CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C. I. M. E.)

MATHEMATICAL PROBLEMS IN RELATIVISTIC HYDRODYNAMICS

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CHAPTER I

FLUID SCHEMAS IN RELATIVISTIC HYDRODYNAMICS

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§ 1. GENERALITIES ON THE RELATIVISTIC DYNAMICS OF FLUIDS

1. The geometric context. – The geometric context of relativistic fluid mechanics is a differentiable manifold V of dimension 4, class C^∞, on which one is given a pseudo-Riemannian structure g of signature $+ - - -$. The geometry of the space-time (V, g) is that of the Riemannian connection that is canonically associated with g.

The metric that is defined by g is said to have *normal hyperbolic* type. It introduces the structure of a flat Minkowski space-time on each tangent vector space $T_x(V)$. In local coordinates (x^α), one has:

$$g = g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta \qquad (\alpha, \beta = 0, 1, 2, 3). \tag{1.1}$$

The tensor $g_{\alpha\beta}$, which is called the *fundamental tensor of gravitation*, is required to verify a system of second-order partial differential equations that generalizes the Laplace-Poisson equations and gives rise to the conservation conditions. Those equations are the ten Einstein equations:

$$S_{\alpha\beta} = \chi\, T_{\alpha\beta}\,, \tag{1.2}$$

in which $S_{\alpha\beta}$ depends upon only the Riemannian structure g of space-time, $T_{\alpha\beta}$ has a purely-mechanical significance, and χ is a constant factor.

The tensor $T_{\alpha\beta}$, which is called the *impulse-energy tensor* of the fluid, must describe, at best, the energy distribution in space-time. The tensor $S_{\alpha\beta}$ is restricted by the following two conditions:

1. The $S_{\alpha\beta}$ depend upon only the $g_{\alpha\beta}$ and their derivatives of the first two orders, and they are linear in the second-order derivatives.

2. $S_{\alpha\beta}$ is conservative; i.e.:

$$\nabla_\alpha S^\alpha_\beta = 0. \tag{1.3}$$

One can show ([1]) that one necessarily has:

$$S_{\alpha\beta} = h\,[R_{\alpha\beta} - \tfrac{1}{2}(R + k)\, g_{\alpha\beta}],$$

([1]) E. CARTAN, J. Math. pures et appliquées **1** (1922), 141-203.

in which $R_{\alpha\beta}$ is the Ricci curvature, R is the scalar curvature of (V, g), and h and k are two arbitrary constants. k is the cosmological constant, which plays no role in the description of fluids; one can then suppose that $k = 0$. On the other hand, upon suppressing the extraneous factor h, one can take the left-hand side of the Einstein equations to be:

$$S_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2} R\, g_{\alpha\beta}\,, \tag{1.4}$$

and S will be called the *Einstein tensor*.

Since the Einstein tensor $S_{\alpha\beta}$ is conservative, the same thing will be true for the impulse-energy tensor $T_{\alpha\beta}$. The equations:

$$\nabla_\alpha T^{\alpha\beta} = 0 \tag{1.5}$$

then express the conservation of impulse energy and define the evolution of the fluid.

2. The impulse-energy tensor. – In any relativistic theory of fluids, the first step consists of choosing an expression for the impulse-energy tensor $T_{\alpha\beta}$. Each expression for $T_{\alpha\beta}$ defines a fluid schema. If one wishes to satisfy the Einstein equations then $T_{\alpha\beta}$ must be symmetric. However, in order for $T_{\alpha\beta}$ to be able to describe a physical fluid, it is necessary that there must exist a unit vector field u^α that is time-like:

$$g_{\alpha\beta}\, u^\alpha\, u^\beta = +\,1, \tag{2.1}$$

and for which the scalar $T_{\alpha\beta}\, u^\alpha\, u^\beta$ is positive. u^α is called the *unit velocity vector* of the fluid, and its trajectories define *streamlines*.

Indeed, real fluids are endowed with various properties. The forces of internal constraint that play a fundamental role in the dynamical study translate into the proper stress tensor. Caloric phenomena introduce a scalar θ that is called the *proper temperature field*. The electromagnetic properties can be represented by two antisymmetric tensor fields $H_{\alpha\beta}$, $G_{\alpha\beta}$, as one knows. On the other hand, it is appropriate to study the thermodynamic evolution of the fluid. These various properties can be envisioned in a geometric decomposition of the impulse-energy tensor.

One is then led to put $T_{\alpha\beta}$ into the form:

$$T_{\alpha\beta} = \rho\, u_\alpha\, u_\beta - \pi_{\alpha\beta} - Q_{\alpha\beta} + \tau_{\alpha\beta}\,, \tag{2.2}$$

in which ρ is a positive scalar that represents the proper density of ponderable matter-energy, $\pi_{\alpha\beta}$, the proper pressures, $Q_{\alpha\beta}$, the thermal exchanges due to conduction, and $\tau_{\alpha\beta}$ is the electromagnetic energy tensor. If one neglects some of the properties then the corresponding terms will not appear in the decomposition. Similarly, one can introduce some new terms in order to study new properties.

Each expression for $T_{\alpha\beta}$ will then correspond to a fluid schema. In each case, the evolution of the fluid will be defined by the conservation equations (1.5), and when one takes the unit vector character of u^α into account, they will lead to the following equations:

(2.3) $$u_\beta \nabla_\alpha T^{\alpha\beta} = 0 ,$$

(2.4) $$(g_\rho^\beta - u^\beta u_\rho) \nabla_\alpha T^{\alpha\rho} = 0 .$$

(2.3) is called the *continuity equation*, and (2.4) constitutes the differential system for the streamlines.

One might possibly add some other equations to those ones, such as the equations of thermodynamics and the equations of the electromagnetic field. In that way, one will obtain the fundamental system of equations for the schema considered. Therefore, the pure fluid schema has been the subject of numerous studies that have become classical, and in particular, the ones by L. P. Eisenhart and A. Lichnerowicz. The thermodynamic fluid schema was studied by C. Eckart and the author in his thesis in 1954. The electromagnetic field schema was the subject of the work by A. Lichnerowicz, and those of the author dating back to 1955, which have since provoked numerous papers, and notably those of G. Pichon. He was led, in a special case, to relativistic magnetohydrodynamics, which was the subject under study in the very beautiful work of Y. Choquet-Bruhat and A. Lichnerowicz.

The mathematical study of some of those schemas constitutes the topic of this conference.

3. Proper frame. – One calls an orthonormal frame $(V_{\lambda'})$ at a point x in space-time (V_4 , g) a *proper frame* when the first vector $V_{0'}$ coincides with the unit velocity vector u and the other three vectors $V_{i'}$ define the space that is associated with the time direction u.

One can refer the space-time in the neighborhood of any point to a proper frame field that one supposes to be differentiable (but not necessarily integrable). The world-metric will then take the canonical form:

(3.1) $$g = \eta_{\lambda'\mu'}\, \omega^{\lambda'} \otimes \omega^{\mu'} = \omega^{0'} \otimes \omega^{0'} - \omega^{1'} \otimes \omega^{1'} - \omega^{2'} \otimes \omega^{2'} - \omega^{3'} \otimes \omega^{3'},$$

in which $\omega^{\lambda'}$ are the dual 1-forms to the vector fields $V_{\lambda'}$. Hence, $<\omega^{\lambda'}, V_{\mu'}> = \delta_{\mu'}^{\lambda'}$, where $\delta_{\mu'}^{\lambda'}$ is the Kronecker symbol, which is equal to 1 when $\lambda' = \mu'$ and 0 when $\lambda' \neq \mu'$. The $\omega^{\lambda'}$ thus constitute four linearly-independent Pfaff forms.

The consideration of the proper frame can be quite useful. Indeed, since the tangent vector space $T_x (V)$ has the structure of Minkowski space-time, the proper frame $V_{\lambda'}$ must be identified with a local Galilean frame in which the fluid has zero velocity. If one knows the components of a tensor t relative to the proper frame then its components in an arbitrary frame (e_α) can be deduced from the latter by known transformation formulas.

Indeed, if $(A_\alpha^{\lambda'})$ is the matrix of the passage from the frame (e_α) to the frame $(V_{\lambda'})$ and $(A_{\lambda'}^\alpha)$ is the inverse matrix then one will have:

(3.2) $$A^{\alpha}_{0'} = u^{\alpha}, \qquad A^{\alpha}_{i'} = V_{(i')}{}^{\alpha},$$

(3.2′) $$A^{0'}_{\alpha} = u_{\alpha}, \qquad A^{i'}_{\alpha} = -V^{(i')}{}_{\alpha}.$$

If t is a tensor of order 2 then its components $t_{\alpha\beta}$ in the frame (e_α) can be deduced from its components $t_{\lambda'\mu'}$ in the proper frame by the formulas:

(3.3) $$t_{\alpha\beta} = A^{\lambda'}_{\alpha} A^{\mu'}_{\beta} t_{\lambda'\mu'} .$$

In particular, one has:

(3.4) $$g_{\alpha\beta} = u_\alpha u_\beta - V_{(1')\alpha} V_{(1')\beta} - V_{(2')\alpha} V_{(2')\beta} - V_{(3')\alpha} V_{(3')\beta} .$$

Hence, in order to determine the expression for the impulse-energy tensor of a pure fluid, one must first refer it to the proper frame. In it, the fluid is characterized by its proper matter-energy density ρ and its partial pressure tensor $\pi_{i'j'}$. Its impulse-energy tensor has components:

$$T_{0'0'} = \rho, \qquad T_{i'j'} = -\pi_{i'j'}$$

in the proper frame. Now refer space-time to local coordinates x^α, so one will have $\omega^{\lambda'} = A^{\lambda'}_{\alpha} dx^{\alpha}$, and an application of formulas (3.3) will give:

(3.5) $$T_{\alpha\beta} = \rho\, u_\alpha u_\beta - \pi_{\alpha\beta} ,$$

in which $\pi_{\alpha\beta} = \sum_{i',j'} \pi_{i'j'} A^{i'}_{\alpha} A^{j'}_{\beta}$ satisfy the identities:

(3.6) $$\pi_{\alpha\beta}\, u^\alpha = 0.$$

One sees that in the case of a pure fluid, the impulse-energy tensor decomposes relative to u^α into a temporal component $\rho\, u_\alpha u_\beta$ and a spatial component $\pi_{\alpha\beta}$.

Definition. – One says that the fluid is *perfect* if the pressure quadric in the proper frame is a sphere ; i.e., if $\pi_{i'j'} = p\ \delta_{i'j'}$, where p is called the *scalar pressure* of the fluid.

For a perfect fluid, one has $\pi_{\alpha\beta} = p \sum_{i'} A^{i'}_{\alpha} A^{j'}_{\beta}$, so upon taking (3.2) and (3.4) into account, $\pi_{\alpha\beta} = p\,(g_{\alpha\beta} - u_\alpha u_\beta)$. Therefore, the impulse-energy tensor of a perfect will be given by:

(3.7) $$T_{\alpha\beta} = (\rho + p)\, u_\alpha u_\beta - p\, g_{\alpha\beta} .$$

Call the orthonormal frame (W_λ) at x such that each of its vectors W_λ is a proper vector of the matrix ($R_{\alpha\beta}$) with respect to the matrix ($g_{\alpha\beta}$) the *principal frame*. The directions that are defined by W_λ are nothing but the principal Ricci directions. Now, by virtue of the Einstein equations, the W_λ are also proper vectors of the matrix ($T_{\alpha\beta}$) relative to the matrix ($g_{\alpha\beta}$) .

One can express the components of $T_{\alpha\beta}$ by starting from the proper values and proper vectors in the form:

$$T_{\alpha\beta} = s_0 W_{(0)\alpha} W_{(0)\beta} - \sum_{i=1}^{3} s_i W_{(i)\alpha} W_{(i)\beta} . \tag{3.8}$$

One sees that, in general, the proper frame of a charged thermodynamic fluid is different from the principal frame, except in the case of a pure fluid, where:

$$T_{\alpha\beta} = \rho u_\alpha u_\beta - \sum_{i',j'} \pi_{i'j'} V_{(i')\alpha} V_{(i')\beta} ,$$

for which $\rho = s_0$, and one can perform a rotation of the spatial 3-plane in such a manner that $V_{i'}$ goes to $W_{i'}$. s_i will then be the proper values of the matrix ($\pi_{i'j'}$) .

§ 2. THE THERMODYNAMIC FLUID

4. The perfect fluid and thermodynamic variables. – The impulse-energy tensor of a perfect fluid that is not a heat conductor is:

$$T_{\alpha\beta} = (\rho + p)\, u_\alpha u_\beta - p\, g_{\alpha\beta} , \tag{4.1}$$

It is clear that u^α is a time-like proper vector, and ρ is the corresponding proper value of ($T_{\alpha\beta}$) . Any proper frame of that fluid will coincide with a principal frame that is indeterminate, due to the multiplicity of the triple proper value $-p$.

With an eye towards the energetic study, one decomposes the proper density ρ into the sum of a matter density r and a kinetic energy density (*densité d'énergie vitesse*) $r\varepsilon$, in which ε is the specific internal energy:

$$\rho = r\,(1 + \varepsilon) . \tag{4.2}$$

One is then led to introduce the index f of a fluid, which is defined by:

$$f = 1 + \varepsilon + \frac{p}{r} . \tag{4.3}$$

In these formulas, and in what follows, the physical units have been chosen in such a manner that the limiting velocity c is equal to 1. Otherwise, one must replace ε, p with εc^{-2}, $p c^{-2}$.

The impulse-energy tensor for a perfect fluid that is not a heat conductor then takes the form:

$$T_{\alpha\beta} = r\, f\, u_\alpha\, u_\beta - p\, g_{\alpha\beta}. \tag{4.4}$$

From the thermodynamic viewpoint, the proper temperature θ and the proper specific entropy S can be defined as in classical hydrodynamics by the relation:

$$\theta\, dS = d\varepsilon + p\, d\tau, \tag{4.5}$$

in which $\tau \neq 1 / r$ is the specific volume. Upon taking (4.3) into account, and taking f, s, p to be (not independent) thermodynamic variables, one can write (4.5) in the equivalent form:

$$r\; \theta\, dS = r\, df - dp\,. \tag{4.6}$$

These relations express the idea that among the variables r, θ, f, S, p, there exist only two independent variables, which one often chooses to be f and S or S and p.

If one takes f and S to be the independent variables and if one is given p as a function of f and S then the relation (4.6) will imply that:

$$r = \frac{\partial p}{\partial f}, \qquad r\; \theta = -\frac{\partial p}{\partial S}.$$

The first relation defines the equation of state of the fluid in the form:

$$r = r\,(f, S),$$

and the second relation defines the temperature.

When one applies the conservation conditions to the impulse-energy tensor (4.4), one can infer the continuity equation and the differential system for the streamlines:

$$\nabla_\alpha (r\, f\, u^\alpha) - u^\alpha\, \partial_\alpha\, p = 0\,, \tag{4.7}$$

$$r\, f\, u^\alpha\, \nabla_\alpha\, u^\beta - (g^{\alpha\beta} - u^\alpha\, u^\beta)\, \partial_\alpha\, p = 0. \tag{4.8}$$

If one takes (4.6) into account then one can write the continuity equation in the form:

$$f\, \nabla_\alpha (r\, u^\alpha) + r\; \theta\, u^\alpha\, \partial_\alpha\, S = 0, \tag{4.7′}$$

from which, one can deduce that:

Theorem:

For a perfect fluid, saying that matter is conserved is equivalent to saying that the entropy is constant along the streamlines; i.e.:

$$\nabla_\alpha (r\, u^\alpha) = 0 \qquad \Leftrightarrow \qquad u^\alpha\, \partial_\alpha S = 0\,.$$

A fluid such that $u^\alpha\, \partial_\alpha S = 0$ is called *adiabatic*.

