THE EXTREMUM OF

DEFORMATION WORK

TREATISE FOR THE ACQUISITION

OF THE *VENIA LEGENDI*

AT THE KGL. TECHNISCHEN HOCHSCHULE IN HANNOVER

PRESENTED BY

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EDITED AND INTRODUCED BY

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K. K.: The title page was taken from the (handwritten) title page of the version that was in the library of E. Stein. It included no reference to the year. The belated addition of a year was not followed through on, since, in all probability, the numerous handwritten corrections and extensions did not go back to 1916, which was the year of his Habilitation.

TRANSLATOR'S PREFACE

This translation was based on a book that was edited and introduced by Klaus Knothe. The book, in turn, was based on unpublished typewritten documents with handwritten annotations, and it included Prange's Habilitationsschrift, along with an extended bibliography and some supporting documents of a biographical nature. Since the latter documents were not available to the translator, as they are stored in the archives of various German universities, the translator decided to omit them from the translation, along with Knothe's references to them. The italicized annotations and footnotes were typically due to Knothe, and often reflect the fact that some of the handwritten comments in the original manuscript were ultimately illegible.

There are three aspects of Prange's treatise that make it distinctive among other treatments of the subject in question:

1. The parallel treatment of the deformation of finite structures and the deformation of continuous media.

2. The duality between force and displacement.

3. The duality between the equilibrium conditions for forces (stresses) and the compatibility conditions for strains.

The first of those concepts is particularly useful in dealing with the role of homology in the appearance of internal stresses that are not due to external loads. The second one has the effect of leading to the duality of the principle of virtual forces and the principle of virtual displacements. The third one relates to essentially a Legendre transformation for statics and the Mayer reciprocity theorem for isoperimetric problems in the calculus of variations.

It is the translator's belief that those three aspects of Prange's treatise make it sufficiently compelling as to justify an English version of it.

David Delphenich, Spring Valley, OH, USA, July 2022.

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EDITOR'S INTRODUCTION

0.1 – A brief biography of Georg Prange.

HEINRICH FRIEDRICH WILHELM GEORG PRANGE was born on January 1, 1885, in Hannover as the son of the merchant Georg Prange and died there on February 3, 1941, at the age of fifty-six.

From Easter of 1891 on, Prange attended the preschool at the Höheren Bürgerschule I that existed back then, and from Easter of 1894 on, he attended the humanistic Gymnasium (Lyzeum II) in Hannover. He passed the matriculation examination at that school in Spring of 1903. He then decided to study mathematics: From the Summer semester of 1903 to the Summer semester of 1904, he studied at the University of Göttingen, in the Winter semester of 1904/05 and the Summer semester of 1905, he studied at the University of Munich, and in the Winter semester of 1905/06, as well as the Summer semester of 1906, he once more studied at Göttingen. We then know whose lectures Prange attended in the Winter semester of 1905/06 and the Summer semester of 1906: He studied projective geometry with Klein, number theory and algebra with Minkowski, as well as integral equations with Hilbert. However, he also studied things related to the natural sciences and engineering sciences, such as thermodynamics with Voigt, graphical methods in mechanics and physics with Runge, and surveying with Wiechert, and finally psychology with Müller. Prange listed his academic teachers in his Curriculum Vita, which is added to his dissertation [**110**]: In Göttingen, in addition to the aforementioned Hilbert, Klein, Minkowski, Runge, and Wiechert, they included professors Baumann, Liebisch, Mollwo, G. F. Müller, Peter, Riecke, Schilling, Schulthess, Schwarzschild, Voigt, and Zermelo. In Munich, they included von Bayer, von Braunmühl, Doehlmann, Lindemann, Lipps, Pringsheim, Roentgen, and E. von Weber.

From Summer of 1906 to 1910, Prange had to suspend his studies, which actually lacked only a formal diploma, due to a serious illness (probably pulmonary tuberculosis). It was only in June 1912 that Prange could complete the test for a higher teaching position in the subjects of pure and applied mathematics, as well as physics, before the test committee in Göttingen.

From Fall of 1910 on, Georg Prange was employed as a helper (teaching assistant) with the lectures of C. H. Müller ([1]), who had been previous appointed briefly as a professor of mathematics at the Technische Hochschule in Hannover. After passing his test in Göttingen, Prange took up a position as an assistant in mathematics at the Technische Hochschule in Hannover under C. H. Müller in September 1912, which he held until 1921, when he taught supplementary mathematics and physics at the Bismarck School in Hannover during World War I from August 1915 to Christmas 1918.

At the beginning of his assistantship, Prange addressed the relationship between engineering mathematics and mathematics, and in particular, the variational principles of the theory of elasticity. He next provided the necessary mathematical tools in his dissertation that he submitted to the University of Göttingen: *Die Hamilton-Jacobische Theorie für Doppelintegrale (mit einer Übersicht der Theorie für einfachen Integralen)*. His main dissertation reviewer was Hilbert,

([1]) Conrad H. Müller was a coeditor of the *Encyklopädie der mathematischen Wissenschaften* with Felix Klein.

whose influence upon that work cannot be known. The oral examination took place on December 21, 1914, with a half-hour test in each of applied mathematics (Runge), physics (Voigt), and mathematical analysis (Hilbert). The dissertation was published in 1915 as a university document [**110**].

Already in 1916, Prange presented a Habilitationsschrift to the University of Hannover on the topic of "Das Extremum der Formänderungsarbeit," in which he presented his own work on:

> "the Menabrea-Castigliano school of thought, which was highly controversial in engineering mechanics, and… (presented) the theory of frameworks and the theory of three-dimensional bodies completely in parallel."

The supervisor of the Habilitationsschrift was C. H. Müller. Based upon his successful habilitation, Prange became Privatdozent in mathematics at the Technische Hochschule in Hannover on July 22, 1916. He held two-hour lectures on the following semester on a mathematical subject, and two-hour lectures on a topic in applied mathematics from engineering mechanics [**113**] and replaced Salkowski on the subject of "graphical and numerical methods in analysis."

In 1920, Prange concerned himself with a rehabilitation at the University of Halle. In December 1920, Prange was issued the required habilitation certificate for an inaugural lecture before the philosophy faculty that he presented on February 26, 1921 on the topic of "W. R. Hamilton's Bedeutung für die geometrischen Optik und die Mechanik" ([2]). On February 26, 1923, Prange was consequently assigned a teaching position in applied mathematics with the philosophy faculty at the University of Halle that he took up in the Summer semester of 1921.

In the Summer semester of 1921, Prange was called to the Technische Hochschule in Hannover, and his appointment as a professor of mathematics there became effective on October 1, 1921. He held that position up to his death despite being invited to three other places [viz., Brünn (Brno today), Dresden, and Karlsruhe].

The biographical facts that were established here were taken from the *Catalogus Professorum 1831-1981* [**134**] at the University of Hannover, the Curriculum Vita that is included in the dissertation [110], as well as the Curriculum Vita on the occasion of his appointment at Halle, and information that was stored in the university archives at the universities of Göttingen [**57**], Halle-Wittenberg [**47**], and Hannover [**138**].

0.2 – Prange's Habilitation.

The editor was already aware of Prange at the end of his sixtieth year. In 1966, a two-part article by Orovas and McLean [**105**, **106**] on the topic of "Historical Development of Energetical Principles in Elastomechanics" appeared in the journal Applied Mechanics Review, in which Part I treated the period from Heraclitus to Maxwell, and Part II treated the period from Cotterill to

([2]) The inaugural lecture might have been closely connected with the book *Allgemeine Mechanik* [**102**] that appeared in 1923 and his editing and commentary on W. R. Hamilton's *Systems of rays* [**51**]. Prange was concerned with translating Hamilton's work in 1915 and 1916 [**117**] and planned to publish a work entitled "Bedeutung Hamiltons und die Entwicklung der geometrischen Optik und der Mechanik im neunzehnten Jahrhundert "[**113**].

Prange. Prange's work in the field of elastomechanics was first full appreciated in that survey article (see pages 928-930 in [**106**]).

> "Prange gave the first complete fundamental classification and unified representation of the entire conceptual framework of the classic variational principles in elasticity, in which he linked the methods of the fundamental calculus to the variational principles of elastomechanics."

Somewhat later, it made explicit reference to the Habilitation:

> "Prange gave, in his habilitation dissertation, an integrated representation of the entire spectrum of variational principles of elastomechanics from a purely-mechanical point of view, excluding thermodynamic considerations."

In summation, Orovas and McLean finally stated:

> "Prange was able to demonstrate the analogy existing between the fundamental relations in mechanics of deformable solids and rigid-body mechanics: the Menabrea Principle corresponds to the Principle of Least Action; the Equations of Equilibrium and Compatibility correspond to the Differential Equations of Motion; the Cotterill-Castigliano theorem corresponds to the derivation of Hamilton's Varying Action."

The basic ideas in Prange's work include analogies between the calculus of variations in the realm of analytical mechanics, to which HAMILTON-JACOBI theory belongs, in particular, and the variational calculations in the problems of elasticity theory. An adaptation of Hamilton-Jacobi theory to problems in the theory of elasticity had already been carried out in the dissertation of MAX BORN [**10**] and by HELLINGER [**53**].

In a dissertation by ARWED WALTER in 1868 [**147**] that was mentioned by TRUESDELL and TOUPIN [**141**], which is currently accessible in the library at Humboldt University, just like KIRCHHOFF ([**58**], Vorl. 11, § **5**), he did not go into the adaptation of HAMILTON-JACOBI theory to the problems in the theory of elasticity, but only the extension of HAMILTON's principle to the dynamical problems for continuous media.

Prange wished to publish his Habilitationsschrift as a book, but that never came to pass [**112**, **50**], due to the first World War. In the library at the University of Hannover, up until recently, there was a photocopy of the typewritten draft of it with numerous handwritten entries, but it was lost without a trace in the meantime. It, or another version of it, must have been available to Orovas and McLean [**105**, **106**]. Passages from the second part of their survey paper are obviously English translations of Prange's formulations.

In 1970, the editor had prepared a copy of the text from the library of what was the Technische Hochschule in Hannover at the time, and that text was used as the basis for a word processor version. The handwritten formulas that were inserted were obviously flawed in some places. Comments to the effect of "text unclear," as well as missing passages, showed that the handwritten insertions into the typewritten copy could not have originated with Prange himself. One might therefore assume that the transcript, or at least the insertion of the corrections, first came about after Prange's death (so between 1941 and, say, 1960). The identity of whomever it was that inserted the handwritten formulas into the text is not known.

A copy of another typewritten document with handwritten corrections and insertions exists in the library of Professor Erwin Stein in Hannover, which came to light during the editing of this work. That copy was most graciously placed at the editor's disposal. It involved a copy of the original in the library of the Institute for Mathematics at the Technischen Universität Hannover (Inv.-Nr. 5324) That version is far and away more complete than the version in university library. It is also quite certain that the numerous handwritten insertions and annotations in Stein's version came from the pen of Prange himself. The original copy that exists in the university library is probably a copy of the version in the Institute B for mathematics. The passages that are missing from the university library version were almost exclusively handwritten insertions on the backs of the individual pages, some of which were just barely legible.

The Stein version was employed by the editor to correct the word processor version on the basis of that judgement, and in so doing, almost all of the handwritten insertions were incorporated. The present text then comes relatively close to the version that Prange presented.

The typescript of Prange's work then includes not only an introduction, but two chapters:

Chapter 1: The framework
Chapter 2: The continuous elastic body

In a treatise that was directly connected with the latter, Prange attempted to also explain the theory of beams mathematically. That paper was published in 1919 in the Zeitschrift für Architektur- und Ingenieurwesen under the title "Die Theorie des Balkens in der technischen Elastizitätslehre" [**112**]. That structure of that article was largely parallel to that of Chapters 1 and 2 in the Habilitationsschrift.

0.3 – Prange's later work.

Besides the Habilitation from 1916, which is published for the first time here, and the publication that was closely connected with it that appeared in the Zeitschrift für Architektur- und Ingenieurwesen with the title "Die Theorie des Balkens in der technischen Elastizitätslehre" in 1919, it is not known whether there were other works by Prange on continuum mechanics and structural mechanics. The theoretical foundations from the field of the calculus of variations that are required by the Habilitation were made available in the dissertation that appeared in 1915 as a university publication on the subject of "Die Hamilton-Jacobische Theorie für Doppelintegrale (mit einer Übersicht der Theorie für einfache Integrale)."

In a handwritten Curriculum Vita on the occasion of his application to Halle [**113**], Prange referred to an already-completed "essay" that was concerned with the significance of Hamilton on the development of geometrical optics and mechanics in the Nineteenth Century that was to appear in the Göttinger Abhandlungen. However, its publication did not seem to result, and that work is lost without a trace.

In 1923, CONRAD H. MÜLLER and GEORG PRANGE published a book with the title *Allgemeine Mechanik. Grundlegende Ansätze und elementare methoden der Mechanik des Punktes under der Punktsysteme* [**102]**. (A review of the book by H. REISSNER is in [**119**].) Prange appeared as the translator, editor, and commentator in a 1933 publication on HAMILTON's *Abhandlungen über Strahlenoptik* [**51**]. The translation already existed before 1920 [**114**]. In 1935, a contribution by Prange on the topic of "Die allgemeinen Integrationsmethoden der analytischen Mechanik," whose composition he took up in place of P. Stäckel, who had died, was included in the edition of the *Encyklopädie der mathematischen Wissenschaften* [**114**]. (Review by HAMEL in [**49**]) A final book by Prange that appeared after his death was edited by WERNER VON KOPPENFELS: *Vorlesungen über Integral- und Differentialrechnung (Band 1: Funktionen einer reellen Veränderlichen)* [**116**]. In his planned multi-volume work, he dealt with the publication of his lectures on higher mathematics at the Technischen Hochschule Hannover. Further transcripts of his lectures are in the university library at Hannover. One will likewise find a later unpublished work from 1939 on the subject of "Geodätischen Linien" [**115**] in the Hannover university library.

0.4 – Is the publication of Prange's Habilitationsschrift justified?

In regard to any plans to publish Prange's Habilitationsschrift, which was written between 1914 and 1916 and was available only in the form of copies of two differing typewritten documents, one must ask what grounds there might be for publishing such a thing eighty years later. Why did Prange himself not publish a revised version of the Habilitationsschrift later, especially since he was active in science and publishing up to his death? The list of monographs that were cited above make it clear that Prange was by no means confined to mathematical questions during his academic employment, but he also dealt with mechanical problems. The *Allgemeine Mechanik* that he published with C. H. Müller in 1923 and the survey article in the *Encyklopädie der mathematischen Wissenschaften* in 1935 are evidence of that fact, even if neither of the two works are concerned with the field of continuum mechanics.

The question of what grounds there might have been for Prange's reluctance to publish his Habilitationsschrift cannot be answered unambiguously as long as no further documents emerge from Prange's estate. One can only speculate. It is conceivable that in his later years, Prange himself did not attach the same significance to his Habilitationsschrift that he did to his survey article in 1966. It is also conceivable that he was disenchanted by the lack of any reaction to his paper on the "Theorie des Balkens in der technischen Elastizitätstheorie" that was published in a civil engineering journal. Only HAMEL cited Prange's paper in his *Theoretischen Mechanik* [**50**]. The Habilitationsschrift and its abstract first became known to the international community in the survey paper by Orovas and McLean [**105**, **106**] and some later publications (e.g., [**5**, **13**, **104**]). Finally, it is also conceivable that in his later years, Prange had a revised edition of his

Habilitationsschrift in mind but was prevented from succeeding in that ambition by the second World War, if not his death. The last viewpoint seems to be supported by the numerous insertions and additions in Prange's hand to the typescript in the Institute B for Mathematics and a memorandum in the archives of Springer Verlag.

In a visit to Springer Verlag on August 27, 1942, ARNO SCHLEUSNER had accordingly been encouraged to edit Prange's Habilitationsschrift. Indeed, his title did not coincide with the title in the typescript (in the memorandum, the title was *Die Grundlagen der Elastizitätstheorie*), and the page count is somewhat less than the 40-50 pages that were given, but there can hardly be any doubt that he was dealing with Prange's Habilitationsschrift. Prange's wife gave her approval to the publication, as she was obviously in possession of the manuscript. An interesting aspect of the memorandum was that HAMEL supported the project and that in a lecture series ([3]) at the extension university to the Technische Hochschule Berlin, MARGUERRE, HAMEL, GRAMMEL, and KLOTTER would have referred to the fundamental significance of the paper and its relevance. Finally, it is also interesting that a handwritten memorandum by a colleague at Springer Verlag on August 28, 1942, suggested that a paper permit was hardly to be expected. Obviously, Schleusner also had no success in his attempt to interest the O.K.W. (Oberkommando der Wehrmacht) in his plan. It is an irony of fate that the publication of the Habilitationsschrift failed twice because of shortages that were related to the war.

Whatever reasons SCHLEUSNER might have had for editing Prange's Habilitationsschrift are unknown. One is also reduced to speculation here, as well. It is conceivable that Schleusner had participated in the lecture series and that the inducement to publish came from Hamel, especially since Hamel referred to Prange ([4]) in his book on mechanics [**50**] in connection with the lecture series. In a paper from 1938 [**133**], Schleusner had already pursued the goal of presenting the various forms in which the energy principles were applied, along with their mutual relationships and differences ([5]). Indeed, the canonical transformation did not appear in Schleusner, as opposed to Prange and later Hamel [**50**], but it is conceivable that a suggestion by Hamel might have fallen on fertile ground.

([3]) It is very likely that it was treated in that lecture series, which were published by MARGUERRE in extended form with the title of "Neuere Festigkeitsprobleme des Ingenieurs" in 1950 [**71**]. In the Introduction, Marguerre wrote:

"Neuere Festigkeitsprobleme des Ingenieurs" was the title of a lecture series that were held in the Winter of 1941 before the engineers of the large Berlin companies…

The publication was still scheduled during the war, but the printing house fell victim to a bombing raid in 1944. The book included contributions by FLÜGGE, GRAMMEL, KLOTTER, MARGUERRE, and MESMER. The only thing missing from the book was a lecture by HAMEL on the Ansätze for the theory of elasticity for large deformations, since Hamel had included his thoughts on that in his new book on mechanics [**50**]. Prange was not mentioned in the book.

([4]) In Chapter VII, § **9**, "Die Minimalprinzipe der Elastizitätstheorie," of his book [**50**], Hamel remarked: "From a lecture that was given at the extension university of the Technischen Hochschule." In the concluding section of that book, he said: "The basic ideas that we employ go back to Hilbert in a lecture during the Winter semester of 1905/06 and were developed further by BORN in his Göttinger Preisschrift of 1906. PRANGE had used them in his Hannover Habilitationsschrift, which has unfortunately been published only partially. (GEORG PRANGE: 'Das Extremum der Formänderung')." In saying that, Hamel had obviously started from the assumption that the publication in 1919 [**112**] was part of the Habilitation.

([5]) For the sake of completeness, let it be mentioned at this point that Zweiling asserted in a monograph [**153**] that he was the author of the Schleusner paper, which the former could not publish since he was being politically persecuted by the Third Reich, so Schleusner agreed that he would publish it under his own name.

For the editor, there are two reasons that justify the publication of Prange's Habilitation, even though it is more than eighty years since its writing and more than fifty-five years since the death of Prange. One of them is the *historical significance* of the Habilitationsschrift, which brought the development of energy principles in the theory of elasticity to a convincing conclusion. However, the second one is the *interdisciplinary character* of the work, in which problems of mathematics (in particular, the calculus of variations) are coupled with questions of continuum mechanics (the theory of elasticity) and structural statics (the engineering strength of materials and structural mechanics). In order to elucidate both aspects of the problem, an analysis of the historical context and contents of the Habilitationsschrift would be required.

0.5 – The historical context and contents of Prange's Habilitationsschrift.

The mathematical background: Felix Klein [23, 140] and David Hilbert [118].

Prange came from the Göttingen School of Klein, Hilbert, Minkowski, and Runge, and it was the first two that impressed him the most, whether directly or indirectly.

FELIX KLEIN (1849-1925) studied in Bonn, Göttingen (under Clebsch), and Berlin. Both his conferral of a doctorate in 1868 and Habilitation in 1871 took place in Bonn, and in connection with that, he became a Privatdozent at Göttingen in 1871/72. Already in 1872 (when he was thirty-three years old), Klein was proposed to the University at Erlangen on the initiative of his teacher Clebsch. In his Inaugural Address there, he presented his conceptual picture that would prove decisive for his later activities in regard to organizing the sciences.

> "One cannot lose sight of the unity of all science and the ideal of a total picture in one's specialized studies. Hence, the humanistic and mathematical-natural scientific picture belong together and should not be put into opposition. On the other hand, along with pure mathematics, applied mathematics must also be cultivated in order to preserve their connections with the ancillary domains in science, such as physics and engineering…
>
> Regularly-repeated elementary lectures and lectures on special topics should be held for a small number of interested parties that are both supported by exercises and seminars…" [**140**]

In 1875, he was called to the Technische Hochscule in Munich. While there, Klein, along with A. Brill, reorganized the mathematical curriculum for the engineering sciences along the lines of his conception of things. A series of individual lectures were combined into a four-semester lecture on higher mathematics. His concepts were adopted by numerous other Technische Hochschules and has left its imprint to this day in mathematical education for the engineering sciences.

In his inaugural lecture at Leipzig (1880-86), Klein, in turn, placed *the relationship of new mathematics to its applications* at the center of his considerations. In organizational terms, he generally first presented corresponding arguments after his call to Göttingen (1886). He himself held lectures on not only mathematics, but also a variety of engineering applications (mechanics, potential theory, theory of tops). In terms of publications, he was active as the editor of the journal *Mathematische Annalen* and the *Encyklopädie de mathematische Wissenschaften mit Einschluß ihere Anwendungen*, for which he was especially involved as the editor of the volume on mechanics.

Motivated by his experiences during a sabbatical in the USA (1893), Klein put forth his arguments for the integration of mathematics, physics, and engineering into a *Neuen Göttinger Programm* and in the following years concerned himself with gaining the support of industry for the institutes of applied mathematics, applied mechanics, and applied physics. An essential step towards the achievement of that goal was the founding of *Göttinger Vereinigung zur Förderung der angewandten Physik und Mathematik*. In 1902, an institute for geophysics was founded, and in 1905, institutes for the applied study of electricity, as well as applied mathematics and mechanics, were founded.

In parallel with his ambition to integrate mathematics, natural science, and engineering, Klein was making a concerted effort to attract the best minds in pure mathematics to Göttingen. They already included David Hilbert, since 1895, and later Minkowski (1902), Runge (1904), and Landau (1909).

DAVID HILBERT (1862-1943) had studied in Königsberg and Heidelberg, and in 1885, he was conferred a doctorate in Königsberg. After a brief visit with Klein in Leipzig and Paris, Hilbert was habilitated at Königsberg in 1886, and there he was first a Privatdozent and then an extraordinary professor from 1886 to 1895. In 1895, he was called to Göttingen as a successor to Weber on the initiative of Klein.

In Göttingen, Hilbert first concerned himself with investigations in number theory and the axiomatic foundations of geometry. In 1889, he turned to the *Dirichlet problem*, and then to the calculus of variations. In September of 1889, he presented his proof of the existence of a solution to the Dirichlet problem [**55**] at the annual meeting of the German Mathematical Society and thus contributed to its revival. From 1899 to 1901, Hilbert held lectures on the calculus of variations at Göttingen, which gave rise to a variety of advanced research in its neighboring fields.

- In the year 1906, the prize-winning dissertation of the future Nobel laureate MAX BORN (1882-1970) was published with the title of *Untersuchungen über die Stabilität der elastischen Linie in Ebene und Raum unter verschiedenen Grenzbedingungen* ([6]). In the context of his dissertation, Born addressed Dirichlet's stability principle for the elastic line. The foregoing is closely analogous to the HAMILTONIAN transformation in analytical mechanics ([7]). In a footnote to his dissertation, Born referred to the fact that the relevant part of the Appendix to it was closely based upon a lecture that HILBERT gave on the mechanics of discrete masses in the Winter semester of 1905/06.

([6]) Although published, Born's dissertation, like Prange's Habilitation, has been largely forgotten today. Born was not mentioned in a publication in 1984 [**44**] that treated a very similar problem.

([7]) Consequently, OROVAS also spoke of the BORN-PRANGE canonical variational principle in the introduction to [**13**].

- In the year 1909, a paper by WALTHER RITZ appeared with the title "Über eine neue Methode zur Lösung gewisser Variationsprobleme der mathematischen Physik," [**121**,**122**], in which, building upon Hilbert's investigations into Dirichlet's variational principle, Ritz developed what he called the numerical method for solving boundary-value problems ([8]). Ritz died in July of 1909, since he (and presumably Prange, as well) had been afflicted with tuberculosis, and against the advice of his doctor he declined to go to a sanitorium, but feverishly continued his scientific work.

- In 1914, a former assistant to Hilbert named ERNST HELLINGER (1883-1950), while a Privatdozent in Marburg, published a survey article on "Die allgemeinen Ansätze der Mechanik der Kontinua" in the *Encyklopädie der mathematischen Wissenschaften* that was edited by Klein and C. H. Müller. On one page of that article, in reference to Born, it was shown how, in the general case of three-dimensional continua, by adapting the *canonical transformation of dynamics*, the principle of the minimum potential energy could next be converted into canonical form in which the deformations and stresses would appear as unknown state quantities, and how one would arrive at the principle of Menabrea-Castigliano by introducing the equilibrium conditions as auxiliary conditions.

- GEORG PRANGE's dissertation (published in 1915) and Habilitation of 1916 are in close proximity to those works.

Felix Klein's program of integration, in which he sought to connect mathematics, natural science, and engineering, and the fundamental investigations of David Hilbert on the calculus of variations represents the nutrient medium for Prange's scientific work in his dissertation and habilitation.

The elasticity-theoretical background [68, 139, 137]: Cauchy, Saint-Venant, Clebsch, Voigt, and Love.

The foundations of the linear theory of elasticity had already been laid [**68**, **139**, 1**37**] by the time that Prange undertook his studies (1903-1906). In an 1822 publication, AUGUSTIN LOUIS CAUCHY (1789-1857) had created all of the essential building blocks for the theory of elasticity: He introduced the stress tensor as an extension of the hydrostatic pressure that EULER had employed and presented the equilibrium conditions (equations of motion, resp.) in terms of the stress components. The kinematical connection between the displacements and strains, and therefore, the strain tensor, was likewise presented. Cauchy also gave the material law between the stresses and strains, for which he required two constants in the isotropic case. In that way, it was, above all, debatable as to how many constants were necessary for one to formulate the law of elasticity in the isotropic case and the more general anisotropic one. LOUIS MARIE HENRI

([8]) According to a remark in the *Lexikon der Mathematik* [**42**], that numerical method was one component of Ritz's Habilitation.

NAVIER (1785-1836), based upon atomistic considerations, was a proponent of the theory that one elastic constant would suffice. Cauchy vacillated in his book in 1822, but gave two constants, which generally made the associated derivation unclear at times [**137**]. In the first textbook on the theory of elasticity [**65**], GABRIEL LAMÉ (1795-1850) likewise worked with two constants. The final proof of the "multi-constant" theory was first implied by VOIGT's experimental studies.

WOLDEMAR VOIGT (1850-1919) taught Prange at Göttingen. Born in Leipzig, after the German-French War of 1870/71, he went to Königsberg and studied physics under Franz Neumann. In 1874, he was conferred a doctorate for work that he did on the elastic properties of rock salt. After teaching at Königsberg, in 1883, he was called to Göttingen, where he built up their course offering in theoretical physics. His special interest was in the elastic properties of crystals. In 1910, he published his *Lehrbuch der Krystallphysik* [**144**]. His extensive experimental studies left no doubt that two constants were required for the description of the elastic behavior of materials in the isotropic case and twenty-one in the anisotropic (or aelotropic) case.

For the mathematician, the problem was basically solved as soon as the existence of a solution was proved. However, the practitioner (i.e., the engineer) was also interested in giving a concrete form to the solution in terms of known functions, since no numerical processes for finding approximate solutions were available then, as opposed to the current situation. There were only a few results in regard to solutions for three-dimensional problems [**137**]. The essential contributions to the development of the theory of elasticity then came from the theory of one and two-dimensional continua. In two papers that added up to over 400 pages, in 1855 and 1856, BARRÉ DE SAINT-VENANT (1797-1886) clarified how the foundations of the three-dimensional theory of elasticity could be ascertained for the displacements and stresses in a prismatic beam under stretching, bending, shear bending, and torsion without making Bernoulli's assumption. CLEBSCH presented Saint-Venant's theory seven years later in his monograph on the theory of elasticity.

ALFRED CLEBSCH (1833-1872) was born in Königsberg and while there, like Voigt, he studied under Neumann and was conferred a doctorate in 1854. Already in 1858, he was called to the Polytechnicum in Karlsruhe as a professor of mechanics. During that activity, he wrote his book on the *Theorie der Elasticität fester Körper* [**18**], which was the second monograph on the theory of elasticity after that of Lamé. Barré de Saint-Venant, along with Alfred-Aimé Flamant, translated Clebsch's book into French [**126**], and added a wealth of remarks and commentaries. Whereas the work of Clebsch, which was reported on by Szabó [**137**], for example, is hardly accessible anymore, Saint-Venant's adaptation of it [**126**] recently appeared in 1966 as a reprint. The trail of Clebsch also led to Prange: In 1863, Clebsch was called to Giessen as a professor of pure mathematics, and in 1868 he went to Göttingen. In 1872, he was chosen to be Rector at Göttingen. He died of diphtheria in the same year [**139**]. Already in 1868, after the death of his teacher in Bonn, Klein would come into contact with Clebsch, who was supposed to edit a geometric work from Plücker's estate, and part of that task fell upon Klein, who developed it into the theme of his dissertation. In 1868, Klein was one of Clebsch's students at Göttingen, who proposed Klein for his call to Erlangen. After the death of Clebsch, Klein undertook the task of instructing Clebsch's "orphaned" students. One of several attempts to bring Klein to Göttingen as the immediate successor to Clebsch failed, which was "fortunate," as Courant assessed it in a memorial address in 1925 [**23**].

It is somewhat surprising from Prange's close connection to Clebsch and Klein, as well as Göttingen, that no reference to Clebsch's *Elastizitätstheorie* showed up in Prange's work, since Prange based it almost exclusively upon the monograph of LOVE [**67**, **68**], which appeared somewhat later and was generally more comprehensive.

A. E. H. LOVE (1863-1940) was one of the exceptional proponents of the theory of elasticity and geophysics in England, who first studied at Cambridge and also did scientific work there from 1887 to 1899. From 1899 on, he was a high school teacher at Oxford. The first edition of his main work *Treatise on the Mathematical Theory of Elasticity* appeared in two volumes in 1892/93. In the second edition (1906), whose German translation by Timpe has appeared already, addressed questions of engineering science more rigorously. That version was also employed by Prange ([9]). To this day, Love's *Theory of Elasticity* is still the undisputed standard text on the theory of elasticity. Love's historical introduction is also interesting, which gives a delightful glimpse into the story of the theory of elasticity up to around 1900.

As would emerge from his Habilitation, Prange was eminently familiar with the foundations of the theory of elasticity that had been created during the Nineteenth Century. He based them predominantly on the German translation of Love's textbook [67]. No influence of Clebsch is recognizable. Even the influence of Voigt, under whom Prange had studied, is not immediately noticeable.

The structural-analytic background [111, 139, 136, 106]: Castigliano, Müller-Breslau, and Mohr.

We shall follow Straub [**136**] in his assessment that "in the first half of the Nineteenth Century…*structural analysis* had split off from theoretical mechanics as a special topic." Straub referred to NAVIER as the "creator" of structural analysis, since he was the first to give the differential equations of the bending of beams their form that is still customary [**137**]. Navier's book [**103**] is a compilation and application of many laws and methods from the realm of continuum mechanics that were known before him, and in particular, the application of the theory of elasticity to the practical problems of construction.

If one would like to interpret the difference between structural analysis and the mathematical theory of elasticity then one might fall back on a statement by Love, even if it was concerned with the difference between the mathematical theory of elasticity and engineering mechanics [**68**]:

> "The history of the mathematical theory of elasticity shows clearly that the development of the theory has not been guided exclusively by considerations of its utility in technical mechanics. Most of the men by whose researches it has been founded and shaped have been more interested in natural philosophy than in material progress, in trying to understand the world than in trying to make it more comfortable. From this attitude of mind, it may possibly have resulted that the

([9]) In his treatment of the theory of beams [**112**], Prange did not start from Saint-Venant's continuum-mechanically consistent solution, as it was presented by Love and Clebsch, but worked under the hypothesis that the cross-section would remain planar, as with almost all of Navier's structural analysis; confer page 2.74.

> theory has contributed less to the material advance of mankind than it might otherwise have done. Be that as it may, the intellectual gains which have accrued from the work of those men must be estimated very highly…"

One driving force for the development of structural analysis in Europe in the Nineteenth Century was the industrial revolution, and in particular, the construction of railroad bridges [**62**]. CULMANN's 1864 book *Graphische Statik* [**25**] ([10]) summarized the procedures for calculating with frameworks, and WINKLER's book *Lehre von der Elastizität und Festigkeit* summarized the analytical methods for calculating with beams and arch structures for civil engineers that were employed in construction.

Graphical statics was an impressive instrument with which one could treat a multitude of technically-important constructions relating to bridges and steel structures. It must be extended by analytical and numerical procedures as soon as one has to deal with frameworks or beams (arch structures, resp.) that are statically indeterminate. Castigliano was the first to do that convincingly, as one sees that in the abundance of his predecessors ([11]).

CARLO PIO ALBERTO CASTIGLIANO (1847-1884) was born in Asti, Italy. After three years of working as a teacher, in 1870 he was next a student at the university and then at the polytechnic school [Reale Scuola d'Applicatione degli Ingegnen] in Turin [**64**]. His thesis included the theorems that are still connected with his name and some applications to the theory of structures. It was published in extended form in Italy [**14, 15**] along with a later 1875 paper. The international professional community was first made aware of his work in the French version of his book and its translations into German and English [**11, 12, 13**]. At the point in time when the French edition appeared, Castigliano was employed by the Upper Italian railroad as an engineer. For the hall roof constructions that were cited as examples in the book, one might recall the typical train stations of the second half of the Nineteenth Century. The theoretical part of the book, which takes up more than half of it, takes up the arguments of Saint-Venant and Clebsch, even though those names did not show up in Castigliano's book. Starting from the three-dimensional theory of elasticity, Castigliano also gave the stress distribution in beam sections for various loads at their ends, and from there he arrived at expressions for the dependency of the "deformation work" (density of deformation energy) in beams on the forces on the beam sections, which included the effects of torsional moments and lateral forces.

Castigliano's work was propagated in Germany most zealously by HEINRICH MÜLLER-BRESLAU (1851-1925), who was a follower of Emil Winkler who was called to the Chair of Building Construction and Iron Bridges at the Technische Hochschule in Berlin [**52**]. In 1886, the first edition of his book appeared with the title *Die neueren Methoden der Festigkeitslehre und der*

([10]) An interesting side note to this is that Culmann held that in order to properly apply graphical statics, one required a basic knowledge of projective geometry, and despite the strong objections of his colleagues in the German Technischen Hochschulen, he then moved away from that viewpoint in the second edition [**123**]. Prange must have been aware of the fundamental conflict between a method that were purely oriented towards applications, whose proponents were, above all, Mohr and Müller-Breslau, and a method that would capture the mathematical foundations since he studied projective geometry with Klein and graphical statics with Runge in parallel in the Winter semester of 1905/06.

([11]) For this, one might confer the historical remarks in Prange's Habilitation and the survey article by Oravas and McLean [**106**].

der Statik der Bauconstruktionen ausgehend von dem Gesetze der virtuellen Verrückungen und den Lehrsätzen über die Formänderungsarbeit [**97**]. The concepts that were developed for calculating with statically-indeterminate constructions are still valid to this day.

Even before Müller-Breslau, OTTO MOHR (1835-1918) had already addressed the problem of calculating with statically-indeterminate frameworks in Germany. From 1868 to 1873, Otto Mohr was a professor in Stuttgart, and was then called to teach the subjects of graphical statics, railroad engineering, and hydraulic engineering at the Technische Hochschule in Dresden. In 1874 and 1875, he published some contributions to the theory of frameworks [**80, 81, 83**]. When a first paper of Müller-Breslau appeared in 1883, in which Castigliano's method was referred to as equivalent to that of Mohr [**94**], Mohr reacted indignantly [**85**]. Mohr's riposte was the starting point for a controversy that was partially expressed in a polemical tone of voice. From the initial publication of Müller-Breslau's main work [**97**], in which he answered with a critical overview of Mohr's critique, he revived the "slugfest" [**87, 98**]. The objective background for that controversy is hard to understand today. Greatly simplified, it took the form: Mohr employed a formulation based upon principles that corresponded to the principle of virtual forces. Müller-Breslau placed Castigliano's theorems, in which the concept of "deformation work" played a decisive role, on a par with the latter. The better part of the controversy revolved around the concept of deformation work, whose "extensibility" Mohr reprimanded against, and the associated minimal requirement.

Above all, what was most contentious was whether both principles were equivalent, or could they be replaced in various problem statements (temperature, support lowering, mounting stresses, large deformations, nonlinear-elastic material behavior, etc.), and what sort of conception of work that one would employ in such situations. It was also unclear what sort of connection existed between the statements of virtual work and the energy principles that were known to mechanics. Finally, it was unclear whether an extension of the methods of structural analysis to more general two or three-dimensional continua would be possible.

The controversy was revived once more in connection with an article by WEINGARTEN in the year 1901 [**148**], but Müller-Breslau and Mohr did not generally participate in it, since it mostly involved WEINGARTEN, WEYRAUCH, HERTWIG, MERTENS, and FÖPPL. A presentation of the details of the entire controversy would go beyond the scope of this introduction. One is referred to Prange's historical remarks, the discussions in *Jahrbuch für die Fortschritte der Mathematik* in the year 1889 [**60**], the survey articles of Grüning [**45**] and Dohmke [**26**], the treatise of Orovas and McLean [**106**], and two articles by Kurrer [**63, 64**].

As was just mentioned, the controversy was discussed in the 1889 *Jahrbuch für die Fortschritte der Mathematik* [**60**]. The reviewer [F. K., that is, Friedrich KÖTTER (Berlin)] first addressed the book by Müller-Breslau [**97**] and connected it with two papers by MOHR and MÜLLER-BRESLAU in Civilingenieur [**87, 97**]. In the second phase of the controversy, the last two contributions came from the mathematician WEINGARTEN and the engineer WEYRAUCH in the year 1909 and appeared in the *Nachrichten der königlichen Gesellschaft der Wissenschaften zu Göttingen*. It is then most likely that the controversy was also known to Klein and Hilbert, who lectured their students on problems from the domains of physics and engineering in their seminars. Born's dissertation [**10**] in is the background to that controversy, as well as the brief discussion in the survey paper by Hellinger [**53**], and finally Prange's dissertation and Habilitation.

Prange had touched upon the various principles and laws of structural analysis in the context of general laws of variation in the theoretical study of elasticity in his Habilitationsschrift and in the 1919 article that appeared with the title "Die Theorie der Balkens in der technischen Elastizitätslehre" [**112**]. In the Habilitationschrift, frameworks and beams were treated in parallel, while beams were treated more extensively in the 1919 paper. The definitive idea in that is the application of the canonical transformation of Hamilton-Jacobi theory in the calculus of variations that was known to analytical mechanics to the principle of the extremum of total energy. Displacements (Prange called them *Verrückungen*) appear in the *principle of the extremum of potential energy* as unknown state quantities. In the corresponding *canonical variational problem* that arises after the canonical transformation, displacement quantities and force quantities are the unknown varied state quantities. That new variational principle (which is currently connected with the names of Born, Hellinger, Prange, and Reissner) includes the principle of the extremum of potential energy, as well as another extremal principle that is referred to as the *Menabrea-Castigliano principle*.

