \ldots, q_n, Q_1, \ldots, Q_n, t) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \Omega_k(q_1, \ldots, q_n, Q_1, \ldots, Q_n, t) = 0 \end{array}\right. \tag{504}$$

between the p_ρ, Q_ρ, and t. One can then solve the transformation formulas (492) for $n - k$ of the P_ρ, say, for $P_1, \ldots, P_{n-k}$, and the $p_1, \ldots, p_n$, the $P_1, \ldots, P_n$, and ultimately W^* will be functions in which the $P_{n-k+1}, \ldots, P_n$ can appear, in addition to the p_ρ, Q_ρ, and t. However, one sees immediately from the formulas (493) that one must have:

$$\frac{\partial W^*}{\partial P_{n-k+1}} = 0 , \quad \ldots, \qquad \frac{\partial W^*}{\partial P_n} = 0 ,$$

such that the $P_{n-k+1}, \ldots, P_n$ cannot appear in the W^*, but rather W^* will also be a function of only the $q_1, \ldots, q_n, Q_1, \ldots, Q_n, t$ now. If one now considers the auxiliary conditions (504) using the method of **Lagrange** factors then one will generally obtain a canonical transformation when one

([421]) One can also arrange the calculation in such a way that this solution will be possible. Cf., **G. Morera**, *loc. cit.* ([420]).

is given the function W^*, as well as the functions $\Omega_1, \ldots, \Omega_n$, arbitrarily as functions of the $q_1, \ldots, q_n, Q_1, \ldots, Q_n, t$ and writes out the equations:

(505)
$$\begin{cases} p_\rho = \dfrac{\partial W^*}{\partial q_\rho} + \lambda_1 \dfrac{\partial \Omega_1}{\partial q_\rho} + \cdots + \lambda_k \dfrac{\partial \Omega_k}{\partial q_\rho}, \\ P_\rho = -\left(\dfrac{\partial W^*}{\partial Q_\rho} + \lambda_1 \dfrac{\partial \Omega_1}{\partial Q_\rho} + \cdots + \lambda_k \dfrac{\partial \Omega_k}{\partial Q_\rho} \right), \end{cases}$$

which will establish the transformation, in conjunction with equations (504). In that way, the function H will be replaced with the new function ([422]):

(505.a)
$$K = H + \frac{\partial W^*}{\partial t} + \lambda_1 \frac{\partial \Omega_1}{\partial t} + \cdots + \lambda_k \frac{\partial \Omega_k}{\partial t}.$$

The transformation (492) will be especially simple when one takes the *function* W^* *to be identically zero* (cf., no. **34**) in the expression (493), so one tries to determine the transformation in such a way that it makes the relation:

(506)
$$\sum_{\rho=1}^{n} p_\rho \, dq_\rho - \sum_{\rho=1}^{n} P_\rho \, dQ_\rho - (H - K)\, dt = 0$$

into an identity. The transformation formulas (505) then simplify to:

([422]) **E. Schering**, "Verallgemeinerung der Poisson-Jacobischen Störungsformeln," Gött. Abhandl. **19** (1874), pp. 3 = *Werke I*, proceeded in this case in such a way [cf., ([417.c])] that he converted equation (493) by an elementary canonical transformation into:

$$\sum_{\rho=1}^{n} p_\rho \, dq_\rho - \sum_{\sigma=1}^{n-k} P_\rho \, dQ_\rho + \sum_{\tau=n-k+1}^{n} Q_\tau \, dP_\tau = d\left(W^* + \sum_{\tau=n-k+1}^{n} P_\tau \, Q_\tau \right) = dS,$$

in which $S = W^* + \sum_{\tau=n-k+1}^{n} P_\tau \, Q_\tau$ is then expressed as a function of the $q_1, \ldots, q_n, Q_1, \ldots, Q_{n-k}, P_{n-k+1}, \ldots, P_n$. The transformation formulas for the canonical transformation will then be:

$$\begin{aligned} p_1 &= \frac{\partial S}{\partial q_1}, \quad \cdots \quad p_{n-k} = \frac{\partial S}{\partial q_{n-k}}, \quad p_{n-k+1} = \frac{\partial S}{\partial q_{n-k+1}}, \quad \cdots \quad p_n = \frac{\partial S}{\partial q_n}, \\ P_1 &= -\frac{\partial S}{\partial Q_1}, \quad \cdots \quad P_{n-k} = -\frac{\partial S}{\partial Q_{n-k}}, \quad Q_{n-k+1} = -\frac{\partial S}{\partial P_{n-k+1}}, \quad \cdots \quad Q_n = -\frac{\partial S}{\partial P_n}. \end{aligned}$$

(507)
$$\begin{cases} p_\rho = \quad \lambda_1 \dfrac{\partial \Omega_1}{\partial q_\rho} + \cdots + \lambda_k \dfrac{\partial \Omega_k}{\partial q_\rho}, \\ P_\rho = -\left(\lambda_1 \dfrac{\partial \Omega_1}{\partial Q_\rho} + \cdots + \lambda_k \dfrac{\partial \Omega_k}{\partial Q_\rho} \right), \end{cases}$$

which takes the canonical system (483) to a canonical system with the function:

(507.a)
$$K = H + \lambda_1 \frac{\partial \Omega_1}{\partial t} + \cdots + \lambda_k \frac{\partial \Omega_k}{\partial t}.$$

In particular, should the transformation be completely free of t, so it will possess the form (497), then one will have to assume that the functions Ω_i are free of t :

(508)
$$\begin{cases} \Omega_1 (q_1, \ldots, q_n, Q_1, \ldots, Q_n) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Omega_k (q_1, \ldots, q_n, Q_1, \ldots, Q_n) = 0, \end{cases}$$

such that:

(508.a)
$$K = H.$$

The formula that emerges from (506):

(509)
$$\sum_{\rho=1}^{n} p_\rho \, dq_\rho - \sum_{\rho=1}^{n} P_\rho \, dQ_\rho = 0$$

will again lead to the formulas (507) for the transformation, but the independent variable t will no longer appear explicitly in the expressions on the right-hand side ([423]). Formulas (509) and (507) show that multiplying the p_ρ by a factor will imply multiplying the P_ρ by the same factor. In that case, the p_ρ in the first group of transformation formulas (497) must be homogeneous of degree one in the $P_1, \ldots, P_n$, while the q_ρ in the second group must be homogeneous of order zero in $P_1, \ldots, P_n$. Therefore, one ordinarily refers to those transformations as *homogeneous canonical transformations*.

If the number k of equations (508) is equal to n then the old position coordinates q_ρ can be calculated as functions of the new position coordinates Q_ρ using those n equations:

([423]) **E. Mathieu** started from that condition for the canonical transformation in "Mémoire sur les équations différentielles canoniques de la mécanique," J. de math. (2) **19** (1874), pp. 265. Finally, he also gave the condition a more general form by adding a complete differential, but he did not arrive at the general form (493). It is remarkable insofar as he only started from the bilinear covariant, which he adopted from **Lagrange**, in his application of that to the perturbation calculation. Instead of the general case of k relations (508), he only considered the case in which k = 1.

(510.a) $$q_\rho = \varphi_\rho (Q_1, \ldots, Q_n) ,$$

and the canonical transformation will degenerate into an *extended point transformation*. From (509), the new and old impulse components are coupled by the linear relations:

(510.b) $$P_\rho = \sum_{\sigma=1}^{n} p_\sigma \frac{\partial \varphi_\rho}{\partial Q_\sigma}$$

[cf., also the formulas (431) in no. **29**].

33. Conditions for a transformation to be canonical. – Should a transformation:

(511) $$\begin{cases} p_\rho = \varphi_\rho(P_1, \ldots, P_{n+1}, Q_1, \ldots, Q_{n+1}), \\ q_\rho = \psi_\rho(P_1, \ldots, P_{n+1}, Q_1, \ldots, Q_{n+1}) \end{cases}$$

take *every* canonical system to another canonical system, then from what was developed in the precious section, the (abbreviated) bilinear covariant of the new system $\sum_{\rho=1}^{n} (\delta P_\rho \, dQ_\rho - dP_\rho \, \delta Q_\rho)$ must emerge from transforming the bilinear covariant $\sum_{\rho=1}^{n} (\delta p_\rho \, dq_\rho - dp_\rho \, \delta q_\rho)$ of the original system. Now, by means of (511), one will have the relation:

(512) $$\begin{aligned} &\sum_{\rho=1}^{n} (\delta p_\rho \, dq_\rho - dp_\rho \, \delta q_\rho) \\ &= \sum_{\sigma,\tau=1}^{n} \left[\sum_{\rho=1}^{n} \left(\frac{\partial \varphi_\rho}{\partial P_\sigma} \frac{\partial \psi_\rho}{\partial P_\tau} - \frac{\partial \psi_\rho}{\partial P_\sigma} \frac{\partial \varphi_\rho}{\partial P_\tau} \right) \right] (\delta P_\rho \, dQ_\rho - dP_\rho \, \delta Q_\rho) \\ &+ \sum_{\sigma,\tau=1}^{n} \left[\sum_{\rho=1}^{n} \left(\frac{\partial \varphi_\rho}{\partial Q_\sigma} \frac{\partial \psi_\rho}{\partial Q_\tau} - \frac{\partial \psi_\rho}{\partial Q_\sigma} \frac{\partial \varphi_\rho}{\partial Q_\tau} \right) \right] (\delta Q_\rho \, dQ_\rho - dQ_\rho \, \delta Q_\rho) \\ &+ \sum_{\sigma,\tau=1}^{n} \left[\sum_{\rho=1}^{n} \left(\frac{\partial \varphi_\rho}{\partial P_\sigma} \frac{\partial \psi_\rho}{\partial Q_\tau} - \frac{\partial \psi_\rho}{\partial P_\sigma} \frac{\partial \varphi_\rho}{\partial Q_\tau} \right) \right] (\delta P_\rho \, dQ_\rho - dP_\rho \, \delta Q_\rho) . \end{aligned}$$

In order for the right-hand side to be equal to $\sum_{\rho=1}^{n} (\delta P_\rho \, dQ_\rho - dP_\rho \, \delta Q_\rho)$, the following three classes of equations must be satisfied ([424]).

([424]) The first two are identities when $\sigma = \tau$.

$$
(513) \qquad \left\{
\begin{aligned}
&\sum_{\rho=1}^{n}\left(\frac{\partial\varphi_\rho}{\partial P_\sigma}\frac{\partial\psi_\rho}{\partial P_\tau}-\frac{\partial\psi_\rho}{\partial P_\sigma}\frac{\partial\varphi_\rho}{\partial P_\tau}\right)=0,\\
&\sum_{\rho=1}^{n}\left(\frac{\partial\varphi_\rho}{\partial Q_\sigma}\frac{\partial\psi_\rho}{\partial Q_\tau}-\frac{\partial\psi_\rho}{\partial Q_\sigma}\frac{\partial\varphi_\rho}{\partial Q_\tau}\right)=0,\\
&\sum_{\rho=1}^{n}\left(\frac{\partial\varphi_\rho}{\partial P_\sigma}\frac{\partial\psi_\rho}{\partial Q_\tau}-\frac{\partial\psi_\rho}{\partial P_\sigma}\frac{\partial\varphi_\rho}{\partial Q_\tau}\right)=\begin{cases} 0 & (\sigma\neq\tau),\\ 1 & (\sigma=\tau).\end{cases}
\end{aligned}\right.
$$

However, the sums are precisely the ***Lagrange*** *brackets* of the functions (511) that were introduced in no. **12**, such that one can write the conditions (513) in the form ([424.a]):

$$
(513.a) \qquad \left\{
\begin{aligned}
&[P_\sigma, P_\tau]=0,\\
&[Q_\sigma, Q_\tau]=0,\\
&[P_\sigma, Q_\tau]=\begin{cases} 0 & (\sigma\neq\tau)\\ 1 & (\sigma=\tau)\end{cases}
\end{aligned}\right.
$$

([424.a]) It follows immediately from this that the functional determinant of a canonical transformation (511):

$$
D=\begin{vmatrix}
\dfrac{\partial\varphi_1}{\partial P_1} & \cdots & \dfrac{\partial\varphi_n}{\partial P_1} & \dfrac{\partial\psi_1}{\partial P_1} & \cdots & \dfrac{\partial\psi_n}{\partial P_1}\\
\vdots & \cdots & \vdots & \vdots & \cdots & \vdots\\
\dfrac{\partial\varphi_1}{\partial P_n} & \cdots & \dfrac{\partial\varphi_n}{\partial P_n} & \dfrac{\partial\psi_1}{\partial P_n} & \cdots & \dfrac{\partial\psi_n}{\partial P_n}\\
\dfrac{\partial\varphi_1}{\partial Q_1} & \cdots & \dfrac{\partial\varphi_n}{\partial Q_1} & \dfrac{\partial\psi_1}{\partial Q_1} & \cdots & \dfrac{\partial\psi_n}{\partial Q_1}\\
\vdots & \cdots & \vdots & \vdots & \cdots & \vdots\\
\dfrac{\partial\varphi_1}{\partial Q_n} & \cdots & \dfrac{\partial\varphi_n}{\partial Q_n} & \dfrac{\partial\psi_1}{\partial Q_n} & \cdots & \dfrac{\partial\psi_n}{\partial Q_n}
\end{vmatrix}
$$

is always non-zero, because one easily finds that:

$$
D^2=\begin{vmatrix}
[P_1,Q_1] & \cdots & [P_1,Q_n] & [P_1,P_1] & \cdots & [P_1,P_n]\\
\vdots & \cdots & \vdots & \vdots & \cdots & \vdots\\
[P_n,Q_1] & \cdots & [P_n,Q_n] & [P_n,Q_1] & \cdots & [P_n,P_n]\\
[Q_1,Q_1] & \cdots & [Q_1,Q_n] & [Q_1,P_1] & \cdots & [Q_1,P_n]\\
\vdots & \cdots & \vdots & \vdots & \cdots & \vdots\\
[Q_1,Q_1] & \cdots & [Q_n,Q_n] & [Q_n,P_1] & \cdots & [Q_n,P_n]
\end{vmatrix}=1.
$$

Cf., the corresponding calculation ([342]), in which the **Poisson** brackets are introduced in place of the **Lagrange** ones.

by appealing to the notations that were defined by (96).

