"Saggio di Geometria differenziale dei complessi di rette," Ann. di Mat. pura ed appl. (3) **17** (1910), 179-223.

A sampling of the differential geometry of line complexes.

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Introduction

In the paper "Nuova esposizione della geometria infinitesimale delle congruenze rettilinee" ([1]) and two successive notes ([2]), I based the study of ray congruences upon two quadratic differential forms in two variables, one of which represents the square of the angle between two infinitely-close rays of the congruence, while the other represents the moment of the rays. In that way, I achieved the goal that was proposed by KUMMER ([3]): Construct a theory of rectilinear congruences in parallel to the theory of surfaces that was founded by GAUSS.

FIBBI ([4]) and FUBINI ([5]) also used two quadratic differential forms to represent a ray congruence in a space of constant curvature; *however, nothing that was done along those lines has been attempted for line complexes.*

In this paper, I will give a first taste of that, while taking all of the theory of those quadratic forms that was used for congruences as its basis. It will be ternary, since the position of a ray in a complex can depend upon the values of three independent parameters; however, the first form will necessarily be reducible, since it depends upon only the *directions* of the rays of the complex. There are at most ∞^2 of these directions, since one can suppose that the first form has already been reduced to contain no more than two essential parameters. That hypothesis will obviously produce no loss of generality. Indeed, it will leave three inessential parameters in the first form, which runs contrary to the nature of the geometric entity that the form represents (from the geometric viewpoint).

In the course of my work, it often occurred to me to invoke the results that were contained in the cited paper in the *Annali*; it is assumed that they are known to the reader, and the citations that refer to them will be preceded by the letter *A*.

([1]) Annali di Matematica, (3) **15** (1909), 143.

([2]) "Nuove formole utili per lo studio delle congruence rettilinee e Sull'inviluppata media di una congruenze di retta," Atti della R. Accademia delle Scienze di Torino **44** and **45** (1909-10).

([3]) "Allgemeine Theorie der geradlinigen Strahlensysteme," Crelles Journal **57** (1859).

([4]) "I sistemi doppiamente infiniti di raggi negli spazi di curvature costante," Annali della R. Scuola Normale Superiore di Pisa **8** (1895).

([5]) "Il parallelismo di Clifford negli spazi elliptici, *ibid.*, **9** (1900).

From the sheer size of the subject, we must limit ourselves in this first sampling to explaining only the fundamental concepts of the method and not go beyond the field of general complexes, while omitting the properties that are specialized to particular complexes. Above all, I hope that what I have to say will be sufficient to show the efficacy of the new method, which has all of the advantages that the so-called *intrinsic* methods present.

The complexes that will considered will be general, algebraic or transcendental (but real) ones, so all of the functions that will be introduced will be real functions of real variables. In regard to those functions, we shall say, once and for all, that they will be assumed to *finite and continuous, along with all of the derivatives that occur*, and will also be limited to a suitable region of their domain of existence.

In the course of work, the reader will encounter some known geometric properties that I did not believe would be conveniently omitted from a systematic exposition. However, the entirely new ideas consist of the study of the singularities of complexes and consideration of *bi-singular* rays, which can shed new light on the study of singularities. Moreover, the complexes that are defined by explicit expressions in the coordinates of a general line as functions of three independent parameters will be studied for the first time. In fact, up to now, only the complexes that are defined by one equation in the line coordinates have been studied systematically (with special regard to the algebraic complexes) ([1]).

Definitions.

1. Fix an arbitrary point M along any line in space to be its *origin* and fix a positive sense. A line will then be specified by the coordinate x, y, z of the point M and the direction cosines X, Y, Z of its positive sense with respect to three orthogonal Cartesian axes. Therefore, in order to define a *complex* or *system of* ∞^3 *rays* analytically, it will suffice to give:

$$x, y, z, X, Y, Z$$

as functions of three independent parameters u, v, w.

The rays of the complex that have a given direction (X_0, Y_0, Z_0) correspond to the solutions of the system:

$$X(u, v, w) = X_0\,, \qquad Y(u, v, w) = Y_0\,, \qquad Z(u, v, w) = Z_0\,.$$

Now, these equations are not independent, since:

$$X^2 + Y^2 + Z^2 = X_0^2 + Y_0^2 + Z_0^2 = 1,$$

so the Jacobian matrix:

([1]) If we limit ourselves to the systematic treatments then we shall cite KOENIGS, "Sur les propriétés infinitesimales de l'éspace reglé," Thése, 1882. ZINDLER, *Liniengeometrie*, Bd. II. JESSOP, *A treatise on the line complex*, Cambridge, 1903.

$$\begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} & \frac{\partial X}{\partial w} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} & \frac{\partial Y}{\partial w} \\ \frac{\partial Z}{\partial u} & \frac{\partial Z}{\partial v} & \frac{\partial Z}{\partial w} \end{vmatrix}$$

will have the characteristic 0, 1, or 2. If the characteristic is 1 then *X*, *Y*, *Z* will be reducible to functions of just one essential parameter, so there will be ∞^1 directions for the rays of the complex, and the complex will be composed of lines that lie along a given curve at infinity. If the characteristic is 0 then *X*, *Y*, *Z* will be constants, and the complex will degenerate into an improper star.

We shall exclude these complexes from our considerations ([1]); i.e., *we shall suppose that the preceding matrix has characteristic two.*

It then follows that *X*, *Y*, *Z* will be functions of only two essential parameters; we suppose that they have already been reduced to functions of those parameters, which we denote by *u* and *v*. On the contrary, *x*, *y*, *z* will be functions of not only *u* and *v*, but also a third parameter *w*.

We say that *u*, *v*, *w* are the *(internal) coordinates* of the ray of the complex that they specify.

Let *u*, *v* be essential parameters for *X*, *Y*, *Z*, so the matrix:

$$\begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} & \frac{\partial Z}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} & \frac{\partial Z}{\partial v} \end{vmatrix}$$

will have characteristic two in all of a two-dimensional domain in the variables *u* and *v*, so the square of its horizontals (*quadrato per orizzontali*) will be essentially positive ([2]):

$$\sum\left(\frac{\partial X}{\partial u}\right)^2 \cdot \sum\left(\frac{\partial X}{\partial v}\right)^2 - \left(\sum \frac{\partial X}{\partial u}\frac{\partial X}{\partial v}\right)^2 > 0. \tag{1}$$

In order for a complex to not degenerate into a congruence, one must assume, in addition, that for all values of *u*, *v* in the domain considered:

([1]) Our treatment will demand those exclusions, but not because they can be condemned *a priori*, but because the excluded complexes are of little interest. Moreover, we are not lacking in examples of analogous methods that present analogous exceptions: The developable surfaces are excluded from the study of surfaces by means of *Gauss's spherical representation.* KUMMER's method (*loc. cit.*) for the study of ray congruences does not consider the ones whose rays have only ∞^1 directions.

([2]) The symbol $\sum$ will always represent a sum of three terms that are deduced from the first one by changing *x*, *X* into *y*, *Y*, and then into *z*, *Z*.

$$\frac{\partial x}{\partial w},\ \frac{\partial y}{\partial w},\ \frac{\partial z}{\partial w}$$

are not simultaneously zero.

If the functions X, Y, Z assume the values X_0, Y_0, Z_0 for two values $u = u_0$, $v = v_0$ in that domain then there will be ∞^1 rays in the complex that have the direction (X_0, Y_0, Z_0) and must pass through the points whose coordinates are:

$$x(u_0, u_0, w), \quad y(u_0, u_0, w), \quad z(u_0, u_0, w),$$

where w is arbitrary. Therefore, *there exist* ∞^2 *cylinders in the complex.*

Thus, if (X_0, Y_0, Z_0) is a direction that has been assigned *a priori* then there will be cylinders in the complex whose generators have that direction, and there will exist pairs u_0, v_0 of values for u, v that satisfy the equations:

$$X(u, v) = X_0\,, \quad Y(u, v) = Y_0\,, \quad Z(u, v) = Z_0\,,$$

only two of which will be independent.

For a fixed finite point P (x_0, y_0, z_0) through which rays of the complex pass, it will be possible to determine triples of real values for u, v, w such that the following system of two equations is satisfied:

$$\frac{x(u,v,w)-x_0}{X(u,v)} = \frac{y(u,v,w)-y_0}{Y(u,v)} = \frac{z(u,v,w)-z_0}{Z(u,v)}.$$

If there exists a solution u_0, v_0, w_0 of the system, and if not all of the second-order minors of a certain functional matrix are zero for those values then the system will define two of the variables as functions of the third one (e.g., u and v as functions of w in a neighborhood of w_0, and such that for $w = w_0$, one will have $u = u_0$, $v = v_0$). In that sense, one can say that if a ray of the complex passes through a finite point P then ∞^1 of them will pass through it, in general ([1]), which constitute a cone, namely, the *complex cone* that relates to the point P.

If one draws the ray g' through the origin O of the Cartesian axes parallel to the ray $g(u, v, w)$ of the complex, and one intersects it with the sphere that has O for its center and a radius 1 then one will obtain the point $M'(X, Y, Z)$ that is the *spherical image* of g. If one varies g in the complex then g' will describe a star with center O an infinitude of times, while M' will describe the sphere or a region of it an infinitude of times.

2. A system of ∞^2 rays that are chosen from the ones in a complex constitute a *congruence* of the complex. We shall always consider only congruences that are defined analytically by an equation:

$$f(u, v, w) = 0$$

([1]) From now on, when we say that a certain property is verified *in general*, we intend that to mean that it is verified *provided that certain conditions that translate into equations between the coordinates u, v, w are satisfied.*

between the coordinates *u*, *v*, *w*, where *f* is a function that is finite and continuous in a suitable domain, along with its first derivatives. Furthermore, suppose that *f* is such that the preceding equation will permit us to consider one of the variables as a (finite, continuous, and differentiable) function of the other two.

For example, if it is possible to give the form:

$$w = \varphi(u, v)$$

to the equation of the congruence (or a region of it) then one can assert that there are ∞^2 distinct points at infinity of the rays of the congruence, and also that the spherical image of the congruence will cover an entire region of the sphere whose center is *O* and whose radius is 1. An example of that would be a *congruence w*; i.e., the equation w = constant.

By contrast, if one can give the form:

$$u = \varphi(v), \qquad v = \psi(u)$$

to the equations of the congruence then there will be just ∞^1 distinct points at infinity of the rays of the congruence; i.e., the spherical image of the congruence will reduce to a line. That congruence will elude the general treatment of paper *A* (as well as that of KUMMER). With ZINDLER (*loc. cit.*), we shall say *cylindrical congruence* when we mean one that obviously can be generated by giving a continuous motion to a cylinder whose rectilinear generators (*infletta*) stay rigid during the motion. The *congruences u* or *v* (i.e., the equations u = const. or v = const.) are such congruences.

The congruences *u*, *v*, and *w* are the *coordinate congruences* of the complexes.

Two congruences of the complex:

$$\varphi(u, v, w) = 0, \qquad \varphi\psi(u, v, w) = 0$$

cut along a *ruled surface* (or simply a *ruling*) of the complex. If φ and ψ are independent of *w* then the ruling will be a cylinder (or system of cylinders) that is the intersection of two cylindrical congruences. For example, the *ruling w*, along which only *w* varies (i.e., with the equations u = const., v = const.) is a cylinder.

The *rulings u* or *v* (i.e., along which only *u* or *v* varies) are not cylinders.

The rulings *u*, *v*, *w* are the *coordinate rulings* of the complex. Any ray of the complex will pass through each system.

Finally, note that, by right of the hypotheses that were made, *it is legitimate to perform an arbitrary change of variables of the type:*

$$u' = u'(u, v), \quad v' = v'(u, v), \quad w = w'(u, v),$$

with

$$\frac{\partial(u',v')}{\partial(u,v)} = 0, \qquad \frac{\partial w'}{\partial w} = 0,$$

but no others.

Fundamental quadratic forms.

3. Consider two infinitely-close rays of the complex:

$$g\ (u, v, w), \qquad g'\ (u + du, v + dv, w + dw),$$

and suppose, first of all, that du and dv are not simultaneously zero.

The angle between g and g' is measured by the distance ds' between their spherical images:

$$(X, Y, Z), \qquad (X + dX, Y + dY, Z + dZ),$$

and is given by the formula:

$$ds'^2 = dX^2 + dY^2 + dZ^2. \tag{2}$$

If one introduces the variables u, v then one will have:

$$ds'^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2, \tag{3}$$

in which E, F, G are known functions of u, v:

$$E = \sum\left(\frac{\partial X}{\partial u}\right)^2, \qquad F = \sum\frac{\partial X}{\partial u}\frac{\partial X}{\partial v}, \qquad G = \sum\left(\frac{\partial X}{\partial v}\right)^2. \tag{4}$$

By right of the hypotheses that were made in § 1, the functions E and G are essentially positive, and from (1), the function $EG - F^2$ will also be positive, and its positive square root will always be denoted by Δ. It will then follow that the binary quadratic differential form (3) will be positive-definite: We call it the *first fundamental form* of the complex.

