"Ueber das Princip des kleinsten Zwangs und die damit zusammenhängenden mechanischen Principe," Arch. Math. Phys. **6** (1845), 238-269.

On the principle of least constraint and the mechanical principle that is connected with it

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In the fourth volume of Crelle's Journal, pp. 232, **Gauss** presented a new general fundamental law of mechanics that was classified among the dynamical principles by the name above ("principle of least restraint") in an English textbook (**Earnshaw**, *Dynamics*, Cambridge, 1839). As far as I know, that is the only place in the literature where it was cited, but without examples, moreover, and the same can be said for **Gauss**. The purpose of this treatise is, after a few historical remarks on the treatment of the principle in the two aforementioned papers, to first test that principle in a pair of simplest-possible examples and to then undertake the general analytical treatment and to thus prove its connection with the remaining principles of mechanics.

§ 1.

With the expression that its author gave to it, the principle reads:

> "The motion of a system of material points that are always coupled to each other in some way, and whose motions are, at the same time, always constrained by external restrictions will take place at each moment *with the greatest possible coincidence with the free motion or with the smallest possible constraint*, in which the measure of constraint that the entire system suffers at every point in time is considered to be the sum of the products of the square of the deviation of each point from its free motion with its mass."

Gauss and **Earnshaw** proved that law in different ways by reducing it to other mechanical principles.

I. – First of all, Gauss derived it from d'Alembert's principle, in conjunction with the principle of virtual velocities, it in the following way: For any point of the system let (Tab. IV, Fig. 1 [†]) m be its mass, let A be its location at the time t, let B be the location that it would assume after an infinitely-small time interval τ as a result of the forces that act upon it and the velocity that it would achieve at time t if it were completely free, let C be the actual location that would correspond to the time interval as a result of the system constraint, and finally, let D be any other location that is compatible with the system constraint, which one understands to be infinitely close to the points A and C. $\sum m \cdot \overline{BC}^2$ will then be a minimum when:

$$\sum m \cdot \overline{BD}^2 - \sum m \cdot \overline{BC}^2 > 0 \,.$$

However, if θ is the angle between CB and CD then one will have:

$$\overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2 - 2\overline{BC} \cdot \overline{CD} \cdot \cos\theta$$

for the triangle BCD, and as a result:

$$\sum m \cdot \overline{BD}^2 - \sum m \cdot \overline{BC}^2 = \sum m \cdot \overline{CD}^2 - 2\sum m \cdot \overline{BC} \cdot \overline{CD} \cdot \cos\theta \,, \tag{a}$$

but from the principle of virtual velocities, when applied to the equilibrium of lost forces that is required by d'Alembert's principle, one has:

$$\sum m \cdot \overline{BC} \cdot \overline{CD} \cdot \cos\theta = 0, \tag{b}$$

and as a result:

$$\sum m \cdot (\overline{BD}^2 - \overline{BC}^2) = \sum m \cdot \overline{CD}^2 > 0 \,.$$

In fact, the product $m \cdot \overline{BC} \cdot \overline{CD} \cdot \cos\theta$ is the virtual moment that belongs to the material point m, insofar as $m \cdot \overline{BC}$ can be considered to be its lost force. In order to justify that, as well as to show the essential homogeneity of the equation (a) in regard to the infinitesimals, I shall permit myself to add the following remarks to the Gaussian proof:

The product $m \cdot \overline{BC}$ is, first of all, the product of the mass with a space or path that corresponds to the infinitely small time interval τ. However, one can replace the acceleration in it p that corresponds to a unit time with $\frac{1}{2} p\tau^2$, insofar as merely $\frac{1}{2}\tau^2$ will enter then as a common factor in the summation sign in (b). That is because even though the velocities v that are required at time t that enter into the expression of the principle above yield paths of the form $v\tau$ that will therefore not include the square of τ, the lost forces, or the paths BC that are substituted for them, will,

(†) Translator: The tables and figures referred to were not available to me at the time of translation.

however, not in fact depend upon those velocities *v*, which will be explained completely by the proof of the lost force in *BC*. To that end, (Tab. IV, Fig. 2), starting from the location *A* that corresponds to time *t*, let *AE* be the path that is traversed in the same time interval as a result of the newly-added force, while *AC*, as above, means the actual path: When one extends *AEC* to a parallelogram, *EC* = *AG* will represent the actual component of the accelerating force *AF*, and when one extends *AGF*, *FG* = *AG* will represent the lost component, and finally, the diagonal *AB* in the parallelogram at *AE* and *AF* will represent the path that is freely traversed in the time interval τ. However, that is, at the same, the diagonal in the parallelogram at *AC* and *AH*, and therefore the deviation *BC* from the free path that is parallel and equal to the lost force *AH*, which already illuminates the fact that *BC*, like *AF*, is a quantity of the form $\frac{1}{2}p\tau^2$. However, more directly, it shows the independence of the quantity *BC* (or *AH*) from the velocity that is required at time *t* thus: If AC' is equal and opposite to *AC* then *AH* will be the resultant of *AB* and AC'. However, if one decomposes them into their components, which are AE' and AG' for *AE* and *AF*, resp. (i.e., the quantities that are equal and opposite to *AE* and *AG*, resp.), then *AE* and AE' will cancel as components of *AH*. – Now, since $BC = \frac{1}{2}p\tau^2$, from the cited principle, $\sum m\cdot\overline{BC}\cdot\overline{CD}\cdot\cos\theta = \frac{1}{2}\tau^2\sum mp\cdot\overline{CD}\cdot\cos\theta$ will generally be zero. However, since *BC* is, at the same time, a second-order infinitesimal, *BD* and *CD* must also prove to be such things. Hence, let (*a*) be homogeneous, because otherwise the term in equation (*b*) would drop out as a higher-order infinitesimal, independently of the principle of virtual velocities. However, if *D* is a point of the same kind as *C* then one can repeat the previous construction for it when one replaces *C* with *D* everywhere, and starting from *A*, lets the given initial velocity *AE* agree with *AD*, just as it does as with *AC*, but lets any other accelerating force enter in place of *AF*. In that way, *BD* will prove to be a quantity of the form $\frac{1}{2}p'\tau^2$, where p' is any other acceleration that only corresponds to the conditions on the system that would lead *m* from *B* to *D*, instead of *C*, in the same time interval τ, and then *CD* will also be a quantity of the same form, since it is the resultant of *CB* and *BD*, namely, when λ is the opposite angle to *CD* in the triangle *BCD*, $\overline{CD}^2 = \frac{1}{4}\tau^4\,(p^2 + p'^2 - 2pp'\cos\lambda)$.

II. – **Earnshaw** sought a different type of proof by coupling the static and dynamic properties of the center of mass with d'Alembert's principle by a mechanical construction, to which I cannot, however, ascribe the least evidence for my complete understanding of it, but I can also get around that fact. Start from the fact the resultant of the lost forces – or, as is aptly expressed, the constraint forces (forces of restraint, restraining pressures) – that act upon *m* in the direction *BC*, so *BC* is the amount by which the resultant makes the particle *m* deviate in the infinitely small time interval τ: Thus, if one now imagines all material points of the system as being removed from their constraints and combined at any point β in space in the free state, and pushed away from it by pressures with the same magnitudes and directions as the constraint forces then any particle *m* will move from β to a point γ such that $\beta\gamma$ is equal and parallel to *BC*. Now, since the forces of constraint are such that they will produce equilibrium in the system, according to d'Alembert's principle, they can have no influence on the motion of the center of mass, according to the principle of the conservation of the center of mass, so the center of mass of the particles *m* will remain where it was found for the points γ. However, from a known property of the center of mass, $\sum m\,\overline{\beta\gamma}^2$ will be a minimum, and as a result $\sum m\,\overline{BC}^2$, as well; i.e., the value of that quantity will be smaller than when the constraint forces do not produce equilibrium, as they should according to d'Alembert's principle.

Now, this law of the center of mass is so deeply and subtly rooted in the nature of things that the sum of the products of each mass with the square of its distance from the center of mass must be a minimum: It seems to me that there is much to be done regarding the mechanical picture of constraint, due to the way that the law of conservation of the center of mass is applied. That is because there are such great demands imposed upon mathematical abstraction when one thinks of an arbitrary mass being concentrated a point or likewise united along a line or surface that it occurs to me that the demand that several isolated material points (or masses that are thought of as concentrated into points) in a free state (i.e., uncoupled) such that each of them can move as if one could imagine that it is united into a point, or even more, that several material points of that type could define a system is quite unnatural, if not a *contradictio in adjecto*, and one must therefore first prove that β is the center of mass of that system before one can understand how it remains there. However, the fact that β is actually the center of mass of the material points that are found at the point γ follows from a purely static law that **Lagrange** cited in *Mécanique analytique* in section five of *Statics* (t. 1, page 107) as being due to **Leibnitz** and derived using his own formulas. The law consists of the fact that when several forces are in equilibrium at a point and lines are drawn from that point that represent those forces in magnitude and direction, the point in question can be the center of mass of just as many (as the number of forces present) equal masses that are attached to the endpoints of those lines, and one can then extend that to: If one represents each of the static forces P that are in equilibrium at a point by a product mp, the one factor of which m can be considered to be a mass, while the other one p can be considered to be an acceleration, or as a path that is traversed in a certain time interval, then each point will be the center of mass of the masses m that are placed at distances p from it, while those distances are taken in the direction of the given forces, and $\sum mp^2$ will then be a minimum, from a second law of statics that **Earnshaw** cited. I shall now intertwine the proof of the law that was thus posed, as well as the last one mentioned, with the justification for the English proof of our principle, which consists of the following steps:

1. As a result of the principle of the conservation of the center of mass, the constraint forces that are in equilibrium in the system due to d'Alembert's principle have no influence on the motion of the center of mass, but rather they must be in equilibrium at it, so at any point β in space at all (*). Therefore, if one sets $BC = \beta\gamma = \frac{1}{2}\,p\,\tau^2$, as in no. **I**, and if a, b, c are the angles that the line $\beta\gamma$ makes with the three rectangular axes of x, y, z, resp., then the equations:

(*) Or, more simply: Since the constraint forces produce equilibrium in the given system, from d'Alembert's principle, they must, in particular, cancel the advancing motion of the center of mass, and thus be in equilibrium at a point.

$$(\alpha) \qquad \sum mp\cos a = 0, \qquad \sum mp\cos b = 0, \qquad \sum mp\cos c = 0$$

will express the equilibrium of the constraint forces $m\cdot\overline{BC}$ at the point β.

