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On the analytical representation of the constraint on a material system in general coordinates

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The subject of the following lines is defined by the transformation of the expression for the constraint on a material system from its analytical representation by means of rectangular coordinates into one that uses general coordinates. That problem was solved for the first time by **Lipschitz** (“Bemerkungen zu dem Princip des kleinsten Zwanges,” Borch. Journ., Bd. 82, pp. 316) and in recent times in a very elegant way by **Wassmuth** (Sitzber. kais. Akad. Wiss. Wien, Bd. CIV, Abt. II.a). Nonetheless, the specification of a method of proof that deviates from the path to solution that was employed in cited papers would not be lacking in theoretical interest.

Let a system of n mass-points be given. The position of each individual point might be determined by specifying its three coordinates relative to a rectangular coordinate system, where the ν^{th} mass-point will be denoted $x_{3\nu-2}, x_{3\nu-1}, x_{3\nu}$ when one fixes a definite sequence of axes. In a corresponding way, the components of the resultant of all forces that act upon the mass-point relative to the three coordinates axes will then be represented by the symbols $X_{3\nu-2}, X_{3\nu-1}, X_{3\nu}$ while the equivalent symbols $m_{3\nu-2} = m_{3\nu-1} = m_{3\nu}$ will be chosen for the mass of the point in question in order to achieve unity in our notation. The system of mass-points considered might be assumed to be such that its connections can be described by specifying $(3n - k)$ equations:

$$f_i(x_1, x_2, \ldots, x_{3n}) = 0 \qquad (i = 1, 2, \ldots, 3n - k) \tag{1}$$

between the $3n$ coordinates of the system points analytically, in which it will be assumed that these equations do not contain time explicitly, for the sake of greater simplicity.

Under those assumptions, one can also determine the position of the individual system points by specifying k independent variables $p_1, p_2, \ldots, p_k$ that are connected with the rectangular coordinates by $3n$ equations:

$$x_\mu = \varphi_\mu(p_1, p_2, \ldots, p_k) \qquad (\mu = 1, 2, \ldots, 3n) . \tag{2}$$

The state of the system in question at a particular moment in time can be regarded as being given when one specifies the position and velocity of the individual mass-points, such that the values of the coordinates x_μ and their first derivatives $\dot{x}_\nu$ can be regarded as having known values for the moment in time considered. The accelerations of the system points are restricted from the outset by the constraint on the system in such a way that they cannot assume any arbitrary value b_ν, but only ones for which the equations:

$$\text{(I)} \qquad \sum_{\kappa=1}^{3n} \frac{\partial f_i}{\partial x_\kappa} b_\kappa = - \sum_{\lambda=1}^{3n} \sum_{\mu=1}^{3n} \frac{\partial^2 f_i}{\partial x_\lambda \, \partial x_\nu} \dot{x}_\lambda \, \dot{x}_\nu \qquad (I = 1, 2, \ldots, 3n - k)$$

are satisfied. The right-hand side of those equations can be regarded as constant, since the given values of those quantities at the moment in time considered can be substituted in them for the coordinates and their first derivatives with respect to time. With the chosen system connections between the *possible* accelerations b_ν, the accelerations that *actually* occur for a given force configuration can be selected using Gauss's principle using the condition that the function:

$$Z = \sum_{\tau=1}^{3n} m_\tau \left\{ b_\kappa - \frac{X_\tau}{m_\tau} \right\}^2$$

must be a minimum for all possible values of the accelerations that might actually occur.

