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The general equations of mechanics in the case of non-holonomic constraints.

By **Jean Quanjel** (Halle Gate)

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Translated by D. H. Delphenich

1. – Suppose that one would like to express the idea that a body C rolls without slipping on a surface S at the moment t. One must write out the idea that the velocity of the point of contact of C with S is zero at that moment, which will give equations of constraint that express the rolling of C on S. **Korteweg** has shown, in a general manner (*), that those equations cannot be integrated. Following **Hertz**, one gives the name of *non-holonomic* constraints to all of the constraints that are expressed by non-integrable differential equations, whereas one calls constraints *holonomic* when they can be expressed by finite or integrable equations.

In what follows, we shall examine what the general equations of analytical mechanics will become in the case of non-holonomic constraints.

I. – The Lagrange equations.

2. – **Neumann** (**), **Vierkandt** (***), and **Korteweg** (†) have shown that the Lagrange equations must be modified in the case of non-holonomic constraints.

Appell gave a proof of that fact in tome II of his *Mécanique rationelle*.

We shall reproduce it, while generalizing it slightly.

3. – In all of what follows, we shall suppose that:

(*) **Korteweg**, "Über eine ziemlich verbreitete unrichtige Behandlungsweise eines Problemes der rollenden Bewegung, etc.," Nieuw Archief voor Wiskunde, Tweede Reeks, Deel IV (1900), 130-155.

(**) **C. Neumann**, "Grundzüge der analytischer Mechanik, insbesondere der Mechanik starrer Körper (Zweiter Artikel), Ber. Verh. Kgl. Sächs. Ges. Wiss. Leipzig **40** (1888), 22-88, esp. pp. 32-56.

(***) **Vierkandt**, "Über gleitende und rollende Bewegung," Mon. Math. und Physik **3** (1892), 31-54, esp., pp. 47.

(†) *Loc. cit.*

1. **D'Alembert**'s principle is applicable.

2. It is possible to replace the geometric constraints that are imposed upon the system with forces of reaction.

3. The work done by those forces of reaction is zero for any displacement that is compatible with the constraints.

4. – Suppose that the coordinates of an arbitrary point of the systems are expressed by means of n parameters $q_1, q_2, \ldots, q_n$, and time t in such a way that:

$$(1) \qquad \begin{cases} x = f_1(q_1, q_2, \ldots, q_n, t), \\ y = f_2(q_1, q_2, \ldots, q_n, t), \\ z = f_3(q_1, q_2, \ldots, q_n, t), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

In addition, assume that the n parameters are coupled by k distinct non-holonomic relations:

$$(2) \qquad \begin{cases} A_1\,\delta q_1 + A_2\,\delta q_2 + \cdots + A_n\,\delta q_n = 0, \\ B_1\,\delta q_1 + B_2\,\delta q_2 + \cdots + B_n\,\delta q_n = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ L_1\,\delta q_1 + L_2\,\delta q_2 + \cdots + L_n\,\delta q_n = 0. \end{cases}$$

A virtual displacement that is compatible with the constraints that exist at the moment t is determined by equations of the form:

$$(3) \qquad \begin{cases} \delta x_i = \dfrac{\partial x_i}{\partial q_1}\,\delta q_1 + \cdots + \dfrac{\partial x_i}{\partial q_n}\,\delta q_n, \\ \delta y_i = \dfrac{\partial y_i}{\partial q_1}\,\delta q_1 + \cdots + \dfrac{\partial y_i}{\partial q_n}\,\delta q_n, \\ \delta z_i = \dfrac{\partial z_i}{\partial q_1}\,\delta q_1 + \cdots + \dfrac{\partial z_i}{\partial q_n}\,\delta q_n \end{cases} \qquad (i = 1, 2, \ldots, n).$$

If the δq are coupled by the k equations (2) then one can deduce, e.g., the last k of the δq as homogeneous linear functions of the first $n - k$ of them, which will be arbitrary. If one eliminates the k dependent displacements from (3) then one will obtain values for δx_i, δy_i, δz_i that take the form:

$$\text{(4)} \qquad \left\{ \begin{aligned} \delta x_i &= a_1\,\delta q_1 + a_2\,\delta q_2 + \cdots + a_p\,\delta q_p\,, \\ \delta y_i &= b_1\,\delta q_1 + b_2\,\delta q_2 + \cdots + b_p\,\delta q_p\,, \\ \delta z_i &= c_1\,\delta q_1 + c_2\,\delta q_2 + \cdots + c_p\,\delta q_p \end{aligned} \right. \qquad (p = n - k).$$