All of that is illustrated in the graphical overview in Fig. 1. For Prange, the starting point was the principle of virtual displacements. He then showed the equivalence of that principle with equilibrium conditions and the static boundary conditions, and then went on to the principle of the extremum of potential energy. The principle of virtual displacements is the varied form of that extremal principle. A new expression $\mathcal{E}$ for the potential ([12]) is obtained from the potential energy E with the canonical transformation. If one demands that the displacement quantities and the strains that belong to the force quantities must be compatible then one will once more get the the force quantities should fulfill the equilibrium conditions then that will imply the Menabrea-Castigliano principle, which is itself again equivalent to the compatibility conditions. It is surprising that the principle that is currently known as the principle of virtual force does not emerge clearly from the dual construction of the varied form of the Menabrea-Castigliano principle in the diagram.

Prange defined various expressions for the deformation work and showed that those expressions would be implied inevitably when one started with the canonical transformation with corresponding assumptions.

Up to the time that Prange composed his Habilitationsschrift, procedures in which force quantities appeared as unknowns (*Kraftgrößenverfahren*) were employed almost exclusively in practice. Works in which the displacement components appeared in the solution to the problem were not available to Prange. A corresponding procedure for calculating the auxiliary stresses in frameworks was already being used in 1880 by MANDERLA [**69**] and in 1892 by MOHR [**88**]. Today, that group of procedures is referred to as *deformation methods* (or *displacement quantity methods*). A comprehensive presentation of the deformation methods was first found in 1914 by BENDIXEN [**6**] and in 1926 in the book by OSTENFELD [**107**]. It speaks for Prange's foresight that the fact that the variational-theoretic foundations of the displacement-quantity procedures and the force-quantity procedures are completely on a par with each other emerges clearly from his theoretical presentation, along with the dual construction of both processes. One first finds a

([12]) Prange employed that new symbol for only frameworks and the three-dimensional continuum, but not for beams.

similarly-clear presentation of that duality, although with a view to matrix statics, in ARGYRIS [**2, 3**].

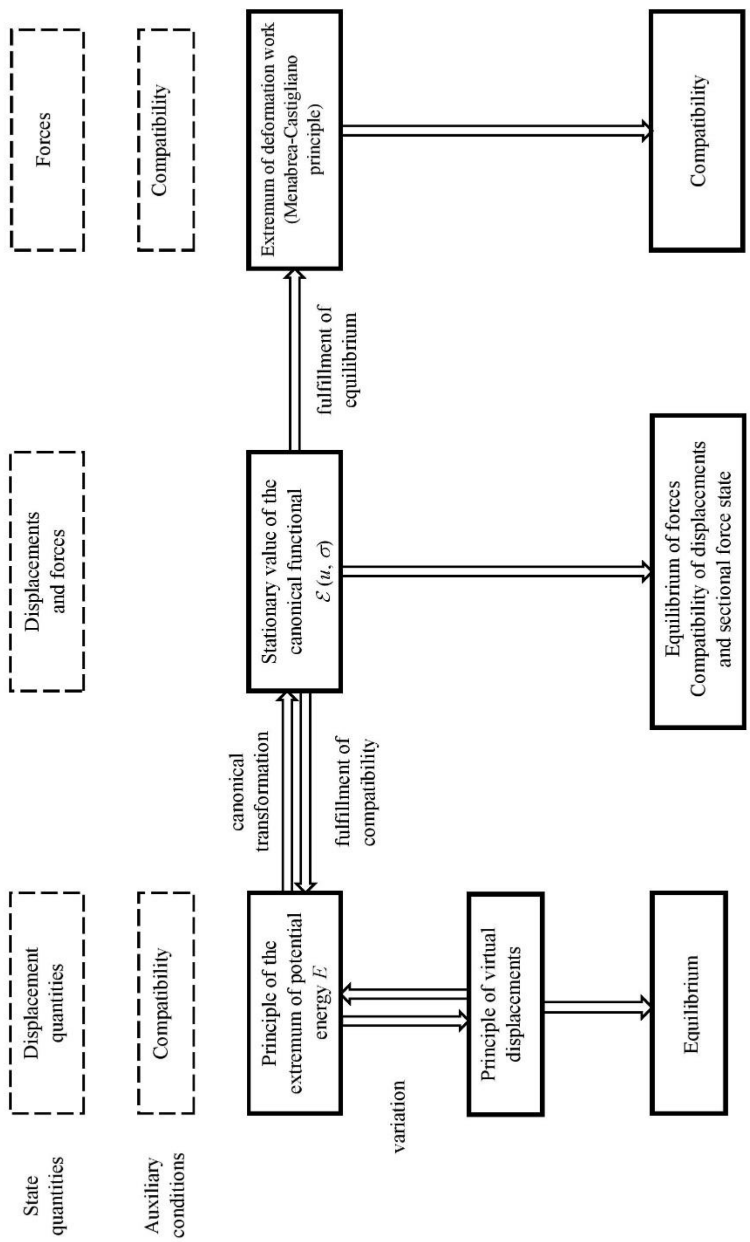

Figure 1. Connections between the principles that Prange considered.

In his Habilitationsschrift and in the related publication on the theory of beams, Prange had managed to create a clear, unified framework in which all of the principles, laws, and procedures that were employed in structural analysis for frameworks and beam structures could be organized by starting from the Hamilton-Jacobi theory. An adaptation of the conceptual framework to three-dimensional continua was possible in complete analogy using the tools of the calculus of variations.

Prange had solved all of the open problems regarding the foundations of classical structural analysis in his Habilitationsschrift. The fact that his Habilitationsschrift was not known to the scientists of the era that were working in the field of structural analysis should not be surprising, since it was never published. However, not even his related work in 1919 on the theory of beams [**112**], which appeared in the *Zeitschrift für Architektur und Ingenieurwesen*, like many of the papers of Müller-Breslau and Mohr, found any actual resonance. One can only speculate about why that would be true. Was the 1919 article too "mathematical" for civil engineers? Did the civil engineers consider the problems that had been raised in the controversy between Müller-Breslau and Mohr to have been solved? At the very least, the latter seems rather unlikely. In the year 1938, an article by SCHLEUSNER appeared on the topic of "Das Prinzip der virtuellen Verrückungen und die Variationsprinzipien der Elastizitätstheorie" [**133**], in which the *principle of virtual displacements* appeared along with the *principle of virtual force*. For Schleusner, as for Prange, it was indisputable that both principles could be regarded as two forms of variational principles. Since Schleusner obviously did not know of Prange's work at that point in time, he did not employ the canonical transformation in order to take each principle to the other, so that transition was less clear. Schleusner, as an engineer, and parallel to him, MARGUERRE, as a mechanician [**70, 72**], were the two people of that era who recognized the necessity of a precise formulation of the *energy principle* most clearly. At the very least, for Schleusner, who was also a practicing structural engineer, the principle of virtual forces was still the most important one from the standpoint of applications.

Even to this day, for many students of construction and mechanical engineering (and not just for students), everything else regarding the intrinsic connection between Castigliano's laws and the principle of virtual work is clear. However, at the beginning of this century, despite persistent ambiguities, in structural analysis, one was in a position to perform calculations for statically-indeterminate structures in which hardly anyone was interested in whether one employed the "Mohr method" or the "Castigliano method." The issue of achieving clarity in the fundamental questions was not especially compelling to the engineers as long as no practical necessity for it existed. The statement that KÖTTER made in his discussion of Müller-Breslau's main work [**60**] still rings true that "*the intended audience for the book is generally not so much inclined towards a theoretical discussion of it as towards the practical utility of the methods that are developed in it.*"

Today we know that there is essentially more to be found in Prange's work. Hellinger's and Prange's adaptation of the "canonical transformation" from the realm of many-body mechanics to the realm of continuum mechanics opened the door to the development of further variational principles, but the time was not ripe for that in 1916. More than thirty years later, E. REISSNER [**120**] rediscovered the Hellinger-Prange variational principle and also used it in practice. In the meantime, since the first electronic computing systems became available, the possibilities that were given by those "mixed variational principles" could be exhausted. A wealth of advanced developments came about in the years that followed in the context of finite-element formulations, among which, the "hybrid" formulation that was given by PIAN [**108, 4**], in particular, has proved to be exceptionally fruitful.

0.6. Appendix to the Introduction.

(Knothe's editing of Prange's Habilitationsschrift included a number of documents of a largely administrative nature, along with various correspondences, that were not available to the translator. If one wishes to confer them, they are included in Knothe's German version of this book.)

0.7. Comments on the editing of the text.

The main source for the following reproduction of Prange's Habilitationsschrift was the copy of the typescript in the library of E. Stein (Institut für Baumechanik und numerische Mechanik at the University of Hannover). The numerous handwritten corrections and additions are mainly due to Prange, as a comparison to the handwritten Curriculum Vita [**113**] or his letters [**117**] will show, but some of them were barely legible. The passages that were unambiguously associated with Prange were included in the following text, but the illegible passages have been pointed out.

The following guidelines were adhered to in the adaptation of the handwritten version to a printed version:

- Prange's spelling is the spelling that is appropriate for the current era, as was also the case in the publication of 1919 [**112**]. Since no unified rules for punctuation were known at the time, the currently-accepted rules were followed. The formulas were included in the punctuation throughout. That also corresponds to the publication of 1919.

- Text that was underlined in the typescript was set in *cursive (italic)* in the edited version. Names that were underlined in the text were reproduced in the edited version in SMALL CAPITALS. Footnotes in the typescript were written in smaller type in the edited version, in which the numbering was correlated with the chapters, and not the pages, as in the typescript. The names of authors in the footnotes were only partially underlined in the typescript, while in the edited version, they were written in SMALLER CAPITALS throughout. Comments and additions by the editor were written as *cursive footnotes.*

- For the structure of the chapters, Prange employed the *paragraph* format, e.g., Chapter I, § **1**. For the word processor, the *section* format was employed instead, so e.g., 1.1. Correspondingly, the chapter number was included in the numbering of the equations, so e.g., (2.30), instead of (30), which is what Prange would have written.

TABLE OF CONTENTS

INTRODUCTION

In the engineering study of elasticity – in particular, in the statics of buildings – an important role is played by the "minimum deformation work," which is an expression that has given rise to much confusion due its vagueness, because that "minimum" can be regarded in different ways, such that misunderstandings would be inevitable. When the variables upon which the expression that is to be made a minimum depends change, that will seldom give the neighboring state, compared to which a minimum is to occur. The type of variation that is carried out in each individual case will not be established more precisely, such that the same expression can exhibit different interpretations.

That lack of exactness was perpetuated by the fact that the engineering theory of elasticity was influenced essentially by the study of frameworks, which is where it was first developed.

In the theory of frameworks, the various conceptions that can appear with the "minimum deformation work" have such a close kinship that one can easily combine them into a general concept that cannot be rigorously classified. With frameworks, one deals with the extremum of an ordinary function, in particular, a quadratic form in finitely-many variables, in which only the coordinates will change. If one adapts that argument to continuous bodies then the extremum of a triple integral will enter in place of that function that must be treated by the calculus of variations. In that way, the dependency of the values of the integral will lead to different problems that one can pose, on the one hand, from selecting functions in the interior, and on the other, from selecting the "boundary values." In engineering, one never has the opportunity to stress that clearly because the basic problems in elasticity for continuous bodies cannot be solved by the present means of analysis in a way that admits numerical calculation, which is all that would be of interest in engineering. Therefore, in the engineering theory of continuous bodies, in particular, for the most important special case of beams, one confines oneself to the study of loads that take the form of isolated forces and gets around the complications in the integration by representing the stress components as linear functions of the isolated forces by making special assumptions about those components. The deformation work will then become a quadratic function of the isolated forces, such that the difference between continuous bodies and frameworks will be blurred somewhat by that.

In what follows, we shall attempt to give a presentation of that entire sphere of thought that comes under consideration from the standpoint of the mathematical theory of elasticity. In particular, we shall make a clear distinction between the various intersecting lines of reasoning, and in that way simultaneously point out the intrinsic connection between them. With the use of the methods that were developed to a great extent for the purpose of analytical mechanics, that will arrive at the so-called Hamilton-Jacobi theory of the calculus of variations.

The starting point for the discussion is the Ansatz of the principle of virtual displacements. By the generality of its character, it will govern any problem in statics. The associated variational formula will be interpreted as the extremal condition for the total potential energy, i.e., the sum of the external forces and the deformation work. We apply the Legendre transformation to that extremal problem, and in that way give rise to a general canonical variational problem whose extremal conditions will give not only the equilibrium conditions, but also the compatibility conditions. Whereas the principle of virtual displacements makes that a special case of the general principle when we regard the compatibility conditions as being fulfilled, we will conversely obtain a new principle when we assume, conversely, that the equilibrium conditions are fulfilled. That new principle represents a conception of things that is connected with the so-called Menabrea principle in the engineering theory of elasticity. It demands that the deformation work should be an extremum with the equilibrium conditions as the auxiliary conditions and yields the compatibility conditions as the extremal conditions. The associated extremal value of the deformation work will be considered in the context of its dependency upon the boundary values of the functions that appear in the integral of the variational problem, and the two theorems that are named after Castigliano on the derivatives of those extremal values will be stated. Considering the dependency of the extremum on the boundary leads to the problem of determining the boundary values, which are now regarded as variable, such that one will again obtain the precise extremum (*extremum extremorum*) of all the extreme values. With that, one arrives at a new extremal problem that represents a second way of looking at Menabrea's principle from the engineering theory of elasticity. For that reason, both approaches will be given the same name since they are not essentially different for frameworks. That is shown in more detail in the first chapter of the following presentation. Namely, precisely the same theory that was depicted for continuous bodies above will be given for frameworks in it. One will also arrive there along the path that was just described when one suitably adapts the idea of the Legendre transformation to the realm of the differential calculus.

The arguments that will be given will lead to an extension of the Menabrea-Castigliano sphere of thought for frameworks, as well as continuous bodies, which is introduced by German engineers, in particular, such as considering internal stresses and temperature changes, introducing a general relationship between stress components and the quantities of deformation in place of Hooke's law, and the like. Since all of the new expressions that enter in place of the deformation work in that way will be implied necessarily by the guiding principles of canonical transformations, the apparent arbitrariness in that new picture will vanish, and will become the intrinsic basis for the "extendibility" of the concept of deformation work ([13]).

Each chapter concludes with a section on the historical context of the development of the questions that were treated in it in order to underscore the relationship between the various Ansätze and developments that one finds in the literature and the presentation of the theory that is given here.

([13]) Naturally, it would not be in the spirit of this treatise to pose the questions in the way that would be most convenient to the applications in engineering practice.

CHAPTER 1

THE FRAMEWORK

1.1. The frame member and its deformation. – We understand a *rod* to mean a body whose one dimension (viz., length) exceeds the other two (viz., cross-section) considerably. In particular, a *frame member* ([14]) is a homogeneous rod of constant cross-section that can be compressed or stretched, but not bent or twisted. Such a frame member can admit or transmit only those forces that act in the direction of its axis and are distributed uniformly over its cross-section, such that all longitudinal fibers will be stretched (compressed, resp.) in the same way. Since the individual cross-sections of the rod can displace only as a whole, and thereby remain parallel to themselves, we can ignore the expansion of the cross-section completely and regard a frame member as a *one-dimensional* structure, and indeed as a line that can admit forces that have the rod line, i.e., the centerline of the rod – as their line of action, and whose individual points can be displaced by a deformation only along the rod line.

In order to establish the individual points of the rod, we would like to introduce the coordinate x along the rod line, while the two endpoints might possess the coordinates x_1 and x_2 .

We now imagine that a force P_1 is applied to one endpoint and that the elastic rod has assumed a certain equilibrium position under its influence, in which a stress $\sigma(x)$ might be produced at each of its points. During the transition to that new equilibrium position, the individual points shall be displaced by the segment Δx, which is regarded as a function of x. In particular, the displacements of the two endpoints shall be equal to Δx_1 and Δx_2, resp.

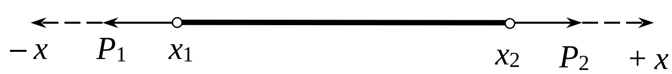

Figure 1.

In order to investigate the equilibrium state further, we superimpose the actual displacement Δx of a point of the rod with a virtual displacement $\delta\Delta x$, which is also a function of x. In particular, let its values at the two endpoints be $\delta\Delta x_1$ and $\delta\Delta x_2$, resp. From the principle of virtual displacements, if equilibrium is to prevail then the work done by the internal and external forces must be equal to each other.

Now, the work done by external forces under that virtual displacement is:

$$P_1 \cdot \delta\Delta x_1 + P_2 \cdot \delta\Delta x_2 .$$

([14]) Here, one imagines the so-called theory of elementary frameworks, in which one deals with rods that are connected by frictionless joints. In reality, frame members will also be subject to bending, since they are riveted, and therefore so-called secondary stresses can appear.

The virtual work done by internal forces arises in such a way that any two points that bound a frame member dx will be displaced by the segment $d\,(\delta\,\Delta x)$ under the virtual displacement. It is therefore the virtual work done by the stress in the rod $\sigma(x)$ for each element dx in the rod.

$$\sigma(x)\cdot d\,(\delta\,\Delta x)\ ,$$

so for the entire rod, it will be:

$$\int_{x_1}^{x_2}\sigma(x)\cdot\frac{d(\delta\,\Delta x)}{dx}\cdot dx\ .$$

In engineering, the virtual work done by the internal forces is refers to as the *deformation work* done by the virtual displacement.

The *equilibrium condition* will then be expressed by the following equation:

$$P_1\ \delta\,\Delta x_1 + P_2\ \delta\,\Delta x_2 - \int_{x_1}^{x_2}\sigma(x)\frac{d(\delta\,\Delta x)}{dx}\,dx\ = 0\ , \tag{1.1}$$

which must exist for every system of displacements $\delta\,\Delta x$.

If we now choose, in particular, the virtual displacements to be equally big for all points along the rod line, $\delta\,\Delta x = c$, then $d\,(\delta\,\Delta x)\,/\,dx = 0$, and we will get:

$$P_1 + P_2 = 0\ ,$$

i.e.:

$$P_2 = -\,P_1 = P\ . \tag{1.2}$$

The two forces must then be in equilibrium.

On the other hand, if we perform the virtual displacement in such a way that we fix the two endpoints of the rod, $\delta\,\Delta x_1 = \delta\,\Delta x_2 = 0$, then it will follow from (1.1) by partial integration that:

$$-\left[\sigma(x)\,d\,(\delta\,\Delta x)\right]_{x_1}^{x_2} + \int_{x_1}^{x_2}\frac{d\sigma(x)}{dx}\,\delta\,\Delta x\,dx\ = \int_{x_1}^{x_2}\frac{d\sigma(x)}{dx}\,\delta\,\Delta x\,dx\ = 0\ ,$$

from which, we further conclude that ([15]):

$$\frac{d\sigma(x)}{dx} = 0\ , \qquad\qquad \sigma(x) = \text{const.} \tag{1.3}$$

The stress will then have a constant value along the rod line. We call that constant stress the *tension in the rod* and denote it by S.

([15]) From the so-called "fundamental lemma of the calculus of variations," O. Bolza, *Vorlesungen über Variationsrechnung*, Leipzig and Berlin (1909), pp. 25.

In order to determine the *magnitude of the tension*, we observe that as a result of (1.2) and (1.3), equation (1.1) will go to:

$$(P - S)\,(\delta\,\Delta x_1 - \delta\,\Delta x_2) = 0\,, \tag{1.4}$$

in which the second factor $\delta\,\Delta x_1 - \delta\,\Delta x_2$ represents the change in length of the entire rod and can be chosen to be non-zero then. Equation (1.4) will then determine the tension to be:

$$S = P\,,$$

i.e., its magnitude is *equal to the magnitude of each of the forces that are applied to the rod*, and it will have the same sign as P_2, so it will have a positive sign when the rod is stretched and a negative sign when the rod is compressed, which should be clear from Fig. 1.

1.2. The framework, and in particular, its equilibrium. – One constructs a *framework* out of frame members in such a way that one puts a certain number of them between n given points in space and connects the ones that come together at each individual point – viz., the so-called *nodes* of the framework – so that they articulate there ([16]). If the framework is to not be in motion then, as one easily convinces oneself ([17]), the n nodes must be connected by at least $(3n - 6)$ rods, while the highest number of possible connecting rods obviously amounts to $n\,(n - 1) / 2$. We assume in what follows that the framework includes r rods, such that:

$$3n - 6 \le r \le \tfrac{1}{2}\,n\,(n-1)\,. \tag{1.5}$$

We imagine that such a framework is loaded with a force $P_1, P_2, \ldots, P_n$ at each of the n nodes, under whose effect a certain equilibrium configuration might occur, and each rod will be subject to a certain tension. We think of those rods as having been enumerated from 1 to r in some way, such that their lengths will be $l_1, l_2, \ldots, l_r$ and $S_1, S_2, \ldots, S_r$. Let the n nodes be given by their coordinates relative to a three-axis, rectangular coordinate system. We would also like to enumerate them, and indeed in such a way that we denote the coordinates of the first point by x_1, x_2, x_3, and correspondingly, those of the k^{th} point by $x_{3k-2}, x_{3k-1}, x_{3k}$, and those of the n^{th} point by $x_{3n-2}, x_{3n-1}, x_{3n}$. We denote the components of the forces $P_1, \ldots, P_n$ that act at the nodes with respect to the coordinate axis analogously, so those of P_1 will be X_1, X_2, X_3, those of P_k will be X_{3k-2}, X_{3k-1}, X_{3k}, and finally those of P_n will be $X_{3n-2}, X_{3n-1}, X_{3n}$. Let the displacements of the nodes that occur as a result of that loading be:

$$\Delta x_1,\ \Delta x_2,\ \Delta x_3,\ \ldots,\ \Delta x_{3n-2},\ \Delta x_{3n-1},\ \Delta x_{3n}\,.$$

([16]) E.g., by a ball joint.

([17]) Cf., e.g., A. FÖPPL, *Vorlesungen über technische Mechanik*, Bd. II, 2nd ed., pp. 268. For the planar framework, the number of necessary rods will amount to $2n - 3$, *ibid.*, pp. 194.

Any rod will connect two nodes; say, the rod h connects nodes μ and ν. Thus, for every rod, there will exist a relation of the following type:

$$f_k = \sqrt{(x_{3\mu-2} - x_{3\nu-2})^2 + (y_{3\mu-2} - y_{3\nu-2})^2 + (z_{3\mu-2} - z_{3\nu-2})^2} - l_h = 0 \qquad (h = 1, 2, \ldots, r). \tag{1.6}$$

The partial derivatives of the function f_k with respect to the coordinates of the node μ:

$$\frac{\partial f_h}{\partial x_{3\mu-2}} = \frac{x_{3\mu-2} - x_{3\nu-2}}{l_h}, \qquad \frac{\partial f_h}{\partial x_{3\mu-1}} = \frac{x_{3\mu-1} - x_{3\nu-1}}{l_h}, \qquad \frac{\partial f_h}{\partial x_{3\mu}} = \frac{x_{3\mu} - x_{3\nu}}{l_h} \tag{1.7}$$

give the direction cosines of the direction of the rod that points from node μ to node ν, and likewise the derivatives of the coordinates of point ν:

$$\frac{\partial f_h}{\partial x_{3\nu-2}} = -\frac{x_{3\mu-2} - x_{3\nu-2}}{l_h}, \qquad \frac{\partial f_h}{\partial x_{3\nu-1}} = -\frac{x_{3\mu-1} - x_{3\nu-1}}{l_h}, \qquad \frac{\partial f_h}{\partial x_{3\nu}} = -\frac{x_{3\mu} - x_{3\nu}}{l_h} \tag{1.7a}$$

give the direction cosines of the opposite direction (which points from node ν to node μ) for the same rod, while the derivatives of f_h with respect to the coordinates of all of the other nodes will vanish since they do not enter into (1.6) at all ([18]):

$$\frac{\partial f_h}{\partial x_\lambda} = 0 \qquad (\lambda \neq 3\mu - 2,\ 3\mu - 1,\ 3\mu,\ 3\nu - 2,\ 3\nu - 1,\ 3\nu)\,. \tag{1.7b}$$

In order to examine the equilibrium state of the framework, we once more appeal to the principle of virtual displacements. We assign a virtual displacement $\delta\Delta x_1, \ldots, \delta\Delta x_{3n}$ to each of the n nodes in the equilibrium configuration and then calculate the virtual work that is done by the external forces and tensions as a result of it. From the principle of virtual displacements, both of them must be equal to each other.

In order to calculate the virtual work done by the r tensions, we observe that the three component sums of the tensions in all rods that radiate from node μ are equal to:

$$\sum_{h=1}^{r} S_h \frac{\partial f_h}{\partial x_{3\mu-2}}, \qquad \sum_{h=1}^{r} S_h \frac{\partial f_h}{\partial x_{3\mu-1}}, \qquad \sum_{h=1}^{r} S_h \frac{\partial f_h}{\partial x_{3\mu}},$$

in which the sums can be extended over all r rods since from (1.7b), the tensions in the rods that do not radiate from node μ will be multiplied by precisely zero. The virtual work that is done by all of the tensions that act node μ is then:

([18]) This type of notation is used (with inessential modifications) by A. FÖPPL, *loc. cit.*, pp. 239.

$$\sum_{h=1}^{r} S_h \left(\frac{\partial f_h}{\partial x_{3\mu-2}} \delta\,\Delta x_{3\mu-2} + \frac{\partial f_h}{\partial x_{3\mu-1}} \delta\,\Delta x_{3\mu-1} + \frac{\partial f_h}{\partial x_{3\mu}} \delta\,\Delta x_{3\mu} \right) ,$$

and the total work done by tension will be found from that by summing over all nodes:

$$\sum_{\lambda=1}^{3n} \left(\sum_{h=1}^{r} S_h \frac{\partial f_h}{\partial x_\lambda} \delta\,\Delta x_\lambda \right) . \tag{1.8}$$

Since the virtual work done by external forces is further equal to:

$$\sum_{\lambda=1}^{3n} X_\lambda\, \delta\,\Delta x_\lambda \ , \tag{1.8a}$$

the principle of virtual displacements will then imply the equation:

$$\sum_{\lambda=1}^{3n} \left(X_\lambda - \sum_{h=1}^{r} S_h \frac{\partial f_h}{\partial x_\lambda} \right) \delta\,\Delta x_\lambda = 0 \tag{1.9}$$

as the *equilibrium condition*.

That work will be referred to as the *deformation work done on the framework* by the virtual displacement.

That equation can be given a somewhat-different form that often proves to be preferable. Namely, varying the relation (1.6) and considering (1.7b) will give:

$$\sum_{\lambda=1}^{3n} \frac{\partial f_h}{\partial x_\lambda} \delta x_\lambda = \delta\, l_h ,$$

and when we choose the variation δx_λ to be the virtual displacement $\delta\,\Delta x_\lambda$ and correspondingly replace $\delta\, l_\lambda$ with $\delta\,\Delta l_\lambda$, that will further give:

$$\sum_{\lambda=1}^{3n} \frac{\partial f_h}{\partial x_\lambda} \delta\,\Delta x_\lambda = \delta\,\Delta l_\lambda , \tag{1.10}$$

in which $\delta\,\Delta l_\lambda$ is the elongation of rod h as a result of the virtual displacement. Equation (1.9) can also be written:

$$\sum_{\lambda=1}^{3n} X_\lambda\, \delta\,\Delta x_\lambda - \sum_{h=1}^{r} S_h\, \delta\,\Delta l_h = 0 \tag{1.11}$$

then ([19]).

([19]) Since $S_h\ \delta\,\Delta l_h$ is the deformation work that is done on the individual rod by the virtual elongation of the rod $\delta\,\Delta l_h$, equation (1.11) expresses the idea that when one calculates the deformation work of the framework, one must

We next choose the virtual displacement as if it could also be performed on a rigid body. The most general such displacement is composed of three parallel displacements along the three coordinate axes and three rotations about the coordinate axes. The elongations $\delta\Delta l_\lambda$ of all frame members will be zero under those displacements, such that equation (1.11) will become simply:

$$\sum_{\lambda=1}^{3n} X_\lambda\,\delta\Delta x_\lambda = 0 \tag{1.12}$$

for them.

For a virtual displacement that is parallel to the x-axis, one has:

$$\delta\Delta x_1 = \delta\Delta x_4 = \ldots = \delta\Delta x_{3n-2} = \text{const.},$$

while all remaining $\delta\Delta x$ will be equal to zero. That implies the condition:

$$\sum_{\mu=1}^{n} X_{3\mu-2} = 0 \tag{1.13}$$

for the external forces. It follows analogously for the virtual displacements that are parallel to the y and z-axis that:

$$\sum_{\mu=1}^{n} X_{3\mu-1} = 0\,, \qquad \sum_{\mu=1}^{n} X_{3\mu} = 0\,. \tag{1.13a}$$

If the virtual displacement consists of a rotation through an angle $\delta\omega$ around the x-axis then:

$$\delta\Delta x_1 = \delta\Delta x_4 = \ldots = \delta\Delta x_{3n-2} = 0\,,$$

and

$$\begin{aligned} \delta\Delta x_2 &= -\,x_3\;\delta\omega\,, \quad \ldots, \quad & \delta\Delta x_{n-1} &= -\,x_n\;\delta\omega\,, \\ \delta\Delta x_3 &= \,x_2\;\delta\omega\,, \quad \ldots, \quad & \delta\Delta x_n &= \,x_{n-1}\;\delta\omega\,. \end{aligned}$$

It will then follow from (1.12) that:

$$\sum_{\mu=1}^{n} (x_{3\mu-1}\,X_{3\mu} - x_{3\mu}\,X_{3\mu-1}) = 0\,, \tag{1.14}$$

and analogously, that will give:

$$\sum_{\mu=1}^{n} (x_{3\mu}\,X_{3\mu-2} - x_{3\mu-2}\,X_{3\mu}) = 0\,,$$

simply take the sum of the virtual deformation works of the individual rods, such that the changes in the angles between the rods can be neglected, since the part of the work that is due to that will be a second-order infinitesimal. (By contrast, the changes in the angles themselves are not by any means of second order.)

$$\sum_{\mu=1}^{n} (x_{3\mu-2}\, X_{3\mu} - x_{3\mu}\, X_{3\mu-2}) = 0 \tag{1.14a}$$

for the rotations around the y and z-axes.

Equations (1.13) and (1.14) together say that *the external forces that act upon the framework must fulfill the equilibrium conditions for the forces on a rigid body.*

In engineering practice, those six condition equations between the external forces are employed in a somewhat-different way. When the framework is intended to be a supporting structure, the motions that it can perform as a rigid body will be inhibited by suitable *fasteners* that might be so arranged that they do not impede the elastic deformation in any way, as we would initially like to assume, for simplicity. Of the six motions that were considered above, we might inhibit the parallel displacements by, say, fixing one node absolutely. A second node that is connected to it by a rod shall further be capable of moving only along a straight line that goes through the fixed node. In that way, only a rotation of the entire framework around that line as its axis will be possible. We inhibit it when we demand of a further node that it must remain in a fixed plane that goes through the aforementioned line. We choose the fixed node of the framework to be the origin of our three-axis coordinate system, with the line that runs through as the x-axis and the plane that goes through it to be the (x, y)-plane.

Corresponding to those *fastening conditions*, we divide the external forces, i.e., the $3n$ components $(X_1, X_2, \ldots, X_{3n})$, into two groups. In the first group, we first include the components of the forces that act upon the fixed node, as well as the y and z-components of the force that acts upon the node that can move only along the x-axis, and finally the z-component of the force that acts upon the points that can move only in the (x, y)-plane. We take the second group to include the components of the remaining $(3n - 6)$ components. We shall call the six components in the first group the *support reaction* of the framework and denote them by $R_1, R_2, \ldots, R_6$, while we shall refer to the $(3n - 6)$ components in the second group as the *loads on the framework* or *external forces* (also active forces), in the more restricted sense. We would now like to change our numbering of the coordinates of the nodes and the components of the external forces that act upon them in such a way that we now number the $(3n - 6)$ components of the second group from 1 to $(3n - 6)$: $X_1, \ldots, X_{3n-6}$. They should always be thought of as given, whereas the reactions $R_1, R_2, \ldots, R_6$ can then be calculated by means of equations (1.13) and (1.14) ([20]). We correspondingly denote the associated nodal displacements by $\Delta x_1, \ldots, \Delta x_{3n-6}$. Due to the fastening conditions, the points of application of the six reactions will either remain unchanged or move perpendicular to

([20]) We have excluded the externally statically-indeterminate frameworks with our choice of fastening conditions and restricted ourselves to the internally statically-indeterminate ones, which will be called statically-indeterminate in what follows, *per se*. In practice, the fastening conditions are not prescribed in such a simple way, either, since all fasteners are elastically compliant, in reality. Those questions, as well as how one reduces externally statically-indeterminate systems to ones that are internal statically indeterminate are treated thoroughly in the engineering literature. In particular, confer O. MOHR, *Abhandlungen aus dem Gebiet der technischen Mechanik*, Berlin, 1906, *passim*, as well as H. MÜLLER-BRESLAU, *Die neueren Methoden der Festigkeitslehre*, 4th ed., Leipzig, 1913, and *Die graphische Statik der Baukonstruktionen*, 3rd ed., Leipzig, 1908.

those reactions. Therefore, the reactions will obviously do no work under a virtual displacement of the framework, and in place of (1.11), one will have:

$$\sum_{\lambda=1}^{3n-6} X_\lambda\,\delta\,\Delta x_\lambda - \sum_{h=1}^{r} S_h\,\delta\,\Delta l_h = 0\,, \tag{1.15}$$

and

$$\sum_{\lambda=1}^{3n-6}\left(X_\lambda - \sum_{h=1}^{r} S_h \frac{\partial f_h}{\partial x_\gamma}\right)\delta\,\Delta x_\lambda = 0\,, \tag{1.15a}$$

in place of (1.9), resp. Since the $\delta\,\Delta x_\lambda$ are arbitrary displacements, that will give the ($3n$ – 6) *equilibrium conditions* for the tensions:

$$\sum_{h=1}^{r} S_h \frac{\partial f_h}{\partial x_\gamma} = X_\lambda \qquad (\lambda = 1\ 2, \ldots, 3n - 6). \tag{1.16}$$

The problem of finding the magnitudes of the tensions in all rods – viz., the so-called *stress problem* – is especially important in engineering. When the number of rods in the framework is as small as possible, i.e., precisely ($3n$ – 6), those ($3n$ – 6) equilibrium conditions will determine all unknown tensions, and the stress problem will be resolved with no further analysis. One therefore calls such frameworks *statically determinate*. However, as soon as the number of rods is greater than ($3n$ – 6), equations (1.16) will no longer suffice to solve the stress problem. The framework will then be called *statically indeterminate*, and the solution of the stress problem will then require that one must appeal to elasticity.

1.3. The intervention of elasticity. Solving the stress problem for statically-indeterminate frameworks. – We now imagine that the individual frame members are elastic and assume that each of them obeys *Hooke's law*, that is, in our case, the elastic rods experience elongations Δl that are proportional to the tensions S :

$$\Delta l = \varepsilon\,S\,, \qquad S = \frac{1}{\varepsilon}\,\Delta l\,. \tag{1.17}$$

The proportionality factor ε is expressed in terms of the so-called *Young modulus E* of the material, the length l and the cross-section F of the rod in the following way:

$$\varepsilon = \frac{l}{E\cdot F}\,. \tag{1.17a}$$

Since all tensions are known from the equilibrium conditions (1.16) for statically-determinate frameworks, one will also know all elongations of the rods from (1.17), and with that, the new lengths that the frame members will assume in the deformed state. Now, a geometrical or kinematical argument will show that the lengths of the rods can be given arbitrarily in a statically-determined framework, but in such a way that the distances between all nodes will be determined by them ([21]). Therefore, the displacements of the nodes that come about as a result of the deformation can also be calculated from the elongations of the rods, and we will also control the elastic deformation of statically-determined frameworks directly.

We now make use of that fact in order to resolve the *stress problem for statically-indeterminate frameworks, as well*. For a statically-indeterminate framework, we can (and generally in many different ways) ignore ($r - 3n + 6$) rods without the remaining system, which should include all n nodes, ceasing to be a figure that does not move. Such a remaining system will include $3n - 6$ rods, so it will be statically determinate. We would then like to call it a *statically-determinate principal system* for the given statically-indeterminate framework.

The distances between all nodes of the given framework are determined from the lengths of the rods in that statically-determinate principle system, and therefore the lengths of its other ($r - 3n - 6$) rods, which are the so-called *superfluous rods*. The dependency might possess the form:

$$\left.\begin{aligned} l_{3n-6+1} &= \varphi_1(l_1, l_2, \ldots, l_{3n-6}), \\ l_{3n-6+2} &= \varphi_2(l_1, l_2, \ldots, l_{3n-6}), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ l_r &= \varphi_{r-3n+6}(l_1, l_2, \ldots, l_{3n-6}). \end{aligned}\right\} \tag{1.18}$$

By taking the differentials of those equations, we will obtain the elongations $\Delta l_{3n-6+1}, \ldots, \Delta l_r$ of the superfluous rods as functions of the elongations $\Delta l_1, \ldots, \Delta l_{3n-6}$ of the statically-determined principal system.

The stress problem for the statically-indeterminate framework is likewise solved when the relations (1.18) are known. Namely, we can introduce the rod elongations Δl into the equilibrium conditions (1.18) in place of the tensions using (1.17). Initially, nothing would be gained by doing that since the $3n - 6$ equations would still not suffice to determine the r unknowns Δl. However, one can eliminate the elongations $\Delta l_{3n-6+1}, \ldots, \Delta l_r$ of the superfluous rods from them by taking the differential of the relation (1.18), and the remaining $\Delta l_1, \ldots, \Delta l_{3n-6}$ will be determined uniquely from them. Nonetheless, if they are known then the elongations of the superfluous rods will also be determined by the differentials of (1.18). We will then know the elongations of all r rods then, and we can easily calculate all tensions from then by using (1.17). Naturally, we will likewise master the deformation of the framework with that.

In the theory of frameworks, one does in fact initially imagine that equations (1.18) have actually been posed in each case ([22]), even though that will already lead to apparently-effortless calculations in some simple cases. However, one will soon see that it would be simpler to directly pose the linear relations for the rod elongations Δl instead of the relations (1.18), which refer to the lengths of the rods themselves, and which one does not need at all for the further calculations.

([21]) Cf., e.g., A. FÖPPL, *Vorlesungen über technische Mechanik*, II, pp. 195.
([22]) Cf., M. LÉVY, *La statique graphique et ses applications aux constructions*, Paris 1874, as well as L. F. MENABREA, Atti della R. accademia dei lincei (2) **2** (1873-1875), pp. 201, *et seq.*

The first to tread that path was MAXWELL ([23]). He succeeded in representing the changes in length of the superfluous rods as linear functions of the changes in length of the rods of the principal system by an argument that again employed concrete representations of the elastic deformation by appealing to the so-called Clapeyron theorem.