Now, consider the *minimum distance* $d\sigma$ between g and g'. Its direction cosines will be given by the formula ([1]):

$$\cos\,(d\sigma, x) = \frac{\left(E\dfrac{\partial X}{\partial v} - F\dfrac{\partial X}{\partial u}\right)du + \left(F\dfrac{\partial X}{\partial v} - G\dfrac{\partial X}{\partial u}\right)dv}{\Delta\sqrt{E\,du^2 + 2F\,du\,dv + G\,dv^2}}, \ \ldots \tag{5}$$

in which, we agree to assume that the value of the radical is positive.

Under the passage from g to g', the point M (x, y, z) that is the origin of g will become the point $(x + dx, y + dy, z + dz)$ of g', so one will obviously have:

$$d\sigma = \sum \cos\,(d\sigma, x)\,dx;$$

it will then follow that the *moment* μ of g and g':

([1]) Cf., BIANCHI, *Lezioni di Geometria Differenziale*, vol. I, § 137.

$$\mu = ds' \, d\sigma$$

is expressed by:

$$\mu = \frac{1}{\Delta}\sum\left[\left(E\frac{\partial X}{\partial v} - F\frac{\partial X}{\partial u}\right)du + \left(F\frac{\partial X}{\partial v} - G\frac{\partial X}{\partial u}\right)dv\right]\left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw\right),$$

and also by:

$$-\mu = D\,du^2 + 2D'\,du\,dv + D''\,dv^2 + 2M\,du\,dw + 2N\,dv\,dw, \tag{6}$$

in which D, D', D'', M, N (and r_0) are known functions of u, v, w that are defined by the following formulas:

$$\left.\begin{aligned}
D &= \frac{F}{\Delta}\sum\frac{\partial X}{\partial u}\frac{\partial x}{\partial u} - \frac{E}{\Delta}\sum\frac{\partial X}{\partial v}\frac{\partial x}{\partial u},\\
D' + \Delta r_0 &= \frac{F}{\Delta}\sum\frac{\partial X}{\partial u}\frac{\partial x}{\partial u} - \frac{E}{\Delta}\sum\frac{\partial X}{\partial v}\frac{\partial x}{\partial u},\\
D' - \Delta r_0 &= \frac{G}{\Delta}\sum\frac{\partial X}{\partial u}\frac{\partial x}{\partial u} - \frac{F}{\Delta}\sum\frac{\partial X}{\partial v}\frac{\partial x}{\partial u},\\
D'' &= \frac{G}{\Delta}\sum\frac{\partial X}{\partial u}\frac{\partial x}{\partial v} - \frac{F}{\Delta}\sum\frac{\partial X}{\partial v}\frac{\partial x}{\partial v},
\end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned}
2M &= \frac{F}{\Delta}\sum\frac{\partial X}{\partial u}\frac{\partial x}{\partial w} - \frac{E}{\Delta}\sum\frac{\partial X}{\partial v}\frac{\partial x}{\partial w},\\
2N &= \frac{G}{\Delta}\sum\frac{\partial X}{\partial u}\frac{\partial x}{\partial w} - \frac{E}{\Delta}\sum\frac{\partial X}{\partial v}\frac{\partial x}{\partial w}.
\end{aligned}\right\} \tag{8}$$

We call the ternary quadratic differential form (6) the *second fundamental form* of the complex.

The two forms (3) and (6), from their geometric significance, are independent, not only of the choice of coordinates u, v, w, but also of the choice of point M that will serve as an origin on g, so changing that point to C will not alter anything.

The inverse formulas follow from (7) and (8):

$$\left.\begin{aligned}
\sum\frac{\partial X}{\partial u}\frac{\partial x}{\partial u} &= \frac{ED' - FD}{\Delta} - Er_0, &\quad \sum\frac{\partial X}{\partial u}\frac{\partial x}{\partial v} &= \frac{ED'' - FD'}{\Delta} - Fr_0,\\
\sum\frac{\partial X}{\partial v}\frac{\partial x}{\partial u} &= \frac{FD' - GD}{\Delta} - Fr_0, &\quad \sum\frac{\partial X}{\partial v}\frac{\partial x}{\partial v} &= \frac{FD'' - GD'}{\Delta} - Gr_0,
\end{aligned}\right\} \tag{7'}$$

$$\frac{\partial X}{\partial u}\frac{\partial x}{\partial w} = 2\,\frac{EN - FM}{\Delta}, \qquad \frac{\partial X}{\partial v}\frac{\partial x}{\partial w} = 2\,\frac{FN - GM}{\Delta}. \tag{8'}$$

4. Now, suppose that $du = dv = 0$. The angle ds' (3) between g and g' will then be zero, if one ignores higher-order infinitesimals, and the two rays can be considered to be parallel. The moment $\mu = ds'\, d\sigma$ will also be zero, but the distance $d\sigma$ between them will not be zero, in general. In that case, the calculations that were made to obtain (6) will not be valid. Nevertheless, they can also continue to be valid in that case, since if one takes $du = dv = 0$ then that will give $\mu = 0$, as it should.

Fundamental theorems.

5. Formulas (4), (7), and (8), which define a complex analytically, permit one to calculate the coefficients of the two fundamental forms (3) and (6) of the complex. We now pose the opposite question: Given two forms of the type (3) and (6), do there exist complexes that admit them as first and second fundamental forms? How many are there? How does one construct them?

First of all, observe that if (3) is to represent the first fundamental form of a complex then it must represent the square of the linear element of the sphere of radius 1, when referred to a system of two curvilinear coordinates (u, v): For this, it is necessary and sufficient that it be a positive-definite form, so:

$$\Delta^2 = E\,G - F^2 > 0,$$

and in addition it must have a curvature equal to + 1 ([1]); i.e., that it is expressed by the equivalence ([2]):

$$\frac{1}{2\Delta}\left\{\frac{\partial}{\partial u}\left(\frac{F}{E\Delta}\frac{\partial E}{\partial v}-\frac{1}{\Delta}\frac{\partial G}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{2}{\Delta}\frac{\partial F}{\partial u}-\frac{1}{\Delta}\frac{\partial E}{\partial v}-\frac{F}{E\Delta}\frac{\partial E}{\partial u}\right)\right\} = 1. \tag{9}$$

We also note the other equivalences ([3]):

$$\left.\begin{aligned}
&\frac{\partial}{\partial v}\begin{Bmatrix}11\\2\end{Bmatrix}-\frac{\partial}{\partial u}\begin{Bmatrix}12\\2\end{Bmatrix}+\begin{Bmatrix}11\\1\end{Bmatrix}\begin{Bmatrix}12\\2\end{Bmatrix}+\begin{Bmatrix}11\\2\end{Bmatrix}\begin{Bmatrix}22\\2\end{Bmatrix}-\begin{Bmatrix}12\\1\end{Bmatrix}\begin{Bmatrix}11\\2\end{Bmatrix}-\begin{Bmatrix}12\\2\end{Bmatrix}^2=E,\\
&\frac{\partial}{\partial u}\begin{Bmatrix}12\\1\end{Bmatrix}-\frac{\partial}{\partial v}\begin{Bmatrix}11\\1\end{Bmatrix}+\begin{Bmatrix}12\\1\end{Bmatrix}\begin{Bmatrix}12\\2\end{Bmatrix}-\begin{Bmatrix}11\\2\end{Bmatrix}\begin{Bmatrix}22\\1\end{Bmatrix}=F,\\
&\frac{\partial}{\partial v}\begin{Bmatrix}12\\2\end{Bmatrix}-\frac{\partial}{\partial u}\begin{Bmatrix}22\\2\end{Bmatrix}+\begin{Bmatrix}12\\1\end{Bmatrix}\begin{Bmatrix}12\\2\end{Bmatrix}-\begin{Bmatrix}11\\2\end{Bmatrix}\begin{Bmatrix}22\\1\end{Bmatrix}=F,\\
&\frac{\partial}{\partial u}\begin{Bmatrix}22\\1\end{Bmatrix}-\frac{\partial}{\partial v}\begin{Bmatrix}12\\1\end{Bmatrix}+\begin{Bmatrix}22\\2\end{Bmatrix}\begin{Bmatrix}12\\2\end{Bmatrix}+\begin{Bmatrix}22\\1\end{Bmatrix}\begin{Bmatrix}11\\1\end{Bmatrix}-\begin{Bmatrix}12\\2\end{Bmatrix}\begin{Bmatrix}22\\1\end{Bmatrix}-\begin{Bmatrix}12\\1\end{Bmatrix}^2=G,
\end{aligned}\right\} \tag{9'}$$

([1]) BIANCHI, *loc. cit.*, § 72.
([2]) BIANCHI, *loc. cit.*, § 43, formula (17).
([3]) *Ibid.*, § 37, formula (IV).

in which the CHRISTOFFEL symbols $\left\{ \begin{matrix} r\,s \\ t \end{matrix} \right\}$ ([1]) are understood to have been formed from the coefficients of the form (2).

It is known ([2]) that if those conditions are satisfied then the search for the three functions *X*, *Y*, *Z* of *u*, *v* that satisfy (4) will depend upon the integration of a differential equation of the RICATTI type.

Therefore, suppose that these functions are known, and then observe that if the determinant:

$$\left. \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} & X \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} & Y \\ \frac{\partial Z}{\partial u} & \frac{\partial Z}{\partial v} & Z \end{vmatrix} = \sqrt{EG - F^2} = \Delta \right\} \tag{10}$$

is non-zero then it will possible to determine, and in just one way, three functions *α*¸ *β*, *γ* of *u*, *v*, *w* such that:

$$\left. \begin{aligned} \alpha \frac{\partial X}{\partial u} + \beta \frac{\partial X}{\partial v} + \gamma X = A, \\ \alpha \frac{\partial Y}{\partial u} + \beta \frac{\partial Y}{\partial v} + \gamma Y = A, \\ \alpha \frac{\partial Z}{\partial u} + \beta \frac{\partial Z}{\partial v} + \gamma Z = C, \end{aligned} \right\} \tag{11}$$

in which *A*, *B*, *C* are given functions of *u*, *v*, *w*.

Set:

$$A = \frac{\partial x}{\partial u}, \qquad B = \frac{\partial y}{\partial u}, \qquad C = \frac{\partial z}{\partial u},$$

and then multiply the resulting equations by:

$$\frac{\partial X}{\partial u}, \frac{\partial Y}{\partial u}, \frac{\partial Z}{\partial u}, \qquad \text{or} \qquad \frac{\partial X}{\partial v}, \frac{\partial Y}{\partial v}, \frac{\partial Z}{\partial v}$$

and upon summing them, one will have, from (4) and (7):

$$E\,\alpha + F\,\beta + g \sum X \frac{\partial X}{\partial u} = \frac{ED' - FD}{\Delta} - E\,r_0\,,$$

$$F\,\alpha + G\,\beta + g \sum X \frac{\partial X}{\partial v} = \frac{FD' - GD}{\Delta} - F\,r_0\,;$$

([1]) *Ibid.*, § 43.
([2]) *Ibid.*, §§ 43, 72.

meanwhile, since:

$$\sum X^2 = 1,$$

it will follow by differentiation that:

$$\sum X \frac{\partial X}{\partial u} = \sum X \frac{\partial X}{\partial v} = 0$$

so the preceding will give:

$$\alpha = \frac{D'}{\Delta} - r_0 , \qquad \beta = -\frac{D}{\Delta}.$$

Thus, set:

$$A = \frac{\partial x}{\partial v}, \qquad B = \frac{\partial y}{\partial v}, \qquad C = \frac{\partial z}{\partial v},$$

and if one operates the same way then one will obtain:

$$\alpha = \frac{D''}{\Delta}, \qquad \beta = -\left(\frac{D'}{\Delta} + r_0\right).$$

Finally, if one sets:

$$A = \frac{\partial x}{\partial w}, \qquad B = \frac{\partial y}{\partial w}, \qquad C = \frac{\partial z}{\partial w},$$

and operates in the same way then, from (4) and (8′), one will find that:

$$E\alpha + F\beta = 2\,\frac{EN - FM}{\Delta}, \qquad F\alpha + G\beta = 2\,\frac{FN - GM}{\Delta},$$

from which, one will get:

$$\alpha = 2\frac{N}{\Delta}, \qquad \beta = -2\,\frac{M}{\Delta}.$$

Collecting these results, one will get the formulas:

$$\left.\begin{aligned}
\frac{\partial x}{\partial u} &= \left(\frac{D'}{\Delta} - r_0\right)\frac{\partial X}{\partial u} - \frac{D}{\Delta}\frac{\partial X}{\partial v} + \gamma X, \\
\frac{\partial x}{\partial v} &= \frac{D''}{\Delta}\frac{\partial X}{\partial u} - \left(\frac{D'}{\Delta} + r_0\right)\frac{\partial X}{\partial v} + \gamma' X, \\
\frac{\partial x}{\partial w} &= 2\frac{N}{\Delta}\frac{\partial X}{\partial u} - 2\frac{M}{\Delta}\frac{\partial X}{\partial v} + \gamma'' X,
\end{aligned}\right\} \tag{12}$$

with analogous formulas for y, Y and z, Z. In them, r_0 is an arbitrary function of u, v, w, and γ, γ', γ'' are three functions that are to be determined.