If we substitute the values (4) in the general **d'Alembert** equations and equate the coefficients of the same variations δq in both sides of them then we will get p equations of the form:

$$\text{(5)} \qquad \left\{ \begin{aligned} &\sum m\left(a_1\frac{d^2x}{dt^2} + b_1\frac{d^2y}{dt^2} + c_1\frac{d^2z}{dt^2}\right) = \sum (a_1\,X + b_1\,Y + c_1\,Z)\,, \\ &\sum m\left(a_2\frac{d^2x}{dt^2} + b_2\frac{d^2y}{dt^2} + c_2\frac{d^2z}{dt^2}\right) = \sum (a_2\,X + b_2\,Y + c_2\,Z)\,, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \end{aligned} \right.$$

which constitute n equations of motion when one includes equations (2).

I say that equations (5) *differ from those of* ***Lagrange****, as a general rule.*

Since the right-hand sides are the same in (5) and the **Lagrange** equations, I must prove that the left-hand sides are different.

One has:

$$\text{(6)} \qquad \left\{ \begin{aligned} x' &= a_1\,q_1' + a_2\,q_2' + \cdots + a_p\,q_p' + a\,, \\ y' &= b_1\,q_1' + b_2\,q_2' + \cdots + b_p\,q_p' + b\,, \\ z' &= c_1\,q_1' + c_2\,q_2' + \cdots + c_p\,q_p' + c \end{aligned} \right. \qquad \left(x' = \frac{dx}{dt}, y' = \cdots\right).$$

The left-hand side of the first of equations (5) can be written:

$$\text{(7)} \qquad \frac{d}{dt}\sum m(a_1\,x' + b_1\,y' + c_1\,z') - \sum m\left(x'\frac{da_1}{dt} + y'\frac{db_1}{dt} + z'\frac{dc_1}{dt}\right).$$

Now, (6) gives:

$$a_1 = \frac{\partial x'}{\partial q_1}, \qquad b_1 = \frac{\partial y'}{\partial q_1}, \qquad c_1 = \frac{\partial z'}{\partial q_1},$$

and since $2T = \sum m(x'^2 + y'^2 + z'^2)$, the first term in (7) can be written $\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right)$, which is nothing but the first term in the **Lagrange** equation. The difference between the second term of the **Lagrange** equation:

$$-\frac{\partial T}{\partial q_1} = -\sum m\left(x'\frac{\partial x'}{\partial q_1} + y'\frac{\partial y'}{\partial q_1} + z'\frac{\partial z'}{\partial q_1}\right)$$

and the second term in the expression (7) is:

$$R_1 = \sum m\left[x'\left(\frac{da_1}{dt} - \frac{\partial x'}{\partial q_1}\right) + y'\left(\frac{db_1}{dt} - \frac{\partial y'}{\partial q_1}\right) + z'\left(\frac{dc_1}{dt} - \frac{\partial z'}{\partial q_1}\right)\right]. \tag{8}$$

Now:

$$\frac{da_1}{dt} = \frac{\partial a_1}{\partial q_1} q_1' + \frac{\partial a_1}{\partial q_2} q_2' + \cdots + \frac{\partial a_p}{\partial q_p} q_p' + \frac{\partial a}{\partial t}$$

and

$$\frac{\partial x'}{\partial q_1} = \frac{\partial a_1}{\partial q_1} q_1' + \frac{\partial a_2}{\partial q_1} q_1' + \cdots + \frac{\partial a_p}{\partial q_1} q_1' + \frac{\partial a}{\partial q_1}.$$

If one takes those values into account then the coefficient of x' in (8) will be:

$$\left(\frac{\partial a_1}{\partial q_2} - \frac{\partial a_2}{\partial q_1}\right) q_2' + \left(\frac{\partial a_1}{\partial q_3} - \frac{\partial a_3}{\partial q_1}\right) q_3' + \cdots + \left(\frac{\partial a_1}{\partial q_p} - \frac{\partial a_p}{\partial q_1}\right) + \left(\frac{\partial a_1}{\partial t} - \frac{\partial a}{\partial q_1}\right). \tag{9}$$

The coefficients of y' and z' have analogous forms.

