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On the undulatory theory of positive and negative electrons

By AL. PROCA

Institut Henri Poincaré, Paris

Translated by D. H. Delphenich

Introduction. – The Dirac theory of "holes" is actually the only one that permits one a glimpse into the behavior of positons. The experimental discovery of the latter has confirmed the fundamental hypothesis and has shown that the proposed equation accounts for the positive electrons, as well as the negative ones. Nevertheless, the difficulties that one encounters remain considerable. Without speaking of infinite proper energies, the structure itself of the theory, in fact, raises other problems, the least inconvenient of which is to render the analysis of the simplest cases quite pathological.

One knows what these difficulties consist of: viz., to describe a positon. One needs to postulate the existence of a uniform, but infinite density, of negative electrons with no mutual interactions. Only the distances from that uniform distribution must be considered to be observables; any unoccupied place on a *negative* energy level – or "hole" – constitutes a positon.

One often emphasizes that from a physical point of view it is difficult to accept the hypothesis of an infinite distribution of electrons. From the mathematical viewpoint, one is constantly obliged to evaluate the differences between the quantities that we know in advance to be infinite. Meanwhile, the results that have been obtained up to the present seem to show quite well that the theory correctly represents reality.

Indubitably, the preceding difficulties are due to the solutions of the Dirac equation that are characterized by a negative kinetic energy. These solutions have no physical sense. In order to relate them to anything that has physical sense, one must make suppositions of a necessarily artificial character, such as, for example, the hypothesis of an infinite density of electrons. It is very probable that if the equation for the electron automatically excludes this type of solution then any difficulty of this type will vanish. We have begun to prove this. Pauli and Weisskopf, in a fundamental memoir ([1]), have remarked that if the wave function of the electron is given by the relativistic *Gordon* equation then none of the preceding difficulties will arise.

Indeed, from the Gordon theory of energy of the elementary particle is always positive, whereas its charge might also be positive or negative. The same equation represents the negatons and positons at the same time, without any need for recourse to the hypothesis of any infinite density of electrons. Moreover, Pauli and Weisskopf have shown that the results that one obtains in the problem of the production of pairs for large

([1]) *Helvetica Physica Acta*, 1934, vol. 7, fasc. 7, pp. 709. Also see a lecture of W. Pauli, *Annales de l'Institut Henri Poincaré* (in press).

energies or in that of the polarization of the vacuum are *the same,* whether one starts with the Gordon equation or the "hole" hypothesis.

On the face of the problem of the positon, two attitudes are therefore possible: That of Dirac, Heisenberg, etc., which takes the Dirac equation as its point of departure (energy with two signs, charge always positive), or that of Pauli and Weisskopf, which advocates the use of another equation that leads to a positive energy and a charge with two signs.

It is clear that the experimental facts will be in favor of the second way of looking at things if *one knows a convenient equation to start with* that is equivalent to that of Dirac in its consequences.

The Gordon equation that was employed by Pauli and Weisskopf is not convenient. It does not account for the spin of the electron, and this is probably its gravest shortcoming. It has the consequence that the treatment that was described by these authors will apply only when the particles in question admit Bose-Einstein statistics, which is obviously incorrect. It is impossible (Pauli, *loc. cit.*) to employ Fermi-Dirac statistics with the Gordon equation, as one would like to do.

If one would then like to proceed like these authors then the first thing to do would consist in finding another equation than that of Gordon that enjoys the same properties as the latter and is more adapted to the description of the electron. There is good reason to recall the argument that led Dirac to establish his equation and to modify it in such a fashion that one obtains another, more satisfactory, one.

2. Conditions that a fundamental equation must satisfy. – In the light of recent experimental discoveries, it seems, *a priori*, impossible to obtain an equation of this type. We then examine the principal conditions that one will have to satisfy.

We first remark that the argument of Dirac that led him to establish his equation in an era when he did not know of the existence and properties of the positon is more convincing today, now that one knows of the phenomena of the production and annihilation of pairs. The corresponding discussion was made by Pauli and Weisskopf, who remarked that:

1. Due to the production of pairs, it is no longer possible to limit quantum mechanics to the one-electron problem. The known experimental results can coincide with the theoretical predictions that are deduced from the solution of a many-body problem.

2. There is no longer any sense in speaking of a density of *particles*. By contrast, the charge density, as well as the charge itself, preserve a precise physical sense and are observable quantities. The charge that is contained in a given volume, when considered as an operator, has proper values that are proportional to the *positive* and *negative* integer numbers.

It results from these remarks that there is no longer any valid reason to postulate the particular form:

$$\sum \psi_r^* \psi_r \tag{1}$$

for the charge density, as Dirac did in order to establish his equation. An immediate consequence of this is that the fundamental equation must not necessarily be *linear* in $\partial \psi_r / \partial t$; the Gordon equation certainly falls into this category. In the following paragraph, we will show the manner in which one may utilize the Dirac argument in order to establish the new equation.