Similarly, if one takes the thermodynamic equation into account then the differential system for the streamlines can be written:

$$f\, u^\alpha\, \nabla_\alpha u^\beta - (g^{\alpha\beta} - u^\alpha u^\beta)\, \partial_\alpha f + \theta\, g^{\alpha\beta}\, \partial_\alpha S = 0 \tag{4.8′}$$

in terms of the variables f, S.

One then deduces that:

Theorem:

If the motion of the fluid is isentropic (viz., S = const.) then the differential system for the streamlines will reduce to:

$$u^\alpha\, \nabla_\alpha u^\beta - (g^{\alpha\beta} - u^\alpha u^\beta)\, \frac{\partial_\alpha f}{f} = 0 \tag{4.9}$$

We shall show that there exists a principal extremal for the streamlines of such an isentropic fluid.

5. Viscous fluids. – In order to characterize the local deformation of the fluid, we shall introduce the Lie derivative of the metric tensor g with respect to the unit velocity vector **u** :

$$(\mathrm{L}_{\mathbf{u}}\, g)_{\alpha\beta} = \nabla_\alpha u_\beta + \nabla_\beta u_\alpha$$

and set:

$$\varepsilon_{\alpha\beta} = \tfrac{1}{2}\, \gamma_\alpha^\rho \cdot \gamma_\beta^\mu\; (\nabla_\rho u_\mu + \nabla_\mu u_\rho)\,, \tag{5.1}$$

in which $\gamma_{\alpha\beta}$ is the spatial projector.

The laws of stress-deformation are assumed to be linear, so if the medium is isotropic then the phenomena that pertain to viscosity will be described by the tensor:

$$\sigma_{\alpha\beta} = C_{\alpha\beta}{}^{\rho\mu}\, \varepsilon_{\rho\mu}\,, \tag{5.2}$$

in which:

$$C_{\alpha\beta\rho\mu} = \lambda\, \gamma_{\alpha\beta}\, \gamma_{\rho\mu} + \mu\, (\gamma_{\alpha\rho}\, \gamma_{\beta\mu} + \gamma_{\alpha\mu}\, \gamma_{\beta\rho}) \tag{5.3}$$

or, if one takes into account that $g_{\alpha\beta}\, u^\alpha\, u^\beta = 1$ then:

$$C_{\alpha\beta\rho\mu} = \lambda\, \gamma_{\alpha\beta}\, g_{\rho\mu} + \mu\, (g_{\alpha\rho}\, g_{\beta\mu} + g_{\alpha\mu}\, g_{\beta\rho})\ . \tag{5.3′}$$

The impulse-energy tensor of the viscous homogeneous fluid is then given by:

$$T_{\alpha\beta} = (\rho + p)\, u_\alpha\, u_\beta - p\, g_{\alpha\beta} + \lambda\, (\nabla_\sigma u^\sigma)\, \gamma_{\alpha\beta} + 2\mu\, \varepsilon_{\alpha\beta}\ . \tag{5.4}$$

That expression was utilized by C. Eckart and G. Pichon. A. Lichnerowicz proposed another expression for $\varepsilon_{\alpha\beta}$ in which the vector $C_\alpha = f\, u_\alpha$ entered into the definition of viscosity. The impulse-energy tensor would then be given by:

$$T_{\alpha\beta} = (\rho + p - \lambda\, \nabla_\rho\, C^\rho)\, u_\alpha\, u_\beta - (p - \lambda\, \nabla_\rho\, C^\rho)\, g_{\alpha\beta} + \mu\, \bar{\varepsilon}_{\alpha\beta}\,, \tag{5.5}$$

with

$$2\,\bar{\varepsilon}_{\alpha\beta} = \bar{\nabla}_\alpha C_\beta + \bar{\nabla}_\beta C_\alpha - \bar{C}^\lambda (\bar{\nabla}_\lambda C_\alpha\, C_\beta + \bar{\nabla}_\lambda C_\beta\, C_\alpha)\,.$$

$\bar{\nabla}$ denotes the covariant derivative for the metric $\bar{g} = f^2\, g$.

6. Heat-conducting fluids. – One now takes the exchange of heat by conduction into account. It is defined by a vector q_α that is orthogonal to the vector u^α. It is the expression for q_α and its presence in the impulse-energy tensor that characterizes that viewpoint.

Eckart chose the tensor:

$$T_{\alpha\beta} = (\rho + p)\, u_\alpha\, u_\beta - p\, g_{\alpha\beta} + \theta_{\alpha\beta} - (u_\alpha\, q_\beta + u_\beta\, q_\alpha)\ . \tag{9.1}$$

The equations that govern the evolution of the fluid are given by the conservation conditions on the impulse-energy tensor, the conservation of the matter current, and the defining equation for q_α:

$$\nabla_\alpha T^{\alpha\beta} = 0,$$

$$\nabla_\alpha (r\, u^\alpha) = 0,$$

$$q_\alpha = -\, \kappa(\, g_\alpha^\beta - u_\alpha\, u^\beta)\, (\partial_\beta\, \theta - \theta\, u^\rho\, \nabla_\rho\, u_\beta)\,, \tag{9.2}$$

and a thermodynamic equation.

In 1954, the author proposed the impulse-energy tensor:

$$T_{\alpha\beta} = (\rho + p)\, u_\alpha\, u_\beta - p\, g_{\alpha\beta} - (u_\alpha\, q_\beta + u_\beta\, q_\alpha)\,, \tag{9.3}$$

in which one neglects the viscosity. q_α is defined by:

(9.4) $$q_\alpha = -\kappa(g_\alpha^\beta - u_\alpha u^\beta)\,\partial_\beta\,\theta.$$

The equations of motion consist of conservation conditions $\nabla_\alpha T^{\alpha\beta} = 0$, and the thermodynamic equation is replaced by the conduction equation, which generalizes the Fourier equation:

(9.5) $$\nabla_\alpha q^\alpha = C\,u^\alpha\,\partial_\alpha\,\theta - \frac{l}{\rho}u^\alpha\,\partial_\alpha\,\rho,$$

in which C is the specific heat at constant volume, and l is the heat of dilatation of the fluid. Pichon reprised that model and added the viscosity term $\theta_{\alpha\beta}$.

For Landau and Lifschitz, the impulse-energy tensor of a heat-conducting fluid is identical to that of a perfect fluid, and the heat current vector q_α makes its contribution by way of the conservation equation for a certain vector P_α. They set:

(9.6) $$T_{\alpha\beta} = r\,f\,u_\alpha\,u_\beta - p\,g_{\alpha\beta},$$

(9.7) $$P_\alpha = r\,u_\alpha - q_\alpha.$$

The equations of motion are given by:

$$\nabla_\alpha T^{\alpha\beta} = 0,$$

$$\nabla_\alpha P^\alpha = 0,$$

to which one adds the defining equation of q_α:

(9.8) $$q_\alpha = -\kappa\,\theta^2\,(g_\alpha^\beta - u_\alpha u^\beta)\,\partial_\beta\left(\frac{1+G}{\theta}\right),$$

in which G is the Gibbs function that is defined by:

(9.10) $$G = \varepsilon + \frac{p}{r} - \theta S.$$

Those models are justified by physical and kinetic considerations and have the advantage that they reduce to the classical, non-relativistic description in the limit. The study of the Cauchy problem shows that the systems of equations that give rise to them are mixed and contain a parabolic part that is provided by either the viscosity or the definition of the heat current vector q_α. It leads to an infinite speed of propagation.

In order to eliminate the difficulty that q_α introduces, Cattaneo and Vernotte suggested that one should modify Fourier's hypothesis with a relaxation term. Kranys translated that hypothesis into the language of relativity:

(9.11) $$q^\alpha + \nu\,u^\beta\,\nabla_\beta\,q^\alpha = -\kappa(g^{\alpha\beta} - u^\alpha u^\beta)\,\partial_\beta\,\theta,$$

The vector q_α will no longer be orthogonal to u_α then. Mahjoreb proposed a new theory in that same year while adopting the viewpoint of Landau-Lifschitz and that of Cattaneo-Vernotte-Kranys.

§ 3. THE ELECTROMAGNETIC FIELD

7. **Representation of the electromagnetic field.** – If an electromagnetic field is present then the fluid will be subject to electromagnetic inductions that one can describe with the aid of two 2-forms: The electric field-magnetic induction 2-form H and the electric induction-magnetic field 2-form G. One lets $*H$, $*G$ denote their dual forms in the sense of the Riemannian volume element η of space-time. In components, one has:

$$(*H)_{\alpha\beta} = \tfrac{1}{2}\,\eta_{\alpha\beta\lambda\mu}\, H^{\lambda\mu}, \tag{7.1}$$

$$(*G)_{\alpha\beta} = \tfrac{1}{2}\,\eta_{\alpha\beta\lambda\mu}\, G^{\lambda\mu}. \tag{7.2}$$

One calls the vectors that are defined by the 1-forms:

$$e = i_u H, \qquad d = i_u G, \qquad h = i_u (*G), \qquad b = i_u (*H), \tag{7.3}$$

in which i_u is the interior product by the unit velocity vector u, the *electric and magnetic fields and inductions,* resp. In components, one has:

$$e_\alpha = u^\rho H_{\rho\alpha}, \qquad d_\alpha = u^\rho G_{\rho\alpha}, \qquad h_\alpha = u^\rho (*G)_{\rho\alpha}, \qquad b_\alpha = u^\rho (*H)_{\rho\alpha}. \tag{7.4}$$

Those vectors are orthogonal to u^α.

Conversely, H, G, $*H$, $*G$ are expressed as functions of e, d, h, b by the formulas:

$$H = u \wedge e - *(u \wedge b), \qquad G = u \wedge d - *(u \wedge h), \tag{7.5}$$

$$*H = u \wedge b + *(u \wedge e), \qquad *G = u \wedge h + *(u \wedge d)\,. \tag{7.6}$$

In the last two relations, the + sign comes from the fact that the * will satisfy the relation $*^2 = \varepsilon_g\,(-1)^{p\,(n-p)}$ on a Riemannian manifold, in which $n = \dim V$, p = degree of the form, and ε_g is the sign of $\det g$. One then deduces the following relations, which are given by:

$$H_{\alpha\beta} = u_\alpha\, e_\beta - u_\beta\, e_\alpha - \eta_{\alpha\beta\lambda\mu}\, u^\lambda\, b^\mu,$$

$$G_{\alpha\beta} = u_\alpha\, d_\beta - u_\beta\, d_\alpha - \eta_{\alpha\beta\lambda\mu}\, u^\lambda\, h^\mu$$

in component form.

In Maxwell's theory of electromagnetism, the inductions depend upon the field linearly. In the isotropic case, where the fluid has a dielectric permittivity λ and a magnetic permeability μ, one will have:

(7.7) $$d = \lambda\, e, \qquad b = \mu\, h .$$

The two relations (7.5) then give:

(7.8) $$G = \frac{1}{\mu}H + \frac{\lambda\mu - 1}{\mu}u \wedge i_u H \;;$$

namely, in components:

(7.8′) $$G_{\alpha\beta} = \frac{1}{\mu}H_{\alpha\beta} + \frac{\lambda\mu - 1}{\mu}\,(u_\alpha\, u^\rho\, H_{\rho\beta} - u_\beta\, u^\rho\, H_{\rho\alpha}) \,,$$

which one can put into the form:

(7.9) $$G_{\alpha\beta} = \frac{1}{\mu}\,\varepsilon_{\alpha\beta}{}^{\rho\sigma}\,H_{\rho\sigma}\,,$$

in which:

$$\varepsilon_{\alpha\beta}{}^{\rho\sigma} = \bar{g}_\alpha^{\,\rho}\,\bar{g}_\beta^{\,\sigma} - \bar{g}_\alpha^{\,\sigma}\,\bar{g}_\beta^{\,\rho}\,, \qquad \text{with} \qquad \bar{g}_\mu^{\,\lambda} = g_\mu^{\,\lambda} - (1 - \lambda\mu)\,u^\lambda\,u_\mu\,.$$

The electromagnetic induction *H*, *G* satisfies the Maxwell equations:

(7.10) $$dH = 0$$

(7.11) $$\delta G = J,$$

in which δ is the codifferential, and *J* is a 1-form whose associated vector defines the electric current. The equation (7.10) signifies that the 2-form *H* is locally exact; i.e., that there locally exists a 1-form ϕ such that $H = d\phi$. ϕ is called the *electromagnetic potential vector*, so upon remarking that $\delta^2 = 0$, (7.11) will give:

(7.12) $$\delta J = 0,$$

which is an equation that expresses the conservation of electric current.

In components, equations (7.10), (7.11), (7.12) are written:

$$\tfrac{1}{2}\,\eta^{\alpha\beta\gamma\delta}\,\nabla_\alpha H_{\beta\gamma} = 0,$$

$$\nabla_\alpha G^{\alpha\beta} = J^{\beta},$$

$$\nabla_\alpha J^{\alpha} = 0\,.$$

One decomposes the electric current *J* into a convection current that is collinear to *u* and a conduction current Γ that is orthogonal to *u*. Γ can be defined by Ohm's law $\Gamma = \sigma e$, where σ is the *electric conduction* of the fluid. One will then have:

(7.13) $$J^{\alpha} = \gamma u^\alpha + \sigma e^\alpha,$$

in which γ is called the *charge density*.

8. The electromagnetic energy tensor. – Starting from $H_{\alpha\beta}$, $G_{\alpha\beta}$, one can construct the electromagnetic impulse-energy tensor $\tau_{\alpha\beta}$ whose divergence will give the electromagnetic force density that acts upon the fluid. Upon generalizing a known result from the non-inductive case (namely, $\lambda = \mu = 1$), one will obtain the tensor that Minkowski gave:

(8.1) $$\tau_{\alpha\beta} = g_{\alpha\beta}\,(G_{\rho\sigma}\,H^{\rho\sigma}) - G_{\rho\alpha}\,H^{\rho}{}_{\beta}\,.$$

In order to interpret that tensor, one must express it with the aid of the vectors e, d, h, b. One gets:

(8.2) $$\tau_{\alpha\beta} = (e_\rho\,d^{\rho} + h_\rho\,b^{\rho})\,(u_\alpha\,u_\beta - \tfrac{1}{2}g_{\alpha\beta}) - (e_\alpha\,d_\beta + h_\alpha\,b_\beta) + (P_\alpha\,u_\beta - u_\alpha\,Q_\beta)\,,$$

in which:

(8.3) $$P_\alpha = \eta_{\alpha\lambda\mu\nu}\,e^{\lambda}\,h^{\mu}\,u^{\nu}\,, \qquad Q_\alpha = \eta_{\alpha\lambda\mu\nu}\,d^{\lambda}\,b^{\mu}\,u^{\nu}\,.$$

P_α is the Poynting vector, and $Q_\alpha = \lambda\mu\,P_\alpha$. One can see the significance of each group of terms in (8.2).

$\tau_{\alpha\beta}$ is not symmetric. One can take the expression for it that Abraham proposed:

(8.4) $$\tau_{\alpha\beta} = -\,(e_\rho\,d^{\rho} + h_\rho\,b^{\rho})\,(u_\alpha\,u_\beta - \tfrac{1}{2}g_{\alpha\beta}) - (e_\alpha\,d_\beta + h_\alpha\,b_\beta) + (P_\alpha\,u_\beta - u_\alpha\,Q_\beta)\,.$$

One might think of symmetrizing it, but the physical reasons for doing that are somewhat obscure.