Thus, form the integrability conditions of the system (12) and their analogues:

$$\frac{\partial}{\partial v}\left(\frac{\partial x}{\partial u}\right)=\frac{\partial}{\partial u}\left(\frac{\partial x}{\partial v}\right),\quad \frac{\partial}{\partial w}\left(\frac{\partial x}{\partial u}\right)=\frac{\partial}{\partial u}\left(\frac{\partial x}{\partial w}\right),\ \frac{\partial}{\partial w}\left(\frac{\partial x}{\partial v}\right)=\frac{\partial}{\partial v}\left(\frac{\partial x}{\partial w}\right), \tag{13}$$

with analogous expressions in y and z.

The first one leads to the following results ([1]):

$$\gamma=\frac{b_{111}}{\Delta}-\frac{\partial r_0}{\partial u},\qquad \gamma'=-\left(\frac{b_{121}}{\Delta}+\frac{\partial r_0}{\partial v}\right), \tag{14}$$

and

$$\frac{1}{\Delta}\left\{\frac{\partial}{\partial u}\left(\frac{b_{221}}{\Delta}\right)+\frac{\partial}{\partial v}\left(\frac{b_{112}}{\Delta}\right)\right\}=H, \tag{15}$$

where:

$$H=\frac{2FD'-ED''-GD}{EG-F^2} \tag{16}$$

and

$$\left.\begin{aligned} b_{112}&=\frac{\partial D}{\partial v}-\frac{\partial D'}{\partial u}-\begin{Bmatrix}1\,2\\1\end{Bmatrix}D+\left(\begin{Bmatrix}1\,1\\1\end{Bmatrix}-\begin{Bmatrix}1\,2\\2\end{Bmatrix}\right)D'+\begin{Bmatrix}1\,1\\2\end{Bmatrix}D'',\\ b_{221}&=\frac{\partial D''}{\partial u}-\frac{\partial D'}{\partial v}+\begin{Bmatrix}2\,2\\1\end{Bmatrix}D+\left(\begin{Bmatrix}2\,2\\2\end{Bmatrix}-\begin{Bmatrix}1\,2\\1\end{Bmatrix}\right)D'+\begin{Bmatrix}1\,2\\2\end{Bmatrix}D'',\end{aligned}\right\} \tag{17}$$

or

$$\left.\begin{aligned} \frac{b_{112}}{\Delta}&=\frac{\partial}{\partial v}\left(\frac{D}{\Delta}\right)-\frac{\partial}{\partial u}\left(\frac{D'}{\Delta}\right)+\begin{Bmatrix}2\,2\\2\end{Bmatrix}\frac{D}{\Delta}-2\begin{Bmatrix}1\,2\\2\end{Bmatrix}\frac{D'}{\Delta}+\begin{Bmatrix}1\,1\\2\end{Bmatrix}\frac{D''}{\Delta},\\ \frac{b_{121}}{\Delta}&=\frac{\partial}{\partial u}\left(\frac{D''}{\Delta}\right)-\frac{\partial}{\partial v}\left(\frac{D'}{\Delta}\right)+\begin{Bmatrix}2\,2\\1\end{Bmatrix}\left(\frac{D}{\Delta}-2\right)+2\begin{Bmatrix}1\,2\\1\end{Bmatrix}\frac{D'}{\Delta}+\begin{Bmatrix}1\,1\\1\end{Bmatrix}\frac{D''}{\Delta}.\end{aligned}\right\} \tag{17'}$$

The first of (12) gives:

$$\frac{\partial}{\partial w}\left(\frac{\partial x}{\partial u}\right)=\frac{\partial}{\partial w}\left(\frac{D'}{\Delta}-r_0\right)\frac{\partial X}{\partial u}-\frac{\partial}{\partial w}\left(\frac{D}{\Delta}\right)\frac{\partial X}{\partial v}+X\,\frac{\partial\gamma}{\partial w}$$

and the third one gives:

$$\frac{\partial}{\partial u}\left(\frac{\partial x}{\partial w}\right)=2\frac{\partial}{\partial u}\left(\frac{N}{\Delta}\right)\frac{\partial X}{\partial u}-2\frac{\partial}{\partial u}\left(\frac{M}{\Delta}\right)\frac{\partial X}{\partial v}+X\,\frac{\partial\gamma''}{\partial u}+2\frac{N}{\Delta}\frac{\partial^2 X}{\partial u^2}-2\frac{\partial^2 X}{\partial u\,\partial v}+\gamma''\frac{\partial X}{\partial u},$$

namely:

$$\frac{\partial}{\partial u}\left(\frac{\partial x}{\partial w}\right)=\left[2\frac{\partial}{\partial u}\left(\frac{N}{\Delta}\right)+2\begin{Bmatrix}1\,1\\1\end{Bmatrix}\frac{N}{\Delta}-2\begin{Bmatrix}1\,2\\1\end{Bmatrix}\frac{M}{\Delta}+\gamma''\right]\frac{\partial X}{\partial u}$$

([1]) For the proof of this, see *A*, §§ 21, 22, 23.

$$+\left[-\frac{\partial}{\partial u}\left(\frac{M}{\Delta}\right)+\begin{Bmatrix}11\\2\end{Bmatrix}\frac{N}{\Delta}-\begin{Bmatrix}1\,2\\1\end{Bmatrix}\frac{M}{\Delta}\right]\frac{\partial X}{\partial v}+\left(\frac{\partial\gamma''}{\partial u}+2\frac{FM-EN}{\Delta}\right)X\,,$$

by virtue of the identities ([1]):

$$\left.\begin{aligned}\frac{\partial^2 X}{\partial u^2}&=\begin{Bmatrix}11\\1\end{Bmatrix}\frac{\partial X}{\partial u}+\begin{Bmatrix}11\\2\end{Bmatrix}\frac{\partial X}{\partial v}-EX\,,\\\frac{\partial^2 X}{\partial u\,\partial v}&=\begin{Bmatrix}1\,2\\1\end{Bmatrix}\frac{\partial X}{\partial u}+\begin{Bmatrix}1\,2\\2\end{Bmatrix}\frac{\partial X}{\partial v}-FX\,,\\\frac{\partial^2 X}{\partial v^2}&=\begin{Bmatrix}2\,2\\1\end{Bmatrix}\frac{\partial X}{\partial u}+\begin{Bmatrix}2\,2\\2\end{Bmatrix}\frac{\partial X}{\partial v}-GX\,.\end{aligned}\right\}\tag{18}$$

Therefore, the second of (13) will become:

$$\begin{aligned}&\left[\frac{\partial}{\partial w}\left(\frac{D'}{\Delta}-r_0\right)-2\frac{\partial}{\partial u}\left(\frac{N}{\Delta}\right)-2\begin{Bmatrix}11\\1\end{Bmatrix}\frac{N}{\Delta}+2\begin{Bmatrix}1\,2\\1\end{Bmatrix}\frac{M}{\Delta}-\gamma''\right]\frac{\partial X}{\partial u}\\&+\left[2\frac{\partial}{\partial u}\left(\frac{M}{\Delta}\right)-\frac{\partial}{\partial w}\left(\frac{D}{\Delta}\right)+2\begin{Bmatrix}1\,2\\2\end{Bmatrix}\frac{M}{\Delta}-2\begin{Bmatrix}11\\2\end{Bmatrix}\frac{N}{\Delta}\right]\frac{\partial X}{\partial v}\\&+\left(\frac{\partial\gamma}{\partial w}-\frac{\partial\gamma''}{\partial u}-2\frac{FN-EN}{\Delta}\right)X\ =0,\end{aligned}$$

and by virtue of the identities ([2]):

$$\frac{\partial\log\Delta}{\partial u}=\begin{Bmatrix}11\\1\end{Bmatrix}+\begin{Bmatrix}1\,2\\2\end{Bmatrix},\qquad\frac{\partial\log\Delta}{\partial v}=\begin{Bmatrix}1\,2\\1\end{Bmatrix}+\begin{Bmatrix}2\,2\\2\end{Bmatrix},\tag{19}$$

it will be transformed into the other one:

$$\begin{aligned}&\left[\frac{\partial}{\partial w}\left(\frac{D'}{\Delta}-r_0\right)-\frac{2}{\Delta}\frac{\partial N}{\partial u}+2\begin{Bmatrix}1\,2\\1\end{Bmatrix}\frac{M}{\Delta}+2\begin{Bmatrix}1\,2\\1\end{Bmatrix}\frac{N}{\Delta}-\gamma''\right]X\\&+\left[\frac{2}{\Delta}\frac{\partial M}{\partial u}-\frac{\partial}{\partial w}\left(\frac{D}{\Delta}\right)-2\begin{Bmatrix}11\\1\end{Bmatrix}\frac{M}{\Delta}-2\begin{Bmatrix}11\\2\end{Bmatrix}\frac{N}{\Delta}\right]\frac{\partial X}{\partial v}\\&+\left(\frac{\partial\gamma}{\partial w}-\frac{\partial\gamma''}{\partial u}-2\frac{FM-EN}{\Delta}\right)X\,.\end{aligned}$$

([1]) BIANCHI, *loc. cit.*, § 72.
([2]) BIANCHI, *loc. cit.*, § 56.

This equivalence and the two other analogous ones that are obtained by changing X into Y or Z are linear and homogeneous in the coefficients of $\frac{\partial X}{\partial u}$, $\frac{\partial X}{\partial v}$, X with non-zero derivatives, and therefore give:

$$\gamma'' = \frac{1}{\Delta}\frac{\partial D'}{\partial w} - \frac{\partial r_0}{\partial w} - \frac{2}{\Delta}\frac{\partial N}{\partial u} + 2\begin{Bmatrix}1\,2\\1\end{Bmatrix}\frac{M}{\Delta} + 2\begin{Bmatrix}1\,2\\2\end{Bmatrix}\frac{N}{\Delta}, \tag{20}$$

$$\frac{\partial D}{\partial w} = 2\frac{\partial M}{\partial u} - 2\begin{Bmatrix}1\,1\\1\end{Bmatrix}M - 2\begin{Bmatrix}1\,1\\2\end{Bmatrix}N, \tag{21}$$

$$\frac{\partial \gamma}{\partial w} - \frac{\partial \gamma''}{\partial u} = 2\,\frac{FM - EN}{\Delta}. \tag{22}$$

Analogously, one finds that the third of the conditions (13) demands that one have:

$$\gamma'' = -\frac{1}{\Delta}\frac{\partial D'}{\partial w} - \frac{\partial r_0}{\partial w} + \frac{2}{\Delta}\frac{\partial M}{\partial v} - 2\begin{Bmatrix}1\,2\\2\end{Bmatrix}\frac{M}{\Delta} - 2\begin{Bmatrix}1\,2\\2\end{Bmatrix}\frac{N}{\Delta}, \tag{23}$$

$$\frac{\partial D''}{\partial w} = 2\frac{\partial N}{\partial v} - 2\begin{Bmatrix}2\,2\\1\end{Bmatrix}M - 2\begin{Bmatrix}2\,2\\2\end{Bmatrix}N, \tag{24}$$

$$\frac{\partial \gamma''}{\partial v} - \frac{\partial \gamma'}{\partial w} = 2\frac{FN - GN}{\Delta}. \tag{25}$$

(14), (15), (20), …, (25) are necessary and sufficient conditions for the integrability of (12): Suppose that they are satisfied, so (12) will give a triple of functions x, y, z of u, v, w (up to an additive constant) by quadratures. These three functions collectively, along with X, Y, Z, define a unique complex completely (§ 1) that admits the forms (3) and (6) as fundamental forms.

The given conditions can be simplified.