Independently of him, MOHR ([24]) achieved the same representation later, by appealing to the principle of virtual displacements directly. After removing the external forces, he also dropped the superfluous rods from the given framework and considered the remaining, statically-determinate, system. We imagine that two forces of absolute value one are applied to the nodes of the system that were connected by superfluous rods l_{3n-6+2} that have the direction of the superfluous rod that was dropped and are equal and opposite to each other. They will create a stress state $s_1^{(1)}$, $s_2^{(1)}$, …, $s_{3n-6}^{(1)}$ in the principal system that is determined completely by the equilibrium conditions of the principal system, which are linear equations of the form (1.16). (In engineering practice, those stresses s are ascertained by simple graphical methods with a suitable choice of the principal system.) We would like to apply the principle of virtual displacements to equation (1.15) for the equilibrium state of the principal system, and indeed in so doing we would like to think of the elongations of the rods of the principal system Δl as being given arbitrarily, which will also determine the displacements Δx of all nodes. Since only the two unit forces act as external forces, the only two nodal displacements that come under consideration will be the ones to which those unit forces are applied.

One can easily give the work done by the two unit forces under the virtual displacement directly since for each of those displacements, the work will be equal to the projection of the displacement of the node in the direction of the force, i.e., onto the first superfluous rod. Taken together, those two projections are equal to the elongation of the connecting line between those two nodes, i.e., they are equal to Δl_{3n-6+1} . Equation (1.15) for the principle of virtual displacement then implies that:

$$\Delta l_{3n-6+1} - \sum_{h=1}^{3n-6} s_h^{(1)} \, \Delta l_h = 0 , \tag{1.19}$$

with which we have then found the first of the desired conditions for the elongations of the rod. If we also apply the same process to all of the remaining pairs of nodes that are connected by superfluous rods then we will find the further relations:

$$\begin{gathered} \Delta l_{3n-6+1} - \sum_{h=1}^{3n-6} s_h^{(2)} \, \Delta l_h = 0 , \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{gathered} \tag{1.19a}$$

([23]) J. Cl. MAXWELL, "On the calculation of the equilibrium and stiffness of frames," Phil. Mag. **27** (1864), pp. 294, and also *Scientific Papers,* 1, pp. 598.

([24]) O. MOHR, "Beitrag zur Theorie der Bogenfachwerkträger," Zeitschrift des Architekten- und Ingenieurverains zu Hannover (1874), pp. 223. O. MOHR, "Beiträge zur Theorie des Fachwerks," *ibid.* (1874), pp. 509, *ibid.* (1875), pp. 17. Naturally, the Mohr process, which is based upon the abstract principle of virtual displacements, also subsumes the cases (like the so-called temperature stresses) that are initially overlooked in the MAXWELL representation.

$$\Delta l_{3n} - \sum_{h=1}^{3n-6} s_h^{(r-3n+6)} \, \Delta l_h = 0 \, .$$

These (r – 3n + 6) conditions (1.19) and (1.19a) for the rod elongations are referred to as the *compatibility conditions* for the framework in the terminology of the theoretical study of elasticity. In technical practice, one introduces the terms *elasticity conditions* or *condition equations* for them. They directly represent the (r – 3n + 6) equations by means of which we can eliminate enough Δl from the equilibrium conditions (1.16) (by introducing the modified Δl) that the remaining ones will be determined uniquely.

If one only wishes to solve the stress problem then it would be simpler to replace the Δl_h in equations (1.19) and (1.19a) with the tensions S_h by using (1.17). We will then get r – 3n + 6 condition equations or *compatibility conditions for the tensions*. Together with the 3n – 6 equilibrium conditions (1.16), they will determine all r tensions.

A path to solving the stress problem for the statically-indeterminate framework that is simpler in principle was pursued by Castigliano ([25]). Just as we can avoid the compatibility conditions for the deformation components (stress components, resp.) completely in the theory of elasticity by introducing the displacement components in place of the six deformation components, here we can also arrive at a solution to the problem without explicitly exhibiting the elasticity relations by observing that from equation (1.10) the rod elongations must be expressible in terms of the displacements of the nodes of the framework in the form:

$$\Delta l_h = \sum_{\lambda=1}^{3n-6} \frac{\partial f_h}{\partial x_\lambda} \Delta x_\lambda \, ,$$

and therefore, we will also have:

$$S_h = \frac{1}{\varepsilon_h} \sum_{\lambda=1}^{3n-6} \frac{\partial f_h}{\partial x_\lambda} \Delta x_\lambda \tag{1.20}$$

from (1.17).

If we introduce those expressions into the (3n – 6) equilibrium conditions (1.16) then the (3n – 6) unknowns Δx_λ can be calculated from them. However, if the Δx_λ are known then we will find all tensions inversely from (1.20). In practice, that theoretically-simpler solution to the stress problem is always coupled with laborious calculations that do not contribute to the clarification of the question that is essential to the engineer, so it has never been adopted into engineering practice.

1.4. The minimum of total energy. Canonical transformation. Menabrea's principle. – For the following considerations, which are true for both statically-determinate and statically-indeterminate frameworks, we once more appeal to the expression (1.15) for the principle of virtual displacements:

([25]) A. CASTIGLIANO, *Dissertazione di laurea*, Turin, 1873.

$$\sum_{\lambda=1}^{3n-6} X_\lambda \, \delta \, \Delta x_\lambda - \sum_{h=1}^{r} S_h \, \delta \, \Delta l_h = 0 \,, \tag{1.15}$$

and substitute:

$$S_h = \frac{1}{\varepsilon_h} \Delta l_h$$

in the second term using (1.17). It will then read:

$$\sum_{h=1}^{r} \frac{1}{\varepsilon_h} \Delta l_h \, \delta \, \Delta l_h \,,$$

and will then be the variation of the sum of squares:

$$\Psi \, (\Delta l_h) = \tfrac{1}{2} \sum_{h=1}^{r} \frac{(\Delta l_h)^2}{\varepsilon_h} \,. \tag{1.21}$$

In the first term in (1.15), the external forces X_λ are given constants, so we can regard that term as the variation of a linear function of the Δx_λ . If we set ([26]):

$$\Phi \, (\Delta x_\lambda) = \sum_{\lambda=1}^{3n-6} X_\lambda \, \Delta x_\lambda \tag{1.22}$$

then we can write (1.15) as:

$$\delta \, [\Phi \, (\Delta x_\lambda) + \Psi \, (\Delta l_h) \,] = 0 \,, \tag{1.23}$$

or when we introduce:

$$E = \Phi + \Psi, \tag{1.24}$$

more briefly as:

$$\delta E = 0 \,. \tag{1.21a}$$

The equation of the principle of virtual displacements is then equivalent to the one that the function E must be an extremum.

From (1.22), *the function* Φ can be regarded as the *potential* or *potential energy of the given external forces*. On the other hand, the individual terms in the sum of the squares (1.21) are equal to the deformation works for the individual rods that are actually done under the *elastic deformation* of the framework as a result of its loading by the external forces. That is because, as one can also always let those external forces increase from their initial values to their final values, the Δl_h and S_h will always be determined by them at each moment (since one only goes through the equilibrium state). One can also characterize the state that exists at each moment by giving the associated values $\overline{\Delta l_h}$, and therefore by regarding them as independent variables. The $\overline{S_h}$ will then

([26]) One should probably call that function the *virial* of the given external forces.

be functions of the $\overline{\Delta l_h}$, and indeed the relation (1.17) will also be true for each instant while the elastic deformation is being produced. Now since the deformation work that is done on rod h is:

$$\bar{S}_h \, d\,\overline{\Delta l_h} = \frac{\overline{\Delta l_h}}{\varepsilon_h} \, d\,\overline{\Delta l_h}$$

at each instant, it will follow that the total deformation work done on rod h by our elastic deformation will be:

$$\int_0^{\Delta l_h} \frac{\overline{\Delta l_h}}{\varepsilon_h} \, d\,\overline{\Delta l_h} = \frac{\left(\overline{\Delta l_h}\right)^2}{2\varepsilon_h} \, .$$

The *function* Ψ is then equal to the *total deformation work done on the framework. E* itself is then the sum of the potential energy of the external forces and the deformation work (of the potential energy of the internal forces), so *E is the total potential energy of the framework*. The relation (1.23a) says only that the equilibrium state of the framework will be characterized by the fact that the total potential energy will possess an extremal value for it ([27]). *In that way, the function E is regarded as a function of the* $(3n - 6)$ *nodal displacements* Δx_λ . Those nodal displacements vary arbitrarily, and the framework remains connected under the variation, but the varied states are not equilibrium states. Equilibrium will exist only when E is an extremum.

The nodal displacements Δx_λ appear in the two terms Φ and Ψ that comprise E in different ways. Whereas Φ is a linear function of the nodal displacements Δx_λ, Ψ initially depends quadratically upon the rod elongations Δl_h, in which one introduces:

$$\Delta l_h = \frac{\partial f_h}{\partial x_{3\mu-2}}(\Delta x_{3\mu-2} - \Delta x_{3\nu-2}) + \frac{\partial f_h}{\partial x_{3\mu-1}}(\Delta x_{3\mu-1} - \Delta x_{3\nu-1}) + \frac{\partial f_h}{\partial x_{3\mu}}(\Delta x_{3\mu} - \Delta x_{3\nu}) \, , \tag{1.26}$$

from (1.6). Thus, only the 3*r differences* between the nodal displacements appear in that. For the following argument, we would like to regard those differences as independently variable. E will then be a function of, on the one hand, the nodal displacements Δx_λ, and on the other, the differences between the nodal displacements $(\Delta x_\sigma - \Delta x_\rho)$ ([28]):

$$E = E\,[(\Delta x_\sigma - \Delta x_\rho), \Delta x_\lambda] = \Psi\,[(\Delta x_\sigma - \Delta x_\rho)] + \Phi\,(\Delta x_\lambda) \, . \tag{1.27}$$

([27]) On this, cf., L. DONATI, Memorie della R. Accademia di Bologna (4) **10** (1889), pp. 267, *et seq.*, A. FÖPPL, *Vorlesungen über technische Mechanik*, Bd. V, pp. 259.

([28]) This dependency of the function E upon the Δx_λ and their differences $(\Delta x_\sigma - \Delta x_\rho)$ is similar to the way that the Lagrangian function $L = T - \Phi$ (where T is the kinetic energy and Φ is the potential energy) in analytical mechanics depends upon coordinates q and their differential quotients with respect to time, namely, the associated velocities:

$$L = L\left(\frac{dq_1}{dt}, \frac{dq_2}{dt}, \ldots, q_1, q_2, \ldots\right) \, .$$

We would now like to apply a transformation to that function E that represents a type of extension of the so-called *Legendre transformation* in the realm of the differential calculus. We set the derivatives of E with respect to the differences of the Δx equal to new unknowns, say:

$$\begin{aligned} \frac{\partial E}{\partial(\Delta x_{3\mu-2} - \Delta x_{3\nu-2})} &= \Xi_{3h-2} , \\ \frac{\partial E}{\partial(\Delta x_{3\mu-1} - \Delta x_{3\nu-1})} &= \Xi_{3h-1} , \\ \frac{\partial E}{\partial(\Delta x_{3\mu} - \Delta x_{3\nu})} &= \Xi_{3h} , \end{aligned} \tag{1.26}$$

in which the lower indices of the Ξ indicate that they belong to rod h. In order to see the meaning of these new unknowns, we observe that from (1.21) and (1.23):

$$\frac{\partial E}{\partial(\Delta x_{3\mu-2} - \Delta x_{3\nu-2})} = \frac{\partial E}{\partial \Delta l_h} \frac{\partial \Delta l_h}{\partial(\Delta x_{3\mu-2} - \Delta x_{3\nu-2})} = \frac{\Delta l_h}{\varepsilon_h} \frac{\partial f_h}{\partial x_{3\mu-2}} ,$$

such that we will then have:

$$\Xi_{3h-2} = S_h \frac{\partial f_h}{\partial x_{3\mu-2}} , \tag{1.27}$$

and analogously:

$$\Xi_{3h-1} = S_h \frac{\partial f_h}{\partial x_{3\mu-1}} , \qquad \Xi_{3h} = S_h \frac{\partial f_h}{\partial x_{3\mu}} , \tag{1.27a}$$

i.e., the three components of the tension S_h along the three coordinate axes. In the spirit of the Legendre transformation, E is now replaced with the new function H ([29]):

([29]) In analytical mechanics, this transformation corresponds to the so-called *canonical transformation of Poisson (Hamilton, resp.)*, by which the derivatives of the Lagrangian function L with respect to the unknowns $\dot{q}$ are set equal to new unknowns p :

$$\frac{\partial L}{\partial \dot{q}} = p , \tag{a}$$

in which the p are the so-called impulses. In that way, the Hamiltonian function:

$$\dot{q}\, p - L(\dot{q}, q) = H(p, q)$$

will appear in place of the Lagrangian function, which depends upon the impulse p (in addition to the q) (*unreadable word in the original*).

$$\sum_{h=1}^{r} [\Xi_{3h-2}\,(\Delta x_{3\mu-2} - \Delta x_{3\nu-2}) + \Xi_{3h-1}\,(\Delta x_{3\mu-1} - \Delta x_{3\nu-1}) + \Xi_{3h}\,(\Delta x_{3\mu} - \Delta x_{3\nu})] \\ -E\,[(\Delta x_\sigma - \Delta x_\rho), \Delta x_\lambda] = H\,(\Xi_h, \Delta x_\lambda)\,, \tag{1.28}$$

which is to be regarded as depending upon the new variables Ξ (in addition to the Δx_λ). The coordinate differences that appear on the left-hand side are then to be thought of as expressed in terms of the Ξ by means of equations (1.26). The sum extends over all frame members.

Since that sum is equal to $\sum_{h=1}^{r} S_h\,\Delta l_h$, from (1.27) and (1.25), and since the differences of the Δx also occur in E only in the combination Δl_h, it would be convenient from now on to no longer regard the $3r$ differences of the Δx_λ as independent variables, but only their r combinations Δl_h ([30]). In order to perform the Legendre transformation, we will then have to introduce the derivatives:

$$\frac{\partial E}{\partial\,\Delta l_h} = \frac{\Delta l_h}{\varepsilon_h} = S_h \tag{1.26a}$$

as new variables, and indeed in the new function:

$$\sum_{h=1}^{r} S_h\,\Delta l_h - E\,(\Delta l_h, \Delta x_\lambda) = H\,(S_h, \Delta x_l)\,. \tag{1.28a}$$

We will then get [see eqs. (1.26a), (1.21), and (1.22)]:

$$H\,(S_h\,, \Delta x_l) = \sum_{h=1}^{r} S_h\,\Delta l_h - E\,(\Delta l_h, \Delta x_\lambda)\,,$$

$$\sum_{h=1}^{r} \varepsilon_h\,S_h^2 - \tfrac{1}{2}\sum_{h=1}^{r} \varepsilon_h\,S_h^2 + \sum_{\lambda=1}^{3n-6} X_\lambda\,\Delta x_\lambda = H\,(S_h\,, \Delta x_l)\,,$$

or

$$H\,(S_h\,, \Delta x_l) = \tfrac{1}{2}\sum_{h=1}^{r} \varepsilon_h\,S_h^2 + \sum_{\lambda=1}^{3n-6} X_\lambda\,\Delta x_\lambda\,. \tag{1.28b}$$

We would now like to introduce a new function $\mathcal{E}$ with the help of the function H by the Ansatz:

$$\mathcal{E}\,(S_h\,, \Delta x_l) = \sum_{h=1}^{r} S_h\,\Delta l_h - H\,(\Delta l_h, \Delta x_\lambda)$$

([30]) That is analogous to the view that one often finds it more convenient to consider the gradient of a certain function $z = z\,(x, y)$ in a certain direction $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$, instead of the two partial derivatives $\partial z / \partial x$, $\partial z / \partial y$.

$$= \sum_{h=1}^{r} \left[S_h \sum_{\lambda=1}^{3n-6} \frac{\partial f_h}{\partial x_\lambda} \Delta x_\lambda \right] - H(\Delta l_h, \Delta x_\lambda) \tag{1.29}$$

and ask what *the extremum of this function* $\mathcal{E}$ would be when we regard the S_h and Δx_λ in it as independent variables ([31]). The variations of the tensions S_h and the nodal displacements Δx_λ will then prove to be independent of each other. No relation will then exist between the tensions and the rod elongations that are determined by the nodal displacements in the varied state. Such a thing will first be introduced by the extremum requirement. If we next fix the Δx_λ in $\mathcal{E}$ and imagine that only the S_h are variable then the extremum condition will become:

$$\frac{\partial \mathcal{E}}{\partial S_h} = \sum_{\lambda=1}^{3n-6} \frac{\partial f_h}{\partial x_\lambda} \Delta x_\lambda - \frac{\partial H}{\partial S_h} = 0 ,$$

or from (1.28b):

$$\varepsilon_h S_h = \sum_{\lambda=1}^{3n-6} \frac{\partial f_h}{\partial x_\lambda} \Delta x_\lambda = \Delta l_h . \tag{1.30a}$$

That says that the *tensions are expressed in terms of the rod elongations (nodal displacements,* Δx_λ*, resp.) by way of Hooke's law.* The other requirement, viz., that $\mathcal{E}$ will also be an extremum for those Δx_λ, *leads to the equilibrium conditions*. That is because if we introduce the expressions for the tensions S_h that were found above into the function $\mathcal{E}$, eq. (1.29), then it will go to the function E whose extremum will, as we know, lead to the equilibrium conditions, cf., eq. (1.23a). However, the direct Ansatz $\partial\mathcal{E} / \partial\, \Delta x_\lambda$ will also lead to the equilibrium conditions. Namely, it will imply that:

$$\frac{\partial \mathcal{E}}{\partial\, \Delta x_\lambda} = \sum_{h=1}^{r} S_h \frac{\partial f_h}{\partial x_\lambda} - \frac{\partial H}{\partial\, \Delta x_\lambda} = 0 ,$$

or from (1.28b) ([32]):

([31]) The transition from E to $\mathcal{E}$ in analytical mechanics is analogous to the transition from the variational problem for Hamilton's principle:

$$\int_{t_0}^{t_1} L(\dot{q}_1, \dot{q}_2, \ldots, q_1, q_2)\, dt$$

to the canonical variational principle.

([32]) Equations (1.30a) and (1.30b) are the analogues of the canonical equations of mechanics:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} , \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q} ,$$

the first of which expresses the connection between the velocity components and the impulse, while the second one expresses the equations of motion. The equations:

$$\sum_{h=1}^{r} S_h \frac{\partial f_h}{\partial x_\lambda} - \frac{\partial H}{\partial \Delta x_\lambda} = 0 \, . \tag{1.30b}$$

We will then have the following result of our extremum condition on $\mathcal{E}$ (S_h, Δx_λ): *The first group* (1.30a) *implies the relations that exist between the tensions and rod elongations* (the displacements of the nodes, resp.) *on the basis of Hooke's law.* (For statically-indeterminate frameworks, we can derive the compatibility conditions for the tensions from that.) *The second group* (1.30b) *represents the equilibrium conditions.* Collectively, the two groups simultaneously resolve the stress problem, as well as the deformation problem, completely and for statically-determinate, as well as statically-indeterminate frameworks.

As we just saw, the previous requirement (1.23a) that the function E (Δx_λ) must be extremized, which was equivalent to the principle of virtual displacements, will arise from the general requirement of the extremum of the function $\mathcal{E}$ as a function of the S_h and the Δx_λ when we assume that the first group of equations (1.30a) is fulfilled, i.e., the compatibility conditions for the deformations are valid.

It is now natural to assume that the second group of equations (1.30b) is fulfilled and *ask whether there is an extremum problem that will lead to the first group, i.e., the compatibility conditions.*

The function that is to be extremized here is obtained immediately, because from (1.29) and (1.28b), $\mathcal{E}$ will possess the form:

$$\mathcal{E}\,(S_h, \Delta x_\lambda) = \sum_{\lambda=1}^{3n-6} \left[\sum_{h=1}^{r} S_h \frac{\partial f_h}{\partial x_\lambda} - X_\lambda \right] \Delta x_\lambda - \tfrac{1}{2} \sum_{h=1}^{r} \varepsilon_h \, S_h^2 \, .$$

Should the equilibrium conditions (1.30b) be fulfilled, as we assume here, then the first term will drop out, and the extremum of the function $\mathcal{E}$, which will yield the compatibility conditions when one considers the auxiliary conditions, is equivalent to the extremum of the function:

$$\bar{\Psi}(S_h) = \tfrac{1}{2} \sum_{h=1}^{r} \varepsilon_h \, S_h^2 \, . \tag{1.31}$$

$$\frac{\partial E(\Delta x_\lambda)}{\partial \Delta x_\lambda} = 0 \, ,$$

which are the conditions for the minimum of E (Δx_λ), correspond to the Lagrange equations (of the second kind) in mechanics:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \, .$$

That function represents the sum of the deformation works of the individual frame members under elastic deformation when they are caused by the tensions S_h . Naturally, one can speak of an extremum of that function, with equations (1.30b) as the auxiliary conditions, only when one is dealing with a statically-indeterminate framework. That is because for a statically-determinate framework, the equilibrium conditions determine all tensions, and the function (1.31) will assume only a single, well-defined value. That agrees with the fact that indeed no compatibility conditions will exist for the tensions in this case either.

However, for a *statically-indeterminate framework*, we compare the stress states in the individual rods that satisfy the equilibrium conditions for the nodes under that extremum requirement and attempt to characterize those of them (the so-called stress diagram) for which the rods take on lengths under the elastic stresses such that they can be once more combined into the given framework. It is only in the special stress state that we first sought that we could refer to the sum (1.31) of the deformation works in the individual rods as the deformation work done on the framework, strictly speaking. Nonetheless, it is customary in engineering literature to refer to the function $\bar{\Psi}$ as the deformation work done on the framework even for a general choice of the S_h. However, when one speaks of the extremum of the deformation work, one then speaks of the extremum of the function $\bar{\Psi}(S_h)$, with the equilibrium conditions:

$$\sum_{h=1}^{r} S_h \frac{\partial f_h}{\partial x_\lambda} = X_\lambda \qquad (\lambda = 1, 2, \dots, 3n - 6)$$

as auxiliary conditions.

In order to show once more that the extremum of $\bar{\Psi}$ will actually lead to the compatibility conditions for the stresses without referring to the foregoing argument, we consider the auxiliary conditions using the method of Lagrange factors.

If $\mu_1, \mu_2, \dots, \mu_{3n-6}$ are the Lagrange factors then we must look for the extremum of:

$$\tfrac{1}{2}\sum_{h=1}^{r} \varepsilon_h S_h^2 + \sum_{\lambda=1}^{3n-6} \mu_\lambda \left(\sum_{h=1}^{r} S_h \frac{\partial f_h}{\partial x_\lambda} - X_\lambda \right)$$

as a function of the S_h . That immediately leads to the equations:

$$\varepsilon_h S_h + \sum_{h=1}^{r} \mu_h \frac{\partial f_h}{\partial x_\lambda} = 0 , \tag{1.32}$$

which are, in fact, the first group of equations (1.30a) and are equivalent to the compatibility conditions. We further remark that the Lagrange factors that were introduced agree with the displacements of the nodes, up to sign. That latter principle is much-employed in the engineering theory of elasticity. It is what one calls *Menabrea's principle*, after the Italian military engineer MENABREA. Later on, we shall consider it from a somewhat-different perspective.

1.5. Deformation work as a function of external forces. Castigliano's theorems. Maxwell's reciprocity theorem. – One can calculate all of the forces X_1, …, X_{3n-6} from the equilibrium conditions for the nodes and the compatibility conditions for the framework:

$$S_h = \sum_{\lambda=1}^{3n-6} a_\lambda^{(k)} X_\lambda , \tag{1.33}$$

whose constant coefficients depend upon the structure of the framework, and for statically-indeterminate frameworks, also upon the dimensions and elastic properties of the frame members. If one introduces the values (1.33) of the S_h into the function (1.31) $\bar{\Psi}(S_h)$ then it will assume an extremal value that is consistent with the equilibrium conditions and for which the compatibility conditions will be fulfilled, from the previous section. We denote it by $\bar{\Psi}^{(k)}$. The deformation work done on the framework that $\bar{\Psi}^{(k)}$ represents will then be *a quadratic form in the external forces:*

$$\bar{\Psi}^{(k)}(S_h) = \sum_{i,k=1}^{3n-6} c_{ik} X_i X_k \tag{1.34}$$

whose coefficients c_{ik} can be constructed from the $a_\lambda^{(k)}$ in a simple way ([33]).

Along with that expression for the deformation work done on the framework as a function of the external forces, we can pose another one *that determines the dependency of the deformation work on the displacements of the nodes*. Namely, if the displacements of the nodes are given and we demand that an equilibrium state of the framework should occur then we can easily determine the external forces for which the nodal displacements would have the given values. That is because from (1.10), we can calculate all rod elongations Δl_h from the nodal displacements Δx_λ, and therefore also know all tensions S_h for the individual rods from Hooke's law. The equilibrium conditions will then immediately imply the external forces X_λ that must be applied to the nodes in order to justify the deformation of the framework that belongs to the given displacements of the nodes Δx_λ . In order to calculate the deformation work that will be performed under the transition of the framework from the stress-free natural state to the desired equilibrium state, we appeal to the expression (1.21):

([33]) If we would like to pursue the analogy between our representation and analytical mechanics further then this representation of the deformation work would be the analogue of the *principal function* or *varied action* that W. R. HAMILTON introduced, i.e., the action integral whose integration path is an extremal:

$$\int_{t_1, q_1^{(k)}, q_2^{(1)}, \ldots}^{t_1, q_1^{(2)}, q_2^{(2)}, \ldots} L(q_1', q_2', \ldots, q_1, q_2, \ldots, t)\, dt .$$

Just as one must choose precisely the extremal from the manifold of integration paths that can connect two given reference points and prescribe them such that value of the integral is determined by the two limits in that context, here we must choose from all possible S_h that fulfill the equilibrium conditions at the nodes for given external forces X_λ, precisely the ones that make the function an extremum, which will then make that extremal value into a function of the external forces.

$$\Psi = \tfrac{1}{2}\sum_{h=1}^{r} \frac{(\Delta l_h)^2}{\varepsilon_h}, \tag{1.21}$$

and replace the Δl_h in them with the nodal displacements Δx_λ . In order to suggest that we regard the state of the framework that belongs to the given nodal displacements as an equilibrium state that is attained under the action of suitable external forces, we denote the expression (1.21) by $\Psi^{(g)}$. Since the Δl_h are homogeneous and linear in the Δx_λ, *the deformation work* $\Psi^{(g)}$ *done on the framework will be a quadratic form in the* Δx_λ :

$$\Psi^{(g)}(\Delta x_\lambda) = \sum_{i,k=1}^{3n-6} \gamma_{ik}\, \Delta x_i\, \Delta x_k\,, \tag{1.35}$$

in which the γ_{ik} are constants that are easy to calculate.

We will get the following two representations of the quadratic forms (1.34) and (1.35) from EULER's theorem on homogeneous functions:

$$\bar{\Psi}^{(k)}(X_\lambda) = \tfrac{1}{2}\sum_{\lambda=1}^{3n-6} \frac{\partial \bar{\Psi}^{(k)}}{\partial X_\lambda} X_\lambda \tag{1.34a}$$

and

$$\Psi^{(g)}(\Delta x_\lambda) = \tfrac{1}{2}\sum_{\lambda=1}^{3n-6} \frac{\partial \Psi^{(g)}}{\partial \Delta x_\lambda} \Delta x_\lambda\,, \tag{1.35a}$$

which we will need shortly.

The two functions $\bar{\Psi}^{(k)}(X_\lambda)$ and $\Psi^{(g)}(\Delta x_\lambda)$ are two expressions for the actual work done on the framework. We would now like to derive a new expression for the common value of the deformation work. If the two systems X_λ (Δx_λ, resp.) belong to the same state of deformation of the framework then their values will agree. Namely, from Hooke's law (1.17), one has:

$$\Psi^{(g)}(\Delta x_\lambda) = \tfrac{1}{2}\sum_{h=1}^{r} l_h \frac{1}{\varepsilon_h} l_h = \tfrac{1}{2}\sum_{h=1}^{r} S_h\, \Delta l_h$$

and

$$\bar{\Psi}^{(k)}(\Delta X_\lambda) = \tfrac{1}{2}\sum_{h=1}^{r} S_h\, \varepsilon_h\, S_h = \tfrac{1}{2}\sum_{h=1}^{r} S_h\, \Delta l_h\,.$$

If we replace the rod elongations Δl_h with the displacements of the nodes Δx_λ by means of equations (10):

$$\Delta l_h = \sum_{\lambda=1}^{3n-6} \frac{\partial f_h}{\partial \Delta x_\lambda} \Delta x_\lambda$$

then that will give:

$$\Psi^{(g)} = \bar{\Psi}^{(k)} = \tfrac{1}{2}\sum_{\lambda=1}^{3n-6} X_\lambda\, \Delta x_\lambda \,. \tag{1.36}$$

One then has the theorem that *the deformation work is equal to one-half the work that would be done by the external forces if they acted with their final values throughout the total deformation*. That theorem goes by the name of *Clapeyron's theorem.*

If we combine that equation (1.36) with (1.34) and (1.35) then we will get the two relations:

$$\begin{aligned}
&\tfrac{1}{2}\sum_{\lambda=1}^{3n-6} X_\lambda\, \Delta x_\lambda = \bar{\Psi}^{(k)}(X_1, X_2, \ldots, X_{3n-6}),\\
&\tfrac{1}{2}\sum_{\lambda=1}^{3n-6} X_\lambda\, \Delta x_\lambda = \Psi^{(g)}(\Delta x_1, \Delta x_2, \ldots, \Delta x_{3n-6}),
\end{aligned} \tag{1.37}$$

or as we can also write, from (1.34a) [(1.35a), resp.]:

$$\sum_{\lambda=1}^{3n-6} \Delta x_\lambda\, X_\lambda = \sum_{\lambda=1}^{3n-6} \frac{\partial \bar{\Psi}^{(k)}}{\partial X_\lambda} X_\lambda \tag{1.37a}$$

and

$$\sum_{\lambda=1}^{3n-6} X_\lambda\, \Delta x_\lambda = \sum_{\lambda=1}^{3n-6} \frac{\partial \Psi^{(g)}}{\partial \Delta x_\lambda} \Delta x_\lambda \,. \tag{1.37b}$$

In the first of those equations, the X_λ are entirely-arbitrary variables, while in the second one, it is the Δx_λ. We then conclude that:

$$\frac{\partial \bar{\Psi}^{(k)}}{\partial X_\lambda} = \Delta x_\lambda, \qquad \frac{\partial \Psi^{(g)}}{\partial \Delta x_\lambda} = X_\lambda \,. \tag{1.38}$$

Those two formulas include the so-called *theorems of Castigliano* on the derivatives of the deformation done on a framework, which are proved in the manner that was given above. We can express them in words as follows:

If the deformation work done on a framework is represented as a function of the $(3n-6)$ *components* X_λ *of the external forces then its derivative with respect to any of those components will be equal to the displacement* Δx_λ *of the associated node in the associated direction,*

and conversely:

If the deformation work done on a framework is expressed as a function of the $(3n - 6)$ *components* Δx_λ *of the nodal displacements then its derivative with respect to any of those components will be equal to the associated force component* X_λ ([34]).

Both theorems are true for statically-determinate, as well as statically-indeterminate frameworks.

In engineering practice, one often applies them in a slightly-different form. For example, if X_{3n-2}, X_{3n-1}, X_{3n} are the three components of force P_μ that is applied to node μ, i.e.:

$$X_{3n-2} = P_\mu \cos(x, P_\mu)\,, \qquad X_{3n-1} = P_\mu \cos(y, P_\mu)\,, \qquad X_{3n} = P_\mu \cos(z, P_\mu)\,,$$

and we introduce the deformation work $\bar{\Psi}^{(k)}$ in place of the three components of the force P_μ itself then we will have:

$$\frac{\partial\bar{\Psi}^{(k)}}{\partial P_\mu} = \frac{\partial\bar{\Psi}^{(k)}}{\partial X_{3n-2}}\cdot\frac{\partial X_{3n-2}}{\partial P_\mu} + \frac{\partial\bar{\Psi}^{(k)}}{\partial X_{3n-1}}\cdot\frac{\partial X_{3n-1}}{\partial P_\mu} + \frac{\partial\bar{\Psi}^{(k)}}{\partial X_{3n}}\cdot\frac{\partial X_{3n}}{\partial P_\mu}\,,$$

or [cf., (1.38)]:

$$\frac{\partial\bar{\Psi}^{(k)}}{\partial P_\mu} = \Delta x_{3\mu-2}\cos(x, P_\mu) + \Delta x_{3\mu-1}\cos(y, P_\mu) + \Delta x_{3\mu}\cos(z, P_\mu) = \Delta\pi_\mu\,, \tag{1.38a}$$

in which $\Delta\pi_\mu$ is the projection of the displacement of node μ onto the direction of the force P_μ that is applied to that node. When we likewise introduce the displacement Δp_μ of node μ into the expression for the deformation work $\Psi^{(g)}$ in place of the three components $\Delta x_{3\mu-2}$, $\Delta x_{3\mu-1}$, $\Delta x_{3\mu}$ of that displacement, we will get, in an entirely-analogous way:

$$\frac{\partial\Psi^{(g)}}{\partial p_\mu} = \Pi_\mu\,, \tag{1.38b}$$

in which Π_μ is the projection of the force that is applied to node μ onto the direction of the displacement.

Castigliano's theorems, when formulated as in (1.38), give the differential quotients of the deformation work $\bar{\Psi}^{(k)}$ [$\Psi^{(g)}$, resp.] with respect to the for *components* (the *components* of the nodal displacements, resp.). Here we have appealed to the local differential of the deformation work in eqs. (1.38a) and (1.38b), and from that, we defined derivatives with respect to the *resultant* P_μ of the components of the force that acts on node μ (the *resultant* Δp_μ of the components of the displacement of node μ, resp.). In practice, it is often convenient to go a bit further and combine

([34]) *In the original version of this treatise, two handwritten theorems were given on the back of page 27 that logically belonged here. The handwriting, as well as the imprecise way of expressing them, suggest that they were not due to the principal editor (Prange?). We will then forgo repeating them.*

several of the forces that act on the framework into *groups of forces* and then regard those forces as functions of one parameter. In the total differential of the deformation work, the differentials of the X_λ will then be determined by that parameter, and the transition to the (directional) derivatives with respect to that parameter will yield a certain statement about the displacements of the application points for the force group.

We thus come to the many *corollaries to Castigliano's theorems.* As an example, we take the most important case that relates to frameworks. Two forces P_μ and P_ν might act at two nodes μ and ν, which are both equal in absolute magnitude and act along the line that connects the nodes but point in opposite directions. If we now imagine that the deformation work is expressed as a function of the external forces then if only P_μ and P_ν vary, we will have ([35]):

$$\delta\,\bar{\Psi}^{(k)} = \frac{\partial\bar{\Psi}^{(k)}}{\partial P_\mu}\delta P_\mu + \frac{\partial\bar{\Psi}^{(k)}}{\partial P_\nu}\delta P_\nu = \Delta\pi_\mu\,\delta P_\mu + \Delta\pi_\nu\,\delta P_\nu ,$$

in which π_μ is the displacement of node μ and π_ν is the displacement of node ν, both of which have the direction of the connecting line. If we regard those forces P_μ and P_ν as functions of one parameter P such that we set:

$$P_\mu = P\,, \qquad P_\nu = -P\,,$$

i.e., if we vary the two forces in such a way that the two forces remain equal to each other:

$$\delta P_\mu = -\,\delta P = \delta P\,,$$

then the factor $\Delta\pi_\mu - \Delta\pi_\nu$ that appears on the right in the equation above will represent the change in length $\Delta\pi$ of the connecting line between both nodes. If we refer to the given force-group as a "tension" P between the two nodes then the equation:

$$\delta\,\bar{\Psi}^{(k)} = (\delta\pi_\mu - \Delta\pi_\nu)$$

will show that the derivative of the deformation work $\bar{\Psi}^{(k)}$ with respect to the tension P will be equal to the change in length $\Delta\pi$ of the line that connects the two nodes:

$$\frac{\partial\bar{\Psi}^{(k)}}{\partial P} = \Delta\pi\,. \tag{1.39}$$

The system of coefficients of the quadratic form:

([35]) *In the following equations, we will deviate slightly from the original versions, in that, e.g., we will not write* $d\bar{\Psi}^{(k)}$*, but* $\delta\,\bar{\Psi}^{(k)}$*, i.e., the variation symbol* δ *will be written in place of d.*

$$\bar{\Psi}^{(k)} = \sum_{i,k=1}^{3n-6} c_{ik} \, X_i \, x_k \tag{1.49}$$

that represents the deformation work as a function of the external forces is symmetric:

$$c_{ik} = c_{ki} \,,$$

which agrees with the identity:

$$\frac{\partial^2 \bar{\Psi}^{(k)}}{\partial X_i \, \partial X_k} = \frac{\partial^2 \bar{\Psi}^{(k)}}{\partial X_k \, \partial X_i} \,. \tag{1.41}$$

The bilinear form that belongs to the quadratic form:

$$\bar{\Psi}^{(k)}(X_1, \ldots, X_{3n-6}; Y_1, \ldots, Y_{3n-6}) = \sum_{i=1}^{3n-6} \left(\sum_{k=1}^{3n-6} c_{ik} \, X_k \right) Y_i$$

will then be a symmetric bilinear form. Since $c_{ik} = c_{ki}$, one will have:

$$\sum_{i=1}^{3n-6} \left(\sum_{k=1}^{3n-6} c_{ik} \, Y_k \right) X_i = \sum_{i=1}^{3n-6} \left(\sum_{k=1}^{3n-6} c_{ik} \, X_k \right) Y_i \,. \tag{1.42}$$

If we imagine that $Y_1, \ldots, Y_{3n-6}$ are external forces that act upon the framework then that will express the deformation work in the form:

$$\bar{\Psi}^{(k)}(Y_\lambda) y = \sum_{i,k=1}^{3n-6} c_{ik} \, Y_i \, Y_k \,. \tag{1.40a}$$

If we correspondingly denote the nodal displacements that are produced by those forces Y by $\Delta y_1, \Delta y_2, \ldots, \Delta y_{3n-6}$ then from the first Castigliano theorem, along with the relation:

$$\Delta x_i = \frac{\partial \bar{\Psi}^{(k)}}{\partial X_i} = \sum_{k=1}^{3n-6} c_{ik} \, X_k \,,$$

one will also have:

$$\Delta y_i = \frac{\partial \bar{\Psi}^{(k)}}{\partial Y_i} = \sum_{k=1}^{3n-6} c_{ik} \, Y_k \,.$$

Equation (1.42) will then go to the new equation:

$$\sum_{i=1}^{3n-6} \Delta x_i \, Y_i = \sum_{i=1}^{3n-6} \Delta y_i \, X_i \,. \tag{1.43}$$

That is the analytical expression for the so-called *Maxwell reciprocity theorem:*

If two systems of forces that act upon a framework:

$$X_1, \ldots, X_{3n-6} \qquad (Y_1, \ldots, Y_{3n-6}, \text{resp.})$$

produce the nodal displacements:

$$\Delta x_1, \ldots, \Delta x_{3n-6} \qquad (\Delta y_1, \ldots, \Delta y_{3n-6}, \text{resp.})$$

then the product of the forces in the second system of forces with the displacements that are produced by the first system will equal the product of the forces in the first system with the displacements that are produced by the second system ([36]).