We preserve the expression (8.1). Upon taking the divergence of that tensor, we will have:

(8.5) $$\nabla_\alpha\,\tau^{\alpha}{}_{\beta} = \nabla_\alpha\,G^{\alpha\rho}\,H_{\rho\beta} + G^{\alpha\rho}\,\nabla_\alpha\,H_{\rho\beta} + \tfrac{1}{4}(G^{\rho\sigma}\,\nabla_\beta\,H_{\rho\sigma} + H_{\rho\sigma}\nabla_\beta\,G^{\rho\sigma})\,.$$

Now, the first group of Maxwell equations can be written:

$$\nabla_\alpha\,H_{\rho\beta} + \nabla_\rho\,H_{\beta\alpha} + \nabla_\beta\,H_{\alpha\rho} = 0\,.$$

After contracted multiplication with $G^{\alpha\rho}$, one will get:

$$2\,G^{\alpha\rho}\,\nabla_\alpha\,H_{\rho\beta} = -\,G^{\alpha\rho}\,\nabla_\beta\,H_{\alpha\rho}\,.$$

Upon substituting that in (8.4) and taking the definition of J into account, one will then have:

(8.5′) $$\nabla_\alpha\,\tau^{\alpha}{}_{\beta} = J^{\rho}\,H_{\rho\beta} + \tfrac{1}{4}(G^{\rho\sigma}\,\nabla_\beta\,H_{\rho\sigma} - H_{\rho\sigma}\nabla_\beta\,G^{\rho\sigma}).$$

One can transform the parenthesis by using the constraint equations, which will finally give:

$$\nabla_\alpha \tau^\alpha{}_\beta = J^\rho H_{\rho\beta} + (\lambda\mu - 1)\, \nabla_\beta u^\rho P_\rho + \tfrac{1}{2}(e_\rho e^\rho \partial_\beta \lambda + h_\rho h^\rho \partial_\beta \mu)\,, \tag{8.6}$$

in which:

$$J^\rho H_{\rho\beta} = \gamma e_\beta - \sigma(e_\rho e^\rho)\, u_\beta + \sigma\mu P_\beta\,.$$

The significance of the group $J^\rho H_{\rho\beta}$ is clear. The supplementary tensor $(\lambda\mu - 1)\nabla_\beta u^\rho P_\rho$ will be zero if $\lambda\mu = 1$ or $P_\alpha = 0$; i.e., if $\tau_{\alpha\beta}$ is symmetric. It will also be zero if u^α is a field with a vanishing covariant derivative. The supplementary term $\frac{1}{2}(e_\rho e^\rho \partial_\beta \lambda + h_\rho h^\rho \partial_\beta \mu)$ corresponds to the phenomena of magnetostriction and electrostriction. Indeed, λ, μ depend upon the state variables.

9. Case in which the $\tau_{\alpha\beta}$ defined by (8.1) **is symmetric.** – The tensor $\tau_{\alpha\beta}$ is not symmetric, in general. Its antisymmetric part is $(\lambda\mu - 1)(u_\alpha P_\beta - u_\beta P_\alpha)$. Since u_α and P_α are orthogonal, the antisymmetric part will be zero (i.e., $\tau_{\alpha\beta}$ will be symmetric) if:

1. $\lambda\mu = 1$, which is the non-inductive case.

2. $P_\alpha = 0$, which will be true when either $e_\alpha = 0$ or $h_\alpha = 0$.

In the non-inductive case, one takes $\lambda = \mu = 1$, so $H_{\alpha\beta} = G_{\alpha\beta} = F_{\alpha\beta}$. The electromagnetic energy tensor is then written:

$$\tau_{\alpha\beta} = \tfrac{1}{4} g_{\alpha\beta}(F_{\rho\sigma} F^{\rho\sigma}) - F_{\rho\alpha} F^\rho{}_\beta\,, \tag{9.1}$$

and

$$\nabla_\alpha \tau^\alpha{}_\beta = J^\rho H_{\rho\beta}. \tag{9.2}$$

The case $e_\alpha = 0$ corresponds to that of magnetohydrodynamics or fluids whose conductivity is $\sigma = \infty$. Since the electric current must be bounded ($\sigma e < \infty$), one will necessarily have $e = 0$. The impulse-energy tensor will then be written:

$$\tau_{\alpha\beta} = \mu\,\{\,|\,h\,|^2 (u_\alpha u_\beta - \tfrac{1}{2} g_{\alpha\beta}) - h_\alpha h_\beta\}\,. \tag{9.3}$$

The case $h_\alpha = 0$ will lead to:

$$\tau_{\alpha\beta} = \lambda\,\{\,|\,e\,|^2 (u_\alpha u_\beta - \tfrac{1}{2} g_{\alpha\beta}) - e_\alpha e_\beta\}\,. \tag{9.4}$$

One agrees to describe the electron in question as a continuous ball.

In each of the cases above, the impulse-energy tensor of the fluid was obtained by adding a known expression to $\tau_{\alpha\beta}$. One will get an impulse-energy tensor that is symmetric, and as a result one can write down the Einstein equations.

10. The charged perfect fluid with no inductions. – In this case, one will have the total impulse-energy tensor:

(10.3) $$T_{\alpha\beta} = (\rho + p)\, u_\alpha\, u_\beta - p\, g_{\alpha\beta} + \tau_{\alpha\beta}\,,$$

in which:

$$\tau_{\alpha\beta} = \tfrac{1}{4} g_{\alpha\beta}\,(F_{\rho\sigma}\, F^{\rho\sigma}) - F_{\rho\alpha}\, F^{\rho}{}_{\beta}\,.$$

The equations of motion are given by the conservation conditions:

(10.2) $$\nabla_\alpha T^{\alpha\beta} = 0,$$

the thermodynamic equation:

(10.3) $$r\ \theta\, dS = r\, df - dp\,,$$

and the Maxwell equations:

(10.4) $$\tfrac{1}{2}\, \eta^{\alpha\beta\gamma\delta}\, \nabla_\alpha F_{\beta\gamma} = 0\,,$$

(10.5) $$\nabla_\alpha F^{\alpha\beta} = J^{\beta}\,.$$

One supposes that the conductivity $\sigma = 0$, in such a way that $J^{\beta} = \gamma u^{\beta}$, and:

(10.6) $$\nabla_\alpha (\gamma u^\alpha) = 0\,.$$

Upon taking f, S to be thermodynamic variables, the conservation conditions (10.2) will give the continuity equation and the differential system for the streamlines:

(10.7) $$r\, f\, u^\alpha\, \nabla_\alpha u^\beta - (g^{\alpha\beta} - u^\alpha\, u^\beta)\, \partial_\alpha p - \gamma u^\alpha\, F_\alpha{}^{\beta} = 0\,,$$

(10.8) $$f\, \nabla_\alpha (r\, u^\alpha) + r\ \theta\, u^\alpha\, \partial_\alpha S = 0\,.$$

One deduces from this that if the motion is adiabatic ($u^\alpha\, \partial_\alpha S = 0$) then there will be conservation of matter:

(10.9) $$\nabla_\alpha (r\, u^\alpha) = 0\,.$$

One infers from (10.6) and (10.9) that:

$$u^\alpha \frac{\partial_\alpha \gamma}{\gamma} + \nabla_\alpha u^\alpha = 0, \qquad u^\alpha \frac{\partial_\alpha r}{r} + \nabla_\alpha u^\alpha = 0,$$

so taking the difference of these will yield $u^\alpha \partial_\alpha \log(\gamma / r) = 0$. The ratio γ / r is then constant along the streamlines, so one sets:

(10.10) $$K = \frac{\gamma}{r}.$$

The differential system for the streamlines can then be put into the form:

(10.11) $$f\, u^\alpha \nabla_\alpha u^\beta - (g^{\alpha\beta} - u^\alpha u^\beta)\, \partial_\alpha f + \theta g^{\alpha\beta} \partial_\alpha S = K\, u^\alpha F_\alpha{}^\beta .$$

CHAPTER II

THE CAUCHY PROBLEM

For each fluid model, one obtains a fundamental system of partial differential equations that allow one to study the evolution of the fluid. An essential problem is to see the extent to which those equations determine the functions that represent the physical quantities that are being envisioned. Since that is a problem of evolution, the mathematical problem that is posed in that way is the Cauchy problem. The initial data that are carried by a hypersurface Σ in space-time determine those quantities in the neighborhood of Σ.

The only classically-known general theorem that answers that question in the analytic case is the Cauchy-Kowalewski existence and uniqueness theorem for a system of N partial differential equations in N unknown functions whose characteristic polynomial is not identically zero.

The analyticity hypothesis restricts the scope of that theorem in physics considerably. Now, one can do without the analyticity hypothesis for strictly-hyperbolic quasi-linear systems. Leray proved an existence and uniqueness theorem for the non-analytic Cauchy problem for such systems. Any solution of that problem possesses a domain of influence; i.e., the value at a point depends upon only part of the initial data, namely, the data that are found inside of a certain conoid with its summit at that point. It is that notion of strict hyperbolicity and its criterion that we shall present, with an eye towards applying it to the various systems of equations that were found in Chapter I.

§ 1. EXISTENCE AND UNIQUENESS THEOREM FOR STRICTLY-HYPERBOLIC SYSTEMS

1. Strictly-hyperbolic systems. – Let V_n be a differentiable manifold of class C^k (k sufficiently large) and dimension n.

Let a (x, D) be a differential operator of order m that acts on functions. Locally, it will depend upon local coordinates x^α and their partial derivatives ∂_α. For $\xi \in T_x^*(V_n)$, $a(x,\xi)$ is a real polynomial in ξ of degree m. One lets h (x, ξ) denote the principal part of a (x, ξ) ; i.e., the homogeneous part of degree m of a (x, ξ). Let V_x (h) be the projective cone that is defined in $T_x^*(V_n)$ by the equation h $(x, \xi) = 0$.

Definition 1. – The differential operator a (x, D) is called *strictly hyperbolic* at the point $x \in V_n$ if the following hypothesis is verified:

(H): There exist points ξ in $T_x^*(V_n)$ such that any line that issues from ξ and does not pass through the summit of the cone V_x (h) will cut it at m distinct real points.

It that were true then the set of points ξ would form the interior of two opposing non-vacuous convex semi-cones $\Gamma_x^+(a)$ and $\Gamma_x^-(a)$ whose boundaries belong to $V_x(h)$.

Now consider a diagonal matrix differential operator $A(x, D)$ that is sufficiently differentiable at x :

$$A(x, D) = \begin{pmatrix} a_1(x,D) & 0 & \cdots & 0 \\ 0 & a_1(x,D) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_N(x,D) \end{pmatrix},$$

in which the $a_i(x, D)$ are differential operators of order $m(i)$.

Definition 2. – One says that the diagonal differential operator $A(x, D)$ is *strictly hyperbolic* at a point x if:

1) The $a_i(x, D)$ are strictly hyperbolic at x .

2) The two opposing convex semi-cones:

$$\Gamma_x^+(A) = \bigcap_i \Gamma_x^+(a_i), \qquad \Gamma_x^-(A) = \bigcap_i \Gamma_x^-(a_i)$$

have a non-vacuous interior.

In order to define strict hyperbolicity in a (connected open) domain Ω of V_n , introduce the cone $C_x^+(A)$ that is dual to the cone $\Gamma_x^+(A)$. $C_x^+(A)$ is the closure of the set of vectors $X \in T_x(V_n)$ such that $<\xi, X> \geq 0$ for any $\xi \in \Gamma_x^+(A)$. The cone $C_x^-(A)$ that is dual to the cone $\Gamma_x^-(A)$ is defined in an analogous manner. Let:

$$C_x(A) = C_x^+(A) \cup C_x^-(A) \ . \tag{1.3}$$

A differentiable path $\gamma: [0, 1] \to V_n$ is called *time-like relative to A* if the positive semi-tangent at each point of γ is in $C_x^+(A)$. A differentiable hypersurface Σ is called *space-like relative to A* if the tangent vector space $T_x(\Sigma)$ at each point x of Σ is exterior to $C(A)$.

Definition 3. – One says that the operator $A(x, D)$ is *strictly hyperbolic in a domain* $\Omega \subset V_n$ if the following two conditions are satisfied:

1) $A(x, D)$ is strictly hyperbolic at any point $x \in \Omega$.

2) The set of temporal paths that join two arbitrary points x_0 , x_1 of Ω is compact or vacuous for the topology of uniform convergence on the set $\{\gamma: [0, 1] \to V_n ,\ \gamma(0) = x_0 ,\ \gamma(1) = x_1\}$.

If A (x, D) is differentiable at x and A (x, D) is strictly hyperbolic at a point x_0 then one can show that there exists a connected open neighborhood Ω of x_0 that is homeomorphic to a ball in R^n in which A (x, D) is strictly hyperbolic. Such an open set will be called *simple.*

2. Leray systems. – Consider a system of partial differential equations with N equations in N unknowns (u^i) and n variables (x^α) that one writes symbolically:

$$A\,(x, u, D)\, u + B\,(x, u) = 0\,, \tag{2.1}$$

in which A (x, u, D) is a diagonal matrix with elements a_i (x, u, D), $i = 1, \ldots, N$, and $B(x,u)$ is a column matrix with elements b_i (x, u) . The a_i (x, u, D) are differential operators of order m (i).

Associate each unknown u^i with an integer s $(i) \geq 1$ and each equation of rank j with an integer t $(j) \geq 1$ such that:

$$m\,(i) = s\,(i) - t\,(i) + 1, \tag{2.2}$$

in which the integers s (i), t (j) are defined only up to an additive constant.

Definition. – One says that the diagonal system (2.1) is *quasi-linear in the Leray sense* if for any i, the differential operator a_i (x, u, D) is linear with respect to the derivatives of order m (i), the relations (2.2) are verified, and the a_i , b_i are sufficiently regular functions of x^α, u^j, and the derivatives of order $\leq s\,(i) - t\,(j)$, and if $s\,(i) - t\,(j) < 0$ then a_i and b_i are independent of u_j .

That being the case, the Cauchy problem for the system (2.1) is posed in the following manner: Let Ω be a simple domain of V_n . The Cauchy data on a hypersurface Σ that is embedded in Ω consist of the values of the functions u^i and their derivatives of order $< m\,(i)$. There (always) exist functions w^i that admit derivatives of order $\leq s\,(i) +$ 1 that are locally square-integrable, and their traces on Σ are the Cauchy data that one has in mind.

A solution of the Cauchy problem that was posed is then a solution (u^i) of (2.1) whose derivatives of order $\leq s\,(i)$ are locally square-integrable and coincide with those of w^i on Σ. Leray proved an existence and uniqueness proof for that problem that we shall state without proof.

Theorem:

If the Cauchy data on Σ are defined by functions w^i that verify the hypotheses:

1) *The differential operator A (x, w, D) is strictly hyperbolic in Ω and the hypersurface Σ is space-like relative to A (x, u, D) .*

2) *The* a_i (x, w, D) $w + b_i$ (x, w) *are annulled on* Σ, *along with their derivatives up to order* t $(i) - 1$.