If one takes a function r_0 of u, v, w arbitrarily then (14) will give γ and γ'. One will then have two distinct expressions (20) and (23) for γ'', which must then give equal values for γ'', so one must have:

$$\frac{\partial D'}{\partial w} = \frac{\partial M}{\partial v} + \frac{\partial N}{\partial u} - 2\begin{Bmatrix}1\,2\\1\end{Bmatrix}M - 2\begin{Bmatrix}1\,2\\2\end{Bmatrix}N. \tag{26}$$

If one takes one-half the sum of the two expressions for γ'' then one will have, more simply:

$$\gamma'' = \frac{1}{\Delta}\left(\frac{\partial M}{\partial v} - \frac{\partial N}{\partial u}\right) - \frac{\partial r_0}{\partial w}. \tag{27}$$

If one substitutes the values (14) and (27) of γ, γ', and γ'' into (22) and (25) then one will have:

$$\frac{1}{\Delta}\frac{\partial b_{112}}{\partial w}+2\frac{EN-FM}{\Delta} = \frac{\partial}{\partial u}\left[\frac{1}{\Delta}\left(\frac{\partial M}{\partial v}-\frac{\partial N}{\partial u}\right)\right], \tag{28}$$

$$-\frac{1}{\Delta}\frac{\partial b_{211}}{\partial w}+2\frac{FN-GM}{\Delta} = \frac{\partial}{\partial v}\left[\frac{1}{\Delta}\left(\frac{\partial M}{\partial v}-\frac{\partial N}{\partial u}\right)\right]. \tag{29}$$

In summary: The unknown functions γ, γ', and γ'' have the values (14) and (27), and there are six relations that couple the coefficients of the two fundamental forms: viz., (15), (21), (24), (26), (28), and (29). However, one can prove that the last two are superfluous – i.e., they can be deduced from the other four. Indeed, the first of (17) will give:

$$\frac{\partial b_{112}}{\partial w} = \frac{\partial}{\partial v}\left(\frac{\partial D}{\partial w}\right)-\frac{\partial}{\partial u}\left(\frac{\partial D'}{\partial w}\right)-\begin{Bmatrix}12\\1\end{Bmatrix}\frac{\partial D}{\partial w}+\left(\begin{Bmatrix}11\\1\end{Bmatrix}-\begin{Bmatrix}12\\2\end{Bmatrix}\right)\frac{\partial D'}{\partial w}+\begin{Bmatrix}11\\1\end{Bmatrix}\frac{\partial D''}{\partial w},$$

so from (21), (24), and (26):

$$\begin{aligned}\frac{\partial b_{112}}{\partial w} = {} & \frac{\partial}{\partial v}\left(\frac{\partial D}{\partial w}\right)-\frac{\partial^2 M}{\partial u\,\partial v}-\frac{\partial^2 N}{\partial u^2} + 2\left[\begin{Bmatrix}11\\1\end{Bmatrix}+\begin{Bmatrix}12\\2\end{Bmatrix}\right]\left(\frac{\partial M}{\partial v}+\frac{\partial N}{\partial u}\right)\\ & + 2\left[\frac{\partial}{\partial u}\begin{Bmatrix}12\\1\end{Bmatrix}-\frac{\partial}{\partial v}\begin{Bmatrix}11\\1\end{Bmatrix}+\begin{Bmatrix}12\\1\end{Bmatrix}\begin{Bmatrix}12\\2\end{Bmatrix}-\begin{Bmatrix}11\\2\end{Bmatrix}\begin{Bmatrix}22\\1\end{Bmatrix}\right]M\\ & + 2\left[\frac{\partial}{\partial u}\begin{Bmatrix}12\\1\end{Bmatrix}-\frac{\partial}{\partial v}\begin{Bmatrix}11\\2\end{Bmatrix}+\begin{Bmatrix}12\\1\end{Bmatrix}\begin{Bmatrix}11\\2\end{Bmatrix}-\begin{Bmatrix}11\\1\end{Bmatrix}\begin{Bmatrix}12\\2\end{Bmatrix}+\begin{Bmatrix}12\\2\end{Bmatrix}^2-\begin{Bmatrix}11\\2\end{Bmatrix}\begin{Bmatrix}22\\2\end{Bmatrix}\right]N\,.\end{aligned}$$

Now, one easily sees that this coincides with (28) if one keeps (9′) and the first of (19) in mind.

One proves that (29) can be omitted in the same way. Nevertheless, (28) and (29) can be useful, as well as the following:

$$\frac{2}{\Delta}\left\{\frac{\partial}{\partial u}\left(\frac{FN-GM}{\Delta}\right)+\frac{\partial}{\partial v}\left(\frac{FM-EN}{\Delta}\right)\right\} = \frac{\partial H}{\partial w}, \tag{30}$$

which one deduces by differentiating (28) with respect to u and (29) with respect to v, and then subtracting them, while bearing (15) in mind.

One can summarize the foregoing as the following fundamental theorem:

Suppose one is given two quadratic differential forms, one of which is in two variables u and v, is definite, and has curvature +1:

$$ds'^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2, \tag{α}$$

and one of which is in three variables u, v, w, and has the type:

$$-\mu = D\,du^2 + 2D'\,du\,dv + D''dv^2 + 2M\,du\,dw + 2N\,dv\,dw. \qquad (\beta)$$

In order for there to exist a complex that admits them as its first and second fundamental forms, respectively, it is necessary and sufficient that the following four relations exist between their coefficients:

$$\left.\begin{aligned}
\frac{1}{2}\frac{\partial D}{\partial w} &= \frac{\partial M}{\partial u} - \begin{Bmatrix}11\\1\end{Bmatrix} M - \begin{Bmatrix}11\\2\end{Bmatrix} N, \\
\frac{\partial D'}{\partial w} &= \frac{\partial M}{\partial v} + \frac{\partial N}{\partial u} - 2\begin{Bmatrix}12\\1\end{Bmatrix} M - 2\begin{Bmatrix}12\\2\end{Bmatrix} N, \\
\frac{1}{2}\frac{\partial D''}{\partial w} &= \frac{\partial N}{\partial v} - \begin{Bmatrix}22\\1\end{Bmatrix} M - \begin{Bmatrix}22\\2\end{Bmatrix} N,
\end{aligned}\right\} \qquad \text{(I)}$$

$$\frac{1}{\Delta}\left\{\frac{\partial}{\partial u}\left(\frac{b_{221}}{\Delta}\right) + \frac{\partial}{\partial v}\left(\frac{b_{112}}{\Delta}\right)\right\} = H, \qquad \text{(II)}$$

in which H has the value (15), *and* b_{112}, b_{221} *have the values* (17) *or* (17′).

The complex is unique (up to a spatial motion).

In order to construct it, it is enough to know the direction cosines X, Y, Z of one of its generic lines as functions of u, v, and the coordinates x, y, z of a point of that line as a function of u, v, w: The first one is obtained by integrating a RICATTI equation, while the second is calculated by quadrature using the formulas:

$$\left.\begin{aligned}
\frac{\partial x}{\partial u} &= \left(\frac{D'}{\Delta} - r_0\right)\frac{\partial X}{\partial u} - \frac{D}{\Delta}\frac{\partial X}{\partial v} + \left(\frac{b_{112}}{\Delta} - \frac{\partial r_0}{\partial u}\right) X, \\
\frac{\partial x}{\partial v} &= \frac{D''}{\Delta}\frac{\partial X}{\partial u} - \left(\frac{D'}{\Delta} + r_0\right)\frac{\partial X}{\partial v} - \left(\frac{b_{221}}{\Delta} + \frac{\partial r_0}{\partial v}\right) X, \\
\frac{\partial x}{\partial w} &= 2\frac{N}{\Delta}\frac{\partial X}{\partial u} - 2\frac{M}{\Delta}\frac{\partial X}{\partial v} + \left[\frac{1}{\Delta}\left(\frac{\partial M}{\partial v} - \frac{\partial N}{\partial u}\right) - \frac{\partial r_0}{\partial w}\right] X,
\end{aligned}\right\} \qquad \text{(III)}$$

and the analogous expressions in y, Y and z, Z will give r_0 *as an arbitrary function of u, v, w.*

(α) and (β) might well be called the *intrinsic equations* of the complex that they specify. From now on, we will always suppose that a complex is defined by its intrinsic equations. All of the geometric elements that we will encounter will then be expressed in terms of their coefficients and their derivatives, and any particular property of a complex will be translated into one or more equivalences between the coefficients and their derivatives that are distinct from the fundamental relations (I) and (II).

Directions in spatial rulings.

6. Consider two infinitely-close rays of the complex:

$$g(u, v, w), \qquad g'(u + du, v + dv, w + dw),$$

and suppose that one does not have $du = dv = 0$.

The plane Π that is determined by g and the direction of the minimum distance $d\sigma$ between g and g' is called the *central plane* of g relative to g', and the point Q at which $d\sigma$ encounters g is called the *central point* of g relative to g'. There are central points and planes of g for any ruled surface that passes through g and g', and Q is a point of the *line of striction* for any such ruling. It is known that they are all *connected* along g (i.e., they contact at every point of g): The tangent plane that is common to the point Q is the central plane, and at any other point of g it will be determined from the known law of HAMILTON ([1]) on the *distribution of tangent planes:*

$$\tan \psi = \frac{t}{p}, \tag{31}$$

in which t is the abscissa of the point with the respect to the central point Q, ψ is the angle that the corresponding tangent plane makes with the central plane Π, and p is the *distributor parameter:*

$$p = \frac{d\sigma}{ds}. \tag{32}$$

If $p = 0$ then g and g' will meet at the central point Q, the tangent plane will be stationary along g and will be the plane gg' that is perpendicular to the central plane (except for the point Q, at which it is indeterminate). The ruling that passes through g and g' will then behave like a developable surface at g.

In any case, the tangent plane to the point at infinity of g, which is the *asymptotic plane* to g relative to g', will be perpendicular to the central plane. If $p = 0$ then it will be the plane gg'.

The correspondence between the points and planes of g that is determined by HAMILTON's law is a (CHASLES) projectivity. If $p = 0$ then it will degenerate into the *singular point* $gg' \equiv Q$ and the *singular plane* gg'.

7. The central plane of g relative to g' is specified by the fact that it passes through g and contains the direction of $d\sigma$, whose direction cosines (5) are known. The distributor parameter p (32) is measured by the ratio (with the signed changed) of the two fundamental forms of the complex:

([1]) This law is commonly attributed to CHASLES, but in reality, it was given for the first time by HAMILTON, as prof. SEGRE observed in the note: "Monge e le congruenze generali di rette," Biblioteca Matematica, 1907-8, pp. 321.

$$p = -\frac{D\,du^2 + 2D'du\,dv + D''dv^2 + 2M\,du\,dw + 2N\,dv\,dw}{E\,du^2 + 2F\,du\,dv + G\,dv^2}. \tag{33}$$

Finally, in order to construct the central point Q, it is enough to know its abscissa r with respect to the origin M (x, y, z) of g, which is (A, § 13):

$$r = -\frac{\sum dx\,dX}{\sum dX^2}.$$

Thus, if one introduces the variables u, v, w and recalls formulas (2), (3), (7′), and (8′) then one will have:

$$\left.\begin{aligned} r - r_0 &= \frac{(FD - ED')\,du^2 + (GD - ED'')\,du\,dv + (GD' - FD'')\,dv^2}{\Delta(E\,du^2 + 2F\,du\,dv + G\,dv^2)} \\ &\quad + 2\frac{(FM - EN)\,du + (GM - FN)\,dv}{\Delta(E\,du^2 + 2F\,du\,dv + G\,dv^2)}\,dw. \end{aligned}\right\} \tag{34}$$

That formula will give precisely the distance $r - r_0$ from the central point Q to a certain point of g whose abscissa is r_0, and whose geometric significance we shall discuss in what follows (§ 12).

8. It is easy to give a formula that will allow one to calculate the angle between the two planes that pass through g and are the central planes of g relative to two infinitely-close lines:

$$g'\ (u + du, v + dv, w + dw), \qquad g''\ (u + \delta u, v + \delta v, w + \delta w),$$

respectively.

Indeed, that angle is equal to the one that is defined by the directions of the minimum distances $d\sigma$ and $\delta\sigma$ from g to g' and g'', resp. However, $d\sigma$ and $\delta\sigma$ are orthogonal to the corresponding spherical linear elements $d\sigma'$ and $\delta\sigma'$, respectively, and all four of them are orthogonal to g, so the angle between $d\sigma$ and $\delta\sigma$ will be equal to the one between ds' and $\delta s'$, so ([1]):

$$\cos(d\sigma, \delta\sigma) = \frac{E\,du\,\delta u + F\,(du\,\delta v + dv\,\delta u) + G\,dv\,\delta v}{\sqrt{(E\,du^2 + 2F\,du\,dv + G\,dv^2)(E\,\delta u^2 + 2F\,\delta u\,\delta v + G\,\delta v^2)}}. \tag{35}$$

In particular, the angle ω between the central planes of g relative to the ruling coordinates u and v that pass through g is given by the formula:

$$\cos\omega = \frac{F}{\sqrt{EG}}, \qquad \sin\omega = \frac{\Delta}{\sqrt{EG}}. \tag{36}$$

([1]) BIANCHI, *loc. cit.*, § 42.

Furthermore: The angle θ, which is found between 0 and 360°, that the central plane Π of an arbitrary ruling that passes through g and g' makes with that of the ruling u that passes through g is given by the formulas:

$$\cos\,\theta = \frac{1}{\sqrt{E}}\left(E\frac{du}{ds'} + F\frac{dv}{ds'}\right), \qquad \sin\,\theta = \frac{\Delta}{\sqrt{E}}\cdot\frac{dv}{ds'}. \tag{37}$$

We also note the inverse formulas:

$$\frac{du}{ds'} = \frac{\Delta\cos\theta - F\sin\theta}{\Delta\sqrt{E}}, \qquad \frac{dv}{ds'} = \frac{\sqrt{E}}{\Delta}\sin\theta. \tag{37′}$$

9. Now, suppose that one has $du = dv = 0$. The rays g (u, v, w) and g' $(u, v, w + dw)$ are then parallel (§ 3), the central plane Π of g relative to g' is the plane gg', and is also the tangent plane at all points of g to all rulings that pass through g and g', which are rulings that behave like cylinders at g. If the distributor parameter p is infinite then the central point Q will be indeterminate.