Maxwell's theorem can be given a series of special versions that one often finds expressed in the literature by specializing the two systems of forces. (Moreover, we would come to the same theorem if we had started from the expression for the deformation work in terms of nodal displacements and had applied the second Castigliano theorem.)

If we would like to pursue the analogy between our representation and analytical mechanics further then that representation of the deformation work as a function of the external forces would be the analogue of the varied action that W. R. Hamilton introduced, i.e., the action integral, when its integration path is an extremal:

$$\int_{t_1, q_1^{(1)}, q_1^{(1)}}^{t_2, q_1^{(2)}, q_1^{(2)}} L\left(q_1', q_2', \ldots, q_1, q_2, \ldots, t\right) dt\,.$$

No matter how one chooses precisely that extremal from the manifold of all integration paths that connect two given limit points and prescribes it such that the value of the integral is determined by the two limits, here we will have to choose from all possible values of the S_h that fulfill the equilibrium conditions at the nodes for given external forces X_λ precisely that value that makes the function an extremum, which will the make that extremal value a function of the external forces.

1.6. A different perspective on Menabrea's principle. – For *statically-determinate frameworks,* all ($3n - 6$) undetermined tensions will be determined as linear functions of the external forces by the equilibrium conditions (1.16) alone. In engineering practice, the solution of those ($3n - 6$) linear equations in ($3n - 6$) unknowns can always be easily achieved by well-developed graphical processes, so the representation of the deformation work $\bar{\Psi}$ as a function of

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([36]) As its derivation will show, that reciprocity theorem is based essentially upon the identity (1.41). From the analogy with analytical mechanics that we have repeatedly appealed to, it then seems to be the analogue of a large class of reciprocity theorems that were first obtained by Hamilton by differentiating the "varied action" twice with respect to the same variables, but in different orders.

the external forces X_λ can always be achieved for the statically-determined framework with relative ease. (Of course, one can then no longer speak of an extremum of the deformation work, since the tensions S_h will be determined completely by the equilibrium conditions, so it cannot be varied in any way.) We then drop the index k from $\bar{\Psi}$ accordingly.

In that way, one can imagine separating a statically-determined principal system from a *statically-indeterminate framework with r* rods such that we mentally cut $(r - 3n + 6)$ of its rods in two when they have been chosen suitably ([37]). The tensions in the principal system thus-determined will then keep the values that they had in the original (statically-indeterminate) framework when we imagine that the principal system is loaded with not only the external forces of the original framework on the lips of the cut, but also the *tensions* in the rods that were cut (which are, of course, initially unknown). We now decompose the function $\bar{\Psi}$ for the indeterminate framework:

$$\bar{\Psi} = \sum_{h=1}^{r} \varepsilon_h \, S_h^2 ,$$

i.e., the sum of the deformation works done on the individual rods (which is not, however, the deformation work done on the framework initially), into two summands, one of which refers to the rods of the statically-determinate system, while the other refers to the superfluous rods, so for a suitable numbering of the rods:

$$\bar{\Psi} = \sum_{h=1}^{3n-6} \varepsilon_h \, S_h^2 + \sum_{h=3n-5}^{r} \varepsilon_k \, S_k^2 = \bar{\Psi}^{(1)} + \bar{\Psi}^{(2)} . \tag{1.44}$$

As we just said, it is now possible to express the tensions S_h in the first terms as functions of the external forces and the tensions $- S_{3n-5}, \ldots, - S_r$ that are applied to the associated principal system by only solving the equilibrium equations. We must choose negative signs when we calculate the tensions in the superfluous rods in a statically-indeterminate framework as a result of the external forces. We consider the external forces X_λ to be given as fixed, while the tensions $S_{3n-5}, \ldots, S_r$ are regarded as being capable of varying arbitrarily. That variation is possible with no further analysis as long as we include only the equilibrium conditions in our calculations, i.e., immediately when we restrict our consideration to the statically-determinate principal system such that the two lips of the cut can be displaced arbitrarily with respect to each other. If we imagine the entire statically-indeterminate framework then that will say that a variation of the tensions in the superfluous rods will produce changes in lengths for those rods such that they will no longer fit together in the framework. That will first be the case when we remove or add material to the lips of the cut that we have in mind. That is because for given tensions $S_{3n-5}, \ldots, S_r$, on the one hand, the deformation of the statically-determined principal system will be completely determined, so the distance between two nodes that were originally connected by a superfluous rod will have a well-defined length, while on the other hand, the associated superfluous rod will have a well-

([37]) The statically-indeterminate framework will then be contrasted from the statically-determinate one in a manner that is similar to how multiply-connected surfaces or bodies relate to the simply-connected ones in geometry.

defined length when it is stressed with the given tension that does not, however, coincide with the first distance. We would like to call the removal of material that is necessary to make the frame member fit into the deformed principal system a *distortion* by adapting an expression that V. VOLTERRA first applied to continuous bodies to frameworks.

If we have now represented the first term in the function (1.44) as a function of the external forces and the tensions in the superfluous rods that are calculated from the external forces:

$$\bar{\Psi}^{(1)} = \bar{\Psi}^{(1)}(X_\lambda, -S_{3n-5}, \ldots, -S_r)$$

then we can apply the derived corollary to the first Castigliano theorem to it and obtain the relation:

$$\frac{\partial \bar{\Psi}^{(1)}}{\partial(-S_k)} = \Delta l'_k \qquad \text{or} \qquad \frac{\partial \bar{\Psi}^{(1)}}{\partial(S_k)} = -\,\Delta l'_k \qquad (k = 3n-5, \ldots, r),$$

in which $(-\Delta l'_k)$ is the change in the distance between those two nodes of the principal system that were originally connected by the superfluous rod k as a result of loading the principal system with the original external forces X_λ and the tensions $S_{3n-5}, \ldots, S_r$. The derivation of the second term in (1.44) with respect to S_k will give $\varepsilon_k S_k$, and will be equal to the change in length $\Delta l''_k$ that the superfluous rod will experience when it is stressed by the tension S_k. The derivation of the function $\bar{\Psi}$ in (1.44) itself will then be:

$$\frac{\partial \bar{\Psi}^{(1)}}{\partial(S_k)} = \Delta l''_k - \Delta l'_k\ ,$$

so it will be equal to the change in length that one must give to the rod k in order to make it fit into the principal system (with the prescribed stress). We would like to denote that distortion by λ_k and we will then have the relation:

$$\frac{\partial \bar{\Psi}}{\partial S_k} = \lambda_k\,, \tag{1.45}$$

which we can express in the following theorem:

If we decompose a statically-indeterminate framework into a statically-determinate principal system and a number of superfluous rods and express the tensions in the rods of the statically-determined principal system in terms of the given external forces by means of the equilibrium conditions in the function $\bar{\Psi} = \sum \varepsilon_h S_k^2$ *and the arbitrarily-taken tensions in the superfluous rods then the derivatives of* $\bar{\Psi}$ *with respect to tensions in the superfluous rods will be equal to distortions that must be performed on the superfluous rods in order for the original framework to exhibit a stress state such that the tensions in the superfluous rods would have the prescribed values.*

Now it is easy to ascertain the tensions in the superfluous rods that actually occur in the given framework with the help of that theorem. That is because, in reality, no distortions have been performed, so all of the λ_k must be precisely zero. If we once more express the function $\bar{\Psi}$ as a function of the given external forces and the temporarily-unknown tensions in the superfluous rods by means of the equilibrium conditions then those (r – 3n + 6) unknown tensions must satisfy the (r – 3n + 6) equations:

$$\frac{\partial\bar{\Psi}}{\partial S_k} = 0 \qquad\qquad (k = 3n - 5, \ldots, r). \tag{1.46}$$

If those equations were combined with the (3n – 6) then we would have r equations, in total, which would determine *all* r tensions S_h .

That process suggests itself when one is dealing with a so-called *externally statically-indeterminate framework*, i.e., when only the precise number of (3n – 6) rods are in fact present, but also the previously-given rigidity conditions for the motion of the nodes, such that free elastic deformation of the framework would be prevented.

If μ such conditions are present (in which three conditions fix a point and the demand that it must move along a curve will define two conditions, while the demand that it must move on a surface will require one) then we can proceed with the solution of the stress problem such that we can ignore those superfluous conventions and apply associated external forces to the nodes in question, namely, the statically-indeterminate reactions $P_1, \ldots, P_\mu$. If the deformation work $\bar{\Psi}$ were expressible as a function of the given external forces X_λ and those statically-indeterminate reactions then from the first Castigliano theorem, the derivatives of $\bar{\Psi}$ with respect to the P_k would imply the displacements ρ_k of the associated nodes in the direction of the reactions P_k :

$$\frac{\partial\bar{\Psi}}{\partial P_k} = \rho_k . \tag{1.45a}$$

However, due to the rigidity conditions, all of the ρ_k must drop out, since they are zero, and therefore the desired statically-indeterminate reactions will be determined in such a way that the derivatives drop out:

$$\frac{\partial\bar{\Psi}}{\partial P_k} = 0 . \tag{1.46b}$$

One easily sees how the stress problem for an internally, as well as externally, statically-indeterminate framework can be solved by combining both groups of formulas (1.46) and (1.46a).

Equations (1.46) and (1.46a) say that the functions $\bar{\Psi}$, as functions of the stresses in the superfluous rods (superfluous reactions, resp.) should be an *extremum*. We have therefore once more arrived at a principle that expresses an extremal property for the function $\bar{\Psi}$ of the tensions in the state that actually occurs. Insofar as $\bar{\Psi}$ seems to be a function of the tensions whose extremum is sought here, as it was in the formulation of Menabrea's principle above, *this new* *principle is also referred to as Menabrea's principle* in engineering literature ([38]), although the earlier formulation and this new one are not kept strictly distinct. In fact, both of them are essentially equivalent for frameworks, since then, as now, due to the existence of equilibrium conditions as auxiliary conditions under the variation of the tensions in the rods, that would only be possible if we regarded the tensions in (r – 3n + 6) rods as variable, while the tensions in the remaining (3n – 6) rods are determined by the auxiliary conditions. The single difference is that there we did not demand that those (3n – 6) rods should define a statically-determinate framework by themselves ([39]).

However, if we adapt those two ways of expressing Menabrea's principle to continuous elastic bodies, as we will do on the second chapter, then that will lead to two viewpoints that are intrinsically different. (The greater part of the ambiguity in what Castigliano did seems to come from the fact that he had not given enough consideration to that fact, at least in his representation of it.)

1.7. Proper stresses in statically-indeterminate frameworks. – The arguments up to now were based upon the assumption that an unloaded framework is free of stresses. However, in a statically-indeterminate framework the rods can be stressed even when it is not loaded. That is because in such a framework, as we said, the distances between any two nodes is determined by the lengths of the rods in the chosen statically-determinate principal system. Now, if the superfluous rods do not have precisely the lengths that are thus determined, i.e., distortions are present, with the terminology that was just used, then the entire framework will be shifted into a state of stress by the addition of the superfluous rods, even when it is not loaded by external forces. We would like to call those stresses *proper stresses* ([40]) and the corresponding tensions S_h^* that are present in the unloaded framework, *proper tensions*. They are easy to determine when the distortions λ_{3n-5}, …, λ_r in the superfluous rods are known. That is because we can use the equilibrium conditions for the nodes, which take the form:

$$\sum_{h=1}^{r} S_h^* \frac{\partial f_h}{\partial x_\lambda} = 0 \qquad (\lambda = 1, 2, \ldots, 3n-6) \tag{1.47}$$

([38]) *A handwritten insertion led to an unclear formulation that the editor corrected accordingly.*

([39]) We would also like to pursue the analogy with analytical mechanics for the argument in this paragraph. We find it in the theory of "hidden" coordinates.

When we know a number of integrals of the Euler-Lagrange equations that are produced by the variational problem for the action integral $\int L(q_1', q_2', \ldots, q_1, q_2, \ldots)\,dt$, we can express a corresponding number of the functions $q(t)$ in terms of the remaining ones. Considering the varied action will then give a means at hand for replacing the original variational problem with another one that includes fewer unknown functions in the form of the Routh-Helmholtz transformation, and there are as many fewer as the number of integrals of the Euler-Lagrange equations that one knows.

Analogously, we have the extremum problem for the function $\bar{\Psi}$ as a function of the r tensions S_h. The equilibrium conditions between the tensions correspond to the integrals of the Lagrangian equations, and the tensions in the statically-determinate principal system play the role of hidden coordinates We eliminate them and then have an extremum problem that relates to only the tensions in the superfluous rods.

([40]) The terms *self stresses, initial stresses, and mounting stresses* are also useful.

for the proper stresses, we can express the function:

$$\bar{\Psi}^* = \sum_{h=1}^{r} \varepsilon_h \, S_h^{*2}$$

in terms of only the proper tensions in the superfluous rods $S_{3n-5}^*, \ldots, S_r^*$. From (1.46a), one then has:

$$\frac{\partial \bar{\Psi}^*}{\partial S_k^*} = \lambda_k \qquad (k = 3n - 5, \ldots, r). \tag{1.48}$$

When the λ are given, the proper stresses in the superfluous rods can be found from that, and the equilibrium conditions (1.47) will yield the remaining proper tensions directly.

We can interpret equations (1.48) by saying:

For the proper tensions that enter as a result of given distortions λ_k, the function:

$$F^* = \bar{\Psi}^* - \sum_{k=3n-5}^{r} \lambda_k \, S_k^{*2} \tag{1.49}$$

will be an extremum when it is regarded as a function of the proper stresses.

In so doing, the proper tensions in the rods of the principal system are thought of as being expressed in F^* in terms of the proper tensions in the superfluous rods by means of the equilibrium conditions (1.47). When we substitute the proper tensions that actually occur in the function F^*, we will have:

$$F^* = \tfrac{1}{2} \sum_{k=1}^{r} S_k^* \, \Delta l_h^* - \sum_{k=3n-5}^{r} \lambda_k \, S_k^{*2} .$$

The elongations of the rods for the *rods in the principal system* are expressed in terms of the nodal displacements in the older way by:

$$\Delta l_h^* = \sum_{h=1}^{r} \frac{\partial f_h}{\partial x_\lambda} \Delta x_\lambda^* \qquad (h = 1, 2, \ldots, 3n - 6). \tag{1.50}$$

By contrast, one now has:

$$\Delta l_h^* - \lambda_r = \sum_{\lambda=1}^{r} \frac{\partial f_h}{\partial x_\lambda} \Delta x_\lambda^* \qquad (h = 3n - 5, \ldots, r) \tag{1.50a}$$

for the *superfluous rods*, so one will have:

$$F^* = \tfrac{1}{2} \sum_{\lambda=1}^{3n-6} \Delta x_\lambda^* \left(\sum_{h=1}^{r} S_h^* \frac{\partial f_h}{\partial x_\lambda} \right) - \tfrac{1}{2} \sum_{k=3n-5}^{r} \lambda_k \, S_k^* \ .$$

However, since the equilibrium conditions (1.47) are valid for the proper stresses, the first term will drop out, and one will have:

$$F^* = -\tfrac{1}{2} \sum_{k=3n-5}^{r} \lambda_k S_k^* = \tfrac{1}{2} \sum_{k=3n-5}^{r} \lambda_k (-S_k^*), \tag{1.51}$$

i.e., the work that is done by the proper tensions S_k^* in the superfluous rods that one calculates from the external forces $(-S_k^*)$ that would be required for the distortions λ_k that are present in the framework to appear. Thus, *the function F^* that one defines with actual values of the proper tensions represents the deformation work that is actually performed on the framework in that way.*

An entirely-similar argument will lead to that conclusion when the framework that carries proper stresses as a result of given distortions is loaded with external forces X_λ. We must then express the stresses in the rods of the principal system in terms of the given external forces and the arbitrarily-chosen tensions in the superfluous rods in the function:

$$\bar{\Psi} = \sum_{h=1}^{r} \varepsilon_h S_h^2 .$$

From the theorem (1.45) in the previous section, we will then have:

$$\frac{\partial \bar{\Psi}}{\partial S_k} = \lambda_k \qquad (k = 3n - 5, \dots, r),$$

and since the λ_k are given, we can calculate the tensions in the superfluous rods from those equations, and then the ones in the rods of the principal system from the equilibrium conditions.

We can again give the following version of that result:

If we have expressed the tensions in the rods of the principal system in the function:

$$F = \bar{\Psi} - \sum_{k=3n-5}^{r} \lambda_k S_k \tag{1.52}$$

in terms of the given external forces and the arbitrarily-chosen tensions in the superfluous rods, and then regard F as a function of the latter tensions then F will be an extremum for the values of those tensions that actually occur.

We employ equations (1.49) and (1.50) in order to see the meaning of the function F for the values of the stresses in the rods that actually occur, and when we recall the equilibrium conditions, we will get ([41]):

([41]) *The notation for the equations in what follows deviates from what was originally used, which was flawed.*

$$F = \tfrac{1}{2}\sum_{\lambda=1}^{3n-6} X_\lambda\, \Delta x_\lambda - \tfrac{1}{2}\sum_{k=3n-5}^{r} \lambda_k\, S_k\,. \tag{1.53}$$

Therefore, *the extremal value of F will be the total deformation work that is done under the transformation of the framework into its ultimate state (including the production of proper stresses).*

1.8. Temperature stresses. Extensions to nonlinear laws of deformation. Engesser's work done by extension. – There are no conditions on the lengths of the rods in a statically-determinate framework, so we can change the temperatures in the individual rods arbitrarily without creating a state of stress in the framework. Things are different for a statically-indeterminate framework, since in that case, there are conditions on the lengths of the superfluous rods. If the individual rods in such a thing are free of proper stresses at a certain temperature t_0 then that will not generally be the case at a different temperature t ([42]). All of the rods will experience elongations:

$$\Delta l_h = l_h\, \alpha\, (t_h - t_0) = l_h\, \alpha\, t'_h$$

under that change in temperature as a result of the coefficient of thermal expansion for the material. The superfluous rods in the principal system will not fit together with their new lengths when its rods have been lengthened in that way. Should they be introduced, then distortions will be produced, and proper stresses will be created by them. One can take up the determination of those proper stresses, which are also called *temperature stresses*, using the method in the previous section. That is because since we know all of the new lengths, we can calculate the distortions that would occur. Nonetheless, here we would once more like to pursue the same path as before to the specification of the extremal principle and likewise take the case in which the framework is still loaded with external forces.

The elongation Δl_h in an individual rod is determined from its tension and its temperature change, and indeed in the follow way:

$$\Delta l_h = \varepsilon_h\, S_h + l_h\, \alpha\, t'_h\;. \tag{1.54}$$

We can regard the t'_h in that as given constants when we direct our attention to their dependencies tension S_h , and that is why we can interpret (1.54) as a relation between rod elongations and tensions that does not have the simple form that was used up to now, which made the two proportional to each other. In order to encompass the most general possibility, *we generalize the relation between the rod elongation and the tension by the Ansatz:*

$$\Delta l_h = \varphi_h\, (S_h)\,, \tag{1.55}$$

([42]) The temperatures in the individual rods do not need to coincide with each other.

in which φ_h is a given single-valued function (i.e., a generalization of Hooke's law). Let:

$$S_h = \psi_h\,(\Delta l_h) \tag{1.55a}$$

be the solution of (1.55).

In order to repeat our argument above for these relations (1.55) and (1.55a), we return to the *general Ansatz* (1.15) *of the principle of virtual displacements:*

$$\sum_{\lambda=1}^{3n-6} X_\lambda\,\Delta x_\lambda - \sum_{h=1}^{r} S_h\,\Delta l_h = 0\,. \tag{1.56}$$

From (1.55a), one has:

$$S_h\,\delta\,\Delta l_h = \psi_h\,(\Delta l_h)\;\delta\,\Delta l_h$$

in that, in which the right-hand side is the variation of the integral:

$$\int_0^{\Delta l_h} \psi_h(\overline{\Delta l_h})\,d\overline{\Delta l_h}\;,$$

i.e., the deformation work done on the framework. Therefore, if we now generalize (1.21) and define the deformation work done on the framework by ([43]):

$$\Psi\,(\Delta l_h) = \sum_{h=1}^{r} \int_0^{\Delta l_h} \psi_h(\overline{\Delta l_h})\,d\overline{\Delta l_h} \tag{1.57}$$

then we can regard the Ansatz (1.56) of the principle of virtual displacements as the vanishing of the variation of the function:

$$E = \Phi + \Psi, \tag{1.58}$$

in which Φ has the older meaning (1.22). *That function E would once more be the total potential energy. Its extremum characterizes the equilibrium state.* Therefore, as before, the Δl_h in $\Psi\,(\Delta l_h)$ are thought of as being replaced by the nodal displacements by means of:

$$\Delta l_h = \sum_{\lambda=1}^{3n-6} \frac{\partial f_h}{\partial x_\lambda}\,\Delta x_\lambda\;.$$

However, in the expression E, for the sake of further analysis, we have then considered the differences in the Δx_λ, along with the latter themselves, to be *independent* variables, instead of the Δl_h , which are composed from them directly. Here, we shall again perform the Legendre transformation, so the derivatives of E with respect to the Δl_h will be set equal to new variables. Differentiation will show that the derivatives are also precisely the tensions here, as well:

([43]) *This is a reasonable extrapolation of an unclear handwritten insertion.*

$$\frac{\partial E}{\partial \Delta l_h} = \frac{\partial \Psi}{\partial \Delta l_h} = \psi(\Delta l_h) = S_h . \tag{1.59}$$

As a result, one further has [cf., (1.28a)]:

$$H(S_h, \Delta x_\lambda) = \sum_{\lambda=1}^{3n-6} X_\lambda \Delta x_\lambda - \Psi(\varphi_h(S_h)) + \sum_{h=1}^{r} \varphi_h(S_h) S_h , \tag{1.60}$$

and that will then imply [cf., (1.29)] that:

$$\mathcal{E} = \sum_{h=1}^{r} \left(S_h \sum_{\lambda=1}^{3n-6} \frac{\partial f_h}{\partial x_\lambda} \Delta x_\lambda \right) - \sum_{\lambda=1}^{3n-6} X_\lambda \Delta x_\lambda + \Psi(\varphi_h(S_h)) - \sum_{h=1}^{r} \varphi_h(S_h) S_h . \tag{1.61}$$

The extremum of $\mathcal{E}$, which is regarded as a function of the nodal displacements Δx_λ and the tensions S_h, will yield the equilibrium conditions, as well as the compatibility conditions. On has, in fact:

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial \Delta x_\lambda} &= \sum_{h=1}^{r} S_h \frac{\partial f_h}{\partial x_\lambda} - X_\lambda = 0 , \\ \frac{\partial \mathcal{E}}{\partial S_h} &= \sum_{\lambda=1}^{3n-6} \frac{\partial f_h}{\partial x_\lambda} \Delta x_\lambda + \psi_h(\varphi_h(S_h)) \varphi_h'(S_h) - S_h \varphi_h'(S_h) - \varphi_h(S_h) = 0 . \end{aligned} \tag{1.62}$$

However, from (1.55) and (1.55a):

$$\psi_h(\varphi_h(S_h)) = S_h ,$$

so when we further consider (1.55), the second group of equations will become:

$$\sum_{\lambda=1}^{3n-6} \frac{\partial f_h}{\partial x_\lambda} \Delta x_\lambda = \Delta l_h .$$

They actually represent the compatibility conditions then.

If we now imagine that the equilibrium conditions:

$$\sum_{h=1}^{r} S_h \frac{\partial f_h}{\partial x_\lambda} = X_\lambda$$

have been fulfilled, in order to arrive at *Menabrea's principle* from this general picture, as well, then the first two terms in (1.61) will drop out, and we will get the function to be extremized:

$$F(S_h) = \psi_h(\varphi_h(S_h)) - \sum_{h=1}^{r} S_h\,\varphi_h(S_h)\ , \tag{1.63}$$

while the equilibrium conditions will appear as auxiliary conditions. From the foregoing, the conditions for the extremum are, in fact, the compatibility conditions.

The function F can be further converted. From (1.57), one has:

$$\Psi(\varphi_h(S_h)) = \sum_{h=1}^{r} \int_0^{S_h} \bar{S}_h \cdot \varphi_h'(S_h)\, d\bar{S}_h\ .$$

It follows from that upon partial integration that:

$$\Psi(\varphi_h(S_h)) = \sum_{h=1}^{r} \left[S_h\,\varphi_h(S_h) - \int_0^{S_h} \varphi_h(\bar{S}_h)\, d\bar{S}_h \right]\ .$$

If we further change the sign, which makes no difference for the extremal, then we will have:

$$F(S_h) = \sum_{h=1}^{r} \int_0^{S_h} \varphi_h(\bar{S}_h)\, d\bar{S}_h\ .$$

Engesser found that function along a different route, and he called it the *work done by extension* ([44]). When the law of proportionality is true for the relationship between tension and rod elongations, it will go to the function $\bar{\Psi}(S_h) = \frac{1}{2}\sum_{h=1}^{r} \varepsilon_h\, S_h^2$ that appears in Menabrea's principle above (§ **5**).

In order to actually exhibit the conditions for an extremum, as well, we consider the auxiliary conditions with the help of the Lagrangian factors. We will then have:

$$\sum_{h=1}^{r} \int_0^{S_h} \varphi_h(\bar{S}_h)\, d\bar{S}_h + \sum_{\lambda=1}^{3n-6} \left[\mu_\lambda \left(\sum_{h=1}^{r} S_h \frac{\partial f_h}{\partial x_\lambda} - X_\lambda \right) \right]$$

as the function of S_h to be extremized, which will, in fact, imply the compatibility conditions:

$$\varphi_h(S_h) + \sum_{\lambda=1}^{3n-6} \left(\mu_\lambda \frac{\partial f_h}{\partial x_\lambda} \right) = 0\ ,$$

([44]) From (1.63), F is the *difference* between the actual deformation work $\Psi(\varphi_h(S_h)) = \sum_{h=1}^{r} \int S_h\, d\Delta l_k$ and the *virtual* deformation work $\sum_{h=1}^{r} S_h\, \Delta l_k$, and therefore it is referred to as the work done by extension.

and the Lagrangian factors μ_λ will again be seen to be the negative values of the nodal displacements.

As we can easily show, the *first of the Castigliano theorems* (§ **5**) *is also true for this function* F. We will know the tensions in all rods as functions of the external forces by solving the stress problem.

$$S_h = S_h\,(X_1, \ldots, X_{3n-6})\,. \tag{1.64}$$

If we replace the S_h in (1.63) with those functions then F will assume the extremal value, and we will have expressed it as a function of the external forces.

Along with that, we shall now exhibit a different expression for the extremal value of $F\,(X_1, \ldots, X_{3n-6})$. Namely, we know from the compatibility conditions that the r rod elongations Δl_h must be expressions in terms of the $(3n-6)$ nodal displacements Δx_λ . If the deformation of the framework has been established then the nodal displacements will be known as functions of the external forces:

$$\Delta x_\lambda = \chi_\lambda\,(X_1, \ldots, X_{3n-6})\,,$$

and we will have the following expressions for the rod elongations as functions of the external forces:

$$\Delta l_h = \sum_{\lambda=1}^{3n-6} \frac{\partial f_h}{\partial x_\lambda}\,\chi_\lambda\,(X_1, \ldots, X_{3n-6})\,,$$

and so, from (1.55):

$$\varphi_h\,(S_h) = \sum_{\lambda=1}^{3n-6} \frac{\partial f_h}{\partial x_\lambda}\,\chi_\lambda\,(X_1, \ldots, X_{3n-6})\,.$$

Thus, the differential will be:

$$\begin{aligned} \varphi_h\,(S_h)\,dS_h &= \sum_{\mu=1}^{3n-6}\left[\left(\sum_{\lambda=1}^{3n-6} \frac{\partial f_h}{\partial x_\lambda}\,\chi_\lambda\right)\frac{\partial S_h}{\partial X_\mu}\,dX_\mu\right] \\ &= \sum_{\mu=1}^{3n-6}\left[dX_\mu\left(\sum_{\lambda=1}^{3n-6}\frac{\partial}{\partial X_\mu}\left(S_h\,\frac{\partial f_h}{\partial x_\lambda}\right)\chi_\lambda\right)\right], \end{aligned}$$

and therefore, the expression F itself will be:

$$\begin{aligned} F^{(k)} &= \sum_{h=1}^{r}\int_0^{S_h} \varphi_h\,(\bar{S}_h)\,d\bar{S}_h \\ &= \sum_{\mu=1}^{3n-6}\left\{\int_0^{\chi_\mu} d\bar{X}_\mu\left[\sum_{\lambda=1}^{3n-6}\chi_\lambda\,\frac{\partial}{\partial X_\mu}\left(\sum_{h=1}^{r}\bar{S}_h\,\frac{\partial f_h}{\partial x_\lambda}\right)\right]\right\}, \end{aligned}$$

in which:

$$\bar{S}_h = S_h\,(\bar{X}_1, \ldots, \bar{X}_\mu)\,.$$

However, from the equilibrium condition, one has:

$$\sum_{h=1}^{r} S_h \frac{\partial f_h}{\partial x_\lambda} = X_\lambda \,,$$

and since the X_λ are independent variables, it will follow from this that:

$$\frac{\partial}{\partial X_\mu}\left(\sum_{h=1}^{r} S_h \frac{\partial f_h}{\partial x_\lambda}\right) = \begin{cases} 0 & \lambda \neq \mu \\ 1 & \lambda = \mu. \end{cases}$$

That will finally give:

$$F^{(k)} = \sum_{h=1}^{r} \int_0^{X_\mu} \chi_\mu \; d\bar{X}_\mu \,,$$

and it will then follow from this that:

$$\frac{\partial F^{(k)}}{\partial X_\mu} = \chi_\mu \,, \tag{1.65}$$

i.e., *the derivative of the extremal value of the work done by extension* $F^{(k)}$ *with respect to one of the external forces will give the associated nodal displacement.*

By contrast, the *second Castigliano theorem* (which has, however, less significance for practical purposes) *does not refer to the work done by extension, but to the deformation work itself:*

$$\Psi = \sum_{h=1}^{r} \int_0^{\Delta l_h} \bar{S}_h \; d\,\overline{\Delta l}_h \,.$$

If we introduce the nodal displacements here then we will have:

$$S_h = S\,(\Delta x_1, \ldots, \Delta x_{3n-6}) = \psi_h\left(\sum_{\lambda=1}^{3n-6} \frac{\partial f_h}{\partial x_\lambda}\,\Delta x_\lambda\right) ,$$

and

$$S_h \, d\,\Delta l_h = \sum_{\lambda=1}^{3n-6} S_h \frac{\partial f_h}{\partial x_\lambda}\, d(\Delta x_\lambda) \,,$$

so:

$$\Psi^{(g)}\,(\Delta x_\lambda) = \int_{\lambda=1}^{\Delta x_\lambda} \left[\sum_{\lambda=1}^{3n-6}\left(\sum_{h=1}^{3n-6} S_h \frac{\partial f_h}{\partial x_\lambda}\right)\right] d\,\overline{\Delta x_\lambda} \,,$$

and as a result ([45]):

([45]) These two theorems are in A. HERTWIG, Zeitschrift für Architekten und Ingenieurwesen **52** (1906), pp. 509.

$$\frac{\partial \Psi^{(g)}}{\partial \Delta x_\lambda} = \sum_{h=1}^{r} S_h \frac{\partial f_h}{\partial x_\lambda} = X_\lambda \, . \tag{1.66}$$

In just the same way as before, we will find the following *corollary* to the first Castigliano theorem:

The derivative of the work done by extension $F^{(k)}$ *with respect to one of the tensions that appear as a result of the external forces is equal to the change in distance between the two points of application of the tension.*

We will also find *the second conception of Menabrea's principle* here again when we replace the deformation work with the extension work. In order to show that, we again imagine that we have chosen a statically-determinate principal system from the statically-indeterminate framework. Here, as well, the equilibrium conditions will imply the tensions in the rods of the principal system as linear functions of the external forces and the tensions in the superfluous rods. If we then decompose the function F into two summands:

$$F\,(S_h) = \sum_{h=1}^{3n-6} \int_0^{\bar{S}_h} \varphi_h(\bar{S}_h)\, d\bar{S}_h + \sum_{k=3n-6}^{r} \int_0^{S_k} \varphi_k(\bar{S}_k)\, d\bar{S}_k = F_1 + F_2$$

and replace the tensions in the rods of the principal system in the first term F_1 with their expressions in terms of the external forces and the tensions S_{3n-5}, …, S_r in the superfluous rods then $\partial F_1 / \partial S_k$ will be equal to the change in the distance between the two nodes of the principal system that were connected by the k^{th} superfluous rod in the old framework. On the other hand, the derivative of the second term with respect to S_k will be equal to the change in length $\varphi_k\,(S_k)$ of the k^{th} rod. Therefore, we also have the theorem here:

The derivative of F with respect to the tension in one of the superfluous rods (which enters into F in two ways) *is equal to the distortion that must performed on that rod in order for it to fit into the deformed framework with the choice of tensions in the superfluous rods that was made.*

Now, those distortions must be zero for the tensions that actually occur. Thus, the following equations will exist for them:

$$\frac{\partial F}{\partial S_k} = 0 \qquad\qquad (k = 3n - 5, \ldots, r), \tag{1.67}$$

i.e., *when the extension work is regarded as a function of the tensions in the superfluous rods, it must be an extremum.*

Now, in order to return to the *influence of temperature* on the stress problem, we need only observe that from (1.54) and (1.55), we will have:

$$\varphi_h\,(S_h) = l_h\,\alpha_h\,t'_h + \varepsilon_h\,S_h$$

in this case. Therefore, the function F will be:

$$F = \sum_{h=1}^{r} \int_0^{S_h} (l_h\, \alpha_h\, t'_h + \varepsilon_h\, \bar{S}_h)\, d\bar{S}_h$$

here, or when we perform the integration:

$$F = \tfrac{1}{2}\sum_{h=1}^{r} \varepsilon_h\, S_h^2 + \sum_{h=1}^{r} l_h\, \alpha_h\, t'_h\, S_h\ .$$

That expression, which was first exhibited by MELAN ([46]), and which MÜLLER-BRESLAU ([47]) had called the *ideal deformation work*, will be the foundation for all investigations into deformation work as soon as the influence of temperature comes into question.

1.9. The historical development. – The statics of rigid bodies suffices for the treatment of the simplest problems that are posed in engineering practice. Nonetheless, problems can also appear directly (such as, perhaps, the determination of the effects of the supports on a beam when there are more than two of them or their effects on the vertical supports of a horizontal plate when there are more than three) in whose solution the statics of rigid bodies will break down. Even though engineering theory has long sought to preserve the picture of the *rigid body* for the materials that they employ, nowadays that sort of problem in statics is addressed in such a way that one replaces the picture of the rigid body with that of an *elastic solid body*. The fact that the introduction of that new concept has happened relatively slowly is probably based in the fact that, on the one hand, the most important building materials of the current era, such as wood and stone, require exceptionally delicate tools for the investigation of their elastic deformation. On the other hand, one can also master the elastic deformation theoretically only in simple cases, like the stretching of a wire, the bending of rod, and even then, only in such a way that one poses special Ansätze in the individual cases, while the general study of the elastic strain in a three-dimensional body, in particular the determination of the stress distribution, was first achieved by NAVIER (1821) and CAUCHY (1821/23).

Even though one then seeks to preserve the picture of the rigid body, naturally one immediately sees that ordinary statics cannot suffice for the treatment of such problems. That is because according to its laws, to once more appeal to the problem of the plate with more than three supports, infinitely-many distributions of pressures are equally possible where the plate acts upon the individual supports, while observation teaches that a well-defined pressure distribution will be produced. One then seeks to remedy that by adding special hypotheses in the individual "statically-indeterminate" problem in each case that would remove the indeterminacy. For the treatment of

([46]) J. MELAN. *The original text did not include any further citation for this. The citations [76, 77] in the Bibliography were partially taken from Orovas [106] and partially a result of my own research.*

([47]) H. MÜLLER-BRESLAU, *Die neueren Methoden der Festigkeitslehre*, Leipzig, 1886, pp. 56.

the plate with more than three supports, EULER ([48]) introduced an assumption that has remained the accepted one in other cases for a long time now, even though it was also contradicted by various arguments ([49]).

That necessarily led to the attempt to combine those individual assumptions into a single unified principle. It was also approached from different angles. In engineering, the *principle of least resistance* of the English engineer MOSELEY has enjoyed a certain reputation for a long time that it could solve all of the problems that remained indeterminate in the statics of rigid bodies such as, e.g., the case in which the sliding of two rigid bodies over each other is impeded by friction ([50]). Of course, the analytical Ansätze and calculations that MOSELEY connected with his principle are completely senseless, as one of his rivals, EARNSHAW, immediately pointed out. The "principle" probably owes its reputation to the fact that for many practical purposes, it can be regarded, not as a "principle," but probably as a *construction rule*. Thus, e.g., in the aforementioned case of contact with friction between two bodies, it will say that one can completely exploit the influence of friction for the determination of friction. One then clearly sees the guiding idea behind MOSELEY's formulation of his principle: In his view of things, the statically-indeterminate pressures must have values such that the system will be in a state on the boundary between motion and rest. Since that made good sense in his study of the effects of friction, one has, for some time, preferred to orient the distribution of internal forces on the basis of the principle of least resistance, e.g., in the theory of arches. That situation gave rise to SCHEFFLER's ([51]) attempt to breathe new life into that principle by giving a different foundation.

One can even adopt the viewpoint (if only with some difficulty) that all bodies should be regarded as elastically yielding. Thus, e.g., since 1853, FAGNOLI ([52]), who was unaware of the work of MOSELEY and SCHEFFLER, sought to exhibit a principle that would be similar to the principle of least resistance (even though he also restricted himself to the problem of the plate with more than three supports in its implementation) in such a way that the problem would belong to the theory of elasticity, despite the fact that the reference to POISSON was probably known to him.

Another attempt to combine the hypotheses that were made in the individual statically-indeterminate problems into a general principle goes back to the considerations of VÉNE ([53]). His starting point was similar to that of MOSELEY, namely, the opinion that the statically-indeterminate reactions that were applied to a rigid body, such as the support reactions on the plate, do not have the same character as the external forces that are applied with no further analysis. The ordinary statics of rigid bodies, with its equilibrium conditions, must be extended by some principle to a "dynamique latente" if one is to calculate those reactions, in a manner that is similar to how one connects dynamics with statics using d'Alembert's principle. That suggests that such a new principle can be expressed by an extremum requirement but leaves the question of what sort of function would be extremized unanswered.

([48]) L. EULER, Novi Comment., Acad. Petroplit. T.**18** (17).

([49]) One can find a summary of the relevant literature in G. FAGNOLI, Memorie della Accademia di Bologna **4** (1853).

([50]) MOSELEY, Philosophical Magazine.

([51]) H. SCHEFFLER, Crelle's Journal für die Baukunst. (1867), pp. 31.

([52]) G. FAGNOLI, Memorie di Bologna **4** (1853).

([53]) For this, one is directed to SERVOIS in the Bulletin de Férrussac **9** (1828).

That was the conclusion of a paper by COURNOT ([54]), who wished to give such a function. Indeed, it did not solve the problem, since its conception and implementation were also flawed, but it was still of great importance due to the fact that MENABREA in particular referred to it, and it was also the source of the flaws in MENABREA's arguments.

Cournot considered a rigid body that was loaded with a number of forces F, F', F'', … and fixed at a number of points, and he then imagined that the reactions are replaced with external forces P, P', P'', … He applied the principle of virtual velocities to that freely-moving rigid body and obtained the relation:

$$F\,\delta f + F'\,\delta f' + \cdots - P\,\delta p - P'\,\delta p' - \cdots = 0\,, \tag{1.68}$$

in which $\delta f, \delta f', \delta f''$, … ($\delta p, \delta p', \delta p''$, …, respective) are the projections of the virtual displacements onto the points of application of the forces F, and the P mean then associated directions of those forces.