In order for the **Lagrange** equation to apply to the parameter q_1, it is necessary that (9) and the other two analogous expressions must be identically zero, which it not generally true. (*See:* **Appell**, *Les mouvements de roulement en Dynamique*, pp. 41.)

The theorem is then proved.

5. Remarks. –

I. No matter what the constraints might be, one can write the left-hand sides of the **Lagrange** equations, on the condition that one must add expressions like R_1 to their left-hand sides. In any case, one will then have:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) - \frac{\partial T}{\partial q_1} = Q_1 + R_1 . \tag{10}$$

We shall point out (no. **7**) a differential equation that R_1 must satisfy.

II. If the expressions (4) are integrable then one will see immediately that $R_1 = 0$, and that equations (10) reduce to those of **Lagrange**.

III. First write out the terms $a_1\ \delta q_1$, $b_1\ \delta q_1$, $c_1\,\delta q_1$ in the right-hand sides of formulas (4) and (6). Define two parts of the terms in terms of δq_2, δq_3, …, q_2', q_3', …, and the term α: The first one contains q_1, while second one does not. Therefore, set:

$$a_2 = \alpha_2 + A_2\,, \quad a_3 = \alpha_3 + A_3\,, \quad \ldots, \qquad b_2 = \beta_2 + B_2\,, \quad \ldots, \qquad c_2 = \gamma_2 + C_2\,, \quad \ldots,$$

in which the Greek symbols denote quantities that include the parameter q_1, while the upper-case Latin characters do not. Formulas (4) and (6) can be written:

$$(4') \qquad \begin{cases} \delta x = (a_1\,\delta q_1 + \alpha_2\,\delta q_2 + \cdots + \alpha_p\,\delta q_p) + (A_2\,\delta q_2 + \cdots + A_p\,\delta q_p), \\ \delta y = (b_1\,\delta q_1 + \beta_2\,\delta q_2 + \cdots + \beta_p\,\delta q_p) + (B_2\,\delta q_2 + \cdots + B_p\,\delta q_p), \\ \delta z = (c_1\,\delta q_1 + \gamma_2\,\delta q_2 + \cdots + \gamma_p\,\delta q_p) + (C_2\,\delta q_2 + \cdots + C_p\,\delta q_p), \end{cases}$$

$$(6') \qquad \begin{cases} x' = (a_1\,q_1' + \alpha_2\,q_2' + \cdots + \alpha_p\,q_p' + \alpha) + (A_2\,q_2' + \cdots + A_p\,q_p' + A), \\ y' = (b_1\,q_1' + \beta_2\,q_2' + \cdots + \beta_p\,q_p' + \beta) + (B_2\,q_2' + \cdots + B_p\,q_p' + B), \\ z' = (c_1\,q_1' + \gamma_2\,q_2' + \cdots + \gamma_p\,q_p' + \gamma) + (C_2\,q_2' + \cdots + C_p\,q_p' + C). \end{cases}$$

Let us prove:

If the first parentheses in the right-hand sides constitute exact differentials then the ordinary ***Lagrange*** *formulas will be applicable to the parameter* q_1 .

By hypothesis, one has $\frac{\partial a_1}{\partial q_2} = \frac{\partial \alpha_2}{\partial q_1}$, and since $\frac{\partial A_2}{\partial q_1} = 0$, one can write:

$$\frac{\partial a_1}{\partial q_2} = \frac{\partial(\alpha_2 + A_2)}{\partial q_1} \qquad \text{or} \qquad \frac{\partial a_1}{\partial q_2} = \frac{\partial a_2}{\partial q_1}.$$

One has the same equalities for the other $a, b, c, \ldots$, so $R_1 = 0$. **Appell** ("Les mouvements de roulements en Dynamique," pp. 44) gave another proof of that theorem and reproduced the application that **Ferrers** made (1872) to the construction of the equations of motion of the hoop.

Note. – It is pointless to show that the preceding theorem will always apply when the coefficients in (4) or (6), other than a_1, do not contain q_1, and if a_1 does not contain $q_1, q_2, \ldots, q_p, t$.

IV. It might happen that some of equations (2) are integrable. Let us assume that s of those equations are found in that case. Upon integration, we will obtain s equations that couple the parameters, which will make the number of independent parameters drop from n to $n - 1$.