2) Thus, if ξ, η, ζ are the coordinates of the point β and x, y, z are those of any point γ then:

$$\overline{\beta\gamma}^2 = \tfrac{1}{4}p^2\tau^4 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$$

and

$$\cos a = \frac{x-\xi}{\frac{1}{4}p^2\tau^4}, \qquad \cos b = \frac{y-\eta}{\frac{1}{4}p^2\tau^4}, \qquad \cos c = \frac{z-\zeta}{\frac{1}{4}p^2\tau^4}.$$

Therefore, from the foregoing equations (α):

$$(\beta) \qquad \sum m(x-\xi) = 0, \qquad \sum m(y-\eta) = 0, \qquad \sum m(z-\zeta) = 0,$$

and those will give:

$$\xi = \frac{\sum mx}{\sum m}, \qquad \eta = \frac{\sum my}{\sum m}, \qquad \zeta = \frac{\sum mz}{\sum m},$$

so the point β whose coordinates are ξ, η, ζ will be the center of mass of the material point m whose coordinates are x, y, z ; i.e., the masses m that are found at the points γ (*).

3) The first part of equations (β) are the partial derivatives of the functions $\frac{\tau^4}{4}\sum m\,p^2$ with respect to ξ, η, ζ, but taken with the opposite sign, and since the second part, namely, the one where one differentiates with respect to those variables a second time, reduces to the essentially positive quantity $\sum m$, while the others, in which one differentiates with respect to two different variables, all reduce to zero. Hence, all conditions are fulfilled that make that function – thus, $\sum m\cdot\overline{BC}^2$ – a minimum with respect to ξ, η, ζ ; i.e., it is smaller then when ξ, η, ζ do not have the values that correspond to the center of mass, so smaller than when the constraint forces $m\cdot\overline{BC}$ are not in equilibrium at a point, so not in equilibrium for the system, either (**).

(*) Now, since one can also set $P = mq$, $P' = mq'$, …, instead of $mp = P$, $m'p' = P'$, etc. (i.e., assume that all of the masses are equal), if n is the number of those masses (or forces P) then:

$$\xi = \tfrac{1}{2}\sum x, \qquad \eta = \tfrac{1}{2}\sum y, \qquad \zeta = \tfrac{1}{2}\sum z,$$

which is the form of this result that Lagrange derived in the cited place as an expression for Leibnitz's law.

(**) One can exhibit this law of the center of mass without differential calculus, namely, that if r_0 is the distance from a material point m to the center of mass then $\sum mr_0^2$ will be a minimum: Let $x_0 = x - \xi$, $y_0 = y - \eta$, $z_0 = z - \zeta$ be the coordinates of m relative to the center of mass, so $r_0^2 = x_0^2 + y_0^2 + z_0^2$, and likewise $r^2 = x^2 + y^2 + z^2$ and $\rho^2 = \xi^2 + \eta^2 + \zeta^2$, and one will have:

III. – If we now compare the two proofs then both of them start from the fundamental principle of dynamics – namely, the equilibrium of the constraint forces – but they differ in that **Gauss**'s proof brings about that equilibrium by means of the general condition of equilibrium, which consists of the vanishing of the virtual moments, while that of **Earnshaw** uses a partial condition that has merely the cancellation of the advancing motion as a consequence. Now, the relationship of our minimum to the one that gives the center of mass seems so intimate and fragile that this proof is in no way general and is initially suited merely to free systems, in which the forces of constraint must fulfill the condition that they are in equilibrium at a point, which will no longer be the case when the system includes a fixed point or a fixed axis, such that $\sum m\cdot\overline{BC}^2$ should reduce to $\sum m r_0^2$ (where r_0 is the distance from the mass to the center of mass) in all cases, so that must happen in the stated cases, in particular, or more generally, but then it could happen only by means of the principle of virtual velocities. Thus, one can assert that the English proof is merely a proof of the principle in one example (although it includes many other cases), as we will show in § **2** in another example of a general type, but that **Gauss**'s general proof must be carried out using the general formula of dynamics, namely, the coupling of d'Alembert's principle with that of virtual velocities. The consequences of the principle of least constraint are then exhibited in just enough generality that it can equally serve as the basis for the derivation of all the equations of motion of a given system, just as the principle of virtual velocities is applied to the equilibrium of lost forces, since the Gaussian proof explains directly that one can conversely arrive at that general formula of dynamics from the principle of least constraint.

Just as one can derive a static formula from any dynamic one by means of d'Alembert's theorem, and conversely, according to **Gauss**, equilibrium is only a special case of the general theorem into which merely the points A themselves enter, corresponding to the equilibrium position, instead of the points C, and $\sum m\cdot\overline{BA}^2$ is a minimum. In fact, that means the same thing as: Set everything in the general formula of dynamics that refers to the actual motion at the time t equal to zero in the case where the system of applied forces is itself in equilibrium. Furthermore, the proof can also be carried out in just the same way, since D corresponds to a virtual position of the material point m, such that AD is equivalent to AB relative to their magnitudes, so the triangle BAD, like BCD before it, will imply the relation:

$$(c)\qquad \sum m\cdot(\overline{BD}^2-\overline{BA}^2) = \sum m\cdot\overline{AD}^2-2\sum m\cdot\overline{BA}\cdot\overline{AD}\cdot\cos\varphi,$$

which is similar to (a), in which φ is the angle between the direction of the free motion AB or that of the force and that of the virtual path AD, and with:

$$\sum m(r^2-r_0^2) = \sum m[(x_0+\xi)^2-x_0^2+(y+\eta)^2-y_0^2+(z+\zeta)^2-z_0^2] = 2\xi\sum m x_0+2\eta\sum m y_0+2\zeta\sum m z_0+\rho^2\sum m.$$

As a result, since:

$$\sum m x_0 = \sum m y_0 = \sum m z_0 = 0,\qquad \text{one will have}\qquad \sum m(r^2-r_0^2) = \rho^2\sum m > 0,$$

which also might be the coordinate origin.

$$\sum m \cdot \overline{BA} \cdot \overline{AD} \cdot \cos\varphi = 0,$$

the expression:

$$\sum m \cdot \overline{BD}^2 - \sum m \cdot \overline{BA}^2$$

will reduce to the essentially positive quantity:

$$\sum m \cdot \overline{AD}^2 .$$

As is already suggested by the demand that *AD* should be equivalent to *AB* in magnitude, it should be remarked here that since $AB = \frac{1}{2} p\tau^2$, if p is, in turn, the accelerating force that acts upon m and τ is the infinitely small time interval then the virtual path *AD* must also be a second-order infinitesimal relative to τ, which is, in fact, also justified by the fact that the quantity *AD* can be thought of as the path that is traversed during the time interval τ when other accelerating forces q enter in place of the forces p that are in equilibrium at the point *A*. (Cf., moreover, § **2**, III, and § **3**, I.)

§ 2.

We shall now test the principle of least constraint directly in some of the simplest examples of motion, as well as equilibrium.

I. – In order to take the simplest possible case of the motion of a system, it is required that all points *A*, *B*, *C*, *D*, *E* for the material points fall along the same line. We therefore assume that two unequal masses m, m' are at the ends of a weightless and absolutely flexible and inextensible string that goes around a weightless pulley, and establish the arrangement of points *ABCDE* for the mass m and the corresponding one $A'B'C'D'E'$ for the mass m' (Tab. IV, Fig. 3): Therefore, from the nature of the system, all distances from the remaining points to *E* on the one side and the distances to E', on the other, will be equal and have the opposite senses, except for *EB* and $E'B'$, which will both have the sense of gravity. Now, one then has:

$$\overline{BC} = \overline{EB} - \overline{EC}, \qquad \overline{B'C'} = \overline{EB} + \overline{EC},$$

$$\overline{BD} = \overline{EB} - \overline{EC} - \overline{CD}, \qquad \overline{B'D'} = \overline{EB} + \overline{EC} + \overline{CD},$$

$$\overline{BD}^2 - \overline{BC}^2 = \overline{CD}^2 - 2 \cdot \overline{CD} \cdot (\overline{EB} - \overline{EC}),$$

$$\overline{B'D'}^2 - \overline{B'C'}^2 = \overline{CD}^2 + 2 \cdot \overline{CD} \cdot (\overline{EB} + \overline{EC}),$$

and as a result, one has the quantity:

$$\sum m\cdot(\overline{B'D'}^2-\overline{B'C'}^2) = (m+m')\cdot\overline{CD}^2+2(m+m')\cdot\overline{CD}\cdot\overline{EC}-2(m-m')\cdot\overline{CD}\cdot\overline{EB}$$
$$= (m+m')\cdot\left\{\overline{CD}^2+2\cdot\overline{CD}\cdot(\overline{EC}-\frac{m-m'}{m+m'}\cdot\overline{EB})\right\}.$$

However, if, as usual, g is the acceleration of gravity and τ is a time interval that can have an arbitrary finite magnitude in our case with all of the foregoing lines then:

$$\overline{EB} = \tfrac{1}{2}g\tau^2, \qquad \overline{EC} = \tfrac{1}{2}\frac{m-m'}{m+m'}g\tau^2,$$

so

$$\overline{EC}-\frac{m-m'}{m+m'}\cdot\overline{EB} = 0,$$

and as a result:

$$\sum m\cdot\overline{BD}^2-\sum m\cdot\overline{BC}^2 = (m+m')\overline{CD}^2 = \sum m\cdot\overline{CD}^2 > 0,$$

which might also be arbitrary line CD, as long as one only takes it to have the direction of the string, and therefore:

$$m\overline{BC}^2+m'\overline{B'C'}^2$$

is a minimum.

If one introduces the values of EB, EC into the expression that is to be a minimum then when one makes $\frac{m-m'}{m+m'} = \mu$, one will have:

$$\overline{BC} = \tfrac{1}{2}(1-\mu)\,g\tau^2, \qquad \overline{B'C'} = \tfrac{1}{2}(1+\mu)\,g\tau^2,$$

and as a result:

$$\sum m\cdot\overline{BC}^2 = [m\,(1-\mu)^2+m'(1+\mu)^2]\,\frac{g^2\tau^4}{4},$$

and one will obviously go from this to $\sum m\cdot\overline{BD}^2$ when one substitutes another acceleration $(\mu+\alpha)\,g$ for $\mu\,g$, in which α is an arbitrary (positive or negative) quantity, from which one gets:

$$\sum m\cdot\overline{BD}^2-\sum m\cdot\overline{BC}^2 = (m+m')\frac{\alpha^2g^2\tau^4}{4},$$

i.e., $\sum m \cdot \overline{CD}^2$, so one has just $CD = \frac{1}{2}\alpha g \tau^2$. However, that also explains the fact that, in regard to proving the minimum by differential calculus, one must differentiate with respect to μ, and since the first two derivatives of the quantity above with respect to μ are:

$$[- m\,(1 - \mu) + m'\,(1 + \mu)]\frac{g^2\tau^4}{4} \qquad \text{and} \qquad (m + m')\frac{g^2\tau^4}{4},$$

the first of which will be identically zero, from the value of μ, while the second one is essentially positive, so the minimum is also confirmed by this argument.