So far, everything is in order with his argument when one chooses the virtual displacements in such a way that they will perform a rigid motion. However, since a rigid body in motion possesses only six degrees of freedom, one can obtain nothing but the six equilibrium conditions for a rigid body from that relation. Cournot wished to fix the fixed points under the displacement, so he set $\delta p = \delta p' = \ldots = 0$, and what remained was:

$$F\,\delta f + F'\,\delta f' + \cdots = 0\,. \tag{1.69}$$

However, in so doing, he overlooked the fact that the δf in (1.69) can now be only the displacements that belong to a *rigid* body, and under which the fixed points are not displaced. However, as soon as more than six fixed points are present, the body will be immobile. i.e., all δf will be zero, and equation (1.69) will be trivial. If less than six fixed points are present then that will lead to only those of the six equilibrium conditions that are still required by the constraint that exists to impede the motion.

By contrast, Cournot was of the opinion that the displacements in equation (1.68) were of a general nature, and in particular, made the mistake of identifying the displacement in (1.69) with the one in (1.68) directly. He virtually asserted that equation (1.68) split in equations (1.69) and:

$$P\,\delta p + P'\,\delta p' + \cdots = 0\,, \tag{1.70}$$

while preserving the displacements in (1.68) ([55]).

Now in order to utilize (1.70), he made the assumption that the unknown reactions P, P', … are proportional to the paths p, p', …, resp. that would be laid through the fixed points "at the first moment" when the links through the fixed points are broken. As one can see, that hypothesis is

([54]) A. COURNOT, Bulletin de Férrussac **9** (1828), pp. 19.

([55]) It would seem that this mistake was created by a misunderstanding in the interpretation of a correct argument of an entirely-different nature that was in POISSON's *Mécanique*. Confer S. D. POISSON, *Traité de mécanique*, 2nd ed., t. I, pps. 669 and 671.

nothing but a neat way of introducing elasticity into the body, although admittedly it also seems to initially take the form of only an analogy with the corresponding displacements of an actual elastic body. On the basis of that assumption, upon assuming that the proportionality factor was the same for all points, he went from equation (1.70) to:

$$p\,\delta p + p'\,\delta p' + \cdots = 0 \tag{1.71a}$$

or

$$P\,\delta P + P'\,\delta P' + \cdots = 0\,, \tag{1.71b}$$

resp., but for which the sum of the squares:

$$p^2 + p'^2 + \cdots \tag{1.72a}$$

or

$$P^2 + P'^2 + \cdots, \tag{1.72b}$$

resp., will be an extremum, which made the equilibrium conditions take the form of auxiliary conditions. The fact that he had replaced the picture of a rigid body with that of an elastic solid body with his argument (if only unintentionally) is shown by the fact that he essentially based his argument upon the fact that it was also applicable to elastic bodies. If m, m', … are the elastic coefficients that belong to the individual bodies then equation (1.70) will go to:

$$\frac{p}{m}\,\delta p + \frac{p'}{m}\,\delta p' + \cdots = 0\,. \tag{1.72}$$

It must be true independently of the special magnitudes of the elastic coefficients, but also in the limit when the elastic coefficients vanish, i.e., the elastic body becomes a rigid body. Of course, in so doing, he ignored the fact that this passage to the limit lacked any precise meaning, as CASTIGLIANO had already pointed out.

Cournot's investigations were taken up by DORNA again ([56]). However, he gave up the idea of the absolutely-rigid body and imagined that it was "elastic in the neighborhood of the points of application of the statically-indeterminate reactions." The body might be, say, fixed at a point and connected to a short elastic rod at that point that articulates with the fixed point and the body. He intended that the lengths and cross-sections of those fictitious rods could be chosen to be the same for all of the fixed points, while their elastic coefficients would be equal to the elastic coefficients of the material in the body "in the neighborhood" of the fixed point. He also started from the principle of the virtual displacements:

$$\sum P\,\delta p + \sum Q\,\delta q = 0 \tag{1.73}$$

in which P were the given forces and Q were the undetermined reactions, and he wished to split that equation into the two equations:

([56]) A. DORNA, Memorie della R. Accademia di Torino (2) **18** (1859), pp. 281.

$$\sum P\,\delta p = 0\,, \qquad \sum Q\,\delta q = 0\,, \tag{1.74}$$

but that served as the grounds for a consideration of orders of magnitude. He imagined that the δq were second-order infinitesimals since the points of application of the reactions were virtually fixed, and that equation (1.69) would next be true, and therefore (1.70), as well, due to (1.68).

Upon appealing to Hooke's law, that further made $Q = E \cdot q$, and he thus found that one must have:

$$\sum \frac{Q\,\delta Q}{E} = 0 \tag{1.75}$$

when the equilibrium conditions of the rigid body are auxiliary conditions, i.e., one must have $\sum Q^2 / E$ = extremum under just those auxiliary conditions.

That change of viewpoint was essentially a break from the picture of the rigid body, and in its place one found the picture of the elastic body. In the following development of the theory, bodies will always be regarded as *elastic*, even when static indeterminacy is present. One first finds that clearly expressed in MENABREA ([57]). He gave the problem statement *the precise phrasing that one must determine the tensions in a statically-indeterminate framework under the assumption of elastic rods.*

Like Dorna and Cournot, he likewise started from the principle of virtual displacements, whose Ansatz he wrote in the form:

$$\sum_{\lambda=1}^{3n-6} X_\lambda\,\delta\,\Delta x_\lambda - \sum_{h=1}^{r} S_h\,\delta\,\Delta l_h = 0 \tag{1.76}$$

for frameworks and made an attempt to show that the order of magnitude of the first term was of second order compared to the second one in the case of a statically-indeterminate system by an argument that was naturally completely unsuccessful. He then concluded that one must have:

$$\sum_{h=1}^{r} S_h\,\delta\,\Delta l_h = 0$$

for it or

$$\sum_{h=1}^{r} \frac{1}{\varepsilon_h} S_h\,\delta S_h = 0\,, \quad \text{or} \quad \sum_{h=1}^{r} \varepsilon_h\,\Delta l_h\,\delta\,\Delta l_h = 0\,, \quad \text{resp.} \tag{1.77}$$

That is equivalent to saying that one must have:

([57]) L. P. MENABREA, Comptes rendus **46** (1858), pp. 1056.
Idem., Memorie della R. Accademia di Torino (2) **25** (1871), pp. 141.
Idem., Atti della R. Accademia dei Lincei (2) **2** (1873-75), pp. 201.
Idem., Comptes rendus **98** (1854).

$$\sum_{h=1}^{r} \frac{1}{\varepsilon_h} S_h^2 = \text{extremum}, \qquad \sum_{h=1}^{r} \varepsilon_h (\Delta l_h)^2 = \text{extremum}, \qquad \text{resp.,} \tag{1.78}$$

in which the equilibrium conditions for the nodes must be considered to be auxiliary conditions.

Indeed, he then asserted that this extremum requirement would lead to the supplementary equations (viz., compatibility conditions) that are required by the geometric conditions, but he proved that assertion only to the extent that he verified in some isolated examples that his process would produce the same solution as exhibiting those geometric conditions directly in those cases, which is easy.

His contemporaries raised objections to that process, and it seemed as if Menabrea himself had sensed that flaw in his argument, because he gave yet another proof of equation (1.10), namely, his *elasticity principle*, but it was also flawed. In it, he formed the variations of the $(3n - 6)$ equilibrium conditions that existed at each isolated node:

$$\sum_{h=1}^{r} \delta S_h \frac{\partial f_h}{\partial x_\lambda} = 0 . \tag{1.79}$$

He then replaced the variations of the tensions in it with those of the rod elongations:

$$\delta S_h = \varepsilon_h \, \delta \Delta l_h ,$$

but he neglected to notice that the rods could not be combined into the framework with their new lengths. He then multiplied the individual equations by the virtual nodal displacements Δx_λ, and after adding all equations, he got:

$$\sum_{\lambda=1}^{3n-6} (\Delta x_\lambda) \cdot \sum_{h=1}^{r} \delta \Delta l_h \frac{\partial f_h}{\partial x_\lambda} = 0 ,$$

or since one has:

$$\sum_{\lambda=1}^{3n-6} \Delta x_\lambda \frac{\partial f_h}{\partial x_\lambda} = \Delta l_h ,$$

he also got:

$$\sum_{h=1}^{r} \varepsilon_h \Delta l_h \, \delta \Delta l_h = 0 , \tag{1.80}$$

and finally, upon reintroducing the tensions:

$$\sum_{h=1}^{r} \frac{1}{\varepsilon_h} S_h \, \delta S_h = 0 . \tag{1.81}$$

One can thank CASTIGLIANO ([58]) for *the first clear foundation of Menabrea's principle*. He first showed in his dissertation that the conditions for the extremum of the function $\sum_{h=1}^{r} \varepsilon_h S_h^2$, with the equilibrium conditions as the auxiliary conditions, would coincide with the compatibility conditions. He used the method of Lagrange factors in his proof and remarked that they would agree with the components of the displacements of the nodes, as we showed on pp. 20. From that, he soon found the first of his two theorems on the derivatives of the actual deformation work with respect to the external forces, and only later did he then pose his second theorem on the derivatives with respect to the components of the displacement, which is less important in practice. On the basis of that theorem, he then arrived at the conception of Menabrea's principle that we referred to as the second one and presented in § **6**. That conception was always referred to as "the" Menabrea principle by him and his followers. Moreover, he also brought proper stresses and distortions under consideration and also recognized that Maxwell's reciprocity theorem could be obtained from his first theorem in the manner of § **5**.

We mention in passing that in a polemic against Menabrea, V. CERUTTI ([59]) had the idea of arriving at Menabrea's elasticity equation by reducing it to the general law of the principle of virtual displacements, as we introduced it in § **1**:

$$\sum_{\lambda=1}^{3n-6} X_\lambda \, \delta \Delta x_\lambda - \sum_{h=1}^{r} \int_0^{l_h} \sigma \frac{d\,\delta\,\Delta x}{dx} dx \; .$$

In so doing, he wished to arrange the variations such that the external forces would do no work, so:

$$\sum_{h=1}^{r} \int_0^{l_h} \sigma \frac{d\,\delta\,\Delta x}{dx} dx = 0 \; .$$

He overlooked the fact that under partial integration of that integral:

$$\int_0^{l_h} \sigma(x) \; \frac{d\,\delta\,\Delta x}{dx} dx = \left[\sigma(x)\,\delta\,\Delta x\right]_0^{l_h} - \int_0^{l_h} \frac{d\sigma}{dx} \delta\,\Delta x\,dx \, ,$$

the boundary term will vanish, since indeed the endpoints of the rod can experience no displacements, such that the foregoing equation would imply only the result that:

$$\frac{d\,\delta\,\Delta x}{dx} = 0$$

([58]) A. CASTIGLIANO, *Dissertazione di Laurea*, Turin, 1873.
Idem., Atti della R. Accademia di Torino **10** (1874/75), pp. 380 and **11** (1875/70), pp. 127.
Idem., Atti della R. Accademia di Torino **17** (1881/82), pp. 705.
Those treatises, with the exception of the last one, are collected in the book A. CASTIGLIANO, *Théorie de l'équilibre des systèmes élastiques*, Turin 1879, German translation (parts of which are flawed) by Hauff.

([59]) V. CERUTTI, Atti della R. Accademia dei Lincei (2) **2** (1873/75), pp. 570.

for each rod.

In the face of such ever-recurring misunderstandings, DONATI ([60]) then gave a clarification of the foregoing relationships in three treatises that generally referred to essentially the same theory for continuous bodies, which we will then discuss in the second chapter. He strongly emphasized that with the Ansatz of the principle of virtual displacements, one could deal with only an extremum of the total potential energy and characterized the neighboring states to the extremum, in that case, as well as for Menabrea's principle, along with pointing out its mechanical meaning.

For the calculation of statically-indeterminate frameworks (to the extent that its problems require such a thing to begin with), German engineering has been helped along by the fact that it has been able to appeal to the geometric conditions for frameworks, i.e., the relations that must exist between the lengths of the rods. MOHR simplified that problem greatly and nurtured it by his use of the principle of virtual displacements. Nonetheless, he was probably inclined to formulate calculations for the statically-indeterminate system as an extremum problem using Moseley's principle of least resistance, which was well-known in Germany, despite it flaws. That was shown by an example in a treatise by WINKLER ([61]), in which he indeed treated the statically-indeterminate problem by the method that Mohr gave but gave an extremum property for the corresponding problem involving full-wall beams.

Somewhat later, FRÄNKEL ([62]) discovered the principle of the extremum of the deformation work, and completely independently of the Italians. His derivation is correct because he started from the fact that variations δS_h of the tensions define a system in equilibrium by themselves. He then employed the principle of virtual displacements while using the actual nodal displacements as the virtual displacements. He then obtained:

$$\sum_{\lambda=1}^{3n-6}\left[\left(\sum_{h=1}^{r}\delta S_h\,\frac{\partial f_h}{\partial x_\lambda}\right)\Delta x_\lambda\right] = \sum_{h=1}^{r}\delta S_h\left(\sum_{\lambda=1}^{3n-6}\frac{\partial f_h}{\partial x_\lambda}\,\Delta x_\lambda\right) = \sum_{h=1}^{r}\Delta l_h\,\delta S_h = \sum_{h=1}^{r}\varepsilon_h\,S_h\,\delta S_h = 0\,.$$

MÜLLER-BRESLAU ([63]) then drew the attention of the German engineers to CASTIGLIANO's book and treated a series of important engineering problems, on the one hand, by the method of the principle of virtual displacements that was founded by Mohr, and on the other hand, by appealing to Menabrea's principle (in its second conception) and showed that one could arrive at the long-known solutions to those problems in many ways.

MOHR ([64]) responded to that article in the same volume by rejecting Menabrea's principle, since the use of the principle of virtual displacements would lead to a complete resolution of all questions that might be posed for statically-indeterminate frameworks in a simpler and more direct

([60]) L. DONATI, Memorie della R. Accademia e dell'Istituto di Bologna (4) **9** (1886), pp. 345; *ibid.* (4) **10** (1889), pp. 267 and *ibid.* (5) **4** (1894), pp. 449.

([61]) H. WINKLER, Zeitschrift für Architekten und Ingenieurvereins zu Hannover, Neue Folge **25** (1879), pp. 199.

([62]) C. FRÄNKEL, Zeitschrift für Architekten und Ingenieurvereins zu Hannover, Neue Folge **28** (1882), pp. 63.

([63]) MÜLLER-BRESLAU, Wochenblatt für Architekten- und Ingenieur, **5**, pp. 87.

([64]) O. MOHR, Wochenblatt für Architekten- und Ingenieur, **5**, pp. 171.

way ([65]). Moreover, there and in later works ([66]), he dealt with the essence of Menabrea's principle, and in particular, the type of variation that would make things very precise. He emphatically stressed that a variation of the stress state can occur only if distortions are present in the rods, and that one must also include the proper stresses in the framework in one's considerations. We wished that only the distortions should actually be varied under that variation. The fact that can regard the tensions that they produce to be independent variables in the question of the extremum, instead of those distortions, requires a special proof ([67]). MÜLLER-BRESLAU replied that the Menabrea-Castigliano arguments could also provide the answers to all questions that might be posed in the calculations with a statically-indeterminate framework. In particular, Mohr's objection that one can arrive at the necessary representation of the deformation work in terms of the external forces directly from the equilibrium conditions only in a statically-indeterminate framework is quite correct, but it does not seem conclusive since having a solution to the stress problem is a prerequisite for tackling the deformation problem. MÜLLER-BRESLAU further stressed that he, with Castigliano, used the second conception of Menabrea's principle and regarded only the tensions in the superfluous rods as variable. Finally, as far as proper stresses (the influence of temperature, resp.) is concerned, to him it was possible to encompass all of that by introducing new expressions for the deformation work in place of the original one. Against the criticism of the "extensibility" of the concept of deformation work, it should be said that, as we have shown, those various expressions for it are implied naturally by a unifying idea, namely, the canonical transformation ([68]).

As we saw, the expression for the *extension work* that ENGESSER introduced follows from that. It is the natural generalization of the deformation work when one wishes to extend Menabrea's principle to nonlinear deformation laws. The Ansätze of WEYRAUCH ([69]) and KRIEMLER ([70]) were next applied to it. By contrast, the *extended deformation work* ([71]) that ENGESSER introduced later really had more of a formal analogy to it.

However, WEINGARTEN ([72]) obtained that same expression by the following line of reasoning:

When temperature variations are present, the rod elongations will be caused by the tensions that are created by external forces S_h and the temperature variations t'_h according to the formula (1.54):

$$\Delta l_h = \varepsilon_h S_h + l_h \alpha t'_h .$$

([65]) That question of preference is ignored in our presentation completely, since only an explanation of the theoretical aspects is intended here.

([66]) O. MOHR, Civilingenieur **31** (1885), pp. 289.

([67]) In MOHR's presentation, that parallel to our previous considerations did not emerge clearly, since he did not introduce distortions as we did by removing (adding, resp.) material, but in a more concrete way by changing the temperature of the rods. He then spoke of temperature variations, instead of distortions.

([68]) The entire bibliography of that controversy is summarized in the *Encyklopädie der mathematischen Wissenschaften*, Bd. IV, article 29a (M. GRÜNING).

([69]) J. WEYRAUCH, *Theorie elastischer Körper*, Leipzig, 1884.

([70]) Zeitschrift für Architekten und Ingenieurwesen, (1905), pp. 311 (also separately).

([71]) ENGESSER, Zentralblatt der Bauverwaltung (1907), pp. 606.

([72]) J. WEINGARTEN, Zeitschrift für Architekten und Ingenieurwesen, 53, Neue Folge **12** (1907), pp. 453.

When no temperature variations are present, one can imagine that those rod elongations are caused by *fictitious tensions:*

$$S'_h = S_h + \frac{l_h\,\alpha}{\varepsilon_h} t'_h \ ,$$

which can be produced by external forces X'_λ in their own right that can be calculated by means of the equilibrium conditions ([73]).

The study of frameworks that are loaded with external forces X_h and the study of the temperature variations t'_h that the individual rods experience will then coincide with the study of frameworks that are loaded with fictitious external forces X'_λ. All of the arguments that we made when elastic deformation at *constant* temperature were present will remain valid in the case where temperature variations appear as long as we just introduce fictitious external forces and the fictitious tensions that belong to them everywhere. The expression for the deformation work that belongs to those fictitious tensions will be:

$$\begin{aligned}\bar{\Psi}_f &= \tfrac{1}{2}\sum_{h=1}^{r} \varepsilon_h\, S_h'^{\,2} = \tfrac{1}{2}\sum_{h=1}^{r} \varepsilon_h \left(S_h + \frac{l_h\,\alpha}{\varepsilon_h}\cdot t'_h\right)^2 \\ &= \tfrac{1}{2}\sum_{h=1}^{r} \varepsilon_h\, S'_h + \sum_{h=1}^{r} S_h\, l_h\, \alpha\, t'_h + \tfrac{1}{2}\sum_{h=1}^{r} \frac{1}{\varepsilon_h}(l_h\,\alpha\, t'_h)^2 \ .\end{aligned}$$

That expression for the fictitious deformation work differs from MÜLLER-BRESLAU’s ideal deformation work by the inclusion of the last term, which is constant since the temperature variations must be given when one treats the stress problem. It is then plausible that WEINGARTEN also found that expression for Menabrea’s principle since the additive constant certainly drops out under differentiation.

However, he encountered some contradictions from the engineering side of things ([74]) since he wanted to regard $\bar{\Psi}_f$ as the *actual* deformation work as soon as temperature variations came into play. That is because corresponding to the definition that we gave to begin with, engineers are accustomed to referring to only the excess work done beyond that of internal forces as the deformation work. By contrast, Weingarten counted one part of the heat supplied as the deformation work by adding the third term.

The actual added heat can, in fact, be split into two parts. One part serves to raise the temperature in the rod by the desired amount ([75]) *when its thermal dilatation is prevented.* From the mechanical theory of heat, the second part is equivalent to the work that would be required in order to produce the thermal dilatation that occurs in reality *at constant temperature.* However, WEINGARTEN counted that second part with the deformation work.

([73]) MELAN had already proposed that idea.

([74]) J. WEYRAUCH, Zeitschrift für Architekten und Ingenieurwesen, 54, Neue Folge **12** (1907), pp. 91.

([75]) For an ideal gas, the first process would be that of bringing the gas to the higher temperature by adding heat at constant volume. The second part consists of an isothermal return to the original pressure.

CHAPTER TWO

THE CONTINUOUS ELASTIC BODY

2.1. The elastic deformation of a continuum. The principle of virtual displacements and the equilibrium conditions. – In this chapter, we shall consider a continuous elastic body under the influence of given external forces. As is known, they are subdivided into the so-called body forces and the tractions on the surface of the elastic body. In the engineering theory of elasticity, one also ignores the body forces and considers only the surface stresses ([76]). We shall adhere to that conception of things in what follows.

We imagine that the surface tractions are distributed continuously over the surface of the body. Their components p_x, p_y, p_z ([77]) along the three axes of a rectangular coordinate system might then be given as continuous functions of position on the surface. The elastic body will deform under the action of those surface forces, under which the individual points might experience an *infinitely-small displacement* with the components u (x, y, z), v (x, y, z), w (x, y, z), and might assume a certain equilibrium configuration. A stress state will then arise in its interior.

It was CAUCHY who first showed that an arbitrary stress state can be characterized analytically by being given six functions, namely, the so-called stress components σ_x, σ_y, σ_z, τ_x, τ_x, τ_x, the first three of which are the normal components perpendicular to the x, y, and z-axes, resp,, while the other three are shear stresses.

In order to investigate the equilibrium state, we shall appeal to the *principle of virtual displacements*. It says that the work done by external forces must be equal to the work done by internal forces if the body is to actually be found in equilibrium. If we denote a virtual displacement by δu, δv, δw, in which those variations should be continuous functions of position, then the work done by surface tractions will be given by the surface integral:

$$\iint_O (p_x\,\delta u + p_x\,\delta u + p_x\,\delta u)\,d\omega\,.$$

The work done by internal forces is given by the volume integral:

$$\iiint_V \left\{\sigma_x \frac{\partial \delta u}{\partial x} + \sigma_y \frac{\partial \delta v}{\partial y} + \sigma_z \frac{\partial \delta w}{\partial z} + \tau_x \left(\frac{\partial \delta v}{\partial z} + \frac{\partial \delta w}{\partial y}\right) + \tau_y \left(\frac{\partial \delta w}{\partial x} + \frac{\partial \delta u}{\partial z}\right) + \tau_z \left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x}\right)\right\} d\kappa\,,$$

in which the factors of the six stress components are the variations of the six *quantities of deformation* of the body:

([76]) Thus, e.g., in the engineering theory of beams, one always imagines that the proper weight of the beam has been replaced by a fictitious load.

([77]) The choice of notation is the usual one that is made in the statics of building construction. Cf., the seminar paper by M. GRÜNING, *Enzyklopädie der mathematischen Wissenschaften*, Bd. IV, Article 29.a.

$$\varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_y = \frac{\partial v}{\partial y}, \qquad \varepsilon_z = \frac{\partial w}{\partial z},$$
$$\gamma_x = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \qquad \gamma_y = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \qquad \gamma_z = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{2.1}$$

That volume integral is referred to as the *deformation work done by the virtual displacement*. The principle of virtual displacements then finds its analytical expression in the equation:

$$0 = \iint_O (p_x\,\delta u + p_x\,\delta u + p_x\,\delta u)\,d\omega$$
$$- \iiint_V \left\{ \sigma_x \frac{\partial \delta u}{\partial x} + \sigma_y \frac{\partial \delta v}{\partial y} + \sigma_z \frac{\partial \delta w}{\partial z} + \tau_x \left(\frac{\partial \delta v}{\partial z} + \frac{\partial \delta w}{\partial y} \right) + \tau_y \left(\frac{\partial \delta w}{\partial x} + \frac{\partial \delta u}{\partial z} \right) + \tau_z \left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} \right) \right\} d\kappa, \tag{2.2}$$

which must be fulfilled by the three arbitrary functions δu, δv, δw for any system.

If we choose the variations δu, δv, δw in the Ansatz (2.2) for the principle of virtual displacements such that variations of the quantities of deformation vanish then the spatial integral will drop out, and we will obtain the condition equations for the surface tractions alone. Such displacements are the ones that a rigid body can experience. One of them possesses six degrees of freedom of motion, so we can displace it parallel to the three coordinate axis, as well as rotate it around the three coordinate axes. If we introduce the corresponding expressions for δu, δv, δw then we will see that *the surface tractions must fulfill* the six conditions:

$$\iint_O p_x\,d\omega = 0, \qquad \iint_O p_y\,d\omega = 0, \qquad \iint_O p_z\,d\omega = 0,$$
$$\iint_O (z\,p_y - y\,p_z)\,d\omega = 0, \qquad \iint_O (x\,p_z - z\,p_x)\,d\omega = 0, \qquad \iint_O (y\,p_x - x\,p_y)\,d\omega = 0, \tag{2.3}$$

i.e., they must fulfill the six conditions for a system of forces that keeps a rigid body in equilibrium.

In order to now examine the *equilibrium of an elastic body*, we partially-integrate the spatial integral in equation (2.2) and get:

$$\iint_O \{[p_x - \sigma_x \cos(x,n) - \tau_z \cos(y,n) - \tau_y \cos(z,n)]\,\delta u$$
$$+ [p_y - \tau_z \cos(x,n) - \sigma_y \cos(y,n) - \tau_x \cos(z,n)]\,\delta v$$
$$+ [p_z - \tau_y \cos(x,n) - \tau_x \cos(y,n) - \sigma_z \cos(z,n)]\,\delta w\}\,d\omega$$
$$+ \iiint_V \left\{ \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_z}{\partial y} + \frac{\partial \tau_y}{\partial z} \right) \delta u + \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_z}{\partial y} + \frac{\partial \tau_y}{\partial z} \right) \delta u + \left(\frac{\partial \tau_z}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_x}{\partial z} \right) \delta v + \left(\frac{\partial \tau_y}{\partial x} + \frac{\partial \tau_x}{\partial y} + \frac{\partial \sigma_z}{\partial z} \right) \delta w \right\} d\kappa = 0, \tag{2.4}$$

in which n is the direction of the *outward-pointing* normal to the surface.

We can choose the virtual displacements such that they will vanish individually on the surface O, so the spatial integral in that equation must also vanish by itself. From the known argument in the calculus of variations, one will then find that one must have *the equilibrium conditions:*

$$\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_z}{\partial y} + \frac{\partial \tau_y}{\partial z} = 0\,, \\
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_z}{\partial y} + \frac{\partial \tau_y}{\partial z} = 0\,, \\
&\frac{\partial \tau_y}{\partial x} + \frac{\partial \tau_x}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0
\end{aligned} \tag{2.5}$$

for every interior point. Equation (2.4) simplifies as a result of the fact that only the surface integral still remains, as we can then further deduce, by the same argument, that *for any point of the surface, the equilibrium conditions:*

$$\begin{aligned}
p_x &= \sigma_x \cos(x, n) + \tau_z \cos(y, n) + \tau_y \cos(z, n)\,, \\
p_y &= \tau_z \cos(x, n) + \sigma_y \cos(y, n) + \tau_x \cos(z, n)\,, \\
p_z &= \tau_y \cos(x, n) + \tau_x \cos(y, n) + \sigma_z \cos(z, n)
\end{aligned} \tag{2.6}$$

must be valid.

The six undetermined stress components are by no means determined by the three equilibrium conditions (2.5) for the interior, to which the three boundary conditions (2.6) must be added. Rather, three of them can be assumed to be arbitrary, and only then can the other three be determined ([78]). Upon appealing to the terminology that was introduced in our treatment of frameworks, we can call the stress problem *three-fold functionally undetermined*, which just says that it is only after we have established the three functions that the solution to the stress problem will even be possible by means of the equilibrium conditions ([79]).

([78]) That somewhat-indeterminate way of expressing things can be formulated more precisely by introducing the Maxwell stress functions. On that topic, cf., A.E.H. LOVE, *Lehrbuch der Elastizität*, Ger. trans. by A. TIMPE, Leipzig and Berlin, 1907, pp. 103.

([79]) Editor's note: *For the question of how many times undetermined the three-dimensional continuum is, the difference between the number of stresses and the number of equilibrium conditions is what is crucial, and not the number of compatibility conditions. Whereas there is no difference between the two numbers for frameworks and frame constructions, but also for the disc, plate, and torsion problem, things are different for the three-dimensional continuum. For the three-dimensional continuum, the number of unknown internal stresses is six, but the number of equilibrium conditions is three. The three-dimensional continuum is therefore three-fold functionally undetermined. The equilibrium conditions will be fulfilled identically by the introduction of three stress functions. If one eliminates the stresses with the help of elasticity laws and introduces the distortions, expressed in terms of the stress functions, into the six compatibility conditions in equation* (2.9) *then that will give six equations for the three stress functions. The six compatibility conditions cannot be mutually independent then, cf., e.g.,* [**128**].

For the introduction of stress functions for the fulfillment of the equilibrium conditions, one should refer to the classical works of AIRY [**1**], MAXWELL [**74, 75**], *and* MORERA [**91, 90**], *the textbook of* LOVE [**68**], *the survey of* TRUESDELL and TOUPIN in *Handbuch der Physik* [**141**], pp. 582-594, *as well as the works of* SCHÄFER [**128, 129, 130, 131**], KRÖNER [**61**], GÜNTHER [**46**], *and* MARGUERRE [**73**].

2.2. The intervention of elasticity. – In order to set aside that indeterminacy in the problem, we shall now assume that the elastic body, which might be geometrically a *simply-connected* body, moreover, obeys *Hooke's law* and is *free from initial stresses*, i.e., that no stresses are present in the body in the absence of external loads (p_x, p_y, p_z).

In this case, the stress components σ_x, σ_y, σ_z, τ_x, τ_y, τ_z *are coupled with the quantities of deformation* ε_x, ε_y, ε_z, γ_x, γ_y, γ_z *by the linear relations:*

$$\begin{aligned} \sigma_x &= \lambda\, e + 2\,\mu\,\varepsilon_x\,, \qquad & \tau_x &= \mu\,\gamma_x\,, \\ \sigma_y &= \lambda\, e + 2\,\mu\,\varepsilon_y\,, \qquad & \tau_y &= \mu\,\gamma_y\,, \\ \sigma_z &= \lambda\, e + 2\,\mu\,\varepsilon_z\,, \qquad & \tau_z &= \mu\,\gamma_z\,, \end{aligned} \tag{2.7}$$

in which λ and μ are the two (Lamé) elasticity constants (80) of the solid body, and ε is an abbreviation for the cubic dilatation:

$$\varepsilon = \varepsilon_x + \varepsilon_y + \varepsilon_z\,. \tag{2.8}$$

If one were to introduce the six quantities of deformation in place of the stress components in the equilibrium conditions (2.5) and (2.6) then the problem would remain three-fold functionally undetermined, precisely as before. Meanwhile, the indeterminacy will vanish when we observe that the six quantities of deformation can be expressed in terms of the *three* components of the displacement u, v, w using equations (2.1). The quantities of deformation are not six arbitrary functions then, but they must satisfy the so-called *compatibility conditions for the quantities of deformation*, which possess the form (81):

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^2 \gamma_x}{\partial y\,\partial z}\,, \qquad 2\,\frac{\partial^2 \varepsilon_x}{\partial y\,\partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_x}{\partial x} + \frac{\partial \gamma_y}{\partial y} + \frac{\partial \gamma_z}{\partial z}\right), \tag{2.9}$$
$$\cdots \qquad\qquad\qquad \cdots$$

If we replace the quantities of deformation in those compatibility conditions by solving equation (2.7) for the stress components then we will get six corresponding condition equations

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(80) Those Lamé elastic constants are connected with the otherwise-often-used elastic constants in the following way:

The so-called *Young modulus* is:

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}\,,$$

and the so-called *Poisson constant* that is employed in German engineering literature is:

$$m = \frac{2(\lambda + \mu)}{\lambda}\,,$$

while often (e.g., LOVE) it is the reciprocal value $\sigma = \lambda / 2\,(\lambda + \rho)$ that is referred to as the Poisson constant. m is the so-called *shear modulus*, for which the symbol G is most customary in the German engineering literature.

(81) A.E.H. LOVE, *loc. cit.*, pp. 59.

for the stress components that will be referred to as the *compatibility conditions for the stress components* of the elastic body. Together with the equilibrium conditions (2.5) and (2.6), they determine the stresses that the body experiences as a result of the load uniquely. One most conveniently gives them the form of the six so-called *stress equations of the theory of elasticity*, which BELTRAMI first gave; they read ([82]):

$$\Delta_2\,\sigma_x + \frac{2(\lambda+\mu)}{3\lambda+2\mu}\frac{\partial^2\Theta}{\partial x^2} = 0\,, \qquad \Delta_2\,\tau_x + \frac{2(\lambda+\mu)}{3\lambda+2\mu}\frac{\partial^2\Theta}{\partial y\,\partial z} = 0\,, \tag{2.10}$$

$$\ldots \qquad\qquad \ldots$$

in which one sets:

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} = \Delta_2$$

and

$$\sigma_x + \sigma_y + \sigma_z = \Theta\,, \tag{2.11}$$

to abbreviate.

In the final analysis, the foregoing argument is based upon the fact that the quantities of deformation, and therefore the stress components, in their own right, are determined by the three components of the displacement. It then seems much simpler, even in the expression (2.7) for Hooke's law, to introduce the expressions (2.1) for the quantities of deformation into the derivatives of the displacement components u, v, w, so to express the stress components in terms of the three displacement components directly and introduce the expressions thus-obtained into the equilibrium conditions (2.5). In that way, we will get the second-order differential equations for u, v, w that are known as the *fundamental equations of the theory of elasticity:*

$$\begin{aligned}
\mu\,\Delta_2\,u+(\lambda+\mu)\,\frac{\partial e}{\partial x} &= 0\,,\\
\mu\,\Delta_2\,v+(\lambda+\mu)\,\frac{\partial e}{\partial y} &= 0\,,\\
\mu\,\Delta_2\,w+(\lambda+\mu)\,\frac{\partial e}{\partial z} &= 0\,.
\end{aligned} \tag{2.12}$$

The boundary conditions (2.6) now read ([83]):

$$\begin{aligned}
p_x &= \lambda\,e\cos(x, n) + \left[\frac{\partial u}{\partial n}+\frac{\partial u}{\partial x}\cos(x,n)+\frac{\partial v}{\partial x}\cos(y,n)+\frac{\partial w}{\partial x}\cos(z,n)\right],\\
p_y &= \lambda\,e\cos(y, n) + \left[\frac{\partial v}{\partial n}+\frac{\partial u}{\partial y}\cos(x,n)+\frac{\partial v}{\partial y}\cos(y,n)+\frac{\partial w}{\partial y}\cos(z,n)\right],
\end{aligned} \tag{2.13}$$

([82]) A.E.H. LOVE, *loc. cit.*, pp. 160.
([83]) A.E.H. LOVE, *loc. cit.*, pp. 157, 158.

$$p_z = \lambda\, e \cos(z, n) + \left[\frac{\partial w}{\partial n} + \frac{\partial u}{\partial z}\cos(x,n) + \frac{\partial v}{\partial z}\cos(y,n) + \frac{\partial w}{\partial z}\cos(z,n)\right] .$$

Equations (2.12), with the given boundary conditions (2.13), do not determine the *displacement components* u, v, w uniquely. However, two systems of solutions u_1, v_1, w_1 and u_2, v_2, w_2 can differ by only expressions that give the displacement of a *rigid* body, so one must have:

$$\begin{aligned} u_1 &= u_2 + a + q\,z - r\,y , \\ v_1 &= v_2 + b + r\,x - p\,z , \\ w_1 &= w_2 + c + p\,y - q\,x , \end{aligned} \tag{2.14}$$

in which a, b, c, p, q, r are constants ([84]). By contrast, if we were not given the external forces p_x, p_y, p_z on the surface of the body, which would seem most natural from our standpoint, but the values of the displacements themselves on that surface, then equations (2.12) would determine the displacements *uniquely* ([85]). The *quantities of displacement*, and therefore the *stress components*, will then be determined uniquely in both cases, because when we compute the quantities of deformation, i.e., the expressions (2.1), from the displacement components, the additional terms in (2.14) will once more drop out precisely.

We shall come back to the actual process of integrating (2.12) and (2.13) at the conclusion of this chapter.

2.3. The minimum of total energy. – If we introduce the quantities of deformation instead of the derivatives of the displacement components into the volume integral (2.1) using the Ansatz of the principle of virtual displacements then we will get:

$$\left.\begin{aligned} 0 = \quad & \iint_O (p_x\,\delta u + p_y\,\delta v + p_z\,\delta w) \\ & - \iiint_V (\sigma_x\,\delta\varepsilon_x + \sigma_y\,\delta\varepsilon_y + \sigma_z\,\delta\varepsilon_z + \tau_x\,\delta\gamma_x + \tau_y\,\delta\gamma_y + \tau_z\,\delta\gamma_z)\,d\kappa . \end{aligned}\right\} \tag{2.15}$$

We would like to replace the stress components in the integrand of the volume integral with the quantities of deformation using Hooke's law (2.7). It will then become the variation of the following function, which is quadratic in the quantities of deformation:

$$\begin{aligned} & \psi(\varepsilon_x, \varepsilon_y, \varepsilon_z, \tau_x, \tau_y, \tau_z) \\ & \quad = \tfrac{1}{2}[(\lambda + 2\mu)(\varepsilon_x + \varepsilon_y + \varepsilon_z)^2 + \mu(\gamma_x^2 + \gamma_y^2 + \gamma_z^2 - 4\varepsilon_y\,\varepsilon_x - 4\varepsilon_z\,\varepsilon_y - 4\varepsilon_x\,\varepsilon_z)] , \end{aligned} \tag{2.16}$$

([84]) *Editor's remark: The original version of these lectures indeed includes a reference to a footnote, but no actual text in the footnote.*

([85]) A.E.H. LOVE, *loc. cit.*, pp. 201.

and when we replace the quantities of deformation in that with the derivatives of the displacement components, that will make:

$$\psi = \tfrac{1}{2}\left\{(\lambda+2\mu)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)^2\right.$$

$$\left.+\left[\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2-4\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}-4\frac{\partial w}{\partial z}\frac{\partial u}{\partial x}-4\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right]\right\} . \tag{2.16.a}$$

The volume integral is then the variation of the following volume integral:

$$\Psi = \iiint_V \psi\, d\kappa . \tag{2.17}$$

Moreover, since the surface tractions $p_x(\omega)$, $p_y(\omega)$, $p_z(\omega)$ are given functions of position on the surface, the surface integral in (2.15) the negative of the variation of:

$$\Phi = -\iint_O [p_x(\omega)\cdot u + p_y(\omega)\cdot v + p_z(\omega)\cdot w]\, d\omega = \iint_O \varphi\, d\omega . \tag{2.18}$$

The integrands ψ and φ both include the three functions u, v, w. The values of the integrals Ψ and Φ can first be calculated when that system of functions is chosen, and naturally the values of those integrals will change with the choice of those functions. They are *functionals* ([86]) of the three functions $u(x, y, z)$, $v(x, y, z)$, $w(x, y, z)$.

