"Das Princip der kleinsten Arbeit der verlorenen Kräfte als ein allgemeines Princip der Mechanik," Zeit. Math. und Phys. **24** (1879), 206-220.

# The principle of least work done by lost forces as a general principle of mechanics

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See Tab. III, Fig. 1 (†)

Translated by D. H. Delphenich

**§ 1.** – Before we begin the task of setting down what we understand the principle of least work done by lost forces to mean and deriving the equations of motion for a system of material points from it, we would like to prove a theorem from pure analysis.

Let:

$$X\,\delta x + Y\,\delta y + Z\,\delta z + \ldots \tag{1}$$

be a function that is homogeneous and linear in the infinitely small quantities $\delta x, \delta y, \delta z, \ldots$, where $X, Y, Z, \ldots$ are functions of $x, y, z, \ldots$ In addition, let:

$$A\,\delta x + B\,\delta y + C\,\delta z + \ldots, \qquad A_1\,\delta x + B_1\,\delta y + C_1\,\delta z + \ldots, \tag{2}$$

and

$$U\,\delta x + V\,\delta y + W\,\delta z + \ldots, \qquad U_1\,\delta x + V_1\,\delta y + W_1\,\delta z + \ldots \tag{3}$$

be given functions of $\delta x, \delta y, \delta z, \ldots$, in which the coefficients of $\delta x, \delta y, \delta z, \ldots$ are functions of $x, y, z, \ldots$ .

(†) Translator: The table and figure were not available to me at the time of translation.

We would like to see what conditions the coefficients $X, Y, Z, \ldots$ of the functions (1) and the coefficients $A, B, \ldots, A_1, B_1, \ldots, U, V, \ldots, U_1, V_1, \ldots$ of the functions (2) and (3) must fulfill in order for the function (1) to not assume any positive values for those infinitely small $\delta x, \delta y, \delta z, \ldots$ that make the functions (2) positive or equal to zero and the function (3) equal to zero.

If we assume that the number of functions (2) and (3) is smaller than that of the variables $x, y, z, \ldots$ then we shall add just as many arbitrary homogeneous linear functions $\delta\Omega, \delta\Omega_1, \ldots$ of the $\delta x, \delta y, \delta z, \ldots$ as it takes to make the number of functions (2) and (3), along with the arbitrary $\delta\Omega, \delta\Omega_1, \ldots$, equal to the number of infinitely small $\delta x, \delta y, \delta z, \ldots$

Since the functions $\delta\Omega, \delta\Omega_1, \ldots$ are arbitrary and different from (2) and (3), they can be either positive, negative, or zero for those values of $\delta x, \delta y, \delta z, \ldots$ that make the functions (2) positive or equal to zero, and the functions (3) equal to zero.

No matter what the coefficients of the functions (1), (2), and (3) might also be, we can always set:

$$
\begin{aligned}
X\,\delta x + Y\,\delta y + Z\,\delta z + \ldots &+ \lambda\,(A\,\delta x + B\,\delta y + C\,\delta z + \ldots) \\
&+ \lambda_1\,(A_1\,\delta x + B_1\,\delta y + C_1\,\delta z + \ldots) + \ldots \\
&+ \mu\,(U\,\delta x + V\,\delta y + W\,\delta z + \ldots) \\
&+ \lambda_1\,(U_1\,\delta x + V_1\,\delta y + W_1\,\delta z + \ldots) + \ldots \\
&+ \omega\,\delta\Omega + \omega_1\,\delta\Omega_1 + \ldots = 0
\end{aligned} \tag{4}
$$

for all arbitrary values of $\delta x, \delta y, \delta z, \ldots$, in which we understand $\lambda, \lambda_1, \ldots, \mu, \mu_1, \ldots, \omega, \omega_1, \ldots$ to mean undetermined quantities. In fact, since $\delta x, \delta y, \delta z, \ldots$ are always completely arbitrary and $\lambda, \lambda_1, \ldots, \mu, \mu_1, \ldots, \omega, \omega_1, \ldots$ remain undetermined, we will have to set the coefficients that enter into equation (4) next to the latter equal to zero. In that way, one will get just as many equations as quantities $\lambda, \lambda_1, \ldots, \mu, \mu_1, \ldots, \omega, \omega_1, \ldots$, and from those equations, we can ascertain the values of $\lambda, \lambda_1, \ldots, \mu, \mu_1, \ldots, \omega, \omega_1, \ldots$ that equation (4) makes possible for all arbitrary $\delta x, \delta y, \delta z, \ldots$

We would now like to present equation (4) in the form:

$$
\begin{aligned}
X\,\delta x + Y\,\delta y + \ldots &= -\lambda\,(A\,\delta x + B\,\delta y + \ldots) - \lambda_1\,(A_1\,\delta x + B_1\,\delta y + \ldots) + \ldots \\
&\quad - \mu\,(U\,\delta x + V\,\delta y + \ldots) - \mu_1\,(U_1\,\delta x + V_1\,\delta y + \ldots) + \ldots \\
&\quad - \omega\,\delta\Omega - \omega_1\,\delta\Omega_1 - \ldots
\end{aligned} \tag{5}
$$

