"Des lois géométrique qui régissent les déplacements d'un système solide dans l'espace…," J. Math. pures appl. **5** (1840), 380-440.

On the geometric laws that govern the displacements of a solid system in space and the variation of the coordinates that are produced by those displacements, when considered independently of the causes that might produce them.

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The general idea of the translation and rotation of a solid system

1. – I intend a solid system to mean an arbitrary assemblage, whether continuous or discontinuous, of points that are invariably linked with each other in such a way that when three of those points that are not in a straight line are given, along with their distances to all other points of the system, the position of that system will be invariably determined for each change in position of the triangle that is formed by those three points.

Indeed, that must be the case, since one can construct *only one* pyramid on a given triangular base with given distances that is identical to or superimposable with another given pyramid. Thus, when three points of a solid that are not in a straight line are fixed, no displacement of that solid will be possible.

However, if there are only two given points that must remain immobile in the system then the invariability of the distances to any other point from those two would initially imply the immobility of all points with which it is in a straight line. That line will become a *fixed axis*, around which any other point can only turn around a circumference that is concentric and normal to the axis, and since all points of the system are invariably coupled with one of them and the fixed axis, so the *rotation* of one implies the rotation of all of them, and the *amplitude* of that rotation is equal for all points of the system.

Any displacement of a *solid* around two fixed points will then reduce to a *rotation* of equal amplitude *and in the same sense* for all points of the system around an axis that is defined those two fixed points.

This would be the place to remark that any given rotation can be replaced with a rotation in the opposite sense with an amplitude that is the complement of the first one in 400°.

Different rotations around the same axis can be combined into one rotation that is equal to their sum, when one takes care to assign opposite signs to the amplitudes of rotations that are performed in opposite senses, while the order of succession of the rotations will remain arbitrary, moreover.

If the amplitudes of the rotations are infinitely small then the arcs that are described by the points of a solid that are displaced to a finite distance from the axis will agree with their chords, which are variously inclined with respect to each other according to the angles between the radii that are drawn from the axis to the points of the system.

However, if one supposes that the axis is invariably coupled to the system that one considers to be nonetheless infinitely distant then an infinitely-small rotation of an infinitesimal order that is *reciprocal* to that of the distance to the axis of that system will have the effect of making all of its points describe finite lines that are *equal* and *parallel* in such a manner that the system will be subject to only a *translation*, which is how one then refers to the displacement of a solid that results from the equal transport of all points parallel to a given direction.

Therefore, any *translation* of a system can be rigorously considered to be a rotation of an infinitely-small amplitude around a fixed axis that is infinitely distant and normal to the direction of that translation.

One would not be surprised then to ultimately find that all of the properties of *translations* are included in those of rotations, just as the properties of the straight line are included in those of the circumference, and we will nonetheless stop there, in particular.

However, first we shall complete this general discussion of displacements around fixed axes with the following theorem, which will become intuitive by inspecting the figure, and whose consequences will help us in what follows.

On the transport of the axis of rotation parallel to itself

The rotation of a solid system around a fixed axis can be replaced with an equal rotation around another axis that is parallel to the first one *followed by* a *translation* of that system that is equal and parallel to the chord of the arc that is described by a point on the *second* axis around the first one, by virtue of the first rotation, or what amounts to the same thing, except for the sense of the translation, which is changed, to the chord of the arc that will be described by a point on the first axis around the second one.

2. – Let us return to displacement by *translation.*

If two positions of the same solid are such that the chords that connect each point of the solid in one of those position to its *correspondent* in the other are equal and parallel then it will be clear that in order for the solid to arrive at the second position from the first one, one will only need to slide it in some way parallel to itself along one of those chords, whose magnitude and direction will be a measure of the magnitude and direction of the translation of that system, and *vice versa*. Nothing would be simpler than the translation of a solid system with the direction and magnitude of a given translation.

General law of the composition of successive translations

If the system is subjected to several consecutive translations that are different in direction, as well as extent, then all of those translations can obviously be composed into a single translation that is equal and parallel to the line that connects a point in the first position to its correspondent in the position that defines the system. It is the line that closes the polygon that is traced by the successive translations of that point, whose magnitude and direction depend, as one knows, upon only the magnitudes and directions of the various edges of that polygon, and *no matter what their order of succession might be, moreover.*

The projections of that *resultant translation* onto three rectangular coordinate axes will be equal to the sums of the projections of the given translations onto the same axes and will thus determine its magnitude and direction by an analytical process that is independent of the graphical process of constructing the polygon.

That is the law of composition of the successive translations of a solid system by means of which one can conversely decompose any given translation into a sequence of translations that differ in magnitude and direction and are subject to only the condition that the sum of their projections onto three rectangular axes is equal to that of the projections of the given translation onto the same axis, or more generally, onto an *arbitrary* axis.

One can call that law of composition the *law of the polygon of the translations.*

On the displacement of a system around a fixed point

3. – Two different positions of the same system are given around a point that remains immobile under that displacement. One arbitrarily chooses a triangle in that solid whose summit will be the fixed point, and one considers the two *corresponding* positions of that triangle, and in particular, the angles that are formed by the *corresponding* edges of the two triangles that start from the same common summit. If one draws two planes through that summit that are normal to the respective planes of those angles then they will be separated likewise. It is obvious that the intersection of those two planes will enjoy the property that each of its points can be considered to be the common summit of two *identical* or *superimposable* pyramids that have the two triangles in question as their bases in such a manner that the intersection is supposed to be invariably coupled with the displaced system, so it will necessarily remain immobile under the supposed displacement. That displacement itself will then reduce to a displacement by rotation around a fixed axis that one determined the construction that was just explained.

In the singular case (which one can always avoid, moreover) in which the two bisecting planes, whose intersection serves to determine the axis of rotation, coincide in just one plane, the axis will obviously be nothing but the line of intersection of the planes of the two triangles in question. Any other line than that one, although it might belong to two bisecting plane that coincide in just one, will form symmetric, non-superimposable, trihedral angles with the edges of the triangles considered.

Therefore, any displacement of a system around a fixed point will amount to a displacement along a path of rotation around a fixed axis that passes through that point, or more generally, it can be obtained by an equal rotation around a different axis that is parallel to the first one and located wherever one desires, provided that the rotation is followed by a displacement that takes the form of a parallel translation with an extent that is equal to the chord of the arc that the fixed point will describe around that new axis when it becomes mobile with a sense of translation that is opposite to that of the chord.

That generalization of a displacement around a fixed point is a consequence of the theorem that relates to the transport of the axis of rotation parallel to itself (no. **1**).

On the arbitrary displacement of a solid system in space

4. – Finally, consider two entirely-arbitrary positions of the same solid and seek to represent the mode of displacement that will take the solid from one of those positions to the other in the simplest manner.

Imagine that one draws lines through an arbitrary point of that solid in its first position that are equal and parallel to the ones that join the *corresponding* points in the second position to those of all the other points of the displaced solid. One will have thus constructed an assemblage of points that is a solid that is entirely identical to the one in question, but whose intermediate position between the two given positions will be derived from the first one by means of a certain rotation around a fixed axis that passes through that chosen point in order to refer the displacement of the system to it, while in turn, the second position of the solid will be derived from that intermediate position by means of a translation whose direction and extent are equal to those of the line that joins the chosen point and its correspondent.

If one observes, moreover, that by virtue of the theorem that relates to the transport of the rotation around a given axis to another axis that is parallel to it then nothing will prevent one from supposing that the intermediate position of the solid will result from an equal rotation with the same sense around an axis that passes through an arbitrary point of the solid, followed by a translation that is equal and parallel to the chord of the arc that is described by a new point that is chosen in order to refer it to the displacement of the system around the first axis in in the first rotation that was admitted, since that translation, when composed with the one that acts on the intermediate solid in the defining position, will reduce to a translation that is equal and parallel to the line that joins that new origin to its corresponding points in the defining situation, and finally, since there is only one possible axis of rotation for each origin of the displacement. One ultimately finds that one has proved the following theorem completely, which is indisputably one of the most beautiful in geometry, and which must be considered to be the fundamental basis for the geometric laws of motion.

Fundamental theorem

No matter how a solid has been transported from one place to another, one can always consider that displacement to be the result of two consecutive displacements by rotation and translation, where the rotation is performed around a fixed axis that is drawn through an arbitrary point of the solid in the first position and parallel to a certain direction that is invariably determined by the two positions of the solid being considered, as well as the sense and amplitude of the rotation. The translation then operates parallel to the line that joins a point on that axis to its correspondent in the second position of the system, so the magnitude of that line will be a measure of the translation.

The order of those displacements can be inverted: The translation can precede the rotation, but it will then be performed around an axis that passes through the point in the second position that corresponds to the one in the first position to which the displacement is referred. Furthermore, the direction of the axis of rotation, the amplitude, and the sense of rotation are the same for all points of the system before or after the translation.

In that succession of displacements, one remarks that the line that joins an arbitrary point of the solid in its first position to its correspondent in the second, i.e., the line that is *actually* traversed by that point, will form the third edge of a triangle such that the first one, which varies for each point of the system. is normal to the axis of rotation, and the second one, which is constant for all points of the system, is a measure of the translation of the system relative to the origin of the displacement.

The projection of that line onto the axis of rotation is then constant for all points of the solid and can be constant only with respect to that same direction of the axis of rotation because that projection onto any direction is sum of two projections, namely, that of the chord of the arc of rotation and that of the line that is traversed by translation. The first of those projections is variables, and the second is constant. Their sum cannot be constant then for any direction other than the one that makes the first of those projections zero, namely, the direction of the axis of rotation itself.

All of the points of a solid system that displaces in an arbitrary manner are then transported *equally relative* to the direction of the axis of rotation.

If that constant projection is zero then the displacement will reduce to a rotation without translation around a certain axis (which is easy to determine), so the line that is traversed by each point in a plane normal to the common direction to all of the *axes* of rotation will be found to be the base of an isosceles triangle that is located in that plane, whose angle at the summit will be equal to the amplitude of that rotation, and whose summit will belong to the desired axis.

In general, that constant projection is a measure of the *absolute* translation of the system, which is minimum for all of the ones that are relative to the various origins of the displacement. It is the translation itself of the points of the system that will be transported parallel to the axis of rotation. In order for those points considered to be origins of the displacement, the axis of rotation, which we call the *central axis of the displacement* in order to distinguish it, is at the same time the axis of translation, in such a manner that the displacement of the system is referred to that central axis. It reduces to turning around that axis while *sliding* parallel to its direction in such a way that the motion can be compared to that of a screw in its nut, which is the simplest expression for the fundamental theorem in which the two displacements of rotation and translation are found to be performed orthogonal to each other (*).

5. – However, one must find that central axis, i.e., one must determine the points of the system whose defining displacement reduces to a translation parallel to the direction of the axis of rotation. Now that will happen by the following construction:

Draw a perpendicular to the axis of rotation that is equal in extent to the projection of the translation onto that perpendicular through an arbitrary point of the solid and in the plane of the two axes of rotation and translation *relative to the point* that is considered to be the origin of the displacement. Construct an isosceles triangle whose summit angle is equal to the amplitude of the rotation of the system whose base is along that perpendicular and normal to the plane of the two

(*) I believe that beautiful theorem in geometry belong to Chasles, who published it in the *Bulletin universel des Sciences* **14** (1830).

axes and whose summit belong to the *central axis* if the isosceles triangle is placed in the sense of rotation relative to the plane of the two axes, moreover.

Conversely, if one considers the displacement of the origin of the two given relative axes with respect to the axis of rotation, which is drawn through that summit, then one will see that by virtue of the rotation around the *central axis*, the point will first arrive at the extremity of the base of the isosceles triangle that was indicated above, and that base measures the displacement of the point perpendicular to the central axis of rotation, and then by virtue of the translation parallel to that same axis, it will finally arrive at its defining situation at the extremity of the relative axis of the supposed translation.

That construction will indeed show that the entire displacement reduces to a simple rotation around the central axis when the axis of relative translation is normal to the axis of rotation, so the summit of the isosceles triangle will return to its first position, by definition.

Moreover, whenever the various points of a solid remain in a parallel plane under the displacement of the solid, that displacement will reduce to a rotation around a certain fixed axis that is normal to those planes.

Displacing a planar figure in its plane

The displacement of a planar figure in its plane always reduces to a rotation around a fixed center in that plane then, which is easy to establish directly.

If one considers the quadrilateral that is composed of the line that joins two corresponding positions of a point of the solid, the central axis, and the perpendiculars that are based at those two positions along that axis then one will find that the line that joins to the midpoint of the first side of that quadrilateral to the one that is opposite to it will be perpendicular to those two lines, which is a property that gives another construction of the central axis when one is given only two points of the system, their correspondents, and the direction of the axis of rotation.

One passes a line through the midpoints of each of the lines that join one of the given points to its correspondent that is parallel to the given direction of the axis of rotation and a normal to those two lines, so the planes that are formed at each midpoint by that normal and the line parallel to the axis of rotation will cut along locus of the central axis itself (*).

6. – We must now state the main corollaries that one can deduce from the fundamental theory that relate to the particular displacements of the points, lines, and planes in a solid system.

1. The distances that separate each point of the solid in its first position from its correspondent in the second one all project equally onto the direction of the axis of rotation. That common projection is a measure of the *absolute* translation of the system.

2. Any line that belongs to a displaced system can only turn *with respect to its direction* around the axis of rotation, so a very simple relationship will result between the angle that the line

(*) That construction was pointed out to me by my friend Lévy.

forms with the axis of rotation and the one that it forms with its original position as a result of its displacement and the amplitude of the rotation, namely:

The sine of ***one-half*** *the* ***angular*** *displacement of any line that belongs to a displaced solid system is equal to the sine of one-half the rotation of that system, multiplied by the sine of the angle between that line and the axis of rotation.*

That proposition, which will become clear when one inspects the figure, serves to establish, in the most elementary manner, one of the most remarkable properties of the motion of the Earth, which is the almost-absolute invariability of the poles of the globe, since the *angular* motion of the terrestrial axis is infinitely less than that of the rotation of the Earth.

3. Any line parallel to the axis of rotation is transported parallel to itself, while the angular displacement of any line normal to that axis is equal to the amplitude of that rotation.

4. Any plane figure that is invariably linked with the displaced system and whose plane is normal to the axis of rotation is then transported in a plane parallel to the first one to a distance that is equal to the absolute translation of the system.

Any plane figure that is parallel to the axis is, on the contrary, transported in a plane that is inclined with respect to its original position through an angle that is equal to the amplitude of the rotation.

5. The midpoint of the line that joins an arbitrary point of the system to its correspondent is the point on that line that is closest to the central axis of the displacement.

6. The midpoints of all lines that join all points of a plane figure to their correspondents after an arbitrary displacement of that figure are in the same plane as the midpoints of all the lines that join any point that is *external* to that plane figure to its *symmetric* correspondent.

That plane makes an equal angle with the planes of the two plane figures, as well as with the *corresponding* lines in the two figures, whether in their plane or outside of it, but *symmetrically* inclined.

7. Now that we have presented the fundamental geometric law for the passage of a solid from a given position to another likewise-given one in an arbitrary manner, we must look for the law of successive composition of those displacements by means of which if we are given the elements of those successive displacements then we can construct or calculate the element of the defining position of the solid, the position of its defining central axis, the amplitude of the rotation, and the length of the translation.

We have already discussed the law of composition of *translation*, so we shall give that of rotations around different fixed axes, and finally that of arbitrary displacements that each result from a combined translation and a rotation.

We deduce an important transformation of the fundamental theorem from that law of composition of rotations around different axes, namely:

Any displacement of a solid system can be represented by a succession of two rotations of that system around two fixed, non-concurrent axes in an infinitude of way. The product of the sine of one-half its rotation, multiplied by the sine of the angle between those two planes, and the shortest distance between them is equal, for all ***pairs*** *of* ***conjugate*** *axes, to the product of the sine of one-half the rotation of the system around the* ***central*** *axis of the displacement, multiplied by one-half the absolute translation of the system.*

Otherwise, the volume of the tetrahedron whose opposite edges are situated in an arbitrary manner along each of the conjugate axes and whose magnitudes are proportional to the sines of one-half the respective rotations will be constant for all pairs of conjugate rotations that are suitable for representing a given displacement

Therefore, any displacement of a solid system will, by definition, reduce to turning around one or both fixed axes.

In the case where one of those axes is parallel to the *central* axis, it will result from the law of composition of rotations that its conjugate will be situated at infinity, that the rotation that corresponds to it will become infinitely small, and that it will be thus transformed into a simple rotation, which leads back to the first statement of the fundamental theorem, which is a statement that is only a special case of the one that we just gave.

On the composition of successive rotations of a solid around two convergent axes

8. – We shall now present the law of composition of successive rotation of a solid around different axes. We begin by consider only two convergent axes and then seek to determine the *resultant* axis of those two rotations, which is, by definition, the one around which one will find that the given solid has turned in order to arrive at its final position from its initial one.