We can convince ourselves that ψ is the *total deformation work of the elastic body* that is done by the deformation that is determined by the displacements u, v, w. That is because we can characterize any current state of deformation that is created during the deformation by the associated values $\bar{u}(x, y, z)$, $\bar{v}(x, y, z)$, $\bar{w}(x, y, z)$ of the displacements, and therefore regard the values of those functions for any volume element of the body as independent variables. The deformation work that is done on the volume element when it goes from those values to the neighboring values $\bar{u}+d\bar{u}$, $\bar{v}+d\bar{v}$, $\bar{w}+d\bar{w}$ is:

$$d\kappa(\bar{\sigma}_x\, d\bar{\varepsilon}_x + \bar{\sigma}_y\, d\bar{\varepsilon}_y + \bar{\sigma}_z\, d\bar{\varepsilon}_z + \bar{\tau}_x\, d\bar{\gamma}_x + \bar{\tau}_y\, d\bar{\gamma}_y + \bar{\tau}_z\, d\bar{\gamma}_z) .$$

([86]) In mathematics, one understands a *functional* to mean an expression that depends upon the *total course of one or more functions*, such that a value for it can be calculated whenever one knows those functions. The functions upon which the functional depends then play the role of independent variables upon which the value of the function depends. For example, the arc-length of the curve $y(x)$ that connects two points x_1, y_1 and x_2, y_2 in the plane:

$$s = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$$

is a functional of the function $y(x)$.

We will get the deformation work done by the total deformation of the volume element from that when we integrate $\bar{u}$, $\bar{v}$, $\bar{w}$ from 0 to their final values *u*, *v*, *w*. Now, since we have:

$$d\kappa \int_{0,0,0}^{u,v,w} \{\bar{\sigma}_x \, d\bar{\varepsilon}_x + \bar{\sigma}_y \, d\bar{\varepsilon}_y + \bar{\sigma}_z \, d\bar{\varepsilon}_z + \bar{\tau}_x \, d\bar{\gamma}_x + \bar{\tau}_y \, d\bar{\gamma}_y + \bar{\tau}_z \, d\bar{\gamma}_z\} = d\kappa \, \psi ,$$

ψ will be, in fact, the deformation work per unit volume at the point *x, y, z*, and the volume integral:

$$\psi = \iiint_V \psi \, d\kappa$$

will be the deformation work done on the entire body. On the other hand:

$$\varphi = -(p_x\, u + p_y\, v + p_z\, w) \tag{2.18a}$$

is obviously the potential energy per unit area of surface tractions, so Φ is the *potential energy of the surface tractions for the total body. The sum of both of them:*

$$E = \Phi + \Psi = \iint_O \varphi \, d\omega + \iiint_V \psi \, d\kappa \tag{2.19}$$

is then the total potential energy of the elastic body.

The Ansatz (2.15) of the principle of virtual displacements reads:

$$\iint_O \delta\varphi \, d\omega + \iiint_V \delta\psi \, d\kappa = 0 \tag{2.15a}$$

with the new notation, so it can be summarized as:

$$\delta \left[\iint_O \varphi \, d\omega + \iiint_V \psi \, d\kappa \right] = \delta E = 0 , \tag{2.20}$$

i.e., the *total potential energy E will be an extremum.*

Conversely, *the extremum of the expression E*, which can be regarded as a functional of the displacements *δu, δv, δw, characterizes the equilibrium state*.

In fact, when we look for the extremum of the sum of a volume integral and an integral over the bounding surface of that volume, as represented by:

$$E = \iint_O \varphi \, d\omega + \iiint_V \psi \, d\kappa ,$$

using the rules of the calculus of variations, we will have the three Euler-Lagrange equations:

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial u_x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial u_y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial u_z}\right) = 0\,,\\
&\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial v_x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial v_y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial v_z}\right) = 0\,,\\
&\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial w_x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial w_y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial w_z}\right) = 0\,,
\end{aligned}
\tag{2.21a}
$$

as the conditions for the three unknown functions u, v, w, initially for the interior, with:

$$
u_x = \frac{\partial u}{\partial x}\,, \qquad u_y = \frac{\partial u}{\partial y}\,, \qquad u_z = \frac{\partial u}{\partial z}\,, \qquad \ldots,
$$

and then the three equations for the surface:

$$
\begin{aligned}
&\frac{\partial\varphi}{\partial u}+\frac{\partial\psi}{\partial u_x}\cos(x,n)+\frac{\partial\psi}{\partial u_y}\cos(y,n)+\frac{\partial\psi}{\partial u_z}\cos(z,n) = 0\,,\\
&\frac{\partial\varphi}{\partial v}+\frac{\partial\psi}{\partial v_x}\cos(x,n)+\frac{\partial\psi}{\partial v_y}\cos(y,n)+\frac{\partial\psi}{\partial v_z}\cos(z,n) = 0\,,\\
&\frac{\partial\varphi}{\partial w}+\frac{\partial\psi}{\partial w_x}\cos(x,n)+\frac{\partial\psi}{\partial w_y}\cos(y,n)+\frac{\partial\psi}{\partial w_z}\cos(z,n) = 0\,.
\end{aligned}
\tag{2.21b}
$$

If we introduce the expressions (2.16a) [(2.18a), resp.] for ψ and φ into those equations then they will go directly to the equilibrium conditions (2.12) and the boundary conditions (2.13).

As was mentioned before, in the theoretical study of elasticity, one often cares to pose the problem in a form such that one does not prescribe the tractions on the surface, but the displacements $u(\omega)$, $v(\omega)$, $w(\omega)$ themselves. Therefore, the three functions $p_x(\omega)$, $p_y(\omega)$, $p_z(\omega)$ in the surface integral Φ are not immediately known in the total energy E. However, since the volume integral Ψ remains unchanged, the requirement that E should be an extremum once more leads directly to equations (2.21a) [the equations (2.12) that coincide with them, resp.]. Moreover, since those equation determine the three functions $u(x, y, z)$, $v(x, y, z)$, $w(x, y, z)$ uniquely when their values are given on the surface, we will know the total deformation of the body. Equations (2.1) immediately imply the quantities of deformation (2.7), and then the stress components, and from them we will find the stresses on the surface, and therefore ultimately the external forces that would be in a position to justify the required deformation.

We see that in this way of looking at things, the term Φ in the sum of the integrals E plays no role. We would arrive at the same result if we had looked for only the extremum of the integral:

$$
\Psi = \iiint_V \psi\, d\kappa
$$

for prescribed displacements on the surface, instead of an extremum of the potential energy E. If we introduce the solutions of the Euler-Lagrange equation for the functions u, v, w in ψ then Ψ will assume the desired extremal value that belongs to the equilibrium state and will be denoted by $\Psi^{(e)}$. It represents the actual deformation work done during the transition to the new deformation state. That extremal value of $\Psi^{(e)}$ will be completely determined for every choice of surface displacements $u(\omega)$, $v(\omega)$, $w(\omega)$, so it will be a *functional* of the surface displacements:

$$\Psi^{(e)} = \Psi^{(e)}\,[u(\omega), v(\omega), w(\omega)]\,. \tag{2.22}$$

2.4. Canonical transformation. Menabrea's principle. – In the foregoing section, we used the principle of virtual displacements in order derive the theorem that for given surface tractions p_x, p_y, p_z, an elastic body will be in equilibrium when the displacement components $u(x, y, z)$, $v(x, y, z)$, $w(x, y, w)$ extremize the total potential energy:

$$E = \iint_O \varphi\, d\omega + \iiint_V \psi\, d\kappa\,, \tag{2.19}$$

which is regarded as a functional of those functions. We would now like to apply a *canonical transformation* to that variational problem, and in that way go over to a new formulation.

Since φ does not include the derivatives of u, v, w at all, only the volume integral will come into question for the canonical transformation. We must set the partial differential coefficients of ψ with respect to the nine derivatives of u, v, w, namely, u_x, u_y, u_z, …, w_z, equal to new variables. Since, from (2.16a), the derivatives u_y and v_x enter into ψ only in the combination $(u_y + v_x)$, we will have:

$$\frac{\partial\psi}{\partial u_y} = \frac{\partial\psi}{\partial(u_y + v_x)} = \frac{\partial\psi}{\partial v_x}\,,$$

and analogously:

$$\frac{\partial\psi}{\partial v_z} = \frac{\partial\psi}{\partial w_y}\,, \qquad \frac{\partial\psi}{\partial w_x} = \frac{\partial\psi}{\partial u_z}\,.$$

Of the nine new unknown functions to be introduced, only six of them are distinct from each other then. Upon performing the differentiation, we will see that, from (2.1) and (2.7), they will give precisely the previously-considered six stress components:

$$\begin{gathered}\frac{\partial\psi}{\partial u_x} = \sigma_x\,, \qquad \frac{\partial\psi}{\partial u_x} = \sigma_x\,, \qquad \frac{\partial\psi}{\partial u_x} = \sigma_x\,, \\ \frac{\partial\psi}{\partial v_z} = \frac{\partial\psi}{\partial w_y} = \tau_x\,, \qquad \frac{\partial\psi}{\partial w_x} = \frac{\partial\psi}{\partial u_z} = \tau_y\,, \qquad \frac{\partial\psi}{\partial u_y} = \frac{\partial\psi}{\partial v_x} = \tau_z\,.\end{gathered} \tag{2.23}$$

We calculated the partial derivatives u_x, …, w_z as functions of σ_x, σ_y, …, τ_z from those equations. Naturally, we cannot calculate nine unknowns from six equations, but we will once more make the problem determinate when we do not calculate v_x and w_y individually, but only in the combinations $(v_z + w_y)$, and likewise $(w_x + u_z)$, $(u_y + v_x)$.

In the spirit of the theory of the canonical transformation, we must now replace the function ψ with the function:

$$H(\sigma_x, \sigma_y, \sigma_z, \tau_x, \tau_y, \tau_z) = u_x \frac{\partial\psi}{\partial u_x} + u_y \frac{\partial\psi}{\partial u_y} + u_z \frac{\partial\psi}{\partial u_z} + (v_z + w_y)\frac{\partial\psi}{\partial(v_z + w_y)} + (w_x + u_z)\frac{\partial\psi}{\partial(w_x + u_z)} + (u_y + v_x)\frac{\partial\psi}{\partial(u_y + v_x)} - \psi, \tag{2.24}$$

in which the derivatives u_x, …, $(u_y + v_x)$ are expressed in terms of the nine variables. Once one has done that ([87]), one will get:

$$H = \frac{1}{2}\left[\frac{1}{\mu}\frac{\lambda+\mu}{3\lambda+2\mu}(\sigma_x+\sigma_y+\sigma_z)^2 - \frac{1}{\mu}(\sigma_y\,\sigma_x + \sigma_z\,\sigma_x + \sigma_x\,\sigma_y - \tau_x^2 - \tau_y^2 - \tau_z^2)\right]. \tag{2.24b}$$

According to the laws of the canonical transformation, the variational problem (2.19) will be replaced with the *new variational problem of extremizing the expression:*

$$\mathcal{E} = \iint_O \psi\, d\omega + \iiint_V \left\{\frac{\partial u}{\partial x}\sigma_x + \frac{\partial v}{\partial y}\sigma_y + \frac{\partial w}{\partial z}\sigma_z + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\tau_x + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\tau_y + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\tau_z - H(\sigma_x, \sigma_y, \sigma_z, \tau_x, \tau_y, \tau_z)\right\} d\kappa, \tag{2.25}$$

and indeed when it is regarded as a functional of the nine functions u, v, w ; σ_x, σ_y, σ_z, τ_x, τ_y, τ_z.

In order to determine the nine functions that actually *extremize* $\mathcal{E}$, we next vary only the σ_x, …, τ_z. As the extremal conditions for $\mathcal{E}$, the *first group of Euler-Lagrange equations* will give the relations:

([87]) When one is doing that, it is convenient for one to observe that ψ is a quadratic function in u_x, v_y, …, $(u_y + v_x)$, such that from Euler's theorem, one will have:

$$u_x \frac{\partial\psi}{\partial u_x} + u_y \frac{\partial\psi}{\partial u_y} + u_z \frac{\partial\psi}{\partial u_z} + \cdots + (u_y + v_x)\frac{\partial\psi}{\partial(u_y + v_x)} = 2\psi,$$

so

$$H = \psi, \tag{2.24a}$$

where one imagines that the stress components are introduced into ψ.

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial H}{\partial \sigma_x} = \frac{1}{\mu}\left[\frac{\lambda+\mu}{3\lambda+2\mu}(\sigma_x+\sigma_y+\sigma_z)-(\sigma_y+\sigma_z)\right], & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} &= \frac{\partial H}{\partial \tau_x} = \frac{1}{\mu}\tau_x , \\
\frac{\partial u}{\partial x} &= \frac{\partial H}{\partial \sigma_y} = \frac{1}{\mu}\left[\frac{\lambda+\mu}{3\lambda+2\mu}(\sigma_x+\sigma_y+\sigma_z)-(\sigma_z+\sigma_x)\right], & \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z} &= \frac{\partial H}{\partial \tau_y} = \frac{1}{\mu}\tau_y , \\
\frac{\partial w}{\partial z} &= \frac{\partial H}{\partial \sigma_z} = \frac{1}{\mu}\left[\frac{\lambda+\mu}{3\lambda+2\mu}(\sigma_x+\sigma_y+\sigma_z)-(\sigma_z+\sigma_x)\right], & \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} &= \frac{\partial H}{\partial \tau_z} = \frac{1}{\mu}\tau_z .
\end{aligned} \tag{2.26}$$

The variation of u, v, w then leads to the following *two groups of extremum conditions*: For the interior of the body, the Euler-Lagrange equations must be valid:

$$\begin{aligned}
\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_z}{\partial y}+\frac{\partial \tau_y}{\partial z} &= 0 , \\
\frac{\partial \tau_z}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_x}{\partial z} &= 0 , \\
\frac{\partial \tau_y}{\partial x}+\frac{\partial \tau_x}{\partial y}+\frac{\partial \sigma_z}{\partial z} &= 0 ,
\end{aligned} \tag{2.27a}$$

and for the surface, one has the conditions:

$$\begin{aligned}
p_x &= \sigma_x \cos(x, n) + \tau_z \cos(y, n) + \tau_y \cos(x, n) , \\
p_y &= \tau_z \cos(x, n) + \sigma_y \cos(y, n) + \tau_x \cos(x, n) , \\
p_z &= \tau_y \cos(x, n) + \tau_x \cos(y, n) + \sigma_z \cos(x, n) .
\end{aligned} \tag{2.27b}$$

The *first group* (2.26) of those equations says that the *six* stress components can be expressed in terms of the *three* functions u, v, w. We can then derive the *compatibility conditions* for the stress components from them, which are completely equivalent to the latter equations. The *second group* (2.27a) and (2.37b) consists of the *equilibrium conditions for the interior and the surface.*

In this variational problem, viz., of extremizing the functional $\mathcal{E}$, we have regarded all nine functions that go into $\mathcal{E}$ as variables. Mathematically, the facts are simple, but of course their physical meaning for elastic bodies in not entirely obvious. We have regarded the displacement and the stress state as mutually-independent and freely-varying. *Finding the extremum of $\mathcal{E}$ will then imply the compatibility equations, as well as the equilibrium conditions*.

When we regard the first our first group of equations (2.26) as having been fulfilled, we will then return to the *variational principle of extremizing E* (which arose immediately from the principle of virtual displacements) that was treated in the previous section. At the time, we varied in such a way that *compatibility was preserved*, i.e., the body remained completely connected during the variation. *The neighboring states that one compares to the extremum of E are no longer equilibrium states.* The unique equilibrium state will just be characterized by the extremum of *E* precisely.

On the other hand, we can *specialize the variational problem of $\mathcal{E}$* in such a way that we do not assume that the compatibility conditions (2.26) are fulfilled, *but only the equilibrium conditions* (2.27a) *and* (2.27b). That would mean that we require of the stress components that they satisfy the equilibrium conditions for every volume element and every surface element, but no longer require that they satisfy the compatibility conditions. In order to actually produce that stress state, we can probably achieve the correct stress state for every volume element, but the deformed volume elements would no longer fit together, such that they could no longer constitute the original body either. We will get the condition for a construction to be possible that would make precisely that stress state exist, so the compatibility conditions would be satisfied, from the extremum requirement.

If we partially integrate the first term in the volume integral in the expression (2.25) for $\mathcal{E}$, and in so doing observe that the equilibrium conditions (2.27a) and (2.27b) are fulfilled now, then that will reduce the variational problem for $\mathcal{E}$ to that of extremizing:

$$- \iiint_V H(\sigma_x, \sigma_y, \sigma_z, \tau_x, \tau_y, \tau_z)\, d\kappa \ ,$$

with the equilibrium conditions (2.27a) and (2.27b) as auxiliary conditions. From (2.24b), $H = \psi(\sigma_x, \sigma_y, \sigma_z, \tau_x, \tau_y, \tau_z)$ is identical to the deformation work per unit volume. Thus:

$$- \iiint_V H(\sigma_x, \ldots, \tau_z)\, d\kappa = - \iiint_V \psi(\sigma_x, \ldots, \tau_z)\, d\kappa$$

is, up to sign (which does not matter for the extremum requirement), the sum of the deformation works for the individual volume elements. *The extremum requirement then demands that this sum of the deformation works over the individual volume elements of the body:*

$$\bar{\Psi} = \iiint_V \bar{\psi}(\sigma_x, \sigma_y, \sigma_z, \tau_x, \tau_y, \tau_z)\, d\kappa \tag{2.28}$$

must be extremized, while the equilibrium conditions (2.27a) *and* (2.27b) *exist as auxiliary conditions.* That is *Menabrea's principle (in its first conception) for the continuous elastic body.*

In order to confirm that it will lead to the compatibility condition, we would like to vary (2.28), and in so doing, consider the auxiliary conditions for the interior and boundary using the method of Lagrange factors. Therefore, let $\pi(x, y, z)$, $\rho(x, y, z)$, $\chi(x, y, z)$ be three Lagrange factors for the interior, and let $\pi'(\omega)$, $\rho'(\omega)$, $\chi'(\omega)$ be three such things for the surface. We will then have to extremize the expression:

$$\iiint_V \left\{ \bar{\psi} + \pi\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_z}{\partial y} + \frac{\partial \tau_y}{\partial z}\right) + \rho\left(\frac{\partial \tau_z}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_x}{\partial z}\right) + \chi\left(\frac{\partial \tau_y}{\partial x} + \frac{\partial \tau_x}{\partial y} + \frac{\partial \sigma_z}{\partial z}\right) \right\} d\kappa \tag{2.29}$$

$$+\iint_O \{ \quad \pi'[\sigma_x \cos(x,n) + \tau_z \cos(y,n) + \tau_y \cos(z,n) - p_x]$$
$$\rho'[\tau_z \cos(x,n) + \sigma_y \cos(y,n) + \tau_x \cos(z,n) - p_y]$$
$$\chi'[\tau_y \cos(x,n) + \tau_x \cos(y,n) + \sigma_z \cos(z,n) - p_z]\}\, d\omega$$

with no auxiliary conditions. *In the interior*, that leads to the six conditions:

$$\frac{\partial\pi}{\partial x} = + \frac{\partial\bar{\psi}}{\partial\sigma_x}, \qquad \frac{\partial\rho}{\partial y} = + \frac{\partial\bar{\psi}}{\partial\sigma_y}, \qquad \frac{\partial\chi}{\partial z} = + \frac{\partial\bar{\psi}}{\partial\sigma_z},$$
$$\frac{\partial\chi}{\partial y} + \frac{\partial\rho}{\partial z} = \frac{\partial\bar{\psi}}{\partial\tau_x}, \qquad \frac{\partial\pi}{\partial z} + \frac{\partial\chi}{\partial x} = \frac{\partial\bar{\psi}}{\partial\tau_y}, \qquad \frac{\partial\rho}{\partial x} + \frac{\partial\pi}{\partial y} = \frac{\partial\bar{\psi}}{\partial\tau_z}. \tag{2.30}$$

The variation of the integral (2.26) then reduces to the surface integral here:

$$0 = \iint_O \{ \quad (\pi' - \pi)[\delta\sigma_x \cos(x,n) + \delta\tau_z \cos(y,n) + \delta\tau_y \cos(z,n)]$$
$$(\rho' - \rho)[\delta\tau_z \cos(x,n) + \delta\sigma_y \cos(y,n) + \delta\tau_x \cos(z,n)]$$
$$(\chi' - \chi)[\delta\tau_y \cos(x,n) + \delta\tau_x \cos(y,n) + \delta\sigma_z \cos(z,n)]\}\, d\omega,$$

and due to the arbitrariness in the variations, we then conclude that:

$$\pi'(\omega) = \pi(\omega), \qquad \rho'(\omega) = \rho(\omega), \qquad \chi'(\omega) = \chi(\omega). \tag{2.30a}$$

We will then see that the six stress components must be capable of being expressed in terms of three functions when the extremum exists, i.e., that *the compatibility conditions must be fulfilled*. The individual deformed volume elements will then fit together, and we can construct the body from them. We denote the extremal value of the integral (2.28) by $\bar{\Psi}^{(e)}$, so it will then represent the deformation work of the elastic body. By the way, we should point out that the Lagrange factors prove to be precisely the displacement components ([88]).

2.5. The extremum of the deformation work as a functional of the surface displacements and the surface tractions. Castigliano's theorems. The Betti reciprocity theorem. – The equilibrium state of an elastic body is characterized by the simultaneous existence of compatibility conditions and equilibrium conditions. Both of them together find their simplest analytical expression in basic equations of elasticity (2.12), i.e., the Euler-Lagrange equations for the integral:

([88]) For this section, cf., the presentation by E. HELLINGER in *Encyklopädie der mathematischen Wissenschaften*, Bd. IV, Art. 31 (4 Teilband, pp. 654).

$$\Psi = \tfrac{1}{2}\iiint_V \left\{(\lambda+2\mu)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)^2\right.$$
$$\left. + \ \mu\left[\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial z}\right)^2-4\frac{\partial w}{\partial z}\frac{\partial u}{\partial x}-4\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right]\right\}d\kappa \tag{2.31}$$

that represents the deformation work $\int \psi \, d\kappa$ of the body, in which ψ has been replaced with its expression in (2.16a)

If we introduce precisely the solutions of the Euler-Lagrange equations into that integral for the functions *u*, *v*, *w*, i.e., we introduce the "extremal," in the terminology of the calculus of variations, then it will assume its extremal value that characterizes equilibrium. We shall call it the *extremal integral:*

$$\Psi^e = \iiint_{\mathcal{E}} \psi \, d\kappa\,, \tag{2.32}$$

in which the symbol $\mathcal{E}$ should suggest that while performing the integration, the *u*, *v*, *w* are replaced with the values for the extremal, so Ψ^e will be the actual value of the deformation with that is done by the elastic deformation. It is given by the variational problem for *E*, in which one seeks equilibrium under the assumption of compatibility in the form $\Psi^{(g)}$, and by Menabrea's principle, in which one seeks compatibility under the assumption of equilibrium in the form $\Psi^{(k)}$. Now, as we already mentioned before, the extremal is determined uniquely when the boundary values $u(\omega)$, $v(\omega)$, $w(\omega)$ are given, and when the surface tractions $p_x(\omega)$, $p_y(\omega)$, $p_z(\omega)$ are prescribed, it includes only the indeterminacy that nonetheless makes the integrand in (2.32) uniquely determined. *The extremal integral then takes on a uniquely-determined value in both cases, so it is itself determined uniquely by surface displacements (tractions, resp.).* However, its value depends upon all of the values that the functions $u(\omega)$, $v(\omega)$, $w(\omega)$ [$p_x(\omega)$, $p_y(\omega)$, $p_z(\omega)$, resp.] assume on the surface, so it not a function, but a functional, of those quantities. In order to distinguish both of those pictures, we shall write:

$$\Psi^e = \Psi^e\,[u(\omega), v(\omega), w(\omega)]\,, \tag{2.33a}$$

or

$$\Psi^{*e} = \Psi^e\,[p_x(\omega), p_y(\omega), p_z(\omega)]\,, \tag{2.33b}$$

resp. We would like to denote the functional derivatives ([89]) with respect to the functions upon which Ψ^e depends by Ψ_u^e, $\Psi_{p_x}^{*e}$, resp. In any event, they are functionals of the functions, but they also depend upon the location ω on the surface at which the functional derivatives are constructed:

([89]) *Editor's remark: In a handwritten footnote, Prange wrote* "funkt. Ableitungen." *Apparently, it should have read* "on that, cf., my Dissertation, Göttingen, 1915."

$$\Psi_u^e = \Psi_u^e\ [u\ (\omega), v\ (\omega), w\ (\omega)] \tag{2.34a}$$

or

$$\Psi_{p_x}^{*e} = \Psi_{p_x}^{*e}\ [p_x\ (\omega), p_y\ (\omega), p_z\ (\omega)]\ , \tag{2.34b}$$

resp.

According to Volterra's fundamental theorem, the changes in the functional under a variation of the functions upon which it depends is expressed by means of the functional derivatives in the form:

$$\delta\,\Psi^e = \iint_O (\Psi_u^e\,\delta u + \Psi_v^e\,\delta v + \Psi_w^e\,\delta w)\, d\omega \tag{2.35a}$$

or

$$\delta\,\Psi^{*e} = \iint_O (\Psi_u^{*e}\,\delta u + \Psi_v^{*e}\,\delta v + \Psi_w^{*e}\,\delta w)\, d\omega\ , \tag{2.35b}$$

resp.

We would next like to regard the *deformation work as a functional of the surface tractions*. We would like to compare its variation (2.35a) with the variation of the extremal integral (2.32), as it is implied by the boundary formula from the calculus of variations ([90]):

$$\delta\,\Psi^e = \iint_O \left(\frac{\partial\psi}{\partial u_n}\delta u + \frac{\partial\psi}{\partial v_n}\delta v + \frac{\partial\psi}{\partial w_n}\delta w\right) d\omega\ , \tag{2.36}$$

in which u_n, v_n, w_n are the derivatives of the extremal with respect to the outward-pointing normal to the surface of the body.

Since the variations δu, δv, δw are completely arbitrary on the surface, it would follow from (2.35a) and (2.36) that their coefficients must coincide, and we will then find that the functional derivatives are:

$$\begin{aligned}
\Psi_u^e &= \frac{\partial\psi}{\partial u_n} = \frac{\partial\psi}{\partial u_x}\cos(x,n) + \frac{\partial\psi}{\partial u_y}\cos(y,n) + \frac{\partial\psi}{\partial u_z}\cos(z,n)\ ,\\
\Psi_v^e &= \frac{\partial\psi}{\partial v_n} = \frac{\partial\psi}{\partial v_x}\cos(x,n) + \frac{\partial\psi}{\partial v_y}\cos(y,n) + \frac{\partial\psi}{\partial v_z}\cos(z,n)\ ,\\
\Psi_w^e &= \frac{\partial\psi}{\partial w_n} = \frac{\partial\psi}{\partial w_x}\cos(x,n) + \frac{\partial\psi}{\partial w_y}\cos(y,n) + \frac{\partial\psi}{\partial w_z}\cos(z,n)\ .
\end{aligned} \tag{2.37}$$

However, from equations (2.21b), the right-hand sides of those equations, in which the extremal is introduced for u, v, w, are equal to exactly the surface tractions p_x, p_y, p_z that must be applied to the surface in order to produce the deformation that belongs to the given surface displacements. We then have:

$$\Psi_u^e = p_x\ , \qquad \Psi_v^e = p_y\ , \qquad \Psi_w^e = p_z\ , \tag{2.38}$$

([90]) On that, cf., my Dissertation, Göttingen, 1915.

which we express in the form of a theorem:

If the extremal value of the deformation work is expressed as a functional of the surface displacements then the functional derivatives with respect to the functions $u(\omega)$, $v(\omega)$, $w(\omega)$ – viz., the components of the surface displacements – *at every point of the surface will be equal to the associated components of the surface tractions that must be applied in order to produce the distorted state.*

This theorem is the analogue of the second of the two Castigliano theorems that were stated for frameworks. Here, we would also like to refer to it as the *second theorem of Castigliano* for continuous bodies.

We would like to construct the expression:

$$\iint_O (\Psi_u^e\,\delta u + \Psi_v^e\,\delta v + \Psi_w^e\,\delta w)\,d\omega = \iint_O (p_x\,u + p_y\,v + p_z\,w)\,d\omega \tag{2.39}$$

using the functional derivatives, which are indeed functions of position on the surface. In that expression, u, v, w are once more the given values of the displacements on the surface from which $\Psi\,[u, v, w]$ was constructed. That expression no longer depends upon the individual locations ω on the surface, but only on the totality of values of the three functions $u(\omega)$, $v(\omega)$, $w(\omega)$. In order to calculate its value, we substitute the expression (2.37) for the functional derivatives and obtain the surface integral:

$$\begin{aligned}\iint_O \quad &\left\{\left(\frac{\partial\psi}{\partial u_x}u + \frac{\partial\psi}{\partial v_x}v + \frac{\partial\psi}{\partial w_x}w\right)\cos(x, n)\right.\\ &+\left(\frac{\partial\psi}{\partial u_y}u + \frac{\partial\psi}{\partial v_y}v + \frac{\partial\psi}{\partial w_y}w\right)\cos(y, n)\\ &\left.+\left(\frac{\partial\psi}{\partial u_z}u + \frac{\partial\psi}{\partial v_z}v + \frac{\partial\psi}{\partial w_z}w\right)\cos(z, n)\right\}d\omega.\end{aligned}$$

From Gauss's integral theorem, that surface integral is equal to the following volume integral:

$$\begin{aligned}\underset{\varepsilon}{\iiint_V} \quad &\left\{\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial u_x}u + \frac{\partial\psi}{\partial v_x}v + \frac{\partial\psi}{\partial w_x}w\right)\right.\\ &+\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial u_y}u + \frac{\partial\psi}{\partial v_y}v + \frac{\partial\psi}{\partial w_y}w\right)\\ &\left.+\frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial u_z}u + \frac{\partial\psi}{\partial v_z}v + \frac{\partial\psi}{\partial w_z}w\right)\right\}d\kappa,\end{aligned}$$

in which the symbol $\mathcal{E}$ once more suggests that the functions *u*, *v*, *w* that belong to the extremal have been substituted. If we perform the differentiations in it and observe that the functions *u*, *v*, *w* satisfy the equations (2.21a) then that will give:

$$\underset{\mathcal{E}}{\iiint_V} \left\{\frac{\partial\psi}{\partial u_x}u_x + \frac{\partial\psi}{\partial u_y}u_y + \frac{\partial\psi}{\partial u_z}u_z\right\} d\kappa\,,$$

or since (2.16a) implies that ψ is homogeneous of order two in the derivatives $u_x, \ldots, w_z$, from (2.32), we will have:

$$2\underset{\mathcal{E}}{\iiint_V} \psi\, d\kappa = 2\,\Psi^e.$$

We then find the relation:

$$\begin{aligned}\Psi^e &= \tfrac{1}{2}\iint_O (\Psi^e_u \cdot u + \Psi^e_v \cdot v + \Psi^e_w \cdot w)\, d\omega \\ &= \tfrac{1}{2}\iint_O (p_u\, u + p_v\, v + p_w\, w)\, d\omega\,,\end{aligned} \tag{2.40}$$

which expresses *Clapeyron's theorem* for continuous bodies:

This actual deformation work is one-half as great as the work done by the surface tractions that would maintain the deformation that would be achieved if they had those final values during the entire process.

Furthermore, equation (2.40) can be interpreted from the standpoint of the functional calculus. One can easily conclude from the definition [eq. (2.32)] of the extremal integral as the functional:

$$\Psi^e\,[u(\omega), v(\omega), w(\omega)\,;\,\omega]$$

that Ψ^e is an "entire" functional of the arguments, with FRÉCHET's terminology ([91]). If one multiplies all functions upon which an entire functional depends by a constant *c* and multiplies the value of the functional when has substituted those new functions by precisely c^n then the "entire" functional will be homogeneous of order *n* ([92]). If we were to now multiply the boundary values $u(\omega)$, $v(\omega)$, $w(\omega)$ that establish the extremal in the present case by an arbitrary constant *c* then, due to the homogeneous and linear character of the Euler-Lagrange equations (2.12), the functions *u* (*x*, *y*, *z*), *v* (*x*, *y*, *z*), *w* (*x*, *y*, *z*) would also be multiplied by that constant *c*. The expression (2.32) ([93]) for the extremal integral shows that it is then multiplied by c^2:

([91]) M. FRÉCHET, Ann. sci. de l'E.N.S. (3) **27** (1910), pp. 193.
([92]) M. FRÉCHET, *loc. cit.*
([93]) *Editor's remark: In conjunction with (2.31).*

$$\Psi^e\,[c\,u(\omega)\,,\,c\,v\,(\omega),\,c\,w\,(\omega)] = c^2\;\Psi^e\,[\,u(\omega)\,,\,v\,(\omega),\,w\,(\omega)]\,. \tag{2.41}$$

The deformation work Ψ^e *is therefore an “entire” homogeneous functional of order two of the three arguments. One can then regard Clapeyron’s equation (2.40) as an adaptation of Euler’s theorem to entire homogeneous functionals.*

FRÉCHET gave a representation for all functionals that are continuous functions of their arguments ([94]). In our case, it would read:

$$\Psi^e\,[\,u(\omega)\,,\,v\,(\omega),\,w\,(\omega)]$$

$$\begin{aligned}= \lim_{n=\infty}\iint\iint \quad &\{K_n(\omega,\omega^*)\,u(\omega)\,u(\omega^*) + 2L_n(\omega,\omega^*)\,u(\omega)\,v(\omega^*)\\ &+M_n(\omega,\omega^*)\,v(\omega)\,v(\omega^*) + 2N_n(\omega,\omega^*)\,u(\omega)\,w(\omega^*)\\ &+2O_n(\omega,\omega^*)\,v(\omega)\,w(\omega^*) + P_n(\omega,\omega^*)\,w(\omega)\,w(\omega^*)\}\,d\omega\,d\omega^*.\end{aligned} \tag{2.42}$$

When the individual functions $K_n(\omega,\omega^*)$, etc., converge uniformly with increasing n to the functions $K\,(\omega,\omega^*)$, etc., that are defined on the entire surface, one can simplify the expression to:

$$\begin{aligned}\Psi^e = \iint\iint \quad &\{K(\omega,\omega^*)\,u(\omega)\,u(\omega^*) + 2L(\omega,\omega^*)\,u(\omega)\,v(\omega^*)\\ &+M(\omega,\omega^*)\,v(\omega)\,v(\omega^*) + 2N(\omega,\omega^*)\,u(\omega)\,w(\omega^*)\\ &+2O(\omega,\omega^*)\,v(\omega)\,w(\omega^*) + P(\omega,\omega^*)\,w(\omega)\,w(\omega^*)\}\,d\omega\,d\omega^*.\end{aligned} \tag{2.43}$$

Later, we will see that there is, in fact, an expression with that simple form by solving the boundary value problem for the fundamental equations of elasticity (2.12). We would then like to remark, however, that it also lets us work with the expression (2.42) unchanged, by the following directly-relevant argument:

It emerges from this representation the first functional derivatives of Ψ^e:

$$\begin{aligned}\Psi^e_u &= 2\iint\{K(\omega,\omega^*)\,u(\omega^*) + L(\omega,\omega^*)\,v(\omega^*) + M(\omega,\omega^*)\,w(\omega^*)\}\,d\omega^*\,,\\ \Psi^e_v &= 2\iint\{L(\omega,\omega^*)\,u(\omega^*) + M(\omega,\omega^*)\,v(\omega^*) + O(\omega,\omega^*)\,w(\omega^*)\}\,d\omega^*\,,\\ \Psi^e_w &= 2\iint\{N(\omega,\omega^*)\,u(\omega^*) + O(\omega,\omega^*)\,v(\omega^*) + P(\omega,\omega^*)\,w(\omega^*)\}\,d\omega^*\end{aligned} \tag{2.44}$$

have the form of entire first-order functionals of the surface displacements, as indeed they should, from (2.40).

([94]) M. FRÉCHET, *loc. cit.*

If we take the functional derivatives of those expressions (2.44), i.e., the second functional derivatives of the deformation work Ψ^e with respect to the functions $u(\omega)$, $v(\omega)$, $w(\omega)$ at an arbitrary location ω^* on the surface, then we will see that they will be:

$$\begin{aligned}
&\Psi^e_{uu} = 2K(\omega,\omega^*), \quad \Psi^e_{uv} = \Psi^e_{vu} = 2L(\omega,\omega^*), \quad \Psi^e_{uw} = \Psi^e_{wu} = 2N(\omega,\omega^*), \\
&\Psi^e_{vv} = 2M(\omega,\omega^*), \quad \Psi^e_{vw} = \Psi^e_{wv} = 2O(\omega,\omega^*), \quad \Psi^e_{ww} = 2P(\omega,\omega^*),
\end{aligned}$$

so here they will no longer depend upon the special choice of the functions $u(\omega)$, $v(\omega)$, $w(\omega)$ at all. They are functions of only the two points ω, ω^* on the surface and are determined by only the nature of Ψ^e, i.e., by the form of the elastic body and its elastic properties. From the general laws of the functional calculus ([95]), they must be symmetric functions of their two arguments ω and ω^*.

