Projective Line geometry

By

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Foreword

The present book is an attempt to interest a broader circle of younger mathematicians in line geometry. It should not serve as a more or less complete summary of the old and new results in line geometry, but only a selection of the topics that promise to attract some lasting interest due to their simple and "intuitive" content. The main emphasis shall reside in the geometric content, not in the analytical tools; that viewpoint will serve especially for the comparison of models that are constructed from discrete lines.

As the title of the book suggests, it will be essentially projective questions that are treated. This restriction was necessary in order to not exceed the scope of the book; however, it is also factually based, since metric line geometry finds its most natural representation in terms of entirely different analytical tools (viz., Study's dual vectors).

The larger part of the book relates to differential line geometry; the part on algebraic line geometry will present only the must essential basic concepts, and some examples that are occasionally interspersed throughout (e.g., third-degree ruled surface, third and fourth-order space curves, quadratic system of lines, complexes of lines, etc). The projective differential geometry of curves and surfaces will be employed in the treatment of torses and parabolic systems of lines, since line coordinates are better adapted to that problem than point and plane coordinates. In that sense, the book is an extension of volume 22 of this collection, in which E. Salkowski has presented the differential geometry of curves and surfaces.

The analytical tools are the same ones that W. Blaschke applied to Lie's sphere geometry in volume 3 of his *Differentialgeometrie* (Berlin, 1929), which was revised by G. Thomsen. One will find many of the theorems of differential line geometry that will be treated in what follows in that content-rich book (by way of analogy with sphere geometry). The same is true for *Geometria proiettiva differenziale* of G. Fubini-E. Čech (Bologna, 1926), which one might do better to read in the French version (Paris, 1931).

The applications to mechanics (e.g., frameworks, Ball's theorem of screws, stress distributions in membranes) take up a relatively sizable space. Many problems will be dealt with in them, and in the theory of infinitesimal surface bending, that are first posed in their metric formulation, but will nevertheless remarkably lead to projective aspects. Unfortunately, questions of integral geometry must be passed over.

I would like to thank Herren Dr. O. Baier and Prof. Dr. J. Lense, as well as Herrn Dr. Lennertz, for their friendly assistance in the correcting and for many worthwhile suggested improvements.

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Introduction

If one employs a line as the basic element of spatial geometry in place of the point or the plane following the process of **J. Plücker** (1801-1868) then one can speak of *line geometry* ([1]). We will be concerned with that kind of geometry here. We will thus mainly investigate the relationships that remain preserved under the group of projective maps, which implies *projective* line geometry. All considerations will relate to real, three-dimensional projective space – i.e., to the space of Euclidian geometry, when it is extended by the imaginary structures of projective geometry. Analytically, we will then be dealing with the development of *the invariant theory of projective transformations in line coordinates* in a manner that is similar to the way that one presents the motion invariants in Cartesian point coordinates in elementary analytical geometry.

After we have learned about the necessary basic concepts of *algebraic* line geometry, we will turn to *differential* projective line geometry. We will then successively treat the differential geometry of 1, 2, and 3-parameter sets of lines (*families, systems, and complexes of lines,* resp.) in the space of lines, which depends upon four parameters, just as one examines 1 and 2-parameter point-sets (viz., curves and surfaces) in a point space that depends upon three parameters in point geometry. It will be shown in that way that the *projective differential geometry of curves and surfaces* will be handled at the same time.

As **S. Lie** has remarked, line geometry is closely connected with a geometry that employs the sphere as the basic element. Whoever would wish to learn about this very intriguing relationship could confer, e.g., **W. Blaschke-G. Thomsen:** *Differentialgeometrie III* (Berlin, 1929) ([2]).

Before we take up our line-geometric investigations, we shall summarize the necessary basic notions of analytic geometry and the differential geometry of curves and surfaces (or at least, the main terms) in the first two paragraphs. In addition, in order to simplify the analytical tools, we shall assume once and for all that:

All functions will be assumed to be regular; i.e., they can be developed in convergent power series in the domains in question of the independent variables.