That resultant axis must be placed in such a way that when one has accomplished the two indicated rotations around convergent axes, it will return to its first position. Therefore, if one draws a plane through each of the given axes that makes an angle with those two axes that is equal to one-half the rotation relative to that axis, so the intersection of those two planes is the desired resultant axis, then by virtue of the first rotation, it will arrive at its *symmetric* position with respect to the plane of the two axes and return to its original position by the second rotation.

At the same time, one sees that the angle between those two planes is a measure of one-half the composed rotation, since the first axis, which is immobile during the first rotation, is displaced only by the second one and is found to describe an angle around the resultant axis (which is determined as was just said) that is twice the angle between the two planes. Observe that one-half the rotation around each axis can be measured indifferently by the internal or external angle between the two planes that go through that axis, while the sense of rotation is all that will change according to whether one adopts one or the other measure, since any rotation that is accomplished in one sense around an axis is equivalent to the opposite rotation with an amplitude that is complementary to the first one with respect 400°.

Depending upon the order of succession of those rotations, it will then happen that the axis that is *symmetric* to the one that was just determined with respect to the plane of the two given axes will correspond, as the resultant axis, to the same rotations when they are accomplished in an order that is inverse to the one that is supposed around the same axes. Hence, one sees that the value of the resultant rotation will not depend upon the order of the rotations, but the position of the resultant axis will depend upon it essentially, and that in the composition of rotations of a solid system, the order of those rotations cannot be modified without altering the position of the resultant axis and the value itself of the resultant rotation if there are more than two axes to be successively composed.

That is the characteristic difference to be pointed out between the composition of successive rotations and that of successive translations. Furthermore, an analogy exists between those two types of composition and the properties of the rectilinear triangle and those of the spherical triangle, and if one compares the translations that are parallel to the three edges of a rectilinear triangle to the sines of the half-rotations that are accomplished around those three edges of a trihedral angle then the values of the translations and those of the sines will be likewise proportional to the sines of the angles that are opposite to the respective edges in the rectilinear triangle and the trihedral angle.

Composition of infinitely-small rotations

9. – However, those two resultant symmetric axes that correspond to the same rotations in two different orders of succession will coincide in the plane of the two axes if the rotations become infinitely small, and two important consequences will result:

The first one is that the order of succession of infinitely-small rotations around two convergent axes, and as a result, around any axis that one desires, will be irrelevant. The second one is that the axis and the amplitude of the infinitely-small rotation that is the resultant of the succession of two infinitely-small rotations around two convergent axes is determined in the same manner as the axis and magnitude of the translation that will result from two successive translations that are proportional to the given rotations and parallel to their axes, and since upon moving the axes of rotation to infinity, one will transform the infinitely-small rotations into finite translations that are perpendicular to those axes and inclined with respect to each other like the axes themselves, so one will encompass the full generality of the law of composition of finite rotations, which also encompasses the law of composition of translations by the intermediary of infinitely-small rotations.

On the composition of rotations around two parallel axes

10. – All of the points of the system that displace as a result of two consecutive rotations around two parallel axes will remain in planes normal to those axes. The displacement will then reduce to a simple rotation around a certain axis that is parallel to the first one. Having said that, the method for determining and constructing that axis that was presented for two convergent axes will appy likewise to two parallel axes and will give the result that an axis that is parallel to the first one and a composed rotation will be equal to the sum or the difference of the given rotations according to whether they are performed in the same sense or in opposite senses, resp.

Pairs of parallel rotations

However, a very remarkable case presents itself here, which is that of two rotations that are equal and have opposite sense, so the composed rotation is zero and the resultant axis will be situated at infinity, which is a situation in which the displacement will come down to a simple translation. Indeed, as a result of two equal, but opposite, rotations around two parallel axes, each point of the displaced solid will have traversed, by definition, a line that is constant in magnitude and direction for all points of the system. The direction of the line defines an angle with the normal to the plane of the two axes that is equal to one-half the supposed rotation around each axis, at the same time as it is normal to those two axes. Its magnitude is equal to the product of the distance between the two axes with twice the sine of one-half the rotation.

The order of succession of the rotations is not irrelevant. The symmetric position of the direction of the translation relative to the normal to the plane of the two axes corresponds to a change in the order of succession of the rotations.

All of that results easily from a comparison of similar triangles, and when the rotations are infinitely small, one will once more see that the order of succession is irrelevant, so the translation will then operate along the normal to the plane of the two axes.

Therefore, any *pair of parallel rotations* is equivalent to a simple translation, and conversely, any *translation* can be replaced with a pair of that type in an infinitude of ways.

Those pairs of parallel rotations, which are then composed and decomposed according to the law of translations in an arbitrary order of succession, can occupy all positions in space that correspond to the same translation in magnitude and direction.

Those compositions and decompositions are obtained by replacing the pairs with the translations that they represent.

One thus finds the generalization to *pairs of finite rotations* of the law of composition that Poinsot was, I believe, the first to point out for pairs of *infinitely-small* rotations.

11. – Any displacement of a solid system reduces to a rotation followed by a translation, and that translation can always be replaced with a *pair of rotations* for which one of the axes will intersect the given axis of rotation of the system, so the rotations around those two convergent axes will compose into just one rotation, and that will give the proof of the transformations of the fundamental theory that was asserted above, namely: Any displacement of a solid system can always be produced by the succession of two rotations around two non-convergent fixed axes, and in an infinitude of ways.

On the composition of rotations around an arbitrary number of non-convergent fixed axes

12. – Finally, one must deal with the composition of rotations around an arbitrary number of non-convergent fixed axes. Take a point in space in order to *refer* all of those rotations to it. We have seen that any rotation around a fixed axis can be replaced with another equal rotation that is accomplished around another axis that is parallel to the first one, followed by a translation that is equal to the chord of the arc that is described by a point on the new axis around first one as a result of the given first rotations. We have also seen that a translation followed by rotation that is accomplished around an axis that passes through the extremity of the axis of translation can, on the contrary, be preceded by it of the axis of rotation passes through the origin of the *axis* of translation. Having said that, if one passes axes that are parallel to the given non-convergent axes through the point that is chosen so that one can refer all of the rotations to it then the displacement of the system will successively take place around those axes by means of the transport of the rotations to the convergent axes that are drawn parallel to the first ones and the successive exchange of the rotations around axes that pass through the extremities of the lines of translation with rotations around axes that pass through their origin. Then the displacement of the system will then split into a sequence of rotations that are equal to the given rotations, respectively, which are accomplished successively around convergent axes that are parallel to the first ones, followed by a series of translations that result from the chords that are *successively* traversed by the chosen point around the given non-convergent axes *in the order that is indicated by the rotations.*

The composition of rotations around convergent axes and that of translations takes place according to the methods that were indicated above. By definition, the displacement of the system will come down to a rotation and a translation relative to two axes of rotation and translation that pass through the point that is chosen to be the origin of the displacement.

One sees by that construction that the elements of the defining rotation of the system depend upon only the amplitude and direction of the rotations, and not at all on the distribution of the axes of rotation, whereas the magnitude and direction of the translation depend upon both the rotations, their directions, and the *position* of the axes of those composed rotations, insofar as the chords that are successively described by the point that is chosen in order to refer the displacement of the system to it will vary in magnitude and direction according to the successive positions that the displaced point will take relative to the various axes of the given rotation.

If one finds consecutive axes among the system of those axes that form *pairs of parallel rotations* then it will be obvious that those *pairs* will not provide any element in the determination of the direction and amplitude of the resultant rotation and that they will influence only the magnitude and direction of the result translation, so as a result of those *pairs of rotations*, the point where the successive rotations determine that translation will be found to describe the equivalent translation for each respective pair when it happens.

Examination of the particular case of two non-convergent fixed axes

13. – If we consider two non-convergent axes and the shortest distance between them, and we take the origin of the displacement to be the extremity of that shortest distance along the first axis of rotation, i.e., along the axis of rotation that the first one must perform, while passing an axis parallel to the given second one through that origin then we will have combined the two convergent axes into a third one, which will be the *axis of rotation* of the displacement relative to that origin, and we will have the axis of the relative translation in the form of the chord of the arc that the origin describes by virtue of the rotation to be performed around the second given axis. The projections of that chord into the relative axis of rotation, thus-determined, are a measure of the absolute translation of the displacement. It is equal to the sum of the projections of the two sides of the isosceles triangle that is its base. Each of those edges is equal to the shortest distance between the two given non-convergent axes, and it is easy to assure oneself, in addition, that each of those edges is equally inclined with respect to the composed relative axis. Indeed, that shortest distance is normal to the plane of the two convergent axes. Now, upon considering the angle that is formed by that normal and the line that is *symmetric* to the resultant axis relative to that plane, one will see that the angle will not change when one supposes it to be mobile and implied by the second rotation that makes the *symmetric* line that we spoke of coincide with the resultant axis. However, under that rotation, the normal will be inclined over a plane that is perpendicular to the second axis in such a manner as to be parallel to the second edge of the isosceles triangle that we consider, and since it is obvious that the angle between the normal and the resultant axis is supplementary to the one that it forms with its *symmetric* position, we will see that the resultant axis is, effectively, as we just said, equally inclined with respect to the two edges of that isosceles triangle. Hence, we finally conclude that the absolute translation of a solid system that is produced by the succession of two rotations around two non-convergent fixed axes is equal to twice the distance between those two axes when they are projected onto the direction of the axis of the composed or resultant axis of rotation.

However, it is obvious that the cosine of the angle between that distance and the composed axis is equal to the sine of the angle between that same axis and the plane of the two component axes, and as a result of the law of the proportionality of the sines of the semi-rotations and those of the angles that are found between the opposite axes, it is found to be equal to the product of the sines of the given semi-rotations with the sine of the angle between the two axes, divided by that of the composed semi-rotation. We have thus succeeded in completing the fundamental theorem that we have already stated, thus-transformed, namely, that *any displacement of a solid system can always be produced in an infinitude of ways by the succession of two rotations around two non-convergent fixed axes*, provided that *the product of the sines of the successive semi-rotations, multiplied by the distance between the two conjugate axes and the sine of the angle between those axis, is equal to the product of the absolute semi-translation of the displaced system, multiplied by the sine of the resultant semi-rotation.*

On the composition of successive displacements of a system of combined rotations and translations

14. – We are now in a position to solve the following general problem completely, in which we consider the succession of arbitrary displacements of the same solid.

If one is given the axes of rotation and translation, as well as the amplitude and extent of those rotations and translations for each successive displacement of a system then one wishes to construct the axes of rotation and translation of that system relative to a given origin.

The solution of that problem is obviously the same as that of the preceding problem, in which one deals with only rotations around fixed axes since the translations can be replaced with pairs of rotations around fixed axes. We have neatly indicated that fact in the last paragraph to no. **12**. There is therefore no reason to stop to discuss it further.

Examination of the particular case of successive infinitely-small displacements

That solution will be simplified appreciably when one considers only infinitely-small displacements. Firstly, as far as the given rotations and their result are concerned, that is because the order of those rotations is irrelevant, and the composition of an arbitrary number of them around convergent axes will work like that of translations that are proportional to those rotations and parallel to those axes. Then, as far as the composed translation is concerned, it is because the order of the successive rotations and translations that are performed by the origin of the displacement is likewise irrelevant, and each of those rotations and translations can be established directly and separately as if the point being displaced were being displaced only alternately, and not successively, which amounts to saying that the space swept out by each of those displacements is infinitely small. The composition of those partial translations that results from the distribution of the given axes, or that of the translations themselves that are coupled with the rotations, obeys the same law as that of the rotations.

15. – We must now apply calculations to the geometric laws that we just discussed relative to the arbitrary displacements of a solid system, and first deduce the formulas for the variation of the coordinates of a solid system, which occupy a distinguished place in analytical mechanics.

Determining the formulas for the ***variation*** *of the coordinates of a solid system that are produced by an arbitrary displacement*

Let $x, y, z, \ldots, x + \Delta x, y + \Delta y, z + \Delta z$ be the coordinates of two arbitrary points that correspond in the two positions of the displaced system. Furthermore, let x, y, z be the coordinates of the midpoint of the line that connects them in such a way that one will have:

$$\mathrm{x} = x + \tfrac{1}{2}\Delta x\,, \quad \mathrm{y} = y + \tfrac{1}{2}\Delta y\,, \quad \mathrm{z} = z + \tfrac{1}{2}\Delta \mathrm{z}\;.$$

Finally, let g, h, l be the angles that are formed by the direction of the axis of rotation with the coordinate axes, let θ be the amplitude of the rotation, let t be the magnitude of the absolute translation, and let X, Y, Z be the coordinates of an arbitrary point on the central axis of displacement.

Consider the rectangular triangle whose hypotenuse is formed from the line that joins two corresponding points and whose other two sides are given, in one case, by the chord of the arc that is described by the first point by virtue of the rotation θ, and in the other, by the line that is traversed by translating that same point after the rotation has been performed. It is clear that the variations Δx, Δy, Δz will be equal to the respective projections of that hypotenuse, i.e., the sum of the projections of the other two sides of that rectangular triangle onto the respective coordinate axes. Now, the side that is equal and parallel to the absolute translation t gives the three projections $t \cos g$, $t \cos h$, $t \cos l$. The other side is equal to $2u \tan \frac{1}{2}\theta$, where u denotes the distance from the central axis to the side to which it is perpendicular. That distance is obviously the same as the distance from the point (x, y, z) to that central axis, since the direction of that side is the same as that of a line that is drawn through the point (x, y, z) perpendicular to the central axis and at the distance from that point to the axis. If we let G, H, L denote the angles between that line and the coordinate axes then we will first have the following expressions:

$$\Delta x = t \cos g + 2u \tan \tfrac{1}{2}\theta \, \cos G ,$$
$$\Delta y = t \cos h + 2u \tan \tfrac{1}{2}\theta \, \cos H ,$$
$$\Delta z = t \cos l + 2u \tan \tfrac{1}{2}\theta \, \cos L ,$$

and since we necessarily have:

$$\cos g \cos G + \cos h \cos H + \cos l \cos L = 0 ,$$

we will deduce that:

$$\Delta x \cos g + \Delta y \cos h + \Delta z \cos l = t ,$$

$$\Delta x^2 + \Delta y^2 + \Delta z^2 = t^2 + 4u^2 \tan^2 \tfrac{1}{2}\theta ,$$

which would also result from the figure (†). The first terms $t \cos g$, $t \cos h$, $t \cos l$ represent the part of the variations that produce the absolute displacement by translation. The second terms are due to the rotation that is performed by the displacement. Upon comparing those first and second terms to each other, relative to each coordinate axis, one will find that the first ones, which measure the effect or *moment* of the translation of the system relative to each coordinate axis, have values that equal the projections relative to each coordinate axis of a triangle whose summit is the midpoint of the line that is, by definition, traversed by the point in question and whose base is a line segment that is included along the central axis and has a length of $4 \tan \frac{1}{2}\theta$. In the case of an infinitely-small displacement, that midpoint and the point itself will coincide, and it will then result that for a point of the system that has been displaced by infinitely little, the *effect* of the rotation relative to a given direction is equal to twice the projection relative to that direction of a triangle whose summit at that point and whose base is taken along the central is equal to the rotation of the system.

That fact shows how the theory of projections is attached to the laws of translation by way of linear projections and to the laws of rotation of solids by way of projections of area. Let us pursue that idea.