One can also express the condition in terms of ***Poisson** brackets*, instead of the **Lagrange** brackets. In order to do that, one needs only to observe that from the arguments in no. **26**, the **Poisson** bracket that is formed from two arbitrary functions:

(514) $$(F, G) = \sum_{\rho=1}^{n}\left(\frac{\partial F}{\partial p_\rho}\frac{\partial G}{\partial q_\rho} - \frac{\partial F}{\partial q_\rho}\frac{\partial G}{\partial p_\rho}\right)$$

will transform contragrediently to the bilinear differential form $\sum_\rho (\delta p_\rho\, dq_\rho - dp_\rho\, \delta q_\rho)$. Therefore, one has that:

$$\sum_{\rho=1}^{n}(\delta p_\rho\, dq_\rho - dp_\rho\, \delta q_\rho) = \sum_{\rho=1}^{n}(\delta P_\rho\, dQ_\rho - dP_\rho\, \delta Q_\rho)$$

for the transformation (511), so if the transformation (511) takes the functions:

$$F(p_1, \ldots, p_n, q_1, \ldots, q_n) \quad \text{and} \quad G(p_1, \ldots, p_n, q_1, \ldots, q_n)$$

to

$$\bar{F}(p_1, \ldots, p_n, q_1, \ldots, q_n) \quad \text{and} \quad \bar{G}(p_1, \ldots, p_n, q_1, \ldots, q_n),$$

resp., then the relation ([425]):

(515) $$(F, G) = (\bar{F}, \bar{G})$$

must also be true, i.e., the transformation (511) must take the **Poisson** bracket expression (514) of two arbitrary functions to the **Poisson** bracket of the transformed function. Now, since one has:

(516) $$\begin{aligned}\frac{\partial \bar{F}}{\partial P_\rho}\frac{\partial \bar{G}}{\partial Q_\rho} - \frac{\partial \bar{F}}{\partial Q_\rho}\frac{\partial \bar{G}}{\partial P_\rho} &= \sum_{\sigma,\tau=1}^{n}\left(\frac{\partial F}{\partial p_\sigma}\frac{\partial G}{\partial p_\tau} - \frac{\partial F}{\partial p_\tau}\frac{\partial G}{\partial p_\sigma}\right)\left(\frac{\partial \varphi_\sigma}{\partial P_\rho}\frac{\partial \varphi_\tau}{\partial Q_\rho} - \frac{\partial \varphi_\sigma}{\partial Q_\rho}\frac{\partial \varphi_\tau}{\partial P_\rho}\right) \\ &+ \sum_{\sigma,\tau=1}^{n}\left(\frac{\partial F}{\partial q_\sigma}\frac{\partial G}{\partial q_\tau} - \frac{\partial F}{\partial q_\tau}\frac{\partial G}{\partial q_\sigma}\right)\left(\frac{\partial \psi_\sigma}{\partial P_\rho}\frac{\partial \psi_\tau}{\partial Q_\rho} - \frac{\partial \psi_\sigma}{\partial Q_\rho}\frac{\partial \psi_\tau}{\partial P_\rho}\right) \\ &+ \sum_{\sigma,\tau=1}^{n}\left(\frac{\partial F}{\partial p_\sigma}\frac{\partial G}{\partial q_\tau} - \frac{\partial F}{\partial q_\tau}\frac{\partial G}{\partial p_\sigma}\right)\left(\frac{\partial \varphi_\sigma}{\partial P_\rho}\frac{\partial \psi_\tau}{\partial Q_\rho} - \frac{\partial \varphi_\sigma}{\partial Q_\rho}\frac{\partial \psi_\tau}{\partial P_\rho}\right)\end{aligned}$$

when one sums the **Poisson** brackets of the function in (511) over ρ, it will follow that:

([425]) Also cf., on this, **S. Kantor**, "Über einen neuen Gesichtpunkt in der Theorie des Pfaffschen Problem…," Wien Sitzungsber. **110** (1901), II[a], pp. 1147, esp., pp. 1161, *et seq*. Conversely, the invariance of the bilinear covariant also follows from the relation (515).

(516.a)
$$(\bar{F},\bar{G}) = \sum_{\sigma,\tau=1}^{n}\left(\frac{\partial F}{\partial p_\sigma}\frac{\partial G}{\partial p_\tau}-\frac{\partial F}{\partial p_\tau}\frac{\partial G}{\partial p_\sigma}\right)(\varphi_\sigma,\varphi_\tau)$$
$$+\sum_{\sigma,\tau=1}^{n}\left(\frac{\partial F}{\partial q_\sigma}\frac{\partial G}{\partial q_\tau}-\frac{\partial F}{\partial q_\tau}\frac{\partial G}{\partial q_\sigma}\right)(\psi_\sigma,\psi_\tau)$$
$$+\sum_{\sigma,\tau=1}^{n}\left(\frac{\partial F}{\partial p_\sigma}\frac{\partial G}{\partial q_\tau}-\frac{\partial F}{\partial q_\tau}\frac{\partial G}{\partial p_\sigma}\right)(\varphi_\sigma,\psi_\tau).$$

Should the right-hand side of this reduce to the **Poisson** expression (F, G) then the relations ([426]):

(517)
$$(\varphi_\sigma, \varphi_\tau) = 0, \qquad (\psi_\sigma, \psi_\tau) = 0, \qquad (\varphi_\sigma, \psi_\tau) = \begin{cases} 0 & (\sigma \neq \tau) \\ 1 & (\sigma = \tau) \end{cases}$$

would have to be true, which express the conditions for (511) to be a canonical transformation with the help of the **Poisson** brackets. In that form, they say that the functions φ_σ, ψ_σ are *the canonical basis for a function group* (cf., no. **28**). Thus, if a number of functions φ_σ and ψ_σ are given, say:

(518)
$$\begin{cases} \varphi_1 & \cdots & \varphi_\nu, \\ \psi_1 & \cdots & \psi_\mu, \end{cases}$$

that satisfy the conditions (517) then $(n - \nu)$ functions φ and $n - \mu$ functions ψ can be determined in such a way that that they will define a canonical transformation, along with the given functions (518) ([427]).

If the more general transformation enters in place of (511):

(519)
$$\begin{cases} p_\rho = \varphi_\rho(P_1,\dots,P_n,Q_1,\dots,Q_n,t), \\ q_\rho = \psi_\rho(P_1,\dots,P_n,Q_1,\dots,Q_n,t) \end{cases}$$

then one will immediately come back to the results that were achieved above when one regards it as a transformation of the $2(n + 1)$ variables and combines the original coordinates p_ρ, q_ρ with:

(520)
$$q_{n+1} = t, \qquad p_{n+1} = -H,$$

corresponding to (500), and analogously combines the new coordinates with:

(520.a)
$$Q_{n+1} = T = t = q_{n+1}, \qquad P_{n+1} = -K.$$

([426]) Cf., **E. Bour**, "Sur l'intégration des équ. diff. part. du premier et du sec. ordre," J. Éc. Polyt. **22**, cah. **39** (1862), pp. 149, esp., pp. 156, *et seq.*

([427]) Such extensions of incomplete given canonical transformations to complete ones were treated by **E. Schering**, *loc. cit.*, ([422]), namely, *Werke I*, pp. 257, *et seq.*

The transformation formulas (519) can then be replaced with:

(521)
$$\begin{cases} p_\rho = \varphi_\rho(P_1, \dots, P_n, Q_1, \dots, Q_n, Q_{n+1}), & (\rho = 1, \dots, n), \\ p_{n+1} = \varphi_{n+1}(P_1, \dots, P_n, Q_1, \dots, Q_n, Q_{n+1}), & \\ \qquad = P_{n+1} + E(P_1, \dots, P_n, Q_1, \dots, Q_n, Q_{n+1}), & \\ q_\rho = \psi_\rho(P_1, \dots, P_n, Q_1, \dots, Q_n, Q_{n+1}), & (\rho = 1, \dots, n), \\ q_{n+1} = \psi_{n+1}(Q_{n+1}) = Q_{n+1}, & \end{cases}$$

in which the function $E(P_1, \dots, P_n, Q_1, \dots, Q_n, t)$ is added, corresponding to (494.b). For that transformation, one must then have:

(522)
$$\sum_{\rho=1}^{n} (\delta p_\rho\, dq_\rho - dp_\rho\, \delta q_\rho) - (\delta H\, dt - dH\, \delta t)$$
$$= \sum_{\rho=1}^{n} (\delta P_\rho\, dQ_\rho - dP_\rho\, \delta Q_\rho) - (\delta K\, dt - dK\, \delta t),$$

i.e.:

(522.a)
$$\sum_{\rho=1}^{n+1} (\delta p_\rho\, dq_\rho - dp_\rho\, \delta q_\rho) = \sum_{\rho=1}^{n+1} (\delta P_\rho\, dQ_\rho - dP_\rho\, \delta Q_\rho).$$

The conditions for the fulfillment of that relation can be written in the following way with the help of the **Poisson** brackets: On the one hand, since P_{n+1} appears only in the function φ_{n+1}, the relations (517) for the **Poisson** brackets of the functions (519) must remain valid. On the other hand, one must add the further conditions:

(523)
$$\begin{cases} (\varphi_\rho, \varphi_{n+1}) = 0, \quad (\psi_\rho, \varphi_{n+1}) = 0, & (\rho = 1, \dots, n) \\ (\varphi_\rho, \psi_{n+1}) = 0, \quad (\psi_\rho, \psi_{n+1}) = 0, & \\ (\varphi_{n+1}, \psi_{n+1}) = 0, & \end{cases}$$

in which the **Poisson** brackets are thought of as being formed from the 2 (n + 1) variables. From (521), the second and third row in that:

(524)
$$(\varphi_\rho, \psi_{n+1}) = 0, \qquad (\psi_\rho, \psi_{n+1}) = 0, \qquad (\varphi_{n+1}, \psi_{n+1}) = 1$$

are fulfilled identically, and therefore do not need to be mentioned explicitly. By contrast, the equations of the first row imply the conditions ([428])

(525)
$$\frac{\partial \varphi_\rho}{\partial t} = (\varphi_\rho, E), \qquad \frac{\partial \psi_\rho}{\partial t} = (\psi_\rho, E),$$

([428]) Cf., **E. Schering**, *loc. cit.* ([412]), cf., esp., *Werke I*, pp. 237.

that must be added to (517), in which the **Poisson** brackets are once more thought of as being formed from the $2n$ variables $P_1, \ldots, P_n, Q_1, \ldots, Q_n$.

If one chooses the **Lagrange** brackets, instead of the **Poisson** brackets, for the conditions then in addition to the conditions (513.a), which are true unchanged for the functions (519), one must add the further relations ([429]):

$$(526)\qquad \begin{cases} \displaystyle\sum_{\rho=1}^{n}\left(\frac{\partial\varphi_\rho}{\partial Q_\sigma}\frac{\partial\psi_\rho}{\partial t}-\frac{\partial\psi_\rho}{\partial Q_\sigma}\frac{\partial\varphi_\rho}{\partial t}\right)=[Q_\sigma,t]=-\frac{\partial E}{\partial Q_\sigma}, \\ \displaystyle\sum_{\rho=1}^{n}\left(\frac{\partial\varphi_\rho}{\partial P_\sigma}\frac{\partial\psi_\rho}{\partial t}-\frac{\partial\psi_\rho}{\partial P_\sigma}\frac{\partial\varphi_\rho}{\partial t}\right)=[P_\sigma,t]=-\frac{\partial E}{\partial P_\sigma}. \end{cases}$$

If not all of the functions are given, but only some of the functions φ_ρ, ψ_ρ, and possibly the function E, then they must satisfy the condition equations (517), (525) that one can define with them. Then and only then can the system of functions be extended to a complete canonical transformation ([430]).

34. Connection between canonical transformations and contact transformations. – **S. Lie** ([431]) had already recognized the relation between canonical transformations and the *contact transformations* that he had studied systematically quite early on [cf., III D 7 (**H. Liebmann**), esp. no. **6**, in which the connection between canonical transformations and contact transformations is already referred to]. In order to bring the connection between the two domains into view, one can start from the special case (497) of the canonical transformations and extend the ideas that were developed by **W. R. Hamilton** (cf., no. **13**) in order to explain the contact transformations of the special form that comes into play here in the following way: One understands an *element* in the R_n of the $(q_1, \ldots, q_n)$ to mean the pairing of a point $(q_1, \ldots, q_n)$ with a vector $(p_1, \ldots, p_n)$ that belongs to it. According to **Lie**, such an element is united with its neighboring element $(q_1 + dq_1, \ldots, q_n + dq_n)$ and $(p_1 + dp_1, \ldots, p_n + dp_n)$ when an occupancy z $(q_1, \ldots, q_n)$ of the R_n of the $(q_1, \ldots, q_n)$ can be given that possesses the vectors at the two neighboring points as gradients, i.e., when one has:

$$(527)\qquad p_1\, dq_1 + \ldots + p_n\, dq_n = dz\ .$$

If one now considers a transformation of the element (p_ρ, q_ρ) into the correspondingly-defined element [point $(Q_1, \ldots, Q_n)$ and vector $(P_1, \ldots, P_n)$] in the R_n of the $(Q_1, \ldots, Q_n)$:

$$(528)\qquad p_\rho = \varphi_\rho\,(P_1, \ldots, P_n, Q_1, \ldots, Q_n)\ , \qquad q_\rho = \psi_\rho\,(P_1, \ldots, P_n, Q_1, \ldots, Q_n)$$

([429]) Cf., **E. Schering**, *loc. cit.* ([412]), cf., esp., *Werke I*, pp. 239.

([430]) **E. Schering**, "Verallgemeinerung der Poisson-Jacobischen Störungsformeln…," *Werke I*, pp. 258.

([431]) **S. Lie**, "Die Störungstheorie und die Berührungstransformationen," Arch. for Math. og Naturvid. **2** (1877), pp. 129 = *Werke III*, pp. 295.

then that transformation will be a contact transformation if and only if it takes two united elements into two elements that are once more united, i.e., two elements ($Q_1, \ldots, Q_n, P_1, \ldots, P_n$) and ($Q_1 + dQ_1, \ldots, Q_n + dQ_n, P_1 + dP_1, \ldots, P_n + dP_n$) whose vectors $P_1, \ldots, P_n$ [$P_1 + dP_1, \ldots, P_n + dP_n$, resp.] can be regarded as the gradients of an occupancy function Z ($Q_1, \ldots, Q_n$) in the R_n of the ($Q_1, \ldots, Q_n$). It must follow from (527) then, by way of (528), that:

$$P_1\, dQ_1 + \ldots + P_n\, dQ_n = dZ\ ,$$

resp., which amounts to the same thing as saying that the transformation (528) must make the equation:

$$P_1\, dQ_1 + \ldots + P_n\, dQ_n = \rho\, [dz - (p_1\, dq_1 + \ldots + p_n\, dq_n)] \tag{529}$$

an identity, in which ρ can initially be a function of the ($2n + 1$) variables $z, p_1, \ldots, p_n, q_1, \ldots, q_n$. Meanwhile, since the transformation formulas (528) are free of z ([432]), it will follow that Z must possess the form:

$$Z = A\, z + U\, (p_1, \ldots, p_n, q_1, \ldots, q_n) \tag{530}$$

as a function of $z, p_1, \ldots, p_n, q_1, \ldots, q_n$, in which A, which is identical to ρ, proves to be a constant ([432.a]). One can set the constant A equal to 1 with no further discussion, because that only amounts to introducing $A\, z$ in place of z and correspondingly introducing $A\, p_\rho$ in place of p_ρ, such that (529) will go to:

$$P_1\, dQ_1 + \ldots + P_n\, dQ_n = p_1\, dq_1 + \ldots + p_n\, dq_n + dU\, (p_1, \ldots, p_n, q_1, \ldots, q_n)\ . \tag{530.a}$$

However, that is precisely the relation that defines the canonical transformation in the special case in which the time remains unchanged and the transformation of the (q_ρ, p_ρ) does not enter in, according to no. **32** [cf., eq. (488)]. Hence, when one adds the relation ([432.b]):

$$z = Z + W\, (P_1, \ldots, P_n, Q_1, \ldots, Q_n)$$

those canonical transformations, those canonical transformations (497) will then be *contact transformations of the* (x, p), with **Lie**'s terminology. In that way, with **S. Lie**, one will not refer to the quantity z as an occupancy of the R_n of the ($q_1, \ldots, q_n$), but as a coordinate that is on a par with the $q_1, \ldots, q_n$ and is fundamental to the interpretation of the transformation of the R_{n+1} of the ($z, q_1, \ldots, q_n$). The individual *element* that now belongs to the coordinates $z, q_1, \ldots, q_n, p_1, \ldots, p_n$

([432]) **S. Lie** (cf., *Theorie der Transformationsgruppen II*, Leipzig 1890, pp. 125) used the name "contact transformation in the x, p" for these special contact transformations. Namely, he denoted the coordinates of the point by x_ρ, instead of q_ρ.