Be that as it may, the new equation must satisfy these conditions:

1. – *It must present relativistic and electromagnetic (i.e., gauge) invariance.*

2. – *The wave function must have four components.*

Indeed, in the Dirac theory, four components suffice to account for the known phenomena, and there is no reason to exceed that number. In addition, it seems easy to establish a theory with a large number of components, but it is clear that a similar attempt will have no interest in the context of experiments, so we will not impose that requirement.

3. – *The passage to the case in which an external field is present must be done, as always, by adding the operator* Φ *– viz., the potential of the field – to the operator* $\frac{h}{i}\frac{\partial}{\partial x_r}$.

There is no reason to renounce that condition, which will be imposed for reasons of correspondence.

4. – *One must be able to form a world vector that represents the current and charge density.*

5. – *This current must satisfy a conservation law by virtue of the fundamental equation*, as well as in the presence of an external field.

6. – *The temporal component of this vector – viz., the charge density – must be capable of being positive or negative.*

This condition is essential.

7. – *One must be able to form a symmetric tensor of second rank that represents the tensor density of energy-quantity of motion.*

8. – *This tensor must satisfy a conservation law by virtue of the fundamental equation* that has the form $\sum_r \frac{\partial T_{rk}}{\partial x_r} = 0$ in vacuo or $\sum_r \frac{\partial T_{rk}}{\partial x_r} = -\sum_r j_r \cdot F_{rk}$, where j_r is the current and F_{rk} is the external field.

9. – *The energy density must be capable of being positive or zero, so its expression must be a positive definite form*, which must be true in the presence or absence of an external field.

This condition is essential.

10. – *One must be able to exhibit the existence of a spin and a magnetic moment.*

3. Indications regarding the choice of a fundamental equation. – The argument by which one arrives at the fundamental equation might not have an absolute character; in fact, it serves only to guide our choice. Once this choice has been made, one postulates the equation thus found and then justifies this postulate by the results obtained. This is the case in the Dirac theory, moreover, and some of the reasoning that was employed on that occasion, when conveniently modified, may serve again.

Our point of departure is condition 9, according to which the energy density must be represented by a positive-definite form:

$$\sum_r \Phi_r^* \Phi_r \,,$$

where the Φ_r may be either the components of a function that call the "wave function" or, more generally, *functions of it*. The conservation of energy will be expressed by:

$$\frac{d}{dt}\int \sum \Phi_r^* \Phi_r = \int \sum \left(\frac{\partial \Phi_r^*}{\partial t} \Phi_r + \Phi_r^* \frac{\partial \Phi_t}{\partial t} \right) dv = 0,$$

the integral being taken over a convenient volume or even all of space. One may deduce from this, as in the Dirac theory ([1]), that at a well-defined instant one may simultaneously assign arbitrary values to:

$$\partial \Phi_r^* / \partial t \,, \qquad \partial \Phi_r / \partial t, \qquad \text{and} \qquad \Phi_r^* \,, \, \Phi_r \,.$$

Therefore, the Φ satisfy equations that linear equations in $\partial / \partial t$, and, by symmetry, also linear in $\partial / \partial x_r$.

On the other hand, the fundamental relation:

$$\left(\frac{W}{c} \right)^2 = p_1^2 + p_2^2 + p_3^2 + m^2 c^2$$

leads to the second-order equation:

([1]) PAULI, *Handbuch der Physik*, vol. 19, pp. 217.

$$\Box\Phi_r - \frac{m^2c^2}{h^2}\Phi_r = 0, \tag{2}$$

and it is natural to postulate, as in the Dirac theory, that this is satisfied in the case where a field is absent.

We must then linearize (2) – i.e., we must find a system of equations that have (2) as their consequence. Now, only the Dirac solution leads to a true *linearization,* and that is not convenient in our problem. One deduces from this that the Φ_r are not "wave functions," or rather that the expression for energy does not *uniquely* contain terms that depend directly upon the wave functions ψ_s . There will also be terms that are necessarily combinations of ψ_s and $\partial / \partial x_r$, and they collectively permit a sort of "linearization;" for example, it will make it easier to understand what it consists of. We take the linearized equation to be:

$$\Box\,\Phi_r - k^2\,\Phi_r = 0 \qquad (r = 0, 1, 2, 3). \tag{2}$$

The system:

$$\Phi_1 = \frac{\partial\psi}{\partial x}, \qquad \Phi_2 = \frac{\partial\psi}{\partial y}, \qquad \Phi_3 = \frac{\partial\psi}{\partial z}, \qquad \Phi_0 = \frac{1}{c}\frac{\partial\psi}{\partial t}, \tag{3}$$

$$\frac{\partial\Phi_1}{\partial x} + \frac{\partial\Phi_2}{\partial y} + \frac{\partial\Phi_3}{\partial z} - \frac{1}{c}\frac{\partial\Phi_0}{\partial t} = k^2\psi, \tag{4}$$

where ψ is a scalar, has the relation (2) as a consequence; it suffices to substitute (3) in (4) and differentiate. Meanwhile, it is clear that the fundamental equation remains the second-order one that defines ψ:

$$\Box\;\psi = k^2\psi; \tag{5}$$

the Φ_r are deduced from it by means of first-order equations.