The chord $2u \tan \frac{1}{2}\theta$ is normal to both the central axis and the perpendicular that is based at the point (x, y, z) on that axis, which is equal to u, which is equal to:

(†) Translator: I have not succeeded in locating the figures that are alluded to in this treatise.

$$u \cos G = (\mathrm{y} - Y) \cos l - (\mathrm{z} - Z) \cos h\,,$$
$$u \cos H = (\mathrm{z} - Z) \cos g - (\mathrm{x} - X) \cos l\,,$$
$$u \cos L = (\mathrm{x} - X) \cos h - (\mathrm{y} - Y) \cos g\,,$$

and as a result:

$$\Delta x = t \cos g + 2u \tan \tfrac{1}{2}\theta \cos G = \mathrm{A} + p\,\mathrm{y} - n\,\mathrm{z}\,,$$
$$\Delta y = t \cos h + 2u \tan \tfrac{1}{2}\theta \cos H = \mathrm{B} + m\,\mathrm{z} - p\,\mathrm{x}\,,$$
$$\Delta \mathrm{z} = t \cos l + 2u \tan \tfrac{1}{2}\theta \cos L = \Gamma + n\,\mathrm{x} - m\,\mathrm{y}\,,$$

A, B, Γ, *m*, *n*, *p* denote six constants that depend upon the position of the central axis of the displacement, the magnitude of the translation, and the amplitude of the rotation, so it will follow that:

$$\mathrm{A} = t \cos g + 2 \tan \tfrac{1}{2}\theta \;(Z \cos h - Y \cos l)\,,$$
$$\mathrm{B} = t \cos h + 2 \tan \tfrac{1}{2}\theta \;(X \cos l - X \cos g)\,,$$
$$\Gamma = t \cos l + 2 \tan \tfrac{1}{2}\theta \;(Y \cos g - X \cos h)\,,$$

$$m = 2u \tan \tfrac{1}{2}\theta \cos g\,, \quad n = 2u \tan \tfrac{1}{2}\theta \cos h\,, \quad p = 2u \tan \tfrac{1}{2}\theta \cos l\,.$$

If one denotes the variations of the coordinates of the origin of the coordinate axes by α, β, γ then one will have the following relations:

$$\alpha = \mathrm{A} + \tfrac{1}{2}\,(p\,\beta - n\,\gamma)\,,$$
$$\beta = \mathrm{B} + \tfrac{1}{2}\,(m\,\gamma - p\,\alpha)\,,$$
$$\gamma = \Gamma + \tfrac{1}{2}\,(n\,\alpha - m\,\beta)\,,$$

by means of which one can replace the constants A, B, Γ with their values as functions of the variations α, β, γ, if one so desires, and one will then have:

$$\Delta x = \alpha + 2 \tan \tfrac{1}{2}\theta \;[(\mathrm{y} - \tfrac{1}{2}\beta) \cos l - (\mathrm{z} - \tfrac{1}{2}\gamma) \cos h]\,,$$
$$\Delta y = \beta + 2 \tan \tfrac{1}{2}\theta \;[(\mathrm{z} - \tfrac{1}{2}\gamma) \cos g - (\mathrm{x} - \tfrac{1}{2}\alpha) \cos l]\,,$$
$$\Delta \mathrm{z} = \alpha + 2 \tan \tfrac{1}{2}\theta \;[(\mathrm{x} - \tfrac{1}{2}\alpha) \cos h - (\mathrm{y} - \tfrac{1}{2}\beta) \cos g]\,,$$

whose first terms α, β, γ express the *moments* of the translation *relative* to the origin of the coordinates, whose values is $\sqrt{\alpha^2 + \beta^2 + \gamma^2}$, and the ones that are affected by the rotation express the *moments* of that rotation around the axis *relative* to the origin. We can establish those formulas directly, like the preceding ones that we constructed along the *central* axis.

Equations of the central axis

16. – The equations of the central axis are, in turn, deduced from the formulas above with the greatest simplicity, because for all points of that axis, the effect of the rotation is zero, so one will have $\Delta x = t \cos g$, $\Delta y = t \cos h$, $\Delta z = t \cos l$. The coordinates x, y, z will belong to that axis, and upon suppressing the line, one will get the desired equations expressed in terms of the elements of the variations:

$$p\,y - n\,z + \mathrm{A} = t\cos g = \frac{m(\mathrm{A}\,m + \mathrm{B}\,n + \Gamma\,p)}{m^2 + n^2 + p^2} = \frac{m(\alpha\,m + \beta\,n + \gamma\,p)}{m^2 + n^2 + p^2},$$

$$m\,z - p\,x + \mathrm{B} = t\cos h = \frac{n(\mathrm{A}\,m + \mathrm{B}\,n + \Gamma\,p)}{m^2 + n^2 + p^2} = \frac{n(\alpha\,m + \beta\,n + \gamma\,p)}{m^2 + n^2 + p^2},$$

$$n\,x - m\,y + \Gamma = t\cos l = \frac{p(\mathrm{A}\,m + \mathrm{B}\,n + \Gamma\,p)}{m^2 + n^2 + p^2} = \frac{p(\alpha\,m + \beta\,n + \gamma\,p)}{m^2 + n^2 + p^2},$$

and even more simply by eliminating the constants A, B, Γ:

$$\frac{x - \frac{1}{2}\alpha - \dfrac{p\,\beta - n\,\gamma}{m^2 + n^2 + p^2}}{m} = \frac{y - \frac{1}{2}\beta - \dfrac{m\,\gamma - p\,\alpha}{m^2 + n^2 + p^2}}{n} = \frac{z - \frac{1}{2}\gamma - \dfrac{n\,\alpha - m\,\beta}{m^2 + n^2 + p^2}}{p},$$

which are equations in which the coordinates that are subtracted from the general coordinates x, y, z are precisely those of the summit of the isosceles triangle that is normal to the plane of the two axes that relates to the translation and rotation that pass through the origin of the coordinate axes and is raised over a base that is perpendicular to the axis that relates to the rotation, and is cut by one-half the axis of relative translation and a magnitude that is equal to the translation projected onto its appropriate direction with the summit angle being equal to the amplitude of the rotation θ. That agrees well with the construction that was just given.

Those equations can be further written as follows, by introducing the rotation and the inclinations of the axis of rotation:

$$\frac{x - \frac{1}{2}\alpha - \frac{1}{2}\cos\frac{1}{2}\theta(\beta\cos l - \gamma\cos h)}{\cos g}$$
$$= \frac{y - \frac{1}{2}\beta - \frac{1}{2}\cos\frac{1}{2}\theta(\gamma\cos g - \alpha\cos l)}{\cos h}$$
$$= \frac{z - \frac{1}{2}\gamma - \frac{1}{2}\cos\frac{1}{2}\theta(\alpha\cos h - \beta\cos g)}{\cos l}.$$

General equation of the central axis

However, one can represent those three equations for the projections of central axis by just one equation with undetermined coefficients, namely:

$$\cos a\,\Delta x + \cos b\,\Delta y + \cos c\,\Delta z = t \cos (t, a, b, c)\,,$$

in which:

$$\cos (t, a, b, c) = \cos a \cos g + \cos b \cos h + \cos c \cos l\,,$$

in which a, b, c denote three angles that are formed arbitrarily by an arbitrary direction and the coordinate axes. Due to the indeterminacy in those angles a, b, c, that single equation will resolve into three others $\Delta x = t \cos g$, $\Delta y = t \cos h$, $\Delta z = t \cos l$, and it expresses the most general property of the displacement of points that are located along the central axis, which is that it must be equal to the absolute translation of the system without rotation relative to an arbitrary direction and estimated along that direction.

Examination of the case of infinitely-small variations

17. – If one considers only the variations of the coordinates of a solid system that are produced by an infinitely-small displacement in which the constants α, β, γ are infinitely small of the same order as those variations when the symbol Δ is exchanged for the usual symbol δ, and one neglects the second-order terms in the formulas that were given above, then one will have the following expressions for the variations:

$$\delta x = \alpha + p\,y - n\,z\,, \qquad \delta y = \beta + m\,z - p\,x\,, \qquad \delta z = \gamma + n\,x - m\,y\,,$$

$$m = \theta \cos g\,, \qquad n = \theta \cos h\,, \qquad p = \theta \cos l\,,$$

and for the equations of the central axis:

$$\frac{\theta\,x + \gamma \cos h - \beta \cos l}{\cos g} = \frac{\theta\,y + \alpha \cos l - \gamma \cos g}{\cos h} = \frac{\theta\,z + \beta \cos g - \alpha \cos h}{\cos l}\,.$$

18. – Let us return to the general formula:

$$\begin{aligned}
\Delta x &= \mathrm{A} + p\,\mathrm{y} - n\,\mathrm{z} = \alpha + p\,(\mathrm{y} - \tfrac{1}{2}\beta) - n\,(\mathrm{x} - \tfrac{1}{2}\gamma)\,,\\
\Delta y &= \mathrm{B} + m\,\mathrm{z} - p\,\mathrm{x} = \beta + m\,(\mathrm{z} - \tfrac{1}{2}\gamma) - p\,(\mathrm{z} - \tfrac{1}{2}\alpha)\,,\\
\Delta z &= \Gamma + n\,\mathrm{x} - m\,\mathrm{y} = \gamma + n\,(\mathrm{x} - \tfrac{1}{2}\alpha) - m\,(\mathrm{y} - \tfrac{1}{2}\beta)\,,
\end{aligned}$$

in which:

$$m = 2 \tan\tfrac{1}{2}\theta\, \cos g\,, \quad n = 2 \tan\tfrac{1}{2}\theta\, \cos h\,, \quad p = 2 \tan\tfrac{1}{2}\theta\, \cos l\,.$$

One then infers that:

$$(\Delta x - \alpha)^2 + (\Delta y - \beta)^2 + (\Delta x - \gamma)^2$$

$$= 4\tan^2 \tfrac{1}{2}\theta\{(\mathrm{x} - \tfrac{1}{2}\alpha)^2 + (\mathrm{y} - \tfrac{1}{2}\beta)^2 + (\mathrm{z} - \tfrac{1}{2}\gamma)^2 - [(\mathrm{x} - \tfrac{1}{2}\alpha)\cos g + (\mathrm{y} - \tfrac{1}{2}\beta)\cos h + (\mathrm{z} - \tfrac{1}{2}\gamma)\cos l]\}^2,$$

which is an equation that expresses the idea that the chord that is described by a rotation around the relative axis that passes through the origin is equal to twice the tangent of one-half the rotation, multiplied by the distance from the origin to the midpoint of that chord, which is estimated perpendicularly to the axis of rotation, which is a distance that is equal to the distance between the midpoints of the lines that are, by definition, traversed by the point in question and by the origin.

Upon starting from that property, which the figure will explain, of the equality of the projections of all of the lines that are traversed, by definition, onto the direction of the axis of rotation and the invariability of the distance from an arbitrary point to the origin, one will immediately have the following three equations, from which one will easily deduce the value for the variations Δx, Δy, $\Delta \mathrm{z}$ that were found before, namely:

$$(\Delta x - \alpha)^2 + (\Delta y - \beta)^2 + (\Delta \mathrm{z} - \gamma)^2$$

$$= 4\tan^2 \tfrac{1}{2}\theta\{(\mathrm{x} - \tfrac{1}{2}\alpha)^2 + (\mathrm{y} - \tfrac{1}{2}\beta)^2 + (\mathrm{z} - \tfrac{1}{2}\gamma)^2 - [(\mathrm{x} - \tfrac{1}{2}\alpha)\cos g + (\mathrm{y} - \tfrac{1}{2}\beta)\cos h + (\mathrm{z} - \tfrac{1}{2}\gamma)\cos l]\}^2,$$

$$(\Delta x - \alpha)\cos g + (\Delta y - \beta)\cos h + (\Delta \mathrm{z} - \gamma)\cos l = 0,$$

$$(\Delta x - \alpha)(\mathrm{x} - \tfrac{1}{2}\alpha) + (\Delta y - \beta)(\mathrm{y} - \tfrac{1}{2}\beta) + (\Delta \mathrm{z} - \gamma)(\mathrm{z} - \tfrac{1}{2}\gamma) = 0.$$

However, one will finally succeed in expressing those variations as functions of the variables x, y, z by eliminating the coordinates x, y, z by means of their values $x + \tfrac{1}{2}\Delta x$, $y + \tfrac{1}{2}\Delta y$, $z + \tfrac{1}{2}\Delta z$, respectively. With that elimination, one will have:

$$\begin{aligned}
\Delta x - \alpha &= 2\tan\tfrac{1}{2}\theta\,(y\cos l - z\cos h) + \tan\tfrac{1}{2}\theta\,[(\Delta y - \beta)\cos l + (\Delta \mathrm{z} - \gamma)\cos l],\\
\Delta y - \beta &= 2\tan\tfrac{1}{2}\theta\,(z\cos g - x\cos l) + \tan\tfrac{1}{2}\theta\,[(\Delta \mathrm{z} - \gamma)\cos g + (\Delta x - \alpha)\cos l],\\
\Delta \mathrm{z} - \gamma &= 2\tan\tfrac{1}{2}\theta\,(y\cos l - y\cos g) + \tan\tfrac{1}{2}\theta\,[(\Delta x - \alpha)\cos h + (\Delta y - \beta)\cos g],
\end{aligned}$$

so one will finally have these definitive formulas:

$$\begin{aligned}
\Delta x &= \alpha + \sin\theta\,(y\cos l - z\cos h) + 2\sin^2\tfrac{1}{2}\theta\,[\cos g\,(x\cos g + y\cos h + z\cos l) - x],\\
\Delta y &= \beta + \sin\theta\,(z\cos g - x\cos l) + 2\sin^2\tfrac{1}{2}\theta\,[\cos h\,(x\cos g + y\cos h + z\cos l) - y],\\
\Delta \mathrm{z} &= \gamma + \sin\theta\,(y\cos l - y\cos g) + 2\sin^2\tfrac{1}{2}\theta\,[\cos l\,(x\cos g + y\cos h + z\cos l) - z],
\end{aligned}$$

and only the first terms will survive when one passes from finite variations to infinitely-small variations.

Expressions for the finite variations as rational functions of the six arbitrary constants $\alpha, \beta, \gamma, m, n, p$

If one preserves the original constants m, n, p in those formulas, and infers the values of Δx, Δy, Δz directly from the equations:

$$\begin{aligned}
\Delta x &= \alpha + p\,(y + \tfrac{1}{2}\Delta y - \tfrac{1}{2}\beta) - n\,(z + \tfrac{1}{2}\Delta z - \tfrac{1}{2}\gamma)\;, \\
\Delta y &= \beta + m\,(z + \tfrac{1}{2}\Delta z - \tfrac{1}{2}\beta) - p\,(x + \tfrac{1}{2}\Delta x - \tfrac{1}{2}\alpha)\,, \\
\Delta z &= \gamma + n\,(x + \tfrac{1}{2}\Delta x - \tfrac{1}{2}\alpha) - m\,(y + \tfrac{1}{2}\Delta y - \tfrac{1}{2}\beta)
\end{aligned}$$

then one will have the following expressions for them as *rational* functions of the six constants α, β, γ, m, n, p :

$$\begin{aligned}
\Delta x &= \alpha + \frac{p\,y - n\,z + \frac{1}{2}m\,(m\,x + n\,y + p\,z) - \frac{1}{2}(m^2 + n^2 + p^2)\,x}{1 + \frac{1}{4}(m^2 + n^2 + p^2)}\;, \\
\Delta y &= \beta + \frac{m\,z - p\,x + \frac{1}{2}n\,(m\,x + n\,y + p\,z) - \frac{1}{2}(m^2 + n^2 + p^2)\,y}{1 + \frac{1}{4}(m^2 + n^2 + p^2)}\;, \\
\Delta z &= \gamma + \frac{n\,x - m\,y + \frac{1}{2}p\,(m\,x + n\,y + p\,z) - \frac{1}{2}(m^2 + n^2 + p^2)\,z}{1 + \frac{1}{4}(m^2 + n^2 + p^2)}\;.
\end{aligned}$$

Important consequence relating to the transformation formulas for rectangular coordinates

Upon comparing those expressions to the ones that one would obtain by considering the transformation of rectangular coordinates, as will be pointed out in no. **26**, one will obtain a method for reducing the nine coefficients that enter into the formulas for that transformation of three independent variables m, n, p that is free of any radicals, which has not been done before, I believe. Upon denoting, as is customary, those nine coefficients by a, b, c, a', b', c', a'', b'', c'', and letting $\cos(x\,x')$, $\cos(y\,y')$, etc. denote the cosines of the angles between the old and new axes, one will easily find that:

$$\begin{aligned}
a &= \cos(x\,x') = \frac{1 + \frac{1}{4}(m^2 - n^2 - p^2)}{1 + \frac{1}{4}(m^2 + n^2 + p^2)}\,, & a' &= \cos(y\,x') = \frac{\frac{1}{2}m\,n - p}{1 + \frac{1}{4}(m^2 + n^2 + p^2)}\,, \\
b &= \cos(x\,y') = \frac{\frac{1}{2}mn + p}{1 + \frac{1}{4}(m^2 + n^2 + p^2)}\,, & b' &= \cos(y\,y') = \frac{1 + \frac{1}{4}(n^2 - m^2 - p^2)}{1 + \frac{1}{4}(m^2 + n^2 + p^2)}\,, \\
c &= \cos(x\,z') = \frac{\frac{1}{2}m\,p - n}{1 + \frac{1}{4}(m^2 + n^2 + p^2)}\,, & c' &= \cos(y\,z') = \frac{\frac{1}{2}n\,p + m}{1 + \frac{1}{4}(m^2 + n^2 + p^2)}\,,
\end{aligned}$$

$$a'' = \cos(z\,x') = \frac{\frac{1}{2}\,p\,m + n}{1 + \frac{1}{4}(m^2 + n^2 + p^2)}\,, \qquad b'' = \cos(z\,y') = \frac{\frac{1}{2}\,n\,p - m}{1 + \frac{1}{4}(m^2 + n^2 + p^2)}\,,$$

$$c'' = \cos(z\,z') = \frac{1 + \frac{1}{4}(p^2 - m^2 - n^2)}{1 + \frac{1}{4}(m^2 + n^2 + p^2)}\,,$$

and conversely:

$$\tfrac{1}{4}m^2 = \frac{1 + a - b' - c''}{1 + a + b' + c''}\,, \qquad \tfrac{1}{4}n^2 = \frac{1 + b' - a - c''}{1 + a + b' + c''}\,, \qquad \tfrac{1}{4}p^2 = \frac{1 + c'' - a - b'}{1 + a + b' + c''}\,,$$

$$m = \frac{2(c' - b'')}{1 + a + b' + c''}\,, \qquad n = \frac{2(a'' - c)}{1 + a + b' + c''}\,, \qquad p = \frac{2(b - a')}{1 + a + b' + c''}\,.$$

Upon eliminating m, n, p from the values of the coefficients a', b, c', a'', b'', c, one will obtain the Monge formulas in the form of *irrational* functions of the three constants a, b', c'' (*).