Also cf., the presentation by **L. P. Eisenhart**, "Contact transformations," Annals of Math. (2) **30** (1929), pp. 211.

([432.a]) Cf., **S. Lie**, *loc. cit.* ([432]).

([432.b]) In which W is thought of as arising from ($- U$) in (530) by the transformation (528).

determines a point in that R_{n+1} with a planar M_n that goes through it, and two manifolds *contact* at a point when they have an element in common there. The term *contact transformation* shall correspondingly express the idea that two manifolds that have an element in common will always go to two manifolds that have the transformed element in common under the transformation. In that sense, the argument that was posed will show that a canonical transformation in $2n$ variables like (497) can be regarded as a special contact transformation of an R_{n+1}, and that one can also conversely interpret a contact transformation in the (x, p) in an R_{n+1} as a canonical transformation in $2n$ variables.

The *general contact transformation*, as it is established by (529), can also be interpreted as a *canonical transformation*, but generally not as one in $2n$ variables. Rather, in order to interpret it, one must go to $(2n + 2)$ variables by multiplying the relation (529) by λ / ρ, in which λ shall represent a new variable. If one then sets ([432.c]):

$$\frac{\lambda}{\rho} = -P_0 , \qquad Z = Q_0 ,$$
$$\lambda = -p_0 , \qquad z = q_0$$

then (529) will go to:

$$P_0\,dQ_0 + P_1\,dQ_1 + \ldots + P_n\,dQ_n = p_0\,dq_0 + p_1\,dq_1 + \ldots + p_n\,dq_n ,$$

and one will see that one has a *homogeneous canonical transformation* in $(2n + 2)$ variables. Conversely, it will follow that the canonical transformation (497) can be regarded as a contact transformation in the R_n of the $(q_1, \ldots, q_n)$ if and only if it is a *homogeneous* canonical transformation.

Accordingly, the general canonical transformation (477), under which time is also transformed, will not generally represent a contact transformation of the R_{n+1} of the $(q_1, \ldots, q_n, t)$, either. It is only when one has $dW \equiv 0$ in the relation (485.b), so when the function W is a constant, that one will have a contact transformation of the R_{n+1} ([433]). That is because one can then give (485.b) the form:

$$dt - \sum_{\rho=1}^{n} \frac{p_\rho}{H}\,dq_\rho = \frac{K}{H}\left(dT - \sum_{\rho=1}^{n} \frac{P_\rho}{H}\,dQ_\rho\right),$$

and that will be identical to (529) when one takes:

$$z = t , \qquad Z = T , \qquad \rho = \frac{K}{H}$$

([432.c]) One always has $\rho \neq 0$.

([433]) **C. Carathéodory**, "Les transformations canoniques de glissement et leur application à l'optique géométrique," Roma Linc. Rend. (6) **12**^{2} (1930), pp. 353.

and replaces p_ρ with p_ρ / H and P_ρ with P_ρ / H . As **C. Carathéodory** showed, the general transformation (477) can easily be converted into the special one for which one has W = const. identically. Namely, if one imagines that one has determined the solution of the canonical system (483) that assumes the values $p_\rho = p_\rho^*$, $q_\rho = q_\rho^*$ for $t = t^*$:

(531)
$$\begin{cases} p_\rho = p_\rho\,(t, p_1^*, \dots, p_n^*, q_1^*, \dots, q_n^*), \\ q_\rho = q_\rho\,(t, p_1^*, \dots, p_n^*, q_1^*, \dots, q_n^*) \end{cases}$$

and adds the relation:

(531.a)
$$t = \varphi(t^*, p_1^*, \dots, p_n^*, q_1^*, \dots, q_n^*) \ ,$$

which is initially completely arbitrary, then when one introduces latter into (531), one will obtain a canonical transformation:

(532)
$$\begin{cases} p_\rho^* = f_\rho\,(t, p_1, \dots, p_n, q_1, \dots, q_n), \\ q_\rho^* = q_\rho\,(t, p_1, \dots, p_n, q_1, \dots, q_n), \\ t^* = \ h(t, p_1, \dots, p_n, q_1, \dots, q_n) \end{cases}$$

that takes the individual element $(q_\rho^*, t^*, p_\rho^*)$ to the element (q_ρ, t, p_ρ) by displacing along the extremal of the variational problem that the canonical system (483) belongs to. Due that sliding of the element along the extremal, **C. Carathéodory** referred to those special transformations as *sliding transformations*. When S is the principal function of the variational problem in question, the relation:

(532.a)
$$\sum_{\rho=1}^{n} p_\rho\, dq_\rho - H\, dt \ = \sum_{\rho=1}^{n} p_\rho^*\, dq_\rho^* - (H)^*\, dt^* + dS$$

will be true for them. The sliding transformations are then certain canonical transformations.

One can derive a canonical transformation from the two canonical transformations (477) and (532) that couples the p_ρ^*, q_ρ^*, t^* and P_ρ, Q_ρ, T_ρ with each other and for which one will get:

$$\left(\sum_{\rho=1}^{n} p_\rho^*\, dq_\rho^* - (H)^*\, dt^*\right) - \left(\sum_{\rho=1}^{n} P_\rho\, dQ_\rho - K\, dT\right) = d\,(W - S)$$

when one subtracts (532.a) from (485.b). One then needs only to choose the arbitrary function h in (532) such that one continually has:

$$d\,(W - S) = 0 \ ,$$

and one will have the relation:

(533) $$\left(\sum_{\rho=1}^{n} p_\rho^* \, dq_\rho^* - (H)^* \, dt^*\right) - \left(\sum_{\rho=1}^{n} P_\rho \, dQ_\rho - K \, dT\right) = 0 ,$$

which says that the transformation:

(533.a) $$\begin{cases} p_\rho^* = \Phi_\rho \, (P_1, \ldots, P_n, Q_1, \ldots, Q_n, T), \\ q_\rho^* = \Psi_\rho \, (P_1, \ldots, P_n, Q_1, \ldots, Q_n, T), \\ t^* = \; \mathrm{X} \, (P_1, \ldots, P_n, Q_1, \ldots, Q_n, T) \end{cases}$$

is a contact transformation. *The general canonical transformation* (477) *the arises by composing a sliding transformation of the canonical system* (483) *with a contact transformation* ([339.a]).

The incorporation of the canonical transformations into the theory of contact transformations that **Lie** constructed systematically draws attention to the property of the canonical transformations that the set of all of them defines an infinite group of transformations. Now, when **Lie** picked out a one-parameter group from that infinite group, he showed that for the simplest case of the contact transformations in x, p, as they are represented by the transformations (528), together with (530.a), the differential equations that establish those infinitesimal transformations will possess precisely the canonical form:

(534) $$\begin{cases} \delta q_\rho = \dfrac{\partial \Omega}{\partial p_\rho} \delta\alpha , \quad \delta p_\rho = -\dfrac{\partial \Omega}{\partial q_\rho} \delta\alpha , \\ \Omega = \Omega \, (p_1, \ldots, p_n, q_1, \ldots, q_n). \end{cases}$$

Conversely, every such canonical system of equations with an arbitrarily-chosen function Ω ($p_1, \ldots, p_n, q_1, \ldots, q_n$) will represent a one-parameter group of contact transformations in the (x, p) (canonical transformations in $2n$ variables whose infinitesimal transformation it represents, resp.).

Conversely, according to **Lie**, one can think of every finite contact transformation as arising from the "infinite repetition" of a suitable infinitesimal transformation, i.e., the P_ρ, Q_ρ, to which the original quantities p_ρ, q_ρ will go under the finite contact transformation, are the solutions to (534):

([339.a]) The canonical transformation (492), which is distinguished from the general canonical transformation by the fact that time remains untransformed ($t = T$), can naturally be also interpreted in the given way as the composition of a sliding transformation and a contact transformation. Meanwhile, since time must be transformed, in addition, under the contact transformation, one has made no use of such an interpretation. Rather, one cares to regard t as a parameter that remains unchanged under that transformation. In place of (491), one will then have the relation:

$$\sum_{\rho=1}^{n} p_\rho \, dq_\rho - \sum_{\rho=1}^{n} P_\rho \, dQ_\rho = d^* W$$

(in which the differential d^* refers to only the variables P_ρ^*, Q_ρ^*), and one sees from the analogy with (530.a) that with this way of looking at things, the transformation is a contact transformation of the x, p, with **Lie**'s terminology.

(534.a)
$$\begin{cases} P_\rho = g_\rho(\alpha, p_1, \ldots, p_n, q_1, \ldots, q_n), \\ Q_\rho = h_\rho(\alpha, p_1, \ldots, p_n, q_1, \ldots, q_n) \end{cases}$$

that belong to a suitable parameter value $\alpha = \alpha$ and assume the values p_ρ (q_ρ, resp.) for $\alpha = 0$ ([434]). If the relations (534.a) are identical to the transformation formulas (528) when $\alpha = \bar{\alpha}$, in that sense, then the function U in (530.a) must be constrained by the function Ω (534) and conversely. One will get that connection immediately when one defines the *principal function* (cf., no. **16**):

(535) $V(q_1, \ldots, q_n, Q_1, \ldots, Q_n)$

$$= \mathcal{E}\int_0^{\bar{\alpha}} \left[p_1 \frac{\delta q_1}{\delta \alpha} + \cdots + p_n \frac{\delta q_n}{\delta \alpha} - \Omega(p_1, \ldots, p_n, q_1, \ldots, q_n) \right] \delta\alpha$$

of the variational problem that belongs to the canonical system (534) ([435]), in which the contact transformation (534.a) will next take on the representation:

(535.a)
$$P_\rho = \frac{\partial V}{\partial Q_\rho}, \qquad p_\rho = -\frac{\partial V}{\partial q_\rho} \qquad (\rho = 1, \ldots, n)$$

when one sets $\alpha = \bar{\alpha}$. If one solves the second n of those equations for $Q_1, \ldots, Q_n$ ([436]) and introduces the values thus-obtained:

$$Q_\rho = h_\rho(\bar{\alpha}, p_1, \ldots, p_n, q_1, \ldots, q_n)$$

into V ($\partial V / \partial Q_\rho$, resp.) then the first n of those equations will take the form:

$$P_\rho = g_\rho(\bar{\alpha}, p_1, \ldots, p_n, q_1, \ldots, q_n) .$$

At the same time, the relation:

$$\sum P_\rho \, dQ_\rho - \sum p_\rho \, dq_\rho = dV$$

([434]) In this, the transformation, which is to be regarded as a passive transformation (viz., the introduction of new variables), has been reinterpreted as an active transformation that associates an element p_ρ, q_ρ with a new element P_ρ, Q_ρ. However, that naturally serves only to clarify the connect between finite and infinitesimal transformations. One must always establish that the transformation should serve to introduce new coordinates here (for unvaried integral curves) in its own right.

([435]) It should be remarked in passing that the function V in (533) satisfies the two partial differential equations:

$$\Omega\left(\frac{\partial V}{\partial Q_1}, \ldots, \frac{\partial V}{\partial Q_n}, Q_1, \ldots, Q_n\right) = C, \qquad \Omega\left(-\frac{\partial V}{\partial q_1}, \ldots, -\frac{\partial V}{\partial q_n}, q_1, \ldots, q_n\right) = C.$$

([436]) Which is assumed to be possible, for the sake of simplicity. One can easily free oneself from that assumption.

which is equivalent to (535.a), will go to the relation (530.a). For the variation of $\sum_\rho p_\rho \, dq_\rho$ under the infinitesimal transformation (534), one will correspondingly obtain:

$$\delta\left(\sum_{\rho=1}^{n} p_\rho \, dq_\rho\right) = d\ \delta V = d\left(p_1 \frac{\partial\Omega}{\partial p_1} + \cdots + p_n \frac{\partial\Omega}{\partial p_n} - \Omega\right),$$

in which the right-hand side includes the differential of the integrand of the variational problem that belongs to the canonical system (534) and whose principal function is the function V.

Entirely-analogous arguments can be presented in the case when the independent variable t also enters into the transformation, except that the condition (530.a) will then take the form:

$$(536) \qquad P_1 \, dQ_1 + \ldots + P_n \, dQ_n = p_1 \, dq_1 + \ldots + p_n \, dq_n + \left(dW - \frac{\partial W}{\partial t} dt\right),$$

because t is regarded as a parameter that is held constant under the transformation. If one again thinks of a finite contact transformation as being generated by the infinite repetition of the infinitesimal transformation of a one-parameter group, in the spirit of **Lie**, then its infinitesimal transformation will also be further given by a canonical system:

$$(537) \qquad \delta q_\rho = \frac{\partial\Omega}{\partial p_\rho} \delta\alpha, \qquad \delta p_\rho = -\frac{\partial\Omega}{\partial q_\rho} \delta\alpha,$$

except that now the parameter t (which is kept constant under the transformation) also appears in $\Omega = \Omega(p_1, \ldots, p_n, q_1, \ldots, q_n, t)$ (437).

An associated finite contact transformation will again be represented by the solutions to the canonical system (537):

$$(538) \qquad \begin{cases} P_\rho = g_\rho(\alpha, p_1, \ldots, p_n, q_1, \ldots, q_n, t), \\ Q_\rho = h_\rho(\alpha, p_1, \ldots, p_n, q_1, \ldots, q_n, t), \end{cases}$$

in which one must set $\alpha = \alpha$. On the other hand, one can appeal to its representation by the principal function:

$$(539) \qquad V(q_1, \ldots, q_n, Q_1, \ldots, Q_n, t)$$

$$= \mathcal{E}\int_0^{\bar{\alpha}} \left[p_1 \frac{\delta q_1}{\delta\alpha} + \cdots + p_n \frac{\delta q_n}{\delta\alpha} - \Omega(p_1, \ldots, p_n, q_1, \ldots, q_n, t)\right] \delta\alpha$$

(437) The transformations of the one-parameter group in the phase-R_{2n+1} of the $(p_1, \ldots, p_n, q_1, \ldots, q_n, t)$, which arise from an integral of the equations of motion (cf., no. **25**), are then *contact transformations* in the R_{n+1} of the $(q_1, \ldots, q_n, t)$ [cf., no. **18**.c].

of the variational problem that the canonical system (537) belongs to, which will make it take the form:

(539.a) $$P_\rho = \frac{\partial V}{\partial Q_\rho}, \qquad p_\rho = -\frac{\partial V}{\partial q_\rho}.$$

Those are $2n$ formulas that one can combine into the relation:

(539.b) $$\sum P_\rho\, dQ_\rho - \sum p_\rho\, dq_\rho = dV - \frac{\partial V}{\partial t}.$$

The function that appears in (536):

$$W(p_1, \ldots, p_n, q_1, \ldots, q_n, t)$$

will be obtained from $V(Q_1, \ldots, Q_n, q_1, \ldots, q_n, t)$ when one solves the second group of equations (539.a) for $Q_1, \ldots, Q_n$ and introduces the values thus-found in V. If one combines (439.b) with the identity:

$$H\,dt - \bar{H}\,dt \equiv 0,$$

in which $\bar{H}$ might arise from H by the transformation (539.a), then that will give:

(540) $$\sum P_\rho\, dQ_\rho - \left(\bar{H} - \frac{\partial V}{\partial t}\right) dt - \left(\sum p_\rho\, dq_\rho - H\,dt\right) = dV,$$

and one will once more see that the function $K(P_1, \ldots, P_n, Q_1, \ldots, Q_n, t)$ of the transformed canonical system is coupled with the function $H(p_1, \ldots, p_n, q_1, \ldots, q_n, t)$ of the original canonical system by ([438]):

(540.a) $$K = \bar{H} - \frac{\partial V}{\partial t}.$$

On the other hand, if $\partial V / \partial t$ is:

(540.b) $$\frac{\partial V}{\partial t} = -E(P_1, \ldots, P_n, Q_1, \ldots, Q_n, t),$$

when one expresses the q_ρ in it in terms of P_ρ, Q_ρ, t then, as one will infer from (539.a), the function $V(p_1, \ldots, p_n, Q_1, \ldots, Q_n, t)$ will satisfy the partial differential equation:

([438]) In the theory of perturbations, Ω was obviously the **Hamilton**ian function of the unperturbed motion, while K represented the perturbing function. If the transformation formulas, as in (528), are independent of t then K will be the function that arises from H itself by the transformation.