Be that as it may, if one is given the complex *scalar* ψ which satisfies (5) then one may deduce the Φ from it by means of (3), and consequently, write a second-rank tensor whose temporal component:

$$\Phi_0^*\Phi_0 + \sum_{r=1}^{3}\Phi_r^*\Phi_r + k^2\psi^*\psi \tag{6}$$

will be a positive-definite form. This shows that one may establish a theory of a particle with positive energy in that fashion.

The fundamental equation (5) is the *Gordon equation* and the corresponding theory was developed by Pauli and Weisskopf. It indeed satisfies the conditions of the preceding paragraph, *except 2 and 10.* Upon generalizing the preceding process, we find another system that also satisfies the last two conditions.

In the preceding example, the wave function ψ is a scalar. Now, we have need for a magnitude with four complex components; one may take two spinors ψ_r, χ_s . One may take a vector complex vector that was formed by combining the four components of the two vectors a_{rs} , b_{rs} . One may also take the $2\times2 = 4$ complex components of a spinor of second rank g_{rs} and stop with the simplest possibilities ([1]).

The solution by two spinors is the Dirac solution, which is not convenient. As a result, being given such a g_{rs} amounts to being given an anti-symmetric tensor of second rank with real components ([2]) and two invariants. Finally, the latter solution, which takes the form of a wave vector, is the simplest; this is why we shall adopt it.

The preceding example is completely characterized when one is given the corresponding Lagrange function, which is, following Gordon:

$$L = h^2c^2\sum_{\mu=1}^{4}\frac{\partial\psi^*}{\partial x_\mu}\frac{\partial\psi}{\partial x_\mu} + m^2c^4\psi^*\psi \qquad (\mu = 1, \ldots, 4).$$

The energy density tensor:

$$T_{\mu\nu} = h^2c^2\left(\frac{\partial\psi^*}{\partial x_\mu}\frac{\partial\psi}{\partial x_\nu} + \frac{\partial\psi^*}{\partial x_\nu}\frac{\partial\psi}{\partial x_\mu}\right) + L\cdot\delta_\mu ,$$

the field equations, etc., are deduced from L immediately. We look for the possible L in the case of a wave function that is the world vector ψ_r . L is an invariant; it appears additively in the expression T_{44} , and consequently will be comprised, like energy, of a sum of terms of the form A^*A. Among them, one will have:

1. Terms that contain the components of the function ψ explicitly.

The simplest invariant combination that one may form from it is:

$$\sum\psi_r^*\psi_r$$

and:

2. Terms that are formed by combining $\partial / \partial x$ and ψ_r .

The simplest combination will be the divergence $A = \partial\psi_r / \partial x_r$; one confirms that it does not work. The second degree of complexity will be A = a second-rank tensor of type $\partial\psi_r / \partial x_s$, and more particularly, the one that has the minimum number of components, namely, *the rotation of the vector* ψ_r :

$$A_{rs} = \frac{\partial\psi_s}{\partial x_r} - \frac{\partial\psi_r}{\partial x_s}.$$

The simplest Lagrangian will then be of the form:

([1]) One may nonetheless take magnitudes with several indices by imposing certain symmetry conditions on them; for example, g_{rst} , when it is anti-symmetric in s and t, has only four non-zero components.

([2]) Cf., LAPORTE and UHLENBECK, *Physical Review* (1931), pp. 1552.

$$L = \sum_{rs} A^*_{rs} A_{rs} + k^2 \sum_r \psi^*_r \psi_r \,,$$

where k is a constant that we take to be proportional to the proper mass, for reasons that will become apparent in the sequel. By hypothesis, for the case of no external field we explicitly write:

$$L = \frac{h^2c^2}{2}\left(\frac{\partial \psi^*_s}{\partial x_r} - \frac{\partial \psi^*_r}{\partial x_s}\right)\left(\frac{\partial \psi_s}{\partial x_r} - \frac{\partial \psi_r}{\partial x_s}\right) + m^2c^4 \psi^*_r \psi_r \,,$$

with the usual summation convention. The Lagrangian for the case of an external field will be deduced from the preceding one by the known rule (condition 3).

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4. Notations. Lagrange functions. – For reasons that will become obvious shortly, we do not adopt notations that are analogous to the ones that were employed in the Dirac theory; this should not introduce any difficulty, but will offer no immediate advantage, either.

Therefore, consider a vector $\psi_r = a_r + ib_r$ and its complex conjugate $\psi^* = a_r - ib_r$. We may develop the theory in a universe that is defined by $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_0 = ct$. For the symmetry of the formulas, it is meanwhile advantageous to allow a fourth imaginary coordinate. Nevertheless, in that case, one must take certain precautions in order to obtain the correct reality conditions.