([1]) Let some of the following older presentations of line geometry be mentioned:

G. Koenigs, *La géométrie reglée et ses applications*, Paris, 1895.

K. Zindler: *Liniengeometrie I, II*, Leipzig, 1902, 1906.

F. Klein: Volume 1 of his *Gesammelte Abhandlungen*, Berlin, 1921.

Metric differential line geometry is treated by e.g., **W. Blaschke**: *Differentialgeometrie I*, Berlin, 1930; one finds many details about projective differential line geometry in **Fubini-Čech**: *Geometria proiettiva differenziale*, Bologna, 1926 (or better yet, Paris, 1931) and **W. Blaschke and G. Thomsen:** *Differentialgeometrie III*, Berlin, 1929.

Among the large number of individual papers on the subject, let us mention:

G. Sannia: Ann. di mat. (3) **17** (1910).

G. Thomsen: Math. Zeit. (1924) and Hamburger Abhandlungen (1925).

W. Haack: Monatshefte für Math. u. Phys. **36** (1929); **44** (1936); Math. Zeit. **33** (1931); **35** (1932); **40** (1935); **41** (1936).

The algebraic questions were treated by **E. A. Weiss** in the sense of **E. Study** in *Einführung in die Lineiengeometrie und Kinematik*, Leipzig-Berlin, 1935.

([2]) Cf., also, **L. Bieberbach:** *Einführung in die höhere Geometrie*, Leipzig-Berlin, 1933.

§ 1. Basic features of analytic geometry.

1. Vectors. Rectangular point and plane coordinates. We start with three-dimensional *Euclidian space* and assume that the concept of a *vector* is known, as well as the laws of vector addition and subtraction. We shall denote vectors by large German symbols and scalar quantities by small Latin ones. $|\mathfrak{X}|$ means the absolute value (i.e., the "length") of the vector $\mathfrak{X}$.

Linear dependency of vectors:

n vectors $\mathfrak{X}^1, \mathfrak{X}^2, \ldots, \mathfrak{X}^n$ are called *linearly dependent* when at least one equation ([3]):

$$a_1\mathfrak{X}^1 + a_2\mathfrak{X}^2 + \ldots + a_n\mathfrak{X}^n = 0$$

exists, without all coefficients a_i having to vanish.

2 (3, resp.) vectors are linearly dependent if and only if they are parallel to a line (plane, resp.); this is the case especially when one of the vectors is the null vector.

More than three vectors will always be linearly dependent.

Rectangular point coordinates:

A rectangular coordinate system is established by an initial point O and three pair-wise perpendicular vectors $\mathfrak{E}^1, \mathfrak{E}^2, \mathfrak{E}^3$ (viz., *basis vectors*) of the same length that define a *right-handed system*. The rectangular point coordinates x_1, x_2, x_3 of the point P are defined by decomposing the position vector $\mathfrak{X} = \overrightarrow{OP}$ along the three linearly-independent basis vectors:

$$\mathfrak{X} = x_1\mathfrak{E}^1 + x_2\mathfrak{E}^2 + x_3\,\mathfrak{E}^3.$$

Product definitions:

Inner product (scalar!):

$$\mathfrak{X}\mathfrak{Y} = x_1\,y_1 + x_2\,y_2 + x_3\,y_3 = \mathfrak{Y}\mathfrak{X};$$

in particular:

$$\mathfrak{X}\mathfrak{X} = x_1x_1 + x_2\,x_2 + x_3\,x_3 = |\,\mathfrak{X}\,|^2.$$

Outer product (vector!):

$$\mathfrak{X}\times\mathfrak{Y} = \begin{vmatrix} \mathfrak{E}^1 & \mathfrak{E}^2 & \mathfrak{E}^3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = -\,\mathfrak{Y}\times\mathfrak{X}\ .$$

The vectors $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{X}\times\mathfrak{Y}$ define a right-hand system.

([3]) We sometimes employ upper indices 1, 2, …, n and sometimes lower ones. The basis for that will first be given when we introduce the concept of a tensor (§ 20).