(540.c) $$\frac{\partial V}{\partial t} + E\left(\frac{\partial V}{\partial Q_1}, \ldots, \frac{\partial V}{\partial Q_n}, Q_1, \ldots, Q_n, t\right) = 0 ,$$

from which one can determine it when E is given ([439]). However, it will follow from this that a canonical transformation that depends upon t :

$$p_\rho = \varphi_\rho (t, P_1, \ldots, P_n, Q_1, \ldots, Q_n) ,$$
$$q_\rho = \psi_\rho (t, P_1, \ldots, P_n, Q_1, \ldots, Q_n) ,$$

i.e., a family of canonical transformations with t as the parameter of the family, will represent a solution of the canonical system ([439.a]):

(540.d) $$\frac{dq_\rho}{dt} = \frac{\partial E}{\partial p_\rho}, \qquad \frac{dp_\rho}{dt} = -\frac{\partial E}{\partial q_\rho}.$$

Up to now, the contact transformation was thought of as being determined by the function W (V or U, resp.), so the function K was then calculated from H. However, even in his earliest investigation, **S. Lie** could already characterize a contact transformation in the x, p by saying that the transformation functions defined the canonical basis for a function group by appealing to the **Poisson** brackets ([440]). He arrived at that notion when he posed the problem in such a way that he did not give W to begin with, but demanded that there should be a canonical transformation that takes a canonical system with a prescribed function H :

(541) $$\frac{dq_\rho}{dt} = \frac{\partial H}{\partial p_\rho}, \qquad \frac{dp_\rho}{dt} = -\frac{\partial H}{\partial q_\rho}$$

to another canonical system with the prescribed function K:

(542) $$\frac{dQ_\rho}{dt} = \frac{\partial K}{\partial P_\rho}, \qquad \frac{dP_\rho}{dt} = -\frac{\partial K}{\partial Q_\rho}.$$

That problem will become especially simple when the independent variable t does not appear in the two functions H and K. Namely, one will get ([440.a]) such a contact transformation when one defines two canonical function groups (cf., no. **28**): on the one hand, the function group:

([439]) Cf., also **G. Morera**, "Sulla trasformazione delle equaz. diff. di Hamilton, Nota I," Roma Linc. Rend. (5) **12**[1] (1903), pp. 113, esp., pp. 118.

([439.a]) That way of looking at things in in **S. Lie**, *loc. cit.* ([431]), § 2 = *Werke III*, pp. 303.

([440]) "Kurzes Résumé mehrerer neuer Theorien," Christiania Forhandlingar (1873), pp. 24 = *Werke III*, pp. 1.

([440.a]) Cf., **S. Lie**, *loc. cit.* ([431]), esp., *Werke III*, pp. 302.

(541.a)
$$\begin{cases} H_1, H_2, \ldots, H_n, & H_\rho = H_\rho(p_1, \ldots, p_n, q_1, \ldots, q_n), \\ G_1, G_2, \ldots, G_n, & G_\rho = G_\rho(p_1, \ldots, p_n, q_1, \ldots, q_n), \end{cases}$$

in which H_1 coincides with the function H in (541) ([441]), and on the other hand, the function group:

(542.a)
$$\begin{cases} K_1, K_2, \ldots, K_n, & K_\rho = K_\rho(p_1, \ldots, p_n, q_1, \ldots, q_n), \\ L_1, L_2, \ldots, L_n, & L_\rho = L_\rho(p_1, \ldots, p_n, q_1, \ldots, q_n), \end{cases}$$

in which one should have $K_1 = K$ ([442]), and then set ([443]):

(543)
$$\begin{cases} H_\rho(p_1, \ldots, p_n, q_1, \ldots, q_n) = K_\rho(P_1, \ldots, P_n, Q_1, \ldots, Q_n), \\ G_\rho(p_1, \ldots, p_n, q_1, \ldots, q_n) = L_\rho(P_1, \ldots, P_n, Q_1, \ldots, Q_n). \end{cases}$$

([441]) Obviously, due to the fact that:

$$(H_1, H_\rho) = (H, H_\rho) = 0, \qquad (H_1, G_\rho) = (H, G_\rho) = 0 \qquad (\rho = 2, \ldots, n),$$

the functions $H_2, \ldots, H_n, G_2, \ldots, G_n$ will be integrals of the canonical system (541). Along with $H_1 = H$ = const., one $G_1 - t$ = const. as the last integral.

([442]) Therefore, $K_2, \ldots, K_n, L_2, \ldots, L_n, L_1 - t$ are integrals of the canonical system (542), along with K_1 (= K).

([443]) That is because if one considers:

$$p_\rho^* = H_\rho(p_1, \ldots, p_n, q_1, \ldots, q_n), \qquad q_\rho^* = G_\rho(p_1, \ldots, p_n, q_1, \ldots, q_n)$$

to be a coordinate transformation then it will be a canonical (contact, resp.) transformation, and indeed, it will take the system (541) to:

(544)
$$\begin{cases} \dfrac{dq_1^*}{dt} = 1, & \dfrac{dp_1^*}{dt} = 0, \\ \dfrac{dq_\rho^*}{dt} = 0, & \dfrac{dp_\rho^*}{dt} = 0 \end{cases} \qquad (\rho = 2, \ldots, n).$$

Likewise:

$$P_\rho^* = K_\rho(P_1, \ldots, P_n, Q_1, \ldots, Q_n), \qquad Q_\rho^* = L_\rho(P_1, \ldots, P_n, Q_1, \ldots, Q_n)$$

is a contact transformation that takes the system (542) to:

(544.a)
$$\begin{cases} \dfrac{dQ_1^*}{dt} = 1, & \dfrac{dP_1^*}{dt} = 0, \\ \dfrac{dQ_\rho^*}{dt} = 1, & \dfrac{dP_\rho^*}{dt} = 0 \end{cases} \qquad (\rho = 2, \ldots, n).$$

However, the transformation:

$$p_\rho^* = P_\rho^*, \qquad q_\rho^* = Q_\rho^*,$$

takes the canonical system (544) to the canonical system (544.a).

If the independent variable t appears explicitly in the function H (K, resp.) in the canonical systems (541) and (542) then one can proceed in the same way as long as one only introduces the new quantities:

$$(545) \qquad \begin{cases} v = H(p_1, \dots, p_n, q_1, \dots, q_n, t), \quad u = t, \\ H^*(p_1, \dots, p_n, q_1, \dots, q_n, u) = -v + H(p_1, \dots, p_n, q_1, \dots, q_n, u), \end{cases}$$

in order to convert the canonical system (541) into ([444]):

$$(545.a) \qquad \begin{cases} \dfrac{dq_\rho}{dt} = \dfrac{\partial H^*}{\partial p_\rho}, \quad \dfrac{dp_\rho}{dt} = -\dfrac{\partial H^*}{\partial q_\rho}, & (\rho = 1, \dots, n) \\ \dfrac{dv}{dt} = \dfrac{\partial H^*}{\partial u}, \quad \dfrac{du}{dt} = -\dfrac{\partial H^*}{\partial v} (= 1) \end{cases}$$

and correspondingly introduces the new quantities:

$$(546) \qquad \begin{cases} V = K(P_1, \dots, P_n, Q_1, \dots, Q_n, U), \quad U = t, \\ K^*(P_1, \dots, P_n, V, Q_1, \dots, Q_n, U) = -V + K(P_1, \dots, P_n, Q_1, \dots, Q_n, U) \end{cases}$$

in order to convert the canonical system (542) into:

$$(546.a) \qquad \begin{cases} \dfrac{dQ_\rho}{dt} = \dfrac{\partial K^*}{\partial P_\rho}, \quad \dfrac{dP_\rho}{dt} = -\dfrac{\partial K^*}{\partial Q_\rho}, & (\rho = 1, \dots, n) \\ \dfrac{dV}{dt} = \dfrac{\partial K^*}{\partial U}, \quad \dfrac{dU}{dt} = -\dfrac{\partial K^*}{\partial V} (= 1). \end{cases}$$

One must then define the two canonical function groups:

$$(547) \qquad H_1, \dots, H_n, \; H^*, \qquad G_1, \dots, G_n, \; G^* \; (= u),$$

and

$$(548) \qquad K_1, \dots, K_n, \; K^*, \qquad L_1, \dots, L_n, \; L^* \; (= U),$$

in which the $H_1, \dots, H_n, G_1, \dots, G_n$, must be independent of v (the $K_1, \dots, K_n, L_1, \dots, L_n$ must be independent of V, resp.) ([445]). The transformation then be given by the Ansatz:

([444]) Cf., **G. Morera**, "Sulla trasformazione delle equaz. diff. di Hamilton, Nota I," Roma Linc. Rend. (5) **12**[1] (1903), pp. 113, esp., pp. 119.

([445]) If one considers the fact that one has:

(549)
$$\begin{cases} H_\rho(p_1,\dots,p_n,q_1,\dots,q_n,t) = K_\rho(P_1,\dots,P_n,Q_1,\dots,Q_n,t), \\ G_\rho(p_1,\dots,p_n,q_1,\dots,q_n,t) = L_\rho(P_1,\dots,P_n,Q_1,\dots,Q_n,t), \\ (\rho = 1,\dots,n), \end{cases}$$

in which one already considers the fact that:

(549.a)
$$u = U = t .$$

That must be combined with the relation:

$$H^*(p_1, \dots, p_n, v, q_1, \dots, q_n, u) = K^*(P_1, \dots, P_n, V, Q_1, \dots, Q_n, U)$$

or ([446]):

(549.b)
$$-v + H(p_1, \dots, p_n, q_1, \dots, q_n, u) = -V + K(P_1, \dots, P_n, Q_1, \dots, Q_n, U) ,$$

resp.

Those arguments once more make it clear [cf., ([419])] that *the problem of integrating a canonical system can also be regarded as a problem in canonical transformation.* In fact, the integration of the given system (541) will indeed be complete when it can be converted into a canonical system (542) whose function K does not depend upon the $P_1, \dots, P_n, Q_1, \dots, Q_n$, so it will either be identically constant or a function of only the variable t. That is because the system (542) will then assume the form ([447]):

(550)
$$\frac{dP_\rho}{dt} = 0 , \qquad \frac{dQ_\rho}{dt} = 0 .$$

$$(H^*, u) = -\frac{\partial H^*}{\partial v} = 1$$

[the **Poisson** brackets are formed from $(2n + 2)$ independent variables when H^* appears in them] then one can infer the following relations from the canonical form of the function group:

$$(H^*, H_\rho) = \frac{\partial H_\rho}{\partial t} + (H, H_\rho) = 0 ,$$
$$(H^*, G_\rho) = \frac{\partial G_\rho}{\partial t} + (H, G_\rho) = 0 ,$$

i.e., those functions H_ρ, G_ρ determine a system of $2n$ integrals:

$$H_\rho = c_\rho , \qquad G_\rho = \gamma_\rho \qquad (\rho = 1, \dots, n)$$

of the canonical system (541). Naturally the same thing is true of K_ρ, L_ρ with respect to the canonical system (542).

([446]) Cf., also, **S. Lie**, *loc. cit.* ([431]), esp., *Werke III*, pp. 308.

([447]) Cf., e.g., **E. T. Whittaker**, *Analytical Dynamics*, pp. 310.

Finally, in order to show that the most general transformation that takes an *individual* canonical system to another canonical system is not a contact transformation ([448]), **S. Lie** likewise appealed to the relative integral invariants. As before:

(551) $$\int \sum p_\rho \, dq_\rho$$

is a relative integral invariant of the original system, just as:

(552) $$\int \sum P_\rho \, dQ_\rho$$

is a relative integral invariant of the transformed system. Now, (552) will *no longer* arise from the transformation of (551) ([449]), but rather from a different first-order relative integral invariant, which might possess the form:

(553) $$\int \sum_\rho [L_\rho(p_1, \ldots, p_n, q_1, \ldots, q_n) \, \delta p_\rho + M_\rho(p_1, \ldots, p_n, q_1, \ldots, q_n) \, \delta q_\rho] \; .$$

Now, in order to determine the most general form of that relative integral invariant (553), **Lie** ([450]) imagined introducing new variables into the canonical system:

(554) $$\frac{dq_\rho}{dt} = \frac{\partial H}{\partial p_\rho}, \qquad \frac{dp_\rho}{dt} = -\frac{\partial H}{\partial q_\rho} \qquad [H = H(p_1, \ldots, p_n, q_1, \ldots, q_n)],$$

and determining a function group for $H_1 = H$ with the canonical basis ([451]):

(555) $$H_1, \ldots, H_n, \quad G_1, \ldots, G_n \qquad (H_1 \equiv H)$$

and performing the coordinate transformation:

(555.a) $$p_\rho^* = H_\rho, \qquad q_\rho^* = G_\rho .$$

The canonical system (554) will then take the form:

([448]) Such that when one applies it to other canonical systems, it will not generally produce a system of canonical form.

([449]) That would lead to the canonical transformations in the proper sense.

([450]) **S. Lie**, *loc. cit.* (431), esp., *Werke III*, pp. 313.

([451]) Whose functions are then integrals of (554), except for G_1 .