19. – Starting from the geometric consideration of the displacement of a solid system, we first deduced the characteristic properties or general laws of that displacement, which always reduced to a rotation and a subsequent translation, or even to just a pair of rotations around two fixed parallel or non-parallel axes, and the displacement will reduce to a simple translation in the first case if the rotations are equal and in the opposite sense, moreover. From those properties, we just concluded the analytical expressions for the finite or infinitely-small variations of the coordinates of a solid system that is displaced in an arbitrary manner.

We shall now deduce the laws of composition of rotations and translation that we first discussed synthetically from the expressions for those variations, and we will then establish those same formulas for the variation of the coordinates of a solid system directly by following three distinct analytical procedures, while deducing the invariability of the distances between the points of that system uniquely.

Laws of composition for successive translations and rotations, deduced analytically from the formulas for the variations of the coordinates of the displaced system

Let the symbols Δ', Δ'' denote the successive variations of the coordinates of the displaced solid, let the symbol Δ denote the composed variations that result from the succession of displacements, let x′, y′, z′, x″, y″, z″ denote the coordinates of the midpoints of the lines that are traversed in succession by the point (x, y, z), and as always, let x, y, z be those of the midpoint of the line that is, by definition, traversed by that point, in such a way that one has:

$$\mathrm{x}' = x + \tfrac{1}{2}\Delta' x\,, \qquad \mathrm{y}' = y + \tfrac{1}{2}\Delta' y\,, \qquad \mathrm{z}' = z + \tfrac{1}{2}\Delta' z\,,$$

(*) Lacroix, tome I, page 533.

$$\mathrm{y}'' = x + \Delta'x + \tfrac{1}{2}\Delta''x\,, \qquad \mathrm{y}'' = y + \Delta'y + \tfrac{1}{2}\Delta''y\,, \qquad \mathrm{z}'' = z + \Delta'\mathrm{z} + \tfrac{1}{2}\Delta''\mathrm{z}\,,$$
$$\mathrm{x} = x + \tfrac{1}{2}\Delta x\,, \qquad \mathrm{y} = y + \tfrac{1}{2}\Delta y\,, \qquad \mathrm{z} = z + \tfrac{1}{2}\Delta\mathrm{z}\,,$$

$$\Delta x = \Delta'x + \Delta''x\,, \qquad \Delta y = \Delta'y + \Delta''y\,, \qquad \Delta\mathrm{z} = \Delta'\mathrm{z} + \Delta''\mathrm{z}\,.$$

Moreover, let $t,\ t'$, $\theta,\ \theta'$, $g, h, l,\ g',\ h',\ l'$ denote the absolute translations, the rotations, and the inclinations of the axes of rotation, resp., for each of the two consecutive displacements that are being considered, and let T, Θ, G, H, L, resp., denote the analogous elements of the defining or composed displacement.

Now examine the composition of the simple translations and the rotations without translation separately. First suppose that the displacements reduce to translations, so one will have:

$$\Delta'x = t\cos g\,, \qquad \Delta''x = t'\cos g'\,, \qquad \Delta x = T\cos G = t\cos g + t'\cos g'\,,$$
$$\Delta'y = t\cos h\,, \qquad \Delta''y = t'\cos h'\,, \qquad \Delta y = T\cos H = t\cos h + t'\cos h'\,,$$
$$\Delta'\mathrm{z} = t\cos l\,, \qquad \Delta''\mathrm{z} = t'\cos l'\,, \qquad \Delta\mathrm{z} = T\cos L = t\cos l + t'\cos l'\,,$$

from which one will see that the length and direction of the resultant translation of two successive translations are nothing but the third side of a triangle that is defined by a point of the system that successively traversed it by virtue of the two given translations, and one can easily generalize that method of composition by passing from the triangle to the polygon of translations.

Now suppose that one is dealing with the composition of two rotations without translation around two convergent axes at the origin of the coordinates. One will have, successively:

$$\Delta'x = 2\tan\tfrac{1}{2}\theta\,(\mathrm{y}'\cos l - \mathrm{z}'\cos h)\,, \qquad \Delta''x = 2\tan\tfrac{1}{2}\theta'\,(\mathrm{y}''\cos l - \mathrm{z}''\cos h')\,,$$
$$\Delta'y = 2\tan\tfrac{1}{2}\theta\,(\mathrm{z}'\cos g - \mathrm{x}'\cos l)\,, \qquad \Delta''y = 2\tan\tfrac{1}{2}\theta'\,(\mathrm{z}''\cos g' - \mathrm{x}''\cos l')\,,$$
$$\Delta'\mathrm{z} = 2\tan\tfrac{1}{2}\theta\,(\mathrm{x}'\cos h - \mathrm{y}'\cos g)\,, \qquad \Delta''\mathrm{z} = 2\tan\tfrac{1}{2}\theta'\,(\mathrm{x}''\cos h' - \mathrm{y}''\cos g')\,,$$

$$\Delta x = \Delta'x + \Delta''x = 2\tan\tfrac{1}{2}\Theta\,(\mathrm{y}\cos L - \mathrm{z}\cos H)\,,$$
$$\Delta y = \Delta'y + \Delta''y = 2\tan\tfrac{1}{2}\Theta\,(\mathrm{z}\cos G - \mathrm{x}\cos L)\,,$$
$$\Delta\mathrm{z} = \Delta'\mathrm{z} + \Delta''\mathrm{z} = 2\tan\tfrac{1}{2}\Theta\,(\mathrm{x}\cos H - \mathrm{y}\cos G)\,.$$

One must determine the elements Θ, G, H, L as functions of the given elements $\theta, g, h, l,\ \theta',\ g',\ h',\ l'$ for each of the rotations being composed.

Now, it is appropriate to first eliminate the variables x', y', z', x'', y'', z'' from those relations by means of the relations that were given above. Indeed, one easily concludes that:

$$\mathrm{x}' = \mathrm{x} - \tfrac{1}{2}\Delta''x\,, \qquad \mathrm{x}'' = \mathrm{x} + \tfrac{1}{2}\Delta'x\,,$$
$$\mathrm{y}' = \mathrm{y} - \tfrac{1}{2}\Delta''y\,, \qquad \mathrm{y}'' = \mathrm{y} + \tfrac{1}{2}\Delta'y\,,$$
$$\mathrm{z}' = \mathrm{z} - \tfrac{1}{2}\Delta''\mathrm{z}\,, \qquad \mathrm{z}'' = \mathrm{z} + \tfrac{1}{2}\Delta'\mathrm{z}\,,$$

and then, in turn:

$$\begin{aligned}
\Delta' x &= 2\tan\tfrac{1}{2}\theta\,(\mathrm{y}'\cos l - \mathrm{z}'\cos h) \;\; + \tan\tfrac{1}{2}\theta\,(\cos h\,\Delta''\mathrm{z} - \cos l\,\Delta'' y)\,,\\
\Delta'' x &= 2\tan\tfrac{1}{2}\theta'\,(\mathrm{y}\cos l' - \mathrm{z}\cos h') + \tan\tfrac{1}{2}\theta'(\cos l'\,\Delta' y - \cos h'\Delta'\mathrm{z})\,,\\
\Delta' y &= 2\tan\tfrac{1}{2}\theta\,(\mathrm{z}\cos g - \mathrm{x}\cos l) \;\; + \tan\tfrac{1}{2}\theta\,(\cos l\,\Delta'' x - \cos g\,\Delta''\mathrm{z})\,,\\
\Delta'' y &= 2\tan\tfrac{1}{2}\theta'\,(\mathrm{z}\cos g' - \mathrm{x}\cos l') + \tan\tfrac{1}{2}\theta'(\cos g'\,\Delta'\mathrm{z} - \cos l'\Delta' x)\,,\\
\Delta' \mathrm{z} &= 2\tan\tfrac{1}{2}\theta\,(\mathrm{x}\cos h - \mathrm{y}\cos g) \;\; + \tan\tfrac{1}{2}\theta\,(\cos g\,\Delta'' y - \cos h\,\Delta'' x)\,,\\
\Delta'' \mathrm{z} &= 2\tan\tfrac{1}{2}\theta'\,(\mathrm{x}\cos h' - \mathrm{y}\cos g') + \tan\tfrac{1}{2}\theta'(\cos h'\,\Delta' x - \cos g'\Delta' y)\,.
\end{aligned}$$

One then deduces the following values for the partial variations $\Delta' x$, $\Delta'' x$, $\Delta' y$, $\Delta'' y$, $\Delta'\mathrm{z}$, $\Delta''\mathrm{z}$:

$$\Delta' x = \frac{2\tan\frac{1}{2}\theta\,(\mathrm{y}\cos l - \mathrm{z}\cos h) + 2\tan\frac{1}{2}\theta\tan\frac{1}{2}\theta'[\mathrm{x}\cos\nu - \cos g'(\mathrm{x}\cos g + \mathrm{y}\cos h + \mathrm{z}\cos l)]}{1 - \tan\frac{1}{2}\theta\tan\frac{1}{2}\theta'\cos\nu}\,,$$

$$\Delta'' x = \frac{2\tan\frac{1}{2}\theta'\,(\mathrm{y}\cos l' - \mathrm{z}\cos h') + 2\tan\frac{1}{2}\theta\tan\frac{1}{2}\theta'[\mathrm{x}\cos\nu - \cos g\,(\mathrm{x}\cos g' + \mathrm{y}\cos h' + \mathrm{z}\cos l')]}{1 - \tan\frac{1}{2}\theta\tan\frac{1}{2}\theta'\cos\nu}\,,$$

and similarly for the other ones, in which ν represents the angle between the two rotational axes ([*]). One forms the complete variations Δx, Δy, Δz from those expressions by addition, and upon comparing their terms with their expressions as functions of the desired elements Θ, G, H, L, one will arrive at the following relations:

$$\begin{aligned}
\tan\tfrac{1}{2}\Theta\cos G &= \frac{\tan\frac{1}{2}\theta\cos g + \tan\frac{1}{2}\theta'\cos g' + 2\tan\frac{1}{2}\theta\tan\frac{1}{2}\theta'(\cos h\cos l' - \cos l\cos h')}{1 - \tan\frac{1}{2}\theta\tan\frac{1}{2}\theta'\cos\nu}\,,\\
\tan\tfrac{1}{2}\Theta\cos H &= \frac{\tan\frac{1}{2}\theta\cos h + \tan\frac{1}{2}\theta'\cos h' + 2\tan\frac{1}{2}\theta\tan\frac{1}{2}\theta'(\cos l\cos g' - \cos g\cos l')}{1 - \tan\frac{1}{2}\theta\tan\frac{1}{2}\theta'\cos\nu}\,,\\
\tan\tfrac{1}{2}\Theta\cos L &= \frac{\tan\frac{1}{2}\theta\cos l + \tan\frac{1}{2}\theta'\cos l' + 2\tan\frac{1}{2}\theta\tan\frac{1}{2}\theta'(\cos g\cos h' - \cos h\cos g')}{1 - \tan\frac{1}{2}\theta\tan\frac{1}{2}\theta'\cos\nu}\,,
\end{aligned}$$

from which, one will derive the ultimate value for the resultant rotation:

$$\cos\tfrac{1}{2}\Theta = \cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta' - \sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta'\cos\nu\,,$$

and the inclinations of the resultant axis:

([*]) One has $\cos\nu = \cos g\cos g' + \cos h\cos h' + \cos l\cos l'$.

$$\sin\tfrac{1}{2}\Theta\cos G = \sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta'\cos g + \sin\tfrac{1}{2}\theta'\cos\tfrac{1}{2}\theta\cos g' + \sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta'(\cos h\cos l' - \cos l\cos h'),$$
$$\sin\tfrac{1}{2}\Theta\cos H = \sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta'\cos h + \sin\tfrac{1}{2}\theta'\cos\tfrac{1}{2}\theta\cos h' + \sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta'(\cos l\cos g' - \cos g\cos l'),$$
$$\sin\tfrac{1}{2}\Theta\cos L = \sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta'\cos l + \sin\tfrac{1}{2}\theta'\cos\tfrac{1}{2}\theta\cos l' + \sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta'(\cos g\cos h' - \cos h\cos g').$$

One first remarks that the order of succession of the rotations θ, θ' in those formulas is irrelevant to the amplitude of the resultant rotation, but it is not irrelevant to the direction of the axis of that rotation, because the transposition of the elements $\theta\theta'$, $g\ h\ l$, $g'h'l'$, which will not alter the value of the rotations, will alter those of the cosines of the angles of its axis when changing the signs of the terms of second order with respect to those rotations in the expression for the cosine, unless those terms do not disappear for infinitely-small values of θ, θ'.

The expression for $\cos\tfrac{1}{2}\Theta$ is then well-suited to the angle of a spherical triangle whose opposite arc is equal to ν and whose other two angles are equal to $\tfrac{1}{2}\theta$, $\tfrac{1}{2}\theta'$. Meanwhile, in order to specify the position of the resultant axis with respect to the component axes, suppose (as is always possible) that:

$$\cos l = \cos l' = 0, \qquad \cos h = 0, \qquad \cos g = 1, \qquad \cos g' = \cos\nu, \qquad \cos h' = \sin\nu,$$

so one will have:

$$\sin\tfrac{1}{2}\Theta\cos G = \sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta' + \sin\tfrac{1}{2}\theta'\cos\tfrac{1}{2}\theta\cos\nu,$$
$$\sin\tfrac{1}{2}\Theta\cos H = \sin\tfrac{1}{2}\theta'\sin\tfrac{1}{2}\theta\sin\nu,$$
$$\sin\tfrac{1}{2}\Theta\cos L = \sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta'\sin\nu.$$

In case the order of the rotations is changed, the sign of $\cos L$ will become negative, and that is the only alteration that the change will produce in those formulas, so one sees that the new axis that results from that change in the order of rotation will be placed in a position that is *symmetric* to that of the first one relative to the plane of the two given axes.

If one lets H' denote the angle that is formed by the resultant axis and the axis of the rotation θ' then one will have:

$$\cos H' = \cos H + \cos h' + \cos G\cos g' = \frac{\sin\tfrac{1}{2}\theta'\cos\tfrac{1}{2}\theta + \sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta'\cos\nu}{\sin\tfrac{1}{2}\Theta}.$$

Moreover, one will have:

$$\sin^2\tfrac{1}{2}\Theta = \begin{cases} \sin^2\tfrac{1}{2}\theta\sin^2\nu + (\sin\tfrac{1}{2}\theta'\cos\tfrac{1}{2}\theta + \sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta'\cos\nu)^2, \\ \sin^2\tfrac{1}{2}\theta'\sin^2\nu + (\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta' + \sin\tfrac{1}{2}\theta'\cos\tfrac{1}{2}\theta\cos\nu)^2, \end{cases}$$

and consequently:

$$\sin^2 G = \frac{\sin^2\tfrac{1}{2}\theta'\sin^2\nu}{\sin^2\tfrac{1}{2}\Theta}, \qquad \sin^2 H' = \frac{\sin^2\tfrac{1}{2}\theta\sin^2\nu}{\sin^2\tfrac{1}{2}\Theta},$$

which are equations that establish the inverse proportionality between the sines of the semi-rotations and those of the angles that are defined by the resultant axis and the those of the corresponding component axes, and that will then lead to the construction that we first indicated for the composition of rotations.

The terms of second order relative to the rotations would make the formulas that one would obtain by following an analogous path for the composition of rotations around a finite number of converging axes very complicated. We shall not stop to dwell upon that then.