Determinant product (scalar!):

$$<\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}> = \mathfrak{X}\,(\mathfrak{Y} \times \mathfrak{Z}) = \mathfrak{Y}\,(\mathfrak{Z} \times \mathfrak{X}) = \mathfrak{Z}\,(\mathfrak{X} \times \mathfrak{Y}) = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix};$$

when $<\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}> > 0$ (< 0, resp.), the vectors $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ will define a right-handed (left-handed, resp.) system.

Identities:

$$(\mathfrak{X} \times \mathfrak{Y}) \times \mathfrak{Z} = (\mathfrak{X}\mathfrak{Z})\,\mathfrak{Y} - (\mathfrak{Y}\mathfrak{Z})\,\mathfrak{X}\,;$$

(1)

$$(\mathfrak{X} \times \mathfrak{Y})\,(\mathfrak{Z} \times \mathfrak{T}) = (\mathfrak{X}\mathfrak{Z})\,(\mathfrak{Y}\mathfrak{T}) - (\mathfrak{X}\mathfrak{T})\,(\mathfrak{Y}\mathfrak{Z}).$$

Vanishing products:

$\mathfrak{X}\mathfrak{Z} = 0$: The vectors $\mathfrak{X}$, $\mathfrak{Y}$ are perpendicular.

$$\left.\begin{array}{ll} \mathfrak{X}\times\mathfrak{Y} = 0: & \text{The vectors } \mathfrak{X}, \mathfrak{Y} \\ <\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}> = 0: & \text{The vectors } \mathfrak{X}, \mathfrak{Y}, \mathfrak{Z} \end{array}\right\} \text{are linearly dependent.}$$

Rectangular plane coordinates:

The equation of a plane that does not go through *O* can be brought into the form:

$$\mathfrak{W}\mathfrak{X} + 1 = 0.$$

The uniquely-determined vector $\mathfrak{W}$ is perpendicular to the plane. The *rectangular plane coordinates* w^1, w^2, w^3 are defined by the decomposition of $\mathfrak{W}$ along $\mathfrak{E}^1$, $\mathfrak{E}^2$, $\mathfrak{E}^3$. The plane coordinates of a plane that includes the point *O*:

$$\mathfrak{W}\mathfrak{X} = 0$$

are determined only up to an arbitrary proportionality factor; $\mathfrak{W}$ is any altitude vector over the plane.

2. Projective point and plane coordinates. Euclidian space is extended to *projective space* by the addition of the imaginary elements: From now on, "intersect" shall mean "intersect," as well as "be parallel," and "cone" shall mean "cone" with a real vertex, as well as "cylinder." We will assume that the definition of *homogeneous projective point coordinates* x_i and *plane coordinates* w^i (i = 1, 2, 3, 4) in terms of double ratios from the coordinate tetrahedron by means of a *unit point* and a *unit plane* are known. The united position of a point and a plane is expressed by:

$$x_i\, w^i = 0,$$

in which the summation sign $\sum_{i=1}^{4}$ is implied. We agree, once and for all, that:

Any index that appears twice in a product – once above and once below – is to be summed over.

If the imaginary plane is taken to be the plane $x_4 = 0$ then the ratios x_i / x_4 (w^i / w^4, resp.) ($i = 1, 2, 3$) will be *affine coordinates.* In addition, if the three coordinate axes 41, 42, 43 (Fig. 2) are pair-wise perpendicular and the unit point determines equal units of length on them then the affine coordinate system will specialize to a *rectangular* one. In that case, we shall call the x_k and w^k ($k = 1, 2, 3, 4$) *homogeneous affine (homogeneous rectangular, resp.) coordinates.*

In what follows, we shall call the point with the projective coordinates x_i "the point x," for brevity, and the plane with the coordinates w^i, "the plane w." The *double ratio d* of four points p, q, r, s is given by:

$$d \equiv (p, q, r, s) = \frac{(pr)(qs)}{(qr)(ps)}, \tag{2}$$

with:

$$(pr) = p_i\, r_k - p_k\, r_i, \qquad (qs) = q_i\, s_k - q_k\, s_i, \qquad \text{etc.}$$

3. Projective maps. One distinguishes between *a*) *collinear* and *b*) *correlative or dual* projective maps [*a*) *collineations* and *b*) *correlations*, resp.]. They are given by:

$$\tilde{x}_i = \alpha_1^1 x_1 + \ldots + \alpha_1^4 x_4 \equiv \alpha_1^k x_k \tag{3a}$$

or

$$\tilde{w}^i = \beta^{i\,1} x_1 + \ldots + \beta^{i\,4} x_4 \equiv \beta^{ik} x_k, \tag{3b}$$

resp., in projective point (plane, resp.) coordinates, with non-vanishing coefficient determinants α (β, resp.).