For that, it will suffice to write the coordinates in the form:

$$x_1 = x, \qquad x_2 = y, \qquad x_3 = z, \qquad x_4 = \varepsilon ct, \tag{7}$$

ε being a complex unit such that $\varepsilon^2 = -1$, it commutes with the other imaginary unit i that appears in the expression for the ψ_r . It will then be incorrect to write $\varepsilon\, i = -1$, but one will have $\varepsilon\, i = i\, \varepsilon$. *The asterisk that indicates the complex conjugate will change the sign of i, but not that of ε.* The spatial components of ψ_r will be $a_r + ib_r$, a_r, b_r real (r = 1, 2, 3), and the temporal component will be $\psi_4 = \varepsilon a'_4 + i\varepsilon b'_4$, with a'_4, b'_4 real. The complication that is introduced by the use of two complex units is compensated by the advantage of symmetry. Moreover, ε always disappears when one passes to the variable t. In addition, for greater clarity, we avoid the use of the asterisk for *functions of* ψ^*; we use different letters to indicate that function and its complex conjugate.

Finally, set:

$$\partial_r = \frac{\partial}{\partial x_r} \qquad (r = 1, 2, 3, 4).$$

Having done this, consider the case of *no external field.*

We set:

$$F'_{rs} = \partial_r \psi^*_s - \partial^*_s \psi_r \,, \qquad G'_{rs} = \partial_r \psi_s - \partial_s \psi_r \,. \tag{9}$$

F'_{rs} and G'_{rs} are complex conjugate; they contain an ε as a factor whenever one of the indices r or s is equal to 4.

By hypothesis, the Lagrangian of the problem will be:

$$L = \frac{h^2c^2}{2} F'_{rs} G'_{rs} + m^2 c^4 \psi_r^* \psi_r \,, \tag{10}$$

in this case, h being Planck's constant divided by 2π.

One then passes to *the case where a field exists* that is defined by a potential:

$$\Phi_1, \Phi_2, \Phi_3, \Phi_4 = \varepsilon \Phi_0 \qquad (\Phi_0, \ldots, \Phi_3 \text{ real}) \tag{11}$$

by the substitutions ([1]):

$$\frac{\partial \psi_r}{\partial x_s} \to \frac{\partial \psi_r}{\partial x_s} - \frac{ie}{hc}\Phi_s \psi_r \,, \qquad \frac{\partial \psi_r^*}{\partial x_s} \to \frac{\partial \psi_r^*}{\partial x_s} + \frac{ie}{hc}\Phi_s \psi_r^* \,. \tag{12}$$

To simplify the notation, we set:

$$A_s = \frac{e}{hc}\Phi_s \,, \tag{13}$$

$$k = \frac{mc}{h} \,. \tag{14}$$

We further set:

$$\left.\begin{aligned} F_{rs} &= (\partial_r + iA_r)\psi_s^* - (\partial_s + iA_s)\psi_r^* \,, \\ G_{rs} &= (\partial_r - iA_r)\psi_s - (\partial_s - iA_s)\psi_r \,; \end{aligned}\right\} \tag{15}$$

F_{rs}, G_{rs} are anti-symmetric tensors of second order. The Lagrange function in the case of an external field is the written, with the usual summation convention:

$$\boxed{L = \frac{h^2c^2}{2} F_{rs} G_{rs} + m^2 c^4 \psi_r^* \psi_r \,.} \tag{16}$$

5. Fundamental equations. – The momenta are:

$$\frac{\partial L}{\partial(\partial_r \psi_s)} = h^2 c^2 F_{rs} \,, \qquad \frac{\partial L}{\partial(\partial_r \psi_s^*)} = h^2 c^2 G_{rs} \,.$$

One has, in turn:

([1]) We adopt the convention of Pauli and Weisskopf for the sign of e; cf., *loc. cit.*, pp. 722.

$$\frac{\partial L}{\partial \psi_s} = m^2c^4\psi_s^* - ih^2c^2A_rF_{rs}, \qquad \frac{\partial L}{\partial \psi_s^*} = m^2c^4\psi_s + ih^2c^2A_rG_{rs},$$

$$\frac{\partial L}{\partial A_s} = ih^2c^2(\psi_rF_{rs} - \psi_s^*G_{rs}).$$

The fundamental equations:

$$\partial^r\left[\frac{\partial L}{\partial(\partial_r\psi_s)}\right] = \frac{\partial L}{\partial \psi_s}, \qquad \partial^r\left[\frac{\partial L}{\partial(\partial_r\psi_s^*)}\right] = \frac{\partial L}{\partial \psi_s^*} \tag{20}$$

take the form:

$$\boxed{(\partial_r + iA_r)F_{rs} = k^2\psi_s^*, \qquad (\partial_r - iA_r)G_{rs} = k^2\psi_s} \tag{21}$$

to which, one agrees to add equations (14), which define the intermediary magnitudes F_{rs} and G_{rs} .

These have the form of Maxwell's equations (for an imaginary "potential"), completed by elements that represent the influence of an external field. One may then deduce a certain number of consequences from this fact that we will pass over, for the moment.