In order for the first part of this equation to not assume positive values for any values of the finitely small quantities that make the functions (2) positive or zero and the functions (3) equal to zero, one must be able to determine the quantities $\delta x, \delta y, \delta z, \ldots$ as being equal to zero. Otherwise, one would be able to choose only those values for the infinitely small $\delta x, \delta y, \delta z, \ldots$ that make the functions (2) equal to zero, and the function (1) would be equal to a sum of arbitrary functions $\delta\Omega, \delta\Omega_1, \ldots$ that could also prove to be positive, but the latter consequence would contradict the basic assumption. As a result, one must have:

$$
\begin{aligned}
X\,\delta x + Y\,\delta y + \ldots &= -\lambda\,(A\,\delta x + B\,\delta y + \ldots) \\
&\quad - \lambda_1\,(A_1\,\delta x + B_1\,\delta y + \ldots) + \ldots
\end{aligned} \tag{6}
$$

$$-\mu\,(U\ \ \delta x + V\ \ \delta y + \ldots)$$
$$-\mu_1\,(U_1\ \delta x + V_1\ \delta y + \ldots) + \ldots = 0$$

for all arbitrary values of $\delta x$, $\delta y$, $\delta z$, …, or what amounts to the same thing:

$$(7)\qquad \left\{\begin{array}{l} X + \lambda A + \lambda_1 A_1 + \cdots + \mu U + \mu_1 U_1 + \cdots = 0, \\ Y + \lambda B + \lambda_1 B_1 + \cdots + \mu V + \mu_1 V_1 + \cdots = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{array}\right.$$

which are the equations into which the expression (6) decomposes. Furthermore, it is clear that the coefficients $\lambda$, $\lambda_1$, …, must be determined to be positive from equations (7), in order to make the first part of equation (5) assume no positive values for those values of the infinitely quantities that make the functions (2) positive or equal to zero and functions (3) equal zero. Otherwise, the first part of equation (5) would prove to have negative values $\lambda$ if one were to one choose values for $\delta x$, $\delta y$, $\delta z$, … that made the function:

$$A\ \delta x + B\ \delta y + C\ \delta z + \ldots$$

positive and the other functions (2) and (3) equal to zero.

We have then found the following conditions that must be fulfilled in order for the function (1) to assume no positive values for those values of $\delta x$, $\delta y$, $\delta z$, … that make the functions (2) positive or equal to zero and the functions (3) equal to zero.

1. Equation (6) must be valid for all arbitrary values of $\delta x$, $\delta y$, $\delta z$, …; i.e., when the function (1) is added to the functions (2) and (3), each of which is multiplied by an undetermined factor, that sum must be equal to zero for all arbitrary values of $\delta x$, $\delta y$, $\delta z$, …

2. The factors $\lambda$, $\lambda_1$, … must be positive. That condition is not only *necessary*, but also *sufficient*, since the function (1) will assume no positive values when it is fulfilled for all values of $\delta x$, $\delta y$, $\delta z$, … that make the functions (2) positive or equal to zero and the functions (3) equal to zero.

Since $\delta x$, $\delta y$, $\delta z$, … are arbitrary, their coefficients in equation (6) must be set equal to zero. One will then get just as many equations (7) from that as there are infinitesimals $\delta x$, $\delta y$, $\delta z$, ... If one eliminates the undetermined factors $\lambda$, $\lambda_1$, … $\mu$, $\mu_1$, … from them then one will get the expressions that represent the stated mutual dependency of the coefficients of the functions (1), (2), and (3) upon each other.

One must combine the equations that are obtained in that way with the inequalities that should express the idea that the factors that are ascertained from equations (7) must always remain positive, no matter what numerical values they might also assume.

**§ 2.** – We shall now go on to the explanation of what we understand the principle of least work done by lost forces to mean and the derivation of the equations of motion for a system of material points from it. In order to be able to ascertain the motion of a system of material points, one must first determine the latter; i.e., one must show what displacements are possible for the system considered. The forces that act upon the material points must then be given. It is known that a system of material points is determined analytically by means of the following conditions: Displacements will be possible that make certain linear functions of the displacements equal to zero or equal to zero and positive. In the latter case, the aforementioned functions will change signs only when one goes from possible displacements to impossible ones. The system whose motion is to be ascertained consists of $n$ material points whose masses are $m_1, m_2, \ldots, m_i, \ldots, m_n$, and whose coordinates at the end of time $t$ are:

$$(x_1, y_1, z_1), \qquad (x_2, y_2, z_2), \quad \ldots, \quad (x_i, y_i, z_i), \quad \ldots, \quad (x_n, y_n, z_n) .$$

Let those displacements of the system:

$$(x_1, y_1, z_1), \qquad (x_2, y_2, z_2), \quad \ldots, \quad (x_i, y_i, z_i), \quad \ldots, \quad (x_n, y_n, z_n)$$

be possible that make the linear functions of them:

$$(8) \qquad \left\{ \begin{array}{l} \sum \left\{ \dfrac{\partial L'}{\partial x_i} \delta x_i + \dfrac{\partial L'}{\partial y_i} \delta y_i + \dfrac{\partial L'}{\partial z_i} \delta z_i \right\} + \dfrac{\partial L'}{\partial t} \delta t, \\ \sum \left\{ \dfrac{\partial L''}{\partial x_i} \delta x_i + \dfrac{\partial L''}{\partial y_i} \delta y_i + \dfrac{\partial L''}{\partial z_i} \delta z_i \right\} + \dfrac{\partial L''}{\partial t} \delta t, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{array} \right.$$

positive or equal to zero, and make the functions:

$$(9) \qquad \left\{ \begin{array}{l} \sum \left\{ \dfrac{\partial M'}{\partial x_i} \delta x_i + \dfrac{\partial M'}{\partial y_i} \delta y_i + \dfrac{\partial M'}{\partial z_i} \delta z_i \right\} + \dfrac{\partial M'}{\partial t} \delta t, \\ \sum \left\{ \dfrac{\partial M''}{\partial x_i} \delta x_i + \dfrac{\partial M''}{\partial y_i} \delta y_i + \dfrac{\partial M''}{\partial z_i} \delta z_i \right\} + \dfrac{\partial M''}{\partial t} \delta t, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{array} \right.$$

equal to zero. In the cited expression, $x_i$, $y_i$, $z_i$ mean the coordinates of a point $m_i$ of the system, and the $i$ are assigned all whole number values from 1 to $n$.