For the purpose of representing that minimum condition with the use of general coordinates, it is important to first derive a lemma. If one denotes the accelerations that actually occur by g_ν then one can decompose any acceleration b_ν into a sum:

$$\text{(3)} \qquad b_\nu = g_\nu + z_\nu \, .$$

Physically, that corresponds to the decomposition of any possible motion of the system for a well-defined initial position and a given velocity configuration into the actually-occurring motion plus a possible motion for the same initial positions of all points when they are all at rest, under the assumptions that were made. One sees that this argument corresponds to the fact that the supplementary acceleration z_ν cannot be chosen arbitrarily. Moreover, they must satisfy the conditions (I), in which all of the initial velocities are set equal to zero, such that the accelerations z_ν are chosen to correspond with the $(3n - k)$ equations:

$$\text{(II)} \qquad \sum_{\kappa=1}^{3n} \frac{\partial f_i}{\partial x_\kappa} z_\kappa = 0 \qquad (i = 1, 2, \ldots, 3n - k) \, .$$

If one substitutes the decomposition of the possible accelerations b_ν that was encountered above into the expression for the constraint then one will get it in the form:

$$Z = \sum_{\tau=1}^{3n} m_\tau \left\{ g_\kappa - \frac{X_\tau}{m_\tau} + z_\tau \right\}^2$$

$$= \sum_{\tau=1}^{3n} m_\tau \left\{ g_\kappa - \frac{X_\tau}{m_\tau} \right\}^2 + 2\sum_{\tau=1}^{3n} m_\tau \left\{ g_\kappa - \frac{X_\tau}{m_\tau} \right\} z_\tau + \sum_{\tau=1}^{3n} m_\tau\, z_\tau^2 \,.$$

The required condition that this function Z should be a minimum for all possible choices of the accelerations for the choice $b_\nu = g_\nu$ can also be replaced with the condition that the expression:

$$2\sum_{\tau=1}^{3n} m_\tau \left\{ g_\kappa - \frac{X_\tau}{m_\tau} \right\} z_\tau + \sum_{\tau=1}^{3n} m_\tau\, z_\tau^2 \;> 0$$

for all admissible [i.e., compatible with the conditions (II)] choices of the supplementary accelerations z_τ. That condition can be expressed in yet another form. Namely, if z'_τ means any system of supplementary accelerations z_τ that are compatible with the equations (II) then $\rho\, z'_\tau$ will also be such a thing, in which the numerical factor ρ can be chosen arbitrarily. Hence, the inequality:

$$2\sum_{\tau=1}^{3n} m_\tau \left\{ g_\kappa - \frac{X_\tau}{m_\tau} \right\} z'_\tau + \rho \sum_{\tau=1}^{3n} m_\tau\, z_\tau'^2 \;> 0$$

must then be fulfilled for every positive value of ρ. However, that is possible only when the sum:

$$S = 2\sum_{\tau=1}^{3n} m_\tau \left\{ g_\kappa - \frac{X_\tau}{m_\tau} \right\} z'_\tau$$

is either zero or positive.

Nevertheless, since any choice z'_τ of the supplementary accelerations can also be associated with the choice of $z_\tau = - z'_\tau$, for which the sum considered can change sign under the assumption that it is positive, the expression S can only be zero.

The condition that is contained in Gauss's principle that the constraint on the system should be a minimum for the motion that actually occurs can then be replaced with the condition that the sum will be:

(III) $$\sum_{\tau=1}^{3n} m_\tau \left\{ g_\kappa - \frac{X_\tau}{m_\tau} \right\} z_\tau = 0,$$

and in fact for all choices of the numbers z_τ that correspond to the conditions (II).

It then seems that the **d'Alembert**'s principle is the expression for not only the sufficient, but also necessary, condition for the constraint on the system will be a minimum for the actually-occurring motion compared to all possible motions. However, since **d'Alembert**'s principle can also be regarded as a combination of **Lagrange**'s differential equations of order 1 or 2, it is also permissible to regard the fulfillment of the latter equations as the necessary and sufficient condition for the function Z to have the minimum property that is required by the principle of least constraint.