10. All of the curves in space that pass through a point P have the tangent o in common with an infinitely-close point P', which one calls their *direction.*

All of the rulings that pass through a line g and an infinitely-close line have all of the geometric elements that we recalled in § 6 in common with an infinitely-close line g'. By analogy, one says that they have *the direction of the spatial ruling* in common at g, and that direction *passes through g.*

Only one direction in a spatial ruling that *belongs to the ruling* will pass through any ray of a ruled surface, in general, ∞^1 directions that belong to a congruence will pass through any ray of a congruence, in general, and ∞^2 directions that belong to a complex will pass through any ray of a complex.

Fix a direction in the spatial ruling that passes through a ray g of a complex and belongs to the complex – i.e., fix a line g' that is infinitely-close to g in the complex. One will then know the central plane Π (and therefore the angle θ that it makes with central plane of the ruling u that passes through g), the central point Q (and therefore its abscissa r with respect to the point M that is the origin of g), and the distributor parameter p. In order to determine them, it is enough to apply formulas (37), (33), and (34).

Conversely, if one knows θ, r, and p then one will know the central plane Π, the central point Q, and in general, all of the geometric elements that were pointed out in § 6 – i.e., one will know a direction of the spatial ruling that passes through g.

We then call the three numbers θ, r, p the *coordinates* of the direction considered and denote its direction by (θ, r, p).

None of this is true when $du = dv = 0$ – i.e., when g and g' are parallel (§§ 3 and 9). In that case, one says that g and g' determine a *cylindrical singular direction* that passes through g.

We then call a direction that is specified by two incident neighboring rays a *conical singular direction*; $p = 0$ for such a direction.

Congruences in a complex.

11. A non-cylindrical congruence (§ 2) is specified by two binary quadratic differentials that are called the *first and second fundamental forms* of the congruence. The first one represents the square of the angle of two infinitely-close rays of the congruence, while the second one represents the moment of those two rays (*A*, §§ 2, 4, and 21).

If one is given the equation of the congruence (§ 2):

$$w = w(u, v) \tag{38}$$

then the two forms will be obtained from the forms (α) and (β) of the complex by replacing w and dw in them with the expression in (38) and:

$$dw = \frac{\partial w}{\partial u}du + \frac{\partial w}{\partial v}dv,$$

resp.

However, the form (α) does not depend upon w and dw, so: *All non-cylindrical complexes of the congruence have the first fundamental form in common, which is also the first fundamental form* (α) *of the complex.*

It then follows that: *An arbitrary non-cylindrical congruence of an assigned complex is specified by its second fundamental form.*

12. Therefore, a coordinate congruence w is specified by its second fundamental form:

$$D\,du^2 + 2D'\,du\,dv + D''dv^2, \tag{39}$$

which is obtained from (β) by taking the value for w that is constant along the congruence considered and taking $dw = 0$.

The coefficients of the two forms (α) and (39) of the congruence w are related by a unique relation, namely, (II) (*A*, § 21).

Formulas (33) and (34) become:

$$r - r_0 = \frac{(FD - ED')\,du^2 + (GD - ED'')\,du\,dv + (GD' - FD'')dv^2}{\Delta(E\,du^2 + 2F\,du\,dv + G\,dv^2)}, \tag{40}$$

$$p = -\frac{D\,du^2 + 2D'\,du\,dv + D''dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}, \tag{41}$$

and together with (37), they give the coordinates θ, r, p of the ∞^1 directions of the spatial ruling that passes through the ray $g(u, v, w)$ of the congruence w and belongs to the congruence.

r_0 is the abscissa of the *mean point* of g in the congruence w. (A, § 14).

r can vary between two extreme values that correspond to two fixed points of g that are called *limit points*, so the central point Q of the ∞^1 directions that pass through g and belong to the congruence w cannot leave the segment that is found between those two points.

p also admits an absolute maximum p_1 and an absolute minimum p_2 whose sum is the *mean parameter* H (16), and whose product is the *absolute parameter:*

$$K = \frac{DD'' - D'^2}{EG - F^2} \tag{42}$$

of the congruence w for g.

One finds two points – called *foci* – on g at which all of the rulings that pass through g and belong to the congruence w will contact. They are placed symmetrically with respect to the mean point of g, and their distance from that point is $\pm\sqrt{-K}$. Therefore, they are real and distinct, real and coincident, or complex-conjugate according to whether one has:

$$K <, = , > 0$$

for g, resp. g is then called a *hyperbolic, parabolic, or elliptic* ray of the congruence w, respectively.

Without demanding anything further, we can assert that all of the formulas of paper A are integrally applicable to a congruence w of the complex.

13. More generally: A congruence of the complex whose equation is (38) is specified by its second fundamental form:

$$\left(D + 2M\frac{\partial w}{\partial u}\right)du^2 + \left(D' + M\frac{\partial w}{\partial v} + N\frac{\partial w}{\partial u}\right)du\,dv + \left(D'' + 2N\frac{\partial w}{\partial v}\right)dv^2, \tag{43}$$

and all of the formulas in paper A will be applicable to it *as long as one replaces D, $2D'$, D'' in those formulas with the coefficients in* (43).

14. In order to apply the preceding, we propose to *seek the surface whose normals are rays of a given complex.* That search was begun for the first time by MALUS and then reprised by TRANSON in 1861 ([1]).

One should observe that it is equivalent to *the search for the normal congruence of a complex*, and that a congruence is normal only when its mean parameter H is zero (A, § 19). It is therefore enough to equate the numerator of the expression (16) for H to zero,

([1]) *Journal of the l'École Polytechnique*, 38th letter, pp. 195.

in which one has substituted the coefficients of (43) in place of D, $2D'$, D'', in order to obtain *the differential equation of the normal congruences of a complex:*

$$2\,(FN - GM)\frac{\partial w}{\partial u} + 2\,(FM - EN)\,\frac{\partial w}{\partial v} + (2FD' - ED'' - GD) = 0. \qquad (44)$$

It is linear in the first partial derivatives of the unknown function w of u, v, so: *There are always an infinitude of normal congruences in a complex* (which depend upon an arbitrary function).

Any solution of (44) contains an arbitrary constant that defines a family of ∞^1 normal congruences of the complex such that one such congruence will pass through any ray of the complex (or a region of it), in general. One can therefore say, with TRANSON, that: *Any complex can always be sliced into ∞^1 normal congruences (and in an infinitude of ways).*

If a normal congruence (38) of the complex is known, and therefore, if its two fundamental forms (α) and (43) are known then in order to construct the ∞^1 parallel surfaces that are orthogonal to its rays, it will be enough to apply the process that was explained in *A*, § 27.

In conclusion, recall that DARBOUX, in two recent notes ([1]), has proved a noteworthy theorem, which he announced at the end of 1870 ([2]): *If one knows a first family of surfaces that admit the lines of a complex for their normals then one can determine all of the other ones without integrating.*

15. Some other noteworthy surfaces that KLEIN ([3]) sought are the ones whose tangents to the asymptotes of a system are lines of a given complex.

The search for those surfaces is equivalent to the search for the *parabolic congruences* of the complex (*A*, § 17), which are characterized by annulling the absolute parameter K. Therefore, if one equates the numerator of the expression (42) for K to zero, after substituting the coefficients in (43) for D, $2D'$, D'', then one will obtain *the differential equation of the parabolic congruences of a complex:*

$$\left(M\frac{\partial w}{\partial v} - N\frac{\partial w}{\partial u}\right)^2 + 2(D'N - DM)\frac{\partial w}{\partial u} + 2(D'M - D''M)\frac{\partial w}{\partial v} = D\,D'' - D'^2. \qquad (45)$$

The requested surfaces are the focal surfaces (with coincident sheets), and can be constructed as was explained in *A*, § 24 ([4]).

([1]) *Comptes rendus*, 15 and 22 November 1909.
([2]) *Bulletin des Sciences Mathématiques*, 1870, pp. 348.
([3]) *Mathematische Annalen*, v. V.
([4]) Other noteworthy congruences of a complex (that were researched by COSSERAT) are the *isotropic* ones of RIBAUCOUR: We shall not occupy ourselves with them, since it can be shown (cf., ZINDLER, *loc. cit.*, page 195) that they do not exist in a generic complex.

Ruled surfaces in a complex.

12. A ruled surface in a complex is determined by its finite equations:

$$\varphi(u, v, w) = 0, \qquad \psi(u, v, w) = 0$$

or by its differential equations:

$$\frac{du}{U} = \frac{dv}{V} = \frac{dw}{W}$$

(in which, U, V, W are known functions of u, v, w) and the knowledge of one of the rays that pass through it.

Each of its rays will determine a direction of the spatial ruling that belongs to the complex, and its coordinates θ, r, p will be obtained from (33), (34), and (37) upon replacing du, dv, dw with the proportional quantities U, V, W. In particular, (33) permits one to construct the central point Q of each ray and the line of striction of the ruling.

A ruling will be developable when $p = 0$, so *the differential equation of a developable surface of the complex is:*

$$D\,du^2 + 2D'\,du\,dv + D''dv^2 + 2M\,du\,dw + 2N\,dv\,dw = 0. \tag{46}$$

It is satisfied by the ruling w – i.e., by the ∞^1 cylinders of the complex. If one sets $w = w(u, v)$ – i.e., if one equates the form (43) to zero – then one will have, in particular, the differential equation of the two systems of ∞^1 developables that are contained in the congruence (38).

The edges of regression of the developables are called *the curves of the complex*, since they generate the rays of the complex as their envelope.

Consider one such curve C and the tangent g at one of its point Q. The cone of the complex that has Q for its vertex (§ 1) will contain g as a generator: *The tangent plane to the cone along g is the osculating plane to C at the point Q.* In fact, Q can be regarded as the intersection of the tangent g to the curve with the successive tangents. The plane of those two lines is the osculating plane C to the point Q, and on the other hand, since it is the plane of the two successive generators of the cone whose vertex is Q, it will be the tangent plane to the cone along g.

Other facts about directions.

17. We call the rays of a complex that result when $M = N = 0$ the *bi-singular rays* of the complex. *Unless stated to the contrary* (§ 25), *they will be excluded from our considerations.*

Let $g(u, v, w)$ be a ray of the complex. There will be ∞^2 directions of the spatial ruling that belong to the complex and pass through g, since the coordinates θ, r, p of a direction (§ 10) depend upon the ratios of two of the quantities du, dv, dw to the third one. Therefore, a relation must exist between those coordinates.

In order to find it, set, for simplicity:

$$\frac{du}{ds'} = u_1,\ \frac{dv}{ds'} = v_1,\ \frac{dw}{ds'} = w_1, \tag{47}$$

and recall formulas (33) and (34):

$$- p = Du_1^2 + 2D'u_1v_1 + D''v_1^2 + 2(Mu_1 + Nv_1)\,w_1,$$

$$r - r_0 = \frac{(FD - ED')u_1^2 + (GD - ED'')u_1v_1 + (GD - FD'')v_1^2}{\Delta} + 2\,\frac{(FM - EN)u_1 + (GM - FN)v_1}{\Delta}w_1.$$

If one eliminates w_1 then one will obtain:

$$\begin{aligned}&[(EN - FM)\,u_1 + (FN - GM)\,v_1]\,p - \Delta\,(M\,u_1 + N\,v_1)\,(r - r_0)\\ &= (Du_1^2 + 2D'u_1v_1 + D''v_1^2)\,[(FM - EN)\,u_1 + (GM - FN)\,v_1]\\ &- [(FD - ED')\,u_1^2 + (GD - ED'')\,u_1\,v_1 + (GD' - FD'')\,u_2^2\,]\,(Mu_1 + Nv_1)\,.\end{aligned}$$

However, the right-hand side can be written as:

$$[(D'M - DN)\,u_1 + (D''M - D'N)\,v_1]\ (Eu_1^2 + 2F\,u_1v_1 + G\,v_1^2)\,,$$

i.e.:

$$(D'M - DN)\,u_1 + (D''M - D'N)\,v_1\,,$$

since from (α) and (47), one will have:

$$Eu_1^2 + 2F\,u_1v_1 + G\,v_1^2 = 1\,.$$

Therefore:

$$\begin{aligned}&[(EN - FM)\,u_1 + (FN - GM)\,v_1]\,p - \Delta\,(M\,u_1 + N\,v_1)(r - r_0)\\ &= (D'M - DN)\,u_1 + (D''M - D'N)\,v_1\,,\end{aligned}$$

i.e.:

$$\left.\begin{aligned}&[(FN - FM)du + (FN - GM)\,dv]p - \Delta(M\,du + N\,dv)(r - r_0)\\ &= (D'M - DN)\,du + (D''M - D'N)\,dv,\end{aligned}\right\} \tag{48}$$

and finally, from (37):

$$\left.\begin{aligned}&[(EN - FM)\cos\theta - \Delta\,M\,\sin\theta]p - [(EN - FM)\sin\theta + \Delta\,M\,\cos\theta](r - r_0)\\ &= (D''M - DN)\cos\theta + \frac{1}{\Delta}[(ED'' - FD')M + (FD - ED')N]\sin\theta.\end{aligned}\right\} \tag{48′}$$

(48) *or* (48′) *is the relation that couples the coordinates* θ, r, p *of a generic direction of the spatial ruling that passes through the ray* g (u, v, w) *and belongs to the complex.*

The cylindrical singular directions are the exceptions to this. For them, in fact, one has $du = dv = 0$ (§ 10), so (48) will vanish. If one prefers then one can say that (48′) will be replaced with $p = \infty$ (§ 9) for them.