They indeed present relativistic and electromagnetic invariance; the first three conditions of paragraph 2 are thus satisfied:

Eliminate F_{rs} and G_{rs} ; one has, in general:

$$\left.\begin{aligned}(\partial^r - iA^r)(\partial^s - iA^s) - (\partial^s - iA^s)(\partial^r - iA^r) &= -i(\partial^rA^s - \partial^sA^r),\\ (\partial^r + iA^r)(\partial^s + iA^s) - (\partial^s + iA^s)(\partial^r + iA^r) &= +i(\partial^rA^s - \partial^sA^r).\end{aligned}\right\} \tag{22}$$

Set:

$$\left.\begin{aligned}\Box_+ &= \sum_{\mu=1}^{4}(\partial_\mu + iA_\mu)^2, \quad \Box_- = \sum_{\mu=1}^{4}(\partial_\mu - iA_\mu)^2,\\ J &= \sum_{\mu=1}^{4}(\partial_\mu + iA_\mu)\psi_\mu^*, \quad I = \sum_{\mu=1}^{4}(\partial_\mu - iA_\mu)\psi_\mu,\end{aligned}\right\} \tag{23}$$

$$H_{rs} = \frac{hc}{e}(\partial_rA_s - \partial_sA_r). \tag{24}$$

The fundamental equations become:

$$\left.\begin{aligned}\Box_+\psi_s^* - (\partial_s + iA_s)J &= \left(k^2\delta_{rs} + \frac{ie}{hc}H_{rs}\right)\psi_r^*,\\ \Box_-\psi_s - (\partial_s - iA_s)I &= \left(k^2\delta_{rs} + \frac{ie}{hc}H_{rs}\right)\psi_r.\end{aligned}\right\} \tag{25}$$

In the case of no field, and provided that k – and therefore, the *mass of the particle – is non-zero*, which is an essential condition, one deduces that:

$$I = J = 0. \tag{26}$$

Indeed, in this case (25) becomes:

$$\Box\psi_s^* - \partial_s J = k^2\psi_s^*, \qquad \Box\psi_s - \partial_s I = k^2\psi_s,$$

and upon applying the operator $\sum_s \partial_s$, one deduces the stated result (26).

Therefore, in *the absence of a field the equations become equations of the Gordon type:*

$$\begin{aligned}\Box\psi_s^* &= k^2\psi_s^*,\\ \Box\psi_s &= k^2\psi_s,\end{aligned} \qquad \text{with} \qquad \begin{cases}\partial_s\psi_s^* = \partial_s\psi_s = 0,\\ k = \dfrac{mc}{h}.\end{cases} \tag{27}$$

In the general case, we apply the operator $\sum_s \partial_s$ to (21); one will have:

$$-\frac{ie}{2hc} F_{rs} \cdot H_{rs} = k^2\, J, \qquad -\frac{ie}{2hc} G_{rs} \cdot H_{rs} = k^2\, I.$$

Therefore, if $k = 0$, and only in this case then:

$$J = -\frac{ieh}{2hc}\frac{1}{mc^2} \cdot F_{rs} \cdot H_{rs}, \qquad I = \frac{ieh}{2hc}\frac{1}{mc^2} \cdot G_{rs} \cdot H_{rs}. \tag{28}$$

We immediately point out some other interesting relations.

By virtue of (17), the $\dfrac{\partial L}{\partial(\partial_r\psi_s)}$ are *anti-symmetric* tensors; one thus has:

$$\partial^r\partial^s\left[\frac{\partial L}{\partial(\partial_r\psi_s)}\right] = \partial^r\partial^s\left[\frac{\partial L}{\partial(\partial_r\psi_s^*)}\right] = 0.$$

The fundamental equations permit one to deduce:

$$\partial^s\left(\frac{\partial L}{\partial\psi_s}\right) = \partial^s\left(\frac{\partial L}{\partial\psi_s^*}\right) = 0. \tag{29}$$

6. Existence and conservation of current. – We define the current, as Gordon did, following Mie, by means of the derivative of the Lagrange function with respect to the potential, and with the sign changed:

$$j_s = -\frac{\partial L}{\partial \Phi_s} = -\frac{e}{hc}\frac{\partial L}{\partial A_s}, \tag{30}$$

so, by virtue of (19):

$$j_s = i\, ehc\, (\psi_r^* \, G_{rs} - \psi_r \, F_{rs}). \tag{31}$$

It is clear that (30) is a world-vector.

It always satisfies a conservation law $\partial^s j_s = 0$. Indeed, one has:

$$\partial^s j_s = i\, ehc\, (\partial_s \psi_r^* \cdot G_{rs} + \psi_r^* \cdot \partial_s G_{rs} - \partial_s \psi_r \, F_{rs} - \psi_r \, \partial_s F_{rs}).$$

By virtue of the fundamental equations, this is:

$$= i\, ehc\, [G_{rs} (\partial_s + iA_s) \psi_r^* - F_{rs} (\partial_s - iA_s) \psi_r],$$

and by virtue of the anti-symmetric character of F_{rs}, G_{rs}, this is:

$$= i\, ehc \left[\frac{G_{rs} \cdot F_{rs}}{2} - \frac{F_{rs} \cdot G_{rs}}{2}\right] \equiv 0.$$

There thus exists a vector j_s that satisfies a conservation law. In order for us to be able to confirm that it indeed represents a current, it is further necessary that when it is combined with the electromagnetic field it indeed furnishes the expression for the force that results from the divergence of the energy tensor; we show that this is indeed the case.