With the use of equation (III) that was just developed, the expression for the constraint on the system will go over to the form:

$$Z = \sum_{\tau=1}^{3n} m_\tau \left\{ g_\kappa - \frac{X_\tau}{m_\tau} \right\}^2 + \sum_{\tau=1}^{3n} m_\tau\, z_\tau^2 .$$

The problem of the transformation of that expression into general coordinates will then further reduce to the transformation of the sum:

$$Z' = \sum_{\tau=1}^{3n} m_\tau\, z_\tau^2 ,$$

since the sum:

$$\varphi = \sum_{\tau=1}^{3n} m_\tau \left\{ g_\kappa - \frac{X_\tau}{m_\tau} \right\}^2 ,$$

which exhibits the actually-occurring minimum value of the constraint, necessarily plays the role of a constant in the search for the minimum of the function Z .

The transformation of the sum Z' can be performed in the following form. The κ arbitrarily-chosen numbers β_κ ($\kappa = 1, 2, \ldots, k$) might represent arbitrary accelerations of the coordinates p_κ . Any choice of those numbers β_κ then belongs to a choice of accelerations b_ν , in which the reciprocal relationships are given by the $3n$ equations:

(IV) $$b_\nu = \sum_{\kappa=1}^{k} \frac{\partial \varphi_\nu}{\partial p_\kappa} \beta_\kappa + \sum_{\lambda=1}^{k} \sum_{\mu=1}^{k} \frac{\partial^2 \varphi_\nu}{\partial p_\lambda \, \partial p_\mu} \dot{p}_\lambda \dot{p}_\mu \qquad (\nu = 1, 2, \ldots, 3n),$$

such that all possible choices of the numbers b_ν also seem to be given by all arbitrary choice of the numbers β_κ . Naturally, the quantities p_κ and $\dot{p}_\kappa$ in equations (IV) are to be assigned vales that correspond to the initial values x_ν and $\dot{x}_\nu$. Now, one can once more decompose the accelerations β_κ into sums:

(4) $$\beta_\kappa = \gamma_\kappa + \zeta_\kappa ,$$

in which the numbers γ_κ mean the accelerations of the general coordinates that correspond to the actual motions, while the numbers ζ_κ are supplementary accelerations of the p_κ that one is completely free to choose. Since that decomposition of the accelerations β_κ once more corresponds to the same decomposition of any possible motion of the system into the actually-occurring motion plus a possible motion with the same initial position of all points that can be performed when all of then are completely at rest, which was the case for the decomposition of the accelerations b_ν into the sum $g_\nu + z_\nu$, all of the choices of supplementary accelerations z_ν that are compatible with the conditions (II) must represent all conceivable choices of the values ζ_κ that correspond to equations (IV), when one only sets all of the velocities $\dot{p}_\kappa$ equal to zero in those equations. One will then get the $3n$ equations:

$$\text{(V)} \qquad z_\nu = \sum_{\kappa=1}^{k} \frac{\partial \varphi_\nu}{\partial p_\kappa} \zeta_\kappa \qquad (\nu = 1, 2, \ldots, 3n)$$

between the values z_ν and ζ_κ .

It is easy to replace the supplementary accelerations ζ_κ with other variables. If L means the *vis viva* of the system, and P_κ means the general force components, then it is known that the theorem that the expressions:

$$Q_\kappa = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_\kappa}\right) - \frac{\partial L}{\partial p_\kappa} - P_\kappa \qquad (\kappa = 1, 2, \ldots, k)$$

should vanish for the actual motion will follow from equations (III). The expressions Q_κ contain the accelerations in the first power, and indeed:

$$\text{(VI)} \qquad Q_\kappa = \sum_{\lambda=1}^{k} a_{\lambda,\kappa}\, \beta_\lambda - F_\kappa \qquad (\kappa = 1, 2, \ldots, k),$$

and the F_κ in those equations mean well-defined functions of the coordinates p_κ and their first derivatives $\dot{p}_\kappa$, as well as the general force components P_κ , while the factors $a_{\lambda,\kappa}$ are defined by the equations:

$$a_{\lambda,\kappa} = \sum_{\nu=1}^{3n} m_\nu \frac{\partial \varphi_\nu}{\partial p_\lambda} \frac{\partial \varphi_\nu}{\partial p_\kappa} \qquad \begin{pmatrix} \lambda = 1, 2, \ldots, k \\ \kappa = 1, 2, \ldots, k \end{pmatrix} .$$