Finally, if one would like to apply the principle by first finding the acceleration of the system then let z, z' be the distances from the points A, A', resp., to the horizontal diameter of the pulley at time t, so because the condition on the system is $\Delta z = -\Delta z'$, corresponding to the time interval t, one will have:

$$EC = - E'C' = \frac{d^2z}{dt^2}\cdot\frac{\tau^2}{2} + \ldots,$$

and as a result:

$$BC = -\left(g - \frac{d^2z}{dt^2}\right)\frac{\tau^2}{2} - \ldots, \qquad B'C' = \left(g + \frac{d^2z}{dt^2}\right)\frac{\tau^2}{2} - \ldots,$$

so the quantity that should be a minimum is:

$$\left\{m\left(g - \frac{d^2z}{dt^2}\right)^2 + m'\left(g + \frac{d^2z}{dt^2}\right)^2\right\}\frac{\tau^4}{4} + \ldots$$

Taking its first derivative with respect to the actual acceleration d^2z / dt^2 will then give:

$$\left\{m\left(g - \frac{d^2z}{dt^2}\right) + m'\left(g + \frac{d^2z}{dt^2}\right)\right\}\frac{\tau^4}{4} + \ldots,$$

as before, and since that equation must be independent of the magnitude of the time interval τ, it must follow that:

$$m\left(g - \frac{d^2z}{dt^2}\right) + m'\left(g + \frac{d^2z}{dt^2}\right) = 0,$$

so:

$$\frac{d^2z}{dt^2} = \frac{m - m'}{m + m'}\,g\,.$$

Since one has $d^2 z / dt^2 = 0$ immediately, and the same thing we be true for all of the following time derivatives, the expressions above will reduce to their first terms, and since the second derivative $(m + m')\ \tau^4 / 4$ is essentially positive, it will indicate a minimum.

II. – Furthermore, the principle can be applied to a single material point. There, as well, one has $m \cdot \overline{BC}^2$, so BC is a minimum, up to sign. That is self-explanatory for free motion, where $BC = 0$, but for constrained motion, one can generally understand it in a very simple way: For the motion of a point on a skew plane under the influence of gravity, e.g., in the direct geometric construction (Tab. IV, Fig. 4), BC is the normal to the skew plane, so it is the shortest line from B to the latter, and indeed for an arbitrary finite path or a finite time interval τ. However, that is obviously true for any motion of that kind, which might also be on a surface or a curve, for which the motion of the point is constrained, and which might come into play for forces when the time interval is generally taken to be infinitely small, because the lost force in that case is always a normal pressure on that curve or surface, so BC is normal, and therefore it is the shortest line from B to the curve or surface along which the compatible points D must also lie.

In order to also apply the differential equation for the minimum as a way of finding the acceleration to the simplest case of that kind here, namely, the one is which the prescribed path is rectilinear, let x, y be the coordinates of the point relative to two rectangular axes, where the y axis is taken to have the same sense as gravity, and let s be the path that traverse the skew plane at time t, so if α is the angle between the skew plane and the direction of gravity then $x = s \sin \alpha$, $y = s \cos \alpha$, and:

$$\overline{BC}^2 = \left[\frac{d^2x}{dt^2}\cdot\frac{\tau^2}{2}+\cdots\right]^2+\left[\left(g-\frac{d^2x}{dt^2}\right)\cdot\frac{\tau^2}{2}-\cdots\right]^2$$

$$= \left[\left(\frac{d^2s}{dt^2}\right)^2\sin^2\alpha+\left(g-\frac{d^2s}{dt^2}\cos^2\alpha\right)^2\right]\frac{\tau^4}{4} + \ldots,$$

from which, since the derivative with respect to $d^2 s / dt^2$ must vanish independently of τ, it will follow that:

$$\frac{d^2s}{dt^2}\sin^2\alpha+\left(g-\frac{d^2s}{dt^2}\cos\alpha\right)\cos\alpha = 0,$$

i.e.:

$$\frac{d^2s}{dt^2} = g \cos\alpha,$$

and since the higher derivatives of s with respect to t must vanish, and therefore those of x, y, as well, the second derivative will reduce to the positive quantity:

$$(\sin^2 \alpha + \cos^2 \alpha)\frac{\tau^4}{4}.$$

III. – Now, as far as equilibrium is concerned, for which the principle of least constraint gives a minimum to $\sum m \cdot \overline{AB}^2$, so one establishes that:

$$\sum m(\overline{BD}^2 - \overline{AB}^2) > 0$$

in the first of our examples (Tab. IV, Fig. 3), if we are to change nothing in the figure then we must consider E, E' to be the equilibrium locations, instead of A, A', resp.:

$$\overline{BE} = \overline{B'E'}, \quad \overline{ED} = \overline{E'D'},$$

$$\overline{BD} = \overline{BE} - \overline{ED}, \qquad \overline{B'D'} = \overline{BE} + \overline{ED},$$

and as a result:

$$\sum m \cdot (\overline{BD}^2 - \overline{BE}^2) = m\,[(\overline{BE} - \overline{ED})^2 - \overline{BE}^2] + m'[(\overline{BE} + \overline{ED})^2 - \overline{BE}^2]$$
$$= (m + m') \cdot \overline{ED}^2 - 2\,(m - m') \cdot \overline{ED} \cdot \overline{EB},$$

but in equilibrium one has $m = m'$, so as a result:

$$\sum m \cdot (\overline{BD}^2 - \overline{BE}^2) = 2m \cdot \overline{ED}^2 > 0.$$

A lever (Fig. IV, Tab. 5) will serve as a second example, for which the two masses m, m' will be in equilibrium when they are in the positions A, A', resp., with the lever arms a, a', resp., and the condition for that is $ma = m'a'$. Now, one has, in turn, $\overline{AB} = \overline{A'B'}$, and furthermore, if θ is the angle of rotation of the lever from the position AA' to the position DD' then:

$$AD = 2a \sin \tfrac{1}{2}\theta, \qquad A'D' = 2a' \sin \tfrac{1}{2}\theta,$$

so since the angles at A and A' in the triangles BAD, $B'A'D'$ are $\frac{1}{2}\theta$ and $180^\circ - \frac{1}{2}\theta$, resp.:

$$\overline{BD}^2 = \overline{AB}^2 + 4a^2 \sin^2 \tfrac{1}{2}\theta - 2a \cdot \overline{AB} \cdot \sin\theta,$$

$$\overline{B'D'}^2 = \overline{AB}^2 + 4a'^2 \sin^2 \tfrac{1}{2}\theta - 2a' \cdot \overline{AB} \cdot \sin\theta.$$

As a result:

$$\sum m\cdot(\overline{BD}^2-\overline{BA}^2) = \sum m\cdot\overline{AD}^2 - 2(ma-m'a')\cdot\overline{AB}\cdot\sin\theta ,$$

whose second term must vanish as a result of the equilibrium condition, but the first one will be:

$$\sum m\cdot\overline{AD}^2 = 4\,(ma^2+m'a'^2)\sin^2\tfrac{1}{2}\theta = 4ml\sin^2\tfrac{1}{2}\theta$$

when one denotes the total length of the lever by l and the common rotational moment by μ, so:

$$\sum m\cdot(\overline{BD}^2-\overline{BA}^2) = 4\mu\, l\sin^2\tfrac{1}{2}\theta > 0.$$

In both examples, one has $AB = A'B' = \frac{1}{2}g\tau^2$, so the quantity to be minimized is:

$$\sum m\cdot\overline{AB}^2 = (m+m')\,\frac{g^2\tau^4}{4} ,$$

in which one cannot overlook how and with respect to what it is to be differentiated if one is to confirm a minimum in that way under the assumption of an equilibrium condition, or to derive the equilibrium condition when one assumes a minimum. It is only because that differentiation corresponds to the transition from BA to BD, during which, merely the point A changes, while B remains the same, that one must first introduce the coordinates of the point A in order to be able to complete the differentiation. Therefore, in the first example, let z_0, z'_0 be the vertical ordinates of the equilibrium locations E, E', resp., and let z, z' be those of the points B, B', resp., so:

$$BE = z - z_0 , \quad B'E' = z' - z'_0 ,$$

and as a result:

$$\sum m\cdot\overline{BE}^2 = m\,(z-z_0)^2 + m\,(z-z'_0)^2,$$

and when one differentiates with respect to z_0, the first two derivatives will be:

$$-\,2m\,(z-z_0) - 2m\,(z-z'_0)\,\frac{dz'_0}{dz_0} ,$$

$$2m + 2m'\left(\frac{dz'_0}{dz_0}\right)^2 - 2m'(z'-z'_0)\,\frac{d^2z'_0}{dz_0^2} .$$

However, due to the condition on the system $z_0 + z'_0 = \text{const.}$, it will follow that:

$$\frac{dz'_0}{dz_0} = -1, \qquad \frac{d^2 z'_0}{dz_0^2} = 0,$$

along with all the following derivatives. Therefore, when one now considers that $BE = B'E' = \frac{1}{2} g\tau^2$, the first one will reduce to $-(m - m')\, g\tau^2$ (i.e., to zero, when $m - m'$), and then the second will reduce to the positive quantity $4m$.

If one refers the points A, A', B, B' in the second example to two rectangular axes, viz., the x and y axes, such that the origin lies at the fulcrum of the lever and the positive x axis makes an angle of α with the lever arm a in the equilibrium position, so it makes an angle of $180^\circ + \alpha$ with the other one a', from the condition on the system: Hence, if x_0, y_0 and x, y are the coordinates of m in the positions A, B, resp., and likewise x'_0, y'_0 and x', y' are those of m' in the positions A', B', resp., while AB, as well as $A'B'$ make the angles $90^\circ + \alpha$ and α with the positive x and y axes:

$$\overline{AB}\cdot \sin\alpha = -(x - x_0), \qquad \overline{A'B'}\cdot\sin\alpha = -(x' - x'_0),$$

$$\overline{AB}\cdot\cos\alpha = y - y_0, \qquad \overline{A'B'}\cdot\cos\alpha = y' - y'_0,$$

and

$$\sum m\,\overline{AB}^2 = m\,[(x - x_0)^2 + (y - y_0)^2] + m'\,[(x' - x'_0)^2 + (y' - y'_0)^2].$$

As a result, when one differentiates with respect to α, but only lets the coordinates that refer to the points A, A' vary, the first two derivatives will be:

$$-2m\left[(x - x_0)\frac{dx_0}{d\alpha} + (y - y_0)\frac{dy_0}{d\alpha}\right] - 2m'\left[(x' - x'_0)\frac{dx'_0}{d\alpha} + (y' - y'_0)\frac{dy'_0}{d\alpha}\right],$$

and

$$2m\left[\left(\frac{dx_0}{d\alpha}\right)^2 + \left(\frac{dy_0}{d\alpha}\right)^2\right] + 2m'\left[\left(\frac{dx'_0}{d\alpha}\right)^2 + \left(\frac{dy'_0}{d\alpha}\right)^2\right]$$

$$-2m\left[(x - x_0)\frac{d^2x_0}{d\alpha^2} + (y - y_0)\frac{d^2y_0}{d\alpha^2}\right] - 2m'\left[(x' - x'_0)\frac{d^2x'_0}{d\alpha^2} + (y' - y'_0)\frac{d^2y'_0}{d\alpha^2}\right].$$