(556) $$\begin{cases} \dfrac{dq_1^*}{dt} = 1, \quad \dfrac{dp_1^*}{dt} = 0, \\ \dfrac{dq_\rho^*}{dt} = 0, \quad \dfrac{dp_\rho^*}{dt} = 0, \end{cases} \qquad (\rho = 2, \dots, n)$$

while the relative integral invariant (553) will go to:

(553) $$\int \sum_\rho [L_\rho^*(p_1^*, \dots, q_n^*)\,\delta p_\rho^* + M_\rho^*(p_1^*, \dots, q_n^*)\,\delta q_\rho^*] \; .$$

However, should that be a relative integral invariant of the system (556), then it must give a function $\Phi(p_1^*, \dots, p_n^*, \dots, q_n^*, \dots, q_n^*)$ such that one will have:

(557.a) $$\frac{\partial L_\rho^*}{\partial q_1^*} = \frac{\partial \Phi}{\partial p_\rho^*}, \qquad \frac{\partial M_\rho^*}{\partial q_1^*} = \frac{\partial \Phi}{\partial q_\rho^*},$$

i.e., the functions L_ρ^* (M_ρ^*, resp.) in the relative integral invariant (557) will possess the form:

(557.b) $$\begin{cases} L_\rho^* = \displaystyle\int \frac{\partial \Phi}{\partial p_\rho^*} dq_1^* + l_\rho(p_1^*, \dots, p_n^*, q_2^*, \dots, q_n^*), \\ M_\rho^* = \displaystyle\int \frac{\partial \Phi}{\partial q_\rho^*} dq_1^* + m_\rho(p_1^*, \dots, p_n^*, q_2^*, \dots, q_n^*). \end{cases}$$

Now, since one will obtain the relative integral invariant (553) from the integral invariant (557) that was thus determined by performing the inverse of the transformation (555.a), one will see immediately that it does not need to possess the form:

$$\int \sum_{\rho=1}^{n} p_\rho \, \delta q_\rho \, .$$

That is because since the transformation (555.a) is a contact transformation, the relative integral invariant (557) must also possess the form:

$$\int \sum_{\rho=1}^{n} p_\rho^* \, \delta q_\rho^*$$

in this case, so one must have:

$$L_\rho^* = \frac{\partial \Omega}{\partial p_\rho^*}, \qquad M_\rho^* = \frac{\partial \Omega}{\partial q_\rho^*} + p_\rho^*,$$

while from (557.b) the L_ρ^*, M_ρ^* do not need to possess that special form ([452]).

([452]) Analogous considerations are also found in **G. Morera**, "Sulla trasformazione delle equaz. diff. di Hamilton, Nota II," Roma Linc. Rend. (5) **12**[1] (1903), pp. 149.

CHAPTER VIII

THE EQUIVALENCE PROBLEM AND RELATED TOPICS

35. Transformation of one mechanical problem into another. Concept of equivalence. – One cares to refer to two mechanical systems with the same number of degrees of freedom for which the integration of their **Lagrang**ian equations of motion possesses a certain relationship as *analytically equivalent*. Of course, that concept of equivalence of two mechanical systems is still not established completely by that. The most restricting formulation was given by **P. Stäckel** ([453]), who demanded that the equations of motion of both system, which might be:

(558) $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\rho}\right) - \frac{\partial T}{\partial q_\rho} = Q_\rho \qquad (\rho = 1, \ldots, n),$$

or

(559) $$\frac{d}{dt}\left(\frac{\partial \mathfrak{T}^*}{\partial \dot{\mathfrak{q}}_\rho}\right) - \frac{\partial \mathfrak{T}^*}{\partial \mathfrak{q}_\rho} = \mathfrak{Q}^*_\rho \qquad (\rho = 1, \ldots, n),$$

would have to go to each other under a transformation of the position coordinates:

(560) $$\mathfrak{q}_\rho = \varphi_\rho (q_1, \ldots, q_n) \qquad (\rho = 1, \ldots, n).$$

In so doing, **Stäckel** restricted himself to the case in which the coefficients of the kinetic energies in the two problems:

(561) $$T = \tfrac{1}{2}\sum_{\lambda,\mu} g_{\lambda\mu}\, \dot{q}_\lambda\, \dot{q}_\mu , \quad \text{or} \quad \mathfrak{T}^* = \tfrac{1}{2}\sum_{\lambda,\mu} \mathfrak{g}^*_{\lambda\mu}\, \dot{\mathfrak{q}}_\lambda\, \dot{\mathfrak{q}}_\mu , \quad \text{resp.},$$

did not include time t explicitly, but depended upon only the position coordinates:

(561.a) $$g_{\lambda\mu} = g_{\lambda\mu}(q_1, \ldots, q_n) , \qquad \mathfrak{g}^*_{\lambda\mu} = \mathfrak{g}^*_{\lambda\mu}(\mathfrak{q}_1, \ldots, \mathfrak{q}_n) ,$$

such that it would seem reasonable, in the spirit of no. **6**, to regard the spatial M_n of the $(q_1, \ldots, q_n)$, [$(\mathfrak{q}_1, \ldots, \mathfrak{q}_n)$, resp.] as ***Riemann**ian spaces* whose arc-length elements are:

(561.b) $$ds^2 = \sum_{\lambda,\mu} g_{\lambda\mu}\, dq_\lambda\, dq_\mu , \qquad d\mathfrak{s}^{*2} = \sum_{\lambda,\mu} \mathfrak{g}^*_{\lambda\mu}\, d\mathfrak{q}_\lambda\, d\mathfrak{q}_\mu ,$$

([453]) **P. Stäckel**, "Über die Differentialgleichungen der Dynamik und den Begriff der analytischen Äquivalenz dynamischer Probleme," J. f. Math. **107** (1891), pp. 319.

and the mechanical problems will then be denoted briefly by:

(561.c) $$(ds, Q_\rho)\,, \qquad (d\mathfrak{s}^*, \mathfrak{Q}^*_\rho)\,.$$

Stäckel also imagined that the components of applied forces were functions of only the position coordinates, but it was sometimes necessary to allow them to depend upon the velocity components, while he cared to exclude the explicit appearance of time here, in general ([454]). In order to compare the equations of motion (558) and (559), it seems convenient to introduce the $\mathfrak{q}_\rho$ into (559) in place of the q_ρ by means of the transformation formulas (560), which might make equations (559) go to:

(559.a) $$\frac{d}{dt}\left(\frac{\partial \mathfrak{T}}{\partial \dot{q}_\rho}\right) - \frac{\partial \mathfrak{T}}{\partial q_\rho} = Q_\rho\,,$$

with

$$\mathfrak{T} = \tfrac{1}{2}\sum \mathfrak{g}_{ik}(q_1, \ldots, q_n)\,\dot{q}_i\,\dot{q}_k\,,$$

$$\mathfrak{Q}_\rho\,(q_1, \ldots, q_n) = \sum \mathfrak{Q}^*_\sigma \frac{\partial \varphi_\sigma}{\partial q_\rho}\,,$$

$$d\mathfrak{s}^2 = \sum \mathfrak{g}_{ik}(q_1, \ldots, q_n)\,dq_i\,dq_k\,,$$

and then recalculate (558) and (559.a) in the form:

(562) $$\ddot{q}_\rho + \sum_{\lambda,\mu=1}^{n} \begin{Bmatrix} \lambda\ \mu \\ \rho \end{Bmatrix} \dot{q}_\lambda\,\dot{q}_\mu = Q^\rho \qquad \left(Q^\rho = \sum_\sigma g^{\rho\sigma} Q_\sigma\right)$$

or

(563) $$\ddot{q}_\rho + \sum_{\lambda,\mu=1}^{n} \begin{Bmatrix} \lambda\ \mu \\ \rho \end{Bmatrix}^* \dot{q}_\lambda\,\dot{q}_\mu = \mathfrak{Q}^\rho \qquad \left(\mathfrak{Q}^\rho = \sum_\sigma \mathfrak{g}^{\rho\sigma} \mathfrak{Q}_\sigma\right),$$

resp., in which the $\begin{Bmatrix} \lambda\ \mu \\ \rho \end{Bmatrix}$ are the **Christoffel** three-index symbols of the arc-length element ds, and $\begin{Bmatrix} \lambda\ \mu \\ \rho \end{Bmatrix}^*$ are those of the arc-length element $d\mathfrak{s}$. In order for both mechanical problems to be

([454]) For the transformation of the force components, one should observe that the virtual work:

$$\sum Q_\rho\,\delta q_\rho\,, \qquad \sum \mathfrak{Q}^*_\rho\,\delta \mathfrak{q}^*_\rho\,, \qquad \text{resp.},$$

is an invariant.

analytically equivalent in the **Stäckel** sense, it will then be necessary and sufficient ([455]) that one must have:

$$\left\{\begin{matrix}\lambda\ \mu\\ \rho\end{matrix}\right\} = \left\{\begin{matrix}\lambda\ \mu\\ \rho\end{matrix}\right\}^*, \qquad Q^\rho = \mathfrak{Q}^\rho \qquad (\lambda, \mu, \rho = 1, \ldots, n). \tag{564}$$

The left-hand group in equations (564) generally implies the relation:

$$\mathfrak{g}_{ik} = c\, g_{ik}\,, \tag{565}$$

in which one understands c to mean a constant ([456]). The fact that $\mathfrak{Q}^\rho = Q^\rho$ then further implies the relations:

$$\mathfrak{Q}_\rho = \sum_\lambda \mathfrak{g}_{\rho\lambda}\, Q^\lambda = \sum_\lambda \mathfrak{g}_{\rho\lambda} \sum_\mu g^{\lambda\mu} Q_\mu = c \sum_\mu \left(\sum_\mu g_{\rho\lambda}\, g^{\lambda\mu}\right) Q_\mu = c\, Q_\rho \tag{566}$$

for the covariant force components ([457]).

In contrast to this narrow conception of the notion of equivalence, two mechanical problems suggest an obvious extension of it. Instead of demanding that the *space-time lines* of the motion should go to each other under the transformation (560), one can restrict oneself to the requirement that only the *trajectories* of a mechanical problem should go to each other under the transformation (560) ([458]). Since each of the two systems of equations (558) [(559), resp.] possesses $2n - 1$ integrals that are free of time, that demand can be expressed by saying that $2n - 1$ integrals of

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([455]) Cf., **P. Stäckel**, *loc. cit.* ([453]), pp. 326.

([456]) **P. Stäckel**, *loc. cit.* ([453]), pp. 337. Here, the word "generally" means that the **Riemann** curvature tensor of the quadratic differential form ds^2 should have rank $(n - 1)$. If it had a lower rank then that would be an exceptional case. For example, if the coefficients $g_{\lambda\mu}$ of ds^2 are constant, i.e., the **Riemann** curvature tensor vanishes identically, then in order to have equivalence, it would suffice for the $\mathfrak{g}_{\lambda\mu}$ to be likewise constant, but have entirely arbitrary values, moreover, that are completely different from the $g_{\lambda\mu}$. That would seem to be the most degenerate case compared to (565). In the intermediate cases, the $q_1, \ldots, q_n$ will split into m categories, such that one will have:

$$ds^2 = \sum_{\lambda,\mu=1}^{n_1} g_{\lambda\mu}^{(1)}\, dq_\lambda\, dq_\mu + \sum_{\lambda,\mu=n_1+1}^{n_2} g_{\lambda\mu}^{(2)}\, dq_\lambda\, dq_\mu + \cdots + \sum_{\lambda,\mu=n_{m-1}+1}^{n_m} g_{\lambda\mu}^{(m)}\, dq_\lambda\, dq_\mu\,,$$

$$d\mathfrak{s}^2 = C_1 \sum_{\lambda,\mu=1}^{n_1} g_{\lambda\mu}^{(1)}\, dq_\lambda\, dq_\mu + C_2 \sum_{\lambda,\mu=n_1+1}^{n_2} g_{\lambda\mu}^{(2)}\, dq_\lambda\, dq_\mu + \cdots + C_m \sum_{\lambda,\mu=n_{m-1}+1}^{n_m} g_{\lambda\mu}^{(m)}\, dq_\lambda\, dq_\mu\,,$$

in which the $g_{\lambda\mu}^{(\sigma)}$ depend upon only the $q_{n_{\sigma-1}+1}, \ldots, q_{n_\sigma}$ $(n_1 + n_2 + \ldots + n_m = n)$. Cf., **G. Fubini**, "Ricerche gruppali sulle equazioni della dinamica, Nota III," Roma Linc. Rend. (5) **12**[2] (1903), pp. 145, esp., pp. 146.

([457]) For $Q_\rho = 0$, one also has $\mathfrak{Q}_\rho = 0$ then, which agrees with the fact that the geodetic lines of the two arc-length elements ds and $d\mathfrak{s}$ are identical, from (565). **Stäckel** considered the equivalence of the motion of a material line on a rectilinear surface with the motion of a point on a rectilinear surface as an example of that.

([458]) In the spirit of this requirement, the applied forces shall depend upon only the position coordinates when one establishes the requirement in the manner that was given above.

equations (559) that are free of t should go to $2n - 1$ corresponding integrals of (558) under the transformation (560). That suggests that in order to emphasize the fact that the individual points of two trajectories of (558) and (559) that are associated with each other in that way will be reached at completely-different times, one should use a symbol for time in (559) that is different from the one in (558) and correspondingly replace equations (559) with ([459]):

$$(567)\qquad \frac{d}{dt}\left(\frac{\partial\mathfrak{T}^*}{\partial\dot{\mathfrak{q}}_\rho}\right)-\frac{\partial\mathfrak{T}^*}{\partial\mathfrak{q}_\rho} = \mathfrak{Q}^*_\rho \qquad (\rho = 1, \ldots, n),$$

in which one now has:

$$\dot{\mathfrak{q}}_\rho = \frac{d\mathfrak{q}_\rho}{d\mathfrak{t}}.$$

Here, it would be convenient to employ the transformation (560) in order to replace the $\mathfrak{q}_\rho$ with the q_ρ and give those equations the form:

$$(568)\qquad \frac{d}{dt}\left(\frac{\partial\mathfrak{T}}{\partial\dot{q}^*_\rho}\right)-\frac{\partial\mathfrak{T}}{\partial q_\rho} = \mathfrak{Q}_\rho \qquad \left(\dot{q}^*_\lambda = \frac{dq_\rho}{d\mathfrak{t}}\right),$$

in which:

$$\mathfrak{T} = \frac{1}{2}\left(\frac{d\mathfrak{s}}{d\mathfrak{t}}\right)^2 = \tfrac{1}{2}\sum g_{\lambda\mu}(q_1,\ldots,q_n)\,\dot{q}^*_\lambda\,\dot{q}^*_\mu, \qquad \mathfrak{Q}_\rho = \mathfrak{Q}_\rho(q_1, \ldots, q_n).$$

In the spirit of the requirement that was imposed, in the R_n of the $(q_1, \ldots, q_n)$, the trajectories of the two mechanical problems with the equations of motion (558) [(568, resp.) must be identical ([460]). Now, the individual trajectories will belong to the arc-length ds ($d\mathfrak{s}$, resp.) according to whether

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([459]) Cf., e.g., **P. Appell**, "Sur des transformations de mouvements," J. f. Math. **110** (1892), pp. 37. **P. Appell** had already treated the special case of a point in a plane with the equations of motion:

$$\frac{d^2x}{dt^2} = X, \qquad \frac{d^2y}{dt^2} = Y,$$

and applied the projective transformation:

$$\mathfrak{x} = \frac{a\,x+b\,y+c}{a''\,x+b''\,y+c''}, \qquad \mathfrak{y} = \frac{a\,x+b\,y+c}{a''\,x+b''\,y+c''}$$

to it, along with the transformation of time:

$$K\,d\,\mathfrak{t} = \frac{dt}{(a''\,x+b''\,y+c'')^2}.$$

Cf., **P. Appell**, "De l'homographie en mécanique," Am. J. Math. **12** (1889), pp. 103 and *ibid.* **13** (1890), pp. 153. Further literature on the development of that idea can be found in the cited article (viz., J. f. Math., **110**).