For any $x \in \Sigma$, *the Cauchy problem for* (2.1) *admits at least one solution in a neighborhood of* x . *If* $(\bar{u}^i)$ *and* (u^i) *are two solutions, and if* $\bar{u}^i$ *and* u^i *have derivatives of order* $\leq s$ $(i) + 1$ *that are locally square-integrable then they will coincide.*

Definition. – A quasi-linear system that verifies the preceding hypotheses will be called a *strictly-hyperbolic quasi-linear system* or a *Leray system.*

The quasi-linear systems that one encounters in physics are not always diagonal systems. In order to apply Leray's theorem, one must convert them to diagonal form. One knows that one can always do that. However, the strictly-hyperbolic character must be proved. We shall present the method in one case.

§ 2. APPLICATION TO THE EQUATIONS OF THE HYDRODYNAMICS OF PERFECT FLUIDS

3. Harmonic coordinates. – Harmonic coordinates have been an invaluable tool in the study of the Cauchy problem that relates to the Einstein equations.

Definition. – A local coordinate system (x^ρ) is called *harmonic* if each coordinate function x^ρ is a solution of the Laplace equation:

$$\Delta f = - g^{\alpha\beta} (\partial_{\alpha\beta} f - \Gamma^\gamma_{\alpha\beta} \partial_\gamma f) = 0, \tag{3.1}$$

in which $\Gamma^\gamma_{\alpha\beta}$ are the coefficients of the Riemannian connection.

One remarks that the characteristic manifolds of (3.1) are tangent to the elementary space-time cone C_x at each point.

If (x^ρ) is a local coordinate system then one sets:

$$F^\rho = \Delta x^\rho = g^{\alpha\beta} \Gamma^\rho_{\alpha\beta} , \tag{3.2}$$

in which the F^ρ depend upon $g_{\alpha\beta}$ and their first derivatives. If $F^\rho = 0$ then the local coordinate system will be harmonic.

One associates the F^ρ with the quantities $L_{\alpha\beta}$ that are defined by:

$$L_{\alpha\beta} = \tfrac{1}{2} (g_{\alpha\rho}\ \partial_\beta F^\rho + g_{\beta\rho}\ \partial_\alpha F^\rho) . \tag{3.3}$$

Lemma:

In an arbitrary local coordinate system, the components of the Ricci tensor can be put into the form:

$$R_{\alpha\beta} = R_{\alpha\beta}{}^{(h)} + L_{\alpha\beta}\,, \tag{3.4}$$

in which:

$$R_{\alpha\beta}{}^{(h)} = -\tfrac{1}{2} g^{\lambda\mu}\,\partial_{\lambda\mu}\, g_{\alpha\beta} + F_{\beta\rho}\,(g_{\lambda\mu}\,,\, \partial_{0'}\, g_{\lambda\mu})\,, \tag{3.5}$$

where the $F_{\alpha\beta}$ are regular functions.

In order to prove the lemma, it will suffice to exhibit the second-order derivatives of $g_{\lambda\mu}$ in $L_{\alpha\beta}$ and $R_{\alpha\beta}$. One has (modulo terms in $g_{\lambda\mu}$ and $\partial_{0'}\, g_{\lambda\mu}$):

$$L_{\alpha\beta} \approx \tfrac{1}{2}\left\{\partial_\beta(g_{\alpha\rho}\, g^{\lambda\mu}\Gamma^{\rho}_{\lambda\mu}) + \partial_\alpha(g_{\beta\rho}\, g^{\lambda\mu}\Gamma^{\rho}_{\lambda\mu})\right\}$$

$$\approx \tfrac{1}{2} g^{\lambda\mu}(\partial_\beta[\lambda\mu,\alpha] + \partial_\alpha[\lambda\mu,\beta])\,,$$

namely:

$$L_{\alpha\beta} \approx \tfrac{1}{2} g^{\lambda\mu}(\partial_{\alpha\lambda}\, g_{\beta\mu} + \partial_{\beta\lambda}\, g_{\alpha\mu} - \partial_{\alpha\beta}\, g_{\lambda\mu})\,.$$

On the other hand:

$$R_{\alpha\beta} \approx \partial_\lambda \Gamma^{\lambda}_{\alpha\beta} - \partial_\alpha \Gamma^{\lambda}_{\lambda\beta}$$

$$\approx g^{\lambda\mu}(\partial_\lambda[\alpha\beta,\mu] - \partial_\alpha[\lambda\beta,\mu])\,,$$

namely:

$$R_{\alpha\beta} \approx \tfrac{1}{2} g^{\lambda\mu}(\partial_{\alpha\lambda}\, g_{\beta\mu} + \partial_{\beta\lambda}\, g_{\alpha\mu} - \partial_{\alpha\beta}\, g_{\lambda\mu} - \partial_{\lambda\mu}\, g_{\alpha\beta})\ .$$

One then deduces that:

$$R_{\alpha\beta} \approx -\tfrac{1}{2} g^{\lambda\mu}\partial_{\alpha\beta}\, g_{\lambda\mu} + L_{\alpha\beta}\,,$$

and one therefore has the lemma.

Corollary:

In arbitrary local coordinates, one will have:

$$S_{\alpha\beta} = S^{(h)}_{\alpha\beta} + K_{\alpha\beta}\,, \tag{3.6}$$

in which:

(3.7) $$S^{(h)}_{\alpha\beta} = R^{(h)}_{\alpha\beta} - \tfrac{1}{2} R^{(h)} g_{\alpha\beta} ,$$

(3.8) $$K_{\alpha\beta} = L_{\alpha\beta} - \tfrac{1}{2} L\, g_{\alpha\beta} ,$$

with $R^{(h)} = g^{\lambda\mu} R^{(h)}_{\lambda\mu}$ *and* $L = g^{\lambda\mu} L_{\lambda\mu}$. *If the coordinates are harmonic then* $K_{\alpha\beta} = 0$.

4. Application to the study of the solutions of the Einstein equations. –

Theorem:

Any solution of the system of Einstein equations:

(4.1) $$S_{\alpha\beta} = \chi\, T_{\alpha\beta}$$

in harmonic coordinates is a solution of the system:

(4.2) $$S^{(h)}_{\alpha\beta} = \chi\, T_{\alpha\beta} ,$$

(4.3) $$\nabla_\alpha T^{\alpha}{}_{\beta} = 0 .$$

Conversely, any solution of the system (4.2), (4.3) that satisfies the following conditions on a space-like hypersurface Σ :

(4.4) $$F^{\rho} = 0,$$

(4.5) $$S^0_\alpha = \chi T^0_\alpha$$

will be a solution to the corresponding Cauchy problem for the system (4.1) of Einstein equations.

Indeed, any solution of (4.2) will verify (4.2) when it is written in harmonic coordinates, because $L_{\alpha\beta} = 0$, and as a solution of (4.1), it must verify the conservation conditions $\nabla_\alpha S^{\alpha}{}_{\beta} = 0$; i.e., (4.3).

Conversely, consider a solution of (4.2), (4.3) that corresponds to Cauchy data on Σ that satisfy the conditions (4.4) and (4.5) on Σ. One will have $S^0_\alpha = S^{(h)0}_\alpha + K^0_\alpha$ on Σ, and by virtue of (4.2) and (4.5), one will have:

$$K^0_\alpha = 0 \qquad \text{on } \Sigma .$$

Upon specifying the expression for K^0_α, one will have:

$$K^0_\alpha = \tfrac{1}{2} g_{\alpha\rho}\, g^{0\beta} \partial_\beta F^\rho + \tfrac{1}{2} \partial_\beta F^0 - \tfrac{1}{2} g^0_\alpha\, \partial_\rho F^\rho ,$$

namely, upon taking the condition (4.4) into account, which implies that $\partial_i F^\rho = 0$ on Σ :

$$K^0_\alpha = \tfrac{1}{2} g_{\alpha\rho}\, g^{00} \partial_0 F^\rho = 0.$$

Since Σ is space-like, $g^{00} \neq 0$, and one will necessarily have $\partial_0 F^\rho = 0$.

Hence, the solution of (4.2), (4.3) that one considers will satisfy:

$$\partial_0 F^\rho = 0 \tag{4.6}$$

on Σ. For that solution, one will further have:

$$\nabla_\lambda K^{\lambda\mu} = \nabla_\lambda S^{\lambda\mu} - \nabla_\lambda S^{(h)\lambda\mu}$$

$$= -\chi \nabla_\lambda T^{\lambda\mu} .$$

One deduces by virtue of (4.3) that:

$$\nabla_\lambda K^{\lambda\mu} = 0 . \tag{4.7}$$

Now, differentiation of:

$$K^{\lambda\mu} = L^{\lambda\mu} - \tfrac{1}{2} L\, g^{\lambda\mu} = \tfrac{1}{2}(g^{\lambda\rho}\,\partial_\rho F^\mu + g^{\mu\rho}\,\partial_\rho F^\lambda) - \tfrac{1}{2} g^{\lambda\mu}\,\partial_\rho F^\rho$$

gives:

$$\partial_\lambda K^{\lambda\mu} = \tfrac{1}{2}(g^{\lambda\rho}\,\partial_{\lambda\rho} F^\mu + g^{\mu\rho}\,\partial_{\lambda\rho} F^\lambda - g^{\lambda\mu}\,\partial_{\lambda\rho} F^\rho) + \text{terms linear in } \partial_\rho F^\lambda .$$

It will then result that equation (4.7) can be written:

$$g^{\lambda\rho}\,\partial_{\lambda\rho} F^\mu + A^{\lambda\mu}{}_\rho\,\partial_\lambda F^\rho = 0, \tag{4.8}$$

in which $A^{\lambda\mu}{}_\rho$ are regular functions.

Hence, F^ρ satisfies a hyperbolic linear system that admits an existence and uniqueness theorem. If Σ is space-like then the only solution of (4.8) that satisfies $F^\rho = 0$ and $\partial_0 F^\rho = 0$ on Σ will be the zero solution. It will then result that the solution considered is a solution of the Einstein equations (4.1) when they are written in harmonic coordinates.

5. Formal analysis of the fundamental system of hydrodynamics. – The fundamental system of equations for the hydrodynamics of perfect, adiabatic fluids is composed of the equations:

$$S_{\alpha\beta} = \chi\,(r\, f\, u_\alpha\, u_\beta - p\, g_{\alpha\beta}), \tag{5.1}$$

$$u^\alpha\,\partial_\alpha S = 0, \tag{5.2}$$

(5.3) $$g_{\alpha\beta}\, u^{\alpha}\, u^{\beta} = +\,1,$$

(5.4) $$dp = r\,df - r\,\theta\,dS\,.$$

One takes f and S to be the thermodynamic variables, so $r = r\,(f, S)$ and $p = p\,(f, S)$ will then be known functions of f and S.

One has a system of 16 partial differential equations in 16 unknown functions $g_{\alpha\beta}$, f, S, u^{α}. One must first make a formal analysis of the Cauchy problem. In order to do that, one gives the values of $g_{\alpha\beta}$, $\partial_0\, g_{\alpha\beta}$, S on a hypersurface Σ with local equation $x^0 = 0$ and seeks to determine the solution in the neighborhood of Σ. Suppose that Σ is not tangent to the elementary cones; i.e.:

(5.5) $$g^{00} \neq 0,$$

and that one has:

(5.6) $$F^{\rho} = 0$$

on Σ.

A classical study shows that if $g^{00} \neq 0$ then the quantities $S^{\;0}_{\alpha}$ will be known as functions of the Cauchy data $g_{\alpha\beta}$, $\partial_0\, g_{\alpha\beta}$. The Cauchy data ($g_{\alpha\beta}$, $\partial_0\, g_{\alpha\beta}$, S) must then verify the compatibility conditions:

$$S^{\;0}_{\alpha} = \chi\,(r\,f\,u^0\,u_{\alpha} - p\,g^{0}_{\;\alpha})\,.$$

Suppose, for the moment, that the values of f on Σ are known. The preceding equations can be written:

(5.7) $$\chi\,r\,f\,u^0\,u_{\alpha} = S^{0}_{\alpha} + \chi\,p\,g^{0}_{\;\alpha}\,.$$

Upon taking into account the fact (5.3) that u^{α} is a unit vector, one will infer that:

$$(\chi\,r\,f\,u^0)^2 = [\Omega^0(p)]^2 = g^{\alpha\beta}\,(S^{0}_{\alpha} + \chi\,p\,g^{0}_{\;\alpha})(S^{0}_{\beta} + \chi\,p\,g^{0}_{\;\beta})\,.$$

(5.7) will then give the value of u^0 :

(5.8) $$u^0 = \frac{S^{00} + \chi p g^{00}}{\Omega^0(p)}$$

and then that of:

(5.9) $$\chi\,r\,f = \frac{[\Omega^0(p)]^2}{S^{00} + \chi p g^{00}}\,.$$

Equation (5.9) implicitly defines the possible value(s) of f. One can then write:

(5.9′) $$F(f) \equiv \chi r f (S^{00} + \chi p g^{00}) - g^{\alpha\beta} (S^0_\alpha + \chi p g^0_\alpha)(S^0_\beta + \chi p g^0_\beta) .$$

Upon differentiating with respect to f and taking (5.4) into account, one will get:

$$F'_f = \chi(r + r'_f f)(S^{00} + \chi p g^{00}) + \chi^2 r^2 f g^{00} - 2g^{0\alpha} r (S^0_\beta + \chi p g^0_\beta) ,$$

namely, from (5.7):

$$F'_f = \chi^2 r^2 f \left[g^{00} - \left(1 - \frac{f r'_f}{r}\right) u^0 u^0 \right].$$

One sees that f will be known on Σ if $F'_f \neq 0$; i.e.:

$$g^{00} - \left(1 - \frac{f r'_f}{r}\right) u^0 u^0 \neq 0 .$$

Once f is known on Σ, one can deduce u^α with the aid of (5.7):

$$u^\alpha = \frac{S + \chi p g^{0\alpha}}{\Omega^0(p)}$$

if

$$u^0 \neq 0 .$$

Since that is true, by virtue of the arguments that were made in the preceding paragraph **4**, one can replace the system (5.1), (5.2), (5.3), (5.4) with the system:

(5.10) $$R^{(h)}_{\alpha\beta} = \chi [r f u_\alpha u_\beta - (r f - 2p) g_{\alpha\beta}],$$

(5.11) $$u^\alpha \partial_\alpha S = 0,$$

(5.12) $$\nabla_\alpha (r u^\alpha) = 0,$$

(5.13) $$f r u^\alpha \nabla_\alpha u^\beta - (g^{\alpha\beta} - u^\alpha u^\beta) (\partial_\alpha f - \theta \partial_\alpha S) = 0 ,$$

in which (5.12), (5.13) come from the conservation conditions $\nabla_\alpha T^{\alpha\beta} = 0$ and (5.3). One remarks that (5.13) implies that:

$$u^\alpha \nabla_\alpha u^\beta u_\beta = 0,$$

which shows that if u^α is a unit vector on Σ then it will remain in the neighborhood of Σ.

Suppose that the Cauchy data are given in terms of formal series in the local coordinates and look for formal solutions of the system (5.10), (5.11), (5.12), (5.13).