In any case, the temporal component is:

$$j_4 = i\, ehc\, (\psi_r^* \, G_{r4} - \psi_r \, F_{r4}); \tag{32}$$

it can take on values that are either positive or negative.

7. Energy-quantity of motion tensor and conservation law. – In order to establish both the expression for this tensor and its conservation law, we proceed as Schrödinger did. Differentiate the Lagrange function with respect to x_ρ, ρ arbitrary; one will have:

$$\partial_\rho L = \frac{\partial L}{\partial(\partial_r \psi_s)} \cdot \partial^2_{r\rho} \psi_s + \frac{\partial L}{\partial(\partial_r \psi_s^*)} \cdot \partial^2_{r\rho} \psi_s^* + \frac{\partial L}{\partial \psi_s} \cdot \partial_\rho \psi_s + \frac{\partial L}{\partial \psi_s^*} \cdot \partial_\rho \psi_s^* + \frac{\partial L}{\partial A_s} \cdot \partial_\rho A_s$$

so, by virtue of the fundamental equations:

$$\partial_\rho L = \partial_r (L\, \delta_{r\rho}) = \partial_r \left\{\frac{\partial L}{\partial(\partial_r \psi_s)} \cdot \partial_\rho \psi_s + \frac{\partial L}{\partial(\partial_r \psi_s^*)} \cdot \partial_\rho \psi_s^*\right\} + \frac{\partial L}{\partial A_s} \cdot \partial_\rho A_s .$$

Let H_{rs} be the external electromagnetic field and let j_s be the current:

$$j_s = -\frac{e}{hc}\frac{\partial L}{\partial A_s}. \tag{30}$$

We have proved that $\partial^s j_s = 0$; therefore:

$$H_{\rho s}\, j_s = -\frac{\partial L}{\partial A_s}\cdot\partial_\rho A_s + \partial_s\left(\frac{\partial L}{\partial A_s}A_\rho\right). \tag{33}$$

Add (32) and (33); upon taking into account the fundamental equations:

$$H_{\rho s}\, j_s = \partial_r\left\{\frac{\partial L}{\partial(\partial_r\psi_s)}\cdot\partial_\rho\psi_s + \frac{\partial L}{\partial(\partial_r\psi_s^*)}\cdot\partial_\rho\psi_s^* + \frac{\partial L}{\partial A_s}\cdot A_s - \delta_{r\rho}L\right\}, \tag{34}$$

one finds that:

$$H_{\rho s}\, j_s = \partial^r\{h^2c^2\, F_{rs}(\partial_\rho - iA_\rho)\psi_s + h^2c^2\, G_{rs}(\partial_\rho + iA_\rho)\psi_s^* - \delta_{r\rho}L\}. \tag{35}$$

Now, one easily sees that one has the identity:

$$0 \equiv \partial_r\left\{\frac{\partial L}{\partial(\partial_r\psi_s)}\cdot\partial_s\psi_\rho + \frac{\partial L}{\partial(\partial_r\psi_s^*)}\cdot\partial_s\psi_\rho^* - \frac{\partial L}{\partial\psi_r}\cdot\psi_\rho - \frac{\partial L}{\partial\psi_r^*}\cdot\psi_\rho^*\right\}; \tag{36}$$

it suffices to take (29) into account, along with the fact that the $\frac{\partial L}{\partial(\partial_r\psi_s)}$ are anti-symmetric.

Upon making (36) explicit, with the aid of the fundamental formulas (17) and (18), one has:

$$0 \equiv \partial^r\{h^2c^2\, F_{rs}(\partial_s - iA_s)\psi_\rho + h^2c^2\, G_{rs}(\partial_s + iA_s)\psi_\rho^* - m^2c^4\,(\psi_r^*\psi_\rho + \psi^*\psi_r)\}. \tag{37}$$

Subtract (37) from (35); by virtue of (15), one has:

$$H_{\rho s}\, j_s = \partial^r\{h^2c^2\,(F_{rs}\cdot G_{\rho s} + G_{rs}\cdot F_{\rho s}) + m^2c^4\,(\psi_r^*\psi_\rho + \psi^*\psi_r) - \delta_{r\rho}L\}. \tag{38}$$

namely:

$$\boxed{\partial^r T_{r\rho} = H_{\rho s}\cdot j_s,} \tag{39}$$

where:

$$\boxed{T_{r\rho} = h^2c^2(F_{rs}G_{\rho s} + F_{\rho s}G_{rs}) + m^2c^4(\psi_r^*\psi_\rho + \psi_\rho^*\psi_r) - \delta_{r\rho}L.} \tag{40}$$

This $T_{r\rho}$ is a *symmetric* tensor; (39) shows that its divergence is equal to the Lorentz force. It may thus be chosen to be the energy-quantity of motion tensor.