V. If equations (2) are not immediately integrable then one can demand to know if they do not admit an integration. That question was examined, in part, by **Hamel** in his paper "Die **Lagrange-Euler**'schen Gleichungen der Mechanik," Leipzig, Teubner, 1902.

6. Appell equations. – We shall transform equation (10) in such a manner as to make the quantity R_1 disappear, which is too complicated to define using formula (8).

One has:

$$\frac{\partial T}{\partial q_1'} = \sum m \left(x' \frac{\partial x'}{\partial q_1'} + y' \frac{\partial y'}{\partial q_1'} + z' \frac{\partial z'}{\partial q_1'} \right) ,$$

or, in view of (6):

$$\frac{\partial T}{\partial q_1'} = \sum m \left(a_1 x' + b_1 y' + c_1 z' \right) .$$

As a result:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'} \right) = \sum m \left(a_1 x'' + b_1 y'' + c_1 z'' + x' \frac{da_1}{dt} + y' \frac{db_1}{dt} + z' \frac{dc_1}{dt} \right) .$$

With **Appell**, we set:

$$S = \tfrac{1}{2} \sum m \left(x''^2 + y''^2 + z''^2 \right) ,$$

and we can then write:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'} \right) = \frac{\partial S}{\partial q_1''} + \sum m \left(x' \frac{da_1}{dt} + y' \frac{db_1}{dt} + z' \frac{dc_1}{dt} \right) . \tag{11}$$

Moreover:

$$\frac{\partial T}{\partial q_1} = \sum m \left(x' \frac{\partial x'}{\partial q_1} + y' \frac{\partial y'}{\partial q_1} + z' \frac{\partial z'}{\partial q_1} \right) . \tag{12}$$

If we subtract (12) from (11) then we will duplicate the left-hand side of equation (10), which will become:

$$\frac{\partial S}{\partial q_1''} + \sum m \left[x' \left(\frac{da_1}{dt} - \frac{\partial x'}{\partial q_1} \right) + y' \left(\frac{db_1}{dt} - \frac{\partial y'}{\partial q_1} \right) + z' \left(\frac{dc_1}{dt} - \frac{\partial z'}{\partial q_1} \right) \right] = Q_1 + R_1 .$$

Now, from (8), the second term on the left-hand side is nothing but R_1. One will then have:

$$\frac{\partial S}{\partial q_1''} = Q_1 , \tag{13}$$

and analogous equations that relate to the other independent parameters. Those equations are due to **Appell**, who gave another proof in the Comptes rendus de l’Académie des Sciences de Paris (1899).

Appell started from **d’Alembert**’s principle, instead of transforming the generalized **Lagrange** equation (10), as we did.

7. Remark. –

I. When one follows the procedure that led to equations (13), the following viewpoints would seem advantageous:

1. Equations (13) present themselves quite naturally as a generalization of the **Lagrange** equations.

2. One immediately sees the equivalence of equations (10) and (13). Indeed, one will recover the latter upon performing the transformations on (13) that are inverse to the ones that one made on (10).

3. In addition, one sees that equations (13) are identical to the ordinary **Lagrange** equations for holonomic systems

4. Finally, following the procedure will allow one to easily show the equivalence of equations (13) and the ones that were obtained by the authors **Kortweg** and **Vierkandt**, among others, and we shall say a word about that shortly.

II. If one subtracts (13) from (10) then one will get:

$$R_1 = \frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) - \frac{\partial T}{\partial q_1} - \frac{\partial S}{\partial q_1''}, \qquad R_2 = \ldots \tag{14}$$

Appell established the same equation (Journal de **Jordan**, 1901) and made some relatively interesting remarks in regard to it.

The quantities R_1, R_2 enter into the generalization of **Jacobi**'s theorem that we shall give later on.

8. The Korteweg equations (Nieuw Archief, 1899). – We shall give a simple exposition of the **Korteweg** procedure. Suppose for the moment that we abstract from the constraints (2) by replacing them with some reaction forces that we add to the applied forces. The body is then made free, and we can apply the **Lagrange** equations to it.