If one replaces the accelerations β_λ in the right-hand sides of equations (VI) with the sums $\gamma_\lambda + \zeta_\lambda$, which are equivalent to them, and further observes that the expressions Q_κ must vanish as a result of equations (III) when the possible accelerations β_λ are replaced with the ones that actually occur then one will get the k linear relations:

$$Q_\kappa = \sum_{\lambda=1}^{k} a_{\lambda,\kappa}\, \zeta_\lambda \qquad (\kappa = 1, 2, \ldots, k)$$

between the expressions Q_κ and the supplementary ζ_κ , from which one can conclude the relations:

$$\text{(VII)} \qquad \zeta_\kappa = \frac{1}{D} \sum_{\lambda=1}^{k} A_{\lambda,\kappa}\, Q_\lambda \qquad (\kappa = 1, 2, \ldots, k) .$$

In them, D means the determinant of the k^2 elements $a_{\lambda,\kappa}$, and $A_{\lambda,\kappa}$ means the adjoint of the element $a_{\lambda,\kappa}$. Equations (V) and (VII) imply the equations:

(VIII) $$z_\nu = \frac{1}{D}\sum_{\kappa=1}^{k}\sum_{\lambda=1}^{k}\frac{\partial\varphi_\nu}{\partial p_\kappa}A_{\lambda,\kappa}\,Q_\lambda \qquad (\nu = 1, 2, \ldots, 3n),$$

which define the relations between the supplementary accelerations z_ν of the rectangular coordinates and the expressions Q_λ . Now, with the help of equations (VIII), one will be in a position to exhibit the expression for the function Z' with the use of general coordinates. To that end, one merely has to replace the quantities z_ν^2 in Z' with the Q_λ and obtain:

$$Z' = \frac{1}{D^2}\cdot\sum_{\kappa=1}^{k}\sum_{\lambda=1}^{k}Q_\kappa\,Q_\lambda\cdot\sum_{\rho=1}^{k}\sum_{\sigma=1}^{k}A_{\rho\kappa}A_{\rho\lambda}\sum_{\nu=1}^{3n}m_\nu\frac{\partial\varphi_\nu}{\partial p_\rho}\frac{\partial\varphi_\nu}{\partial p_\sigma}\,.$$

If one observes the defining equations of the quantities $a_{\lambda\kappa}$ and the relation:

$$\sum_{\rho=1}^{k}\sum_{\sigma=1}^{k}A_{\rho\kappa}A_{\rho\lambda}\,a_{\rho\sigma} = D\cdot A_{\lambda,\kappa}\,,$$

which ensues from the property of the determinant that the sum $\sum_{\rho=1}^{k}A_{\rho,\kappa}a_{\rho,\sigma}$ is either zero or D according to whether σ is or is not different from κ, resp., then one will arrive at the formula:

$$Z' = \frac{1}{D}\sum_{\kappa=1}^{k}\sum_{\lambda=1}^{k}A_{\kappa,\lambda}\,Q_\kappa\,Q_\lambda\,.$$

That will then imply the theorem that the expression for the constraint on a material system with the use of general coordinates can be written in the form:

$$Z = \frac{1}{D}\sum_{\kappa=1}^{k}\sum_{\lambda=1}^{k}A_{\kappa,\lambda}\,Q_\kappa\,Q_\lambda + \varphi,$$

in which the function:

$$\varphi = \sum_{\nu=1}^{3n}m_\nu\left\{g_\nu - \frac{X_\nu}{m_\nu}\right\}^2$$

means the value of the constraint for the motion that actually occurs, which plays the role of a constant in the search for the minimum value of the function Z . As one can see from the representation of Z' in rectangular coordinates itself, its smallest value is zero, and that can happen only when all $z_\nu = 0$, or when all Q_λ vanish.