Any linear form in the x_i will be transformed into a linear form in the $\tilde{x}_k$ [$\tilde{w}^k$, resp.] under (3a) [(3*b*), resp.]. As a result, the points x of a plane w will go to the points $\tilde{x}$ of a plane $\tilde{w}$ (the planes $\tilde{w}$ through a point $\tilde{x}$, resp.). One can prescribe that:

$$w^k\, x_k = \tilde{w}^i \tilde{x}_i$$

under collineations, as well as correlations, and then obtain:

$$w^k\, x_k = \alpha_i^k \tilde{w}^i x_k \qquad (w^k\, x_k = \beta^{ik} \tilde{x}_i x_k, \text{ resp.})$$

upon substituting that into (3*a*) [(3*b*), resp.]. One will get the representations:

(4a) $$w^k = \alpha_1^k \tilde{w}^1 + \ldots + \alpha_4^k \tilde{w}^4 \equiv \alpha_i^k \tilde{w}^i$$

or

(4b) $$w^k = \beta^{1k} \tilde{x}_1 + \ldots + \beta^{4k} \tilde{x}_4 \equiv \beta^{ik} \tilde{x}_i ,$$

which are equivalent to (3a), [(3b), resp.], by identifying the coefficients of x_k on both sides of the equation. There is an invertible one-to-one correspondence:

under *a*):	point ↔ point,	plane ↔ plane,
under *b*):	point ↔ plane,	plane ↔ point,
under *a*) and *b*):	line ↔ line; the united position of points, lines, and planes remain preserved.	

There is precisely one collineation (correlation, resp.) that transforms five points, no four of which lie in a plane, into five other points of that kind (five planes, no four of which go through a point).

The so-called *fundamental theorem of projective geometry* (4) states that:

Any map of projective space that takes points to points and rectilinear sequences of points to rectilinear sequences of points in an invertible, one-to-one correspondence is a collineation.

Under any collineation, there is at least one point that will correspond to itself and at least one plane that will correspond to itself (viz., a *fixed point, fixed plane*, resp.). In general (5) there exist four fixed points and four fixed planes that define the vertices and faces of a tetrahedron.

Special projective maps:

a) **Affinity:** One sets $\alpha_4^1 = \alpha_4^2 = \alpha_4^3 = 0$ in (3a). The imaginary plane $x_4 = 0$ is a fixed plane; parallelism will remain preserved.

b) **Polarity:** One sets $\beta^{ik} = \begin{cases} 0 \\ 1 \end{cases}$ for $\begin{cases} i \neq k \\ i = k \end{cases}$ in (3b), $\tilde{w}^i = x_i$ (i = 1, 2, 3, 4). By choosing homogeneous, rectangular coordinates, any point x will be associated with the plane $\tilde{w}$ that arises by reflecting the polar plane of the point x with respect to the unit sphere around O through O.

Equations (3a) and (4a) can be regarded as *coordinate transformations;* we will often make use of that reinterpretation.

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(4) One can find the proof in, e.g., **W. Blaschke-G. Thomsen**, *Differentialgeometrie III*, § 50.

(5) Cf., **R. Baldus**: "Klassification der ebenen und räumlichen Kollineationen," Münchener Berichte, 1928.