We will then write:

$$\left\{\begin{aligned} &\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) - \frac{\partial T}{\partial q_1} = Q_1 + R_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\frac{d}{dt}\left(\frac{\partial T}{\partial q_n'}\right) - \frac{\partial T}{\partial q_n} = Q_n + R_n, \end{aligned}\right. \tag{14}$$

in which $R_1, R_2, \ldots, R_n$ are the components of the reaction forces.

Imagine a displacement $\delta q_1, \delta q_2, \ldots, \delta q_n$ that is compatible with the constraints and suppose that this displacement can be performed in such a manner that the work done by reaction forces is zero (which will be true for rolling motions in particular), i.e.:

$$R_1\, \delta q_1 + R_2\, \delta q_2 + \ldots + R_n\, \delta q_n = 0\,. \tag{15}$$

By hypothesis, equations (2) express *all* of the constraints to which the system is subject. (15) must then a consequence of (2). In other words, the values of the k displacements $\delta q_{p+1}, \ldots, \delta q_n$ that are deduced from (2) as functions of $\delta q_1, \ldots, \delta q_p$ must satisfy (15) for any $\delta q_1, \delta q_2, \ldots, \delta q_p$, which demands that the determinant:

$$\begin{vmatrix} R_1\,\delta q_1 + R_2\,\delta q_2 + \cdots + R_p\,\delta q_p & R_{p+1} & \cdots & R_n \\ A_1\,\delta q_1 + A_2\,\delta q_2 + \cdots + A_p\,\delta q_p & A_{p+1} & \cdots & A_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \cdots & \cdots & \cdots \\ L_1\,\delta q_1 + L_2\,\delta q_2 + \cdots + L_p\,\delta q_p & L_{p+1} & \cdots & L_n \end{vmatrix}$$

must be identically zero. That condition will produce the p equations:

$$\begin{vmatrix} R_1 & R_{p+1} & \cdots & R_n \\ A_1 & A_{p+1} & \cdots & A_n \\ \cdots & \cdots & \cdots & \cdots \\ L_1 & L_{p+1} & \cdots & L_n \end{vmatrix} = 0\,, \qquad \begin{vmatrix} R_2 & R_{p+1} & \cdots & R_n \\ A_2 & A_{p+1} & \cdots & A_n \\ \cdots & \cdots & \cdots & \cdots \\ L_2 & L_{p+1} & \cdots & L_n \end{vmatrix} = 0\,, \quad \ldots \tag{16}$$

If one eliminates the quantities $R_1, R_2, \ldots, R_n$ from equations (16) by means of (14) then one will get the equations of motion.

Remark. – The starting point for **Korteweg** is the same as the one that permitted us to recover the **Appell** equations.

The latter equations are superior to those of **Korteweg**. It is, moreover, possible to show the equivalence of the **Korteweg** equations and those of **Appell** by means of very lengthy calculations.

9. The Vierkandt method. – It is nothing but the method of **Lagrange** multipliers.

Upon substituting the values of x, y, z and $\delta x, \delta y, \delta z$ that are deduced from (1) in the **d'Alembert** equations, it will take the form:

$$\sum \left[\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) - \frac{\partial T}{\partial q_1} - Q_i\right]\delta q_i = 0 \qquad (i = 1, 2, \ldots, n), \tag{17}$$

or

(17) $$\sum (P_i - Q_i)\,\delta q_i = 0 ,$$

to abbreviate.

Suppose that the δq are coupled by the k equations (2).

We add those equations to equations (17) after multiplying the former by the quantities λ_1, λ_2, …, λ_k , respectively (which are undetermined for now). We will get:

(18) $$\left\{\begin{array}{l} (P_1 - Q_1 - \lambda_1 A_2 - \lambda_2 A_1 - \cdots - \lambda_k L_1)\,\delta q_1 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ +(P_n - Q_n - \lambda_1 A_n - \lambda_n A_1 - \cdots - \lambda_k L_n)\,\delta q_n \quad = 0. \end{array}\right.$$

As always, we take the independent parameters to be $q_1, q_2, \dots, q_p$ ($p = n - k$) .