We further assume that forces $F_1, F_2, \ldots, F_i, \ldots, F_n$ act upon the material points of the system that is determined by means of the conditions on the possible displacements (8) and (9):

$$(X_1, Y_1, Z_1), \quad (X_2, Y_2, Z_2), \quad \ldots, \quad (X_i, Y_i, Z_i), \quad \ldots, \quad (X_n, Y_n, Z_n)\,.$$

If one knows the conditions on the possible displacements of a system and the forces that act upon its material points then one will find the conditions (equations) on an actual displacement of the system during an infinitely small time interval $\partial t$, and in that way, ascertain the motion of the system. We would like to say that it is characteristic of a differential that it represents those changes in the coordinates that take place as a result of actual displacements of the material point.

Since the actual displacements:

$$(\partial x_1, \partial y_1, \partial z_1), \quad (\partial x_2, \partial y_2, \partial z_2), \quad \ldots, \quad (\partial x_i, \partial y_i, \partial z_i), \quad \ldots, \quad (\partial x_n, \partial y_n, \partial z_n)$$

of the material points of a system will also belong to its possible displacements then, they must make the functions (8) and (9) equal to zero in the case where the constraints to which those functions refer actual exist as such. The constraints to which the functions (8) refer exclude one part of space for the masses that move in it and leave them free to move in the other part. For the former part of space, the functions (8) will be negative, while they will be positive for the latter. The constraint only comes into effect when it obstructs the transition of a material point from one part of space to the other. As a result, the actual displacements of the system will proceed to the boundary that separates the two spaces from each other, since otherwise the displacements would be independent of the constraints. Now, as far as the functions (9) are concerned, it is clear that the actual displacements that are possible in them are the ones that make the functions (9) equal to zero. We conclude from this that an actual displacement that makes the functions (8) and (9) equal to zero can be determined by means of the following equations:

$$(10) \qquad \left\{ \begin{array}{l} \sum \left\{ \dfrac{\partial L'}{\partial x_i}\delta x_i + \dfrac{\partial L'}{\partial y_i}\delta y_i + \dfrac{\partial L'}{\partial z_i}\delta z_i \right\} + \dfrac{\partial L'}{\partial t}\delta t, \\ \sum \left\{ \dfrac{\partial L''}{\partial x_i}\delta x_i + \dfrac{\partial L''}{\partial y_i}\delta y_i + \dfrac{\partial L''}{\partial z_i}\delta z_i \right\} + \dfrac{\partial L''}{\partial t}\delta t, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{array} \right.$$

$$(11) \qquad \left\{ \begin{array}{l} \sum \left\{ \dfrac{\partial M'}{\partial x_i}\delta x_i + \dfrac{\partial M'}{\partial y_i}\delta y_i + \dfrac{\partial M'}{\partial z_i}\delta z_i \right\} + \dfrac{\partial M'}{\partial t}\delta t, \\ \sum \left\{ \dfrac{\partial M''}{\partial x_i}\delta x_i + \dfrac{\partial M''}{\partial y_i}\delta y_i + \dfrac{\partial M''}{\partial z_i}\delta z_i \right\} + \dfrac{\partial M''}{\partial t}\delta t, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array} \right.$$

**§ 3.** – Since the number of equations (10) and (11) is always smaller than the number of coordinates that determine the position of the material points of a system, the actual displacements must be determined by some other conditions.

Let $O_i$ be the position of a material point of the system at the end of time $t$, let $\overline{O_iA_i}$ represent the direction of the velocity $v_i$ that the point $m_i$ has at the end of time $t$ . $\overline{O_iA_i}$ will then likewise represent the displacement that the point $m_i$ would have as a result of the velocity that it has acquired during the infinitely small time interval $\partial t$ .

Let $\overline{O_iB_i}$ be the displacement that the point $m_i$ would experience during the time $\partial t$ under the action of the force $F_i$ if it did not acquire any velocity and is entirely free. The diagonal line $\overline{O_iC_i}$ of the parallelogram that is constructed from $\overline{O_iA_i}$ and $\overline{O_iB_i}$ would represent the displacement of the material point $m_i$ in the event that the latter were free at time $t$ and moved with the previously acquired velocity $v_i$ under the action of the force $F_i$ . However, since the point in question is not free, it cannot displace along $\overline{O_iC_i}$ and complete an actual displacement $\overline{O_iE_i}$ . We would like to decompose the force $F_i$ that acts along the direction $\overline{A_iC_i}$ into two others that act along the directions $\overline{A_iE_i}$ and $\overline{E_iC_i}$ , respectively, and denote them by $J_i$ and $P_i$ , resp. The first of those forces will produce a displacement $A_i E_i$ that has the actual displacement of the material point as a consequence when it is coupled with the displacement $O_i A_i$ that takes place as a result of the acquired velocity as if that point moved freely. If one carries out that decomposition for all points of the system then one will see that the forces $J_1$ , $J_2$ , …, $J_i$ , …, $J_n$ will produce *actual* displacements as if each of the points were free.

As far as the forces $F_1$ , $F_2$ , …, $F_i$ , …, $F_n$ are concerned, *they cannot seek to generate a displacement that will have a possible displacement as a consequence when it is coupled with an actual one.* The latter condition serves to exhibit the missing equations in the actual displacements. However, we would first like to focus our attention more upon those displacements that *do give possible displacements when they are coupled with the actual ones.* Let:

$$(\Delta x_1 , \Delta y_1 , \Delta z_1), \quad (\Delta x_2 , \Delta y_2 , \Delta z_2), \quad \ldots, (\Delta x_i , \Delta y_i , \Delta z_i), \quad \ldots, (\Delta x_n , \Delta y_n , \Delta z_n) \tag{12}$$

be an arbitrary displacement of a system. If it is coupled with the actual displacement then that will yield the displacements:

$$\begin{gathered}(\Delta x_1 + \partial x_1 , \Delta y_1 + \partial y_1 , \Delta z_1 + \partial z_1), \quad (\Delta x_2 + \partial x_2 , \Delta y_2 + \partial y_2 , \Delta z_2 + \partial z_2), \\ \ldots, (\Delta x_i + \partial x_i , \Delta y_i + \partial y_i , \Delta z_i + \partial z_i), \\ \ldots, (\Delta x_n + \partial x_n , \Delta y_n + \partial y_n , \Delta z_n + \partial z_n) .\end{gathered} \tag{13}$$