On the analytical composition of rotations around non-convergent axes

20. – As for the composition of rotations around non-convergent axes, and generally the composition of given successive arbitrary displacements of a solid system by variations such as $\Delta' x$, $\Delta'' x$, $\Delta''' x$, etc., whose analytical form is known, one will generally have:

$$\Delta x = \Delta' x + \Delta'' x + \Delta''' x + \ldots = \mathrm{A} + 2 \tan \tfrac{1}{2}\Theta\,(\mathrm{y} \cos L - \mathrm{z} \cos H) ,$$
$$\Delta y = \Delta' y + \Delta'' y + \Delta''' y + \ldots = \mathrm{B} + 2 \tan \tfrac{1}{2}\Theta\,(\mathrm{z} \cos G - \mathrm{x} \cos L) ,$$
$$\Delta \mathrm{z} = \Delta' \mathrm{z} + \Delta'' z + \Delta''' z + \ldots = \Gamma + 2 \tan \tfrac{1}{2}\Theta\,(\mathrm{x} \cos H - \mathrm{y} \cos G) .$$

The constants A, B, Γ, … are found to be determined as functions of the analogous constants A′, B′, Γ′, A″, B″, Γ″, … that are appropriate to each of the consecutive displacements that are being composed and the other elements of those displacements. However, it is obvious that the elements Θ, G, H, L of the defining rotation will be only functions of the *rotational* elements of those displacements, as we saw before by geometric considerations. For example, take the calculation of the composition of rotations around two non-convergent fixed axes, one of which coincides with the x-axis, and the other of which is normal to the z-axis and cuts that axis at a distance from the origin that is equal to u. As always, let ν denote the angle between those two axes of rotation, and let θ, θ' denote the amplitudes of the respective rotations. One will first have the values that were given above for the amplitude of the rotation and the direction of the resultant axis, and in order to either complete the variations of the coordinates or fix the position of the central axis, one will only have to calculate the variations of the origin or the coordinates α, β, γ. Now the ones that produce the first rotation around the x-axis are zero. It will then suffice to calculate the ones that produce the rotation θ', for which one will generally have:

$$\Delta'' x = \alpha + 2 \tan \tfrac{1}{2}\theta' [(\mathrm{y}'' - \tfrac{1}{2}\beta) \cos l' - (\mathrm{z}'' - \tfrac{1}{2}\gamma) \cos h'] ,$$
$$\Delta'' y = \beta + 2 \tan \tfrac{1}{2}\theta' [(\mathrm{z}'' - \tfrac{1}{2}\gamma) \cos g' - (\mathrm{x}'' - \tfrac{1}{2}\alpha) \cos l'] ,$$
$$\Delta'' \mathrm{z} = \gamma + 2 \tan \tfrac{1}{2}\theta' [(\mathrm{x}'' - \tfrac{1}{2}\alpha) \cos h' - (\mathrm{y}'' - \tfrac{1}{2}\beta) \cos g'] .$$

Those variations must be zero for all points on the axis of the rotation θ', for which one has:

$$\cos l' = 0 , \quad \cos h' = \sin \nu , \quad \cos g' = \cos \nu , \quad \mathrm{y}'' = \mathrm{x} \tan \nu , \quad \mathrm{z}'' = u .$$

The following values for α, β, γ will then result, which one easily deduces by the same construction:

$$\alpha = u \sin\, \nu \sin\, \theta', \qquad \beta = -\, u \sin\, \nu \sin\, \theta', \quad \gamma = 2u \sin^2 \tfrac{1}{2}\theta'.$$

Hence, one infers the value T of the absolute translation that results from two rotations around two non-convergent fixed axes:

$$T = \alpha \cos G + \beta \cos H + \gamma \cos L = \frac{2u \sin \nu \sin \frac{1}{2}\theta \sin \frac{1}{2}\theta'}{\sin^2 \frac{1}{2}\Theta} = -\,2\alpha \cos L,$$

which conforms to the theorems in no. **13**.

Therefore, the resultant axis and the two given divergent axes will come even closer to being parallel to the same plane as the distance that separates the latter two becomes greater with respect to the absolute translation of the composed displacement.

The equations of the central axis are obtained by replacing α, β, γ, Θ, G, H, L with their values that were given above in the general equations:

$$\frac{x - \frac{1}{2}\alpha - \frac{1}{2}\cot\frac{1}{2}\Theta(\beta\cos L - \gamma\cos H)}{\cos G}$$
$$= \frac{y - \frac{1}{2}\beta - \frac{1}{2}\cot\frac{1}{2}\Theta(\gamma\cos G - \alpha\cos L)}{\cos H}$$
$$= \frac{z - \frac{1}{2}\gamma - \frac{1}{2}\cot\frac{1}{2}\Theta(\alpha\cos H - \beta\cos G)}{\cos L}.$$

In the case of infinitely-small rotations, those formulas simplify considerably, and one will find that the central axis is parallel to the plane of the two component axes and will meet the line of shortest distance between them. Under that hypothesis, upon neglecting the second-order infinitesimals, one will effectively have:

$$\cos G = \frac{\theta + \theta' \cos\nu}{\Theta}, \qquad \cos H = \frac{\theta' \sin^2 \nu}{\Theta}, \qquad \cos L = \frac{\theta\,\theta' \sin\nu}{2\Theta},$$

$$\Theta = \theta^2 + \theta'^2 + 2\theta\,\theta' \cos\nu,$$

$$\alpha = u\,\theta' \sin\nu, \quad \beta = -\, u \sin\theta' \cos\nu, \quad \gamma = 0, \quad T = \frac{u\,\theta\,\theta\,\theta \sin\nu}{2\Theta},$$

and for the equations of the central axis, one will have:

$$y = \frac{x\,\theta' \sin\nu}{\theta + \theta' \cos\nu}, \qquad z = \frac{u\,\theta'(\theta' + \theta\cos\nu)}{\Theta^2}.$$

Composition of successive rotation around three rectangular axes

21. – We conclude our discussion of that subject by giving the following formulas for the composition of three successive relations $\theta, \theta', \theta''$ around the three coordinate axes of *x*, *y*, *z*. The elements of the composed rotation Θ, *G*, *H*, *L* are then expressed by:

$$\cos\tfrac{1}{2}\Theta = \cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta'\cos\tfrac{1}{2}\theta'' - \sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta'\sin\tfrac{1}{2}\theta'',$$

$$\sin^2 G = \frac{1-\cos\theta'\cos\theta''}{2\sin^2\frac{1}{2}\Theta}, \quad \sin^2 H = \frac{1-\cos\theta'\cos\theta''+\sin\theta\sin\theta'\sin\theta''}{2\sin^2\frac{1}{2}\Theta}, \quad \sin^2 G = \frac{1-\cos\theta\cos\theta'}{2\sin^2\frac{1}{2}\Theta}.$$

Those formulas will become symmetric with respect to each coordinate only when the rotations become infinitely small, since the term $\sin\theta\sin\theta'\sin\theta''$ will vanish in $\sin^2 H$, which is hypothesis that makes the order of those rotations irrelevant, moreover, and for which one will find that:

$$\Theta^2 = \theta^2 + \theta'^2 + \theta''^2, \quad \cos G = \frac{\theta}{\Theta}, \quad \cos H = \frac{\theta'}{\Theta}, \quad \cos L = \frac{\theta''}{\Theta},$$

according to the law of composition of infinitely-small rotations.

The inverse problem to the one that we just solved, which has the goal that when we are given a finite rotation and its axis, we must decompose it into three rotations around the coordinate axes, comes down to inferring the values of $\theta, \theta', \theta''$ as functions of the given elements Θ, *G*, *H*, *L* from the equations above, which is an operation that is intractable for finite rotations, but takes on great simplicity for the infinitely-small rotations.

On the composition of the successive infinitely-small displacements of a solid system

22. – We shall now consider the laws of composition and decomposition of successive infinitely-small displacements that are inferred from the analytical expressions for the infinitely-small variations of the coordinates of a solid system directly and within a certain scope.

Those variations are generally expressed *linearly* in the following manner as functions of the infinitely-small elements of the displacement $\alpha, \beta, \gamma, m, n, p$:

$$\delta x = \alpha + p\,y - n\,z, \qquad \delta y = \beta + m\,z - p\,x, \qquad \delta z = \gamma + n\,x - m\,y.$$

It then results that the variations that produce several successive infinitely-small displacements are composed by adding the variations that are due to each of those successive displacements separately when referred to the first position of the system. The *elements* of the composed or defining displacement are formed by the sum of the analogous elements of the partial displacements.

That is why the complete differential of a function of several variables is formed by adding the partial differentials that relate to each of the variables, which are partial differentials for which it is irrelevant whether one takes them on the original function itself or that function when it is successively augmented by infinitely-small increases in each of its variables, since one can neglect the second-order infinitesimals.

If one then lets α', β', γ', m', n', p', α'', β'', γ'', m'', n'', p'', α''', β''', γ''', m''', n''', p''', etc., denote the elements of the successive displacements that one must compose, and one has:

$$\delta' x = \alpha' + p' y - n' z\,, \qquad \delta' y = \beta' + m' z - p' x\,, \qquad \delta' z = \gamma' + n' x - m' y\,,$$
$$\delta'' x = \alpha'' + p'' y - n'' z\,, \qquad \delta'' y = \beta'' + m'' z - p'' x\,, \qquad \delta'' z = \gamma'' + n'' x - m'' y\,,$$
$$\ldots$$

for each of them, when considered separately, then the elements of the defining displacement of the system, which are represented by A, B, C, M, N, P, will be the sums of the given partial elements, respectively. One will have:

$$A = \alpha' + \alpha'' + \alpha''' = \sum \alpha\,, \quad B = \sum \beta\,, \quad C = \sum \gamma\,, \quad M = \sum m\,, \quad N = \sum n\,, \quad P = \sum p\,,$$

and for the defining expressions of the variations of the coordinates of the system:

$$\delta x = A + P\,y - M\,z = A + \Theta\,(y \cos L - z \cos H)\,,$$
$$\delta y = B + M\,z - P\,x = B + \Theta\,(z \cos C - x \cos L)\,,$$
$$\delta z = C + N\,x - M\,y = C + \Theta\,(x \cos H - y \cos G)\,,$$

upon introducing the rotation and the inclinations of its axis. *Conversely*, one can consider any given infinitely-small displacement of a solid system whose elements are A, B, C, M, N, P to be the result of successive displacements whose elements are α', β', γ', m', n', p', etc., which are subject to only the condition that they must be collectively equal to the given elements A, B, C, M, N, P, respectively, and the order of succession of those displacements is entirely arbitrary, moreover. Consequently, it will become obvious that the rotation θ, whose *components* relative to the coordinate axes are equal to m, n, p, respectively, results from *successive* rotations m, n, p, since the translation t, which is parallel to the axis of rotation, can also be considered to be the resultant of three *successive* translations $t \cos g$, $t \cos h$, $t \cos l$ that are parallel to the coordinate axes.

23. – In *mechanics*, those *successive* rotations m, n, p are referred to by the name of *elementary rotations*, and they are considered to be *simultaneous* under the passage from the *geometric laws* to the *mechanical* laws of the displacement of the body, although geometry can account for them only by supposing that they are *successive*, because it is obvious that while the system turns around the axis though a rotation of θ when the angles with the coordinate axes are g, h, l, that rotation is not accomplished *at the same time* as the three rotations m, n, p around those coordinate axes. That will imply four axes of rotation, instead of one (*Mécanique analytique*, t. I, page 52). Here, we have touched upon a fundamental point in the philosophy of mathematics that separates geometry from mechanics, and the objective of this treatise is to confirm all of its importance.

The rotation θ results from the successive composition of the rotations $\theta \cos g$, $\theta \cos h$, $\theta \cos l$, because for each coordinate axis, the variation due to that rotation θ is the sum of the variations that are due to each of those rotations separately. Indeed, if the system turns only around the x-axis with a rotation $\theta \cos g$ then one will have:

$$\delta' x = 0 , \qquad \delta' y = \theta z \cos g , \qquad \delta' z = - \theta y \cos g .$$

On the contrary, it turns around the y-axis with a rotation of $\theta \cos h$, one will have:

$$\delta'' x = \theta z \cos h , \qquad \delta'' y = 0 , \qquad \delta'' z = \theta h \cos h$$

for the second displacement, and finally, the rotation $\theta \cos l$ around the z-axis, when considered to be the only one, will give:

$$\delta''' x = \theta y \cos l , \qquad \delta''' y = - \theta x \cos l , \qquad \delta''' z = 0 .$$

The sum of those composed variations relative to each axis will then give the following expressions for the composed displacement, when it is performed, by definition, as the rotation θ around the axis (g, h, l):

$$\delta x = \theta (y \cos l - z \cos h) , \qquad \delta y = \theta (z \cos g - x \cos l) , \qquad \delta z = \theta (x \cos h - y \cos g) .$$

24. – Let us return to the composition of arbitrarily-given successive displacements for the same system. Conforming to no. **15**, the formulas for the variations can then be written:

$$\begin{aligned}
\delta x &= \alpha + \theta (y \cos l - z \cos h) = t \cos g + \theta u \cos G , \\
\delta y &= \beta + \theta (x \cos g - x \cos l) = t \cos h + \theta u \cos H , \\
\delta z &= \gamma + \theta (x \cos h - y \cos g) = t \cos l + \theta u \cos L ,
\end{aligned}$$

in which u denotes the distance from the point (x, y, z) to the *central* axis of the displacement, and G, H, L are the angles that are formed by the coordinate axes and the direction of the infinitely-small *arc* $u\ \theta$, which describes a *rotation* around that central axis.

One has the general expression for the relative displacement along an arbitrary direction s that makes the angles a, b, c with the coordinate axes:

$$\delta s = t \cos (t, s) + \theta u \cos (t\, u, s) ,$$

in which cos (t, s), cos ($\theta\, u$, s) denote the cosines of the angles between that direction and the central axis, and θ denotes the arc of the infinitely-small rotation.

However, if two lines are given in space then one knows that the distance from a point on one line to the other line is inverse to the sine of the angle that the first line forms with the plane that contains the second one and the point in question, or what amounts to the same thing, that the product of that sine and distance is constantly equal to the product of the shortest distance between those lines with the sine of their inclination. If one then lets D denote the distance from the central axis to the line that is drawn through the point of the system that one considers in the direction s, and lets ν denote the angle between that direction and the axis then one will have:

$$u \cos(\theta u, s) = D \sin \nu \qquad \text{and} \qquad \delta s = t \cos \nu + D\, \theta \sin \nu .$$

$t \cos \nu$ is the *moment* of the translation of the system relative to the direction s. $D\, \theta \sin \nu$ is that of the rotation of the relative to that same direction (*).

Now, if one considers the successive displacement of the system around the central axes whose elements are $t, t', t'', \theta, \theta', \theta'', g, h, l, g', h', l', g'', h'', l''$, etc., and one lets δS denote the composed resultant displacement of a point of the system relative to that same direction s, while T, Θ, G, H, L denote the elements of resultant central axis, then one will have:

$$\delta S = \sum t \cos \nu + \sum D \theta \cos \nu = T \cos V + \Theta\, \mathfrak{D} \sin V ,$$

in which $\mathfrak{D}$ denotes the distance from the resultant axis to the line s, which is drawn through the point (x, y, z).

Due to the indeterminates a, b, c that the equation includes, and since it must be true for all points of the system, it will be equivalent to the following six equations, which give the position of the central axis of the displacement, the translation, and the resultant rotation:

$$\begin{gathered}\sum \alpha - T \cos G + \Theta \mathrm{y} \cos L - \Theta \mathrm{z} \cos H = 0 , \\ \sum \beta - T \cos H + \Theta \mathrm{z} \cos G - \Theta \mathrm{x} \cos L = 0 , \\ \sum \gamma - T \cos L + \Theta \mathrm{x} \cos H - \Theta \mathrm{y} \cos G = 0 ,\end{gathered}$$

$$\Theta \cos G = \sum \theta \cos g , \qquad \Theta \cos H = \sum \theta \cos h , \qquad \Theta \cos L = \sum \theta \cos l ,$$

from which one infers that:

$$T = \cos G \sum \alpha + \cos H \sum \beta + \cos L \sum \gamma ,$$

in which x, y, z denote the coordinates of an arbitrary point of the resultant central axis, because one has:

(*) *That moment is also* a measure of the entire translation that is performed by an arbitrary point on the central axis along that same axis with a length that is equal to the product of the rotation θ with the distance from that point to the line s, so that translation is estimated or projected *orthogonally* to that line s, i.e., along a line that is drawn through the point in question normally to the plane that contains that point and the latter line.

$$\delta S = \cos a \sum \delta x + \cos b \sum \delta y + \cos c \sum \delta z\,,$$

$$\cos V = \cos a\,[(y - \mathrm{y}) \cos L - (z - \mathrm{z}) \cos H] + \cos b\,[(z - \mathrm{z}) \cos G - (x - \mathrm{x}) \cos L] + \cos c\,[(x - \mathrm{x}) \cos H - (y - \mathrm{y}) \cos G]\,.$$

One can rigorously consider only rotations in that analysis, since the translations t, t' can always be represented by pairs of rotations, and one sets simply:

$$\delta S = T \cos V + \Theta\, \mathfrak{D} \sin V = \sum \theta D \sin \nu\,.$$