Upon exhibiting the derivatives $\partial_{00}\, g_{\alpha\beta}$, $\partial_0\, S$, $\partial_0\, f$, $\partial_0\, u^\alpha$ in these equations, one will get:

(5.14) $$-\tfrac{1}{2}\, g^{00}\, \partial_{00}\, g_{\alpha\beta} = (C.\ d.)$$

(5.15) $$u^0\, \partial_0\, S = (C.\ d.),$$

(5.16) $$r\, \partial_0\, u^0 + r'_f\, u^0\, \partial_0 f + r'_s u^0\, \partial_0 f = (C.\ d.),$$

(5.17) $$f\, u^0\, \partial_0\, u^\beta - (g^{0\beta} - u^0\, u^\beta)\, \partial_0 f + g^{0\beta}\, \partial_0 S = (C.\ d.),$$

in which the right-hand sides are known functions of the Cauchy data. (5.14) then gives $\partial_{00}\, g_{\alpha\beta}$ if $g^{00} \neq 0$. If $u^0 \neq 0$ then (5.15) will give $\partial_0\, S$, and (5.16) and (5.17) will then determine $\partial_0\, u^0$, $\partial_0 f$ for $\beta = 0$ if the determinant $g^{00} - \left(1 - \frac{f\, r'_f}{r}\right) u^0\, u^0 \neq 0$. Finally, (5.17) will give $\partial_0\, u^0$ for $\beta = 1$ if $u^0 \neq 0$.

Hence, under the condition that:

$$g^{00} \neq 0, \qquad u^0 \neq 0, \qquad g^{00} - \left(1 - \frac{f\, r'_f}{r}\right) u^0\, u^0 \neq 0,$$

one can calculate $\partial_{00}\, g_{\alpha\beta}$, $\partial_0\, S$, $\partial_0 f$, and $\partial_0\, u^\alpha$. The same conclusions extend to the higher-order derivatives that one obtains by differentiating the various equations with respect to x^0, in such a way that the desired formal series will be defined uniquely.

If the Cauchy problem is analytic then the Cauchy-Kowalewski theorem will imply the following result:

Theorem:

In the analytic case, if the Cauchy data $g_{\alpha\beta}$, $\partial_0\, g_{\alpha\beta}$, S satisfy the conditions:

1) *The form $g_{\alpha\beta}\, X^\alpha\, X^\beta$ is normal hyperbolic,*

2) *The hypersurface Σ that carries the Cauchy data is defined locally by $x^0 = 0$ and is space-like,*

3) *One has $F^\rho = 0$ and $S_\alpha^{\ 0} = \chi T_\alpha^{\ 0}$ on Σ,*

then the Cauchy problem for the system of hydrodynamical equations will admit one and only one solution in a neighborhood of any point $x \in \Sigma$.

Characteristic manifolds. – The preceding study shows that the characteristic manifolds of the Cauchy problem are defined by the following equations:

$$g^{\alpha\beta}\,\partial_\alpha\phi\,\partial_\beta\phi = 0,$$

$$u^\alpha\,\partial_\alpha\phi = 0,$$

$$\left[g^{\alpha\beta} - \left(1 - \frac{f\,r'_f}{r}\right)u^\alpha u^\beta\right]\partial_\alpha\phi\,\partial_\beta\phi = 0\,.$$

The first one defines gravitational waves, the second one defines matter or entropy waves, and the third one defines hydrodynamic waves. A classical calculation will show that these three types of waves propagate with the velocities 1, 0, v, respectively, with:

$$v = \sqrt{\frac{r}{f\,r'_f}}\,.$$

Relativity demands that one must have $(f\,r'_f)/r \geq 1$ in order to have $v \leq 1$, which demands that the characteristic manifolds that define the hydrodynamical waves must be time-like.

6. Existence and uniqueness theorem. – Recall the system that was studied in the preceding paragraph:

(6.1) $$R^{(h)}_{\lambda\mu} = \chi\left[r\,f\,u_\lambda u_\mu - \tfrac{1}{2}(r\,f - 2p)u_\lambda u_\mu\right],$$

(6.2) $$u^\alpha\,\partial_\alpha S = 0,$$

(6.3) $$\nabla_\alpha\,(r\,u^\alpha) = 0,$$

(6.4) $$f\,u^\alpha\nabla_\alpha u^\beta - (g^{\alpha\beta} - u^\alpha u^\alpha)\,\partial_\alpha f - \theta\,g^{\alpha\beta}\,\partial_\alpha S = 0\,.$$

It does not present itself in a diagonal form, except for (6.1) and (6.2). We shall transform it in such a way that we obtain a diagonal system.

For the moment, keep (6.1) and (6.2) as they are.

In order to get an equation in f, take the contracted derivative ∇_β of (6.4) and get:

$$(g^{\alpha\beta} - u^\alpha u^\beta)\,\nabla_\alpha\nabla_\beta f - \theta\,g^{\alpha\beta}\,\nabla_\alpha\nabla_\beta S - f\,u^\alpha\nabla_\beta\,\nabla_\alpha u^\beta$$

$$= h\,(1 \text{ in } g_{\alpha\beta},\ 1 \text{ in } S,\ 1 \text{ in } f,\ 1 \text{ in } u^\alpha),$$

in which the notation on the right-hand side signifies that the terms that are not specified contain derivatives whose maximum order is indicated for each unknown. In order to get $u^\alpha\nabla_\beta\,\nabla_\alpha u^\beta$, consider equation (6.3), which develops into:

$$\nabla_\alpha u^\beta + \frac{r_f'}{r} u^\alpha \nabla_\beta f = 0 .$$

Upon differentiating it along the streamlines and using the Ricci identity:

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\, u^\beta = - R_{\alpha\beta}\, u^\beta,$$

one will have:

$$u^\alpha \nabla_\beta \nabla_\alpha u^\beta + \frac{r_f'}{r}\, u^\alpha u^\beta \nabla_\alpha \nabla_\beta f = k\,(2 \text{ in } g_{\alpha\beta},\ 1 \text{ in } S,\ 1 \text{ in } f,\ 1 \text{ in } u^\alpha) .$$

Upon substituting that into the equation for f, one will get:

(6.5)
$$\left[g^{\alpha\beta} - \left(1 - \frac{f\, r_f'}{r}\right) u^\alpha u^\beta \right] \nabla_\alpha \nabla_\beta f - \theta\, g^{\alpha\beta} \nabla_\alpha \nabla_\beta S$$

$$= l\,(2 \text{ in } g_{\alpha\beta},\ 1 \text{ in } S,\ 1 \text{ in } f,\ 1 \text{ in } u^\alpha) .$$

Apply the operator $u^\gamma \nabla_\gamma$ to that equation:

$$\left[g^{\alpha\beta} - \left(1 - \frac{f\, r_f'}{r}\right) u^\alpha u^\beta \right] u^\gamma \nabla_\gamma \nabla_\alpha \nabla_\beta f - \theta\, g^{\alpha\beta} u^\gamma \nabla_\gamma \nabla_\alpha \nabla_\beta S$$

$$= m\,(3 \text{ in } g_{\alpha\beta},\ 2 \text{ in } S,\ 2 \text{ in } f,\ 2 \text{ in } u^\alpha) .$$

Now, the Ricci identity gives:

(6.6)
$$g^{\alpha\beta} u^\gamma \nabla_\gamma \nabla_\alpha \nabla_\beta S = g^{\alpha\beta} u^\gamma \nabla_\alpha \nabla_\beta \nabla_\gamma S - u^\gamma R_\gamma{}^\rho \nabla_\rho S ,$$

so if one takes into account the fact that $u^\alpha \partial_\alpha S = 0$ then one will have:

(6.6′)
$$g^{\alpha\beta} u^\gamma \nabla_\gamma \nabla_\alpha \nabla_\beta S = n\,(2 \text{ in } g_{\alpha\beta},\ 2 \text{ in } S,\ 0 \text{ in } f,\ 2 \text{ in } u^\alpha) ,$$

and the equation in f will ultimately be written as:

(6.7)
$$\left[g^{\alpha\beta} - \left(1 - \frac{f\, r_f'}{r}\right) u^\alpha u^\beta \right] u^\gamma \partial_{\alpha\beta\gamma} f = F\,(3 \text{ in } g_{\alpha\beta},\ 2 \text{ in } S,\ 2 \text{ in } f,\ 2 \text{ in } u^\alpha) .$$

As for the unknowns u^β, consider the system (6.4), to which we apply the operator $\left[g^{\alpha\beta} - \left(1 - \frac{f\, r_f'}{r}\right) u^\alpha u^\beta \right] \nabla_\alpha \nabla_\beta$:

$$\left[g^{\alpha\beta}-\left(1-\frac{f\,r_f'}{r}\right)u^\alpha u^\beta\right]u^\gamma\nabla_\alpha\nabla_\beta\nabla_\gamma u^\gamma-\left[g^{\alpha\beta}-\left(1-\frac{f\,r_f'}{r}\right)u^\alpha u^\beta\right](g^{\gamma\lambda}-u^\gamma u^\lambda)\,\nabla_\alpha\nabla_\beta\nabla_\gamma f$$

$$+\,\theta g^{\alpha\beta}g^{\gamma\lambda}\,\nabla_\alpha\nabla_\beta\nabla_\gamma S = p\,(3\text{ in }g_{\alpha\beta},\,2\text{ in }S,\,2\text{ in }f,\,2\text{ in }u^\alpha)\,.$$

Apply the operator $(g^{\gamma\lambda}-u^\gamma u^\lambda)\,\nabla_\gamma$ to (6.5):

$$\left[g^{\alpha\beta}-\left(1-\frac{f\,r_f'}{r}\right)u^\alpha u^\beta\right](g^{\gamma\lambda}-u^\gamma u^\lambda)\,\nabla_\alpha\nabla_\beta\nabla_\gamma f-\theta g^{\alpha\beta}(g^{\gamma\lambda}-u^\gamma u^\lambda)\,\nabla_\alpha\nabla_\beta\nabla_\gamma S$$

$$= q\,(3\text{ in }g_{\alpha\beta},\,2\text{ in }S,\,2\text{ in }f,\,2\text{ in }u^\alpha)\,.$$

Upon taking that relation into account, the equation in u^λ will become:

$$\left[g^{\alpha\beta}-\left(1-\frac{f\,r_f'}{r}\right)u^\alpha u^\beta\right]u^\gamma\nabla_\alpha\nabla_\beta\nabla_\gamma u^\lambda-\theta g^{\alpha\beta}(g^{\gamma\lambda}-u^\gamma u^\lambda)\,\nabla_\alpha\nabla_\beta\nabla_\gamma S+\theta g^{\alpha\beta}g^{\gamma\lambda}\nabla_\alpha\nabla_\beta\nabla_\gamma S$$

$$= r\,(3\text{ in }g_{\alpha\beta},\,2\text{ in }S,\,2\text{ in }f,\,2\text{ in }u^\alpha)\,.$$

Now, the fact that $u^\gamma\partial_\gamma S = 0$ implies (6.6), in such a way that one will finally obtain:

$$\left[g^{\alpha\beta}-\left(1-\frac{f\,r_f'}{r}\right)u^\alpha u^\beta\right]u^\gamma\partial_{\alpha\beta\gamma}u^\lambda = U^\lambda\,(3\text{ in }g_{\alpha\beta},\,2\text{ in }S,\,2\text{ in }f,\,2\text{ in }u^\alpha)\,. \tag{6.8}$$

We have thus transformed the system (6.1), (6.2), (6.3), (6.4) into the following diagonal system:

$$-\tfrac{1}{2}g^{\alpha\beta}\partial_{\alpha\beta}g_{\lambda\mu} = G_{\lambda\mu}\,(1\text{ in }g_{\alpha\beta},\,0\text{ in }S,\,0\text{ in }f,\,0\text{ in }u^\alpha)\,, \tag{6.9}$$

$$u^\alpha\partial_\alpha S = 0, \tag{6.10}$$

$$\left[g^{\alpha\beta}-\left(1-\frac{f\,r_f'}{r}\right)u^\alpha u^\beta\right]u^\gamma\partial_{\alpha\beta\gamma}f = F\,(3\text{ in }g_{\alpha\beta},\,2\text{ in }S,\,2\text{ in }f,\,2\text{ in }u^\alpha), \tag{6.11}$$

$$\left[g^{\alpha\beta}-\left(1-\frac{f\,r_f'}{r}\right)u^\alpha u^\beta\right]u^\gamma\partial_{\alpha\beta\gamma}u^\lambda = U^\lambda\,(3\text{ in }g_{\alpha\beta},\,2\text{ in }S,\,2\text{ in }f,\,2\text{ in }u^\alpha)\,, \tag{6.12}$$

in which (6.9) is obtained by starting from (6.1), thanks to the expression for $R^{(h)}_{\lambda\mu}$.

Arrange the set $\{g_{\alpha\beta}, S, f, u^\alpha\}$ by enumerating it from 1 to 16. With some obvious notations, we will have:

$$a\,(\alpha\beta) = g^{\alpha\beta}\,u^{\gamma}\,\partial_{\alpha\beta\gamma}, \qquad m\,(\alpha\beta) = 2, \qquad \alpha, \beta = 0, 1, 2, 3,$$

$$a\,(11) = u^{\alpha}\,\partial_{\alpha}, \qquad m\,(11) = 1,$$

$$a\,(12) = \left[\, g^{\alpha\beta} - \left(1 - \frac{f\,r_f'}{r}\right) u^{\alpha} u^{\beta} \right] u^{\gamma}\,\partial_{\alpha\beta\gamma}, \qquad m\,(12) = 3\,.$$

Associate the following indices with the unknowns and the equations:

$$a\,(N) = \left[\, g^{\alpha\beta} - \left(1 - \frac{f\,r_f'}{r}\right) u^{\alpha} u^{\beta} \right] u^{\gamma}\,\partial_{\alpha\beta\gamma}, \qquad m\,(N) = 3, \qquad N = 13, 14, 15, 16.$$

Associate the following indices with the unknowns and the equations:

$$s\,(\alpha\beta) = 4, \qquad s\,(11) = 3, \qquad s\,(12) = 3, \qquad s\,(N) = 3,$$

$$t\,(\alpha\beta) = 3, \qquad t\,(11) = 3, \qquad t\,(12) = 1, \qquad t\,(N) = 1,$$

and draw a table for the maximum order of differentiation, along with one for the differences $s\,(i) - t\,(j)$:

1	0	0	0
	0		0
3	2	2	2
3	2	2	2

Maximum order of differentiation

1	0	0	0
1	0	0	0
3	2	2	2
3	2	2	2

Matrix $[s\,(i) - t\,(j)]$

We see that the maximum order of differentiation, as well as the orders of the differential operators $a\,(i)$, is compatible with the choice of indices.

We have shown that the system (6.9), (6.10), (6.11), (6.12) is a quasi-linear system in the Leray sense.

Let us show that it is a strictly-hyperbolic system.

The Cauchy data on Σ are, for example, the formal series that were calculated in § **5**. (It suffices to take the sum of the first n terms for a suitable n.) Their derivatives of order $\leq s\,(i) + 1$ are obviously locally square-integrable. We shall suppose, moreover, that the Cauchy data $(g_{\alpha\beta}, \partial_0\, g_{\alpha\beta}, S)$ on Σ satisfies the conditions:

1) The quadratic form $g_{\alpha\beta}\,X^{\alpha}\,X^{\beta}$ is normal hyperbolic.

2) One has $F^\rho = 0$, $S^{0}{}_{\alpha} = \chi\, T^{0}{}_{\alpha}$ on Σ ; the second relation defines an admissible value for f such that $(f\, r_f')/r \geq 1$.

With those givens, one can look for the semi-cones $\Gamma_x^+(a_i)$.