If the displacement (12), in conjunction with the actual one, has a possible displacement as a consequence then $\Delta x_i + \partial x_i$ , $\Delta y_i + \partial y_i$ , $\Delta z_i + \partial z_i$ must make the functions:

(14)
$$\left\{\begin{array}{l} \sum\left\{\frac{\partial L'}{\partial x_i}\delta x_i+\frac{\partial L'}{\partial y_i}\delta y_i+\frac{\partial L'}{\partial z_i}\delta z_i\right\} \\ +\sum\left\{\frac{\partial L'}{\partial x_i}\partial x_i+\frac{\partial L'}{\partial y_i}\partial y_i+\frac{\partial L'}{\partial z_i}\partial z_i\right\}+\frac{\partial L'}{\partial t}\cdot\partial t, \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right.$$

positive or equal to zero when they are substituted in the functions (8) and (9) in place of $\delta x_i$, $\delta y_i$, $\delta z_i$, and make the functions:

(15)
$$\left\{\begin{array}{l} \sum\left\{\frac{\partial M'}{\partial x_i}\delta x_i+\frac{\partial M'}{\partial y_i}\delta y_i+\frac{\partial M'}{\partial z_i}\delta z_i\right\} \\ +\sum\left\{\frac{\partial M'}{\partial x_i}\partial x_i+\frac{\partial M'}{\partial y_i}\partial y_i+\frac{\partial M'}{\partial z_i}\partial z_i\right\}+\frac{\partial M'}{\partial t}\cdot\partial t, \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right.$$

equal to zero.

However, as a result of equations (10) and (11), the expressions (14) and (15) will go to the following homogeneous linear functions:

(16)
$$\sum\left\{\frac{\partial L'}{\partial x_i}\Delta x_i+\frac{\partial L'}{\partial y_i}\Delta y_i+\frac{\partial L'}{\partial z_i}\Delta z_i\right\},\quad \sum\left\{\frac{\partial L''}{\partial x_i}\Delta x_i+\frac{\partial L''}{\partial y_i}\Delta y_i+\frac{\partial L''}{\partial z_i}\Delta z_i\right\},\dots$$

and

(17)
$$\sum\left\{\frac{\partial M'}{\partial x_i}\Delta x_i+\frac{\partial M'}{\partial y_i}\Delta y_i+\frac{\partial M'}{\partial z_i}\Delta z_i\right\},\quad \sum\left\{\frac{\partial M''}{\partial x_i}\Delta x_i+\frac{\partial M''}{\partial y_i}\Delta y_i+\frac{\partial M''}{\partial z_i}\Delta z_i\right\},\dots$$

It is clear from this that all displacements (12) that have *possible* displacements as a consequence in conjunction with actual ones will make the homogeneous linear functions (16) positive and equal to and the functions (17) equal to zero. One also sees that the expressions (16) and (17) represent changes in the functions $L'$, $L''$, …, $M'$, $M''$, … that are *independent* of the changes in time.

Let $\overline{E_i D_i}$ be the aforementioned displacement $\Delta s_i$ of a material point $m_i$ and let $\Delta x_i$, $\Delta y_i$, $\Delta z_i$ be its projections onto the coordinates. From the triangle $D_i\,E_i\,C_i$, one gets:

$$\overline{D_iC_i}^2=\overline{D_iE_i}^2+\overline{E_iC_i}^2-2\overline{D_iE_i}\cdot\overline{E_iC_i}\cdot\cos(\overline{D_iE_i},\overline{E_iC_i}),$$

in which $(\overline{D_iE_i}, \overline{E_iC_i})$ denotes the angle between the displacement and the direction of the lost force $P_i$ . If one multiplies those equations by $m_i$ and sums over all points of the system then one will get:

$$\sum m_i \cdot \overline{D_iC_i}^2 - \sum m_i \cdot \overline{E_iC_i}^2 = \sum m_i \cdot \overline{D_iE_i}^2 - 2\sum m_i \cdot \overline{D_iE_i} \cdot \overline{E_iC_i} \cdot \cos(\overline{D_iE_i}, \overline{E_iC_i}). \tag{18}$$

In the second term on the right-hand side of the last equation, $\overline{E_iC_i}$ denotes the displacement that the lost force $P_i$ would generate for the free motion of the material point in question.

However, that displacement will be determined by the equation:

$$\overline{E_iC_i} = \frac{P_i}{m_i} \cdot \frac{\partial t^2}{2}. \tag{19}$$

If $\overline{E_iC_i}$ were replaced with its value then one would get:

$$2\sum m_i \cdot \overline{D_iE_i} \cdot \overline{E_iC_i} \cdot \cos(\overline{D_iE_i}, \overline{E_iC_i}) = \partial t^2 \cdot \sum P_i \cdot \overline{D_iE_i} \cdot \cos(P_i, \overline{D_iE_i}).$$

However, since the lost forces $P_1, P_2, \ldots, P_i, \ldots, P_n$ cannot generate displacements that would have possible displacements as a consequence in conjunction with the actual ones, and since forces in general have no ambition to generate displacements relative to which the total moment can assume no positive values, we conclude that the total moment of the lost force:

$$\sum P_i \cdot \overline{D_iE_i} \cdot \cos(P_i, \overline{E_iC_i})$$

and as a result, the second term on the right-hand side of equations (18):

$$2\sum m_i \cdot \overline{D_iE_i} \cdot \overline{E_iC_i} \cdot \cos(\overline{D_iE_i}, \overline{E_iC_i})$$

cannot assume positive values and the right-hand sides of the aforementioned equations must always remain positive. In that way, one will come to the conclusion that during the motion of a system of material points, one will have:

$$\sum m_i \cdot \overline{E_iC_i}^2 < \sum m_i \cdot \overline{D_iC_i}^2.$$

That inequality includes the so-called **Gaussian** *principle of least constraint*, although **Gauss** only proved it in the case where the conditions on the system were independent of time. As **Gauss** said:

> "The new principle is the following one: The motion of a system of material points that are coupled with each other in some way, and whose motions are, at the same time, constrained by whatever sort of external restrictions, will take place at each moment with the greatest possible agreement with the free motion or under the smallest possible constraint, when one considers a measure of the constraint

> that the entire system suffers at each time point to be the sum of the products of the square of the deviation of each point from its free motion with its mass."
>
> (**Gauss**, *Werke*, Bd. V, pp. 26)

However, one can consider the inequality (20) from a different angle that expresses its mechanical meaning more precisely. Namely, the expression:

$$\sum m_i \cdot \overline{E_i C_i}^2 ,$$

which assumes its smallest value for the actual displacements, can be transformed into the form:

$$\frac{\partial t^2}{2} \cdot \sum P_i \cdot \overline{E_i C_i}$$

using equation (19), or also into:

$$\frac{\partial t^2}{2} \cdot \sum P_i \, p_i$$

when one sets $E_i \, C_i = p_i$ . However, since $P_i$ denotes the lost force and $p_i$ denotes the displacement that the force would impart to the material point under free motion, $P_i \, p_i$ will represent the work that the lost forces would do under free motion. The meaning of the expression (20) is clear from that: *The work that is done by lost forces under the motion of a system of material points will have its smallest value for free motion. The infinitely small increase in that work done will remain positive for any displacement that has a possible displacement as a consequence when it is combined with the actual one.*

One then sees that **Gauss**'s *principle of least constraint* can be called the *principle of least work done by lost forces*, which will bring one into much closer agreement with the present viewpoint on natural phenomena.

**§ 4.** – We would now like to derive the equations of the actual displacements from the principle of least work done by lost forces.

Let $\Delta \sum (P_i \, p_i)$ denote the change in the lost work that would be produced by the displacements that generate possible displacements when combined with the actual ones. From the principle of least work done by lost forces:

$$\Delta \sum (P_i \, p_i) \tag{21}$$

will always be positive then. If one replaces $P_i$ with its magnitude in equation (19) then one will get:

$$\Delta \sum (P_i \, p_i) = \frac{2}{\partial t^2} \cdot \Delta \sum m_i \, p_i^2 = \frac{2}{\partial t^2} \cdot \sum m_i \cdot \Delta p_i^2 .$$

The coordinates $x_i$ , $y_i$ , $z_i$ determine the position of the material point $m_i$ at the time $t$ . One lets $a_i$, $b_i$ , $c_i$ denote the coordinates of $m_i$ at time $t + \partial t$ in the case where the material point in question moves entirely freely during the time interval $\partial t$, and lets $\xi_i$ , $\psi_i$ , $\zeta_i$ denote the coordinates of that point for the case of the actual displacement of it that takes place. One will then have:

$$p_i^2 = (\xi_i - a_i)^2 + (\eta_i - b_i)^2 + (\zeta_i - c_i)^2 .$$

One lets $\Delta\xi_i$ , $\Delta\psi_i$ , $\Delta\zeta_i$ denote the projections onto the coordinate axes of $D_i E_i$ ; i.e., the changes that the coordinates $\xi_i$ , $\psi_i$ , $\zeta_i$ experience under those infinitely small displacements of the system that have a possible displacement as a consequence when they are combined with the actual one. If one draws a line $O_i G_i$ through the point $O_i$ whose coordinates are $x_i$ , $y_i$ , $z_i$ that is parallel equal to $E_i D_i$ then one will see that:

$$\Delta\xi_i = \Delta x_i , \qquad \Delta\eta_i = \Delta y_i , \qquad \Delta\zeta_i = \Delta z_i ,$$

$$\Delta p_i^2 = 2\{(\xi_i - a_i) \cdot \Delta x_i + (\eta_i - b_i) \cdot \Delta y_i + (\zeta_i - c_i) \cdot \Delta z_i \},$$

$$\Delta \sum (P_i\, p_i) = \frac{4}{\partial t^2} \sum m_i\{(\xi_i - a_i) \cdot \Delta x_i + (\eta_i - b_i) \cdot \Delta y_i + (\zeta_i - c_i) \cdot \Delta z_i\}. \tag{22}$$

From the *principle of the least work done by lost forces*, the right-hand side of the equation above must remain positive for any displacement that has a possible displacement as a consequence when it is combined with the actual one, or what amounts to the same thing, the function:

$$\frac{2}{\partial t^2} \sum m_i\{(a_i - \xi_i) \cdot \Delta x_i + (b_i - \eta_i) \cdot \Delta y_i + (c_i - \zeta_i) \cdot \Delta z_i\}, \tag{23}$$

which is homogeneous and linear in the displacements, must not assume any positive values for those displacements that make the functions (16) positive and equal to zero and the functions (17) equal to zero.