18. (48′) will permit us to calculate one of the three coordinates θ, r, p when one is given the other two, at least, in general. Therefore: *In general, one direction of a spatial ruling (that is not a cylindrical singular one) will pass through a ray of the complex and belong to the complex that is specified by the values of any two of its coordinates, which can be given arbitrarily.*

However, there are exceptional directions:

a) *If one fixes θ arbitrarily then one can also assign an arbitrary value to r (so that p will then be specified) unless:*

$$(FM - EN)\,du + (FN - GM)\,dv = 0, \tag{49}$$

i.e.,:

$$(EN - FM)\cos\theta - \Delta\,M\sin\theta = 0. \tag{49′}$$

In the excluded case – i.e., when θ has the value θ_1 that is defined by the formula:

$$\left.\begin{aligned}\cos\theta_1 &= \frac{\Delta\,M}{\sqrt{E(GM^2 - 2F\,M\,N + EN^2)}},\\ \sin\theta_1 &= \frac{EN - FM}{\sqrt{E(GM^2 - 2F\,M\,N + EN^2)}},\end{aligned}\right\} \tag{50}$$

r can assume only the unique value r_1 that is defined by:

$$r_1 - r_0 = \frac{(FD'' - GD')M^2 + (GD - ED'')MN + (ED' - FD)N^2}{\Delta(GM^2 - 2FMN + EN^2)}, \tag{51}$$

which would result from (48′), while p remains completely arbitrary.

The angle θ_1 defines a plane Π_1 and the abscissa r_1 of a point Q_1 of g ([1]). They are the central plane and central point, respectively, that are common to ∞^1 directions in the spatial ruling that passes through g – i.e., to the directions (θ_1, r_1, p) with p arbitrary.

b) *If one fixes θ arbitrarily then one can also assign an arbitrary value to p (so r is then specified), unless one has:*

([1]) One observes that formulas (50), (51), and the following ones (53) and (54) are always meaningful. In fact, if $EG - F^2 > 0$ then the formula:

$$GM^2 - 2FMN + EN^2,$$

which is quadratic in M and N, is essentially positive, and is therefore annulled only for the rays for which one has $M - N = 0$, which we exclude expressly.

$$M\,du + N\,dv = 0, \tag{52}$$

i.e.:

$$(EN - FM)\sin\theta + \Delta\,M\cos\theta = 0. \tag{52′}$$

In the excluded case – i.e., when θ is given the value θ_1' that is defined by the formula:

$$\left.\begin{aligned} \cos\theta_1' &= \frac{FM - EN}{\sqrt{E(GM^2 - 2FMN + EN^2)}}, \\ \sin\theta_1' &= \frac{\Delta M}{\sqrt{E(GM^2 - 2FMN + EN^2)}}, \end{aligned}\right\} \tag{53}$$

p can assume only the unique value:

$$p_1 = -\frac{D''M^2 - 2D'MN + DN^2}{GM^2 - 2FMN + EN^2}, \tag{54}$$

which would result from (48′), while r remains completely arbitrary.

The angle θ_1' defines a plane Π_1' of g: Π_1' and p_1 are the central plane and distributor parameter, respectively, that are common to ∞^1 directions in the spatial ruling that passes through g – i.e., to the directions (θ_1', r, p_1) with r arbitrary.

c) *If one is given p and r arbitrarily then* θ *will be specified* (*up to* 180°), *as long as one does not have:*

$$\left.\begin{aligned} (EN - FM)\,p - \Delta\,M\,(r - r_0) &= D'M - DN, \\ (FN - GM)\,p - \Delta N(r - r_0) &= D''M - D'N. \end{aligned}\right\} \tag{55}$$

If one solves the system (55) then one will find that:

$$p = p_1, \qquad r = r_1,$$

in which p_1 and r_1 have the values (54) and (51). If one attributes those values to p and r then (48′) will be satisfied identically – i.e., for any θ.

Therefore, the point Q_1 of g and the number p_1 are the central point and the distributor parameter, respectively, that are common to ∞^1 directions of the spatial ruling that passes through r – i.e., the directions (θ, r_1, p_1), where θ is arbitrary.

19. The preceding discussion leads one to distinguish three systems of ∞^1 special directions:

$$(\theta_1, r_1, p), \qquad (\theta_1', r, p_1), \qquad (\theta, r_1, p_1), \tag{56}$$

from among the ∞^1 directions of the spatial ruling that passes through a ray of the complex and belongs to the complex, and to consider a certain point Q_1, along with two particular planes Π_1 and Π_1' through g.

It follows from (50) and (53) that:

$$\theta_1 - \theta_1' \equiv 90^{\circ} \pmod{180^{\circ}} ;$$

i.e., *the planes* Π_1 *and* Π_1' *are mutually perpendicular.*

It then follows that the planes Π_1 and Π_1', which are the central planes to the directions of the first two systems (56), respectively, are also asymptotic planes of the inverses of those systems. It also follows that the first two systems cannot have any common direction, while the first and the third systems have the direction (θ_1, r_1, p_1) in common, and the second and third ones have the direction (θ_1', r_1, p_1) in common.

There is always a singular direction (§ 10) in the first system – viz., $(\theta_1, r_1, 0)$. On the contrary, no direction of the other two systems will be singular at a generic ray of the complex; however, all of the rays for which $p_1 = 0$ will be singular.

The rays for which one has $p_1 = 0$ are called the *singular rays* of the complex.

It follows from § 18, *a*) that any plane Π that passes through g can be associated with any point Q of g: They are the central plane and central point, respectively, of *just one* direction in the spatial ruling that passes through g and belongs to the complex. An exception is the single plane Π_1, which cannot be associated with the point Q_1. Π_1 and Q_1 determine, not one, but ∞^1 directions, namely, (θ_1, r_1, p).

In regard to the plane Π_1', one can observe that *it is the tangent plane along g to the cone of the complex that has the vertex* Q_1. Indeed, Π_1' is the asymptotic plane to any direction of the system (θ_1, r_1, p) and to the singular direction $(\theta_1, r_1, 0)$, in particular, so it will be the osculating plane at Q_1 to the edge of regression for an arbitrary developable of the complex that passes through g and have the direction $(\theta_1, r_1, 0)$ at g; one will deduce the statement from that when one takes into account the observation that was made at the end of § 16.

20. The relation (48′) can be put into the much simpler form:

$$p = r \tan \alpha - u, \tag{57}$$

which was given for the first time by KOENIGS (*loc. cit.*), and in which α is an angle that varies with the direction that one considers, while u is a constant.

If one excludes the directions (θ_1, r_1, p) – i.e., if one suppose that (40) is not verified – then one can given the desired form to (48) by taking:

$$\tan \alpha = \frac{\Delta(M\,du + N\,dv)}{(EN - FM)\,du + (FN - GM)\,dv}, \tag{58}$$

$$n = \frac{\Delta(M\,du + N\,dv)\,r_0 - (D'M - DN)\,du - (D''M - D'N)\,dv}{(EN - FM)\,du + (FN - GM)\,dv}, \tag{59}$$

and choose the point M that is the origin of g (and therefore of r_0) in such a way that n remains constant – i.e., independent of $du : dv$. That will be achieved if one sets:

$$\begin{vmatrix} \Delta M\,r_0 - (D'M - DN) & EN - FM \\ \Delta N\,r_0 - (D''M - D'N) & FN - GM \end{vmatrix} = 0,$$

i.e.:

$$r_0 = \frac{(GD' - FD'')\,M^2 + (ED'' - GD)\,MN + (FD - ED')\,N^2}{\Delta(GM^2 - 2FMN + EN^2)}. \tag{60}$$

If one substitutes this into (59) (in which it is now legitimate to set $du = 0$ or $dv = 0$ to begin with) then one will have:

$$n = \frac{D''M^2 - 2D'M\,N + D\,N^2}{GM^2 - 2FMN + EN^2}, \tag{61}$$

i.e., from (54):

$$n = -p_1\,. \tag{61′}$$

The value (60) that was found for r_0 is the distance from the mean point of the ray of the congruence w that passes through it to the point M that is the origin of the ray (§ 12); however, it coincides – up to sign – with the right-hand side of (51), which the distance from the Q_1 to that mean point, so the origin M that we established earlier is nothing but the point Q_1 . In order to represent the arbitrariness in the origin M of g, it is enough to change r into $r - r_0$ in (57).

On the other hand, by virtue of (37′), (58) can be written:

$$\tan\alpha = \frac{(EN - FM)\sin\theta + \Delta M\cos\theta}{(EN - FM)\cos\theta - \Delta M\sin\theta};$$

i.e., from (50):

$$\tan\alpha = \cot(\theta - \theta_1).$$

In summary, we get that:

The coordinates θ, r, p of a direction of the spatial ruling that passes through a ray of the complex and belongs to the complex are coupled by the relation:

$$p - p_1 = (r - r_1)\cot(\theta - \theta_1). \tag{62}$$

In order to prove this, we must exclude the directions (θ_1, r_1, p). Nevertheless, (62) can remain valid in that case, as well. Indeed, for $r = r_1$ and $\theta = \theta_1$ the right-hand side will not have a well-defined value, so (62) would not give a well-defined value for p if it were illegitimately applied, and according to § 18, *a*), that is exactly what must happen.

The constants θ_1, r_1, p_1 that appear in (62) are the coordinates one of those ∞^1 directions.

21. The geometric elements Π_1, Π_1', θ_1, p_1 that we encountered can be linked with an important element that was introduced by KOENIGS (*loc. cit.*).

A direction of an arbitrary spatial ruling that passes through a line g will determine a (CHASLES) projectivity between the points and planes of g by virtue of (31), and conversely.

The ∞^2 directions of a spatial ruling that pass through a ray g of a complex and belong to the complex will then determine just as many CHASLES projectivities in g. One can show that they constitute a *net*, and that it will be *harmonic* or *involutory* with a fixed projectivity on g that KOENIGS called the *normal projectivity* of the line g of the complex. That will, in turn, determine a direction of the spatial ruling that passes through g: One can then prove that this direction is $(\theta_1', r_1, -p_1)$, with our notation ([1]).

It has Π_1' for its central plane, and thus, Π_1 for its asymptotic plane, while it has Q_1 for its central point and $-p_1$ [i.e., n (61′)] for its distributor parameter. We then call Π_1 the *normal asymptotic plane,* Π_1'*, the normal central plane,* Q_1*, the normal central point, and* $-p_1 = n$*, the normal (distributor) parameter of* g*.*

Observe that *the direction* $(\theta_1', r_1, -p_1)$ *that is specified by the normal projectivity will belong to the complex only if* g *is a singular ray of the complex, and it will then be a singular ray.* Indeed, it will have Π_1' for its central plane, so in order for it to belong to the complex, it must be included among the rays of the system $(\theta_1', r_1, -p_1)$ (§ 18, *b*); that can happen only if $-p_1 = p_1 = 0$; i.e., if g is a singular ray (§ 19), and then the direction in question will be singular (§ 10).

22. In order to give an intuitive picture of the behavior of the directions of a spatial ruling that belongs to the complex and passes through a ray g, we shall make a geometric representation.

Let (θ, r, p) be any of those directions: θ is the angle that its central plane Π makes with that of the ruling u that passes through g, r, and the abscissa of its central plane Q with respect to the origin M of g – i.e., the foot of the minimum distance $d\sigma$ between g and the infinitely-close ray g' that corresponds to the direction considered (§§ 6 and 10).

Starting with Q, we carry the segment $QP = p$ in the direction of $d\sigma$ and in an established direction ([2]). The locus of points P for all of the ∞^2 directions that pass through g will be a surface S. Conversely, any point P of S will determine a direction in the spatial ruling that passes through g and belongs to the complex: viz., the one whose central plane is the plane $Pg \equiv \Pi$, whose central point is the foot Q of the perpendicular

([1]) For that, it is enough to compare (62) with the analogous formula of KOENIGS (*loc. cit.*, page 59) and to take into account the geometric significance of the constants that appear in it.

([2]) For example, in the direction that is fixed by formula (5) of § 3.

that goes from P to g, and whose distributor parameter is the length p of the segment QP (with the exception of any possible points of intersection of S with g).

We then call S the *surface of the directions* of g.

If we adopt cylindrical coordinates, assume that g is the axis, that the origin of g is the normal central point Q_1, and that the plane of the origin in the pencil of axes g is the normal asymptotic plane Π_1 then:

$$\varphi = \theta - \theta_1 , \qquad \rho = r - r_1, \qquad p \tag{63}$$

will be the cylindrical coordinates for P, and (62) will give the equation:

$$p - p_1 = \rho \cot \varphi \tag{64}$$

for the surface S in cylindrical coordinates.