The energy is constantly positive. The energy density is defined by $E = -T_{44}$, and one has, upon writing out L explicitly:

$$E = h^2c^2\left(\frac{F_{r's'}\cdot G_{r's'}}{2} - F_{4s}\cdot G_{4s}\right) + m^2c^2(\psi_{r'}^*\psi_{r'} - \psi_4^*\psi_4)\,, \tag{41}$$

where s varies from 1 to 4 and r', s' vary from 1 to 3; this is, again by writing it out explicitly:

$$\begin{aligned} E = h^2c^2(&F_{23}G_{23} + F_{31}G_{31} + F_{12}G_{12} - F_{14}G_{14} - F_{24}G_{24} - F_{34}G_{34}) \\ &+ m^2c^4(\psi_1^*\psi_1 + \psi_2^*\psi_2 + \psi_3^*\psi_3 - \psi_4^*\psi_4)\,, \end{aligned} \tag{42}$$

a quantity that is essentially positive, since G_{rs} is the complex conjugate of F_{rs} .

8. Electromagnetic moment of the electron. – Let the current be:

$$j_s = i\,ehc\,(\psi_r^*\,G_{rs} - \psi_r\,F_{rs}). \tag{31}$$

One may regard it as a sum of two terms of a particular form. Indeed, upon writing it out explicitly, one has the expression:

$$j_s = i\,ehc[\psi_r\,(\partial_s + i\,A_s)\psi_r^* - \psi_r^*\,(\partial_s - i\,A_s)\psi_r + \psi_s^*\,I - \psi_s\,J] + \partial^r(\psi_r^*\psi_s - \psi_s^*\psi_r)\,. \tag{43}$$

In the case of a zero external field, $I = J = 0$, and the current reduces to:

$$j_s^0 = i\,ehc\,[\psi_r\cdot\partial^r\psi_r^* - \psi_r^*\cdot\partial_s\,\psi_r + \partial^r(\psi_r^*\psi_s - \psi_s^*\psi_r)]. \tag{44}$$

One may thus separate the current density into two parts $j_s = j_s' + j_s''$, just as in the Dirac theory. The second one:

$$j_s'' = i\,ehc\cdot\partial^r(\psi_r^*\psi_s - \psi_s^*\psi_r) = \partial^r m_{rs}$$

presents itself as the current density that is due to an electric or magnetic moment tensor m_{rs} (which does not depend upon the external field explicitly). We may thus assume that the electron *possesses an electromagnetic moment that is given by:*

$$m_{rs} = i\,ehc(\psi_r^*\psi_s - \psi_s^*\psi_r)\,. \tag{45}$$

The remaining one:

$$j_s' = i\,ehc[\psi_r\,(\partial_s + i\,A_s)\psi_r^* - \psi_r^*\,(\partial_s - i\,A_s)\psi_r + \psi_s^*\,I - \psi_s\,J] \tag{46}$$

will be the conduction current. As in the Dirac theory, each of these partial currents satisfies an equation of continuity:

$$\partial^s j'_s = 0 \qquad \text{and} \qquad \partial^s j''_s = 0.$$

9. Spin. – Starting with the energy-quantity of motion tensor T_{rs} , one obtains, in general, the value of the kinetic moment by forming the integral:

$$P_{r\rho} = \frac{1}{\varepsilon c}\int (x_r \cdot T_{4\rho} - x_\rho \cdot T_{4r})\, dV \qquad (r,\ \rho = 1,\ 2,\ 3), \tag{47}$$

in which:

$$\left.\begin{aligned} T_{4\rho} &= h^2c^2(F_{4s}G_{\rho s} + F_{\rho s}G_{4s}) + m^2c^4(\psi_4^*\psi_\rho + \psi_\rho^*\psi_4), \\ T_{4r} &= h^2c^2(F_{4s}G_{rs} + F_{rs}G_{4s}) + m^2c^4(\psi_4^*\psi_r + \psi_r^*\psi_4). \end{aligned}\right\} \tag{48}$$

Consider an electron in the absence of a field; one will have in this case:

$$\begin{aligned} P_{r\rho} &= \frac{1}{\varepsilon c}\int \{h^2c^2F_{4s}(x_rG_{\rho s} - x_\rho G_{rs}) + m^2c^4\psi_4^*(x_r\psi_\rho - x_\rho\psi_r) + \text{conjugate}\}\, dV \\ &= \frac{1}{\varepsilon c}\int \{h^2c^2[F_{4s}\cdot x_r\cdot\partial_\rho\psi_s - F_{4s}\cdot x_r\cdot\partial_s\psi_\rho - F_{4s}\cdot x_\rho\cdot\partial_r\psi_s + F_{4s}\cdot x_\rho\cdot\partial_s\psi_r] \\ &\quad + m^2\, c^4\cdot\ \psi_4^*(x_r\ \psi_\rho - x_\rho\ \psi_r) + \text{conjugate}\}\ dV. \end{aligned}$$