([460]) Cf., **P. Painlevé**, "Mémoire sur la transformation des équations de la dynamique," J. de math. (4) **10** (1894), pp. 5. **Painlevé** refers to such system as *correspondants*.

one regards them as the trajectories of equations (558) or (568), respectively. On the other hand, since one knows the velocity of motion along the trajectory as a function of arc-length for each of the two problems from the equations of motion (cf., no. **6**), the individual points of the trajectory will be associated with the time at which they are reached by relations of the form ([461]):

(569) $$dt = \psi(s)\, ds$$

or

(570) $$d\mathfrak{t} = \chi(\mathfrak{s})\, d\mathfrak{s}\,.$$

In that way, one will have likewise achieved an association of the differentials of time dt ($d\mathfrak{t}$, resp.) for the individual trajectories. On the other hand, when one observes that the individual trajectory will always be well-defined as soon as one gives one of its points and the associated velocity, one will see that this association must have the form ([462]):

(571) $$d\mathfrak{t} = \frac{dt}{f(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)}\,.$$

Obviously, that *transformation of time* must take the equations of motion (568) to the equations of motion (558).

A *very simple case of such equivalent trajectories* for two systems (ds, Q_ρ) and ($d\mathfrak{s}$, $\mathfrak{Q}_\rho$) is when one has:

(572) $$ds = d\mathfrak{s} \qquad (\text{i.e., } g_{ik} = \mathfrak{g}_{ik}), \qquad \mathfrak{Q}_\rho = c\, Q_\rho\,,$$

since one will then need only to set:

(572.a) $$d\mathfrak{t} = \frac{1}{\sqrt{c}}\, dt\,,$$

in order to convert the equations of motion (568) into (558). At the same time, that trivial case still has a certain significance to it, because if one chooses, e.g., $c = -1$, i.e., if one changes the direction of the applied forces, then it will follow that:

$$d\mathfrak{t} = i\, dt\,,$$

([461]) A trajectory will belong to ∞^1 space-time lines since time does not appear explicitly in it (cf., no. **6**).
([462]) Cf., **T. Levi-Civita**, "Sulle trasformazione delle equazioni dinamiche," Ann. di mat. (2) **24** (1896), pp. 255, esp., pp. 268.

Instead of introducing the velocity components $\dot{q}_1, \ldots, \dot{q}_n$ into f, one can also think of introducing the direction of the trajectory $dq_1 : \ldots : dq_n$ and the geodetic curvature K_g (cf., no. **6**).

i.e., from no. **6**: The true motion of the one problem is the conjugate of the other one ([463]). Now, in general a mechanical problem (ds, Q_ρ) will possess only one problem with equivalent trajectories that differ from those of the former only slightly, namely, the problem:

$$(573) \qquad d\mathfrak{s}^2 = C\,ds^2, \qquad \mathfrak{Q}_\rho = c\,Q_\rho, \quad \text{and thus} \quad d\mathfrak{t} = \sqrt{\frac{C}{c}}\,dt,$$

which **P. Painlevé** then referred to as *correspondants ordinaires* ([464]) (i.e., a trivial correspondence). In so doing, one generally assumes that the force components Q_ρ ($\mathfrak{Q}_\rho$, resp.) of the two mechanical problems with equivalent trajectories, on the one hand, depend upon only the position coordinates, and on the other hand, do not arise from potential.

Therefore, as **P. Appell** already showed ([465]), in the case of this trivial correspondence, the vanishing of the force components of the one problem will have the vanishing of the force components of the other problem as a consequence. However, for two force-free mechanical problems, the trajectories are nothing but the geodetic lines of the two arc-length elements ds ($d\mathfrak{s}$, resp.), and the question of the equivalence of the trajectories of the mechanical problems will then become simply the question of when two different arc-length elements that one imprints upon an M_n will lead to the same geodetic lines (when two **Riemann**ian M_n can be mapped to each other in such a way that the geodetic lines of the one go to the geodetic lines of the other, resp.). That question was initially treated for $n = 2$ in the differential geometry of surfaces in ordinary R_3 [cf., III D 6.a (**A. Voss**), no. **9**] and from that point onward, it was adapted to a general n (cf., also no.

([463]) Cf., **P. Appell**, "Sur une interpretation des values imaginaires du temps en mécanique," C. R. Acad. Sci. Paris **87** (1878), pp. 1074, who gave an application to the plane pendulum, in which he could clarify the meaning of the imaginary periods of the elliptic integrals that appeared.

([464]) The *mechanical similarity* of two motions [cf., IV **6** (**P. Stäckel**), no. **8**] falls withing that category. Namely, if λ is the ratio of the lengths and μ is the ratio of the masses then if $q_1, \ldots, q_n$ are introduced as dimensionless quantities then one will have:

$$C = \mu\lambda^2,$$

such that (573.a) will go to:

$$d\mathfrak{t} = \sqrt{\frac{\mu}{c}}\,\lambda\,dt$$

and the relation:

$$(574) \qquad c\tau^2 = \mu\lambda^2$$

will reproduce the mechanical similarity, in which τ is the ratio of the time units. If the q_ρ are dimensionless then the Q_ρ will have the dimension of [force · length], such that if γ denotes the ratio of the forces then one will have:

$$c = \gamma\lambda,$$

and (574) will go to the known formula for mechanical similarity:

$$(574.\text{a}) \qquad \gamma\tau^2 = \mu\lambda.$$

([465]) **P. Appell**, *loc. cit.* ([459]), esp., pp. 40.

36). Now, if two arc-length elements ds and $d\mathfrak{s}$ have the same geodetic lines then one can also give a non-trivial correspondence between two mechanical *problems with forces*. That is because every system of applied forces $Q_\rho(q_1, \ldots, q_n)$ for the problem the arc-length ds can determine a system of applied forces $\mathfrak{Q}_\rho(q_1, \ldots, q_n)$ for the other problem with the arc-length element $d\mathfrak{s}$ [cf., ([483])] in such a way that the two problems will have equivalent trajectories ([466]). It is important in this case that the association (571) of the times must possess the simplified form:

$$d\mathfrak{t} = \lambda(q_1, \ldots, q_n)\, dt\,, \tag{575}$$

in which only the position coordinates ([467]) will appear (cf., no. **36**). The case in which the applied forces Q_ρ arise from a potential:

$$Q_\rho = -\frac{\partial \Phi}{\partial q_\rho} \qquad \Phi = \Phi(q_1, \ldots, q_n) \tag{576}$$

requires special treatment. From no. **10**, the equations of motion will then possess the energy integral:

$$T + \Phi = k\,, \tag{577}$$

and the trajectories can be combined into natural families of ∞^{2n-2}, each of which is characterized by the numerical value of k. The trajectories of such a family can then be regarded as the geodetic lines of the arc-length element:

$$ds^* = \sqrt{2(k - \Phi)}\, ds\,. \tag{578}$$

Now here, as **G. Darboux** ([468]) showed, a mechanical problem (ds, Φ) will have the equivalent trajectories to the problem ($d\mathfrak{s}$, Ψ) for which ([469]):

([466]) Cf., **P. Painlevé**, *loc. cit.* ([460]), pp. 52. Conversely, as **Painlevé** showed there (at least for $n > 2$), the arc-length elements ds and $d\mathfrak{s}$ must possess the same geodetic lines when two mechanical problems (ds, Q_ρ) and ($d\mathfrak{s}$, $\mathfrak{Q}_\rho$) have equivalent trajectories if one preserves the arc-length elements for two different systems of applied force Q'_ρ, $\mathfrak{Q}'_\rho$ (Q''_ρ, $\mathfrak{Q}''_\rho$, resp.).

([467]) If one allows the applied forces Q_ρ ($\mathfrak{Q}_\rho$, resp.) to also depend upon the velocity component then one can always determine a system of applied forces $\mathfrak{Q}_\rho(q_1, \ldots, q_n, \dot{q}_1^*, \ldots, \dot{q}_n^*)$ for a system of applied forces Q_ρ ($q_1, \ldots, q_n$, $\dot{q}_1, \ldots, \dot{q}_n$) when the arc-length elements ds and $d\mathfrak{s}$ are given in such a way that the equations of motion of both mechanical problems will be taken to each other under the prescribed time transformation (575). Cf., **P. Appell**, *loc. cit.* ([459]), pp. 38.

([468]) **G. Darboux**, C. R. Acad. Sci. Paris **108** (1889), pp. 449, as well as **P. Painlevé**, *loc. cit.* ([460]), pp. 10 and 35.

([469]) The associated time association is:

(579) $$d\mathfrak{s} = \sqrt{\alpha\Phi+\beta}\,ds\,, \quad \Psi = \frac{\gamma\Phi+\delta}{\alpha\Phi+\beta}\,.$$

That is because one has:

$$d\mathfrak{s}^* = \sqrt{2(k^*-\Psi^*)}\,d\mathfrak{s} = \sqrt{2\left(k^*-\frac{\gamma\Phi+\delta}{\alpha\Phi+\beta}\right)}\sqrt{\alpha\Phi+\beta}\,ds = \sqrt{\gamma-\alpha k^*}\sqrt{2\left(-\frac{\beta k^*-\delta}{\alpha k^*-\gamma}-\Phi\right)}ds$$
$$= \sqrt{\gamma-\alpha k^*}\,ds^*\,,$$

when one further sets:

(579.a) $$k = -\frac{\beta k^*-\delta}{\alpha k^*-\gamma} \qquad \text{or} \qquad k^* = \frac{\gamma k+\delta}{\alpha k+\beta}\,.$$

That *Darboux transformation* will imply the *correspondants ordinaires* of the given mechanical problem (*ds*, Φ). It will enter in place of the trivial correspondence in the case where a potential exists.

36. Geodetic mapping between two M_n . Correspondence of arc-length elements and a general correspondence between mechanical systems with applied forces. – The investigation of the *non-trivial correspondence* between two mechanical problems began with the consideration of *force-free systems*. One then deals with the correspondence between two arc-length elements, i.e., with a map between two **Riemann**ian M_n with the arc-length elements:

(580) $$ds^2 = \sum_{\lambda,\mu} g_{\lambda\mu}\,dq_\lambda\,dq_\mu$$

and

(580.a) $$d\mathfrak{s}^2 = \sum_{\lambda,\mu} \mathfrak{g}_{\lambda\mu}\,dq_\lambda\,dq_\mu\,,$$

that takes the geodetic lines of the one M_n to the geodetic lines of the other. The time association (571) will then have the simplified form ([470]):

(581) $$d\mathfrak{t} = \frac{dt}{\mu(q_1,\ldots,q_n)}\,,$$

$$\frac{d\mathfrak{t}}{dt} = \frac{d\mathfrak{s}}{ds}\sqrt{\frac{k-\Phi}{k^*-\Psi}} = (\alpha\Phi+\beta)\sqrt{\frac{\alpha k+\beta}{\beta\gamma-\alpha\delta}}\,,$$

or when one eliminates *k* with the help of the energy integral (577):

$$\sqrt{\beta\gamma-\alpha\beta}\,d\mathfrak{t} = (\alpha\Phi+\beta)\sqrt{\alpha T+(\alpha\Phi+\beta)}\,dt\,.$$

It will then have the form (571).

([470]) Cf., **T. Levi-Civita**, "Sulle trasf. delle equ. din.," Ann. di mat. (2) **24** (1896), pp. 255, esp., pp. 273.

and indeed, when one denotes the discriminants of the two quadratic forms (580) [(580.a), resp.] by G ($\mathfrak{G}$, resp.), one will have:

(581.a)
$$\mu = C\left(\frac{G}{\mathfrak{G}}\right)^{\frac{1}{n+1}},$$

in which one understands C to mean a constant. When one appeals to the **Christoffel** three-index symbols, that will give conditions for the correspondence of (580) and (580.a) in the form of the equations ([471]):

(582)
$$\begin{cases} \left\{\begin{matrix} r\ s \\ j \end{matrix}\right\}^* = \left\{\begin{matrix} r\ s \\ j \end{matrix}\right\}, & (j \neq r \text{ and } s) \\ \left\{\begin{matrix} r\ s \\ r \end{matrix}\right\}^* = \left\{\begin{matrix} r\ s \\ r \end{matrix}\right\} - \frac{1}{2}\frac{\partial \ln \mu}{\partial q_s}, & (r \neq s) \\ \left\{\begin{matrix} r\ r \\ r \end{matrix}\right\}^* = \left\{\begin{matrix} r\ r \\ r \end{matrix}\right\} - \frac{1}{2}\frac{\partial \ln \mu}{\partial q_r}. & \end{cases}$$

They will take a simpler form when one introduces the covariant derivatives of the **Ricci** calculus [cf., III D 10 (**R. Weitzenböck**), Part 2, no. **19**] of $(\mu^2\,\mathfrak{g}_{\lambda\mu})$ relative to the differential form (580). They will then read simply ([472]):

(583)
$$(\mu^2\,\mathfrak{g}_{rs})_{(t)} + (\mu^2\,\mathfrak{g}_{st})_{(r)} + (\mu^2\,\mathfrak{g}_{tr})_{(s)} = 0\,,$$

and that will say that:

$$\mu^2 \sum_{\lambda,\rho} \mathfrak{g}_{\lambda\rho}\,\dot{q}_\lambda\,\dot{q}_\rho = \text{const.}$$

or

(584)
$$\left(\frac{G}{\mathfrak{G}}\right)^{\frac{2}{n+1}} \sum_{\lambda,\rho} \mathfrak{g}_{\lambda\rho}\,\dot{q}_\lambda\,\dot{q}_\rho = \text{const.}$$

is a first integral of the differential equations of the geodetic lines of the arc-length element (580) ([473]). The existence of a corresponding arc-length element then implies the existence of a quadratic

([471]) Cf., **T. Levi-Civita**, *loc. cit.* ([470]), pp. 270.
([472]) **T. Levi-Civita**, *loc. cit.* ([470]), pp. 276.
([473]) That theorem was first proved by **P. Painlevé**, *loc. cit.* ([460]), pp. 43, who showed that:

$$\frac{G}{(ds)^{(n+1)/2}} \quad \text{and} \quad \frac{\mathfrak{G}}{(d\,\mathfrak{s})^{(n+1)/2}}$$

are **Jacobi** multipliers (cf., no. **22**) for the equations of the geodetic lines of ds, as well as for those of $d\,\mathfrak{s}$.

integral (cf., no. **29**) ([474]), as **U. Dini** had already shown for $n = 2$ [cf., III 6.a (**A. Voss**), no. **9**]. For $n = 2$, **U. Dini** had likewise determined the arc-length elements of all surfaces that would be mapped to each other in the sense of the correspondence of arc-length elements, i.e., such that the geodetic lines of the one surface will go to the other one ([475]). As a generalization of that argument, **T. Levi-Civita** determined the form of the corresponding arc-length element by the method of the **Ricci** calculus ([476]). Upon introducing a suitable orthogonal system of curve congruences [cf., III D 11 (**L. Berwald**), no. **20**] with the direction cosines ([477]) $\lambda_{(h)\,r}$, the coefficients $g_{\sigma\tau}$ ($\mathfrak{g}_{\sigma\tau}$, resp.) of two arc-length elements can be expressed in full generality in the form:

(585)
$$\begin{cases} g_{\sigma\tau} = \sum_{h=1}^{n} \lambda_{(h)\sigma}\lambda_{(h)\tau}, \\ \mathfrak{g}_{\sigma\tau} = \sum_{h=1}^{n} \rho_h\, \lambda_{(h)\sigma}\lambda_{(h)\tau}, \end{cases}$$

in which the invariants ρ_h are the roots of the equation:

(585.a)
$$|\, \mathfrak{g}_{\sigma\tau} - \rho\, g_{\sigma\tau} \,| = 0 .$$

Naturally, one also has that conversely:

(584.a)
$$\left(\frac{\mathfrak{G}}{G}\right)^{(n+1)/2} \sum_{\lambda,\rho} g_{\lambda\rho}\, \dot{q}_\lambda^*\, \dot{q}_\mu^* = \text{const.}$$

is an integral of the equations of the geodetic lines of the arc-length element $d\,\mathfrak{s}$. One can think of introducing:

$$\dot{q}_\lambda = \frac{dq_\lambda}{ds}, \qquad \dot{q}_\lambda^* = \frac{dq_\lambda}{d\,\mathfrak{s}}$$

into (584) and (584.a).