In order to fulfill those conditions, it will suffice that when the function (23) is added to the functions (16) and (17), each of which is multiplied by a suitable factor, that sum will remain equal to zero for all arbitrary displacements of the system:

$$\begin{aligned} &\frac{2}{\partial t^2} \sum m_i\{(a_i - \xi_i) \cdot \Delta x_i + (b_i - \eta_i) \cdot \Delta y_i + (c_i - \zeta_i) \cdot \Delta z_i\} \\ &\quad + \lambda' \sum \left\{ \frac{\partial L'}{\partial x_i} \cdot \Delta x_i + \frac{\partial L'}{\partial y_i} \cdot \Delta y_i + \frac{\partial L'}{\partial z_i} \cdot \Delta z_i \right\} + \cdots \\ &\quad + \mu' \sum \left\{ \frac{\partial M'}{\partial x_i} \cdot \Delta x_i + \frac{\partial M'}{\partial y_i} \cdot \Delta y_i + \frac{\partial M'}{\partial z_i} \cdot \Delta z_i \right\} + \cdots = 0, \end{aligned}$$

where $\lambda'$, $\lambda''$, …, $\mu'$, $\mu''$, … are the aforementioned factors, of which $\lambda'$, $\lambda''$, … always remain positive. Since $\Delta x_i$ , $\Delta y_i$ , $\Delta z_i$ are arbitrary, the equation above will decompose into the following one:

$$(24)\qquad \begin{cases} \dfrac{2m_i}{\partial t^2}(a_i - \xi_i) + \lambda'\dfrac{\partial L'}{\partial x_i} + \cdots + \mu'\dfrac{\partial M'}{\partial x_i} + \cdots = 0, \\ \dfrac{2m_i}{\partial t^2}(b_i - \eta_i) + \lambda'\dfrac{\partial L'}{\partial y_i} + \cdots + \mu'\dfrac{\partial M'}{\partial y_i} + \cdots = 0, \\ \dfrac{2m_i}{\partial t^2}(c_i - \zeta_i) + \lambda'\dfrac{\partial L'}{\partial z_i} + \cdots + \mu'\dfrac{\partial M'}{\partial z_i} + \cdots = 0, \end{cases}$$

in which $i$ is set to all whole numbers from 1 to $n$ . From the theory of the motion of a material point, one has:

$$a_i = x_i + v_i \cos(v_i, x) \cdot \partial t + \frac{X_i}{m_i} \cdot \frac{\partial t^2}{2},$$

$$b_i = y_i + v_i \cos(v_i, y) \cdot \partial t + \frac{Y_i}{m_i} \cdot \frac{\partial t^2}{2},$$

$$c_i = z_i + v_i \cos(v_i, z) \cdot \partial t + \frac{Z_i}{m_i} \cdot \frac{\partial t^2}{2}.$$

The actual position of the material point $m_i$ whose coordinates at time $t$ are $x_i$ , $y_i$ , $z_i$ will be determined by the equations:

$$\xi_i = x_i + \frac{\partial x_i}{\partial t} + \frac{\partial^2 x_i}{\partial t^2} \cdot \frac{\partial t^2}{2},$$

$$\eta_i = y_i + \frac{\partial y_i}{\partial t} + \frac{\partial^2 y_i}{\partial t^2} \cdot \frac{\partial t^2}{2},$$

$$\zeta_i = z_i + \frac{\partial z_i}{\partial t} + \frac{\partial^2 z_i}{\partial t^2} \cdot \frac{\partial t^2}{2}$$

at time $t + \partial t$. If one introduces the expression for $a_i$ , $b_i$ , $c_i$ , $\xi_i$ , $\eta_i$ , $\zeta_i$ thus obtained into equations (24) and remarks that:

$$v_i \cos(v_i, x) = \frac{\partial x_i}{\partial t}, \quad v_i \cos(v_i, y) = \frac{\partial y_i}{\partial t}, \quad v_i \cos(v_i, z) = \frac{\partial z_i}{\partial t}$$

then one will get the equations of motion:

$$(25)\qquad \begin{cases} X_i - m_i \dfrac{\partial^2 x_i}{\partial t^2} + \lambda' \cdot \dfrac{\partial L'}{\partial x_i} + \cdots + \mu' \dfrac{\partial M'}{\partial x_i} + \cdots = 0, \\ Y_i - m_i \dfrac{\partial^2 y_i}{\partial t^2} + \lambda' \cdot \dfrac{\partial L'}{\partial y_i} + \cdots + \mu' \dfrac{\partial M'}{\partial y_i} + \cdots = 0, \\ Z_i - m_i \dfrac{\partial^2 z_i}{\partial t^2} + \lambda' \cdot \dfrac{\partial L'}{\partial z_i} + \cdots + \mu' \dfrac{\partial M'}{\partial z_i} + \cdots = 0, \end{cases}$$

whose number is known to amount to $3n$. Those equations, in conjunction with the ones for the actual displacements (10) and (11), determine the $3n$ coordinates of the of the material point and the factors $\lambda', \lambda'', \ldots, \mu', \mu'', \ldots$

**§ 5.** – From what was said above, it is easy to show the connection that exists between the *principle of least work done by lost forces* and the *principle of virtual displacements, in conjunction with d'Alembert's principle.*

If one couples the principle of virtual displacements with that of **d'Alembert** and extends it to the case in which the conditions of a system depend upon time then it is known that the principle expresses the idea that the lost forces cannot generate displacements of the system that have a possible one as a result of being combined with the actual one. However, in order for that to be true, it is necessary that the total moment of the lost forces:

$$(26)\qquad \sum P_i \cdot \cos(P_i, \Delta s_i) \cdot \Delta s_i$$

cannot assume a positive value for the displacement $\Delta s_1, \Delta s_2, \ldots, \Delta s_i, \ldots, \Delta s_n$ that might be possible when combined with the actual one. Now, if one decomposes the lost force $P_i$ into two others: viz., the actual force $F_i$ and the force $J_i = m_i \dfrac{\partial^2 z_i}{\partial t^2}$ that generates the actual motion of the material point and is taken with the opposite sign:

$$P_i \cdot \cos(P_i, x) = X_i - m_i \frac{\partial^2 x_i}{\partial t^2},$$

$$P_i \cdot \cos(P_i, y) = Y_i - m_i \frac{\partial^2 y_i}{\partial t^2},$$

$$P_i \cdot \cos(P_i, z) = Z_i - m_i \frac{\partial^2 z_i}{\partial t^2},$$

$$\Delta x_i = \Delta s_i \cdot \cos(\Delta s_i, x), \qquad \Delta y_i = \Delta s_i \cdot \cos(\Delta s_i, y), \qquad \Delta z_i = \Delta s_i \cdot \cos(\Delta s_i, z),$$

$$\cos(P_i, \Delta s_i) = \cos(P_i, x) \cdot \cos(\Delta s_i, x) + \cos(P_i, y) \cdot \cos(\Delta s_i, y) + \cos(P_i, z) \cdot \cos(\Delta s_i, z),$$

then the total moment of the lost forces (26) will assume the form:

$$\sum\left\{\left(X_i - m_i\frac{\partial^2 x_i}{\partial t^2}\right)\cdot\Delta x_i + \left(Y_i - m_i\frac{\partial^2 y_i}{\partial t^2}\right)\cdot\Delta y_i + \left(Z_i - m_i\frac{\partial^2 z_i}{\partial t^2}\right)\cdot\Delta z_i\right\}.$$