We choose the multipliers λ in such a manner that the coefficients of the k dependent displacements are zero. We will then get the k equations:

(19) $$\left\{\begin{array}{l} P_{p+1} - Q_{p+1} - \lambda_1 A_{p+1} - \cdots - \lambda_k L_{p+1} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ P_n \quad - Q_n \quad - \lambda_1 A_n \quad - \cdots - \lambda_k L_n \quad = 0 . \end{array}\right.$$

Equations (18) will become:

(20) $$(P_1 - Q_1 - \lambda_1 A_2 - \cdots - \lambda_k L_1)\,\delta q_1 + \cdots + (P_p - Q_p - \lambda_1 A_p - \cdots - \lambda_k L_n)\,\delta q_p = 0 .$$

The δq that figure in the latter equation are independent, so their coefficients must be zero, which will produce p new equations, and when they are combined with the system (19) and the system (2), that will define a system of $n + k$ equations for the $n + k$ unknowns $q_1, q_2, \dots, q_p$; $\lambda_1, \lambda_2, \dots, \lambda_k$.

Remark. – That procedure is subject to the same criticisms as that of **Korteweg**, which is not fundamentally different from it. Indeed, since (15) is a consequence of (2), one can write:

$$R_1\,\delta q_1 + R_2\,\delta q_2 + \dots + R_n\,\delta q_n$$

$$\equiv \lambda_1 (A_1\,\delta q_1 + \dots + A_n\,\delta q_n) + \lambda_2 (B_1\,\delta q_1 + \dots) + \dots + \lambda_k (L_1\,\delta q_1 + \dots + L_n\,\delta q_n) ,$$

which will demand that:

$$R_i = \lambda_1 A_i + \lambda_2 B_i + \dots + \lambda_k L_i \qquad (i = 1, 2, \dots, n).$$

10. On the improper applications that have been made of the non-generalized Lagrange equations. – Several esteemed authors have made improper applications of the non-generalized **Lagrange** equations, and above all, in the solution to the problem of rolling motion.

Here is how those authors proceed: They form the quantity $2T$ as a function of the n parameters q. They then eliminate k of the parameters from that expression by means of (2) and apply the non-generalized **Lagrange** formulas to the $n - k = p$ remaining parameters, but that would not be permissible, as was shown in no. **4**. [Of those authors, we cite: **Schouten**, "Over de rollende beweging van een Omwentelingslichaam op een vlak," Verslagen der Koninklijke Akademie, Amsterdam (1899); **Ernest Lindelöf**, "Mouvement de roulement d'un corps de revolution sur un plan"].

II. – The canonical equations.

11. – **Painlevé** showed (*Leçons sur l'intégration des équations de la Mécanique*, pp. 140, *et seq.*) how one can write the canonical equations of a *holonomic system* when the quantity $2T$ is not a quadratic form in q_1', q_2'.

We shall see how one must modify the usual canonical equations for *non-holonomic* systems.

The generalized **Lagrange** equation $\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) - \frac{\partial T}{\partial q_1} = Q_1 + R_1$, which has order two, is equivalent to the first-order system:

$$
(21) \qquad \left\{ \begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) - \frac{\partial T}{\partial q_1} &= Q_1 + R_1 , \\ q_1' &= \frac{dq_1}{dt} . \end{aligned} \right.
$$

Apply the **Poisson** transformation to the system (21), which consists of making $p_1 = \frac{\partial T}{\partial q_1'}$, $p_2 = \frac{\partial T}{\partial q_2'}$, etc., and introducing a function K that is defined by:

$$
(22) \qquad K = p_1 q_1' + p_2 q_2' + \cdots - T .
$$

Upon following a path that was mapped out in the classical treatise on mechanics, one will get the system:

$$(23) \qquad \begin{cases} \dfrac{dq_1}{dt} = \dfrac{\partial K}{\partial p_1}, & \dfrac{dp_1}{dt} + \dfrac{\partial K}{\partial q_1} = Q_1 + R_1, \\ \cdots & \cdots \\ \dfrac{dq_{n-k}}{dt} = \dfrac{\partial K}{\partial p_{n-k}}, & \dfrac{dp_{n-k}}{dt} + \dfrac{\partial K}{\partial q_{n-k}} = Q_{n-k} + R_{n-k}, \end{cases}$$

which are the most general canonical equations of motion.