For $r = \rho$, the components P_{rs} are zero; for $r \neq \rho$, one has:

$$x_r\cdot\partial_\rho\,\psi_s = \partial_\rho(x_r\ \psi_s), \quad x_\rho\cdot\partial_r\ \psi_s = \partial_r(x_\rho\ \psi_s), \tag{49}$$

and:

$$\left.\begin{aligned} F_{4s}\cdot x_r\cdot\partial_s\psi_\rho &= F_{4s}\cdot\partial_s(x_r\psi_\rho) - \psi_\rho\cdot F_{4r}, \\ F_{4s}\cdot x_\rho\cdot\partial_s\psi_r &= F_{4s}\cdot\partial_s(x_\rho\psi_r) - \psi_r\cdot F_{4\rho}. \end{aligned}\right\} \tag{50}$$

One may thus write:

$$\begin{aligned} P_{4s} &= \frac{1}{\varepsilon c}\int \{h^2c^2[F_{4s}\cdot\partial_\rho(x_r\psi_s) - F_{4s}\cdot\partial_s(x_r\psi_\rho) - F_{4s}\cdot\partial_r(x_\rho\psi_s) + F_{4s}\cdot\partial_s(x_\rho\psi_r)] \\ &\quad + h^2c^2\ (\psi_\rho\, F_{4r} - \psi_r\, F_{4\rho}) + m^2c^4\ (x_r\,\psi_\rho - x_\rho\ \psi_r) + \text{conjugate}\}\ dV. \end{aligned}$$

Upon integrating by parts and assuming, as one usually does, that the ψ_r are annulled on the boundary, one will have:

$$\begin{aligned} P_{4s} &= \frac{1}{\varepsilon c}\int h^2c^2\{x_\rho(\psi_s\partial_rF_{4s} + \psi_s^*\partial_rG_{4s}) - x_r(\psi_s\partial_\rho F_{4s} + \psi_s^*\partial_\rho G_{4s})\}\, dV \\ &\quad + \frac{1}{\varepsilon c}\int h^2c^2(\psi_\rho F_{4r} - \psi_r F_{4\rho} + \psi_\rho^* G_{4r} - \psi_r^* G_{4\rho})\, dV\ . \end{aligned} \tag{51}$$

One may regard the first integral, which has the form $\int (x_\rho A_r - x_r A_\rho)\, dV$, as equivalent to an "orbital momentum," and the second one, where the x_r , x_ρ no longer appear explicitly, as the proper momentum of the particle.

The "spin" density of the latter will thus be:

$$\mathfrak{M} = h^2c^2\{ \psi_\rho F_{4r} + \psi_r F_{\rho 4} + \psi_\rho^* G_{4r} + \psi_r^* G_{r4}). \tag{52}$$

It is obvious that this decomposition is arbitrary, which corresponds to the nature of things, moreover.

We remark that the relativistic variance of the spin is that of the total momentum (47) that we started with; it cannot be otherwise. This spin differs from the one that one encounters in the Dirac theory. Indeed, it is certainly comprised of the three temporal components of a tensor of the form P_{rsl} , but *it is not completely anti-symmetric*, as in the case of the Dirac equation. It is meanwhile easy to modify the decomposition (51) so as to separate the total momentum into an "orbital momentum" and a spin that is a completely anti-symmetric tensor of rank three. Indeed, one has, for s = 1, 2, 3, 4:

$$m^2c^4\, x_r\, \psi_4\, \psi_\rho^* = h^2c^2 \cdot \psi_4 \cdot \partial_s(x_r F_{s\rho}) - h^2c^2 \cdot \psi_4 \cdot F_{r\rho} ,$$
$$m^2c^4\, x_\rho\, \psi_4\, \psi_r^* = h^2c^2 \cdot \psi_4 \cdot \partial_s(x_\rho F_{sr}) - h^2c^2 \cdot \psi_4 \cdot F_{\rho r} ,$$

so:

$$\left.\begin{aligned} m^2c^4\psi_4(x_r\psi_\rho^* - x_\rho\psi_r^*) - h^2c^2\cdot\psi_4\cdot\partial_s(x_rF_{s\rho} - x_\rho F_{sr}) &= 2h^2c^2\cdot\psi_4\cdot F_{\rho r},\\ m^2c^4\psi_4(x_r\psi_\rho - x_\rho\psi_r) - h^2c^2\cdot\psi_4^*\cdot\partial_s(x_rG_{s\rho} - x_\rho G_{sr}) &= 2h^2c^2\cdot\psi_4^*\cdot G_{\rho r}.\end{aligned}\right\} \tag{53}$$

One then deduces a separation of the total momentum $P_{r\rho}$ into an "orbital moment" and a "spin":

$$\frac{1}{\varepsilon c}\int h^2c^2(\psi_\rho F_{4r} + \psi_r F_{\rho 4} + \psi_4 F_{r\rho} + \text{conjugate})\, dV\ . \tag{54}$$

In this form, the spin density will be composed of the temporal components of a completely anti-symmetric tensor of third rank, exactly as it is in the Dirac theory.

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