The operator $a\ (\alpha\beta)$ corresponds to the cone $g^{\alpha\beta}\, u^\gamma\, \xi_\alpha\, \xi_\beta\, \xi_\gamma \neq 0$, and the semi-cone $\Gamma_x^+(\alpha\beta)$ will be defined by:

$$g^{\alpha\beta}\, \xi_\alpha\, \xi_\beta \geq 0\ .$$

The operator a (11) corresponds to the cone $u^\alpha\, \xi_\alpha = 0$, and the semi-cone $\Gamma_x^+(11)$ will be defined by:

$$u^\alpha\, \xi_\alpha \geq 0\ .$$

The operators a (12) and a (N) correspond to the cone:

$$\left[g^{\alpha\beta} - \left(1 - \frac{f\, r_f'}{r}\right) u^\alpha u^\beta \right] u^\gamma\, \xi_\alpha\, \xi_\beta\, \xi_\gamma = 0,$$

and the semi-cone $\Gamma_x^+(12) = \Gamma_x^+(N)$ will be defined by:

$$\left[g^{\alpha\beta} - \left(1 - \frac{f\, r_f'}{r}\right) u^\alpha u^\beta \right] \xi_\alpha\, \xi_\beta \geq 0\ . \tag{6.12}$$

Under the hypothesis that $(f\, r_f')/r \geq 1$, we see that the intersections of those three semi-cones $\Gamma_x^+(\alpha\beta)$, $\Gamma_x^+(11)$, $\Gamma_x^+(12) = \Gamma_x^+(N)$ is nothing but the semi-cone (6.12), which has a non-vacuous interior.

We have thus proved that the differential operator $A = (a_i)$ is strictly hyperbolic at each point x. Since it is differentiable at x, it will be strictly hyperbolic in a connected open neighborhood U of x. The initial hypersurface Σ was chosen to be space-like relative to A, so it remains to specify the values on Σ of the derivatives with indices 0 of $g_{\alpha\beta}$, S, f, u^α of order $\leq s\ (i) - 1$, namely:

$$g_{\alpha\beta},\quad \partial_0\, g_{\alpha\beta},\quad \partial_{00}\, g_{\alpha\beta},\quad \partial_{000}\, g_{\alpha\beta},$$

$$S,\quad \partial_0\, S,\quad \partial_{00}\, S,$$

$$f,\quad \partial_0\, f,\quad \partial_{00}\, f,$$

$$u^\alpha,\quad \partial_0\, u^\alpha,\quad \partial_{00}\, u^\alpha\ .$$

$g_{\alpha\beta}$, $\partial_0\, g_{\alpha\beta}$, S were given in the formal Cauchy problem in § **5**, and $\partial_{00}\, g_{\alpha\beta}$, S, $\partial_0 S$, f, $\partial_0 f$, u^α, $\partial_0 u^\alpha$ are obtained by solving that problem. In order to get $\partial_{000}\, g_{\alpha\beta}$, $\partial_{00} S$, $\partial_{00} f$, $\partial_{00} u^\alpha$, it suffices to differentiate equations (6.1), (6.2), (6.3), and (6.4).

The hypotheses of Leray's theorem are then satisfied. One can then deduce that the Cauchy problem that was posed for the non-analytic system (6.9), (6.10), (6.11), (6.12) admits a unique solution ($g_{\alpha\beta}$, S, f, u^α) in the neighborhood of the point $x \in \Sigma$.

It remains for us to show that this solution is a solution of the initial system (6.1), (6.2), (6.3), (6.4). Now, if the data are analytic then the solution ($g_{\alpha\beta}$, S, f, u^α) will be analytic, and it will necessarily be the analytic solution to the first problem.

If the data are not analytic then one can approximate the system by analytic systems, and in each case, the solution to the second Cauchy problem will be the solution to the first Cauchy problem. Upon passing to the limit, Leray's solution to the Cauchy problem will again be a solution to the initial system, and we will have proved the existence and uniqueness theorem for the non-analytic Cauchy problem that relates to the fundamental system of equations of the relativistic hydrodynamics of perfect fluids.

CHAPTER III

THE RELATIVISTIC HYDRODYNAMICS OF ISENTROPIC PERFECT FLUIDS

§ 4. THE INVARIANT DIFFERENTIAL FORM

1. Differential system for the streamlines. – A perfect fluid is called *isentropic* if its entropy S is constant. In that case (cf., Chap. I, § **4**), the differential system for the streamlines is written:

(1.1) $$\frac{dx^\alpha}{ds} = u^\alpha,$$

(1.2) $$u^\alpha \nabla_\alpha u_\beta - (g^\alpha_\beta - u^\alpha u_\beta)\frac{\partial_\alpha f}{f} = 0,$$

in which s is the curvilinear abscissa along the streamlines and f is the index of the fluid. One has seen that:

$$f = 1 + \varepsilon + \frac{p}{r}.$$

We propose to exhibit the geometric properties of the motion of the fluid.

2. Variation of an integral. – Let V_n be an n-dimensional differentiable manifold, let $\pi: T(V_n) \to V_n$ be the fiber bundle of tangent vectors at all points of V_n, and let $D(V_n) \to V_n$ be the bundle of tangent directions. $T(V_n)$ is $2n$-dimensional and $D(V_n)$ is $(2n-1)$-dimensional. $T(V_n)$ and $D(V_n)$ are locally trivial and one can choose local charts that are induced by the ones on V_n, so that a point of $T(V_n)$ will be defined in local coordinates by the set (x^α, X^α), in which (x^α) is a point of the open subset U of local coordinates of V_n and (X^α) are the components of a tangent vector at $(x^\alpha) \in U$ relative to those coordinates (x^α). A point of $D(V_n)$ is defined by the set (x^α, u^α), in which u^α are the direction parameters of the direction.

Let $C: [t_0, t_1] \to V_n$ be a differentiable curve in V_n with its origin at $x_0 = x(t_0)$ and its extremity at $x_1 = x(t_1)$. In local coordinates, it is defined by the parametric representation:

(2.1) $$x^\alpha = x^\alpha(t).$$

One sets:

(2.2) $$\dot{x}^\alpha = \frac{dx^\alpha}{dt}.$$

That curve lifts to a curve $L : t \to (x^\alpha(t), \dot{x}^\alpha(t))$ in $T(V_n)$, and if $\dot{x}^\alpha$ is non-zero for every t then it will lift to the curve Γ in $D(V_n)$.

Let $F : \omega(V_n) \to R$ be a function on $T(V_n)$ with given scalar values that is positively homogeneous of degree 1 with respect to χ; i.e., for any fixed x, $F(x, \lambda X) = \lambda F(x, X)$. For any curve $C : [t_0, t_1] \to V_{n'}$, $F(x^\alpha, \dot{x}^\alpha)$ is a function of t, and one calculates the integral:

(2.3) $$I = \int_{t_0}^{t_1} F\left(x^\alpha(t), \dot{x}^\alpha(t)\right) dt.$$

That integral is, in fact, intrinsically attached to F and C, and it does not depend upon the parametric representation.

Let us calculate the variation of that integral for an arbitrary variation of C with non-fixed extremities. Upon supposing that C is in an open subset of local coordinates, one will have:

$$\delta I = F_{t_1} \cdot \delta t_1 - F_{t_0} \cdot \delta t_0 + \int_{t_0}^{t_1} \delta F \, dt,$$

so, from a classical argument in the calculus of variations, one will have:

(2.4) $$\delta I = \langle \omega, \delta x \rangle_{x_1} - \langle \omega, \delta x \rangle_{x_0} - \int_{t_0}^{t_1} \langle P, \delta x \rangle \, dt,$$

in which ω is the 1-form that is defined on $T(V_n)$ by:

(2.5) $$\omega = \frac{\partial F}{\partial \dot{x}^\alpha} dx^\alpha,$$

and P is a covector that is defined in components by:

(2.6) $$P_\alpha = \frac{d}{dt}\frac{\partial F}{\partial \dot{x}^\alpha} - \frac{\partial F}{\partial x^\alpha},$$

in which the P_α are nothing but the left-hand sides of the Euler equations from the calculus of variations. δx is a vector.

If C is not in an open subset with local coordinates then one can cover it with a finite number of local charts, and one can study the variation of the integral I, which is the same thing.

3. Extremal principle for the streamlines. – Apply the preceding results to the case in which V is space-time and:

$$F = f\sqrt{g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta}\,. \tag{3.1}$$

Calculate the variations of the integral:

$$S = \int_{t_0}^{t_1} f\sqrt{g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta}\,dt\,, \tag{3.2}$$

which is assumed to be evaluated along a time-like curve C.

One will have:

$$\frac{\partial F}{\partial\dot{x}^\alpha} = \frac{f\,g_{\alpha\beta}\,\dot{x}^\beta}{\sqrt{g_{\lambda\mu}\,\dot{x}^\lambda\dot{x}^\mu}}\,, \qquad \frac{\partial F}{\partial x^\alpha} = \frac{\partial_\alpha f\,g_{\beta\gamma}\,\dot{x}^\beta\dot{x}^\gamma + \frac{1}{2}f\,\partial_\alpha g_{\beta\gamma}\,\dot{x}^\beta\dot{x}^\gamma}{\sqrt{g_{\lambda\mu}\,\dot{x}^\lambda\dot{x}^\mu}}\,,$$

$$P_\alpha = \frac{d}{dt}\frac{f\,g_{\alpha\beta}\,\dot{x}^\beta}{\sqrt{g_{\lambda\mu}\,\dot{x}^\lambda\dot{x}^\mu}} - \frac{\partial_\alpha f\,g_{\beta\gamma}\,\dot{x}^\beta\dot{x}^\gamma + \frac{1}{2}f\,\partial_\alpha g_{\beta\gamma}\,\dot{x}^\beta\dot{x}^\gamma}{\sqrt{g_{\lambda\mu}\,\dot{x}^\lambda\dot{x}^\mu}}\,.$$

Take the arc length s of the curve to be the parameter in place of an arbitrary parameter t. The vector $\dot{x}^\alpha = dx^\alpha / ds$ is the unit velocity vector, in such a way that one will get:

$$P_\alpha = f\left[u^\alpha\,\nabla_\alpha u_\beta - (g_\beta^\alpha - u^\alpha u_\beta)\frac{\partial_\alpha f}{f}\right], \tag{3.3}$$

$$\omega = f\,u_\alpha\,dx^\alpha \tag{3.4}$$

from an easy calculation.

One thus obtains the formula:

$$\delta S = \langle\omega, \delta x\rangle_{x_1} - \langle\omega, \delta x\rangle_{x_0} - \int_{s_0}^{s_1}\langle P, \delta x\rangle\,ds\,, \tag{3.5}$$

which gives the variation of the integral:

$$S = \int_{s_0}^{s_1} f\,ds \tag{3.6}$$

for non-fixed extremities.

If the variations have fixed extremities then $\delta x_0 = \delta x_1 = 0$, and one will have:

$$\delta S = -\int_{s_0}^{s_1}\langle P, \delta x\rangle\,ds\,. \tag{3.7}$$

In order for S to be an extremum, it is necessary and sufficient that $P = 0$; i.e. :

$$u^\alpha \nabla_\alpha u_\beta - (g^\alpha_\beta - u^\alpha u_\beta) \frac{\partial_\alpha f}{f} = 0,$$

which are identical in form to (1.1). Hence:

Theorem:

For any motion of an isentropic perfect fluid, the streamlines are locally time-like extremals of the integral (3.6) *for variations with fixed extremities.*

Introduce the conformal metric $\bar{g} = f^2 g$. One has:

(3.8) $$\bar{g}_{\alpha\beta} = f^2 g_{\alpha\beta}, \qquad \bar{g}^{\alpha\beta} = f^{-2} g^{\alpha\beta} .$$

In regard to that metric, the arc length of the curve is defined by $d\bar{s} = f\, ds$, in such a way that the streamlines are defined to be extremals of:

(3.9) $$\bar{S} = \int_{\bar{s}_0}^{\bar{s}_1} d\bar{s} .$$

Those extremals are geodesics of $(V_4, \bar{g})$. Upon setting:

(3.10) $$C_\alpha = f\, u_\alpha ,$$

one will see that $\bar{C}_\alpha = f\, u_\alpha$ and $\bar{C}^\alpha = f^{-1} u_\alpha$, in such a way that $\bar{g}_{\alpha\beta} \bar{C}^\alpha \bar{C}^\beta = 1$, and its geodesics will have the equation:

(3.11) $$\bar{C}^\alpha \nabla_\alpha \bar{C}_\beta = 0 .$$

Corollary:

The streamlines of an isentropic perfect fluid are time-like geodesics of $(V, \bar{g})$.

4. The integral invariant of hydrodynamics. – Consider a fluid motion that is defined, for example, by a Cauchy problem. Let $\mathcal{T}$ be a flow tube that is generated by a one-dimensional cycle Γ_0 that is traced on the initial hypersurface Σ (and not tangent to the streamlines), and let Γ_1 be a cycle that is traced on $\mathcal{T}$ and is homotopic to Γ_0 . Each streamline of $\mathcal{T}$ is bounded at $x_0 \in \Gamma_0$ and $x_1 \in \Gamma_1$. We can apply formula (3.5) to each of those streamlines, so $P = 0$, and since the total variation of S will be zero when x_0 describes the cycle Γ_0 , one will get:

(4.2) $$\int_{\Gamma_0} \omega = \int_{\Gamma_1} \omega .$$

The 1-form ω has the expression:

(4.2) $$\omega = C_\alpha \, dx^\alpha ,$$

so the property (4.1) will translate into the following statement, which generalizes a classical theorem on the conservation of circulation.

Theorem:

If one is given a one-dimensional cycle Γ *that is not tangent to the streamlines then the circulation of the current vector along* Γ *will remain invariant when* Γ *is deformed along the flow tube that is defined by* Γ.

If D is a two-dimensional differentiable manifold with boundary ∂D then Stokes's formula will give:

(4.3) $$\int_{\partial D} \omega = \int_D d\omega .$$

The integral of the form:

$$\Omega = d\omega$$

over the submanifold D is preserved when it is deformed in such a manner that each point remains on the same streamline.

In the language of H. Poincaré, Ω defines an integral invariant for the differential system of the streamlines:

$$\frac{dx^\alpha}{ds} = u^\alpha ,$$

and ω defines a relative integral invariant. The 2-form Ω plays a fundamental role in the description of motion. It admits the local expression:

(4.4) $$\Omega_{\alpha\beta} = \partial_\alpha C_\beta - \partial_\beta C_\alpha .$$

Theorem:

The 2-form Ω *is an invariant for the differential system for the streamlines; i.e.:*

(4.5) $$\mathcal{L}_C \Omega = 0.$$

$\mathcal{L}_C$ *is the Lie derivative along C.*

Indeed, if one uses the identity from the calculus of variations:

$$\mathcal{L}_C \Omega = (di_C + i_C d)\,\Omega$$

then since $\Omega = d\omega$, one will have $d\,\Omega = 0$, and all that will remain is $di_C\,\Omega$; now:

$$(i_C\,\Omega)_\beta = C^\alpha\,\Omega_{\alpha\beta} = C^\alpha\,(\partial_\alpha C_\beta - \partial_\beta C_\alpha)\ .$$

Upon introducing the Riemannian connection that is associated with the conformal metric $\bar{g}$, one will get:

$$\mathcal{L}_C \Omega = \bar{C}^\alpha (\bar{\nabla}_\alpha \bar{C}_\beta - \bar{\nabla}_\beta \bar{C}_\alpha)\,.$$

Since $\bar{C}_\alpha$ is a unit vector, $\bar{C}^\alpha\,\bar{\nabla}_\alpha \bar{C}_\beta = 0$, and on the other hand, from (3.11), $\bar{C}_\alpha$ is a geodesic field, so one will indeed have $\mathcal{L}_C\,\Omega = 0$.