Now, one has:

$$x_0 = a\cos\alpha, \qquad \frac{dx_0}{d\alpha} = -a\sin\alpha, \qquad \frac{d^2x_0}{d\alpha^2} = -a\cos\alpha,$$

$$y_0 = a\sin\alpha, \qquad \frac{dy_0}{d\alpha} = a\cos\alpha, \qquad \frac{d^2y_0}{d\alpha^2} = -a\sin\alpha,$$

$$x'_0 = -a\cos\alpha, \qquad \frac{dx'_0}{d\alpha} = a'\sin\alpha, \qquad \frac{d^2x'_0}{d\alpha^2} = a'\cos\alpha,$$

$$y'_0 = -\,a' \cos\alpha, \qquad \frac{dy'_0}{d\alpha} = -\,a' \cos\alpha, \qquad \frac{d^2 y'_0}{d\alpha^2} = a' \sin\alpha\,,$$

which makes the foregoing expressions go to:

$$-\,2m\,a\,[(y - y_0)\cos\alpha - (x - x_0)\sin\alpha] + 2m'a'\,[(y' - y_0^2)\cos\alpha - (x' - x'_0)\sin\alpha]$$

and

$$2\,(ma^2 + m'a'^2) + 2ma\,[(y - y_0)\sin\alpha - (x - x_0)\cos\alpha] - 2m'a'\,[(y' - y_0^2)\sin\alpha + (x' - x'_0)\cos\alpha],$$

but:

$$(y - y_0)\cos\alpha - (x - x_0)\sin\alpha = AB,$$

$$(y' - y'_0)\cos\alpha - (x' - x'_0)\sin\alpha = A'B',$$

$$(y - y_0)\sin\alpha - (x - x_0)\cos\alpha = AB\sin 2\alpha,$$

$$(y' - y'_0)\sin\alpha + (x' - x'_0)\cos\alpha = A'B'\sin 2\alpha\,.$$

Hence, when one further recalls that after the differentiation is completed, one will have $AB = A'B' = \frac{1}{2}g\tau^2$, the first derivative will reduce to $-\,(ma - m'a')\,g\tau^2$, so to zero when $ma - m'a' = 0$, while the second one will reduce to:

$$2\,(ma^2 + m'a'^2) + (ma - m'a')\,g\tau^2 \sin 2\alpha,$$

so under the same condition, to the positive quantity $2\mu\,l$, when one, in turn, sets $ma = m'a' = \mu$, $a + a' = l$.

In this example of the lever, as well, the paths AB, AD, and the time interval τ can be assumed to be finite quantities, as in the foregoing cases of the pulley and the skew plane. Moreover, the difference between the cases is that whereas for the latter, in the application of the principle of virtual velocities the paths that the points of the system traverse under a displacement of it from the equilibrium position and are projected onto the directions of the forces can be finite, that is not true for the lever, but it probably belongs to the cases in which the virtual paths themselves can be taken instead of their projections, and they can then be finite quantities.

§ 3.

Since it results from the remarks in § **1**.III that in order to derive Gauss's principle from d'Alembert's in full generality, one must apply the equilibrium of lost forces that is established by the latter in the most general way (i.e., by means of the principle of virtual velocities), it shall now be shown how the general analytical express for Gauss's theorem can be confirmed by the general formulas of dynamics that expression the vanishing of the sum of the virtual moments of all lost forces. Moreover, that requires some prior discussion that we shall undertake in this paragraph.

I. – It is known that when one refers all points of the system to a rectangular coordinate system and all of the forces that act upon them to their projections onto the axes, d'Alembert's principle will be represented thus:

$$\sum m\left\{\left(X-\frac{d^2x}{dt^2}\right)\delta x+\left(Y-\frac{d^2y}{dt^2}\right)\delta y+\left(Z-\frac{d^2z}{dt^2}\right)\delta z\right\}=0,$$

which we will write as:

(1) $$\sum m\left[\left(X-\frac{d^2x}{dt^2}\right)\delta x\right]=0,$$

in which we abbreviate each of the terms in the expression that correspond to the three coordinates in succession by a single term that we enclose in square brackets. In that formula, x, y, z are the three coordinates of any one of the material points whose mass is m at the time t, so $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$ are its actual accelerations that correspond to the system constraint along the three axes, and X, Y, Z are the projections onto the axes of the accelerating forces that act upon them independently of the system constraints. Hence, $X-\frac{d^2x}{dt^2}$, $Y-\frac{d^2y}{dt^2}$, $Z-\frac{d^2z}{dt^2}$ are its lost forces, and finally, δx, δy, δz are the projections of its so-called virtual velocities onto the three axes, and thus onto the directions of the forces that are to be in equilibrium. We shall treat those quantities in the spirit of the calculus of functions in order to establish their meanings more precisely, since they are ordinarily considered to be infinitesimals when one appeals to the infinitesimal methods. To that end, if we return to the statement of the principle of virtual velocities, which requires the system to be subjected to a displacement from the equilibrium position that is generally infinitely small and compatible with its nature, and we project the paths that the individual points of the system traverse along the directions of the forces that should be in equilibrium at them, or project both of them onto a common direction, then the sum of the products of those two types of projection with the masses – i.e., the sum of the virtual moments – should vanish. Now, the projections onto the axes of the infinitely small paths or those of the infinitely small variations of the coordinates that correspond to the displacement are usually denoted by δx, etc., and considered to be the virtual velocities. However, we must distinguish between the virtual path and the virtual velocity, and for that reason, if Dx denotes an initially finite or infinitely small increment in x (since there also cases in which the virtual paths can be finite) then the next way of expressing the statement of the principle above can be represented by:

(2) $$\sum m\left[\left(X-\frac{d^2x}{dt^2}\right)Dx\right]=0,$$

in which Dx is now a quantity of the same kind as Δx, except that Δx corresponds to the actual motion of the system during the time interval Δt when it is left to itself, while Dx corresponds to an arbitrary, but compatible, displacement, so to a virtual motion. However, one can convert:

$$\Delta x=\frac{dx}{dt}\Delta t+\frac{d^2x}{dt^2}\frac{\Delta t^2}{2}+\dots$$

into Dx by replacing the time derivatives $\frac{dx}{dt}$, $\frac{d^2x}{dt^2}$, etc., that are determined by the actual motion with arbitrary functions of time, but with the restriction that they are constrained by the condition equations of the system, such that when one denotes those arbitrary quantities by δx, δ^2x, etc., as in the calculus of variations, one can set:

$$Dx=\delta x\cdot\Delta t+\delta^2x\cdot\frac{\Delta t^2}{2}+\dots,$$

in which Δt serves merely as an entirely arbitrary amount of advance that can be taken to be finite or infinitely small according to the situation, and thus determine the order of magnitude of Dx. That is entirely consistent with the spirit of the calculus of variations, where one goes from a given function $x = f\,t$ of t to the varied one by setting $x + Dx = \varphi(t, \varepsilon)$, if ε is a new variable that is absolutely independent and φ is an arbitrary function, except that one must have $\varphi(t, 0) = f\,t$, and when one develops $x + Dx$ in powers of ε, one will have:

$$x+Dx=\varphi_0+\left(\frac{d\varphi}{d\varepsilon}\right)_0\varepsilon+\left(\frac{d^2\varphi}{d\varepsilon^2}\right)_0\frac{\varepsilon^2}{2}+\dots=x+\varepsilon\,\delta x+\frac{\varepsilon^2}{2}\delta^2x+\dots$$

However, one can switch ε with Δt here, since ε is nothing but a completely independent quantity that can take on any degree of smallness, just like the time increment Δt, such that one will have $x + Dx = \varphi(t, \Delta t)$, while $x + \Delta x = f(t + \Delta t)$, and one can add that, since that varied function is always subject to the condition $x = \varphi(t, 0)$, in some situations, one can impose the further conditions that $\frac{dx}{dt}=\delta x$, $\frac{d^2x}{dt^2}=\delta^2x$, etc., up to an arbitrary term at which Δx and Dx should coincide.

However, if one develops formula (2) in Δt using the values of Dx that were presented then one will get:

$$\sum m\left[\left(X-\frac{d^2x}{dt^2}\right)\delta x\right]+\tfrac{1}{2}\Delta t\sum m\left[\left(X-\frac{d^2x}{dt^2}\right)\delta^2x\right]+\dots=0$$

after one drops the common factor Δt, and one must now distinguish between the cases in which the displacement of the system *must* be infinitely small and the ones in which it *can* be finite. In the former case, Δt is taken to be infinitely small, and the foregoing equation will reduce to its first term; i.e., to equation (1). In the latter case, where Δt must have only the degree of smallness that makes the sequence converge, because the equation must be true independently of that arbitrary quantity, it will decompose into an infinite set of equations, the first of which is (1), and which will have the general form:

(3) $$\sum m\left[\left(X - \frac{d^2 x}{dt^2}\right)\delta^n x\right] = 0,$$

in which n can be any positive whole number from unity onwards, and the agreement between all of those equations will be just the analytical indicator that the virtual paths can have a finite magnitude for a system in which that is true.

II. – The special equations of motion for the system into which formulas (1) decomposes can be represented by either introducing all of the condition equations of the system into formula (1) with undetermined factors and then treating all coordinates directly as if they were just as many independent variables or by assuming that the coordinates are reduced to the smallest number of independent variables by means of this condition equations, in which it is necessary in some situations to also appeal to the formulas of the coordinate transformation. In the first case, let $L = 0$, $L' = 0$, etc., be those condition equations – i.e., relations between the coordinates of the individual points of the system by which they are constrained during their entire motion – such that $dL / dt = 0$, etc., and likewise $\delta L = 0$, etc., and which can generally include time either merely directly, implicitly, or also explicitly, such that with the exception of the latter case, there are just as many equations $dL / dt = 0$, $dL = 0$, as there are:

$$\left[\frac{dL}{dx_1}\frac{dx_1}{dt}\right] + \left[\frac{dL}{dx_2}\frac{dx_2}{dt}\right] + \ldots = 0 \qquad \text{or} \qquad \sum\left[\frac{dL}{dx}\frac{dx}{dt}\right] = 0,$$

and

$$\left[\frac{dL}{dx_1}\delta x_1\right] + \left[\frac{dL}{dx_2}\delta x_2\right] + \ldots = 0 \qquad \text{or} \qquad \sum\left[\frac{dL}{dx}\delta x\right] = 0\,.$$

If $\lambda, \lambda', \lambda'', \ldots$ are the so-called elimination factors then the equation:

(4) $$\sum m\left[\left(X - \frac{d^2 x}{dt^2}\right)\delta x\right] + \lambda\,\delta L + \lambda'\,\delta L' + \ldots = 0,$$

which now enters in place of (1), will decompose into just as many equations as the number m of points that are present times three, and they will have the form:

$$(5)\qquad \begin{cases} m\left(X-\dfrac{d^2x}{dt^2}\right)+\lambda\dfrac{dL}{dx}+\lambda'\dfrac{dL'}{dx}+\cdots=0, \\ m\left(Y-\dfrac{d^2y}{dt^2}\right)+\lambda\dfrac{dL}{dy}+\lambda'\dfrac{dL'}{dy}+\cdots=0, \\ m\left(Z-\dfrac{d^2z}{dt^2}\right)+\lambda\dfrac{dL}{dz}+\lambda'\dfrac{dL'}{dz}+\cdots=0, \end{cases}$$

and the equations of motion of the system are the result of eliminating all quantities λ between those systems of that form.