Those properties of the functional derivatives of the deformation work are the source of a *reciprocity theorem* that is analogous to Maxwell's theorem for frameworks and was presented by BETTI. We consider two different systems of surface displacements:

$$u'(\omega),\ v'(\omega),\ w'(\omega) \qquad \text{and} \qquad u''(\omega),\ v''(\omega),\ w''(\omega)$$

and construct the actual deformation work done on the body by both of them:

$$\Psi^{e'} = \Psi^e[u'(\omega), v'(\omega), w'(\omega)] \qquad \text{and} \qquad \Psi^{e''} = \Psi^e[u''(\omega), v''(\omega), w''(\omega)] .$$

When we construct the expressions:

$$\iint \{\Psi^{e'}_{u'}(\omega^*)\cdot u''(\omega^*) + \Psi^{e'}_{v'}(\omega^*)\cdot v''(\omega^*) + \Psi^{e'}_{w'}(\omega^*)\cdot w''(\omega^*)\}\, d\omega^*$$

and

$$\iint \{\Psi^{e''}_{u''}(\omega^*)\cdot u'(\omega^*) + \Psi^{e''}_{v''}(\omega^*)\cdot v'(\omega^*) + \Psi^{e''}_{w''}(\omega^*)\cdot w'(\omega^*)\}\, d\omega^*$$

from the functional derivatives, and substitute the expressions (2.44) for the functional derivatives $\Psi^{e'}_{u'}$, etc., then we will see that due to the symmetry of the second derivatives K, …, P, one must have:

$$\iint \{\Psi^{e'}_{u'}\cdot u'' + \Psi^{e'}_{v'}\cdot v'' + \Psi^{e'}_{w'}\cdot w''\}\, d\omega^* = \iint \{\Psi^{e''}_{u''}\cdot u' + \Psi^{e''}_{v''}\cdot v' + \Psi^{e''}_{w''}\cdot w'\}\, d\omega^* \tag{2.45}$$

identically. However, from Castigliano's theorem (2.38), the functional derivatives of Ψ^e are equal to the surface tractions that would maintain the deformation of the body. For two such systems of surface displacements and surface tractions u', v', w', p'_x, p'_y, p'_z and u'', v'', w'', p''_x, p''_y, p''_z, one will then have the reciprocity relation:

([95]) Cf., e.g., V. VOLTERRA, *Leçons sur les fonctions de lignes*, Paris, 1913, pp. 26.

$$\iint \{p'_x u'' + p'_y v'' + p'_z w''\} d\omega^* = \iint \{p''_x u' + p''_y v' + p''_z w'\} d\omega^* . \tag{2.46}$$

That is the form in which one usually expresses *Betti's theorem.*

We now turn to the *representation of the deformation work as a functional of the surface tractions* Ψ^{*e} [p_x (ω), p_y (ω), p_z (ω)]. As we can next establish, that functional is also an "entire" functional. If we then apply the given criterion for homogeneity then we will see that it is likewise homogeneous of degree two. From Fréchet's theorem, it can then be represented by a boundary value that is analogous to (2.42), and as our direct representation will later show, there also exist boundary values for the coefficients, such that we will directly have the representation:

$$\begin{aligned} \Psi^{*e}[p_x(\omega), p_y(\omega), p_z(\omega)] \quad &= \iint\iint \{\bar{\mathrm{K}}(\omega,\omega^*)\, p_x(\omega)\, p_x(\omega^*) \\ &+ \quad 2\bar{\Lambda}(\omega,\omega^*)\, p_y(\omega)\, p_y(\omega^*) \\ &+ \quad \bar{\Pi}(\omega,\omega^*)\, p_z(\omega)\, p_z(\omega^*)\}\, d\omega\, d\omega^*. \end{aligned} \tag{2.47}$$

An application of the adaptation of Euler's theorem for homogeneous functionals leads to the following representation for Ψ^{*e} with the help of its functional derivatives:

$$2\Psi^{*e} = \iint \{\Psi^{*e}_{p_x}[p_x(\omega), p_y(\omega), p_z(\omega)]\, p_x(\omega) + \Psi^{*e}_{p_y}\; p_y(\omega) + \Psi^{*e}_{p_z}\; p_z(\omega)\}\, d\omega\ . \tag{2.48}$$

Along with that representation of the deformation work Ψ^{*e}, we can also represent it by using the Clapeyron equation (2.40):

$$2\Psi^{*e} = \iint \{u(\omega)\, p_x(\omega) + v(\omega)\, p_y(\omega) + w(\omega)\, p_z(\omega)\}\, d\omega\ , \tag{2.49}$$

in which u, v, w are the surface displacements that correspond to given surface tractions. Of course, they are not determined uniquely by the surface tractions, but their general values will emerge from any particular values of $\bar{u}$, $\bar{v}$, $\bar{w}$ when one introduces six constants by the formulas ([96]):

$$u = \bar{u} + a + d\,y - e\,z\,, \qquad v = \bar{v} + b + c\,z - f\,x, \qquad w = \bar{w} + c + f\,x - d\,y\,.$$

Nevertheless, the expression (2.49) is unique since the additional terms will drop out due to the equilibrium conditions (2.4) for the surface tractions.

However, due to those equilibrium conditions (2.4), the three functions p_x, p_y, p_z in (2.49) are not completely arbitrary, as is also the case for equation (2.48), and we cannot therefore conclude that the coefficients of those two integrals are equal to each other. In order to be able to regard the p_x, p_y, p_z as independently varying, we must also consider the auxiliary conditions (2.4) by the

([96]) *Ed. rem.: The formulas are incorrect. Their correct forms must read:*

$$u = \bar{u} + a + e\,z - f\,y\,, \qquad v = \bar{v} + b + f\,x - d\,z\,, \qquad w = \bar{w} + c + d\,y - e\,x\,.$$

method of Lagrange factors, as one learns in the calculus of variations in the treatment of isoperimetric problems. Those Lagrangian factors are constants that we denote by α, β, γ, λ, μ, ν and then get from (2.48) and (2.49) that:

$$\iint \{(\Psi^{*e}_{p_x} - u - \alpha - \mu\, z + \nu\, y)\, p_x + (\Psi^{*e}_{p_y} - v - \beta - \nu\, x + \lambda\, z)\, p_y + (\Psi^{*e}_{p_z} - w - \gamma - \lambda\, y + \mu\, x)\, p_z\}\, d\omega = 0,$$

in which the functions p_x, p_y, p_z can now be regarded as varying arbitrarily. It will then follow from this equation that all three expressions in parentheses must vanish individually ([97]), and that will then yield:

$$\begin{aligned} \Psi^{*e}_{p_x} &= (u + \alpha + \mu\, z - \nu\, y)\, p_x\,, \\ \Psi^{*e}_{p_y} &= (v + \beta + \nu\, x - \lambda\, z)\, p_y\,, \\ \Psi^{*e}_{p_z} &= (w + \gamma + \lambda\, y - \mu\, x)\, p_z\,. \end{aligned} \tag{2.50}$$

Those equations express a theorem that we would like to refer to as the *first Castigliano theorem.* It is the analogue of the theorem for frameworks to which we gave that name and says that:

The functional derivatives of the extremal value of the deformation work, which are represented as functionals of the surface tractions, are equal to displacement components at the associated points of the surface, up to a displacement that displaces the body like a rigid body.

If the body is constrained, say, in such a way that a motion like a rigid body is impossible, while its elastic motion is not obstructed in any way, then we can calculate the values of the six constants in equations (2.50) from the constraining conditions and then have equations that would immediately yield the purely-elastic displacements of the points on the surface. By contrast, it is naturally not possible here to split off part of the surface tractions as forces of reaction, and in that way, to avoid introducing the equilibrium conditions (2.14) as auxiliary conditions. Namely, the constraining conditions would then necessitate reactions that are "isolated forces," and therefore … singularities of the elastic deformations … ([98]).

2.6. A second conception of Menabrea's principle that is valid for multiply-connected bodies. – Up to now, we have considered the deformations of simply-connected bodies when either the displacement components or the surface tractions were given on the surface. The solution to the problem was implied by the fundamental equations of elasticity (2.12), which are the Euler-Lagrange equations for the variational problem defined by the deformation work. We could next vary the deformation while maintaining compatibility (for the varied state, as well). The equilibrium conditions can no longer be fulfilled in the varied state. Rather, they are given by the

([97]) For the fundamental lemma in the calculus of variations for the isoperimetric problem. Cf., BOLZA, *Lehrbuch der Variationsrechnung*, Leipzig and Berlin (1909), pp. 460.

([98]) *Ed. rem.: The places that where … has been written are illegible.*

condition that the deformation work is an extremum. On the other hand, with Menabrea's principle, we have carried out a variation under which we have required that equilibrium should exist in each volume element. The compatibility conditions will no longer be fulfilled in the varied state. It is only in those deformation states that are characterized by the extremum of the deformation work that the compatibility conditions will be fulfilled, such that we can then construct the elastic body in the deformed state from the individual volume elements with no gaps.

A variation for which equilibrium should exist for every volume element in the varied state, as well as fulfilling the compatibility conditions, is not possible, since it is only for a simply-connected body that there is only a single state in which the conditions are fulfilled.

Things are different when we deal with a *multiply-connected body*, and we would like to consider an annular body as the simplest example. We first point out the following important difference between the two types of connectedness. Whereas it is impossible for a stress state to appear in a simply-connected body when all of the surface tractions are zero, that might very well be possible in a multiply-connected one. If we imagine, say, the ring in question as an example, which might be originally stress-free, and make it a simply-connected by cutting through it with a cross-section and apply external surface tractions to the boundary surfaces that were newly created by the cross-section that are equal and opposite at the corresponding points (i.e., originally coincident) then we will have a simply-connected body that is loaded with surface tractions that fulfill the equilibrium conditions (2.4) and will therefore deform the body in a completely-determined way under their influence. The two surfaces that arise from the cross-section will then be either separated from each other such that a gap will be created, or they will try to get back together again. Following **Volterra**, we call such an operation a *distortion*. The newly-created ring is found to be in a stressed state, although it is not loaded with any external forces on the surface. For bodies with a higher degree of connection, (e.g., a sphere with handles), we can proceed likewise, i.e., first make them simply-connected by sufficiently-many cross-sections and then perform a distortion on each cross-section like the one that was just performed on the ring. A stress state can then be created in such a body in many ways without having to load the surface of the body with tractions.

If we then have a multiply-connected body that is originally stress-free and produce a certain stress state by applying some sort of surface tractions then we can vary that state in such a way that the equilibrium conditions, as well as the compatibility conditions, will then be fulfilled. We will only need to do that on the cross-sections that make it simply connected then.

That argument now allows us to reduce the *stress problem for an arbitrarily-loaded stress-free multiply-connected body* to the stress problem for a simply-connected body, which we would now like to consider. In order to do that, we make the body simply-connected by means of the necessary number of cross-sections and apply equal and opposite surface tractions, which are chosen arbitrarily, moreover, to the two surfaces of the individual cross-sections.

Since the forces that are applied to the individual cuts preserve equilibrium, the system of surface tractions that is applied to the resulting simply-connected body will again be in equilibrium precisely. The stress problem for the simply-connected body then leads to a well-defined solution, and we will then obtain a well-defined value for the deformation work Ψ^{*e}, which is a functional of the original surface tractions and the surface tractions that are applied to the two surfaces of each cut.

We assume that we have determined that functional. On the one hand, it depends upon the given surface tractions $p_x(\omega)$, $p_y(\omega)$, $p_z(\omega)$ on the surface of the uncut body, as well as upon the surface tractions $\bar{p}_x(\omega')$, $\bar{p}_y(\omega')$, $\bar{p}_z(\omega')$ and $\bar{p}_x(\omega'')$, $\bar{p}_y(\omega'')$, $\bar{p}_z(\omega'')$ that are introduced on the two surfaces of a cross-section. In that way, one has $\bar{p}_x(\omega') = -\bar{p}_x(\omega'')$, etc., when ω' and ω'' are two points of the two cut-surfaces that would coincide in the uncut body. We can represent the deformation work by way of equation (2.42) or (2.43) as a two-fold double integral that extends over the surface of the cut body, i.e., over the original surface and each of the individual surfaces that are created by each cut. In order to suggest that dependency of Ψ^{*e}, we write:

$$\Psi^{*e} = \Psi^{*e}[p_x(\omega), \bar{p}_x(\omega'), \bar{p}_x(\omega''); \bar{p}_x(\omega), \bar{p}_y(\omega'), \bar{p}_z(\omega''); \bar{p}_x(\omega''), \bar{p}_y(\omega''), \bar{p}_z(\omega'')]\,. \tag{2.51}$$

In it, we would like to leave the surface tractions that are applied to the surface of the uncut body unvaried but vary the forces that are applied to the two surfaces of each cut. That variation shall be performed in such a way that we always perform equal and opposite, but otherwise arbitrary, variations of the surface tractions at corresponding points of the surfaces of each cut. In that way, the equilibrium conditions that must exist for the surface tractions will remain true, such that we will not need to consider any auxiliary conditions. If we then construct the functional derivatives for an associated pair of points ω' and ω'' and subtract them then we will get:

$$\Psi^{*e}_{\bar{p}_x}[p_x(\omega),\ldots,|\,\omega'] - \Psi^{*e}_{\bar{p}_x}[p_x(\omega),\ldots,|\,\omega''] = u(\omega') - u(\omega'')\,, \qquad \text{etc.,} \tag{2.52}$$

from the first Castigliano theorem. The right-hand side of equation (2.52) represents the relative displacement of the two associated points of a cross-section that occurs as a result of elasticity, because the displacement of the elastic body as a rigid-body that is still included in *u* drops out of the difference, since it is the same for both points.

Under the actual deformation of the uncut multiply-connected body that it experiences due to the forces applied to its surface, the body will also remain connected after the deformation at the (imagined) cut-surfaces without having to remove or add material. Therefore, the relative displacement of the connected points of a cut must be equal to exactly zero, and we will then get the equation:

$$\Psi^{*e}_{\bar{p}_x}[p_x(\omega), \bar{p}_x(\omega'), \bar{p}_x(\omega''),\ldots,|\,\omega'] - \Psi^{*e}_{\bar{p}_x}[p_x(\omega), \bar{p}_x(\omega'), \bar{p}_x(\omega''),\ldots,|\,\omega''] = 0 \tag{2.53}$$

as the condition for the surface tractions at the corresponding points of each cross-section.

It is possible to calculate the forces $\bar{p}_x(\omega') = -\bar{p}_x(\omega'')$, etc., that we must apply in order to keep the body connected under the surface tractions, despite the cuts that are made, from those three equations that exist for each point of each cross-section (from general theorems on implicit functionals). That says only that we know the stress that will arise in the body as a result of the given strain in the imagined cross-section. If we know that, then the stress problem for a multiply-connected body will be reduced to one for a simply-connected body, and with that, the problem will be solved with our assumption.

The surface tractions on the surfaces of the individual cross-sections are then to be determined in such a way that the variation of the functional (2.51) that represents the actual deformation work done on the simply-connected body that is created by the cuts will vanish. However, one can also express that vanishing of the variation by saying that the deformation work shall be an extremum. In that way, we have once more characterized the actual state that occurs as a result of an extremal requirement, and indeed it has a superficial similarity to Menabrea's principle that was introduced above and is also referred to as *Menabrea's principle* in the literature. However, one sees here that both principles are essentially quite different for continuous bodies.

Menabrea's principle, in its first conception, relates to the interior of an elastic body. *One considers the dependency of the deformation work on the values of the stress components in the interior* when their values on the surface are prescribed by the boundary conditions. The principle demands that those functions should be determined in the interior in such a way that the deformation work is an extremum. If we would like to express that briefly in the language of the calculus of variations then we would seek the extremal for given boundary values.

Menabrea's principle, in its second conception, compares only states of deformation that fulfill the equilibrium conditions, as well as the compatibility conditions, so they satisfy the extremum conditions for the interior, but the boundary values are not all prescribed now. Rather, we now consider *the dependency of the deformation work on part of the boundary values and shall determine it in such a way that the extremum will again occur from the extremal values (extremum extremorum).* ([99])

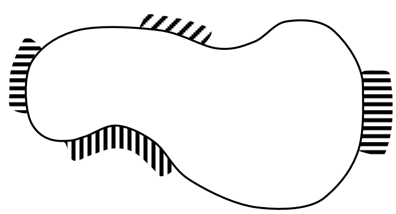

Figure 2.

The second conception of Menabrea's principle finds an application in the problem of *determining the deformation of a simply-connected elastic body that has part of its surface bounded by a rigid body*, such that the displacement will be prevented there, while the free part of the surface is loaded with arbitrary surface tractions [that now no longer need to fulfill the equilibrium conditions (2.4) by themselves].

The problem demands that one must determine a solution to the fundamental elastic equations (2.12) when the boundary conditions consist of specifying the tractions on part of the surface and the displacements on the other parts. That is because we know that the displacement will be zero at all points where the elastic body is bounded by the forced surfaces. Such a problem with "mixed boundary conditions" proves to be much more complicated than the boundary-value problems in the theory of boundary-value problems for partial differential equations that were considered up

([99]) The difference between the two conceptions of the variation is precisely the same as the difference between the so-called Hamilton principle and the principle of varied action in analytical mechanics.

to now ([100]). It would be an important advance if we were to reduce the new problem to the old one in order to determine the state of deformation for given surface tractions.

In order to achieve that, we observe that the new boundary condition will also determine the state of deformation uniquely. After a deformation has occurred, the elastic body will exert well-defined tractions on the rigid surfaces that will be canceled precisely by the reactions $R_x(\omega)$, $R_y(\omega)$, $R_z(\omega)$.

Those reactions and the prescribed surface tractions must collectively fulfill the six equilibrium conditions (2.4).

When we apply those reactions to the elastic body as surface tractions after removing the rigid body, its deformation will not change at all. If we therefore know the reactions then the problem will have been reduced to the stress problem for prescribed surface tractions. It would then come down to determining the reactions. In order to achieve that, we first assume that they are arbitrary and determine the deformation work as a function of the surface tractions that are now present for the elastic body when it is loaded in that way.

The functional derivatives with respect to those surface forces give the displacement of the surface points at each location with the indeterminacy that was given above. However, we know that those displacements will be zero at all points that contact the rigid surfaces, so we will get three equations:

$$\Psi^{*e}_{R_x} = \alpha + \mu z - \nu y , \quad \Psi^{*e}_{R_y} = \beta + \nu z - \lambda x , \quad \Psi^{*e}_{R_z} = \gamma + \lambda y - \mu x , \tag{2.54}$$

for each such point, from which we can ascertain the unknown reactions by solving them. The equilibrium conditions (2.4) that all of the surface forces (viz., given and reactions) must satisfy determine the six constants, which are still arbitrary here ([101]).

If one would like then one can interpret equations (2.54) as conditions for the extremum problem, *so one determines the reactions in such a way that the deformation work will be an extremum*, and indeed in the same sense that we just gave to it for the multiply-connected body.

2.7. Internal stresses. – A free, simply-connected, elastic body will be stress-free when no surface tractions act upon it. However, in the two cases that were treated in the previous section, the body would not need to be stress-free in the absence of surface tractions. We shall employ that

([100]) Cf., e.g., J. HADAMARD, *Leçons sur la propagation des ondes*, Paris (1903), pp. 55.

([101]) *Editor's remark: The statements in the last paragraph are imprecise. When, as was remarked in the preceding paragraph, the "displacements are zero" on the "rigid surfaces," the right-hand sides of eqs. (2.54) must be set to zero.*

However, the deformation of eqs. (2.54) says that a rigid-body displacement is purported for each of the "forced surfaces." [Compare that to the footnote to eq. (45) in Chapter 1.] For the general possibility of prescribing arbitrary displacements on the forced parts of the surface, confer Prange at the end of Section 2.8.

Finally, the (theoretical) case is still conceivable in which the "forced surface" is itself a rigid surface that can displace only like a rigid body. The α, β, γ, λ, μ, ν will then be unknown for each "forced surface." As a supplementary condition, it must then be demanded that the resulting forces and moments of the surface tractions p_x, p_y, p_z that act between the body and the "forced surface" must be equal to the forces and moments that act upon the "forced surface," i.e., that the equilibrium conditions must be fulfilled after cutting the "forced surface" free.

state in order to define the variations there, but assume that the body considered is, in reality, stress-free with no surface tractions acting on it. Meanwhile, our argument would not be modified essentially if internal stresses were present in the body. Naturally, the internal stresses must be known if we are to determine the deformation in the absence of surface tractions. We next ask how one might characterize them.

As Volterra showed, that is relatively simple for multiply-connected bodies. If we were to make it simply connected by means of cross-sections then the two faces of the cut would be moved relative to each other when the body returns to the stressless state. The theorem that the displacements are determined uniquely by the fundamental equations ([102]) tells us that the displacements u', v', w' and u'', v'', w'' of the originally-connected points of the two cut-faces that occur … ([103]) are coupled by the following formulas:

$$\begin{aligned} u'-u'' &= l + r\,y - q\,x\,, \\ v'-v'' &= m + p\,z - r\,x\,, \\ w'-w'' &= n + q\,x - p\,y\,, \end{aligned} \tag{2.55}$$

in which *l*, *m*, *n*, *p*, *q*, *r* are six constants for the … ([104]). The one face has performed a rigid-body motion with respect to the other one. Conversely, we can also produce the most-general internal stresses by moving the surfaces of the cross-sections that make the body simply connected in that way. The motion is known to consist of three translations and three rotations, each of which are characterized by six constants ([105]).

Knowing those distortions is also enough to determine the state of deformation of a multiply-connected body from the surface displacements, as well as the surface tractions.

If we would like to address the stress problem for a multiply-connected body, which is replaced with internal stresses from known distortions, then we can proceed precisely as in the previous section, only we can no longer set the right-hand sides of equations (2.53) equal to zero, but to the right-hand sides of (2.55) that are determined by distortions. However, one can also determine the effects (viz., stress forces) that are produced at the points of the cut from the equations that will arise in that way, which will then reduce the stress problem to the one for a simply-connected body.

The extremal problem that this equation implies consists of extremizing the functional:

$$\Psi^{*e} = \sum \iint \{(u'-u'')\,p_x + (v'-v'')\,p_y + (w'-w'')\,p_z\}\,d\omega\,, \tag{2.56}$$

in which the integrals in the summation extend over the individual cross-sections on which the distortions are performed (the summation is over the number of cross-sections). If one introduces

([102]) V. VOLTERRA, "Sur l'équilibre des corps élastiques multiplement connexe," Ann. sci. de l'É.N.S (3) **24** (1907), 410-517.

([103]) *Ed. rem: The (handwritten) word was illegible.*

([104]) *Ed. rem: The (handwritten) word was illegible.*

([105]) *Editor's remark: Equation (2.55) means that each of the two cut-faces can only displace like a rigid surface (rigid body), which is similar to a beam cross-section, from the validity of the Bernoulli hypothesis. That is by no means the general case.*

the expressions (2.55) into that then we will see that it is precisely the sum of the three components of the resultant of the cross-sectional stresses and their moments with respect to the coordinate axes that will appear in the integrals in the sum. However, in order to take functional derivatives, one must start from the expression (2.56).

If internal stresses are present that are due to forced rigid surfaces in the second case then they cannot be characterized so simply, because when the displacements are obstructed over entire regions of the surface, one cannot succeed in making the body free of internal stresses by means of a finite number of cuts ([106]). In this case, the entire stress state must be given. One can then calculate the displacements that the individual points of the forces regions of the surface had experienced when the internal stresses were produced from that stress state ([107]).

The stress problem will now reduce to the stress problem with given surface tractions, just as it did when no internal stresses were given, except that we have not replaced the functional derivatives of the deformation work with respect to the arbitrarily-given reactions of the forced surfaces in the form (2.54), but we have added the known displacements at the points in question on the right-hand side.

2.8. The influence of temperature on deformation. Engesser's work done by expansion. – Up to now, we have assumed that the elastic body that we subject to deformation possesses equal temperatures everywhere. Such a body will be stress-free in an unloaded state when it is simply connected, no matter what degree its temperature might be. Things are different for a multiply-connected body. If the body is stress-free at constant temperature then it will no longer remain stress-free at a different temperature, even if it is constant over the entire body, but it will exhibit internal stresses in that way. We can ascertain those internal stresses in such a way that we determine the distortions that would produce the same effect as a constant change in temperature. Therefore, the problems that one could address here would reduce to the ones in the previous section. Those phenomena present an analogy to the behavior of the statically-determinate and statically-indeterminate frameworks that we discussed in § **7** of the first chapter ([108]).

([106]) Things are similar for the so-called "casting stresses." For them, a body that is apparently simply connected consists of an (infinite) sequence of multiply-connected layers as a result of a process of successive solidification.

([107]) Conversely, we can determine the values of those displacements from the internal stresses. In order to do that, we must solve the stress problem for the body when the displacements are zero on the forced parts of the surface and the surface tractions are zero on the free parts.

([108]) *Editor's remark: The statements in this paragraph are not correct.*

An elastic body that is stress-free in an unloaded state will also be stress-free after being heated when the following three conditions are fulfilled:

- *All of the material properties of the body must be homogeneous (including the linear, thermal coefficient of expansion), i.e., for isotropic materials, one must have $E(x, y, z) = \text{const.}$ and $\alpha(x, y, z) = \text{const.}$*
- *The body cannot be supported (nicht gelagert sein) (or only statically-determinate).*
- *The temperature can vary only linearly, i.e., $t(x, y, z) = f_0 + f_1\,x + f_2\,x + f_3\,x$.*

In order to prove that, one ascertains the deformation quantities ε_x, ε_y, ε_z, γ_x, γ_y, γ_z for the given temperature distribution from (2.57, 2.57a) under the assumption that the deformed state is stress-free, and therefore the equilibrium conditions are fulfilled. If one puts those deformation quantities into eq. (2.9) then one will see that the compatibility conditions are also fulfilled. Nothing will change for multiply-connected bodies then. As a Gedanken experiment, one first converts the multiply-connected body into a simply-connected one. One then adds additional material and labels the separating surfaces between the original body and the added material. After the deformation, We begin the investigation anew and go back to the generally-valid Ansatz (2.2) of the principle of virtual displacements.

The first thing to do is to exhibit the relationship between the deformation quantities and the stress components. If α is the linear coefficient of thermal expansion of the material that comprises the body then the extensions of the length that result from a temperature change of $t(x, y, z)$ degrees will be $\varepsilon_x = \varepsilon_y = \varepsilon_z = \alpha\, t$, while the changes in angle γ_x, γ_y, γ_z will not occur. In place of the relations (2.7) between the deformation quantities and the stress components that are given by Hooke's law, one will now have the relations:

$$\begin{aligned}
\varepsilon_x &= \frac{1}{2\mu}\sigma_x - \frac{\lambda}{2\mu(3\lambda+2\mu)}(\sigma_x+\sigma_y+\sigma_z)+\alpha\, t\,,\\
\varepsilon_y &= \frac{1}{2\mu}\sigma_y - \frac{\lambda}{2\mu(3\lambda+2\mu)}(\sigma_x+\sigma_y+\sigma_z)+\alpha\, t\,,\\
\varepsilon_z &= \frac{1}{2\mu}\sigma_z - \frac{\lambda}{2\mu(3\lambda+2\mu)}(\sigma_x+\sigma_y+\sigma_z)+\alpha\, t\,,
\end{aligned} \tag{2.57}$$

while the other three will remain:

$$\gamma_x = \frac{1}{\mu}\tau_x\,, \qquad \gamma_y = \frac{1}{\mu}\tau_y\,, \qquad \gamma_z = \frac{1}{\mu}\tau_z\,. \tag{2.57a}$$

Those relations can be regarded as a special case of a more general way of looking at things in which the deformation quantities are any sort of given single-valued functions of the stress components, into which the coordinates x, y, z of the point in question of the body can also enter. Thus, they might have, say, the form:

$$\varepsilon_x = f_1(\sigma_x, \sigma_y, \sigma_z, \tau_x, \tau_y, \tau_z)\,, \qquad \text{etc.,} \tag{2.58}$$

in which the form of the *f will also depend upon the location of the point considered in the body.* We can, perhaps, represent it as a polynomial in the stress components whose coefficients are known functions of the location.

The solution of (2.58) for the stress components might read:

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one separates the deformed, simply-connected body along those labelled separating surfaces again and removes the added material. Since the deformed, simply-connected body is stress-free, the state of deformation will not change by the removal of the added material. A multiply-connected, unsupported body with homogeneous material properties will also remain stress-free under a linearly-varying change in temperature.

The following statements are true for a framework: A framework that is statically determinate when one includes the support conditions will remain stress-free under any change of form that is due to heating. A statically-determinate supported framework that is internally statically indeterminate will remain stress-free under constant heating when all of the rods in the framework possess the same E modulus, the same cross-sectional area F, and the same linear coefficient of thermal expansion α.

$$\sigma_x = \varphi_1 \,(\varepsilon_x,\, \varepsilon_y,\, \varepsilon_z,\, \gamma_x,\, \gamma_y,\, \gamma_z)\;, \qquad \text{etc.,} \tag{2.59}$$

in which the φ are the same type of functions as the f. If we were to develop our theory once more for this more general relationship then we would obtain it for the case of temperature stresses when we introduce the special expressions (2.57) for the f. The functions φ must (from the foundations of the mechanical theory of heat, which we will return to in section 2.9) have the form that makes the integrand in the volume integral:

$$\sigma_x\,\delta\varepsilon_x + \sigma_y\,\delta\varepsilon_y + \sigma_z\,\delta\varepsilon_z + \tau_x\,\delta\gamma_x + \tau_y\,\delta\gamma_y + \tau_z\,\delta\gamma_z$$

take the form of the total differential of a function $\psi\,(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_x, \gamma_y, \gamma_z)$:

$$\sigma_x\,\delta\varepsilon_x + \sigma_y\,\delta\varepsilon_y + \sigma_z\,\delta\varepsilon_z + \tau_x\,\delta\gamma_x + \tau_y\,\delta\gamma_y + \tau_z\,\delta\gamma_z = \delta\psi \tag{2.60}$$

by means of (2.59). The principle of virtual work will then have the form:

$$0 = -\iint_O (p_x\,\delta u + p_y\,\delta v + p_z\,\delta w)\,d\omega + \iiint_V \delta\psi\,d\kappa\,, \tag{2.61}$$

or, when we denote the … ([109]) integrand by $\delta\varphi$ ([110]) using (2.18a):

$$0 = \delta\left[\iint_O \varphi\,d\omega + \iiint_V \delta\psi\,d\kappa\right] = \delta E\,, \tag{2.62}$$

i.e., *E is an extremum. It is then regarded as dependent upon the displacement components u, v, w that are introduced into ψ according to the relations (2.1).*

From the rules of the calculus of variations, that extremum of the sum of integrals E leads to the Euler-Lagrange equations, and for the conditions on the surface, which possess the forms (2.21a) and (2.21b) precisely, but naturally, they will no longer coincide with the equations (2.12) and (2.13) now but are more general in character.

We have converted that variational problem by applying the canonical transformation. The new variables that we introduced in that way are also stress components here, because from (2.60) and (2.59), one has:

$$\begin{aligned} \frac{\delta\psi}{\delta\varepsilon_x} &= \varphi_1\,(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_x, \gamma_y, \gamma_z) = \sigma_x\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \frac{\delta\psi}{\delta\gamma_z} &= \varphi_6\,(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_x, \gamma_y, \gamma_z) = \tau_z\,. \end{aligned} \tag{2.63}$$

([109]) *Ed. rem.: A (handwritten) word could not be deciphered.*
([110]) One should not confuse this φ, which is the potential energy of the external forces, with the symbol in (2.59), so we shall preserve the old notation.

In the spirit of the Legendre transformation, the function:

$$H(\sigma_x, \sigma_y, \sigma_z, \tau_x, \tau_y, \tau_z) = \varepsilon_x \frac{\partial\psi}{\partial\varepsilon_x} + \varepsilon_y \frac{\partial\psi}{\partial\varepsilon_y} + \varepsilon_z \frac{\partial\psi}{\partial\varepsilon_z} + \gamma_x \frac{\partial\psi}{\partial\gamma_x} + \gamma_y \frac{\partial\psi}{\partial\gamma_y} + \gamma_z \frac{\partial\psi}{\partial\gamma_z} \tag{2.64}$$

will enter in place of ψ, in which the deformation quantities on the right-hand side are to be replaced with the stress components according to (2.63). The *canonical variational problem* that thus arises possesses the form:

$$\mathcal{E} = \iint_O \varphi\, d\omega + \iiint_V \left\{ \frac{\partial u}{\partial x}\sigma_x + \frac{\partial v}{\partial y}\sigma_y + \frac{\partial w}{\partial z}\sigma_z + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\tau_x + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\tau_y + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\tau_z \right.$$

$$\left. - H(\sigma_x, \sigma_y, \sigma_z, \tau_x, \tau_y, \tau_z) \right\} d\kappa . \tag{2.65}$$

As the extremal conditions, when we direct our attention to the displacements, we will have the equilibrium conditions for the interior and the surface in the forms (2.27a) and (2.27b) here. By contrast, the further conditions will be the compatibility conditions in the new form:

$$\frac{\partial u}{\partial x} = \frac{\partial H}{\partial \sigma_x}, \ldots, \quad \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial H}{\partial \tau_x}, \ldots \tag{2.66}$$

In order to arrive at *Menabrea's principle,* we impose the equilibrium conditions (2.27a) and (2.27b) as the auxiliary conditions in the canonical variational problem. If we partially-integrate the first six terms in the spatial integral then we will now get the variational problem:

$$\iiint_V H(\sigma_x, \sigma_y, \sigma_z, \tau_x, \tau_y, \tau_z)\, d\kappa = \text{extremum}, \tag{2.67}$$

in which we have ignored the inessential sign. Upon considering the auxiliary conditions using the method of Lagrange factors, that will imply the compatibility conditions (2.66) as the conditions for an extremum, precisely as it did in § **5**.

The function H is precisely the *work done by expansion* per unit volume that ENGESSER ([111]) introduced and for which he gave the representation:

$$\int_0^{\sigma_x, \sigma_y, \ldots} (\bar{\varepsilon}_x\, \delta\bar{\sigma}_x + \bar{\varepsilon}_y\, \delta\bar{\sigma}_y + \bar{\varepsilon}_z\, \delta\bar{\sigma}_z + \bar{\gamma}_x\, \delta\bar{\tau}_x + \bar{\gamma}_y\, \delta\bar{\tau}_y + \bar{\gamma}_z\, \delta\bar{\tau}_z) . \tag{2.68}$$

Namely, from (2.64), one has:

([111]) H. ENGESSER, Zeitschrift des Architeken- und Ingenieurvereins zu Hannover **35** (1889), pp. 733.

$$H = f_1(\varepsilon_x,\ldots,\tau_z)\cdot\sigma_x + f_2\,\sigma_y + f_3\,\sigma_z + f_4\,\tau_x + f_5\,\tau_y + f_6\,\tau_z - \psi(f_1, f_2,\ldots, f_6)\,, \tag{2.69}$$

in which ψ is expressed in terms of the stress components using (2.60), (2.59), and (2.58) as the integral:

$$\int_0^{\sigma_x,\sigma_y,\ldots} (\bar\sigma_x\,\delta\bar f_1 + \bar\sigma_y\,\delta\bar f_2 + \bar\sigma_z\,\delta\bar f_3 + \bar\tau_x\,\delta\bar f_4 + \bar\tau_y\,\delta\bar f_5 + \bar\tau_z\,\delta f_6)\ .$$

If we integrate that partially then we will have:

$$\psi = \sigma_x f_1 + \sigma_y f_2 + \ldots + \tau_z f_6 - \int_0^{\sigma_x,\sigma_y,\ldots} (\bar f_1\,\delta\bar\sigma_x + \bar f_2\,\delta\bar\sigma_y + \bar f_3\,\delta\bar\sigma_z + \bar f_4\,\delta\bar\tau_x + \bar f_5\,\delta\bar\tau_y + f_6\,\delta\bar\tau_z)$$

here, and when we introduce that into (2.69), we will, in fact, get the expression (2.68) for H precisely.

In the special case in which we treat only *the influence of temperature*, we have to introduce the expressions (2.57) for the f, and when we perform the integration, we will then get:

$$H = \frac{1}{2}\left[\frac{1}{\mu}\frac{\lambda+\mu}{3\lambda+2\mu}(\sigma_x+\sigma_y+\sigma_z)^2 - \frac{1}{\mu}(\sigma_y\,\sigma_z + \sigma_z\,\sigma_x + \sigma_x\,\sigma_y - \tau_x^2 - \tau_y^2 - \tau_z^2)\right]$$
$$- \alpha t(\sigma_x+\sigma_y+\sigma_z)\ . \tag{2.70}$$

That expression differs from the expression (2.24b), which represents the deformation work as a function of the stresses when Hooke's law is valid, by the appearance of the second term. MÜLLER-BRESLAU ([112]) gave that form, which he called the *ideal deformation work* per unit volume.

The *work done by expansion* for the entire body, i.e., the extremal integral:

$$B = \iiint_{\mathcal{E}} H(\sigma_x,\sigma_y,\sigma_z,\tau_x,\tau_y,\tau_z)\,d\kappa\,, \tag{2.71}$$

when regarded as a functional of the surface tractions, *enters in place of the deformation work in the case of the first Castigliano theorem* ([113]). We would like to show directly that its functional derivatives are connected with the surface displacements in the manner of equation (2.50).

If we vary the surface tractions then we will get, on the one hand:

$$\delta B = \iint_O (B_{p_x}\,\delta p_x + B_{p_y}\,\delta p_y + B_{p_z}\,\delta p_z)\,d\omega\ , \tag{2.72}$$

([112]) Cf., H. MÜLLER-BRESLAU, Wochenblatt für Architekten und Ingenieure **6** (1864), pp. 373.
([113]) By contrast, the deformation work stays the same for the second Castigliano theorem.

from the basic rules of the functional calculus. On the other hand, if those stress components assume new values in the extremal integral under that variation then the variation of the extremal integral will read:

$$\delta H = \iiint_{\mathcal{E}} \left(\frac{\partial H}{\partial \sigma_x} \delta\sigma_x + \frac{\partial H}{\partial \sigma_y} \delta\sigma_y + \frac{\partial H}{\partial \sigma_z} \delta\sigma_z + \frac{\partial H}{\partial \tau_x} \delta\tau_x + \frac{\partial H}{\partial \tau_y} \delta\tau_y + \frac{\partial H}{\partial \tau_z} \delta\tau_z \right) d\kappa \,,$$

or when we introduce the expressions (2.66) for the coefficients:

$$\delta B = \iiint_{\mathcal{E}} \left[\frac{\partial u}{\partial x} \delta\sigma_x + \frac{\partial v}{\partial y} \delta\sigma_y + \frac{\partial w}{\partial z} \delta\sigma_z + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \delta\tau_x + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \delta\tau_y + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \delta\tau_z \right] d\kappa \,.$$

If we partially-integrate the individual terms in this then the volume integral will drop out, since the variations alone must also fulfill the equilibrium conditions for the interior, and we will get:

$$\delta B = \iint_O \{ u\,(\delta\sigma_x \cos(x,n) + \delta\tau_z \cos(y,n) + \delta\tau_y \cos(z,n) + v(\cdots) + w(\cdots) \}\, d\omega \;.$$

However, the expression in brackets is precisely p_x, from the equilibrium conditions for the surface tractions, and since analogous statements are true for the other terms, we will have a second expression for the variation of B :

$$\delta B = \iint_O (u\ \delta p_x + v\,\delta p_y + w\,\delta p_z)\, d\omega \;. \tag{2.73}$$

By comparing (2.72) and (2.72), while considering the conditions (2.4) using the method of Lagrange factors, we will get relations of the form:

$$B_{p_x} = u + \alpha + \mu z + \nu y, \ldots,$$

which are analogous to (2.50) precisely. However, the second conception of Menabrea's principle and the extensions of it to which we arrived are based upon the relations (2.50). Therefore, the work done by extension will enter in place of the deformation work in all of those arguments.

2.9. Remark on the historical development. – At the time when the theory of elasticity was being first developed, one was accustomed to treating all questions of theoretical physics on the basis of the molecular theory of matter. It was therefore only natural then to look for a way of arriving at continuous elastic bodies upon starting from the theory of frameworks by regarding them as "molecular frameworks." The molecules (which one, with Boscovich, regarded as material points) would be the nodes of the framework, and the mutual forces that they exert upon each other would be created by stressed frame members. The fundamental Ansätze of Navier, Cauchy, and Poisson seemed to show the way by which one could move forward.

On the other hand, in England, under the influence of Green, the direct treatment of the continuum, while abandoning the molecular hypothesis, achieved validity relatively soon. The treatment of the elastic potential (viz., the deformation work) stood at the center of the theory of elastic bodies of that type, and it is that quantity that plays precisely the principal role in all of our questions. It probably explains the remarkable fact that some of the arguments of the Menabrea-Castigliano school for continuous bodies were anticipated by the Englishman Cotterill in a series of papers that have admittedly attracted little attention ([114]).

Cotterill arrived at his considerations by his ambition to give a precise sense to the entirely-vague "principle of least resistance" that was mentioned in the first chapter for the special case of continuous bodies. If one starts from the law of energy then it will say that the compatibility conditions will be implicit when one makes the work done by the internal stress components an extremum and adds the equilibrium conditions, which are considered to be valid only in the interior, as an auxiliary condition for the extremum ([115]). In that way, he had posed Menabrea's principle in its first conception, and he also considered the auxiliary conditions using the method of Lagrange factors. He also wished to represent the dependency of the the extremal value of the deformation work on the external forces, and thus define the extremal integral in our sense. Meanwhile, he restricted himself in so doing to the special case in which the elastic body is loaded with isolated forces. In that way, with no further analysis, he then avoided the complication that is inherent to the integration of the fundamental equations ([116]) when the load consists of isolated forces, and he assumed that the stress components could be represented by linear functions of the isolated forces with constant coefficients. In that case, the deformation work will be a quadratic function of the function of the isolated forces whose derivatives he obtained from the first "Castigliano" theorem. He also expressed the deformation work as a function of the displacements and had also expressed the second "Castigliano" theorem already. He also wrote down the second conception of Menabrea's principle in connection with the first Castigliano theorem.