If one intersects S with a plane that passes through g and has the equation φ = constant then one will get a line s whose equation in Cartesian coordinates p, ρ is (64); therefore, *S is a ruled surface.*

If one transforms the cylindrical coordinates into Cartesian ones x, y, z by setting:

$$x = p \cos \varphi, \qquad y = p \sin \varphi, \qquad z = \varphi,$$

then one will see immediately that *S is a middle of the fourth-order ruled surface:*

$$(x^2 + y^2)\, y^2 = (x\, z + p_1\, y)^2.$$

23. If $p_1 \neq 0$ – i.e., if g is a singular ray of the complex – then s will meet g at a point that varies with φ and will traverse the entire line g. If one intersects S with the plane that is perpendicular to g at Q_1 and has the equation $\rho = 0$ then one will get a circle that has Q_1 for its center and p_1 for its radius. Therefore: *Two plane directors of the ruling S are the ray g and the circle that has g for its axis, Q_1 for its center, and p_1 for its radius.*

When $\varphi = \theta - \theta_1 = 0$, it will result that $\rho = r - r_1 = 0$, and p will remain arbitrary (§ 18, a), so the special directions (θ_1, r_1, p) will have the points of the line perpendicular to g at Q_1 and the plane Π_1 for their images on S. Similarly, the directions (θ_1', r, p_1) will have all of the points of a line that is parallel to g in the plane Π_1' at a distance p_1 for their images on S, and the images of the directions (θ, r_1, p_1) on S will be all of the points of the aforementioned circle.

All of the points of g belong to S and are the only ones for which one will have $p = 0$, so (§ 10) *the singular directions have all of the points of g for their images.*

Observation: One must exclude the cylindrical singular directions from the preceding considerations (cf., the end of § 17). Nevertheless, since one has $p = \infty$ for them, one can say that *their images on S are all points at infinity on S.*

They are also the points at infinity of the cone whose equation is:

$$p = \rho \cot \varphi. \tag{65}$$

That cone, whose vertex is Q_1, is the director cone for the ruling S.

Singular rays.

24. The surface S will reduce to that cone along a singular ray.

One can arrive at the consideration of singular rays of a complex along other paths, one of which is more consistent with the methods of differential geometry. Let g be an arbitrary ray of the complex, let A be one of its points, and consider two arbitrary rulings of the complex that pass through g. The two rulings contact at A (and at any other point of g), so they will have the direction of the spatial ruling at g in common. However, in the contrary case one will have two distinct tangent planes at A, in general. Nevertheless, is it legitimate to demand that there must exist a ray g in the complex on which there exists a point A such that *all* of the rulings of the complex that pass through g contact it?

We begin by observing that if such a ray exists then, in particular, all of the developables of the complex that pass through g must contact at A (excluding the ones for which A is a singular point, and thus do not have a well-defined tangent plane at A). However, the tangent plane to a developable that passes through g will be stationary along g and will be perpendicular to the central plane of the singular direction of the spatial ruling that is determined by the developable. Meanwhile: All of the singular directions in the spatial ruling that passes through g and belong to the complex must have the same central plane.

Now, for a singular direction ($p = 0$), (62) will become:

$$-p_1 = (r - r_1) \cot (\theta - \theta_1),$$

which shows that *if one does not have* $p_1 = 0$ then θ (and therefore the central plane) will vary with r. One must therefore necessarily have $p_1 = 0$ – i.e., g must be a singular ray of the complex. The preceding equation then gives:

$$\theta \equiv \theta_1 + 90^\circ \equiv \theta_1' \qquad (\text{mod } 180^\circ)$$

and proves that the fixed central plane must be the plane Π_1', and therefore that the fixed tangent plane must be the plane Π_1.

Conversely, consider a singular ray g with Π_1 for its normal asymptotic plane and Q_1 for its normal central points, and consider an arbitrary direction in the spatial ruling (θ, r, p) that belongs to the complex and passes through g, and whose central plane is Π and whose central point is Q. The plane Π_1 will be the tangent plane to all of the ruled surfaces that pass through *g along the fixed direction* (θ, r, p) at a certain point A that can be determined from **HAMILTON**'s law (§ 6):

$$p \tan \Psi = Q A.$$

Ψ is the angle between the two central planes Π and Π_1 – i.e., $\Psi = \theta - \theta_1$ – so:

$$Q\,A = p \tan(\theta - \theta_1).$$

However, if $p_1 = 0$ then (62) will give:

$$p = (r - r_1) \cot(\theta - \theta_1),$$

so

$$Q\,A = r - r_1 = Q\,Q_1,$$

and therefore $A \equiv Q_1$. However, Q_1 is a fixed point of g that does not depend upon the fixed direction (θ, r, p). Thus, an arbitrary ruling of the complex that passes through g will touch the plane Π_1 at the point Q_1.

We conclude:

A characteristic property of a singular ray of a complex is that all of the ruled surfaces of the complex that pass through it will contact the same point of the ray. That point will be the normal central point Q_1, *and the common tangent plane will be the normal asymptotic plane* Π_1 .

The point Q_1 and the plane Π_1 of a singular ray are called the *singular point and singular plane* of the ray, respectively.

It follows from (§ 12) that:

The singular point Q_1 *is one of the two foci of the singular ray for all of the congruences of the complexes that pass through it. Therefore, the focal surfaces of that congruence will contact at* Q_1 *and have the plane* Π_1 *for the common tangent plane there.*

Bi-singular rays.

25. We now consider the bi-singular rays of the complex, which are characterized analytically by the equality:

$$M = 0, \; N = 0, \tag{66}$$

which we excluded in § 17.

Only ∞^1 *directions of a spatial ruling that belong to complex will pass through a bi-singular ray.*

Indeed, the right-hand sides of formulas (33), (34), and (37), which give the coordinates θ, r, p of a direction, will depend upon only the ratio $du : dv$ at a bi-singular ray, which is arbitrary.

In other words: The cited formulas coincide with (40), (41), and (37), which give the coordinates of a direction that belongs to the congruence w that passes through g (and indeed any non-cylindrical congruence of the complex that passes through g, any one of which can be taken to be the congruence w). Therefore:

All of the non-cylindrical congruences of the complex that pass through a bi-singular ray g behave the same way at g, in the sense that they have all of the direction of the spatial ruling in common at *g*.

Since a neighborhood of a bi-singular ray *g* coincides with a neighborhood of a ray of the congruence, everything that was said to be true at *A* in §§ 5, 6, 7, 13, 14, 15, 16 will be true for it. In particular:

All of the congruences that pass through g have the same foci Q_1, Q_2 *on g and the same focal planes* Π_1 *and* Π_2 (which can be real or imaginary, and distinct or coincident).

The characteristic geometric property of singular rays that was proved in § 24 will be verified (in fact, doubly) for a bi-singular ray. Indeed, an arbitrary ruling that passes through such a ray will always belong to that congruence of complexes, and will then contact the ray at the two foci Q_1 and Q_2 of that congruence with tangent planes Π_1 and Π_2, resp.

Therefore:

The characteristic property of a bi-singular ray is that all of the rulings of the complex that pass through it will contact at two fixed points of the ray.

Those two points Q_1 and Q_2 are called *singular points* of the ray, and the relative tangent planes Π_1 and Π_2 are called the *singular planes* of the ray.

The focal surfaces (with two sheets) of all of the congruences of the complex that pass through a bi-singular ray contact at two singular points and have the two singular planes of the ray for tangent planes there.

Like the foci on a ray of the congruence, *the singular rays (and the focal planes) of a bi-singular ray can be real and distinct, real and coincident, or complex-conjugate* ([1]).

In the first case, one calls the bi-singular ray *hyperbolic*, in the second case, it is *parabolic*, and in the third, it is *elliptic*.

The analytical character that distinguishes the three cases is:

$$K <, = , > 0,$$

respectively, which is the analytical character that distinguishes the three cases when one considers the ray of the congruence *w* that that passes through it (cf., the end of § 12).

Finally, observe that for an arbitrary ray of the complex, *K* will represent the absolute parameter of the congruence *w* that passes through it (§ 12), so it will be invariant under any transformation of the just the variables *u*, *v*; however, it is *not* invariant under the more general transformations that we defined (cf., the end of § 2).

However, the preceding results permit is to assert that:

([1]) However, one notes that *the singular point and singular plane of a singular ray are always real.*

The sign of K (and therefore the annulling of K) on a bi-singular ray is invariant under all transformations of the variables u, v, w that were pointed out in § 2.

Loci of singular rays.

26. A ray is singular only when $p_1 = 0$ on it; i.e., from (54), when its coordinates *u*, *v*, *w* annul the function:

$$\sigma(u, v, w) = D''M^2 - 2D'M\,N + D\,N^2, \tag{67}$$

but do not annul *M* and *N* simultaneously. However, if $M = N = 0$ then the ray will be bi-singular, but σ will also be zero then. We can then call all of the rays of the complex whose coordinates *u, v, w* satisfy the equation:

$$D''M^2 - 2D'M\,N + D\,N^2 = 0 \tag{68}$$

its *singular rays*; i.e., both the singular rays, properly speaking, and the bi-singular ones. When we wish to distinguish the former from the latter, we will call the former the *ordinary* singular rays.

In regard to the function σ, we observe that it is finite and continuous, along with its derivatives, just like the functions *D*, *D*′, *D*″, *M*, *N* that accompany it (which corrects the explicit statement that was made in the *Introduction*).

27. Do there exist singular rays?

In the theory of algebraic complexes, in which there is no distinction made between real and imaginary, one has the following results ([1]):

Either all of the rays of the complex are singular or none of them are. If one excludes the limiting cases from this then in any complex there will exist ∞^2 singular rays that form a congruence, namely, the *singularity congruence* of the complex. As a consequence, the singular points of the singular rays form a surface (or line) that is one of the two focal sheets of the singularity congruence and which one calls the *singularity surface* of the complex; the other sheet will be called the *accessory surface.*

One notes that if one defines a complex analytically then one can resolve which of the three cases that were just enumerated is verified by an algebraic process.

The handful of authors that addressed the transcendental complexes announced the same results ([2]). Now, it seems to me that this is not legitimate.

If the function σ reduces to a constant *c* then according to whether *c* is zero or not, one can assert that either all rays of the complex are singular or none of them are, resp. However, *if* σ *is an effective function or u, v, w (which is true, in general) then one cannot say that.*

In fact, nothing gives one the right to assert that there exist triples of values of *u*, *v*, *w* that annul the function σ, nor does one have any means of deciding whether one has *at*

([1]) Cf., e.g., ZINDLER, *loc. cit.*
([2]) Cf., e.g., KOENIGS (*loc. cit.*); PICARD, *Traité d'Analyse*, vol. I, chap. XI, no. 16.

least one, and even if there is one then there can be ∞^2 of them, and that would give rise to a congruence, but there can also be just ∞^1 of them, or a finite number; they might or might not vary continuously, and might be completely or partially isolated.

The rest of the results cease to be valid for the same algebraic complexes if one considers only the real rays (as we do here).

It would thus be vain to attempt a classification of the transcendental complexes that is based upon the existence or non-existence of singular rays and their distribution throughout the complex. One can only enumerate and study some of the more noteworthy of the possible cases, limited to conveniently-small regions of the complex, and that is what we shall do. However, a complete study of the singularities must be performed only case-by-case for any individual complex or class of complexes. What we shall now say in general can serve as a useful guide.

28. It can happen that σ reduces to a non-zero constant or that even if it is a function of u, v, w then it still cannot be annulled. *The complex will then have no singular rays.*

However, it might be that σ is identically zero, so *all rays of the complex are singular.* Such a complex is called *special*. One can further prove, by intuitive considerations ([1]), that *a special complex is composed of all tangents to a surface, or all lines that meet a curve, or a curve that is locus of singular points of the rays of the complex.*

We then note that *the rays of a complex cannot all be bi-singular;* i.e., *the functions M, N cannot both be identically zero.* In fact, for $M = N = 0$, (III) of § 5, and the analogues for y and z, will give:

$$\frac{\partial x}{\partial w} = -\frac{X}{\Delta}\frac{\partial r_0}{\partial w}, \qquad \frac{\partial y}{\partial w} = -\frac{Y}{\Delta}\frac{\partial r_0}{\partial w}, \qquad \frac{\partial z}{\partial w} = -\frac{Z}{\Delta}\frac{\partial r_0}{\partial w}.$$

However, r_0 is an arbitrary function of u, v, w, so it is legitimate to assume that $r_0 = 0$; it will then result that:

$$\frac{\partial x}{\partial w} = \frac{\partial y}{\partial w} = \frac{\partial z}{\partial w} = 0,$$

identically. If one prefers, it is impossible for the complex to not degenerate into a congruence (§ 1).

29. Now, if one excludes the two preceding limiting cases then what will remain is a third hypothesis, namely, that there should exist at least one triple of real values u', v', w' of u, v, w that annuls the function σ – i.e., *there exists a complex with at least one singular ray g'* (u', v', w').

One can make various hypotheses about g':

([1]) We will only sketch the proof. One can find a rigorous proof for the first time in KLEIN (*loc. cit.*) for algebraic complexes and in KOENIGS (*loc. cit.*) for transcendental complexes.