12. – When there exists a force function U that is explicitly independent of time and the velocity of the points of the system (in other words, if U depends upon only the parameters $q_1, q_2, \ldots, q_p$), equations (23) will take the form:

$$(24) \qquad \begin{cases} \dfrac{dq_1}{dt} = \dfrac{\partial (K-U)}{\partial p_1}, & \dfrac{dp_1}{dt} + \dfrac{\partial (K-U)}{\partial q_1} = R_1, \\ \cdots & \cdots \end{cases}$$

13. – If, in addition to the conditions on U that were indicated above, the expressions for the coordinates of the various points of the system are explicitly independent of time then one will know that one then has $T = K$, and if one then sets $K - U = T - U = H$, as usual, then the system (24) can be written in the simpler form:

$$(25) \qquad \frac{dq_1}{dt} = \frac{\partial H}{\partial p_1}, \qquad \frac{dp_1}{dt} = -\frac{\partial H}{\partial p_1} = R\,.$$

III. – Jacobi's theorem.

14. – In this subsection, we shall assume that the coordinates x, y, z are expressed explicitly as functions of the parameters and time. We suppose only that U is explicitly independent of time, and we shall continue to denote the quantity $K - U$ by H. The canonical equations will then keep the form (25).

Can the solution of the system (25) depend upon the solution of the ordinary **Jacobi** equation:

$$(26) \qquad \frac{\partial V}{\partial t} + H\left(\frac{\partial V}{\partial q_1}, \frac{\partial V}{\partial q_2}, \frac{\partial V}{\partial q_3}, q_1, q_2, q_3, t\right) = 0$$

in the case of non-holonomic constraints?

(For easy of writing, we shall suppose that there are only three parameters q_1, q_2, q_3.)

If we verify the calculations as **Appell** did (*Mécanique*, tome II) then we will confirm that the values of dq / dt that are inferred by (25) are identical to the ones that (27) implies, but the same thing is not true for the values of the dp / dt. One must then pose the question: *It is possible to modify equation* (26) *in such a manner that* ***Jacobi****'s theorem remains applicable?*

We shall now examine that question.

15. – Let us try to determine a function φ of the parameters such that a solution $V(q_1, q_2, q_3, t, a_1, a_2, a_3)$ (a_1, a_2, a_3 are arbitrary constants) of the equation:

$$\frac{\partial V}{\partial t} + H\left(\frac{\partial V}{\partial q_1}, \frac{\partial V}{\partial q_2}, \frac{\partial V}{\partial q_3}, q_1, q_2, q_3, t\right) + \varphi = 0 \tag{26}$$

will imply the same values for the dq / dt and dp / dt that the canonical equations (25) do, when one sets:

$$\left\{\begin{aligned} &\frac{\partial V}{\partial a_1} = b_1, \quad \frac{\partial V}{\partial a_2} = b_2, \quad \frac{\partial V}{\partial a_3} = b_3, \\ &\frac{\partial V}{\partial q_1} = p_1, \quad \frac{\partial V}{\partial q_2} = p_2, \quad \frac{\partial V}{\partial q_3} = p_3. \end{aligned}\right. \tag{28}$$

First, let us find the form that φ must have in order for the dp / dt to have identical values in (25) and (28). The latter give:

$$\left\{\begin{aligned} &\frac{dp_1}{dt} = \frac{\partial^2 V}{\partial q_1\,\partial t} + \frac{\partial^2 V}{\partial q_1^2}\,q_1' + \frac{\partial^2 V}{\partial q_1\,\partial q_2}\,q_2' + \frac{\partial^2 V}{\partial q_1\,\partial q_3}\,q_3', \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right. \tag{24}$$

If we express the idea that V is a solution of (25) then we will get:

$$\frac{\partial^2 V}{\partial t\,\partial q_1} + \frac{\partial H}{\partial\left(\dfrac{\partial H}{\partial q_1}\right)}\frac{\partial^2 V}{\partial q_1^2}{}_1 + \frac{\partial H}{\partial\left(\dfrac{\partial H}{\partial q_2}\right)}\frac{\partial^2 V}{\partial q_1\,\partial q_2} + \frac{\partial H}{\partial\left(\dfrac{\partial H}{\partial q_3}\right)}\frac{\partial^2 V}{\partial q_1\,\partial q_3} + \frac{\partial H}{\partial q_1} + \frac{\partial \varphi}{\partial q_1} \equiv 0,$$

or, if one recalls (28):

$$\frac{\partial^2 V}{\partial q_1\,\partial t} + q_1'\frac{\partial^2 V}{\partial q_1^2} + q_2'\frac{\partial^2 V}{\partial q_1\,\partial q_2} + q_3'\frac{\partial^2 V}{\partial q_1\,\partial q_3} \equiv -\frac{\partial H}{\partial q_1} - \frac{\partial \varphi}{\partial q_1}. \tag{30}$$