In the second case, let ω, ψ, χ, etc., be the latter geometric independent variables, or just as many mutually independent functions of time, which one can set equal to the coordinates of the points of the system after one considers all of the conditions on it. Those coordinates are then, in turn, generally either purely geometric functions of those independent variables or coupled with them by relations in which time does not enter explicitly besides, such that one will have $x = f(\omega, \psi, \chi, \ldots)$, except in the latter case, and as a result:

$$\frac{dx}{dt} = \frac{dx}{d\omega}\frac{d\omega}{dt}+\frac{dx}{d\psi}\frac{d\psi}{dt}+\ldots,$$

$$\frac{d^2x}{dt^2} = \frac{dx}{d\omega}\frac{d^2\omega}{dt^2}+\frac{d^2x}{d\omega^2}\left(\frac{d\omega}{dt}\right)^2+2\frac{d^2x}{d\omega\,d\psi}\frac{d\omega}{dt}\frac{d\psi}{dt}+\frac{dx}{d\psi}\frac{d^2\psi}{dt^2}+\frac{d^2x}{d\psi^2}\left(\frac{d\psi}{dt}\right)^2+\ldots,$$

$$\begin{aligned}\frac{d^3x}{dt^3} = {} & \frac{dx}{d\omega}\frac{d^3\omega}{dt^3}+\frac{d^3x}{d\omega^3}\left(\frac{d\omega}{dt}\right)^3+3\frac{d^2x}{d\omega^2}\frac{d\omega}{dt}\frac{d^2\omega}{dt^2} \\ & + 3\frac{d^2x}{d\omega\,d\psi}\left(\frac{d\psi}{dt}\frac{d^2\omega}{dt^2}+\frac{d\omega}{dt}\frac{d^2\psi}{dt^2}\right)+3\frac{d^3x}{d\omega^2\,d\psi}\left(\frac{d\omega}{dt}\right)^2\frac{d\psi}{dt}+3\frac{d^3x}{d\omega\,d\psi^2}\frac{d\omega}{dt}\left(\frac{d\psi}{dt}\right)^2 \\ & + \frac{dx}{d\psi}\frac{d^3\psi}{dt^3}+\frac{d^3x}{d\psi^3}\left(\frac{d\psi}{dt}\right)^3+3\frac{d^2x}{d\psi^2}\frac{d\psi}{dt}\frac{d^2\psi}{dt^2}+\ldots,\end{aligned}$$

etc., and likewise:

$$\delta x = \frac{dx}{d\omega}\delta\omega+\frac{dx}{d\psi}\delta\psi+\ldots,$$

$$\delta^2 x = \frac{dx}{d\omega}\delta^2\omega + \frac{d^2x}{d\omega^2}\delta\omega^2 + 2\frac{d^2x}{d\omega\, d\psi}\delta\omega\,\delta\psi + \frac{dx}{d\psi}\delta^2\psi + \frac{d^2x}{d\psi^2}\delta\psi^2 + \ldots,$$

etc., and furthermore:

$$\Delta x \;=\; \frac{dx}{d\omega}\Delta\omega + \frac{dx}{d\psi}\Delta\psi + \cdots + \frac{d^2x}{d\omega^2}\frac{\Delta\omega^2}{2} + \frac{d^2x}{d\omega\, d\psi}\Delta\omega\,\Delta\psi + \frac{d^2x}{d\psi^2}\frac{\Delta\psi^2}{2} + \ldots,$$

$$Dx \;=\; \frac{dx}{d\omega}D\omega + \frac{dx}{d\psi}D\psi + \cdots + \frac{d^2x}{d\omega^2}\frac{D\omega^2}{2} + \frac{d^2x}{d\omega\, d\psi}D\omega\, D\psi + \frac{d^2x}{d\psi^2}\frac{D\psi^2}{2} + \ldots,$$

$$\Delta\omega = + \ldots, \qquad D\omega = \delta\omega\,\Delta t + \ldots, \text{ etc.}$$

Now, just as sometimes the coordinates of all points can enter into the condition equations $L = 0$, but sometimes just some of them, and sometimes just one of them, so can all of the variables, some of them, or all of the independent ones enter into the relations $x = f(\omega, \psi, \ldots)$. Furthermore, if one such independent variable ω is to befit several points of the system in common then that would mean that for each of those points, from any arbitrary time point t onwards, its change $\Delta\omega$ or $D\omega$ would have the same value, while the value that corresponds to the time point t would have the form $\alpha + \omega$, where α is a constant that has different values that are given by the system constraint for different points of the system that depend upon ω. However, in part in order to refer to the variable ω beforehand without distinguishing between all points of the system, and in part in order to also be able to deal with the special case in which if, e.g., ω is an angle then depending upon how many of them there are, $\Delta\omega$ would prove to be positive for the one point and negative for the others, one can establish more generally that for any point of the system, the value of ω at time t will have the form $\alpha + a\omega$, where a is a constant that is similar to α, namely, both of them can be zero, such that the relation between one coordinate x and the independent variables can generally represented by:

$$x = f(\alpha + a\omega, \beta + b\psi, \ldots),$$

instead of by $x = f(\omega, \psi, \ldots)$. Now, since the variations $\delta\omega$, $\delta^2\omega$, … of a geometric independent variable ω are considered to be completely arbitrary, mutually independent functions of time, when one introduces the value of δx into equation (1), it will decompose into just as many equations:

$$(6) \qquad \sum m\left[\left(X - \frac{d^2x}{dt^2}\right)\frac{dx}{d\omega}\right] = 0, \qquad \sum m\left[\left(X - \frac{d^2x}{dt^2}\right)\frac{dx}{d\psi}\right] = 0, \qquad \text{etc.,}$$

as there are independent variables ω, ψ, … They will be equations that all have order two with respect to the time derivatives of the independent variables, by means of the value of $\frac{d^2x}{dt^2}$, and will therefore suffice to determine the motion of the system.

III. – Of the two forms [viz., (5) and (6)] that one can give to the equations of motion for a system, it is the latter that is suitable for our purposes. Just as it can emerge from (1), by introducing independent variables, formula (3) will, in fact, decompose into equations whose general form is:

(7) $$\sum m\left[\left(X - \frac{d^2x}{dt^2}\right)\frac{d^n x}{d\omega^{n'} d\omega^{n''}\cdots}\right] = 0,$$

in which n', n'', … are likewise positive whole numbers, but all of their values can be zero, and they are coupled by the condition $n' + n'' + \ldots = n$. However, since equations (6) already determine the motion completely, all of the equations in formula (7), except for that one, will include equations that are identical to it, which immediately demands that all higher derivatives of the coordinates with respect to ω, ψ, … will vanish or reproduce the first one; i.e., in general, that the coordinates will be either linear functions of the independent variables:

$$\Delta = A + a\omega + b\psi + c\chi + \ldots$$

or exponential functions of the form:

$$\Delta = K\,k^{\alpha\omega + \beta\psi + \gamma\chi + \ldots}$$

with the arbitrary basis k, where K, A, a, b, c, … are constants relative to time that can vary from one point of the system to another, while α, β, γ, … are constants that are independent of the system constraints, like k. Hence, they would be the analytical conditions for the virtual paths to be finite. However, one will get even more coexisting equations in that case by successive differentiation of (7) with respect to t. The first one gives:

$$\sum m\left\{\left(X - \frac{d^2x}{dt^2}\right)\left(\frac{d^{n+1}x}{d\omega^{n'+1}d\psi^{n''}\cdots}\frac{d\omega}{dt} + \frac{d^{n+1}x}{d\omega^{n'}d\psi^{n''+1}\cdots}\frac{d\psi}{dt} + \cdots\right) + \frac{d}{dt}\left(X - \frac{d^2x}{dt^2}\right)\cdot\frac{d^n x}{d\omega^{n'}d\psi^{n''}\cdots}\right\} = 0,$$

from which, since the first component of that expression will vanish due to (7), when the expressions $d\omega / dt$, $d\psi / dt$, … enter before the summation sign, one will have:

(8) $$\sum m\left[\frac{d}{dt}\left(X - \frac{d^2x}{dt^2}\right)\cdot\frac{d^n x}{d\omega^{n'}d\psi^{n''}\cdots}\right] = 0\,.$$

A second one (i.e., differentiating the latter expression) will likewise lead to:

$$\sum m\left[\frac{d^2}{dt^2}\left(X - \frac{d^2x}{dt^2}\right)\cdot\frac{d^n x}{d\omega^{n'}d\psi^{n''}\cdots}\right] = 0,$$

and when one proceeds in that way, one will arrive at the general formula:

(9) $$\sum m\left[\frac{d^i}{dt^i}\left(X - \frac{d^2x}{dt^2}\right)\cdot\frac{d^n x}{d\omega^{n'}d\psi^{n''}\cdots}\right] = 0,$$

in which the foregoing, along with (7), are included for the values i = 0, 1, 2, resp., and in which i can take on all positive whole numbers from zero onwards. If the virtual path is not to be finite, so (7) will not be true, then the total time derivatives of equations (6) will indeed be valid, but they will not decompose into equations of the forms (8) and (9).

We finally infer the following result from this:

1) If we add equations (6) after multiplying by $d\omega / dt$, $d\psi / dt$, …, resp., then since we have restricted ourselves to the case in which x depends upon t only by way of the independent variables ω, ψ, …, we will get an equation:

(10) $$\sum m\left[\left(X - \frac{d^2x}{dt^2}\right)\frac{dx}{dt}\right] = 0$$

that is entirely analogous to (6) and (1) in form, and whose integral is known to include the *vis viva* principle. One will likewise get it when one switches the variations or virtual velocities δx in (1) with the time derivatives – or actual velocities dx / dt. We can also say, by switching the virtual path Dx in (2) with the actual path Δx, which are admissible switches, as well relations between the coordinates and the independent variables or the condition equations of the system that do not include time explicitly, as **Lagrange** expressed it in analytical mechanics. Now, in the cases where Dx can be finite, the same thing must also be true for Δx, and one will then have as a consequence of:

(11) $$\sum m\left[\left(X - \frac{d^2x}{dt^2}\right)\Delta x\right] = 0,$$

along (10), a series of equations with the general form:

(12) $$\sum m\left[\left(X - \frac{d^2x}{dt^2}\right)\frac{d^n x}{dt^n}\right] = 0,$$

which is, in fact, justified by (7) when one introduces the expression for $d^n x / dt^n$ in terms of the independent variables.