Let g′ be an ordinary singular ray; i.e., its coordinates do not annul M and N simultaneously. It will not annul the first partial derivatives of σ, in general, then, so equation (68) will define one of the variables u, v, w as a function of the other two (e.g., w as a function of u, v in a neighborhood of the values u', v', such that for $u = u'$, $v = v'$, one will have $w = w'$).

Therefore:

If there exists an ordinary singular ray g′ in the complex then there will generally exist ([1]) *a congruence that is composed of rays that are all singular and pass through g′.*

One calls it *a congruence of singularities* of the complex.

If all three derivatives of σ are annulled at g' then there is nothing that we can assert.

30. *Let g′ be bi-singular.* Its coordinates satisfy the system:

$$M = 0, \; N = 0; \tag{69}$$

however, in general, not all second-order minors of the functional matrix:

$$\begin{vmatrix} \dfrac{\partial M}{\partial u} & \dfrac{\partial M}{\partial v} & \dfrac{\partial M}{\partial w} \\ \dfrac{\partial N}{\partial u} & \dfrac{\partial N}{\partial v} & \dfrac{\partial N}{\partial w} \end{vmatrix}$$

will be annulled; therefore, the system (69) will define two of the variables u, v, w as functions of the third (e.g., u, v as functions of w in a certain interval that contains w', and such that for $w = w'$, it will result that $u = u'$, $v = v'$).

Therefore:

In general, a ruled surface of a complex that is composed of nothing but bi-singular rays will pass through a bi-singular ray of that complex.

31. We shall now see whether there also exist ordinary singular rays in the neighborhood of a bi-singular ray g'. The three first partial derivatives of σ are annulled for it (which is easy to verify), so the analysis that was carried out in § 29 will no longer be valid, and must be updated. We therefore distinguish three cases:

a) *g′ is elliptic* (§ 25). Therefore, one will have $K > 0$ at g'. However, K is a continuous function, and one will then have $K > 0$, so $DD'' - D'^2 > 0$ ([2]) in an entire neighborhood of g'. It then follows that the function σ, which is trinomial of degree two

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([1]) Cf., the final note in § 1.

([2]) Recall that the denominator $EG - F^2$ in the expression (42) for K is essentially positive (§ 3).

in M and N, will keep a constant sign (which is necessarily common to D and D') in this neighborhood; one can therefore annul then only when both M and N are annulled simultaneously.

Therefore:

There exist no ordinary singular rays in the neighborhood of a bi-singular ray.

b) *g′ is hyperbolic.* One will have $K < 0$ (i.e., $DD'' - D'^2 < 0$) at g', and therefore in an entire neighborhood of g', as well. It will then follow that in that neighborhood it will be possible to decompose the trinomial σ into the product of two linear binomials in M and N:

$$\sigma = (\alpha_1 M + \beta_1 N)(\alpha_2 M + \beta_2 N),$$

whose coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real functions of u, v, w, such that:

$$\alpha_1\alpha_2 = D'', \qquad \alpha_1\beta_2 + \alpha_2\beta_1 = -2D', \qquad \beta_1\beta_2 = D. \tag{70}$$

There will be singular rays in the neighborhood of g' whose coordinates annul one or the other factor of σ. The first factor will be annulled at g', because M and N are annulled, but its three first partial derivatives will not be annulled, in general, so the equation:

$$\alpha_1 M + \beta_1 N = 0$$

will define one of the variables u, v, w as a finite and continuous function of the other two, which will give rise to a congruence C_1 of singular rays that pass through g'. The same statement will apply to the other factor.

Therefore:

Two regions C_1, C_2 of a congruence C that is composed of nothing singular rays will pass through a hyperbolic, singular ray, in general.

C is called a *congruence of singularities* of the complex.

The two regions C_1 and C_2 in the neighborhood of g′ can have only bi-singular rays in common. Indeed, if:

$$(\alpha_1\beta_2 - \alpha_2\beta_1)^2 = 4(DD'' - D'^2) > 0 \tag{71}$$

then the two linear factors of σ will be distinct in the neighborhood of g', so they will be annulled simultaneously only when $M = N = 0$.

c) *g′ are parabolic*, i.e., $K = 0$ at g'. There can then exist a neighborhood of g' in which one constantly has $K > 0$ or $K < 0$. In such a case, the results that were stated in *a*) or *b*), respectively, are valid. However, if no such neighborhood exists then one can say nothing beyond what was said in § 30.

32. In many cases, from the assumed existence of *one* singular ray g', we could deduce the existence of an infinitude of other singular rays that form a congruence, or of an infinitude of bi-singular rays that form a ruling, or both things at once, in a conveniently small neighborhood of the first one. If one then applies that to the new rays that are obtained, and so on, then one can *continue* the congruence or ruling that was constructed.

It might be that one exhausts all of the singular rays of the complex in that way. One will then obtain a unique congruence of singularities, a ruling of bi-singular rays, or both things at once. However, it might also be that some new singular ray is unattainable (when starting with g'). It will then be the starting ray of a new congruence or ruling of singular rays, and so on.

Finally, there might be *isolated* singular rays – i.e., ones in whose neighborhood there exist no other singular rays.

Loci of singular points.

33. We have nothing to say about the existence of singular points, because that question is subordinate to the question of the existence of singular rays to which they would belong.

In regard to their distribution in space, we examine the more noteworthy cases.

Consider a congruence of singularities C of the complex. The singular points of its rays (which are generally ordinary singular ones) will be foci of those rays for any congruence of the complex that passes through it (cf., the end of § 24), and therefore for C, in particular. Therefore:

The locus of singular points of the ordinary singular rays of a congruence of singularities is one of the two sheets of the focal surface of the congruence.

Therefore, that sheet S_1 will be called a *singularity surface* of the complex, while the other one S_2 will be an *accessory surface.* Both of them are said to be *relative* to the congruence of singularities that is being considered. One notes that one or the other of them can degenerate into a (*singularity* or *accessory*, resp.) *curve*.

Moreover, from the theorem at the end of § 24, one has that *the envelope of the singular planes of the ordinary singular rays of the congruence C coincide with the singularity surface* S_1 .

From the same theorem, one also has that *the singularity surface* S_1 *is the envelope of the focal surfaces of all congruence of the complex that pass through the rays of C.*

34. Now, consider a *special* complex. Any congruence C that is contained in it will be a congruence of singularities and one of the two sheets of the focal surface of C will be a singularity surface. If one therefore considers all of the congruences that are contained in the complex, and each of them is considered to be a suitable sheet of the focal surface, then one can assert that each of those sheets is the envelope of all the other ones. Obviously, that can happen only when all of the sheets coincide; i.e., the complex is composed of the tangents to a surface. That proves a theorem that was stated in § 28.

35. *If there is a bi-singular ray g' in a congruence of singularities C then that ray cannot be elliptic*, since there are no ordinary singular rays in a conveniently small neighborhood of an elliptic ray (§ 31, *a*).

Therefore: *The two singular points Q_1' and Q_2' of g' are certainly real* (distinct or coincident). Q_1' and Q_2' are the foci of g' in C, and one can then find one of them on S_1 and the other on S_2. However, on the other hand, nothing will distinguish one point from the other one, so *Q_1' and Q_2' both belong to S_1, as well as S_2*.

36. For those who find little satisfaction in such inductions, we prove that *a point that moves on S_1 that traverses two convenient curves will pass through the points Q_1' and Q_2'*.

If g' is parabolic then Q_1' and Q_2' will coincide, and our original assertion will be true.

Therefore, suppose that g' is hyperbolic, and recall (§ 31, *b*) that in a neighborhood of g', the congruence C to which it belongs can be imagined to have been divided into two regions C_1 and C_2 that intersect along g', and whose equations are:

$$\alpha_1 M + \beta_1 N = 0, \qquad \alpha_2 M + \beta_2 N = 0,$$

respectively.

Now, suppose that an ordinary singular ray g moves in the region C_1 and tends to g' across a ruling that composed of ordinary singular rays. Consequently, the singular point Q_1 of g will move on the surface of singularities S_1, describing a curve and tending to a limiting position L_1 on g'.

The abscissa r_1 of Q_1 on g is given by the formula (51). However, g is constantly contained in the region C_1, so M and N are proportional to β_1, and $-\alpha_1$, resp., and formula (51) will become:

$$r_1 - r_0 = \frac{(FD'' - GD')\beta_1^2 - (GD - ED'')\beta_1\alpha_1 + (ED' - FD)\alpha_1^2}{\Delta(G\beta_1^2 + 2F\beta_1\alpha_1 + E\alpha_1^2)}. \tag{72}$$

One notes that in this formula r_0 is the abscissa of the mean point of g in the congruence w that passes through g.

In the limit as g tends to g', that formula will give the abscissa r_1 of the point L_1 of g' that is the limit of Q_1. Therefore, if one supposes that the functions E, F, G, D, D', D'', α_1, β_1 in (72) have the values that they assume on g' then the right-hand of (72) is the distance $Q_0' L_1$ from the point L_1 of g' to the mean point Q_0' of g' in the congruence w that passes through it.

In order to simply, we specialize the internal coordinates u, v of the complex, and suppose that it is such that the first fundamental form of the complex will assume the *isothermal* form ([1]):

$$ds'^2 = \lambda\,(du^2 + dv^2).$$

Therefore, take:

$$E = G = \lambda, \qquad F = 0,$$

so (72) will become:

$$Q_0' L_1 = r_1 - r_0 = \frac{D'(\alpha_1^2 - \beta_1^2) + (D'' - D')\,\alpha_1\beta_1}{\Delta\,(\alpha_1^2 + \beta_1^2)},$$

and, from (70), it will assume the even simpler form:

$$Q_0' L_1 = \frac{\alpha_2\beta_1 - \alpha_1\beta_2}{2\Delta}.$$

If one switches the indices 1 and 2 then one will get the formula:

$$Q_0' L_2 = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{2\Delta},$$

which is the distance from the mean point Q_0' of g' to the point L_2 to which the singular point Q_1 of a singular ray g tends when it moves in the congruence C_2 and tend to g'.

It then follows that $Q_0' L_1 = - Q_0' L_2$; i.e., that the points L_1 and L_2 are symmetric with respect to Q_0'.

Moreover, from (71) and then from (42), one will have:

$$Q_0' L_1 \cdot Q_0' L_2 = -\frac{(\alpha_1\beta_2 - \alpha_2\beta_1)^2}{4\Delta^2} = \frac{DD'' - D'^2}{\Delta^2} = K,$$

so

$$(Q_0' L_1)^2 = (Q_0' L_2)^2 = -K.$$

That proves (§ 12) that the points L_1 and L_2 are the foci of g' in the congruence w that passes through it, and therefore (§ 25) coincide with the singular points Q_1' and Q_2' of g'.

37. With that, there should be no doubt about the results of § 35. In summary, one has:

If a (not necessarily elliptic) bi-singular ray belongs to a congruence of singularities then it will be bi-tangent to the singularity surface, as well as the accessory surface that relates to the congruence: In the contrary case, those two surfaces will contact at the two singular points of the ray.

([1]) Which is always possible in an infinitude of ways.

In addition:

The intersection of the tangent planes that are common to the two surfaces at the two singular points of the ray is the ray itself.

That is because those tangent planes are the singular planes of the ray (§ 25).

38. It can often happen that a congruence of singularities contains an entire ruling of bi-singular (non-elliptic) rays. The preceding results are then applicable to each of them. Therefore:

If a congruence of singularities contain a ruling of bi-singular rays then the singularity surface and the accessory surface relative to the congruence will contact along the points of a curve.

The consideration of bi-singular rays puts us into close proximity with the noteworthy theorem of VOSS ([1]):

The singularity surface and the accessory surface of an algebraic complex (§ 27) *contact along all points of a curve.*

39. In general, in the neighborhood of a bi-singular, elliptic ray g', there exists a ruling of bi-singular rays that passes through g' (§ 30), and no other singular rays exist (§ 31, *a*). *The rays of that ruling are certainly elliptical* since K is positive on g', and since it is continuous, it will keep its sign in the neighborhood of g'.

The locus of singular points of a ruling of rays that are all elliptic bi-singular is a pair of conjugate imaginary lines.

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([1]) "Ueber Complexe und Congruenzen," Math. Ann. **9** (1876).

CORRECTIONS TO PAPER *A*.

Page	3	line	8	read Q, instead of G
“	10	“	19	delete *focal*
“	10	“	20	add *the focal planes that are perpendicular to it*
“	16	“	12	delete *or asymptotic*
“	24	“	12	read $-\left\{\begin{matrix}1\,2\\1\end{matrix}\right\}$, instead of $+\left\{\begin{matrix}1\,2\\1\end{matrix}\right\}$
“	34	“	12	read $\frac{\partial}{\partial u}(\rho - \rho_0)$, instead of $\frac{\partial \rho}{\partial u}$
“	34	“	1	read $\frac{\partial}{\partial v}(\rho + \rho_0)$, instead of $\frac{\partial \rho}{\partial v}$

Torino, 14 March 1910.