In order to transform a translation t along the direction g, h, l into a pair of rotations, one must solve the following three equations:

$$t \cos g = u_1\, \theta_1 \cos G_1 + u_2\, \theta_2 \cos G_2 = \theta_1\,[(y - \mathrm{y}) \cos l_1 - (\mathrm{z} - \mathrm{z}) \cos h_1] + \theta_2\,[(y - \mathrm{y}') \cos l_2 - (\mathrm{z} - \mathrm{z}') \cos h_2]\,,$$

$$t \cos h = u_1\, \theta_1 \cos H_1 + u_2\, \theta_2 \cos H_2 = \theta_1\,[(\mathrm{z} - \mathrm{z}) \cos g_1 - (x - \mathrm{x}) \cos l_1] + \theta_2\,[(\mathrm{z} - \mathrm{z}') \cos g_2 - (x - \mathrm{x}') \cos l_2]\,,$$

$$t \cos l = u_1\, \theta_1 \cos L_1 + u_2\, \theta_2 \cos L_2 = \theta_1\,[(x - \mathrm{x}) \cos h_1 - (y - \mathrm{y}) \cos g_1] + \theta_2\,[(x - \mathrm{x}') \cos h_2 - (y - \mathrm{y}') \cos g_2]\,,$$

in which x, y, z, x′, y′, z′, etc., represent the coordinates of the axes of rotation, and g_1, h_1, l_1, g_2, h_2, l_2 represent their inclinations. Those equations must be satisfied independently of the variables x, y, z, and will then give the following six equations:

$$\theta_1 \cos g_1 + \theta_2 \cos g_2 = 0\,, \qquad \theta_1 \cos h_1 + \theta_2 \cos h_2 = 0\,, \qquad \theta_1 \cos l_1 + \theta_2 \cos l_2 = 0\,,$$

$$\begin{aligned} t \cos g &= (\mathrm{z} \cos h_1 - \mathrm{y} \cos l_1)\, \theta_1 + (\mathrm{z}' \cos h_2 - \mathrm{y}' \cos l_2)\, \theta_2\,, \\ t \cos h &= (\mathrm{x} \cos l_1 - \mathrm{z} \cos g_1)\, \theta_1 + (\mathrm{z}' \cos l_2 - \mathrm{z}' \cos g_2)\, \theta_2\,, \\ t \cos l &= (\mathrm{y} \cos g_1 - \mathrm{x} \cos h_1)\, \theta_1 + (\mathrm{y}' \cos g_2 - \mathrm{x}' \cos h_2)\, \theta_2\,, \end{aligned}$$

from which one infers that:

$$\theta_2 = -\,\theta_1\,, \qquad \cos g_1 = \cos g_2\,, \qquad \cos h_1 = \cos h_2\,, \qquad \cos l_1 = \cos l_2\,,$$

$$t \cos g = \theta_2\,[(\mathrm{z} - \mathrm{z}') \cos l_1 - (\mathrm{y} - \mathrm{y}') \cos l_2]\,, \qquad \text{etc.},$$

which are equations that are appropriate to exclusively the two parallel axes of rotation in a plane normal to t when the distance between the two axes is equal to the quotient t / θ^2. Let us return to the equation:

$$\delta S = T \cos V + \Theta\, \mathfrak{D} \sin V = \sum \theta D \sin \nu .$$

It then results that, *relative to an arbitrary direction s*, the composition of the rotations $\theta,\ \theta',\ \theta''$ around fixed axes will resolve into a rotation Θ and a translation or *pair* of rotations whose *moment* $T \cos V$ is equal to the excess of the sum of the *moments* of those partial rotations beyond the resultant moment of the composed rotation, where those moments are estimated relative to the same point.

For an arbitrary point that is distinct from the origin of the coordinates, one will have:

$$\delta S - \delta S_0 = \sum \theta D \sin \nu - \sum \theta D_0 \sin \nu_0 = \Theta \sin V (\mathfrak{D} - \mathfrak{D}_0) .$$

Therefore, the difference between the composed *moments* of a solid system that is subject to successively submitting to infinitely-small displacements with respect to points of that system is equal to the sum of the differences between the resultant *moments* for those two points of each of the successive rotations of the system when all of those moments are estimated relative to the same direction.

One knows that those *moments* refer to nothing but the *effect* of the displacement of the system to the point that is it referred to. The translation is common to all points so it cannot alter their relative positions, either under the partial displacements or under the composed displacement. It will then be obvious *a priori* that any *relative displacement*, whether partial or composed, of an independent point of the system with respect to another point of that system can depend upon only the *partial* or *composed rotations* of that system.

Conditions for ***equilibrium*** *for several successive infinitely-small displacements*

24. – We are led to search for the conditions that must be satisfied by the elements of the indicated successive displacements of a solid system in order that the system should successively pass through the various infinitely-close positions that correspond to those elements and return to its original position, which would constitute, by definition, an *equilibrium* state or neutralization of those successive displacements. Now, it is obvious that all of the conditions that one deals with are included in just one equation that is decomposable into six others, due to the indeterminates that it includes, namely:

$$\delta S = 0 ,$$

since that equation expresses the idea that each point of the system returns to its original position. The six equations that are included in the general equation $\delta S = 0$ are:

$$\delta x_0 = \sum \alpha = 0 ,$$
$$\delta y_0 = \sum \beta = 0 ,$$
$$\delta z_0 = \sum \gamma = 0 ,$$

$$\sum \theta \cos g = 0 , \qquad \sum \theta \cos h = 0 , \qquad \sum \theta \cos l = 0 ,$$

in which one lets δx_0 , δy_0 , δz_0 denote the complete variations or the coordinates of the origin. The first three express the *immobility* of the origin of the coordinates, and the other three express the idea that no rotation has taken place in the displaced system, by definition, which is a double condition that excludes the possibility of an arbitrary defining displacement. That double condition immediately results from the relation:

$$\delta S = T \cos V + \Theta \, \mathfrak{D} \sin V = 0 ,$$

which is a relation that can be satisfied for an arbitrary point only if one has:

$$T = 0 , \qquad \Theta = 0 .$$

Now $\Theta = 0$ implies *a fortiori* that following three equations:

$$\sum \theta \cos g = 0 , \qquad \sum \theta \cos h = 0 , \qquad \sum \theta \cos l = 0 ,$$

and when the fundamental equation (no. **15**) is applied to the origin:

$$\delta x_0^2 + \delta y_0^2 + \delta z_0^2 = t^2 + \theta^2 u^2 ,$$

when the rotation is zero with the translation, will give three other ones:

$$\delta x_0 = 0 , \qquad \delta y_0 = 0 , \qquad \delta z_0 = 0 .$$

Therefore, the *equilibrium* condition for a series of successive infinitely-small displacements of the same solid will be six in number, three of which express the idea that the displacement of a system admits no translation, and the other three of which exclude any rotation.

Those six equilibrium equations are analytically included in just one equation that expresses the *general* law of that equilibrium in the simplest manner, namely:

$$\sum \theta D \sin \nu = 0 ,$$

which expresses the idea that the resultant *moment* for an arbitrary point of the system, which is the sum of all *moments* that are produced at that point by the rotations and translations or *pairs* of *rotations* that are successively transmitted to or suffered by the system, is zero with respect to an arbitrary direction. However, if all conditions remain the same then the equilibrium of those successive infinitely-small displacements will continue to prevail, no matter how rapid one supposes the succession of those displacements to be: Upon passing to the limits, one will arrive at the identity of the laws of equilibrium for *successive* causes of infinitely-small displacements, along with the equilibrium of *simultaneous* causes of infinitely-small displacements.

Therefore, the equation:

$$\sum \theta D \sin V = 0$$

expresses the general condition for the *immobility* of an invariable system that is subjected to arbitrary causes that act *simultaneously* in such a way that they describe infinitely-small rotations θ, θ', etc., around given fixed axes.

The analogy between those laws of composition and equilibrium with those of the composition and equilibrium of ***forces*** *applied to an invariable system*

26. – The analogy between that general law and that of the equilibrium of forces that are applied to an invariable is striking. The applied forces along the axes of rotation are supposed to be proportional to those rotations, so the *moment* of a force on that system will be identically proportional to that of the corresponding rotation, and each translation will be replaced by a pair of forces that are applied along the axes of the pair of rotations that is equivalent to that translation. However, the analogy extends to the laws of composition in the same way that it does to equilibrium and can be stated as follows: A given system of successive displacements is to be composed into a defining displacement, and at the same time, a system of forces that that proportional to the given successive rotations for each displacement that are applied along the same axes as those rotation. If one does not suppose that the translations of the successive displacements are implicitly included in the rotations by way of their representation as pairs of rotations then they will be represented in the system of forces considered by pairs of forces whose moments are equal to those of the translations relative to the three coordinate axes, and the system of successive displacements will then come down to a defining displacement that is composed of a rotation and an absolute translation relative to the central axis of rotation. Similarly, the system of forces will resolve by the successive composition of its elements into just one force and just one force-couple that is situated in a plane normal to the resultant force. That resultant force is applied to the central axis of the defining displacement, which will be, at the same time, the *central axis* of the static system. It will be proportional to the resultant rotation, and moment of the couple normal to that force will be proportional to the absolute translation of the system that acts parallel to the central axis. When the axes of rotations to be composed are all parallel and pass through well-defined points, the central axis that results from them will also be parallel and pass through a certain point that corresponds to the center of the parallel forces, and it will always be the same no matter what the direction of the axes of rotation might be, and it is nothing but the center of gravity of some well-defined points along the axes of the component rotations when all of the rotations are equal (*).

(*) Just as any arbitrary number of successive infinitely-small rotations that are performed on a solid around different fixed axes reduces to two conjugate rotations in an infinitude of ways, similarly, any system of forces that are applied to a solid can be represented in an infinitude of ways by two conjugate forces whose product, multiplied by the distance that separates them and the sine of their direction, is constant.

Indeed, the equations of the central axis that result from composing the fixed axes of rotation reduce to:

$$\text{y} \cos l - \text{z} \cos h + \frac{\sum \alpha}{\sum \theta} = 0 , \qquad \text{x} \cos h - \text{y} \cos g + \frac{\sum \gamma}{\sum \theta} = 0 ,$$

in this case, and one will have:

$$T = 0 , \qquad \Theta = \sum \theta = \sum \alpha = \sum \theta \ (\text{Z} \cos h - \text{Y} \cos l) , \qquad \text{etc.,}$$

in which X, Y, Z denote the coordinates of the axis of rotation θ, so the resultant axis will pass through the point whose coordinates are:

$$x = \frac{\sum \theta X}{\sum \theta} , \quad y = \frac{\sum \theta Y}{\sum \theta} , \quad z = \frac{\sum \theta Z}{\sum \theta} .$$

On the determination of the variations of the coordinates of a solid system that are due to an arbitrary displacement of that system, analytically deduced from the conditions of invariability of that system

27. – The consideration of the displacement of coordinate axes that are supposed to be invariably coupled with the displaced system leads immediately to the algebraic expression for the variations of the coordinates of that system and it all comes down to reducing the number of arbitrary constants that are present in the calculations to a minimum, as one will see.

Indeed, if one lets $a, b, c, a', b', c', a'', b'', c''$ denote the cosines of the angles that the displaced coordinate axes form with their original directions then the coordinates $x + \Delta x$, $y + \Delta y$, $z + \Delta z$ of a point of the system after displacement, when referred to the original position of the coordinate axes, can be expressed as functions of the coordinates of that same displaced point when they are referred to the displaced axes, and they will be the same as those of the point considered in the first position, when referred to the first axes according to known formulas for the transformation of the rectangular coordinates. One will then have the following three equations:

$$\begin{aligned} x + \Delta x &= \alpha + a\,x + b\,y + c\,z , \\ y + \Delta y &= \beta + a'x + b'y + c'z , \\ z + \Delta z &= \gamma + a''x + b''y + c''z , \end{aligned}$$

by means of which Δx, Δy, Δz are generally found to be expressed, no matter what the displacement of the solid might be, as linear functions of the coordinates x, y, z that include twelve arbitrary coefficients that actually reduce to six constants, because the first three of them α, β, γ express the variation of the coordinates of the origin, and the other three a, b', c'', by means of which one eliminates the other six cosines using the Monge formulas, define the change of *direction* in the space of the original coordinate axes.

However, the reduction of those twelve constants to six can be achieved even more simply without the use of the Monge formulas, which are complicated and radical, and along a path that, will lead to less-complicated analytical expressions for Δx, Δy, Δz relative to the variables x, y, z. Indeed, introduce the coordinates x, y, z of the midpoint of the line that joins the two corresponding positions of the point x, y, z into the equations above in such a way that one has:

$$x = \mathrm{x} - \tfrac{1}{2}\Delta x\,, \qquad y = \mathrm{y} - \tfrac{1}{2}\Delta y\,, \qquad z = \mathrm{z} - \tfrac{1}{2}\Delta z\,.$$

In order to combine those three equations into just one, multiply them by the three undetermined factors μ, ν, π, and additionally consider the single equation:

$$\tfrac{1}{2}\Delta x\,(a\mu + a'\nu + a''\pi + \mu) + \tfrac{1}{2}\Delta y\,(b\mu + b'\nu + b''\pi + \nu) + \tfrac{1}{2}\Delta z\,(c\mu + c'\nu + c''\pi + \pi)$$

$$= \alpha\mu + \beta\nu + \gamma\pi + (a\mu + a'\nu + a''\pi - \mu)\,\mathrm{x} + (b\mu + b'\nu + b''\pi - \nu)\,\mathrm{y} + (c\mu + c'\nu + c''\pi - \pi)\,\mathrm{z}\,.$$

In order to determine Δx, one must give values to m, n, p in that equation that will annul the coefficients of Δy, Δz, i.e., one must set:

$$b\mu + b'\nu + b''\pi + \nu = 0\,, \qquad c\mu + c'\nu + c''\pi + \pi = 0\,,$$

from which, one infers that:

$$\mu = (1 + b')(1 + c'') - c'b''\,, \qquad \nu = a'' - b(1 + c'')\,, \qquad \pi = bc' - c(1 + b')\,.$$

Now, as one knows, there exist the following relations between the nine cosines a, b, c, a', b', c', a'', b'', c'':

$$\begin{aligned}
a &= b'c'' - c'b''\,, & a' &= cb'' - bc''\,, & a'' &= bc' - cb'\,,\\
b &= c'a'' - a'c''\,, & b' &= ac'' - ca''\,, & b'' &= ca' - ac'\,,\\
c &= c'b'' - b'c''\,, & c' &= ba'' - ab''\,, & c'' &= ab' - ba'\,,
\end{aligned}$$

from which, one deduces these six other ones:

$$\begin{aligned}
&a^2 + a'^2 + a''^2 = 1\,, & &b^2 + b'^2 + b''^2 = 1\,, & &c^2 + c'^2 + c''^2 = 0\,,\\
&ab + a'b' + a''b'' = 0\,, & &ac + a'c' + a''c'' = 0\,, & &bc + b'c' + b''c'' = 0\,,
\end{aligned}$$

which express the idea that the original axes and the new ones are rectangular. The ambiguity in the signs that presents itself in the logic in question is resolved by another condition that is completely necessary in the consideration of the displacement of a solid system, namely: The new system of coordinate axes that is produced by the displacement of the first axes, along with the first one, will always remain subject to the conditions of superposition that are possible for each respective axis and its correspondent.

Having said that, it is clear that due to the relations above, one has:

$$\mu = 1 + a + b' + c'', \qquad \nu = a' - b, \qquad \pi = a'' - c,$$

so one infers that:

$$a\,\mu + a'\,\nu + a''\,\pi = 1 + a + b' + c'' = \mu,$$

which gives:

$$\Delta x - \alpha = \frac{2[(\mathrm{y} - \tfrac{1}{2}\beta)(b - a') - (\mathrm{z} - \tfrac{1}{2}\alpha)(b - a')}{1 + a + b' + c''},$$

and one will similarly have:

$$\Delta y - \beta = \frac{2[(\mathrm{z} - \tfrac{1}{2}\gamma)(c' - b'') - (\mathrm{x} - \tfrac{1}{2}\alpha)(b - a')}{1 + a + b' + c''},$$

$$\Delta \mathrm{z} - \gamma = \frac{2[(\mathrm{x} - \tfrac{1}{2}\alpha)(a'' - c) - (\mathrm{y} - \tfrac{1}{2}\beta)(c' - b'')}{1 + a + b' + c''},$$

which are formulas that will become identical to the ones in no. **15** upon setting:

$$m = \frac{2(c' - b'')}{1 + a + b' + c''}, \qquad n = \frac{2(a'' - c)}{1 + a + b' + c''}, \qquad p = \frac{2(b - a')}{1 + a + b' + c''}.$$

28. – In the case of infinitely-small displacements, upon neglecting the second-order angular displacement, one will have:

$$a = 1, \qquad b' = 1, \qquad c'' = 1$$

directly, which will give:

$$\delta x = \alpha + b\,y + c\,z, \qquad \delta y = \beta + a'\,x + c'\,z, \qquad \delta z = \gamma + a''\,x + b''\,y,$$

and since the distance from an arbitrary point to the origin is invariable under the displacement of the system, one will have:

$$x\,(\delta x - \alpha) + y\,(\delta y - \beta) + z\,(\delta z - \gamma) = 0$$

for any x, y, z, so the following relations will necessarily exist between the cosines b, c, a', c', a'', b'' which are infinitely-small of order one:

$$b + a' = 0, \qquad c + a'' = 0, \qquad c' + b'' = 0,$$

and one will finally have:

$$\delta x = \alpha + p\,y - n\,z, \qquad \delta y = b + m\,z - p\,x, \qquad \delta z = \gamma + n\,x - m\,y$$

then upon setting:

$$m = c' = -\,b'', \qquad n = a'' = -\,c\,, \qquad p = b = -\,a'\,.$$

29. – However, it is interesting to arrive at the same formulas for the finite or infinitesimal variations of the coordinates of a solid system along a purely-algebraic path that is independent of any geometric considerations by starting from the algorithmic expression for the conditions of the problem, i.e., the invariability of the distance between the points of that solid.