2) Since the quantities X and x include the increments ΔX, Δx during the time interval, from formula (1), one will have:

$$\sum m\left[\left(X + \Delta X - \frac{d^2(x+\Delta x)}{dt^2}\right)\delta(x+\Delta x)\right] = 0 \tag{13}$$

[in which one can also write D for δ, which is analogous to formula (3)]. That is, one will have the equilibrium formula for the lost force at time $t + \Delta t$, and since:

$$\Delta X = \frac{dX}{dt}\Delta t + \frac{d^2 X}{dt^2}\frac{\Delta t^2}{2} + \ldots,$$

$$\Delta x = \frac{dx}{dt}\Delta t + \frac{d^2 x}{dt^2}\frac{\Delta t^2}{2} + \ldots,$$

and as a result:

$$\frac{d^2\Delta x}{dt^2} = \Delta\frac{d^2 x}{dt^2} = \frac{d^3 x}{dt^3}\Delta t + \frac{d^4 x}{dt^4}\frac{\Delta t^2}{2} + \ldots,$$

$$\delta\,\Delta x = \Delta\delta x = \frac{d\delta x}{dt}\Delta t + \frac{d^2\delta x}{dt^2}\frac{\Delta t^2}{2} + \ldots,$$

$$X + \Delta X - \frac{d^2(x+\Delta x)}{dt^2} = X - \frac{d^3 x}{dt^3} + \frac{dx}{dt}\left(X - \frac{d^2 x}{dt^2}\right)\Delta t + \ldots,$$

the first part of (13) will define a development in Δt whose individual terms must vanish, due to the arbitrariness in the magnitude of the time interval, from which one will get a consequence of the equations:

$$\sum m\left[\left(X - \frac{d^2 x}{dt^2}\right)\frac{d\delta x}{dt} + \frac{d}{dt}\left(X - \frac{d^2 x}{dt^2}\right)\cdot\delta x\right] = 0,$$

$$\sum m\left[\left(X - \frac{d^2 x}{dt^2}\right)\frac{d^2\delta x}{dt^2} + 2\frac{d}{dt}\left(X - \frac{d^2 x}{dt^2}\right)\frac{d\delta x}{dt} + \frac{d^2}{dt^2}\left(X - \frac{d^2 x}{dt^2}\right)\cdot\delta x\right] = 0,$$

etc.,

and naturally these are nothing but the total derivatives of equations (1) with respect to time t. However, in the case where equation (7) is true, and as a result (9), they will, in turn, decompose into equations of the general form:

(14) $$\sum m\left[\frac{d^i}{dt^i}\left(X-\frac{d^2x}{dt^2}\right)\cdot\frac{d^{i'}\delta x}{dt^{i'}}\right]=0,$$

in which i and i' are positive whole numbers that can take on all values from zero onwards, such that equation (1) is contained in the latter equation for $i + i' = 0$, and for $i + i' = 1$, it will contain these two:

$$\sum m\left[\left(X-\frac{d^2x}{dt^2}\right)\cdot\frac{d\delta x}{dt}\right]=0,\qquad \sum m\left[\frac{d}{dt}\left(X-\frac{d^2x}{dt^2}\right)\cdot\delta x\right]=0,$$

which one can, in fact, justify immediately by means of (9) when one introduces the values of δx and $d\delta x / dt$; i.e.:

$$\frac{d\delta x}{dt}=\frac{dx}{d\omega}\frac{d\delta\omega}{dt}+\left(\frac{d^2x}{d\omega^2}\frac{d\omega}{dt}+\frac{d^2x}{d\omega\, d\psi}\frac{d\psi}{dt}+\cdots\right)\delta\omega+\ldots$$

Finally, since the equations that are included in (4) are also true then, it will likewise follow, just as (14) follows from (1), that:

(15) $$\sum m\left[\frac{d^i}{dt^i}\left(X-\frac{d^2x}{dt^2}\right)\cdot\frac{d^{i'}\delta^n x}{dt^{i'}}\right]=0,$$

which is justified in the same way that (14) is justified by (9).

§ 4.

It can now be shown that the principle of least constraint leads to the same equations as the formula for the equilibrium of the constraint forces, and indeed in such a way that in the cases where the principle of virtual velocities is true for finite displacements of the system, our principle will also be valid for a finite time interval, while in general (i.e., when the virtual path must be taken to be infinitely small), the time interval for which the free motion of the point of the system will be compared with the actual can also be taken to be infinitely small here.

I. – Let A (to preserve the terminology of § **1**) be the position of any material point m of the system, whose rectangular coordinates at time t are x, y, z, and at the time $t + \Delta t$, it will go to the two corresponding positions C and B, whose coordinates are $x + \Delta x$, $y + \Delta y$, $z + \Delta z$, and $x + \xi$, $y + \eta$, $z + \zeta$, resp., in which Δx, Δy, Δz, and likewise ξ, η, ζ, are functions of Δt then. If one then sets:

$$BC = u \qquad \text{and} \qquad \sum mu^2 = U$$

then one will have:

$$u^2 = (\xi - \Delta x)^2 + (\eta - \Delta y)^2 + (\zeta - \Delta z)^2,$$

and the quantity U that is to be a minimum will be:

(*a*) $$U = \sum m\left[(\xi - \Delta x)^2\right] .$$

Now, in order to develop U in Δt, and therefore as a function of Δt , one will set:

$$\xi = \xi_0 + \left(\frac{d\xi}{d\Delta t}\right)_0 \Delta t + \left(\frac{d^2\xi}{d\Delta t^2}\right)_0 \frac{\Delta t^2}{2} + \left(\frac{d^3\xi}{d\Delta t^3}\right)_0 \frac{\Delta t^3}{2\cdot 3} + \ldots,$$

but for $\Delta t = 0$, for time t , one will have:

$$\xi_0 = 0, \qquad \left(\frac{d\xi}{d\Delta t}\right)_0 = \frac{dx}{dt}, \qquad \left(\frac{d^2\xi}{d\Delta t^2}\right)_0 = X,$$

in which, as in § **3**, X, Y, Z are the components along the axes of the accelerating forces that act upon the material point at time t independently of the system constraints. Furthermore, one has:

$$\frac{d^2\xi}{d\Delta t^2} = X + \Delta X = X + \frac{dX}{dt}\Delta t + \frac{d^2X}{dt^2}\frac{\Delta t^2}{2} + \ldots$$

at the end of the time interval Δt, but also, with the foregoing expression for ξ :

$$\frac{d^2\xi}{d\Delta t^2} = \left(\frac{d^2\xi}{d\Delta t^2}\right)_0 + \left(\frac{d^3\xi}{d\Delta t^3}\right)_0 \Delta t + \left(\frac{d^4\xi}{d\Delta t^4}\right)_0 \frac{\Delta t^2}{2} + \ldots,$$

and as a result, upon comparing these two identical developments:

$$\left(\frac{d^3\xi}{d\Delta t^3}\right)_0 = \frac{dX}{dt}, \qquad \left(\frac{d^4\xi}{d\Delta t^4}\right)_0 = \frac{d^2X}{dt^2}, \qquad \text{etc.}$$

One will then have:

$$\xi = \frac{dx}{dt}\Delta t + X\frac{\Delta t^2}{2} + \frac{dX}{dt}\frac{\Delta t^3}{2\cdot 3} + \ldots$$

and

$$\Delta x = \frac{dx}{dt}\Delta t + \frac{d^2x}{dt^2}\frac{\Delta t^2}{2} + \frac{d^3x}{dt^3}\frac{\Delta t^3}{2\cdot 3} + \ldots,$$

and as a result:

$$\xi - \Delta x = \left(X - \frac{d^2x}{dt^2}\right)\frac{\Delta t^2}{2} + \frac{d}{dt}\left(X - \frac{d^2x}{dt^2}\right)\cdot\frac{\Delta t^3}{2\cdot 3} + \ldots,$$

so:

$$(b) \qquad U = \frac{\Delta t^4}{4}\sum m\left[\left(X - \frac{d^2x}{dt^2} + \frac{\Delta t}{3}\frac{d}{dt}\left(X - \frac{d^2x}{dt^2}\right) + \cdots\right)^2\right].$$

From the developments in § **3**, III, one also has:

$$\frac{d^2(\xi - \Delta x)}{d\Delta t^2} = X + \Delta X - \frac{d^2(x+\Delta x)}{dt^2},$$

so the formula (13) there can be represented in the form:

$$(16) \qquad \sum m\left[\frac{d^2(\xi - \Delta x)}{d\Delta t^2}\cdot D(x+\Delta x)\right] = 0$$

when one likewise replaces the virtual velocities with the virtual paths.

II. – Now, in order to show directly by analogy with the Gaussian proof that U is smaller than when one starts from A and takes a compatible point D (i.e., another point of the trajectory) at time $t + \Delta t$, instead of the actual location C, let $x + Dx$, $y + Dy$, $z + Dz$ be the coordinates of m at the position D, where Dx, Dy, Dz are arbitrary changes in the coordinates that are compatible with the state of the system at time t that will take one from A to D (hence, they are quantities of the type in § **3**) along the virtual path that is denoted in this way:

$$Dx = \delta x \cdot \Delta t + \delta^2 x \cdot \frac{\Delta t^2}{2} + \ldots,$$

but with the difference that, from the discussion in regard to the Gaussian proof in § **1**, the velocities dx / dt that are reached at the time t have ξ introduced into Dx, as in Δx ; i.e., one takes $\delta x = dx / dt$, and at the same time $\delta\omega = d\omega / dt$, etc. However, one will then have:

$$Dx - \Delta x = \left(\delta^2 x - \frac{d^2x}{dt^2}\right)\frac{\Delta t^2}{2} + \left(\delta^3 x - \frac{d^3x}{dt^3}\right)\frac{\Delta t^3}{2\cdot 3} + \ldots,$$

and from the expressions for $\frac{d^2x}{dt^2}$, $\delta^2 x$, etc., in terms of the independent variables when one sets $\delta\omega = \frac{d\omega}{dt}$, $\delta\psi = \frac{d\psi}{dt}$, etc., in them:

$$\delta^2 x - \frac{d^2 x}{dt^2} = \frac{dx}{d\omega}\left(\delta^2\omega - \frac{d^2\omega}{dt^2}\right) + \frac{dx}{d\psi}\left(\delta^2\psi - \frac{d^2\psi}{dt^2}\right) + \ldots,$$

$$\delta^3 x - \frac{d^3 x}{dt^3} = \frac{dx}{d\omega}\left(\delta^3\omega - \frac{d^3\omega}{dt^3}\right) + \frac{dx}{d\psi}\left(\delta^3\psi - \frac{d^3\psi}{dt^3}\right) + \ldots$$
$$+\ 3\left(\frac{d^2 x}{d\omega^2}\frac{d\omega}{dt} + \frac{d^2 x}{d\omega\, d\psi}\frac{d\psi}{dt} + \cdots\right)\left(\delta^2\omega - \frac{d^2\omega}{dt^2}\right) + \ldots,$$

etc.