The 2-form Ω is an invariant form for the differential system of the streamlines. We shall look for all of the differential systems that leave it invariant. We must determine all of the vector fields X such that:

$$\Omega_{\alpha\beta}\,X^\alpha = 0.$$

The existence of X depends upon the rank of the preceding system, and since $\Omega_{\alpha\beta}$ is antisymmetric, one will have:

1. If Ω has rank 2 then the characteristic vectors will form a 2-plane Π_x at each point x. The field of 2-planes Π admits 2-dimensional integral manifolds that are generated by the streamlines.

2. If Ω has rank 0 then $\Omega = 0$. Since $\Omega = d\omega$, ω will be a closed 1-form, so there will exist a function ϕ such that $\omega = d\phi$. As a result, $C_\alpha = \partial_\alpha\,\phi$: The streamlines are orthogonal trajectories to the family of hypersurfaces $\phi = \text{const.}$

Those results are important for the study of rotational and irrotational motions of fluids.

§ 2. ROTATIONAL AND IRROTATIONAL MOTIONS

5. Vorticity tensor and Helmholtz equations.

Definition. – One calls the antisymmetric tensor of order 2 that is defined by the invariant 2-form Ω the *vorticity tensor.*

It constitutes the true relativistic extension of the rotation of the velocity that is introduced in classical mechanics. If one recalls the expression for $f = 1 + \varepsilon\,c^{-2} + p\,r^{-1}c^{-2}$, in which c is the speed of light then one will see that $C_\alpha = f\,u_\alpha$ will differ from

u_α by some terms in c^{-2}, and $\Omega_{\alpha\beta} = \partial_\alpha C_\beta - \partial_\alpha C_\beta$ will differ from $\overset{\circ}{\Omega}_{\alpha\beta} = \partial_\alpha u_\beta - \partial_\alpha u_\beta$ by terms in c^{-2}.

Theorem:

The vorticity tensor satisfies the Helmholtz equation:

$$C^\rho \nabla_\rho \Omega_{\alpha\beta} + \nabla_\alpha C^\rho \Omega_{\rho\beta} + \nabla_\beta C^\rho \Omega_{\alpha\rho} = 0 . \tag{5.1}$$

Indeed, a simple calculation shows that these equations are a consequence of the equation $\mathcal{L}_C \Omega = 0$, which expresses the idea that Ω is an invariant form. One performs the calculation in the initial metric (V, g).

Definition. – One says that a fluid motion is *rotational* if $\Omega \neq 0$ and *irrotational* if $\Omega = 0$.

Theorem:

In order for a motion of an isentropic perfect fluid to be irrotational, it is necessary and sufficient that the streamlines should be orthogonal to the same (local) hypersurface.

Indeed, let Σ be a space-like hypersurface such that $\Omega_{\alpha\beta} = 0$ on Σ. One can choose local coordinates such that Σ is represented by $x^0 = 0$ and the streamlines are represented by x^i = const. (i.e., Gaussian coordinates). The Helmholtz equations show that $\partial_0 \Omega_{\alpha\beta} = 0$. It will then result that $\Omega_{\alpha\beta} = 0$ in the neighborhood of Σ.

6. Vorticity vector. – Suppose that the motion is irrotational. Let us study the 2-plane Π_x at the point $x \in V_4$ that is composed of the characteristic vectors X^α ; viz., vectors such that:

$$\Omega_{\alpha\beta} X^\beta = 0 . \tag{6.1}$$

One will already have that the vector u^α in Π_x is tangent to the streamline that passes through x . In order to succeed in determining Π_x , it will suffice for us to look for a second vector that is not collinear with u^α. We choose one such vector θ that is orthogonal to the first one. That vector is defined by the equations:

$$\Omega_{\alpha\beta}\, \theta^\beta = 0, \qquad \theta^\alpha u^\alpha = 0 . \tag{6.2}$$

The vector θ^α is defined only up to a factor, so one will have, by an algebraic calculation:

$$\theta^\alpha = \tfrac{1}{2} \eta^{\alpha\beta\gamma\delta} u_\beta \Omega_{\gamma\delta} , \tag{6.3}$$

in which $\eta_{\alpha\beta\gamma\delta}$ is the Riemannian volume element for (V, g). One remarks that $\theta^\alpha = 0$ implies that $\Omega_{\alpha\beta} = 0$.

Definition. – One gives the name of *vorticity vector* to the vector θ^α that is defined by (6.3), and its trajectories are called *vortex lines.*

From their definition, the vortex lines are orthogonal to the streamlines. On the other hand, the differential system for the vortex lines:

$$\frac{dx^\alpha}{dt} = \theta^\alpha$$

admits the 2-form Ω as an invariant form. One will immediately deduce the following properties:

Theorem:

If one is given a one-dimensional cycle Γ *that is not tangent to the vortex lines then the circulation of the vorticity vector along* Γ *will remain invariant when one deforms* Γ *along the vortex tube that is defined by* Γ.

Let $\mathcal{T}$ be a flow tube, while Γ and Γ' are homotopic cycles on $\mathcal{T}$. Each of those cycles defines a vortex tube, namely, Θ and Θ'. Let Γ_1 be a cycle on Θ that is homotopic to Γ. The streamline that passes through Γ_1 cuts the vortex tube Θ along a cycle Γ_1' that is homotopic to Γ'. Since ω is a relative integral invariant for the streamlines and also for the vortex lines, one will have:

$$\int_{\Gamma_1'} \omega = \int_{\Gamma_1} \omega .$$

That property constitutes the relativistic generalization of a theorem of Helmholtz in classical dynamics.

Finally, the field of 2-planes $x \to \Pi_x$ that is defined by the characteristic system of the form Ω:

$$\Omega_{\alpha\beta} X^\beta = 0$$

is a completely-integrable field. One gives the name of *characteristic manifolds* of Ω to the two-dimensional integral manifolds W_2 . If one then draws the streamlines that pass through the points of a vortex line then the orthogonal trajectories to those streamlines on W_2 will be vortex lines. That amounts to saying that if a fluid line is a vortex line at one instant then it will remain a vortex line at any instant.

§ 3. PERMANENT MOTIONS

7. Stationary space-time. – One says that a space-time (V_4 , g) is *stationary* if there exists a connected one-parameter group of global isometries that does not leave any point of V_4 invariant, has time-like trajectories, and is such that:

1) Each trajectory z is homeomorphic to R.

2) There exists a three-dimensional differentiable manifold V_3 and a diffeomorphism $V_4 \to V_3 \times R$ that maps the trajectories z onto the right factor R.

V_4 takes the form of a trivial fiber bundle with base V_3 and fiber type R. The fibers are the trajectories of the isometries; one calls them *time-lines.* One calls the base manifold V_3 *space*. It is diffeomorphic to the quotient manifold of V_4 by the equivalence relation that is defined by the group of isometries.

If ξ is the infinitesimal generator of the isometry group then it will satisfy the Killing equations:

$$(\mathcal{L}_\xi g)_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha . \tag{7.1}$$

It results from the definition that there exists a local coordinate system (x^0, x^i) such that the x^i are a system of local coordinates on V_3 and x^0 defines the points on the trajectories of ξ, in such a way that the spatial sections x^0 = const. are globally defined and diffeomorphic to V_3 . One says that those local coordinates (x^0, x^i) are *locally adapted* to the isometry group if the infinitesimal generator ξ admits the contravariant components:

$$\xi^0 = 1, \qquad \xi^i = 0. \tag{7.2}$$

If $g_{\alpha\beta}$ are the components of the metric tensor in that coordinate system then the covariant components of ξ will be:

$$\xi_\alpha = g_{0\alpha} .$$

The Killing equations (7.1) translate into:

$$\nabla_\alpha \xi_\beta = \nabla_\alpha g_{0\beta} + \Gamma^\rho_{\alpha 0} \cdot g_{\rho\beta} = [\alpha 0, \beta] ,$$

namely:

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = \partial_0 g_{\alpha\beta} = 0 .$$

Hence, the $g_{\alpha\beta}$ will be independent of x^0 in the adapted coordinates.

Upon decomposing the metric form using the director variable x^0, one will have:

(7.3) $$g = \frac{g_{0\alpha}\, g_{0\beta}\, dx^\alpha \otimes dx^\beta}{g_{00}} + \hat{g}_{ij}\, dx^i \otimes dx^j ,$$

in which:

(7.4) $$\hat{g}_{ij} = g_{ij} - \frac{g_{0i}\, g_{0j}}{g_{00}}$$

defines a negative-definite metric on the spatial sections. It is invariant under any change of adapted coordinate system of the form:

$$x^{0'} = x^0 + \psi(x^i), \qquad x^{i'} = x^i .$$

One then endows V_3 with that metric.

8. Permanent motion. – One says that the motion of an isentropic perfect fluid is *permanent* if the space-time is stationary and the isometry group leaves the index f and the unit velocity vector invariant; i.e.:

(8.1) $$\mathcal{L}_\xi f = 0, \qquad \mathcal{L}_\xi u = 0.$$

If the coordinates are adapted then the conditions (8.1) will translate into:

(8.2) $$\partial_\xi f = 0, \qquad \partial_\xi u = 0.$$

Theorem:

In order for the motion of an isentropic perfect fluid to be permanent, it is necessary and sufficient that space-time should be stationary.

Choose adapted local coordinates (x^0, x^i) and let Σ be the hypersurface with the equation $x^0 = 0$. Σ is time-like. It results from the Cauchy problem that on Σ and the neighboring hypersurface x^0 = const., one will have [Chap. II.**5**, (5.9)]:

$$\chi\, r\, f\,(S^{00} + \chi\, p\, g^{00}) - g^{\alpha\beta}\,(S^\beta_{\ \alpha} + \chi\, p\, g^0_{\ \alpha})(S^\beta_{\ \beta} + \chi\, p\, g^0_{\ \beta}) = 0 .$$

In adapted coordinates, $\partial_0 S^0_{\ \alpha} = 0$, $\partial_0\, g_{\alpha\beta} = 0$. It then results that upon differentiating with respect to x^0 and taking the thermodynamic equation $dp = r\, df - \theta\, dS$ ($dS = 0$) into account, one will get:

$$\chi(f\, r'_f + r)(S^{00} + \chi\, p\, g^{00})\,\partial_0 f + \chi^2 r^2 \partial_0 f\, g^{00} - 2g^{\alpha 0}(S^0_{\ \alpha} + \chi\, p\, g^0_{\ \alpha})\chi\, r\,\partial_0 f = 0,$$

namely, upon taking Chap. II.**5**, (5.7) into account:

$$\left[g^{00}-\left(1-\frac{f\,r'_f}{r}\right)u^0u^0\right]\cdot\partial_0 f=0\,.$$

One then deduces that $\partial_0 f = 0$ on Σ, and then $\partial_0\, u^\alpha = 0$. The motion is therefore permanent.

Theorem:

The scalar function:

$$H = \xi^\alpha C_\alpha \tag{8.3}$$

preserves a constant value along each streamline under any permanent motion of a fluid.

It will suffice for us to show that $\mathcal{L}_C\,(i_\xi\,\omega) = 0$, in which $\omega = C_\alpha\,dx^\alpha$. Now, upon utilizing the identity from the calculus of variations:

$$\mathcal{L}_C\,(i_\xi\,\omega) = (i_C\,d + d\,i_C)\,i_\xi\,\omega\,,$$

we will have:

$$\begin{aligned}\mathcal{L}_C\,(i_\xi\,\omega) &= i_C\,d\,i_\xi\,\omega\\ &= i_C\,(\mathcal{L}_\xi - i_\xi d)\,\omega\,.\end{aligned}$$

Now it is obvious that $\mathcal{L}_\xi\;\omega = 0$, which is to say that the differential system for the streamlines admits the infinitesimal transformation x, so we likewise deduce that $\mathcal{L}_\xi\;\omega = 0$. That will give:

$$\begin{aligned}\mathcal{L}_C\,(i_\xi\,\omega) &= -\,i_C\,i_\xi\,d\omega\\ &= i_\xi\,i_C\,\Omega = 0\,.\end{aligned}$$

Remark. – The differential system for the streamlines admits the invariant form Ω and the infinitesimal transformation ξ. Since $\mathcal{L}_\xi\;\theta = 0$, one will see that the differential system for the vortex lines possesses the same property. One then deduces that $H = C^\alpha\xi_\alpha$ is likewise constant along the vortex lines. H is then constant along each characteristic manifold W_2 of Ω.

One has:

$$dH = \Omega_{\alpha\beta}\,\xi^\beta\,dx^\alpha, \tag{8.4}$$

which is a formula that makes the preceding results obvious.

9. Bernoulli's theorem. – Introduce the spatial magnitude of the vector u^α relative to the time direction x . Let:

$$-v^2 = \hat{g}_{ij}\, u^i\, u^j .$$

By virtue of the unit character of u, one will have:

$$g_{\alpha\beta}\, u^\alpha\, u^\beta = \frac{1}{g_{00}}(g_{00}u^\alpha)^2 + \hat{g}_{ij}\, u^i u^j = 1,$$

in which:

$$(u_0)^2 = g_{00}\,(1 + v^2) . \tag{9.1}$$

The first integral H has the value $C_0 = f\, u_0$ in adapted coordinates. One then deduces:

$$H^2 = f^2\, g_{00}\,(1 + v^2) .$$

Upon setting $U = g_{00}$, one will get:

Theorem:

The permanent motion of an isentropic perfect fluid satisfies:

$$f^2\, U\,(1 + v^2) = \text{const.} \tag{9.2}$$

along each streamline, in which U is the principal gravitational potential.

That theorem generalizes Bernoulli's theorem. Indeed, from the thermodynamic equation, it will give:

$$f = 1 + \int_{p_0}^{p} \frac{dp}{c^2 r} .$$

One deduces that:

$$\tfrac{1}{2} c^2\, U + U\left(\tfrac{1}{2} v^2 + \int_{p_0}^{p} \frac{dp}{r}\right) = \text{const.},$$

up to terms in c^{-2} .

§ 4. SPATIAL PROJECTIONS

10. A problem in the calculus of variations. – One proposes to study the permanent motions in space V_3 . In order to do that, one must study the projections of the geodesics of (V_4 , $\bar{g}$) onto the quotient space (V_3 , $\hat{\bar{g}}$).

Such a problem was solved in the most general case of a Finslerian manifold (V_{n+1} , $\mathfrak{L}$) that is defined by a differential manifold that is endowed with a function $\mathfrak{L}\,(x, X)$ that is positively-homogeneous of degree 1 on the fiber of the directions $D\,(V_{n+1})$. One will suppose that $(V_{n+1}, \mathfrak{L})$ admits a connected group of global isometries that are defined by a vector field x such that:

$$\mathcal{L}_x \mathfrak{L} = 0 .$$

One refers $(V_{n+1}, \mathfrak{L})$ to local coordinates (x^i, x^0) that are adapted to its isometry group, and one denotes the quotient manifold by V_n .

The differential system for the extremals of $\mathfrak{L}$ admits the relative integral invariant:

$$\omega = \partial_{\dot\alpha} \mathfrak{L}\, dx^\alpha, \tag{10.1}$$

in which $\partial_{\dot\alpha} = \partial / \partial \dot{x}^\alpha$. $\mathfrak{L}$ does not depend upon x^0 (viz., $\partial_0\mathfrak{L} = 0$), so one has the first integral that is provided by the Euler equation in x^0 :

$$\partial_{\dot 0} \mathfrak{L} = h , \tag{10.2}$$

in such a way that $\partial_{\dot 0} \mathfrak{L}\, dx^0 = h\, dx^0$ constitutes an integral invariant for the family (E_k) for the extremals that correspond to the value h. It will then result that:

$$\pi = \partial_{\dot\alpha} \mathfrak{L}\, dx^\alpha \tag{10.3}$$

is a relative integral invariant for the family (E_h).