([474]) The quadratic integral will coincide with the trivial quadratic integral $\sum g_{\lambda\rho}\, \dot{q}_\lambda\, \dot{q}_\rho$ = const., only when $\mathfrak{g}_{\lambda\rho} = c\, g_{\lambda\rho}$, i.e., in the case of the trivial correspondence.

([475]) Cf., the presentation by **G. Darboux**, *Théorie des surfaces*, v. III, Book 6, Chap. 3, esp., pp. 49, *et seq.* The arc-length elements of the two surfaces are:

$$ds^2 = (\Phi(q_1) - \Psi(q_2))(\Phi_1^2(q_1)\,dq_1^2 - \Psi_1^2(q_2)\,dq_2^2)$$

or

$$d\mathfrak{s}^2 = \left(\frac{1}{\Psi(q_2)} - \frac{1}{\Phi(q_1)}\right)\left(\frac{\Phi_1^2(q_1)}{\Phi(q_1)}dq_1^2 + \frac{\Psi_1^2(q_2)}{\Psi(q_2)}dq_2^2\right).$$

They will then have the so-called **Liouville** form (cf., no. **19**).

([476]) **T. Levi-Civita**, *loc. cit.* ([470]), pp. 280.

([477]) The $\lambda_{(h)\,r}$ ($\lambda_{(h)}^{r}$, resp.) are covariant (contravariant, resp.) direction cosines relative to the arc-length element ds.

With the help of such a representation, one will then get the conditions for the correspondence of the arc-length elements in the form:

(586)
$$\begin{cases} (\rho_h - \rho_i)\,\gamma_{hij} = 0, & (h, i, j \text{ are distinct}), \\ (\rho_i - \rho_i)\,\gamma_{iji} = \frac{1}{2}\sum_{r=1}^{n} \frac{\partial \rho_i}{\partial q_r}\lambda_{(j)}^{r}, & (i \neq j), \\ \sum_{r=1}^{n} \frac{\partial(\mu\,\rho_i)}{\partial q_r}\lambda_{(j)}^{r} = 0, & (i \neq j), \\ \sum_{r=1}^{n} \frac{\partial(\mu\,\rho_i)}{\partial q_r}\lambda_{(j)}^{r} = -\rho_i \sum_{r=1}^{n} \frac{\partial \mu}{\partial q_r}\lambda_{(i)}^{r}, \end{cases}$$

in which the γ_{hij} are the rotation coefficients [cf., III D 11 (**L. Berwald**), no. **20**] of the orthogonal system of curves.

If the n roots ρ_h of equation (585.a) are all different from each other then the orthogonal system of curves will consist of the curves of intersection of n systems of mutually-orthogonal M_{n-1} such that when one employs the parameters of the family as coordinates, the two arc-length elements will take the form:

(587)
$$ds^2 = \sum_{\sigma=1}^{n} H_\sigma^2\, dq_\sigma^2 \qquad \text{and} \qquad d\mathfrak{s}^2 = \sum_{\sigma=1}^{n} \rho_\sigma\, H_\sigma^2\, dq_\sigma^2$$

in which:

(587.a)
$$\begin{cases} \mu = \dfrac{\psi_1(q_1)\,\psi_2(q_2)\cdots\psi_n(q_n)}{C}, \\ \rho_\sigma = \dfrac{1}{\mu\cdot\psi_\sigma(q_\sigma)}, \\ H_\sigma^2 = V_\sigma^2(q_\sigma)\prod_{\tau=1}^{n}{}' \,|\,\psi_\tau - \psi_\sigma\,|, \end{cases}$$

and the prime on Π suggests that the multiplication index τ cannot assume the value σ in the product ([478]). With a slight generalization, two corresponding arc-length elements can then be put into the form:

(588)
$$\begin{cases} ds^2 = \sum_{\sigma=1}^{n}\left(\prod_{\tau=1}^{n}{}' \,|\,\psi_\tau - \psi_\sigma\,|\right) dq_\sigma^2, \\ d\mathfrak{s}^2 = \dfrac{C}{(\psi_1 + c)(\psi_2 + c)\cdots(\psi_n + c)} \sum_{\sigma=1}^{n}\left(\prod_{\tau=1}^{n}{}' \,|\,\psi_\tau - \psi_\sigma\,|\right) dq_\sigma^2, \end{cases}$$

([478]) **T. Levi-Civita**, *loc. cit.* ([470]), pp. 286. That representation of the arc-length element was already achieved in some special cases by **R. Liouville**, "Sur les équ. de la dynamique," Acta math. **19** (1895), pp. 251, as well as **G. di Pirro**, "Sulle trasformazioni delle equazioni delle dinamica," Palermo Rend. del circ. mat. **9** (1895), as well as *ibid.* **10** (1896), pp. 241, and **G. Picciati**, "Sulla trasformazione delle equazione della dinamica in alcuni casi particolari," Venedig Atti dell'istit. (7) **7** (1896), pp. 175.

in which c is understood to mean a constant. From (584), one will have the following quadratic integral for the geodetic lines of the arc-length element ds:

$$\sum_{\sigma=1}^{n}\left[(\psi_1+c)\cdots(\psi_{\sigma-1}+c)(\psi_{\sigma+1}+c)\cdots(\psi_n+c)\prod_{\tau=1}^{n}{}'\,|\psi_\tau-\psi_\sigma|\right]\dot{q}_\sigma^2 = \text{const.}, \tag{589}$$

and since that relation must exist identically in c, it will imply *n quadratic integrals*.

If the roots of (585.a) are not all different then in the special where one has $(n-m)$ simple roots, $\rho_1, \ldots, \rho_{n-m}$, and one m-fold root ρ_n, one will have only $(n-m)$ families of M_{n-1} that intersect each other orthogonally. Meanwhile, one can add m families of M_{n-1} that are orthogonal to the former $(n-m)$ families. In general, they cannot intersect each other orthogonally. The arc-length element will then take the form:

$$ds^2 = \sum_{\sigma=1}^{n-m} H_\sigma^2\,dq_\sigma^2 + \sum_{\lambda,\mu=n-m+1}^{n} a_{\lambda\mu}\,dq_\lambda\,dq_\mu\,, \tag{590}$$

while the arc-length element $d\mathfrak{s}$ will assume the form:

$$d\mathfrak{s}^2 = \sum_{\sigma=1}^{n-m} \rho_\sigma H_\sigma^2\,dq_\sigma^2 + \rho_n \sum_{\lambda,\mu=n-m+1}^{n} a_{\lambda\mu}\,dq_\lambda\,dq_\mu\,. \tag{590.a}$$

The equations ([479]):

$$\begin{cases} \mu = \dfrac{\psi_1\cdots\psi_{n-m}\,\psi_n}{C}\,, \\[2mm] \rho_\sigma = \dfrac{1}{\psi_\sigma\cdot\mu}\,, \qquad\qquad (\sigma = 1, \ldots, n-m) \\[2mm] \rho_n = \dfrac{1}{\psi_n\cdot\mu} \end{cases} \tag{591}$$

enter in place of the relations (587.a), from which, one will then get:

$$\begin{cases} H_\sigma^2 = V_\sigma^2(q_\sigma)\prod_{\tau=1}^{n-m}{}'\,|\psi_\tau-\psi_\sigma|\,, \\[2mm] a_{\lambda\mu} = K_{\lambda\mu}(q_{n-m+1},\ldots,q_n)\prod_{\tau=1}^{n-m}{}'\,|\psi_\tau-\psi_\sigma|\,, \end{cases} \tag{592}$$

such that when one absorbs $V_\sigma(q_\sigma)$ into q_σ, the arc-length element will take the form:

([479]) **T. Levi-Civita**, *loc. cit.* ([470]), pp. 293.

(593)
$$\begin{cases} ds^2 = \sum_{\sigma=1}^{n-m} \left(|\psi_\tau - \psi_\sigma| \prod_{\tau=1}^{n-m}{}' |\psi_\tau - \psi_\sigma| \right) dq_\sigma^2 + \prod_{\tau=1}^{n-m}{}' |\psi_\tau - \psi_\sigma| \sum_{\lambda,\mu=n-m+1}^{n} K_{\lambda\mu}\, dq_\lambda\, dq_\mu, \\ d\mathfrak{s}^2 = \dfrac{C}{(\psi_1 + c)\cdots(\psi_{n-m} + c)(\psi_n + c)} \left\{ \sum_{\sigma=1}^{n-m} \dfrac{1}{\psi_\sigma + c} \left(\prod_{\tau=1}^{n-m}{}' |\psi_\tau - \psi_\sigma| \right) dq_\sigma^2 \right. \\ \qquad \left. + \dfrac{1}{\psi_\sigma + c} \prod_{\tau=1}^{n-m}{}' |\psi_\tau - \psi_\sigma| \sum_{\lambda,\mu=n-m+1}^{n} K_{\lambda\mu}\, dq_\lambda\, dq_\mu \right\}. \end{cases}$$

One will then get $(n - m + 1)$ quadratic integrals ([480]) from the quadratic integral:

(594)
$$(\psi_1 + c)\cdots(\psi_{n-m} + c)(\psi_n + c) \left\{ \sum_{\sigma=1}^{n-m} \frac{1}{\psi_\sigma + c} \prod_{\tau=1}^{n-m}{}' |\psi_\tau - \psi_\sigma|\, \dot{q}_\sigma^2 \right. \left. + \frac{1}{\psi_n + c} \prod_{\tau=1}^{n-m}{}' |\psi_\tau - \psi_\sigma| \sum_{\lambda,\mu=n-m+1}^{n} K_{\lambda\mu}\, \dot{q}_\lambda\, \dot{q}_\mu \right\} = \text{const.},$$

which is true identically in *c*.

From this point onward, the case in which equation (585.a) has arbitrarily-many multiple roots will be easy to grasp.

The investigation of the *non-trivial correspondence between two mechanical problems with applied forces* has also been successfully addressed, even if it has also still not attracted as much attention as force-free motion. One can initially establish that the relation (571) between the two time differentials $d\mathfrak{t}$ and dt, into which the velocity components also enter here, must have the form ([481]):

(595)
$$d\mathfrak{t}^2 = \frac{dt^2}{\mu^2(q_1, \ldots, q_n)} \left(1 - \sum_{r,s=1}^{n} c_{rs}\, \dot{q}_r\, \dot{q}_r \right),$$

and that the function μ that enters into it mediates the relation between the components of the applied forces on the two systems, which reads:

(596)
$$\mathfrak{Q}^\rho = \mu^2 Q^\rho .$$

On the other hand, the bracketed factor in (595) will lead to a quadratic integral of the equations of motion (558), and indeed that quadratic integral will be ([482]):

([480]) **T. Levi-Civita**, *loc. cit.* ([470]), pp. 297.
([481]) **P. Painlevé**, *loc. cit.* ([460]), pp. 13 and 59, as well as **T. Levi-Civita**, *loc. cit.* ([470]), pp. 272.
([482]) Naturally:

$$\left(\frac{\mathfrak{G}}{G} \frac{1}{\mu^2} \right)^{2/(n+3)} \left(\frac{dt}{d\mathfrak{t}} \right)^2 = \left(\frac{\mathfrak{G}}{G} \frac{1}{\mu^2} \right)^{2/(n+3)} \left(\mu^2 + \sum_{r,s} c_{rs}\, \dot{q}_r^*\, \dot{q}_s^* \right) = \text{const.}$$

is correspondingly a quadratic integral of the equations of motion (568). **P. Painlevé**, *loc. cit.* ([460]), pp. 65.

(597) $$\left(\frac{G}{\mathfrak{G}}\mu^2\right)^{2/(n+3)}\left(\frac{d\mathfrak{t}}{dt}\right)^2 = \left(\frac{G}{\mathfrak{G}}\mu^2\right)^{2/(n+3)}\frac{1}{\mu^2}\left(1-\sum_{r,s=1}^{n} c_{rs}\,\dot{q}_r\,\dot{q}_s\right) = \text{const.}$$

An exception will occur when the left-hand side is to be identically constant. However, one must then have $c_{rs} = 0$, and from (582), one will have:

(598) $$\mu = C\cdot\left(\frac{G}{\mathfrak{G}}\right)^{1/(n+1)},$$

and from (581.a), that means that the two arc-length elements ds and $d\mathfrak{s}$ have the same geodetic lines ([483]).

If the applied forces have a potential:

(599) $$Q_\rho = -\frac{\partial\Phi}{\partial q_\rho}, \qquad \Phi = \Phi(q_1, \ldots, q_n)$$

then the quadratic integral (597) will coincide with the energy integral ([484]):

(599.a) $$T + \Phi = \text{const.}$$

If that occurs then one can perform a **Darboux** transformation that takes the problem (ds, Φ) to another (ds^*, Ψ) such that the arc-length element ds^* of this new problem and the arc-length element $d\,\mathfrak{s}$ of the problem (ds, Φ), which have equivalent paths, have the same geodetic lines. From (579.a), the geodetic lines of $d\,\mathfrak{s}$ will then correspond to one of the natural families of trajectories ([485]) of (ds, Φ).

The problem of exhibiting necessary and sufficient conditions for two mechanical problems with applied forces to have equivalent trajectories was taken up by **J. E. Wright** ([486]), who appealed to **T. Levi-Civita**'s **Ricci** calculus as a paradigm for it. In some special cases, he

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([483]) One then sees immediately from (596) in this case how one can determine a system of applied forces $\mathfrak{Q}_\rho$ that is associated with any system of applied forces Q_ρ in such a way that the two mechanical problems will have equivalent paths.

([484]) **P. Painlevé**, *loc. cit.* ([460]), pp. 67.

([485]) If the forces in both mechanical problems arise from potentials then one can go from one to the other by a **Darboux** transformation in such a way that the two new arc-length elements will have the same geodetic lines. In the two original problems, a well-defined natural family for the one problem will then correspond to a well-defined natural family of the other problem. Including the energy integral, the two mechanical problems will each have three quadratic integrals. Moreover, when one is not dealing with a **Darboux** transformation, there will no longer be a natural family of trajectories of the first problem that individually go to a natural family for the second problem.

([486]) **J. E. Wright**, "Corresponding dynamical systems," Ann. di mat. (3) **16** (1909), pp. 1. Cf., also **J. E. Wright**, "Invariants of quadratic differential forms," Cambridge Tracts **9** (1908), esp., pp. 80, *et seq.*

determined the form that the arc-length elements and the applied forces of two systems with equivalent paths would need to have ([486.a]).