The left-hand side of that equation is nothing but dp_1 / dt [form. (20)]. If we write out that the value is identical to the one that (25) implies then we will have:

$$-\frac{\partial H}{\partial q_1}+R_1\equiv-\frac{\partial H}{\partial q_1}-\frac{\partial\varphi}{\partial q_1}\qquad\text{or}\qquad -R_1=\frac{\partial\varphi}{\partial q_1}.$$

We will find analogous expressions for q_2, q_3 . We will then have the three conditions:

$$(31)\qquad\begin{cases}-R_1=\dfrac{\partial\varphi}{\partial q_1},\\[2mm] -R_2=\dfrac{\partial\varphi}{\partial q_2},\\[2mm] -R_3=\dfrac{\partial\varphi}{\partial q_3}.\end{cases}$$

If there then exists a function φ that admits the quantities $-R_1$, $-R_2$, $-R_3$, as partial derivatives with to q_1, q_2, q_3, resp., then the values of dp / dt that one infers from (25) and (28) will be identical. An easy calculation will show that if φ exists then the values of q' that one infers from the canonical system will also be identical to the ones that are deduced from (28). Therefore:

The generalized Jacobi theorem.

In order for the ***Jacobi*** *theorem to remain applicable to non-holonomic systems, it is necessary and sufficient that one can complete the ordinary* ***Jacobi*** *equation by means of a function φ that admits $-R_1$, $-R_2$, $-R_3$ as partial derivatives with respect to q_1, q_2, q_3, resp.*

16. – The function φ will exist if:

$$(\alpha)\qquad\frac{\partial R_1}{\partial q_2}=\frac{\partial R_2}{\partial q_1},\quad\frac{\partial R_1}{\partial q_3}=\frac{\partial R_3}{\partial q_1},\quad\frac{\partial R_2}{\partial q_3}=\frac{\partial R_3}{\partial q_2},$$

and since:

$$R_1=\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right)-\frac{\partial T}{\partial q_1}-\frac{\partial S}{\partial q_1''},\qquad R_2=\ldots,$$

the condition (α) can be written:

$$(b)\qquad\frac{\partial}{\partial q_2}\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right)-\frac{\partial^2 S}{\partial q_1''\partial q_2}=\frac{\partial}{\partial q_1}\frac{d}{dt}\left(\frac{\partial T}{\partial q_2'}\right)-\frac{\partial^2 S}{\partial q_2''\partial q_1},\qquad\ldots$$

One should remark that it suffices to calculate the first terms in the two sides of (β) in order for one to determine the terms in T that contain the products $q_2\,q_1'$, $q_1\,q_2'$, and in order to calculate the second ones, it will suffice to determine the terms in S that contain the products $q_1''q_2$, $q_2''q_1$.

17. Particular case. – If the quantities $-R_1$, $-R_2$, $-R_3$ are independent of q_1, q_2, q_3 then the first of equations (25) can be written:

$$\frac{dq_1}{dt} = \frac{\partial (H-\varphi)}{\partial p_1} ,$$

and the last one can be written:

$$\frac{dp_1}{dt} = -\frac{\partial (H-\varphi)}{\partial q_1} ,$$

on the condition that one must take φ to be the function that was defined in no. **16**, but with its sign changed.

In that case, if one takes H to have the expression:

$$H = K - U - \varphi$$

then the generalizes **Jacobi** equation will have the ordinary form:

$$\frac{\partial V}{\partial t} + H\left(\frac{\partial V}{\partial q_1}, \frac{\partial V}{\partial q_2}, \frac{\partial V}{\partial q_3}, q_1, q_2, q_3, t\right) + \varphi = 0 .$$

18. – Certain authors have made an incorrect use of the non-generalized **Jacobi** theorem. We shall cite only **Ezio Crescini** ["Sul moto di una sfera che rotola su di un piano fisso," Rend. Acc. Lincei **5**, 1st sem. (1889), 204-209], who applied that theorem in number **3** of his study.

Halle Gate (Brussels), June 1906.

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