“Bemerkungen zur Note des Herrn Philip E. B. Jourdain über das Prinzip der kleinsten Aktion,” Math. Ann. **64** (1907), 156-159.

Remarks on the Note by Herrn Philip E. B. Jourdain on the principle of least action (*).

By

MORITZ RÉTHY in Budapest.

Translation by D. H. Delphenich

I. – Whenever the motion of a material system is to be natural, the identity (4) of Jourdain (pp. 416) (which defines $\delta_1 T$) will follow from equation (5) (*loc. cit.*, pp. 416):

$$(5) \qquad \int_{t_0}^{t_1} \left(2T \frac{d\,\delta t}{dt} + \delta_1 T + \sum_\nu Q_\nu\,\delta q_\nu\right) dt = 0\,,$$

in which q_ν mean free coordinates, and the two assumptions (in the last line on pp. 418) that:

a) the variations δq_ν vanish at the limits t_0 and t_1, and

b) the equation:

$$\delta_1 T = \sum_\nu Q_\nu\,\delta q_\nu$$

is true for all time.

The assertion that was made (on pp. 418, with no justification) that in the case of natural motion, the premises a) and b) would also imply the vanishing of:

$$\delta \int_{t_0}^{t_1} 2T\,dt$$

is, unfortunately, flawed.

Namely, Jourdain’s identity (4) (pp. 416) would imply the equivalent identity:

$$(\dagger) \qquad \delta_1 T\,dt + T\,\delta\,dt \equiv \delta\,(T\,dt) + \left(\sum_\nu q'_\nu \frac{\partial T}{\partial q'_\nu} - 2T\right) d\,\delta t - \frac{\partial T}{\partial t}\,\delta t\,dt\,.$$

(*) Math. Ann., Bd. 62, pp. 413-418.

On the other hand, since his equation (5) will be satisfied by the natural motion under the assumptions a) and b), it will then follow [when one replaces $\sum_\nu Q_\nu\,\delta q_\nu$ with assumption b) regarding $\delta_1 T$ and then applies the identity (†)] that one will have the equation:

$$0 = \int_{t_0}^{t_1} \delta\,(T\,dt) + \int_{t_0}^{t_1} \left(\sum_\nu q'_\nu \frac{\partial T}{\partial q'_\nu} - 2T \right) d\,\delta t - \int_{t_0}^{t_1} \frac{\partial T}{\partial t}\,\delta t\,dt \ .$$

Therefore, $\delta \int_{t_0}^{t_1} T\,dt$ *will certainly not vanish under Jourdain's assumptions* a) *and* b) *with no further conditions*, because we would then be dealing with the special case of Lagrange's equations in which T is a homogeneous quadratic function of q' with coefficients that do not depend upon t explicitly, which is a case that Jourdain nonetheless excluded. Rather, the premises a) and b) are generally extended by the assumption that:

c) The variation δt is defined by the equation:

$$\int_{t_0}^{t_1} \left[\frac{\partial T}{\partial t}\,\delta t + \left(2T - \sum_\nu q'_\nu \frac{\partial T}{\partial q'_\nu} \right) \frac{d\,\delta t}{dt} \right] dt = 0 \ .$$

The fact that the theorem forfeits its simplicity completely with the necessary extension does not need to be emphasized specifically.

II. – It follows from the identities (11) and (11*) on page 173 of my article (which appeared in Bd. 58 of Math. Ann.), and with the notation in (10*), that one has the general identity:

$$\delta\,[(1 + c)\,T\,dt] - d\left[(1+c)\,T\,\delta t + \sum_i^n \delta' q_i \right] = \left[c\,\delta' T + \sum_{i=1}^n \left(\frac{\partial T}{\partial q_i} - \frac{d}{dt}\frac{\partial T}{\partial q'_i} \right) \delta' q_i \right] dt \ ,$$

in which c is an arbitrary constant and:

$$\delta' \cdot \equiv \delta \cdot - \frac{d}{dt}\,\delta t \ .$$

If I therefore introduce the convention in regard to δt that (*):

(*) On page 176 of my cited article (lines 5 and 6), that would be equation (††). Namely, one writes:

$$\delta' T\,dt + d\,(T\,\delta t)$$

in place of $\delta' T\,dt$ on line 6.

(††) $$d\left[(1+c)T\,\delta t+\sum_{i=1}^{n}\frac{\partial T}{\partial q_i'}\delta' q_i\right]+\left[c\,\delta' T+\sum_{i=1}^{n}Q_i\,\delta' q_i\right]dt = 0$$

then the equation will follow:

$$\delta\,[(1+c)\;T\,dt] = \sum_{i=1}^{n}\left(Q_i+\frac{\partial T}{\partial q_i}-\frac{d}{dt}\frac{\partial T}{\partial q_i'}\right)\delta' q_i\,dt\;.$$

Therefore, the equation:

(†††) $$(1+c)\,\delta\int_{t_0}^{t_1}T\,dt = \int_{t_0}^{t_1}\sum_{i=1}^{n}\left(Q_i+\frac{\partial T}{\partial q_i}-\frac{d}{dt}\frac{\partial T}{\partial q_i'}\right)\delta' q_i\,dt$$

will certainly be true only as long as δt_0 and δt_1 are determined by the equation:

(††′) $$\left[(1+c)T\,\delta t+\sum_{i=1}^{n}\frac{\partial T}{\partial q_i'}\delta' q_i\right]+\int_{t_0}^{t_1}\left(c\,\delta' T-\sum_{i=1}^{n}Q_i\,\delta' q_i\right)dt = 0\;.$$

However, if that last equation is fulfilled for an arbitrary time-point t_1, and if the ratio δq_i / δt is subject to the conditions equations for the motion that exist between the q_i' then $\delta\int_{t_0}^{t_1}T\,dt$ will vanish, which would follow from (†††) if and only if the motion of the material system is the natural one. The statement includes my theorem, along with that of **Voss**. Equation (††′) can be divided into two equations when one sets the integrand $c\,\delta' T-\sum Q\,\delta q = 0$.

Jourdain would now show (in the first line of 3. on pp. 417-8) that the quantities $\delta q_i - q_i'\,\delta t$ that are denoted by $\delta' q_i$ can mean completely general virtual displacements only when δt vanishes. I must admit that the basis for that eludes me. In particular, it seems to me that the appeal to Hamilton's principle (pp. 418, line 3-5) is entirely strange and suspicious (*). However, the only thing that is definitive in the context of this question is the fact that the ratio δq_i / δt is defined such that *homogeneous* linear equations with a simple physical meaning for the quantities $\delta' q_i$ emerge from them that impose no restriction at all on the value of δt in its own right. Jourdain might directly (i.e., without the interference of foreign principles) and clearly recognize that there is an intrinsic contradiction in that definition and the consequences that are inferred from it.

______

(*) In the footnote on pp. 75, line 3. in his article that appeared in the Quart. J. of Math. (1904), Jourdain clearly stated that the derivatives of Voss that he spoke of included "two fundamental errors," one of which was the one that was just presented. Since both alleged errors stand and fall together, I would not like to speak of the second one.