That function, which is linear in the displacements, cannot assume any positive value for those displacements that are possible when combined with the actual one; i.e., ones that make the functions (16) positive and equal to zero and the functions (17) equal to zero. However, from the lemma that was discussed at the beginning, it is necessary and sufficient that the equation:

$$\begin{aligned}\sum\left\{\left(X_i - m_i\frac{\partial^2 x_i}{\partial t^2}\right)\cdot\Delta x_i + \left(Y_i - m_i\frac{\partial^2 y_i}{\partial t^2}\right)\cdot\Delta y_i + \left(Z_i - m_i\frac{\partial^2 z_i}{\partial t^2}\right)\cdot\Delta z_i\right\}&\\ + \lambda'\sum\left\{\frac{\partial L'}{\partial x_i}\cdot\Delta x_i + \frac{\partial L'}{\partial y_i}\cdot\Delta y_i + \frac{\partial L'}{\partial z_i}\cdot\Delta z_i\right\} + \ldots&\\ + \mu'\sum\left\{\frac{\partial M'}{\partial x_i}\cdot\Delta x_i + \frac{\partial M'}{\partial y_i}\cdot\Delta y_i + \frac{\partial M'}{\partial z_i}\cdot\Delta z_i\right\} + \ldots = 0&\end{aligned}$$

should be valid for all arbitrary displacements of the system, and therefore $\lambda'$, $\lambda''$, … must be positive. The equation above decomposed into equations (25). If one introduces the values of $a_i$, $b_i$, $c_i$, $\xi_i$, $\eta_i$, $\zeta_i$ into equation (22) then one will get:

$$-\Delta\sum(P_i\,p_i) = 2\sum\left\{\left(X_i - m_i\frac{\partial^2 x_i}{\partial t^2}\right)\cdot\Delta x_i + \left(Y_i - m_i\frac{\partial^2 y_i}{\partial t^2}\right)\cdot\Delta y_i + \left(Z_i - m_i\frac{\partial^2 z_i}{\partial t^2}\right)\cdot\Delta z_i\right\},$$

or

$$-\Delta\sum(P_i\,p_i) = 2\sum P_i\cdot\Delta s_i\cdot\cos(P_i, \Delta s_i).$$

That equation expresses the *connection* between the *principle of virtual velocities, in conjunction with **d'Alembert**'s principle and that of the least work done by lost forces.* From the latter principle, the increase in the lost work is always positive, while the principle of virtual displacements, in conjunction with **d'Alembert**'s, expresses the idea that the total moment of the lost forces:

$$\sum P_i\cdot\Delta s_i\cdot\cos(P_i, \Delta s_i)$$

does not assume any positive value relative to those displacements, as above.

**§ 6.** – We shall now move on to the *principle of least action*, about which an English mathematician expressed the following opinion: *The Principle of Least Action, in the form commonly given, is a meaningless proposition.*

One lets $T$ denote the *vis viva* of a system of material points:

$$T = \tfrac{1}{2}\sum m_i \left( \frac{\partial x_i^2 + \partial y_i^2 + \partial z_i^2}{\partial t^2} \right)$$

and sets:

$$\sum (X_i \, \Delta x_i + Y_i \, \Delta y_i + Z_i \, \Delta z_i) = \Delta U,$$

in which $U$ can generally be a function of time, since $\Delta$ refers to the coordinate changes that are independent of time. If one changes $T$ by means of the symbol $\Delta$ then one will get, in succession:

$$\begin{aligned}
\Delta T &= \sum m_i \left( \frac{\partial x_i}{\partial t^2} \cdot \Delta \, \partial x_i + \frac{\partial y_i}{\partial t^2} \cdot \Delta \, \partial y_i + \frac{\partial z_i}{\partial t^2} \cdot \Delta \, \partial z_i \right) \\
&= \sum m_i \left( \frac{\partial x_i}{\partial t^2} \cdot \partial \Delta \, x_i + \frac{\partial y_i}{\partial t^2} \cdot \partial \Delta \, y_i + \frac{\partial z_i}{\partial t^2} \cdot \partial \Delta \, z_i \right) \\
&= \frac{\partial}{\partial t} \sum m_i \left( \frac{\partial x_i}{\partial t} \cdot \Delta \, x_i + \frac{\partial y_i}{\partial t} \cdot \Delta \, y_i + \frac{\partial z_i}{\partial t} \cdot \Delta \, z_i \right) \\
&\quad - \sum m_i \left( \frac{\partial^2 x_i}{\partial t^2} \cdot \Delta \, x_i + \frac{\partial^2 y_i}{\partial t^2} \cdot \Delta \, y_i + \frac{\partial^2 z_i}{\partial t^2} \cdot \Delta \, z_i \right),
\end{aligned}$$

from which it will follow that:

$$\begin{aligned}
&\sum m_i \left( \frac{\partial^2 x_i}{\partial t^2} \cdot \Delta \, x_i + \frac{\partial^2 y_i}{\partial t^2} \cdot \Delta \, y_i + \frac{\partial^2 z_i}{\partial t^2} \cdot \Delta \, z_i \right) \\
&= \frac{\partial}{\partial t} \sum m_i \left( \frac{\partial x_i}{\partial t} \cdot \Delta \, x_i + \frac{\partial y_i}{\partial t} \cdot \Delta \, y_i + \frac{\partial z_i}{\partial t} \cdot \Delta \, z_i \right) - \Delta T .
\end{aligned} \tag{28}$$