Therefore, let:

$$x_0,\ x_1,\ x_2,\ x, \qquad y_0,\ y_1,\ y_2,\ y, \qquad z_0,\ z_1,\ z_2,\ z$$

be the coordinates of four points that are invariably coupled to each other and belong to the system that one considers, the first three of which are specially chosen so that one can refer the other ones to them. The distances between those four points must remain constant, no matter what the variations of their coordinates as a result of an arbitrary displacement of the system, so when that condition is expressed algorithmically, it will give the following six equations between the variations of those twelve coordinates:

$$\begin{aligned}
(x_1+\Delta x_1-x_0-\Delta x_0)^2+(y_1+\Delta y_1-y_0-\Delta y_0)^2+(z_1+\Delta z_1-z_0-\Delta z_0)^2 &= (x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2\,,\\
(x_2+\Delta x_2-x_0-\Delta x_0)^2+(y_2+\Delta y_2-y_0-\Delta y_0)^2+(z_2+\Delta z_2-z_0-\Delta z_0)^2 &= (x_2-x_0)^2+(y_2-y_0)^2+(z_2-z_0)^2\,,\\
(x_2+\Delta x_2-x_1-\Delta x_1)^2+(y_2+\Delta y_2-y_1-\Delta y_1)^2+(z_2+\Delta z_2-z_1-\Delta z_1)^2 &= (x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\,,\\
(x+\Delta x-x_0-\Delta x_0)^2+(y+\Delta y-y_0-\Delta y_0)^2+(z+\Delta z-z_0-\Delta z_0)^2 &= (x-x_0)^2+(y-y_0)^2+(z-z_0)^2\,,\\
(x+\Delta x-x_1-\Delta x_1)^2+(y+\Delta y-y_1-\Delta y_1)^2+(z+\Delta z-z_1-\Delta z_1)^2 &= (x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2\,,\\
(x+\Delta x-x_2-\Delta x_2)^2+(y+\Delta y-y_2-\Delta y_2)^2+(z+\Delta z-z_2-\Delta z_2)^2 &= (x-x_2)^2+(y-y_2)^2+(z-z_2)^2\,.
\end{aligned}$$

Those six equations serve to determine Δx, Δy, Δz as functions of the variations of the coordinates of the first three points, which must reduce to six constants as a result of the three equations that coupled them.

One essentially comes down to solving the six equations, and although they are of degree two with respect to the variations Δx, Δy, Δz, they will become linear when one replaces the variables x, y, z with these ones:

$$\mathrm{x} = x + \tfrac{1}{2}\Delta x\,, \quad \mathrm{y} = y + \tfrac{1}{2}\Delta y\,, \quad \mathrm{z} = z + \tfrac{1}{2}\Delta z\,,$$

and similarly for the other points: The system of six equations above can then be written as:

$$(\mathrm{x}-\mathrm{x}_0)(\Delta x-\Delta x_0)+(\mathrm{y}-\mathrm{y}_0)(\Delta y-\Delta y_0)+(\mathrm{z}-\mathrm{z}_0)(\Delta z-\Delta z_0) = 0\,,$$

$$\begin{aligned}
&(\mathrm{x}-\mathrm{x}_0)(\Delta x_1-\Delta x_0)+(\mathrm{y}-\mathrm{y}_0)(\Delta y_1-\Delta y_0)+(\mathrm{z}-\mathrm{z}_0)(\Delta z_1-\Delta z_0)\\
+\;&(\mathrm{x}_1-\mathrm{x}_0)(\Delta x-\Delta x_0)+(\mathrm{y}_1-\mathrm{y}_0)(\Delta y-\Delta y_0)+(\mathrm{z}_1-\mathrm{z}_0)(\Delta z-\Delta z_0) = 0
\end{aligned}$$

$$(\mathrm{x}-\mathrm{x}_0)(\Delta x_2-\Delta x_0)+(\mathrm{y}-\mathrm{y}_0)(\Delta y_2-\Delta y_0)+(\mathrm{z}-\mathrm{z}_0)(\Delta z_2-\Delta z_0)$$

$$+\ (x_2 - x_0)(\Delta x - \Delta x_0) + (y_2 - y_0)(\Delta y - \Delta y_0) + (z_2 - z_0)(\Delta z - \Delta z_0) \quad = 0$$

$$(x_1 - x_0)(\Delta x_1 - \Delta x_0) + (y_1 - y_0)(\Delta y_1 - \Delta y_0) + (z_1 - z_0)(\Delta z_1 - \Delta z_0) = 0\ ,$$

$$(x_2 - x_0)(\Delta x_2 - \Delta x_0) + (y_2 - y_0)(\Delta y_2 - \Delta y_0) + (z_2 - z_0)(\Delta z_2 - \Delta z_0) = 0\ ,$$

$$\begin{aligned}&(x_1 - x_0)(\Delta x_2 - \Delta x_0) + (y_1 - y_0)(\Delta y_2 - \Delta y_0) + (z_1 - z_0)(\Delta z_2 - \Delta z_0)\\ +\ &(x_2 - x_0)(\Delta x_1 - \Delta x_0) + (y_2 - y_0)(\Delta y_1 - \Delta y_0) + (z_2 - z_0)(\Delta z_1 - \Delta z_0) = 0\ .\end{aligned}$$

Upon multiplying those six equations by the undetermined factors μ^2, $\mu\,\nu$, $\mu\,\pi$, ν^2, π^2, $\nu\,\pi$, in succession, and adding them, one will form the following complex equation, which includes the system of the first six:

$$\left.\begin{aligned}&[\mu(x-x_0)+\nu(x_1-x_0)+\pi(x_2-x_0)][\mu(\Delta x-\Delta x_0)+\nu(\Delta x_1-\Delta x_0)+\pi(\Delta x_2-\Delta x_0)]\\ +&[\mu(y-y_0)+\nu(y_1-y_0)+\pi(y_2-y_0)][\mu(\Delta y-\Delta y_0)+\nu(\Delta y_1-\Delta y_0)+\pi(\Delta y_2-\Delta y_0)]\\ +&[\mu(z-z_0)+\nu(z_1-z_0)+\pi(z_2-z_0)][\mu(\Delta z-\Delta z_0)+\nu(\Delta z_1-\Delta z_0)+\pi(\Delta z_2-\Delta z_0)]\end{aligned}\right\} = 0\ .$$

Upon choosing μ, ν, π conveniently, one will successively reduce that equation to one that contains only Δx, Δy, Δz, and one can then deduces the values of those variations from it in the simplest manner. Indeed, set:

$$\begin{aligned}\mu(y - y_0) + \nu(y_1 - y_0) + \pi(y_2 - y_0) &= 0\ ,\\ \mu(z - z_0) + \nu(z_1 - z_0) + \pi(z_2 - z_0) &= 0\ ,\end{aligned}$$

so the general equation reduces to:

$$[\mu(x - x_0) + \nu(x_1 - x_0) + \pi(x_2 - x_0)]\,[\mu(\Delta x - \Delta x_0) + \nu(\Delta x_1 - \Delta x_0) + \pi(\Delta x_2 - \Delta x_0)] = 0\ .$$

Now, the first factor of that product cannot be zero unless the second is likewise zero, as will be proved below. One will then have, simultaneously:

$$\begin{aligned}\mu(\Delta x - \Delta x_0) + \nu(\Delta x_1 - \Delta x_0) + \pi(\Delta x_2 - \Delta x_0) &= 0\ ,\\ \mu(y - y_0) + \nu(y_1 - y_0) + \pi(y_2 - y_0) &= 0\ ,\\ \mu(z - z_0) + \nu(z_1 - z_0) + \pi(z_2 - z_0) &= 0\ ,\end{aligned}$$

which are equations that obviously transform into these ones:

$$\begin{aligned}\Delta x_1 - \Delta x_0 &= p\,(y_1 - y_0) - n\,(z_1 - z_0)\ ,\\ \Delta x_2 - \Delta x_0 &= p\,(y_2 - y_0) - n\,(z_2 - z_0)\ ,\\ \Delta x - \Delta x_0 &= p\,(x_1 - x_0) - n\,(z - z_0)\ ,\end{aligned}$$

in which n and p are two constants that are coupled with the variations of the first three points by the first two of those three equations.

The same analysis will give:

$$\Delta y - \Delta y_0 = m'(z - z_0) - p'(x - x_0),$$
$$\Delta z - \Delta z_0 = n'(x - x_0) - m'(y - y_0),$$

and since one has:

$$(x - x_0)(\Delta x - \Delta x_0) + (y - y_0)(\Delta y - \Delta y_0) + (z - z_0)(\Delta z - \Delta z_0) = 0,$$

it will then follow that $m' = m$, $n' = n$, $p' = p$, and that one will finally have the following general formulas:

$$\Delta x = \mathrm{A} + p\,y - n\,z,$$
$$\Delta y = \mathrm{B} + m\,z - p\,x,$$
$$\Delta z = \Gamma + n\,x - m\,y,$$

in which the six constants A, B, Γ, m, n, p are functions of the variations:

$$\Delta x_0, \Delta x_1, \Delta x_2, \qquad \Delta y_0, \Delta y_1, \Delta y_2, \qquad \Delta z_0, \Delta z_1, \Delta z_2.$$

30. – However, it remains for us to prove our assertion that the factor $\mu(x - x_0) + \nu(x_1 - x_0) + \pi(x_2 - x_0)$ cannot be zero in the problem at hand without the second factor:

$$\mu(\Delta x - \Delta x_0) + \nu(\Delta x_1 - \Delta x_0) + \pi(\Delta x_2 - \Delta x_0)$$

being likewise. Indeed, upon eliminating the factors μ, ν, π, the three simultaneous equations:

$$\mu(x - x_0) + \nu(x_1 - x_0) + \pi(x_2 - x_0) = 0,$$
$$\mu(y - y_0) + \nu(y_1 - y_0) + \pi(y_2 - y_0) = 0,$$
$$\mu(z - z_0) + \nu(z_1 - z_0) + \pi(z_2 - z_0) = 0$$

will give a final equation between the coordinates $x_0, x_1, x_2, x, y_0, y_1, y_2, y, z_0, z_1, z_2, z$ that expresses the idea that the midpoints of the lines that are traversed by the four points that we consider are in the same plane. However, that singular condition cannot be fulfilled unless the pyramids that are formed by those four points in their first and second positions will be symmetric, instead of identical or superposable. That second hypothesis is not admissible in the problem to be solved. However, the first one must be examined. Therefore, if the four given points are in the same plane then we will say that we have both:

$$\mu(x - x_0) + \nu(x_1 - x_0) + \pi(x_2 - x_0) = 0, \qquad \mu(\Delta x - \Delta x_0) + \nu(\Delta x_1 - \Delta x_0) + \pi(\Delta x_2 - \Delta x_0) = 0,$$

and similarly for the variables y, z, and the variations Δy, Δz.

Indeed, if the four points considered are in the same plane, as well as their correspondents, then one will have, simultaneously, upon distinguishing the coordinates of the latter by a prime:

$$g\,(x - x_0) + h\,(x_1 - x_0) + l\,(x_2 - x_0) = 0\,, \qquad g'(x' - x_0') + h'(x_1' - x_0') + l'(x_2' - x_0') = 0\,,$$
$$g\,(y - y_0) + h\,(y_1 - y_0) + l\,(y_2 - y_0) = 0\,, \qquad g'(y' - y_0') + h'(y_1' - y_0') + l'(y_2' - y_0') = 0\,,$$
$$g\,(z - z_0) + h\,(z_1 - z_0) + l\,(z_2 - z_0) = 0\,, \qquad g'(z' - z_0') + h'(z_1' - z_0') + l'(z_2' - z_0') = 0\,,$$

in which g, h, l, g', h', l' are coefficients to be eliminated. Since one has:

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = (x' - x_0')^2 + (y' - y_0')^2 + (z' - z_0')^2\,,$$

$$(x - x_0)(x_1 - x_0) + (y - y_0)(y_1 - y_0) + (z - z_0)(z_1 - z_0)$$
$$= (x' - x_0')(x_1' - x_0') + (y' - y_0')(y_1' - y_0') + (z' - z_0')(z_1' - z_0')\,, \text{ etc.,}$$

by the invariability of the distances, and the ratios h / g, h' / g', l / g, l' / g' will be functions that are identical to those same quantities, will then follow that they are equal and that one can suppose that $g' = g$, $h' = h$, $l' = l$, from which, upon introducing the coordinates of the midpoint x = $(x + x') / 2$, etc., and the differences Δx, Δy, etc., one can conclude that one also has:

$$g\,(\mathrm{x} - \mathrm{x}_0) + h\,(\mathrm{x}_1 - \mathrm{x}_0) + l\,(\mathrm{x}_2 - \mathrm{x}_0) = 0\,,$$
$$g\,(\mathrm{y} - \mathrm{y}_0) + h\,(\mathrm{y}_1 - \mathrm{y}_0) + l\,(\mathrm{y}_2 - \mathrm{y}_0) = 0\,,$$
$$g\,(\mathrm{z} - \mathrm{z}_0) + h\,(\mathrm{z}_1 - \mathrm{z}_0) + l\,(\mathrm{z}_2 - \mathrm{z}_0) = 0\,,$$

$$g\,(\Delta x - \Delta x_0) + h\,(\Delta x_1 - \Delta x_0) + l\,(\Delta x_2 - \Delta x_0) = 0\,,$$
$$g\,(\Delta y - \Delta y_0) + h\,(\Delta y_1 - \Delta y_0) + l\,(\Delta y_2 - \Delta y_0) = 0\,,$$
$$g\,(\Delta z - \Delta z_0) + h\,(\Delta z_1 - \Delta z_0) + l\,(\Delta z_2 - \Delta z_0) = 0\,.$$

When the first three equations are compared to the given three in terms of μ, ν, π, that will show the proportionality of the coefficients g, h, l to the latter ones, and that will finally lead to the equation:

$$\mu\,(\Delta x - \Delta x_0) + \nu\,(\Delta x_1 - \Delta x_0) + \pi\,(\Delta x_2 - \Delta x_0) = 0\,,$$

which was to be established.

34. – The two analytical methods that we just presented for determining the formulas for the variations of the coordinates of a solid system are deduced by purely-algebraic processes. The infinitesimal algorithm again provides a simpler proof of those formulas that is close to the one that Lagrange gave in his *Mécanique*, and includes expressions for the finite, as well as infinitely-small, variations in the same analysis. Here is that proof:

One lets δ denote the infinitely-small differences in the coordinates of a point of the system from those of an infinitely-close point in the undisplaced systems, since according to whether one considers a finite or infinitely-small displacement of that system, the symbol Δ or δ for the variations of those coordinates will be due to that displacement itself. One must generally determine the variations Δx, Δy, Δz by means of the following equation, which includes the simplest algorithmic expression for the invariability of the mutual distances between all points of the system:

$$\Delta\,(dx^2 + dy^2 + dz^2) = 0 .$$

Now, upon setting:

$$\mathrm{x} = x + \tfrac{1}{2}\Delta x\,, \quad \mathrm{y} = y + \tfrac{1}{2}\Delta y\,, \quad \mathrm{z} = z + \tfrac{1}{2}\Delta z\,,$$

that equation will become:

$$d\mathrm{x}\; d\,\Delta x + d\mathrm{y}\; d\,\Delta y + d\mathrm{z}\; d\,\Delta \mathrm{z} = 0$$

by means of the rule for the inversion of the symbols δ, Δ, and it must be satisfied in the most-general manner. To that end, consider Δx, Δy, Δz to be functions of the independent variables x, y, z. The equation above will first produce the following integral upon considering dx, dy, dz to be constants:

$$\frac{d\mathrm{x}\,\Delta x + d\mathrm{y}\,\Delta y + d\mathrm{z}\,\Delta z}{\sqrt{d\mathrm{x}^2 + d\mathrm{y}^2 + d\mathrm{z}^2}} = \text{constant},$$

which is the algorithmic translation of the property of the quadrilateral whose two opposite sides are equal that consists of the fact that those sides and the other two will project *equally* onto the line that joins the midpoints of the latter. However, the equation:

$$d\mathrm{x}\; d\,\Delta x + d\mathrm{y}\; d\,\Delta y + d\mathrm{z}\; d\,\Delta \mathrm{z} = 0$$

will become:

$$d\mathrm{x}^2\,\frac{d\Delta x}{d\mathrm{x}} + d\mathrm{y}^2\,\frac{d\Delta y}{d\mathrm{y}} + d\mathrm{z}^2\,\frac{d\Delta \mathrm{z}}{d\mathrm{z}} + d\mathrm{x}\,d\mathrm{y}\left(\frac{d\Delta x}{d\mathrm{y}} + \frac{d\Delta y}{d\mathrm{x}}\right) + d\mathrm{x}\,d\mathrm{z}\left(\frac{d\Delta x}{d\mathrm{z}} + \frac{d\Delta \mathrm{z}}{d\mathrm{x}}\right) + d\mathrm{y}\,d\mathrm{z}\left(\frac{d\Delta y}{d\mathrm{z}} + \frac{d\Delta \mathrm{z}}{d\mathrm{y}}\right) = 0$$

when one substitutes partial differentials for the complete differentials, and due to the independence of the differentials dx, dy, dz, that will give these six other ones:

$$\frac{d\Delta x}{d\mathrm{x}} = 0\,, \qquad \frac{d\Delta y}{d\mathrm{y}} = 0\,, \qquad \frac{d\Delta \mathrm{z}}{d\mathrm{z}} = 0\,,$$

$$\frac{d\Delta x}{d\mathrm{y}} + \frac{d\Delta y}{d\mathrm{x}} = 0\,, \qquad \frac{d\Delta x}{d\mathrm{z}} + \frac{d\Delta \mathrm{z}}{d\mathrm{x}} = 0\,, \qquad \frac{d\Delta y}{d\mathrm{z}} + \frac{d\Delta \mathrm{z}}{d\mathrm{y}} = 0\,.$$

That system of six equation is easily integrated. The first three show that the variations Δx, Δy, Δz are formulated independently of the variables x, y, z, respectively. Consequently, the same thing will be true of their derivatives, so it is easy to conclude that those six derivatives are constants that are pairwise equal and of opposite sign, which will finally produce the expressions to which we arrived before:

$$\Delta x = \mathrm{A} + p\,\mathrm{y} - n\,\mathrm{z}\,, \quad \Delta y = \mathrm{B} + m\,\mathrm{z} - p\,\mathrm{x}\,, \quad \Delta z = \Gamma + n\,\mathrm{x} - m\,\mathrm{y}\,.$$

We shall not return to the transformations that those formulas will experience when we reestablish the variables x, y, z. It suffices for us to have shown how the method of *variations* applies to the search for those formulas and gives the finite and infinitesimal variations of the coordinates of a solid system in the same algorithmic form, with the one difference that the coordinates of the displaced points are replaced with those of the midpoint of the space that is traversed by those points along a straight line in the case of finite variations.