Since, as we said, Cotterill's work has remained completely unknown, Castigliano's work has always been regarded as the starting point for the theory in the literature. In the subsequent developments in the molecular-theoretic Ansätze for the theory of elasticity, it would probably not be impossible to adapt the theory of frameworks, in the given sense, to continuous bodies. However, Castigliano did not invest such precision in his problem at all, but he only alluded to the concept of that adaptation, if only to leave it behind again ([117]). Rather, he straightaway assumed that the fundamental equations of elasticity were known and then imagined ascertaining the dependency of the quantities of deformation work to be determined on either the surface displacements or the surface tractions by integrating those equations. He had therefore not employed Menabrea's principle in its first conception at all, but rather he confined himself to expressing the dependency of the deformation work on the surface values, and in that way arrived

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([114]) H. COTTERILL, Philosophical Magazine (4) **29** (1865), pp. 299-305, 380-389, 430-436.

([115]) In that way, he himself rejected Moseley's principle, since it lacked precision, and he proposed to make his principle the fundamental principle of the theory of elasticity.

([116]) The same remark is true for all of engineering literature, moreover.

([117]) Without stating that clearly.

at the two theorems that bear his name. Menabrea's principle was then appealed to and applied only in its second conception, namely, to the determination of the unknown reactions on forced bodies and multiply-connected bodies.

In following through on those ideas, he probably always had beams and the work done on beams in mind. That would explain the peculiar assumptions that he introduced into his theory. Namely, he imagined that the elastic body was a polyhedron that was bounded by planar surfaces and also assumed that the bounding surfaces could still be regarded as planar after the deformation. Indeed, he then virtually assumed that the bounding surfaces were displaced like rigid surfaces under the entire deformation.

Under that assumption, the displacement of each planar bounding surface can be expressed by the displacement of one of its points – say, the center of mass – and the rotations around three axes that go through it, which we choose to be the normal to the planar piece and the two principal axes of inertia that lie in that plane. If we denote the components of the displacement of the center of mass by ξ, η, ζ and the components of the rotation by Θ_x, Θ_y, Θ_z then the displacements of the points of the planar boundary face (relative to the axis-system that was introduced in order to define them) will be:

$$u = \xi + z\,\Theta_y - y\,\Theta_z\,, \quad v = \eta + x\,\Theta_z - z\,\Theta_x\,, \quad w = \zeta + y\,\Theta_x - x\,\Theta_y\,. \tag{2.74}$$

By solving the fundamental equations of elasticity, he then wished to express the displacements of the interior points as functions of the surface displacements and obtained them as linear functions of all of the ξ, η, ζ, Θ_x, Θ_y, Θ_z for each polyhedral surface (with coefficients that are functions of position). The deformation work itself will then be a quadratic function of those quantities (with constant coefficients) ([118]).

On the other hand, if the surface tractions were given then he next imagined determining the three components X, Y, Z of the resultant and the three components of the resulting moments M_x, M_y, M_z for each boundary surface, and then imagined varying the distribution in such a way that he assumed:

$$p_x = \frac{X}{\Omega} - y\frac{M_z}{2J_x}\,, \qquad p_y = \frac{Y}{\Omega} - x\frac{M_z}{2J_y}\,, \qquad p_z = \frac{Z}{\Omega} + y\frac{M_x}{2J_x} - x\frac{M_y}{2J_y}\,, \tag{2.75}$$

in which:

$$\Omega = \iint dx\,dy$$

is its area, and:

$$J_x = \iint y^2\,dx\,dy \qquad \text{and} \qquad J_y = \iint x^2\,dx\,dy$$

are the two principal moments of inertia of the polyhedral surface. He believed (probably by appealing to "St. Venant's principle") that this variation of the surface tractions would not imply any variation in the deformation of the body. By solving the fundamental equations of elasticity, he then imagined that the components of the displacement in the interior were linear functions and

([118]) Cf., the two sections of the Appendix for this. (*Editor's remark: The Appendix consists of only one section.*)

then determined the deformation work as a quadratic form in X, Y, Z, M_x, M_y, M_z that belongs to the individual surface of the boundary.

He further appealed to Clapeyron's equation, and then expressed the deformation work by the integral:

$$\tfrac{1}{2}\iint (p_x\, u + p_y\, v + p_z\, w)\, d\omega \ .$$

If one substitutes the expressions (2.74) and (2.75) in that then it will go to:

$$\tfrac{1}{2}\sum (\xi\, X + \eta Y + \zeta\, Z + \Theta_x M_x + \Theta_y M_y + \Theta_z M_z)\,. \tag{2.76}$$

He then deduced the theorems about the derivatives of the deformation work from that in the same way as for frameworks:

$$\frac{\partial \bar{\Psi}^e}{\partial X} = \xi\,, \qquad \ldots, \qquad \frac{\partial \bar{\Psi}^e}{\partial M_z} = \Theta_x\,, \tag{2.77a}$$

or

$$\frac{\partial \Psi^e}{\partial \xi} = X\,, \qquad \ldots, \qquad \frac{\partial \Psi^e}{\partial \Theta_z} = M_x\,, \tag{2.77b}$$

resp. Using his assumption, he had then succeeded in shifting the entire problem from the realm of functional calculus to that of the usual infinitesimal calculus.

Actually, Castigliano only applied that argument to the case of beams, and in that way, operated essentially with isolated forces as the given loads, which is an assumption that is still customary in engineering to this day ([119]). The variation then goes to a differentiation by itself. However, at one point in his book ([120]), one sees that Castigliano himself came rather close to that conception of differentiation as a specialized variation. In order to calculate the displacement of the surface of an elastic body to which no isolated loads are applied, he made an arbitrary choice of isolated load at this point, differentiated with respect to it, and then set it equal to zero again. That imitation of the variation made it obvious that he clearly recognized that he had to master the "boundary-value problem" in order to be able to apply his theory.

Since engineers employ only the applications that Castigliano gave of his theory, the foundation and precise formulation of the entire sphere of thought have long remained in darkness. Some degree of clarity was first achieved in the work of DONATI ([121]). In particular, he had clearly distinguished between the principle of total potential energy and Menabrea's principle and also kept the two conceptions of Menabrea's principle apart from each other. Under the influence of Volterra's first work on functional calculus, he also drew attention to the fact that one was dealing with functional derivatives in the two conceptions of Castigliano's theorems.

([119]) Cf., e.g., H. MÜLLER-BRESLAU, *Die neueren Methoden der Festigkeitslehre*, 4th ed., Leipzig, (1913), pp. 271.
([120]) A. CASTIGLIANO, *Théorie de l'équilibre des système élastiques*, pp. 152.
([121]) I. DONATI, Memoria Bologna (4) **9** (1888), pp. 64; *ibid.* (4) **10** (1889), pp. 85, and *ibid.* (5) **4** (1894), pp. 91.

APPENDIX

REPRESENTING THE EXTREMAL VALUE OF THE DEFORMATION WORK AS A FUNCTIONAL OF THE GIVEN SURFACE VALUES

In order to represent the solution to the fundamental equations of elasticity (2.12) for prescribed surface displacements $u(\omega)$, $v(\omega)$, $w(\omega)$, we would like to perform the integration with the help of a system of Green functions. Following Somigliana, the "fundamental solution" that exhibits the characteristic singularity that one employs is the following system of functions:

$u_1(x, y, z\,;\,\xi, \eta, \zeta) =$

$$\begin{aligned}
u_1 &= \frac{1}{4\pi\mu}\left(\frac{1}{r}-\frac{\alpha}{2}\frac{\partial^2 r}{\partial\xi^2}\right), & v_1 &= -\frac{1}{4\pi\mu}\frac{\alpha}{2}\frac{\partial^2 r}{\partial\xi\,\partial\eta}, & w_1 &= -\frac{1}{4\pi\mu}\frac{\alpha}{2}\frac{\partial^2 r}{\partial\xi\,\partial\zeta}, \\
u_2 &= -\frac{1}{4\pi\mu}\frac{\alpha}{2}\frac{\partial^2 r}{\partial\eta\,\partial\xi}, & v_2 &= \frac{1}{4\pi\mu}\left(\frac{1}{r}-\frac{\alpha}{2}\frac{\partial^2 r}{\partial\eta^2}\right), & w_2 &= -\frac{1}{4\pi\mu}\frac{\alpha}{2}\frac{\partial^2 r}{\partial\eta\,\partial\zeta}, \\
u_3 &= -\frac{1}{4\pi\mu}\frac{\alpha}{2}\frac{\partial^2 r}{\partial\zeta\,\partial\xi}, & v_3 &= -\frac{1}{4\pi\mu}\frac{\alpha}{2}\frac{\partial^2 r}{\partial\zeta\,\partial\eta}, & w_3 &= \frac{1}{4\pi\mu}\left(\frac{1}{r}-\frac{\alpha}{2}\frac{\partial^2 r}{\partial\zeta^2}\right),
\end{aligned} \tag{2.78}$$

in which:

$$\alpha = \frac{\lambda+\mu}{\lambda+2\mu}, \quad r = \sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}\,. \tag{2.78a}$$

The functions u, v, w in each of those three systems are finite and continuous and satisfy the fundamental equations of elasticity (2.12) for all ξ, η, ζ, with the exception of $\xi = x$, $\eta = y$, $\zeta = z$. If we define the expressions (2.13) on the surface of the body with those functions then we will get the three associated systems of surface expressions ([122]).

$$\begin{aligned}
&p_x^{(1)}, \quad p_y^{(1)}, \quad p_z^{(1)}, \\
&p_x^{(2)}, \quad p_y^{(2)}, \quad p_z^{(2)}, \\
&p_x^{(3)}, \quad p_y^{(3)}, \quad p_z^{(3)}.
\end{aligned} \tag{2.79}$$

([122]) They *do not* fulfill equations (2.4), because they do not define an equilibrium system in their own right but possess a resultant $1 / 4\pi\mu$, which goes through the point ξ, η, ζ and lies along the directions of the x, y, and z-axes, resp., for the three systems. That corresponds to the fact that the displacements of the system (2.78) can be generated by any isolated forces of magnitude $1 / 4\pi\mu$ that point in the directions of each of the three coordinate axes.

Let $u(\xi, \eta, \zeta)$, $v(\xi, \eta, \zeta)$, $w(\xi, \eta, \zeta)$ be the system of displacements that one seeks, and let p_x, p_y, p_z be the system of surface tractions that belongs to it according to (2.13). We take them, along with each of the three Somigliana systems, and apply the Betti reciprocity theorem (2.46). Due to the singularity in that system, we must then replace the point x, y, z with a small ball along the boundary surface of the body and then pass to the limit in the known way. In that way, we will get the three equations:

$$\begin{aligned} u(\xi, \eta, \zeta) &= \iint_O (p_x u_1 + p_y v_1 + p_z w_1)\, d\omega - \iint_O (p_x^{(1)} u + p_y^{(1)} v + p_z^{(1)} w)\, d\omega, \\ v(\xi, \eta, \zeta) &= \iint_O (p_x u_2 + p_y v_2 + p_z w_2)\, d\omega - \iint_O (p_x^{(2)} u + p_y^{(2)} v + p_z^{(2)} w)\, d\omega, \\ w(\xi, \eta, \zeta) &= \iint_O (p_x u_3 + p_y v_3 + p_z w_3)\, d\omega - \iint_O (p_x^{(3)} u + p_y^{(3)} v + p_z^{(3)} w)\, d\omega, \end{aligned} \tag{2.80}$$

which express the desired displacement of the point x, y, z in terms of the associated surface displacements and surface tractions. In order to eliminate the surface tractions from that, we shall pass from the fundamental solution (2.80) to the system of Green functions. In order to do that, we must determine three systems of solutions to (2.12) that are finite and continuous in the entire body (viz., the so-called compensators), and take the values:

$$\begin{array}{lll} u_1(\xi, \eta, \zeta; \omega), & v_1(\xi, \eta, \zeta; \omega), & w_1(\xi, \eta, \zeta; \omega), \\ u_2(\xi, \eta, \zeta; \omega), & v_2(\xi, \eta, \zeta; \omega), & w_2(\xi, \eta, \zeta; \omega), \\ u_3(\xi, \eta, \zeta; \omega), & v_3(\xi, \eta, \zeta; \omega), & w_3(\xi, \eta, \zeta; \omega) \end{array} \tag{2.81}$$

on the surface ([123]). Let the associated surface tractions be $p_x^{(1)}(\omega)$, etc., which naturally define an equilibrium system. If we apply Betti's theorem to those three systems of compensators and our desired system of displacements then we will obtain the three equations:

$$\begin{aligned} 0 &= \iint_O (p_x u_1 + p_y v_1 + p_z w_1)\, d\omega - \iint_O (\bar{p}_x^{(1)} u + \bar{p}_y^{(1)} v + \bar{p}_z^{(1)} w)\, d\omega, \\ 0 &= \iint_O (p_x u_2 + p_y v_2 + p_z w_2)\, d\omega - \iint_O (\bar{p}_x^{(2)} u + \bar{p}_y^{(2)} v + \bar{p}_z^{(2)} w)\, d\omega, \\ 0 &= \iint_O (p_x u_3 + p_y v_3 + p_z w_3)\, d\omega - \iint_O (\bar{p}_x^{(3)} u + \bar{p}_y^{(3)} v + \bar{p}_z^{(3)} w)\, d\omega. \end{aligned} \tag{2.82}$$

If we subtract them from the corresponding equations (2.80) then the first integrals on the right-hand side will cancel precisely, so the p_x, p_y, p_z, and the displacements in the interior of the elastic body will be expressed in terms of only the surface displacements by means of the equations:

$$u(x, y, z) = \iint_O \{(\bar{p}_x^{(1)} - p_x^{(1)})\, u(\omega) + (\bar{p}_y^{(1)} - p_y^{(1)})\, v(\omega) + (\bar{p}_z^{(1)} - p_z^{(1)})\, w(\omega)\}\, d\omega, \tag{2.83}$$

([123]) To see how to perform that determination with the help of the theory of linear integral equations, cf., perhaps **H. Weyl**, Rendiconti del circolo matematico di Palermo **39** (1915), pp. 1.

and two analogous equations. If we set ([124]):

$$\bar{p}_x^{(1)}(x, y, z; \omega) - p_x^{(1)}(x, y, z; \omega) = P_x^{(1)}(x, y, z; \omega) \;, \tag{2.84}$$

to abbreviate, then they will take the form:

$$u\,(x, y, z) = \iint_O [P_x^{(1)}(x, y, z; \omega)\, u\,(\omega) + P_y^{(1)}(x, y, z; \omega)\, v\,(\omega) + P_z^{(1)}(x, y, z; \omega)\, w\,(\omega)]\, d\omega \,, \tag{2.85}$$

and two analogous equations for v and w.

We would now like to calculate the value of the deformation work (per unit volume) at the point x, y, z from those values (2.85) of the displacement components. In order to do that, we shall refer to their expressions in (2.16). For the derivatives of u, v, w, (2.85) will yield expressions of the form:

$$\frac{\partial u}{\partial x} = \iint_O \left(\frac{\partial P_x^{(1)}}{\partial x} u + \frac{\partial P_y^{(1)}}{\partial y} v + \frac{\partial P_z^{(1)}}{\partial z} w \right) d\omega \,, \tag{2.86}$$

or as we would like to write, for brevity:

$$\frac{\partial u}{\partial x} = \iint_O \left(\sum \frac{\partial P_x^{(1)}}{\partial x} u \right) d\omega \,, \tag{2.87}$$

in which the summation sign means that we sum over three terms that emerge from the one that is written out by replacing the index x with y, and then z, while simultaneously replacing u with v, and then w.

With that abbreviation, we will then write:

$$\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \iint_O \left[\left(\frac{\partial P_x^{(2)}}{\partial z} + \frac{\partial P_x^{(3)}}{\partial y} \right) u + \left(\frac{\partial P_y^{(2)}}{\partial z} + \frac{\partial P_y^{(3)}}{\partial y} \right) v + \left(\frac{\partial P_z^{(2)}}{\partial z} + \frac{\partial P_z^{(3)}}{\partial y} \right) w \right] d\omega$$

as:

$$\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \iint_O \left[\sum \left(\frac{\partial P_x^{(1)}}{\partial x} + \frac{\partial P_x^{(3)}}{\partial y} \right) u \right] d\omega \,. \tag{2.87a}$$

([124]) Those functions $P_x^{(1)}(x, y, z; \omega)$, etc., are the negative values of the surface tractions that belong to the system of Green functions. They are functions of their arguments that are determined by only the form and elastic properties of the given bodies. The functions of the three columns in the matrix of the nine functions satisfy the fundamental system of elastic equations.

Squares and products of derivatives like (2.87) appear in the expression (2.16). We can combine the product of two of the double integrals that will result into a four-fold integral. If we denote the integration element in the one factor by ω^*, instead of ω, then that will give, e.g.:

$$\left(\frac{\partial u}{\partial x}\right)^2 = \iint \iint \left\{ \left(\sum \frac{\partial P_x^{(1)}(x,y,z;\omega)}{\partial x} u(\omega) \right) \left(\sum \frac{\partial P_x^{(1)}(x,y,z;\omega^*)}{\partial x} u(\omega^*) \right) \right\} d\omega\, d\omega^*,$$

or when we write the second factor as:

$$\sum \frac{\partial P_x^{*(1)}}{\partial x} u^*,$$

to abbreviate, we will finally have:

$$\left(\frac{\partial u}{\partial x}\right)^2 = \iint \iint \left(\sum \frac{\partial P_x^{(1)}}{\partial x} u \right) \left(\sum \frac{\partial P_x^{*(1)}}{\partial x} u^* \right) d\omega\, d\omega^*.$$

With that notation, a unit of deformation work (2.16) is expressed in terms of the displacement components in the following way:

$$\begin{aligned}
\psi\left(\frac{\partial u}{\partial x}, \cdots\right) = {} & \tfrac{1}{2} \iint \iint \\
& \left\{ (\lambda + 2\mu) \left[\sum \left(\frac{\partial P_x^{(1)}}{\partial x} + \frac{\partial P_x^{(2)}}{\partial y} + \frac{\partial P_x^{(3)}}{\partial z} \right) u \right] \left[\sum \left(\frac{\partial P_x^{*(1)}}{\partial x} + \frac{\partial P_x^{*(2)}}{\partial y} + \frac{\partial P_x^{*(3)}}{\partial z} \right) u^* \right] \right. \\
& + \mu \left[\left(\sum \left(\frac{\partial P_x^{(2)}}{\partial z} + \frac{\partial P_x^{(2)}}{\partial y} \right) u \right) \left(\sum \left(\frac{\partial P_x^{*(1)}}{\partial z} + \frac{\partial P_x^{*(2)}}{\partial y} \right) u^* \right) \right. \\
& + \left(\sum \left(\frac{\partial P_x^{(3)}}{\partial x} + \frac{\partial P_x^{(1)}}{\partial z} \right) u \right) \left(\sum \left(\frac{\partial P_x^{*(3)}}{\partial x} + \frac{\partial P_x^{*(1)}}{\partial z} \right) u^* \right) \\
& + \left(\sum \left(\frac{\partial P_x^{(1)}}{\partial y} + \frac{\partial P_x^{(2)}}{\partial x} \right) u \right) \left(\sum \left(\frac{\partial P_x^{*(1)}}{\partial y} + \frac{\partial P_x^{*(2)}}{\partial x} \right) u^* \right) \\
& - 4 \left(\sum \frac{\partial P_x^{(2)}}{\partial y} u \right) \left(\sum \frac{\partial P_x^{*(3)}}{\partial z} u^* \right) \\
& \left. \left. - 4 \left(\sum \frac{\partial P_x^{(2)}}{\partial y} u \right) \left(\sum \frac{\partial P_x^{*(3)}}{\partial z} u^* \right) - 4 \left(\sum \frac{\partial P_x^{(2)}}{\partial y} u \right) \left(\sum \frac{\partial P_x^{*(3)}}{\partial z} u^* \right) \right] \right\} d\omega\, d\omega^*,
\end{aligned} \tag{2.88}$$

in which we can obviously split the term in the penultimate row into the following two terms:

$$-\ 2\left(\sum \frac{\partial P_x^{(2)}}{\partial y}\,u\right)\left(\sum \frac{\partial P_x^{*(3)}}{\partial z}\,u^*\right)-2\left(\sum \frac{\partial P_x^{(2)}}{\partial y}\,u\right)\left(\sum \frac{\partial P_x^{*(3)}}{\partial z}\,u^*\right),$$

and we will have corresponding expressions for the next two terms. The factor of $u\,(\omega)\cdot u\,(\omega^*)$ under the integral sign is then equal to:

$$\begin{aligned}
&\tfrac{1}{2}\Bigg\{(\lambda+2\mu)\left(\frac{\partial P_x^{(1)}}{\partial x}+\frac{\partial P_x^{(2)}}{\partial y}+\frac{\partial P_x^{(3)}}{\partial z}\right)\left(\frac{\partial P_x^{*(1)}}{\partial x}+\frac{\partial P_x^{*(2)}}{\partial y}+\frac{\partial P_x^{*(3)}}{\partial z}\right)\\
&\quad+\ \mu\Bigg[\left(\frac{\partial P_x^{(2)}}{\partial z}+\frac{\partial P_x^{(2)}}{\partial y}\right)\left(\frac{\partial P_x^{*(1)}}{\partial z}+\frac{\partial P_x^{*(2)}}{\partial y}\right)\\
&\quad+\left(\frac{\partial P_x^{(3)}}{\partial x}+\frac{\partial P_x^{(1)}}{\partial z}\right)\left(\frac{\partial P_x^{*(3)}}{\partial x}+\frac{\partial P_x^{*(1)}}{\partial z}\right)\\
&\quad+\left(\frac{\partial P_x^{(1)}}{\partial y}+\frac{\partial P_x^{(2)}}{\partial x}\right)\left(\frac{\partial P_x^{*(1)}}{\partial y}+\frac{\partial P_x^{*(2)}}{\partial x}\right)\\
&\quad-\ 2\left(\frac{\partial P_x^{(2)}}{\partial y}\frac{\partial P_x^{*(3)}}{\partial z}+\frac{\partial P_x^{*(2)}}{\partial y}\frac{\partial P_x^{(3)}}{\partial z}\right)\\
&\quad-\ 2\left(\frac{\partial P_x^{(3)}}{\partial z}\frac{\partial P_x^{*(1)}}{\partial x}+\frac{\partial P_x^{*(3)}}{\partial z}\frac{\partial P_x^{(1)}}{\partial x}\right)\\
&\quad-\ 2\left(\frac{\partial P_x^{(1)}}{\partial x}\frac{\partial P_x^{*(2)}}{\partial y}+\frac{\partial P_x^{*(1)}}{\partial x}\frac{\partial P_x^{(3)}}{\partial y}\right)\Bigg]\Bigg\}\ .
\end{aligned}$$

If we associate the expression (2.16), which is a quadratic form in the derivatives $\partial u / \partial x$, etc., with the bilinear form:

$$\begin{aligned}
&\psi\left(\frac{\partial u'}{\partial x};\frac{\partial u''}{\partial x}\right)\ =\ \tfrac{1}{2}\Bigg\{(\lambda+2\mu)\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)\left(\frac{\partial u''}{\partial x}+\frac{\partial v''}{\partial y}+\frac{\partial w''}{\partial z}\right)\\
&+\ \mu\Bigg[\left(\frac{\partial v'}{\partial z}+\frac{\partial w'}{\partial y}\right)\left(\frac{\partial v''}{\partial z}+\frac{\partial w''}{\partial y}\right)+\left(\frac{\partial w'}{\partial x}+\frac{\partial u'}{\partial z}\right)\left(\frac{\partial w''}{\partial z}+\frac{\partial u''}{\partial y}\right)+\left(\frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial x}\right)\left(\frac{\partial u''}{\partial y}+\frac{\partial v''}{\partial x}\right)\\
&-\ 2\left(\frac{\partial v'}{\partial y}\frac{\partial w''}{\partial z}+\frac{\partial v''}{\partial y}\frac{\partial w'}{\partial z}\right)-2\left(\frac{\partial w'}{\partial z}\frac{\partial u''}{\partial x}+\frac{\partial w''}{\partial z}\frac{\partial u'}{\partial x}\right)-2\left(\frac{\partial w'}{\partial z}\frac{\partial u''}{\partial x}+\frac{\partial w''}{\partial z}\frac{\partial u'}{\partial x}\right)\Bigg]\Bigg\}
\end{aligned}\tag{2.89}$$

then we will see that the expression (2.89) is equal to precisely:

$$\psi\left(\frac{\partial P_x^{(1)}}{\partial x};\frac{\partial P_x^{*(1)}}{\partial x}\right),$$

i.e., (2.89) when one makes the following replacements in it:

$$\begin{aligned} u' &= P_x^{(1)}, \quad v' = P_x^{(2)}, \quad w' = P_x^{(3)}, \\ u'' &= P_x^{*(1)}, \quad v'' = P_x^{*(2)}, \quad w'' = P_x^{*(3)}. \end{aligned} \tag{2.90}$$

The coefficients of the other products of surface displacements are expressed analogously, such that we will get the following expression for a unit of deformation work at the point x, y, z:

$$\begin{aligned} \psi\left(\frac{\partial u}{\partial x},\cdots\right) = \iint\iint & \left\{\psi\left(\frac{\partial P_x^{(1)}}{\partial x};\frac{\partial P_x^{*(1)}}{\partial x}\right) u\,u^* + \psi\left(\frac{\partial P_x^{(1)}}{\partial x};\frac{\partial P_y^{*(1)}}{\partial x}\right) u\,v^*\right. \\ & + \psi\left(\frac{\partial P_x^{(1)}}{\partial x};\frac{\partial P_y^{*(1)}}{\partial x}\right) u^* v + \psi\left(\frac{\partial P_y^{(1)}}{\partial x};\frac{\partial P_y^{*(1)}}{\partial x}\right) v\,v^* \\ & + \psi\left(\frac{\partial P_x^{(1)}}{\partial x};\frac{\partial P_z^{*(1)}}{\partial x}\right) u\,w^* + \psi\left(\frac{\partial P_x^{(1)}}{\partial x};\frac{\partial P_z^{*(1)}}{\partial x}\right) u^* w \\ & + \psi\left(\frac{\partial P_y^{(1)}}{\partial x};\frac{\partial P_x^{*(1)}}{\partial x}\right) v\,w^* + \psi\left(\frac{\partial P_y^{(1)}}{\partial x};\frac{\partial P_z^{*(1)}}{\partial x}\right) v^* w \\ & \left. + \psi\left(\frac{\partial P_x^{(1)}}{\partial x};\frac{\partial P_z^{*(1)}}{\partial x}\right) w\,w^*\right\} d\omega\, d\omega^*. \end{aligned} \tag{2.91}$$

Since the factor $u(\omega^*)\cdot v(\omega)$ will go to $u(\omega)\cdot v(\omega^*)$ upon switching ω and ω^*, we can combine those two terms into one. The same thing is true for the products $u \cdot v$ and $v \cdot w$, such that the integrand will reduce to six terms.

We will get the total deformation work from the unit of deformation work when we integrate over the volume of the elastic body. Since the coordinates x, y, z of the individual points of the body appear only in the factors ψ in the integrand of (2.91), that will give an expression for Ψ^e that has the same form as (2.91), except that now the coefficients have been replaced with:

$$\iiint_V \psi\left(\frac{\partial P_x^{(1)}}{\partial x};\frac{\partial P_x^{*(1)}}{\partial x}\right) d\kappa\,. \tag{2.92}$$

That integral exists and yields well-defined functions of ω and ω^*, such that we will, in fact, get the representation for Ψ^e that we deduced from the Fréchet theorem. By partially integrating the integral of the form (2.92) and recalling the property of the system of functions P, we can represent the values of those coefficients in such a way that the second Castigliano theorem will follow by direct calculation. Namely, if we perform that conversion and then, on the one hand, take the functional derivatives of Ψ^e with respect to the surface displacements, and on the other, the functional derivatives of the surface tractions, as in formula (2,13), with respect to the displacement components (2.85), then that will immediately give identities.

We shall now move on to represent the dependency of the displacement components u, v, w (and by determining them, the deformation work) on the surface tractions. As we have mentioned several times, the displacements are not determined uniquely by that dependency, but a displacement of the type that is performed on a rigid body will remain arbitrary. In order to eliminate that arbitrariness, we can prescribe six conditions (of a linear character) for the displacement components, e.g., we can demand that the conditions:

$$\iiint_V (\bar{\bar{u}}_i\, u + \bar{\bar{v}}_i\, v + \bar{\bar{w}}_i\, w)\, dx = 0 \qquad (i = 1, 2, \ldots, 6) \tag{2.93}$$

are satisfied ([125]) when we give the values of six arbitrary constants $\bar{\bar{u}}_i$, $\bar{\bar{v}}_i$, $\bar{\bar{w}}_i$:

$$\begin{array}{lll} \bar{\bar{u}}_1 = C_1, & \bar{\bar{v}}_1 = 0, & \bar{\bar{w}}_1 = 0, \\ \bar{\bar{u}}_2 = 0, & \bar{\bar{v}}_2 = C_2, & \bar{\bar{w}}_2 = 0, \\ \bar{\bar{u}}_3 = 0, & \bar{\bar{v}}_3 = 0, & \bar{\bar{w}}_3 = C_3, \\ \bar{\bar{u}}_4 = 0, & \bar{\bar{v}}_4 = C_4\, z, & \bar{\bar{w}}_4 = -C_4\, y, \\ \bar{\bar{u}}_5 = -C_5\, z, & \bar{\bar{v}}_5 = 0, & \bar{\bar{w}}_5 = C_5\, x, \\ \bar{\bar{u}}_6 = C_6\, y, & \bar{\bar{v}}_6 = -C_6\, x, & \bar{\bar{w}}_6 = 0. \end{array} \tag{2.94}$$

One cares to call equations of the form (2.93) "orthogonality conditions," so (2.93) says that the desired system of displacements must be orthogonal to the six systems of functions (2.94). We remark that the six equilibrium conditions (2.4) for the surface tractions can be reduced to the one condition that the system of functions:

$$p_x(\omega),\ p_y(\omega),\ p_z(\omega)$$

should be orthogonal to the surface values of (2.94):

$$\iint_O [\bar{\bar{u}}_i\, p_x(\omega) + \bar{\bar{v}}_i\, p_z(\omega) + \bar{\bar{w}}_i\, p_z(\omega)]\, d\omega = 0 \qquad (i = 1, 2, \ldots, 6). \tag{2.95}$$

In order to obtain the desired solution for the fundamental equations of elasticity, we can now attempt to go down the same path that we did before. If we employ Somigliana's fundamental solution then we will also arrive at equations (2.80). However, if we now try to drop the second term on the right-hand side by passing to a system of Green functions then we will find that no regular solution to the fundamental equations (2.12) belongs to surface tractions $p_x^{(h)}$, $p_y^{(h)}$, $p_z^{(h)}$ since those tractions do not satisfy the equilibrium conditions, i.e.:

([125]) H. WEYL, Rendiconti del circolo matematico di Palermo **39** (1915), pp. 1.

$$\iint_O [p_x^h \,\bar{\bar{u}}_i + p_y^h \,\bar{\bar{v}}_i + p_z^h \,\bar{\bar{w}}] \neq 0 \ .$$

We can also get around that difficulty in two ways: One of them is to look for a solution of the fundamental equations of elasticity, while preserving the Somigliana fundamental solutions, not for the actual surface tractions $p_x^{(h)}$, $p_y^{(h)}$, $p_z^{(h)}$, but only for the altered values:

$$p_x^{(h)} + a^{(h)} + d^{(h)} y - e^{(h)} z \,, \qquad p_x^{(h)} + b^{(h)} + e^{(h)} z - f^{(h)} x \,, \qquad p_x^{(h)} + c^{(h)} + e^{(h)} x - d^{(h)} y \ .$$

If the constants are determined in such a way that these new surface tractions fulfill the condition (2.95) then such a solution will exist, and with its help, we can also succeed in eliminating the second integral on the right-hand side of (2.80) ([126]).

The other path was the one that WEYL took ([127]). He likewise altered Somigliana's fundamental solution somewhat, such that it would be orthogonal to the system (2.94), but preserve their symmetry properties. The fundamental solutions u_h', v_h', w_h' will no longer satisfy the fundamental equations of elasticity (2.12), but equations of the form ([128]):

$$u \,\Delta_2 \, u_h' + (\lambda + \mu) \frac{\partial \varepsilon'}{\partial x} \ = \ B\big(\bar{\bar{u}}_i \,(x, y, z), \bar{\bar{u}}_i \,(\xi, \eta, \zeta)\big) . \tag{2.96}$$

The surface tractions $p_x^{(h)'}$, $p_y^{(h)'}$, $p_z^{(h)'}$ that belong to them fulfill the equilibrium conditions, so one will have:

$$\iint_O (p_x^{(h)'} \,\bar{\bar{u}}_i + p_y^{(h)'} \,\bar{\bar{v}}_i + p_z^{(h)'} \,\bar{\bar{w}}_i) \ = 0$$

for them. Thus, they belong to a regular solution $\bar{u}_h$, $\bar{v}_h$, $\bar{w}_h$ of the fundamental equations of elasticity. If we take the desired solution *u*, *v*, *w* together with that system of solutions then the Betti reciprocity theorem will imply the relations:

$$\iint_O (p_x^{(h)'} \, u + p_y^{(h)'} \, v + p_z^{(h)'} \, w) \, d\omega \ = \ \iint_O (p_x \, \bar{u}_h + p_y \, \bar{v}_h + p_z \, \bar{w}_h) \, d\omega . \tag{2.97}$$

Of course, since the new fundamental solution no longer satisfies the fundamental equations of elasticity (2.12), but the equations (2.96), the relation (2.80) will no longer be valid for it with no further conditions, so in its place, one will have a relation whose right-hand side also includes a volume integral:

$$\begin{aligned} u, y, x) = & \iint \,(p_x \, u_x' + p_y \, v_x' + p_z \, w_x') \, d\omega \\ & - \iint \,(p_x^{(1)'} \, u + p_y^{(1)'} \, v + p_z^{(1)'} \, w) \, d\omega \end{aligned} \tag{2.98}$$

([126]) J. FREDHOLM, Acta mathematica **23** (1900), pp. 41.
([127]) H. WEYL, *loc. cit.*
([128]) in which *B* represent a bilinear function of its arguments.

$$- \iiint \left[\sum u\, B\left(\bar{\bar{u}}_i(x,y,z), \bar{\bar{u}}_i(\xi,\eta,\zeta) \right) \right] d\kappa \,.$$

However, since the desired solution u, v, w must be orthogonal to the system (2.94), that volume integral will drop out, and relations of the older form will once more be true:

$$u\,(x, y, z) = \iint (p_x\, u'_x + p_y\, v'_x + p_z\, w'_x)\, d\omega - \iint (p_x^{(1)'}\, u + p_y^{(1)'}\, v + p_z^{(1)'}\, w)\, d\omega \;.$$

Upon adding that to (2.97), we will then get the Ansatz:

$$u\,(x, y, z) = \iint \{ (u'_1 + \bar{u}'_1)\, p_x + (v'_1 + \bar{v}'_1)\, p_y + (w'_1 + \bar{w}'_1)\, p_z \}\, d\omega \;,$$

or when we set:

$$u'_1 + \bar{u}'_1 = U_1\,, \quad \text{etc.},$$

we will get:

$$\begin{aligned} u\,(x, y, z) &= \iint (U_1\, p_x + V_1\, p_y + W_1\, p_z)\, d\omega \;, \\ v\,(x, y, z) &= \iint (U_2\, p_x + V_2\, p_y + W_2\, p_z)\, d\omega \;, \\ w\,(x, y, z) &= \iint (U_3\, p_x + V_3\, p_y + W_3\, p_z)\, d\omega \;. \end{aligned} \tag{2.99}$$

One can arrange for this system of Green functions U_h, V_h, W_h to be orthogonal to (2.94), so it will also possess the same symmetry as Somigliana's fundamental solution (2.78), and the columns of the matrix that it defines will satisfy equations (2.96).

In the same way as before when we started with the Ansatz (2.85), if we start from (2.99) then we will get the following expression for the unit of deformation energy at the point x, y, z :

$$\begin{aligned} \psi = \iint \iint \Bigg\{ & \psi\left(\frac{\partial U_1}{\partial x} \frac{\partial U_1^*}{\partial x} \right) u \cdot u^* + \psi\left(\frac{\partial U_1}{\partial x} \frac{\partial U_1^*}{\partial x} \right) u \cdot v^* \\ & + \psi\left(\frac{\partial U_1^*}{\partial x} \frac{\partial V_1}{\partial x} \right) u^* \cdot v + \psi\left(\frac{\partial U_1^*}{\partial x} \frac{\partial V_1}{\partial x} \right) u^* \cdot v \\ & + \psi\left(\frac{\partial V_1}{\partial x} \frac{\partial V_1^*}{\partial x} \right) v \cdot v^* + \psi\left(\frac{\partial U_1}{\partial x} \frac{\partial W_1^*}{\partial x} \right) u \cdot w^* \\ & + \psi\left(\frac{\partial U_1^*}{\partial x} \frac{\partial W_1}{\partial x} \right) u^* \cdot w + \psi\left(\frac{\partial V_1}{\partial x} \frac{\partial W_1^*}{\partial x} \right) v \cdot w^* \\ & + \psi\left(\frac{\partial V_1^*}{\partial x} \frac{\partial W_1}{\partial x} \right) v^* \cdot w + \psi\left(\frac{\partial W_1}{\partial x} \frac{\partial W_1^*}{\partial x} \right) w \cdot w^* \Bigg\} d\omega\, d\omega^* \,, \end{aligned} \tag{2.109}$$

in which we have:

$$U_1 = U_1\,(x, y, z\,;\,\omega)\,, \qquad U_1^* = U_1(x, y, z; \omega^*)\,, \qquad \text{etc.}$$

If we integrate that expression over the entire elastic body then we will get the total deformation work as an expression of the same form, in which the coefficients:

$$\iiint_V \psi\left(\frac{\partial U_1}{\partial x}\right) d\kappa = \bar{K}(\omega, \omega^*), \qquad \text{etc.} \tag{2.101}$$

are just well-defined functions of ω, ω^*. Here, as well, we can use partial integration and recall the properties of the system of Green functions in order to bring the coefficients (2.101) into a form such that the first Castigliano theorem will become a simple identity when we take the functional derivatives with respect to p_x, p_y, p_z and compare them with the expressions (2.99) for the displacement components.

Remarks concerning the bibliography

In the original version by Prange, the references were included only as footnotes, and mostly in a very abbreviated form. Therefore, all of Prange's cited references were compiled into a Bibliography, in which the citations for the majority of Prange's cited references were taken from the survey article by Oravas [**105**, **106**], and were subsequently verified as much as was possible. The Bibliography is therefore almost complete, at least in regard to the German-language works. Furthermore, all additional works that were cited in the Introduction were also included, so works on the history of engineering and natural science.

Translator's remarks

The formatting of the references was changed be more consistent with his other translations. A number of references were completed, since the modern Internet makes access to those publications more immediate than when Knothe edited the book.

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