Now let:

$$BD = u', \qquad \sum m u'^2 = U',$$

so one can show that:

$$U' - U = \sum m(u'^2 - u^2) > 0.$$

However:

$$\begin{aligned} u'^2 &= (\xi - Dx)^2 + (\eta - Dy)^2 + (\zeta - Dz)^2 \\ &= [(\xi - \Delta x - (Dx - \Delta x))^2] \\ &= u^2 + [(Dx - \Delta x)^2] - 2\,[(\xi - \Delta x)\,(Dx - \Delta x)], \end{aligned}$$

and as a result:

(*c*) $$U' - U = \sum m[(Dx - \Delta x)^2] - 2\sum m[(\xi - \Delta x)(Dx - \Delta x)],$$

and one has $U' - U > 0$, so U will be a minimum when the second term in that expression vanishes, or when:

(17) $$\sum m[(\xi - \Delta x)(Dx - \Delta x)] = 0,$$

since $U' - U$ then comes down to the essentially positive expression:

$$\sum m[(Dx - \Delta x)^2] \qquad \text{or} \qquad \sum m \cdot \overline{CD}^2 .$$

However, if one develops the first part of the equation (17) to be proved in terms of the expressions for $\xi - \Delta x$, $Dx - \Delta x$ in terms of Δt, and one puts all variations and time derivatives in front of the summation sign then one will find that each term is affected with a summation that has the form of the ones that define one side of equation (9). Thus, when the system is such that the virtual paths can have finite magnitudes, all of the individual terms in equation (17) will vanish due to equation (9), which will then be true, and those terms might also be Δt as long as only the first part of the series that it defines converges. That is, the principle of least constraint if true for finite time intervals Δt that can have any arbitrary magnitude, moreover, in the case where the coordinates are linear functions of the independent variables, because in the derivatives $\dfrac{d^n x}{d\omega^{n'}\,d\psi^{n''}\cdots}$, all of the terms except for the first one will vanish from $n = 2$ onward, so Δt can no longer fulfill any convergence conditions. However, if the system admits only infinitely small virtual paths, which is generally the case, then equation (9) will be true for only $i = 0$, $n = 1$, or will reduce to equations (6) due to the fact that only the first term in the development of equation (17) in Δt will vanish, so Δt must be taken to be infinitely small in order for that development to come down to the first term, or for (c) to come down to:

(d) $\quad U' - U =$

$$= \frac{\Delta t^4}{4}\left\{\sum m\left[\left(\frac{dx}{d\omega}\left(\delta^2\omega - \frac{d^2\omega}{dt^2}\right) + \frac{dx}{d\psi}\left(\delta^2\psi - \frac{d^2\psi}{dt^2}\right) + \cdots\right)^2\right] - 2\left(\left(\delta^2\omega - \frac{d^2\omega}{dt^2}\right)\sum m\left(X - \frac{d^2\omega}{dt^2}\right)\frac{dx}{d\omega} + \cdots\right)\right\},$$

which likewise shows the homogeneity of that expression.

The development of $Dx - \Delta x$ above can also be done in such a way that one considers that quantity, and correspondingly $D\omega - \Delta\omega = h$, $D\psi - \Delta\psi = i$, etc., to be increments of Δx and $\Delta\omega$, $\Delta\psi$, … when one goes from the point C to the point D; one will then have:

$$Dx - \Delta x = \frac{d\Delta x}{d\Delta\omega}h + \frac{d\Delta x}{d\Delta\psi}i + \cdots + \frac{d^2 x}{d\Delta\omega^2}\frac{h^2}{2} + \frac{d^2\Delta x}{d\Delta\omega\, d\Delta\psi}hi + \frac{d^2 x}{d\Delta\psi^2}\frac{i^2}{2} + \cdots,$$

where:

$$h = D\omega - \Delta\omega = \left(\delta^2\omega - \frac{d^2\omega}{dt^2}\right)\frac{\Delta t^2}{2} + \ldots,$$

$$i = D\psi - \Delta\psi = \left(\delta^2\psi - \frac{d^2\psi}{dt^2}\right)\frac{\Delta t^2}{2} + \ldots,$$

etc.,

and with the expressions for Δx in terms of $\Delta\omega$, $\Delta\psi$, … that were given in § **3** :

$$\frac{d\Delta x}{d\Delta\omega} = \frac{dx}{d\omega} + \frac{d^2 x}{d\omega^2}\Delta\omega + \frac{d^2 x}{d\omega\, d\psi}\Delta\psi + \cdots = \frac{dx}{d\omega} + \left(\frac{d^2 x}{d\omega^2}\frac{d\omega}{dt} + \frac{d^2 x}{d\omega\, d\psi}\frac{d\psi}{dt} + \cdots\right)\Delta t + \cdots,$$

$$\frac{d^2\Delta x}{d\Delta\omega^2} = \frac{d^2 x}{d\omega^2} + \left(\frac{d^3 x}{d\omega^3}\frac{d\omega}{dt} + \frac{d^3 x}{d\omega^2\, d\psi}\frac{d\psi}{dt} + \cdots\right)\Delta t + \cdots,$$

$$\frac{d^2\Delta x}{d\Delta\omega\, d\Delta\psi} = \frac{d^2 x}{d\omega\, d\psi} + \left(\frac{d^3 x}{d\omega^2\, d\psi}\frac{d\omega}{dt} + \frac{d^3 x}{d\omega\, d\psi^2}\frac{d\psi}{dt} + \cdots\right)\Delta t + \cdots,$$

which are formulas that will used later. – Finally, one can suggest the transition from the point C to the point D by increments $D\,(x + \Delta x)$ in the coordinates $x + \Delta x$, such that the increment from D will have the form $x + \Delta x + D\,(x + \Delta x)$, where one next has:

$$D\,(x + \Delta x) = Dx + \Delta Dx = Dx + \frac{dDx}{dt} + 3\frac{d^2 Dx}{dt^2}\frac{\Delta t^2}{2} + \ldots,$$

as in formula (16), and:

$$Dx = \Delta t\ \delta x + \frac{\Delta t^2}{2}\delta^2 x + \ldots,$$

so

$$D\,(x + \Delta x) = \delta x\ \Delta t + \left(\delta^2 x + 2\frac{d\delta x}{dt}\right)\frac{\Delta t^2}{2} + \left(\delta^3 x + 3\frac{d\delta^2 x}{dt} + 3\frac{d^2\delta x}{dt^2}\right)\frac{\Delta t^3}{2\cdot 3} + \cdots,$$

such that this quantity will have the same scale as the square of Δt, and since one must also likewise take $\delta\omega = 0$, $\delta\psi = 0$, etc., one will have:

$$\delta^2 x = \frac{dx}{d\omega}\delta^2\omega + \frac{dx}{d\psi}\delta^2\psi + \cdots,$$

$$\delta^3 x = \frac{dx}{d\omega}\delta^3\omega + \frac{d^2 x}{d\psi^2}\delta^2\psi^2 + 2\frac{d^2 x}{d\omega\, d\psi}\delta^2\omega\,\delta^2\psi + \cdots,$$

etc.

If one then develops the equation that now enters in place of (17):

$$\sum m\,[(\xi - \Delta x)\, D\,(x + \Delta x)] = 0 \tag{18}$$

then it will be easy to see that, like the development of (15), it yields nothing but terms that will vanish due to equation (9) when it is true, whereas for an infinitely small Δt, instead of (d), one will now have:

$$(d') \qquad U' - U = \frac{\Delta t^4}{4}\left\{\sum m\left[\left(\frac{dx}{d\omega}\delta^2\omega + \frac{dx}{d\psi}\delta^2\psi + \cdots\right)^2\right] - 2\left(\delta^2\omega \sum m\left[\left(X - \frac{d^2 x}{dt^2}\right)\frac{dx}{d\omega}\right] + \cdots\right)\right\},$$

which is a homogeneous formula whose second part will vanish due to equations (6), and therefore, due to (1) (*).

III. – Now, in order to also treat the minimum directly by differential calculus, we can start with the form (*b*) for the U that is developed in Δt (as in § **2**, where we were led to that form by the nature of things) and differentiate with respect to the actual accelerations, and thus set equal to zero its first derivatives with respect to each of the independent accelerations (i.e., with respect to the second time derivatives of the independent variables), and thus obtain just as many equations as there are independent variables. When one reduces the development in Δt to its first term, which is affected with Δt^4, as in generally required, and sets:

$$\frac{d^2\omega}{dt^2} = \omega'', \qquad \frac{d^2\psi}{dt^2} = \psi'', \qquad \text{etc.},$$

those equations will be:

$$(19) \qquad \sum m\left[\left(X - \frac{d^2x}{dt^2}\right)\frac{d}{d\omega''}\frac{d^2x}{dt^2}\right] = 0, \qquad \sum m\left[\left(X - \frac{d^2x}{dt^2}\right)\frac{d}{d\psi''}\frac{d^2x}{dt^2}\right] = 0, \qquad \text{etc.},$$

i.e., equations (6), since due to the expression for $\frac{d^2x}{dt^2}$ in terms of the independent variables, one has:

$$\frac{d}{d\omega''}\frac{d^2x}{dt^2} = \frac{dx}{d\omega}, \qquad \frac{d}{d\psi''}\frac{d^2x}{dt^2} = \frac{dx}{d\psi}, \qquad \text{etc.}$$

However, we shall not pursue that aspect of the matter any further, since when ω', ω''', etc., represent the first, third, etc., time derivatives of ω in it, we will have:

(*) Naturally, the variations that correspond to the transition from C to D do not have the same meaning here that they had in formula (*d*), which referred to the transition from A to D. If one distinguishes between the two concepts by putting a prime on δ for now, then:

$$\delta'^2\omega = \delta^2\omega - \frac{d^2\omega}{dt^2}, \qquad \text{etc.},$$

and correspondingly:

$$\delta'^2x = \delta^2 x - \frac{d^2x}{dt^2}.$$

Above all, when one compares formulas (17) and (18), one must have:

$$D'(x + \Delta x) = Dx - \Delta x$$

identically, and a term-wise comparison of the developments of those two expressions in Δt will give the foregoing one, as well as all of the remaining relations between the variations δ and δ'.

$$\frac{d}{d\omega'}\frac{dx}{dt} = \frac{d}{d\omega''}\frac{d^2x}{dt^2} = \frac{d}{d\omega'''}\frac{d^3x}{dt^3} = \ldots = \frac{dx}{d\omega},$$

i.e., it will be equal to the first term in $\frac{d\Delta x}{d\Delta\omega}$, which points to a more general way of carrying out the differentiations that was already suggested in the foregoing, namely, with increments $\Delta\omega$, $\Delta\psi$, … in the independent variables that correspond to time increments Δt, since those differentiations generally correspond to the transition from the point *C* to *D* that was spoken of in no. II, in which we must revert to the first, undeveloped form (*a*) for the function *U*.