32. – To conclude this work, it now remains for us to rapidly deduce the geometric laws of the displacement of a solid body that we first presented synthetically from the expressions for those variations, and to take our initial analysis for the starting point.

The formulas:

$$\Delta x = \mathrm{A} + p\,\mathrm{y} - n\,\mathrm{z}\,, \quad \Delta y = \mathrm{B} + m\,\mathrm{z} - p\,\mathrm{x}\,, \quad \Delta z = \Gamma + n\,\mathrm{x} - m\,\mathrm{y}$$

immediately give the following fundamental relation:

$$m\,\Delta x + n\,\Delta y + p\,\Delta x = \mathrm{A}\,m + \mathrm{B}\,n + \Gamma\,p\,,$$

from which, one sees that the *lines* that are actually traversed by all points of the system upon passing from one position to the other are all equally projected onto the same direction (m, n, p). Upon denoting the angles of that direction by g, h, l, and letting t denote their common projections, one will have:

$$\cos g\,\Delta x + \cos h\,\Delta y + \cos l\,\Delta z = t$$

for all points of the displaced system, and for two different points:

$$\cos g\,(\Delta x - \Delta x') + \cos h\,(\Delta y - \Delta y') + \cos l\,(\Delta z - \Delta z') = 0\,.$$

$\Delta x - \Delta x'$, $\Delta y - \Delta y'$, $\Delta z - \Delta z'$ measure the projections onto the respective coordinate axes of the chord of the arc that is described by the first point around an axis of rotation that is drawn through the second point (x', y', z') parallel to the direction (m, n, p). If one lets θ denote the angle of that rotation and lets u denote the distance from that chord to that axis of rotation then one will obviously have:

$$4u^2 \tan^2 \tfrac{1}{2}\theta = \left\{ \begin{array}{l} (\Delta x - \Delta x')^2 + (\Delta y - \Delta y')^2 + (\Delta z - \Delta z')^2, \\ (m^2 + n^2 + p^2)\{(\mathrm{x} - \mathrm{x}')^2 + (\mathrm{y} - \mathrm{y}')^2 - [(\mathrm{x} - \mathrm{x}')\cos g + (\mathrm{y} - \mathrm{y}')\cos h + (z - z')\cos g]^2\} \end{array} \right.$$

$$u^2 = (\mathrm{x} - \mathrm{x}')^2 + (\mathrm{y} - \mathrm{y}')^2 + (\mathrm{z} - \mathrm{z}')^2 - [(\mathrm{x} - \mathrm{x}')\cos g + (\mathrm{y} - \mathrm{y}')\cos h + (\mathrm{z} - \mathrm{z}')\cos l]^2 .$$

One will then have:

$$4\tan^2 \tfrac{1}{2}\theta = m^2 + n^2 + p^2$$

for any points considered. Therefore, the given displacement of a solid from one position to another can always result from two consecutive displacements by rotation and translation, as was explained at the beginning of this article.

Furthermore, let ν be the amplitude of the angular displacement of a line in the solid, let φ be the angle that it forms with the direction of the axis of rotation, and let x, y, z, x', y', z' be the coordinates of two points on that line. One will have:

$$\cos\varphi = \frac{(x - x')\cos g + (y - y')\cos h + (z - z')\cos l}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} ,$$

and due to the invariability of the mutual distances between the points of that solid:

$$\Delta \cos\varphi = \frac{(\Delta x - \Delta x')\cos g + (\Delta y - \Delta y')\cos h + (\Delta z - \Delta z')\cos l}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} = 0 .$$

The angle φ will remain the same before and after the displacement. As for the angle ν, one will have:

$$\cos\nu = \frac{(x - x')(x - x' + \Delta x - \Delta x') + (y - y')(y - y' + \Delta y - \Delta y') + (z - z')(z - z' + \Delta z - \Delta z')}{(x - x')^2 + (y - y')^2 + (z - z')^2} .$$

Finally, one has the remarkable relation:

$$\sin \tfrac{1}{2}\nu = \sin\varphi \sin \tfrac{1}{2}\theta ,$$

which expresses the theorem that was stated in no. **5**.

The lines parallel to the direction of the axis of rotation are then transported parallel to themselves. Among all of those lines, there is one of them that only slides along itself, and for which the variations Δx, Δy, Δz must obviously be proportional to the cosines of the angles g, h, l. The equations for that line will then be:

$$\frac{\Delta x}{\cos g} = \frac{\Delta y}{\cos h} = \frac{\Delta y}{\cos l} = t\,,$$

and since one has:

$$\mathrm{x} = x + \tfrac{1}{2}\Delta x\,, \quad \mathrm{y} = y + \tfrac{1}{2}\Delta y\,, \quad \mathrm{z} = z + \tfrac{1}{2}\Delta z\,,$$

one will have:

$$p\,\mathrm{y} - n\,\mathrm{z} + \mathrm{A} = t\cos g\,,$$
$$m\,\mathrm{z} - p\,\mathrm{x} + \mathrm{B} = t\cos h\,,$$
$$n\,\mathrm{x} - m\,\mathrm{y} + \Gamma = t\cos l\,,$$
$$\ldots$$

for the equations of the *central* axis of the displacement, which we gave already in no. **16**.

CONCLUSION. – *The general law of statics*.

33. – Geometry considers finite or infinitesimal displacements of solid bodies that are due to the *successive* action of causes or *forces* that are capable of producing them.

Mechanics considers the consecutive displacements of solid bodies, and more generally, those of arbitrary systems of points that are due to the simultaneous and prolonged action of cause or *forces* that are capable of producing them.

Statics is the first branch of mechanics in which one considers only the possibility of infinitely-small – or *virtual* – displacements of those systems that result from the simultaneous and discontinuous action of the same causes.

Geometry teaches that the displacement of a solid body reduces to turning around one or two fixed axes.

It then results that if the forces that act simultaneously on a solid system cannot impose any rotation around an arbitrary fixed axis on it then those forces must be in equilibrium or be neutralized, and the body will remain immobile.

When considered separately, those forces can act in only two manners: Either they tend to make the solid body turn around a fixed axis or they will tend to displace a certain point of the system, or more expressively, they will vary the coordinates of that point. That is the most general manner of considering and examining the action of forces in mechanics.

Nonetheless, the law of equilibrium is identical in those two ways of looking at things, as we will see.

If the arbitrary forces or cause of the displacement tend to impose *elementary* or virtual rotations θ, θ', θ'', … around fixed, well-defined axes on a solid, *in succession* or *simultaneously*, then upon passing to the limits, i.e., upon passing from geometry to mechanics, the law of equilibrium of those forces will consist of saying that the sum of the moments of those rotations should be zero with respect to an *arbitrary axis*.

The equation:

$$\sum \theta D \sin \nu = 0\,,$$

which is the algorithmic translation of that law, implies six specialized equations due to the indeterminacy in that arbitrary axis, which will reduce to three when that solid system reduces to a point.

Let us now examine what happens when the forces that act simultaneously on the solid separately tend to make the coordinates of the various points of that system vary. Any displacement is referred *virtually* to a fixed axis of rotation, so it will suffice to consider the variations of the coordinates of each point that might result from the action of the forces that act on that point *orthogonally* to that same fixed axis, whose resistance will oppose any variation of its other rectangular coordinate axes.

Now, it is obvious that the infinitely-small displacement of a point can be considered to be performed *successively* with respect to each of the rectangular coordinates of that point, so the variation of each of those coordinates is equal to the projection of the total displacement onto the direction of that coordinate, moreover. It then results that for any system in which one finds a fixed axis, the forces that act upon a point of that system can actually only vary the coordinate that point orthogonally to that fixed axis.

On the other hand, due to the invariability of the system, it is obvious that when two *equal* forces act upon the same point, they will be found to be in equilibrium:

1. If they tend to vary its coordinates equally, but in the opposite sense.

2. If they are *applied* to the extremities of an invariable line, but in the opposite sense.

3. If they tend to make a circumference turn in opposite sense whose center is fixed and lies in a plane to which they are applied tangentially.

4. If, more generally, they tend to make a right cylinder turn in contrary senses whose axis is fixed and on whose surface they are orthogonally tangent to its axis.

It results from those proposition that if one considers all forces that tend to displace the various points of a solid system in which one finds a fixed axis along given direction and imposes given *virtual* translations upon them then there will be equilibrium between all of those forces if, upon supposing that they are all applied to points that are equidistant from the fixed axis (which is always possible), the sum of the *moments* of the virtual translations that measure the effect of those forces is zero with respect to that axis.

Upon passing from a well-defined fixed axis to an arbitrary one, it will necessarily follow that the general equation:

$$\sum \theta D \sin \nu = 0$$

is the algorithmic translation of the equilibrium of a system of forces that is capable of producing virtual or infinitely-small translations that are proportional to the rotations θ, θ', θ'', …, those forces being applied to the axes of those rotations *positively* or *negatively*, according to the sign of those rotations.

That explains the very remarkably analogy between the laws of equilibrium, and consequently, those of the composition of infinitely-small rotations, and the laws of equilibrium and composition of forces that are considered in statics.

If one generally denotes those forces by finite quantities $P, P', \ldots$ that are proportional to the virtual translations that they tend to impose then the equation of the equilibrium will be:

$$\sum P\,D \sin \nu = 0 .$$

The general term $P\,D \sin \nu$ expresses the static *moment* of the force P relative to the fixed axis and is equal to the product of the distance from the point of application of that force to the fixed axis, multiplied by the component of that force orthogonal to the same axis.

The conditions for the equilibrium of the forces that are applied to a solid system, namely, that the second consideration of *couples*, however admirably ingenious, which reduces to two conditions in finite terms, is therefore comprised of just one law that is likewise expressed in finite terms, which consists of saying that the sum of the *moments* or the forces on the system is zero with respect to an arbitrary axis.

That law is general, and like the principle of virtual velocities, which applies to only *infinitesimal* transformations, it applies to the equilibrium of any system, whether invariable or not, provided that the conditions of constraint of the points of that system can be replaced by the introduction of forces that permit one to regard those points as entirely free, which are forces that analysis will then determine and eliminate from equations that give the equilibrium of each point.

By means of that introduction, the law of equilibrium of a point immediately implies that of a solid system. We shall not stop to dwell upon that here. We remark only that in this particular case, the forces that were introduced are pairwise equal and opposite in sign, so the sum of the moments of all of the forces that are applied to all points, which must be zero for equilibrium, will contain only those given forces, and will thus express the law of equilibrium of a point or several points that are entirely free, and that of an invariable system as an identity.

On the equation of virtual velocity

If the force P tends to vary the coordinate p then the product $P\,\delta p$ will be equal to $P\,D\,\theta \sin \nu$, and $D\,\theta \sin \nu$ expresses the *moment* of the virtual rotation θ, under the hypothesis that the infinitely-small displacement of the system is arbitrary, which will produce variations of the coordinates of that system that are characterized by the symbol δ.

Indeed, since that infinitely-small displacement reduces *a fortiori* to one of two successive rotations, one must consider it only the first one or its single rotation *virtually*. Moreover, the infinitely-small arc that is described by the point of application of the force P, when projected onto the direction of that force, is found to be equal to the variation itself of the coordinate p along which that force acts, which will give:

$$\delta p = \theta D \sin \nu , \qquad P\,\delta p = \theta P\,D \sin \nu .$$

The general equation of equilibrium of those forces:

$$\sum P\,D\sin\nu = 0$$

is then transformed into this one:

$$\sum P\,\delta p = 0 ,$$

which expresses the idea that *the forces will be in equilibrium in a solid system if that system is perturbed infinitely little from its current position by an arbitrary cause, and the sum of the forces, multiplied by the infinitely-small lengths that are traversed by the points of that system in the direction of those respective forces must be zero, and conversely, which is the statement of the principle of virtual velocities.*

Algorithmically speaking, that equation is indeed superior to the first one, but it is not fundamentally more general. However, it expresses, and as simply as possible, the law of equilibrium of all systems in which the conditions of constraint are capable of being translated into linear equations between the variations of the coordinates of the various points of the system.

Indeed, those conditions are expressed by equations such as:

$$\delta L = 0 , \qquad \delta L' = 0 , \qquad \delta L'' = 0 ,$$

or more generally, by just one equation such as:

$$\lambda\,\delta L + \lambda'\,\delta L' + \lambda''\,\delta L'' + \cdots = 0 ,$$

in which λ, λ', λ'' are arbitrary multipliers, and one denotes the coefficients of the variations δx, δy, δz by X, Y, Z in that equation. If one lets r denote a linear quantity whose direction is (X, Y, Z), while R is a force that is applied to the point (x, y, z) in the direction of that line in which it tends make the point vary, which is equal to $\sqrt{X^2 + Y^2 + Z^2}$, then one will have:

$$\delta r = \frac{X\,\delta x + Y\,\delta y + Z\,\delta z}{\sqrt{X^2 + Y^2 + Z^2}} ,$$

and similarly, for $\delta r'$, $\delta r''$, … The equation above will then take the form:

$$R\,\delta r + R'\,\delta r' + R''\,\delta r'' + \cdots = 0 ,$$

and the two equations:

$$\sum P\,\delta p = 0 , \qquad \sum P\,\delta p + R\,\delta r + R'\,\delta r' + R''\,\delta r'' + \cdots = 0$$

will have the same generality, in which the variations are *constrained* in the first one by the equations of condition and are completely *independent* in that second one. Now, in the latter case, the equation:

$$\sum P\,\delta p + R\,\delta r + R'\,\delta r' + R''\,\delta r'' + \cdots = 0\,,$$

which expresses the equilibrium condition for all points of the system when they are free from any constraint, but subject to the forces that were given originally P, P', P'', …, and to other forces R, R', R'', … that have *statically* replaced the conditions of constraint that were supposed.

The elimination of R, R', R'' from the particular equations that are included in the preceding equation will provide the definitive equations of equilibrium for the first forces, which would result from the constraint on the system. Conversely, if those equations are true then one will have equilibrium because the forces R, R', R'' are, at the same time, found to be determinate, and the equation:

$$\sum P\,\delta p + R\,\delta r + R'\,\delta r' + R''\,\delta r'' + \cdots = 0\,,$$

with independent variations, established the immobility of all points of the system as a result of the action of the given forces and the ones that *statically* equivalent to the given constraint on the various points of the system.

When the system that one considers is continuous, the equations of condition will include definite integrals that represent, in some way, an infinite number of linear conditions between the variations of the coordinates of the system. The arbitrary multipliers appear within that integral sign, so it remains for one to obtain the variations by a method that is entirely analytical and independent of any static consideration. One will easily arrive there by the following general formula:

$$\delta S^n\, U\, dx_1\, dx_2\, dx_3 \cdots dx_n = S^n\, dx_1\, dx_2\, dx_3 \cdots dx_n \left[\delta U + U\left(\frac{d\,\delta x_1}{dx_1} + \frac{d\,\delta x_2}{dx_2} + \cdots + \frac{d\,\delta x_n}{dx_n}\right)\right],$$

in which U is an arbitrary function of the independent variables x_1, x_2, x_3, …, x_n, and the symbol S^n denote a definite multiple integral of order n.

5 December 1840