If $\partial_{\dot 0 \dot 0} \mathfrak{L} \neq 0$ then one can solve (10.2) for $\dot{x}^0$, namely:

$$\dot{x}^0 = \varphi(x^i, \dot{x}^j, h), \tag{10.4}$$

in which φ is a homogeneous function of degree 1 in $\dot{x}^j$. On the other hand, by virtue of the homogeneity of $\mathfrak{L}$, one has:

$$\dot{x}^i\, \partial_i \mathfrak{L} + \dot{x}^0\, \partial_{\dot 0} \mathfrak{L} = \mathfrak{L} .$$

As a result, the form $\dot\pi = \dot{x}^i\, \partial_i \mathfrak{L}$ can be expressed by a function L of the variables x^i, $\dot{x}^j$, h, namely:

$$L(x^i, \dot{x}^j, h) = \mathfrak{L}(x^i, \dot{x}^j, \varphi(x^i, \dot{x}^j, h)) - h\,\varphi(x^i, \dot{x}^j, h), \tag{10.5}$$

and one will have:

$$\partial_i L = \partial_i \mathfrak{L} + \partial_{\dot 0}\mathfrak{L} \cdot \partial_i \varphi - h\, \partial_i \varphi = \partial_i \mathfrak{L} + h\, \partial_i \varphi - h\, \partial_i \varphi = \partial_i \mathfrak{L} .$$

The theorem is thus proved.

Theorem:

The projections onto V_n of the extremals (E_h) for a given value h are the extremals of the function L. They are defined for a differential system that admits the relative integral invariant:

$$\pi = \partial_{\dot{\kappa}} L \, dx^\kappa.$$

11. Case of a Riemannian metric. – Consider the case in which the function $\mathcal{L}$ is defined by:

$$\mathcal{L}^2 = g_{\alpha\beta}\, \dot{x}^\alpha \, \dot{x}^\beta \qquad \alpha, \beta = 0, 1, \ldots, n .$$

Suppose that $g_{00} \neq 0$.

The descent process leads one to form the equation:

$$\tfrac{1}{2}\partial_{\dot{0}} \mathfrak{L}^2 = g_{00}\, \dot{x}^0 + g_{0i}\, \dot{x}^i = h\, \mathfrak{L} \tag{11.}$$

and to eliminate $\dot{x}^0$ from that equation, along with the equation:

$$L = \mathfrak{L} - h\, \dot{x}^0 . \tag{11.2}$$

That elimination will give:

$$L = \sqrt{\left(1 - \frac{h^2}{g_{00}}\right) g_{ij}\, \dot{x}^i \, \dot{x}^j} + h \frac{g_{0i}\, \dot{x}^i}{g_{00}} . \tag{11.3}$$

If $g_{00} = 0$ then one will have:

$$\mathcal{L}^2 = 2 g_{0i}\, \dot{x}^i \, \dot{x}^0 + g_{ij}\, \dot{x}^i \, \dot{x}^j . \tag{11.4}$$

One supposes that $g_{0i}\, \dot{x}^i \neq 0$. The descent process leads one to eliminate $\dot{x}^0$ from (11.2) and the relations:

$$g_{0i}\, \dot{x}^i = h\, \mathfrak{L} ,$$

$$L = \mathfrak{L} - h\, \dot{x}^0 .$$

Elimination gives:

$$L = \frac{g_{0i}\, \dot{x}^i}{2h} + h \frac{g_{ij}\, \dot{x}^i \, \dot{x}^j}{2 g_{0i}\, \dot{x}^i} . \tag{11.5}$$

Application to permanent motions. – It suffices to replace $g_{\alpha\beta}$ with $f^2\, g_{\alpha\beta}$ and obtain the function L whose extremals give the motion in space. There is only one such case, because $g_{00} \neq 0$. One will then obtain:

(11.6)
$$L = \sqrt{\left(1-\frac{h^2}{f^2 g_{00}}\right) f^2 g_{ij}\, \dot{x}^i\, \dot{x}^j} + h\frac{g_{0i}\, \dot{x}^i}{g_{00}} .$$

It would be interesting to develop the calculations.

12. Projection of null-length geodesics. – One considers time-like geodesics as limits. In our problem:

$$\mathfrak{L} = \sqrt{g_{\alpha\beta}\, \dot{x}^\alpha\, \dot{x}^\beta} ,$$

and upon differentiating with respect to $\dot{x}^0$, one will have $h\ \mathfrak{L} = g_{0\alpha}\, \dot{x}^\alpha$, which shows that $h \to \infty$ when $\mathfrak{L} \to 0$, while h keeps the sign of $g_{0\alpha}\, \dot{x}^\alpha$. The extremals of L then coincide with those of L / h .

As a result, the desired extremals that define the projections of the isotropic geodesics of (V_4 , g) will be the extremals of the function:

(12.1)
$$\Lambda = \lim_{h\to\infty} \frac{1}{h} L(x^i, \dot{x}^j, h) .$$

First case: $g_{00} \neq 0$. Passing to the limit gives:

(12.2)
$$\Lambda = \varepsilon\,\varepsilon' \sqrt{-\frac{1}{g_{00}} \hat{g}_{ij}\, \dot{x}^i\, \dot{x}^j} - \frac{g_{0i}\, \dot{x}^i}{g_{00}} ,$$

in which ε' is the sign of $g_{0\alpha}\, \dot{x}^\alpha$ and ε is the sign of g_{00} , and then:

(12.3)
$$\dot{x}^0 = \varepsilon\,\varepsilon' \sqrt{-\frac{1}{g_{00}} \hat{g}_{ij}\, \dot{x}^i\, \dot{x}^j} - \frac{g_{0i}\, \dot{x}^i}{g_{00}} .$$

Second case: $g_{00} = 0$. Passing to the limit gives:

(12.4)
$$L = \frac{g_{ij}\, \dot{x}^i\, \dot{x}^j}{2 g_{0i}\, \dot{x}^i} ,$$

(12.5)
$$\dot{x}^0 = -\frac{g_{ij}\, \dot{x}^i\, \dot{x}^j}{2 g_{0i}\, \dot{x}^i} .$$

We shall apply those results to the study of Fermat's principle.

13. Fermat's principle. – One knows that light rays [**16**] are isotropic geodesics of the Riemannian manifold (V_4 , $\bar{g}$) that is defined by space-time V_4 when it is endowed with the metric:

(13.1) $$\bar{g}_{\alpha\beta} = g_{\alpha\beta} - \left(1 - \frac{1}{\lambda\mu}\right) u_\alpha\, u_\beta\,.$$

Suppose that the motion is permanent. If λ and μ are constant then the isometry group of (V_4 , g) will induce an isometry group on (V_4 , $\bar{g}$).

Choose adapted coordinates.

However, although the trajectories of the isometries of (V_4 , g) are time-like, the trajectories of the induced isometries on (V_4 , $\bar{g}$) can be time-like, space-like, or isotropic. Indeed, if ζ is the infinitesimal generator of the isometry group of (V_4 , $\bar{g}$) then one will have:

$$\zeta^0 = \xi^0 = 1, \qquad \zeta^i = \xi^i = 0$$

for its contravariant components, and the square of that vector will have the value:

(13.2) $$\bar{g}_{00} = g_{00} - (1 - w^2)\, u_0\, u_0\,,$$

in which $w^2 = 1 / \lambda\mu$ is the square of the speed of propagation of light in the fluid.

If we introduce the spatial magnitude of the unit velocity vector u^α relative to the time direction ζ (namely, $\pi^2 = - \hat{g}_{ij}\, u^i\, u^j$) then we will see that:

$$(u_0)^2 \neq g_{00}\,(1 + v^2)\,.$$

Upon substituting that value into (13.2), we will get:

(13.3) $$\bar{g}_{00} = g_{00}\,(v^2\, w^2 + w^2 - v^2)\,.$$

$\bar{g}_{00}$ can change sign.

Upon applying the formulas of the preceding paragraph, one will get the following theorem, which gives the law of propagation of light in space:

Theorem:

If the motion of the fluid is permanent and such that $\bar{g}_{00} \neq 0$ *then the light rays in space will be the extremals of the integral:*

(13.4) $$\int_{x_0}^{x_1} \Lambda\, du = \int_{x_0}^{x_1} \left[\varepsilon\,\varepsilon' \sqrt{-\frac{1}{\bar{g}_{00}}\,\hat{\bar{g}}_{ij}\, \dot{x}^i \dot{x}^j} - \frac{\bar{g}_{0i}\,\dot{x}^i}{\bar{g}_{00}} \right] du\,,$$

in which $\dot{x}^i = dx^i / du$*, for variations with fixed extremities in* V_3 *. The time that a ray takes to go from the point* x_0 *to the point* x_1 *will then be given by:*

$$\int_{x_0}^{x_1} dt \;=\; \int_{x_0}^{x_1}\left[\varepsilon\varepsilon'\sqrt{-\frac{1}{\bar{g}_{00}}\hat{\bar{g}}_{ij}\,\dot{x}^i\dot{x}^j}-\frac{\bar{g}_{0i}\,\dot{x}^i}{\bar{g}_{00}}\right]du\,. \tag{13.5}$$

It is an extremum.

In the case $\bar{g}_{00} = 0$, one will have:

$$\int_{x_0}^{x_1}\Lambda\,du \;=\; \int_{x_0}^{x_1}-\frac{\bar{g}_{ij}\,\dot{x}^i\dot{x}^j}{\bar{g}_{0i}\,\dot{x}^i}\,du\,, \tag{13.6}$$

$$\int_{x_0}^{x_1} dt \;=\; \int_{x_0}^{x_1}-\frac{\bar{g}_{ij}\,\dot{x}^i\dot{x}^j}{\bar{g}_{0i}\,\dot{x}^i}\,du\,. \tag{13.7}$$

It is clear that the results do not depend upon the auxiliary variable u . On the other hand, if space-time is statically orthogonal and its streamlines coincide with the time-lines then one will have the world-metric:

$$g = U\,dx^0\otimes dx^0 + g_{ij}\,dx^i\otimes dx^j$$

and the associated metric:

$$\bar{g} = \frac{U}{n^2}\,dx^0\otimes dx^0 + g_{ij}\,dx^i\otimes dx^j,$$

in which $n^2 = \lambda\mu$. One can then put (13.5) into the form:

$$\int_{x_0}^{x_1} dt = \int_{x_0}^{x_1} n\sqrt{U}\,d\tau\,,$$

in which $d\tau^2 = g_{ij}\,dx^i\,dx^j$ is the line element of (V_3 , $\bar{g}$). In the case of a flat space-time $U = 1$, the preceding theorem translates into:

$$\delta\int_{x_0}^{x_1} n\,d\tau = 0\,.$$

That is how Fermat's principle gets stated in classical optics. The theorem that we proved constitutes the statement of Fermat's principle in general relativity in the case of a fluid in motion. One can likewise prove the equivalence of the principle of least action and the principle of least time with that theorem.

14. Application: relativistic law of the composition of velocities. – We place ourselves in Minkowski space, which is referred to orthonormal coordinates. u^α is the unit velocity world-vector of whose components are classically determined by starting from the spatial velocity **β**, and c is taken to be unity. A simple calculation will give the associated metric, which we will write in the form:

$$d\bar{s}^2 = \frac{V^2-\beta^2}{1-\beta^2}(dx^0)^2 + 2\frac{1-V^2}{1-\beta^2}\beta_i\,dx^0dx^i - \sum_i (dx^i)^2 - \frac{1-V^2}{1-\beta^2}(\beta_i\,dx^i)^2 . \tag{14.1}$$

That metric has hyperbolic normal type. One should note the change of order in the signature when one passes to $V^2 = \beta^2$. One can exhibit that fact by choosing the x^1 axis to be parallel to $\boldsymbol{\beta}$ (i.e., the flow velocity). One will then have:

$$d\bar{s}^2 = \frac{V^2-\beta^2}{1-\beta^2}(dx^0)^2 + 2\frac{1-V^2}{1-\beta^2}dx^0dx^i - \frac{1-V^2}{1-\beta^2}(dx^1)^2 - (dx^2)^2 - (dx^3)^2 , \tag{14.2}$$

which one can put into canonical form by a decomposition into squares. If $V^2 \neq \beta^2$ then one will get:

$$d\bar{s}^2 = \frac{1-\beta^2}{V^2-\beta^2}\left[\frac{V^2-\beta^2}{1-\beta^2}dx^0 + \frac{1-V^2}{1-\beta^2}dx^i\right]^2 - \frac{1-\beta^2}{V^2-\beta^2}(dx^1)^2 - (dx^2)^2 - (dx^3)^2 ,$$

and one sees that for $V^2 > \beta^2$, one will have a signature $+ - - -$, while for $V^2 < \beta^2$, one will have signature $- + - -$. For $V^2 = \beta^2$, one will get:

$$d\bar{s}^2 = 2V\,dx^0\,dx^1 - (1 + V^2)\,(dx^1)^2 - (dx^2)^2 - (dx^3)^2,$$

which still has signature $+ - - -$.

Starting from the associated metric (14.1), we seek to express that theorem by taking the arc length ζ of the ray to be the parameter. We must replace the $\dot{x}^i$ in (13.5) with $\lambda^i = dx^i / d\zeta$, in which $d\zeta^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$. We will then infer that:

$$\frac{dt}{d\zeta} = \frac{1}{W} = \varepsilon\varepsilon'\sqrt{\frac{1-\beta^2}{V^2-\beta^2}V^2 - \beta^2 + (1-V^2)(\beta_i\lambda^i)^2} - \frac{(1-V^2)(\beta_i\lambda^i)}{V^2-\beta^2} .$$

If $V^2 - \beta^2 \neq 0$ then that relation will give:

$$1 - \beta^2 - (1 - \beta^2)\,W^2 - (1 - V^2)\,(1 - W\beta_i\,\lambda^i)^2 = 0 .$$

If one interprets $\boldsymbol{V}$ as the absolute velocity of the propagation of light and $\boldsymbol{W}$ as the relative velocity then one will obviously have:

$$\boldsymbol{V}^2 = \frac{1}{(1+\boldsymbol{W}\cdot\boldsymbol{\beta})^2}\,[\boldsymbol{W}^2 + \boldsymbol{\beta}^2 + 2\,\boldsymbol{W}\cdot\boldsymbol{\beta} + (\boldsymbol{W}\cdot\boldsymbol{\beta})^2 - \boldsymbol{W}^2\,\boldsymbol{\beta}^2] . \tag{14.3}$$

One verifies that this relation will remain valid in the case of $V^2 = \beta^2$ by direct calculation; it is the relativistic formula for the composition of velocities. It is easy to verify that one can put it into the form:

$$\boldsymbol{V} = \frac{1}{1+\boldsymbol{W}_0\boldsymbol{\beta}}\left[\left(1+\frac{\boldsymbol{W}_0\boldsymbol{\beta}}{\boldsymbol{\beta}^2}\right)\boldsymbol{\beta}+\sqrt{1-\boldsymbol{\beta}^2}\left(\boldsymbol{W}-\frac{\boldsymbol{W}\cdot\boldsymbol{\beta}}{\boldsymbol{\beta}^2}\boldsymbol{\beta}\right)\right].$$

One thus obtains a proof of the relativistic law of composition of velocities by starting from Fermat's principle.

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