Since the point *B*, and therefore the quantities ξ, remain unchanged under that transition, the partial derivatives of first and second order of the function *U* with respect to $\Delta\omega$, $\Delta\psi$, … will be:

$$\frac{dU}{d\Delta\omega} = -2\sum m\left[(\xi - \Delta x)\frac{d\Delta x}{d\Delta\omega}\right], \qquad \frac{dU}{d\Delta\psi} = -2\sum m\left[(\xi - \Delta x)\frac{d\Delta x}{d\Delta\psi}\right],$$

etc.,

$$\frac{d^2U}{d\Delta\omega^2} = 2\sum m\left[\left(\frac{d\Delta x}{d\Delta\omega}\right)^2\right] - 2\sum m\left[(\xi - \Delta x)\frac{d^2\Delta x}{d\Delta\omega^2}\right],$$

$$\frac{d^2U}{d\Delta\psi^2} = 2\sum m\left[\left(\frac{d\Delta x}{d\Delta\psi}\right)^2\right] - 2\sum m\left[(\xi - \Delta x)\frac{d^2\Delta x}{d\Delta\psi^2}\right],$$

$$\frac{d^2U}{d\Delta\omega\, d\Delta\psi} = 2\sum m\left[\frac{d\Delta x}{d\Delta\omega}\frac{d\Delta x}{d\Delta\psi}\right] - 2\sum m\left[(\xi - \Delta x)\frac{d^2\Delta x}{d\Delta\omega\, d\Delta\psi}\right],$$

etc.,

and there are then three types of conditions for the minimum, namely:

$$(\alpha) \qquad \frac{dU}{d\Delta\omega} = 0, \qquad \frac{dU}{d\Delta\psi} = 0, \qquad \text{etc.,}$$

$$(\beta) \qquad \frac{d^2U}{d\Delta\omega^2} > 0, \qquad \frac{d^2U}{d\Delta\psi^2} > 0, \qquad \text{etc.,}$$

$$(\gamma) \qquad \frac{d^2U}{d\Delta\omega^2}\cdot\frac{d^2U}{d\Delta\psi^2} < \left(\frac{d^2U}{d\Delta\omega\, d\Delta\psi}\right)^2, \qquad \text{etc.}$$

They will be fulfilled when:

$$(20) \qquad 2\sum m\left[(\xi-\Delta x)\frac{d\Delta x}{d\Delta\omega}\right]=0, \quad 2\sum m\left[(\xi-\Delta x)\frac{d\Delta x}{d\Delta\psi}\right]=0, \qquad \text{etc.,}$$

$$(21) \qquad \sum m\left[(\xi-\Delta x)\frac{d^2\Delta x}{d\Delta\omega^2}\right]=0, \quad \sum m\left[(\xi-\Delta x)\frac{d^2\Delta x}{d\Delta\omega\, d\Delta\psi}\right]=0, \qquad \text{etc.,}$$

since the second derivatives of U will then reduce to their first terms, which are all essentially positive terms of the form $\frac{d^2U}{d\Delta\omega^2}$, as the conditions ($b$) would demand, and the presence of the conditions of the form (γ) is explained immediately by developing its components:

$$\frac{d^2U}{d\Delta\omega^2}\cdot\frac{d^2U}{d\Delta\psi^2}=4\sum m\left[\left(\frac{d\Delta x}{d\Delta\omega}\frac{d\Delta x}{d\Delta\psi}\right)^2\right]+4\sum m\left[\left(\frac{d\Delta x}{d\Delta\omega}\frac{d\Delta y}{d\Delta\psi}\right)^2+\left(\frac{d\Delta x}{d\Delta\psi}\frac{d\Delta y}{d\Delta\omega}\right)^2\right],$$

$$\left(\frac{d^2U}{d\Delta\omega\, d\Delta\psi}\right)^2=4\sum m\left[\left(\frac{d\Delta x}{d\Delta\omega}\frac{d\Delta x}{d\Delta\psi}\right)^2\right]+8\sum m\left[\frac{d\Delta x}{d\Delta\omega}\frac{d\Delta y}{d\Delta\psi}\frac{d\Delta x}{d\Delta\psi}\frac{d\Delta y}{d\Delta\omega}\right],$$

with which things will revert to the theorem that $a^2 + b^2 > 2ab$. However, if one develops the first part of equations (20) in Δt, while (21) are still to be proved, by means of the developments of the derivatives $\frac{d\Delta x}{d\Delta\omega}$, etc., and the quantities $\xi-\Delta x$, above then one will, in turn, find sums in all terms (after the associated splitting off of the factors that are independent of the summations) whose general form is contained in equation (9). Equations (20), (21) will then be true independently of the magnitude of the time interval Δt when all of those sums vanish as a result of equation (9). However, they are only true for $i = 0$, $n = 1$, so the time interval must be taken to be infinitely small, like the virtual path, and equations (20) will all be the same as equations (6), since the former equations will reduce to their first terms for that reason. By contrast, equations (21), whose first terms include the sums:

$$\sum m\left[\left(X-\frac{d^2x}{dt^2}\right)\frac{d^2x}{d\omega^2}\right], \qquad \sum m\left[\left(X-\frac{d^2x}{dt^2}\right)\frac{d^2x}{d\psi^2}\right], \qquad \text{etc.,}$$

will no longer be true, since one would then have to take $n = 2$ in formula (9), but that would also no longer be necessary for the fulfillment of conditions (β), (γ) in that case, because the second parts of the expressions for the second derivatives of U would cancel, due to the infinite smallness itself, since, e.g., $\frac{d^2U}{d\Delta\omega^2}$ would then reduce to:

$$\frac{d^2U}{d\Delta\omega^2} = 2\sum m\left[\left(\frac{dx}{d\omega}\right)^2\right] - \Delta t^2 \sum m\left[\left(X - \frac{d^2x}{dt^2}\right)\frac{d^2x}{d\psi^2}\right],$$

so to $2\sum m\left[\left(\frac{dx}{d\omega}\right)^2\right]$, since $\Delta t = 1 / \infty$, and likewise for the remaining ones.

§ 5.

Here, we shall consider the principle of least constraint in the case of equilibrium.

When the forces whose projections onto the axes are X, Y, Z are themselves in equilibrium in the system, the principle of virtual velocities will imply the equation:

$$(\alpha) \qquad \sum m\,[X\,Dx] = 0,$$

from which one will infer the formulas:

$$(\beta) \qquad \sum m\,[X\,\delta^n x] = 0, \qquad \sum m\left[X\,\frac{d^n x}{d\omega^{n'}\,d\psi^{n''}\cdots}\right] = 0, \qquad \sum m\left[\frac{d^i x}{dt^i}\,\frac{d^n x}{d\omega^{n'}\,d\psi^{n''}\cdots}\right] = 0,$$

when Dx can take on finite values, and one can obtain the latter from the foregoing by differentiating with respect to a variable t that all other ones can be considered to be a function of, or also (what amounts to the same thing) by differentiating with respect to the characteristic δ (except that one must then replace $\frac{d^i X}{dt^i}$ with $\delta^i X$), and that will generally come down to:

$$(\gamma) \qquad \sum m\,[X\,\delta x] = 0, \qquad \sum m\left[X\,\frac{dx}{d\omega}\right] = 0, \qquad \sum m\left[X\,\frac{dx}{d\psi}\right] = 0,$$

so the number of the latter equations will be equal to the number of geometric independent variables that are present, which will then determine the equilibrium position completely. It must now be shown to what extent those equations also express a minimum of the constraint in this case, which can be done in a manner that is completely similar to what was done in § **4**.

Let x, y, z be the coordinates of m in the equilibrium position A, and let $x + \xi$, $y + \eta$, $z + \zeta$, or x', y', z' be the coordinates in the position B where M arrives when it starts from A and moves freely from the equilibrium position under the influence of the forces X, Y, Z that act upon it, like all of the other points of the system. Finally, let $x + Dx$, $y + Dy$, $z + Dz$ be the coordinates of the position D to which A will arrive under an arbitrary displacement of points of the system that is compatible with the system constraints, or rather when one starts from A and perturbs the equilibrium during the time interval Δt during which it would freely come to B and it actually moves under its coupling to the remaining points of the system (such that one can also set Dx, Dy, Dz equal to Δx, Δy, Δz). If one again sets:

$$AB = u, \qquad \sum mu^2 = U,$$

such that U is the function that should be a minimum when equations (β) or (γ) are true. Likewise, if the point D enters in place of A :

$$AB = u', \qquad \sum mu'^2 = U',$$

then

$$u^2 = \xi^2 + \eta^2 + \zeta^2,$$

$$u'^2 = [(\xi - Dx)^2] = u^2 + [Dx^2] - 2\,[\xi\, Dx].$$

Hence:

$$(a) \qquad U = \sum m[\xi^2], \qquad \text{or} \qquad U = \sum m[(x' - x)^2],$$

and

$$(b) \qquad \begin{cases} U' - U = \sum m[Dx^2] - 2\sum m[\xi\, Dx] > 0, \\ \qquad \text{i.e.,} \quad \sum m[\xi\, Dx] = 0, \end{cases}$$

will be the condition for the minimum of U. Alternatively, when one differentiates U in the second form with respect to the independent variables ω, ψ, …, where only the coordinates x of the point A will change, while the coordinates x' of the point B will remain unchanged (as in the example **2**.III) then one will have:

$$(c) \qquad \begin{cases} \dfrac{dU}{d\omega} = -2\sum m\left[\xi\,\dfrac{dx}{d\omega}\right] = 0, \quad \dfrac{dU}{d\psi} = -2\sum m\left[\xi\,\dfrac{dx}{d\psi}\right] = 0, \quad \text{etc.,} \\ \dfrac{d^2U}{d\omega^2} = 2\sum m\left[\left(\dfrac{dx}{d\omega}\right)^2\right] - 2\sum m\left[\xi\,\dfrac{d^2x}{d\omega^2}\right] > 0, \quad \text{etc.,} \\ \dfrac{d^2U}{d\omega^2}\cdot\dfrac{d^2U}{d\psi^2} > \left(\dfrac{d^2U}{d\omega\, d\psi}\right)^2, \quad \text{etc.,} \end{cases}$$

as the system of minimum conditions. Now, since $dx / dt = 0$, so one also has $\delta x = 0$, one will have:

$$\xi = X \cdot \frac{\Delta t^2}{2} + \frac{dX}{dt} \cdot \frac{\Delta t^3}{2 \cdot 3} + \ldots,$$

$$Dx = \delta^2 x \cdot \frac{\Delta t^2}{2} + \delta^3 x \cdot \frac{\Delta t^3}{2 \cdot 3} + \ldots$$

as the developments of those expressions, in which $\delta^2 x$, $\delta^3 x$, etc. have the same values, since one will also have $\delta\omega = \delta\psi = \ldots =$, as in the last development in § **4**.II, namely:

$$\delta^2 x = \frac{dx}{d\omega} \delta^2 \omega + \frac{dx}{d\psi} \delta^2 \psi + \ldots,$$

$$\delta^2 x = \frac{dx}{d\omega} \delta^3 \omega + \frac{d^2 x}{d\omega^2} \delta^2 \omega^2 + 2 \frac{d^2 x}{d\omega\, d\psi} \delta^2 \omega \delta^2 \psi + \ldots,$$

etc. However, those expressions explain the fact that whether Δt is taken to be finite or infinitely small, the minimum conditions in the one form (b) or the other (c) will be justified by the equilibrium conditions (β) or (γ).