As a result of this, equation (27) will assume the form:

$$-\Delta \sum (P_i \; p_i) = 2 \left\{ \Delta U + \Delta T - \frac{\partial}{\partial t} \sum m_i \left( \frac{\partial x_i}{\partial t} \cdot \Delta \, x_i + \frac{\partial y_i}{\partial t} \cdot \Delta \, y_i + \frac{\partial z_i}{\partial t} \cdot \Delta \, z_i \right) \right\}.$$

If one multiplies that equation, which is always considered to be positive, by $\partial t$ and integrates both sides of it between the limits $t = t_0$ and $t = t_1$, which correspond to two well-defined positions of the system in space, such that one will have:

$$\Delta x_i = 0, \qquad \Delta y_i = 0, \qquad \Delta z_i = 0$$

at the limits, then one will get:

$$-\int_{t_0}^{t_1} \Delta \sum (P_i \, p_i) \cdot \partial t \;=\; 2\int_{t_0}^{t_1} (\Delta U + \Delta T) \cdot \partial t \; .$$

However, since Δ refers to a time-independent change, one can put the equation above into the form:

$$-\int_{t_0}^{t_1} \Delta \sum (P_i \, p_i) \cdot \partial t \;=\; 2 \cdot \Delta \int_{t_0}^{t_1} (U + T) \cdot \partial t \; .$$

That equation expresses the connection between the so-called principle of least action and that of least lost work and explains the dynamical meaning of the expression in the right-hand side of the equation above. Since $\Delta\,(P_i\,p_i)$ cannot assume positive value for the displacement that is possible when it is combined with the actual one, we will see that:

$$\Delta \int_{t_0}^{t_1} (U + T) \cdot \partial t$$

cannot assume positive values for that displacement. In that sense, one can call the principle of least action, more precisely, *the principle of greatest action*, although neither of the two statements can be regarded to be strictly rigorous, and that can cause some confusion as to their meaning.

If one considers the expression:

$$\int_{t_0}^{t_1} (U + T) \cdot \partial t$$

by itself, which will experience negative increments for a displacement that is possible in conjunction with the actual one, then one will be in a position to derive the equations of motion of a system of material points regularly even in the case where the conditions depend upon time, and are expressed by means of equations and inequalities. In fact, from what was said above:

$$\Delta \int_{t_0}^{t_1} (U + T) \cdot \partial t \qquad \text{or} \qquad \int_{t_0}^{t_1} (\Delta U + \Delta T) \cdot \partial t$$

cannot assume positive values for the displacements that make the functions (16) positive and equal to zero and the functions (17) equal to zero. If one introduces the value of $\Delta U$ into the integral above, namely:

$$\sum (X_i \, \Delta x_i + Y_i \, \Delta y_i + Z_i \, \Delta z_i) \, ,$$

and the expression for $\Delta T$ in equation (28) then one will get:

(29) $$\int_{t_0}^{t_1}\left\{\sum \left[\left(X_i - m_i\frac{\partial^2 x_i}{\partial t^2}\right)\cdot\Delta x_i + \left(Y_i - m_i\frac{\partial^2 y_i}{\partial t^2}\right)\cdot\Delta y_i + \left(Z_i - m_i\frac{\partial^2 z_i}{\partial t^2}\right)\cdot\Delta z_i\right]\right\}\partial t\,,$$

since the sum:

$$\sum m_i\left(\frac{\partial x_i}{\partial t}\cdot\Delta x_i + \frac{\partial y_i}{\partial t}\cdot\Delta y_i + \frac{\partial z_i}{\partial t}\cdot\Delta z_i\right)$$

will be zero at the limits of the interval. In order for the expression (29) to not assume positive values for the displacements that make the functions (16) positive or equal to zero and the functions (17) equal to zero, it is necessary and sufficient that the equation:

$$\begin{aligned}\int_{t_0}^{t_1=1}\Bigg\{\sum &\left[\left(X_i - m_i\frac{\partial^2 x_i}{\partial t^2}\right)\cdot\Delta x_i + \left(Y_i - m_i\frac{\partial^2 y_i}{\partial t^2}\right)\cdot\Delta y_i + \left(Z_i - m_i\frac{\partial^2 z_i}{\partial t^2}\right)\cdot\Delta z_i\right]\partial t \\ &+ \lambda'\sum\left\{\frac{\partial L'}{\partial x_i}\cdot\Delta x_i + \frac{\partial L'}{\partial y_i}\cdot\Delta y_i + \frac{\partial L'}{\partial z_i}\cdot\Delta z_i\right\} + \ldots \\ &+ \mu'\sum\left[\frac{\partial M'}{\partial x_i}\cdot\Delta x_i + \frac{\partial M'}{\partial y_i}\cdot\Delta y_i + \frac{\partial M'}{\partial z_i}\cdot\Delta z_i\right] + \cdots\Bigg\} + \ldots = 0\end{aligned}$$

should be true for all arbitrary displacements, in which $\lambda'$, $\lambda''$, …, $\mu'$, $\mu''$, … are undetermined factors, the first of which $\lambda'$, $\lambda''$, … always remain positive. Due to the arbitrariness of $\Delta x_i$ , $\Delta y_i$, $\Delta z_i$ the equation above will decompose into $3n$ equations (25). If the expression (29) could assume only *positive* values for the displacements that make the functions (16) positive and equal to zero and the functions (17) equal to zero then we would reach the false conclusion that the factors $\lambda'$, $\lambda''$, … would have to be negative.