37. Mechanical problems whose trajectories go to each other under a group of transformations. – A transformation:

$$q_\rho = \psi_\rho(\mathfrak{q}_1, \ldots, \mathfrak{q}_n) \qquad (\rho = 1, \ldots, n) \tag{600}$$

will take the equations of motion of a mechanical problem:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\rho}\right) - \frac{\partial T}{\partial q_\rho} = Q_\rho \qquad (\rho = 1, \ldots, n) \tag{601}$$

to the new equations:

$$\frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial \dot{\mathfrak{q}}_\rho}\right) - \frac{\partial \bar{T}}{\partial \mathfrak{q}_\rho} = \bar{Q}_\rho \qquad (\rho = 1, \ldots, n), \tag{602}$$

which **P. Painlevé** called *homologous* to (601). Now, should the transformation (600) transform the trajectories of (601) into themselves, then the homologous mechanical problem (602) would need to have the same paths as the problem (601), or in **Painlevé**'s terminology: A transformation (600) will take the trajectories of a mechanical system to themselves when that homologous system of equations (602) that it generates by way of (601) is simultaneously a system of equations that corresponds to (601) ([487]).

It will be especially significant when one does not have a single transformation in (600), but a group of transformations of one or more parameters, since one will then succeed in linking up with the arguments of **S. Lie** that allow one to gain some advantages for the integration of equations from the existence of such groups [cf., II A 4.b (**E. Vessiot**), nos. **13** and **18**]. **Lie** ([488]) himself has already investigated when the geodetic lines of a surface in three-dimensional Euclidian space will

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([486.a]) On the basis of a remark by **P. Stäckel**, "Über Transformationen von Bewegungen," Gött. Nachr. (1898), pp. 157, and in connection with the research of **T. Levi-Civita**, **A. Malipiero**, "Sulla transform. delle equ. della din," Venedig Atti del ist. (8) **3**^{2} (= 60^2) (1901), pp. 469, investigated when two **Riemann**ian M_n could be mapped to each other in such a way that one family of ∞^{2n-3} geodetic lines of the one M_n will go to a family of ∞^{2n-3} geodetic lines of the other M_n.

A question that is related to those arguments is that of the nature of mechanical problems that have a number of common integrals, cf., the conclusion of no. **29**. **J. Drach** has recently reconsidered the **Bertrand** articles that were cited there [cf., ([389]) and ([391])] from the standpoint of rational theories of integration in "Sur les intégrales communes à plusieurs problèmes de mécanique," C. R. Acad. Sci. Paris **157** (1913), pp. 1516 [cf., II A **4**.b **(E. Vessiot**), no. **38**] and extended them. Another type of treatment was given by **G. Pennachietti**, "Sugl'integrali delle equ. della din.," Catania Atti dell'acc. Gionenia (4) **2** (1890), as well as in some other work.

([487]) **P. Painlevé**, "Sur les mouvements des systèmes dont les trajectoires admettent une transformation infinitésimale," C. R. Acad. Sci. Paris **116** (1893), pp. 21.

([488]) **S. Lie**, "Untersuchungen über geodätische Kurven," Math. Ann. **20** (1882), pp. 357.

be transformed to themselves by a group of transformations, and **G. Fubini** ([489]) had carried out the corresponding investigations for a general **Riemann**ian M_n with the arc-length element:

$$ds^2 = \sum_{\lambda,\mu=1}^{n} g_{\lambda\mu}\, dq_\lambda\, dq_\mu\,. \tag{603}$$

If:

$$X f = \xi^1(q_1,\dots,q_n)\frac{\partial f}{\partial q_1} + \dots + \xi^n(q_1,\dots,q_n)\frac{\partial f}{\partial q_n} \tag{604}$$

is the symbol for an infinitesimal transformation ([490]) then the changes that the curly **Christoffel** three-index symbols $\left\{\begin{matrix}\sigma\,\tau\\ \rho\end{matrix}\right\}$ of the arc-length element (603) will experience under that infinitesimal transformation will be given by:

$$\left\{\begin{matrix}\sigma\,\tau\\ \rho\end{matrix}\right\}' = \frac{\partial^2 \xi^\rho}{\partial q_\sigma\,\partial q_\tau} + \sum_{\nu=1}^{n}\left[\left\{\begin{matrix}\sigma\,\nu\\ \rho\end{matrix}\right\}\frac{\partial \xi^\nu}{\partial q_\tau} + \left\{\begin{matrix}\nu\,\tau\\ \rho\end{matrix}\right\}\frac{\partial \xi^\nu}{\partial q_\sigma} - \left\{\begin{matrix}\sigma\,\tau\\ \nu\end{matrix}\right\}\frac{\partial \xi^\nu}{\partial q_\tau}\right] + \sum_{\nu=1}^{n}\xi^\nu \frac{\partial}{\partial q_\nu}\left\{\begin{matrix}\sigma\,\tau\\ \rho\end{matrix}\right\}, \tag{605}$$

and the conditions for the infinitesimal transformation (604) of the geodetic lines of *ds* to go to themselves will read ([491]):

$$2\left\{\begin{matrix}\sigma\,\tau\\ \rho\end{matrix}\right\}' = (\delta_\sigma^\rho + \delta_\tau^\rho)\left\{\begin{matrix}\tau\,\tau\\ \tau\end{matrix}\right\}'. \tag{606}$$

If one knows such infinitesimal transformations then the integration of the geodetic lines will be simplified ([492]).

It is obvious how one might adapt these arguments to mechanical problems. If one restricts oneself to problems in which the applied forces arise from a potential in so doing:

([489]) **G. Fubini**, "Sui gruppi di trasformazioni geodetiche," Turin Mem. della Acc. d. sc. (2) **53** (1903), pp. 261. An overview of all of his relevant investigations was given in **G. Fubini**, "Applicazioni della teoria dei gruppi continui alla geom. diff. e alle equ. di Lagrange," Math. Ann. **66** (1908), pp. 202.

([490]) The ξ^ρ are provided with upper indices in order to emphasize the contravariant character of $\xi^1, \dots, \xi^n$. In what follows, one must observe that $dq_1, \dots, dq_n$ are also the components of a *contravariant* vector, so they are "falsely" indexed then.

([491]) Cf., **G. Fubini**, *loc. cit.* (489), pp. 267. One sets:

$$\delta_\lambda^\mu = \begin{cases} 0 & (\lambda \neq \mu), \\ 1 & (\lambda = \mu) \end{cases}$$

in (606) in the known way.

([492]) For $n = 2$, cf., **S. Lie**, *loc. cit.* ([488]), pp. 431.

(607) $$Q_\rho = -\frac{\partial\Phi}{\partial q_\rho}, \qquad \Phi = \Phi(q_1, \ldots, q_n),$$

so one will then have the energy integral:

(608) $$T + \Phi = k,$$

then one will automatically direct one's gaze to the individual *natural families* of trajectories that are characterized by the numerical value of k, and the analogy with the problem of geodetic lines will become even closer in such a way that one can (cf., no. **10**) speak of the trajectories of the individual natural families as the geodetic lines of the arc-length elements ([493]):

(609) $$ds^* = \sqrt{2(k-\Phi)}\,ds \qquad (ds^2 = 2\,T\,dt^2).$$

Correspondingly, **O. Staude** ([494]) initially posed the question of when a one-parameter group of transformations will take each individual natural family of trajectories into itself for $n = 2$ and then ([495]) for $n = 3$. **P. Stäckel** ([496]) treated the same problem for general n. Now, it will follow immediately from no. **35** that a transformation that takes every natural family to itself must take the mechanical problem (T, Φ) into its **Darboux** transform. Indeed, from no. **35**, the same thing will also be true when one generalizes the problem by no longer demanding that every individual natural family should go to itself, but more generally allowing the transformation to permute the individual natural families with each other as a whole ([497]). From (579). the infinitesimal transformation (604) must take the arc-length element ds to $\sqrt{\alpha\Phi+\beta}\,ds$, so:

(610) $$X\left(\sum g_{\lambda\mu}\,dq_\lambda\,dq_\mu\right) = (\sigma\Phi+\tau)\sum g_{\lambda\mu}\,dq_\lambda\,dq_\mu,$$

whereas, on the other hand, Φ must go to a piecewise-linear function of Φ, which will have the relation:

([493]) In that way, one must observe, moreover, that the analogy between the geodetic problem of the arc-length element (609) and the corresponding mechanical problem (ds, Φ) breaks down in the question of correspondence. An arc-length element $d\mathfrak{s}^*$ that corresponds to ds^* can, in fact, never imply a mechanical problem ($d\mathfrak{s}$, Ψ) that corresponds to (ds, Φ). Cf., **P. Painlevé**, *loc. cit.* ([460]), pp. 77.

([494]) **O. Staude**, "Über die Bahnkurven eines auf einer Oberfläche beweglichen Punktes, welche infinitesimal Transformationen zulassen," Leipzig Berichte **44** (1892), pp. 429.

In conjunction with (609), **A. Kneser** treated the problem in "Das Prinzip der kleinsten Aktion und die infinitesimale Transformation der dyn. Probl.," Dorpat Sitzungsber. d. naturforsch. Ges. **10** (1894), pp. 501.

([495]) **O. Staude**, "Über die Bahnkurven eines in einem Raume von drei Dimensionen beweglichen Punktes, welches infinitesimal Transformationen zulassen," Leipzig Ber. **45** (1893), pp. 511.

([496]) **P. Stäckel**, "Über dynamische Probleme, deren Differentialgleichungen eine infinites. Transf. gestatten," Leipzig Ber. **45** (1893), pp. 331. Cf., also **A. Kneser**, *loc. cit.* ([494]).

([497]) **P. Stäckel** did that in "Anwend. von Lie's Theorie der Transformationsgruppen auf die Differentialgleich. d. Dynamik," Leipzig Ber. **49** (1897), pp. 411. From (579.a), there can be at most two natural families that are transformed into themselves in that way ($k^* = k$).

(611) $$X(\Phi) = \lambda + \mu\,\Phi + \nu\,\Phi^2$$

as a consequence, in which one understands λ, μ, ν to mean constants ([498]), so one will still have:

(611.a) $$\sigma + \nu = 0$$

in particular ([499]). Now, should every individual one of the natural families remain invariant then since the function Φ must transform cogrediently with k, from (579) and (579.a), Φ must also remain invariant, so one must have ([500]):

(612) $$X(\Phi) = 0\,, \qquad \text{i.e.,} \qquad \lambda = \mu = \nu = 0\,,$$

and since one will then have $\sigma = 0$, (610) will simplify to:

(613) $$X(ds^2) = \tau \cdot ds^2,$$

i.e., the arc-length element will be multiplied by a constant under the transformation. In order to exhibit the condition equations for every individual family of trajectories to go to itself under the infinitesimal transformation (604), **P. Stäckel** ([501]) introduced the contravariant vector:

(614) $$\Phi^\rho = \sum g^{\rho\sigma} \frac{\partial\Phi}{\partial q_\sigma}$$

and the expressions ([502]):

([498]) Cf., **G. Fubini**, "Richerche gruppali relative alle equazioni della dinamica, Nota I," Roma Linc. Rend. (5) **12**[1] (1903), pp. 502.

([499]) That is because from the properties of the **Darboux** transformation, one must have:

$$X(\Phi\, ds^2) = (\varepsilon\Phi + \eta)\,ds^2,$$

whereas one must have, on the other hand:

$$X(\Phi\, ds^2) = (\lambda + \mu\Phi + \nu\Phi^2)\,ds^2 + (\sigma\Phi + \tau)\,ds^2,$$

from (610) and (611).

([500]) The orbits of the one-parameter group that is generated by the infinitesimal transformation will then lie on the manifold Φ = const. For $n = 2$, they will coincide with the equipotential curves of the potential on the surface, as **O. Staude** remarked, *loc. cit.* ([494]). He also showed that those equipotential curves are the enveloping curves of a one-parameter family in the ∞^2 trajectories of the natural family and then also that the trajectories could themselves appear.

For this, cf., **A. Kneser**, *loc. cit.* ([494]), § 4.

([501]) **P. Stäckel**, *loc. cit.* ([496]), pp. 336.

([502]) For $\sigma \neq \rho$, $\tau \neq \rho$, they will be the curly three-index symbols $\left\{ \begin{matrix} \sigma\,\tau \\ \rho \end{matrix} \right\}^*$ of the arc-length element ds^* as in (609).

In general, one has:

$$\left(\begin{matrix} \sigma\,\tau \\ \rho \end{matrix} \right) = \left\{ \begin{matrix} \sigma\,\tau \\ \rho \end{matrix} \right\}^* - \frac{1}{2(k-\Phi)}\,\delta^\rho_\sigma \frac{\partial\Phi}{\partial q_\sigma} - \frac{1}{2(k-\Phi)}\,\delta^\rho_\tau \frac{\partial\Phi}{\partial q_\tau}\,.$$

$$(615) \qquad \begin{pmatrix} \sigma\,\tau \\ \rho \end{pmatrix} = \begin{Bmatrix} \sigma\,\tau \\ \rho \end{Bmatrix} - \frac{1}{2(k-\Phi)}\, g_{\sigma\tau}\, \Phi^{\rho}$$

and then defined the $\begin{pmatrix} \sigma\,\tau \\ \rho \end{pmatrix}'$ corresponding to (605), in which he replaced the $\{\ldots\}$ with $(\ldots)$ on the right-hand side of (605). The conditions then took the form:

$$(616) \qquad 2\begin{pmatrix} \sigma\,\tau \\ \rho \end{pmatrix}' = (\delta_{\sigma}^{\rho} + \delta_{\tau}^{\rho})\begin{pmatrix} \tau\,\tau \\ \tau \end{pmatrix}',$$

which is analogous to (606).

A continuous group that takes the trajectories into themselves is finite, and indeed a group with at most n $(n + 2)$ parameters ([503]). **P. Stäckel** had determined normal forms for the dynamical problems with one and two-parameter groups and gave the infinitesimal transformations of the group ([504]). By recasting that line of reasoning, **G. Fubini** ([505]) could determine all groups for n = 3 and also took up the case of a general n already ([506]). Finally, **P. Painlevé** ([507]) made a few remarks about the structure of the group for general n, in which he also considered the fact that the applied forces might not arise from a potential, and he referred to the advantage that the existence of such a group would bring with it in the integration of the equations of motion.

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([503]) By contrast, the group of canonical transformations is an infinite group. That was a source of confusion to **E. Schuntner**, "Über die Äquivalenz und Klassifikation dynamischer Probleme," Ann. di mat. (4) **9** (1931), pp. 307, along with an Addendum in Ann. di mat. (4) **10** (1932), pp. 83, whose reasoning then seemed to be completely faulty. On that, cf., the critique of **W. Wirtinger**, Wien Monatsh. **39** (1932), pp. 241.

([504]) **P. Stäckel**, *loc. cit.* ([496]).

([505]) **G. Fubini**, "Ricerche gruppali sulle equazioni della dinamica, Nota II," Roma Linc. Rend. (5) **12**[2] (1903), pp. 60.

([506]) **G. Fubini**, "Ric. gr…, Nota III," Roma Linc. Rend. (5) **12**[2] (1903), pp. 145.

([507]) **P. Painlevé**, *loc. cit.* ([487]).