4. Point-pairs, curves, and second-order curves. *Point-pairs, curves,* and *second-order surfaces* will be defined in 1, 2, and 3-dimensional projective space, resp., by:

$$c^{ik}\,x_i\,x_k = 0,$$

with real c^{ik}. Let $r \geq 1$ be the rank of the coefficient matrix $\| c^{ik} \|$. The form $c^{ik}\,x_i\,x_k$ can always be transformed into a quadratic form:

$$\tilde{c}^{11}(\tilde{x}_1)^2 + \ldots + \tilde{c}^{rr}(\tilde{x}_r)^2$$

with precisely r coefficients $\tilde{c}^{hh} = \pm 1$. The number of positive coefficients $\tilde{c}^{hh}$ in this, along with the number of negative ones, and thus, the difference s (viz., the *signature*) will be invariant under real coordinate transformations (viz., the *theorem of inertia for quadratic forms*).

Projective classification of point-pairs:

$$r = 2: \quad \begin{cases} s = \ \ 0: & \text{two points,} \\ s = \pm 2: & \text{no point (i.e., two "imaginary" points),} \end{cases}$$

$$r = 1: \quad s = \ \ 1: \ \text{doubly-counted point.}$$

Projective classification of second-order curves (i.e., conic sections):

$$r = 3: \quad \begin{cases} s = \pm 1: & \text{ellipse, hyperbola, parabola,} \\ s = \pm 3: & \text{"null" second-order curve,} \end{cases}$$

$$r = 2: \quad \begin{cases} s = \ \ 0: & \text{line-pairs,} \\ s = \pm 2: & \text{single point (i.e., "imaginary line-pair"),} \end{cases}$$

$$r = 1: \quad s = \ \ 1: \ \text{doubly-counted line.}$$

Projective classification of second-order surfaces:

$$r = 4: \quad \begin{cases} s = \ \ 0: & \text{ruled surfaces (rectilinear hyperboloid and paraboloid),} \\ s = \pm 2: & \text{oval surfaces (ellipsoid, non-rectilinear hyperboloid} \\ & \text{and paraboloid),} \\ s = \pm 4: & \text{"null" second-order surface,} \end{cases}$$

$$r = 3: \quad \begin{cases} s = \pm 1: & \text{cone,} \\ s = \pm 3: & \text{single point (= "null" cone),} \end{cases}$$

$$r = 2: \qquad \begin{cases} s = \ \ 0: & \text{plane-pair,} \\ s = \pm 2: & \text{single line (= "imaginary" plane-pair),} \end{cases}$$

$$r = 1: \qquad s = \ \ 1: \text{doubly-counted plane.}$$

For the sake of brevity, from now on, we shall refer to the ruled surfaces $r = 4$, $s = 0$ as *hyperboloids*, per se; the conic sections with $r < 3$ and the second-order surfaces with $r < 4$ shall be called *singular*.

§ 2. Basic features of the theory of curves and surfaces.

5. Curves and strips. Let a *curve* be given with the parametric representation $\mathfrak{X} = \mathfrak{X}(u)$. The *arc-length* $s = s(u)$ (i.e., the *natural parameter*) is characterized by the fact that $\mathfrak{X}'\mathfrak{X}' = 1$, in which the prime means the derivative d / ds.

Any non-singular point of the curve (i.e., $\mathfrak{X}' \times \mathfrak{X}'' \neq 0$) is associated with a right-hand system of unit vectors $\mathfrak{T}$, $\mathfrak{H}$, $\mathfrak{B}$:

Tangent: $\mathfrak{T} = \mathfrak{X}'$,

Principal normal: $\mathfrak{H}$, which is perpendicular to the plane $\{\mathfrak{T}\mathfrak{B}\}$,

Binormal: $\mathfrak{B}$, which is parallel in the same sense to the vector $\mathfrak{X}' \times \mathfrak{X}''$.

The plane that is spanned by $\mathfrak{T}$ and $\mathfrak{H}$ (with the normal vector $\mathfrak{X}' \times \mathfrak{X}''$) is called the *osculating plane*.

The main problem in the metric theory of curves consists of the presentation and geometric interpretation of the invariants of motion of the vectors $\mathfrak{X}'$, $\mathfrak{X}''$, $\mathfrak{X}'''$, etc., up to derivatives of higher-order. For that, one starts with the *differential equations* (viz., the *Frenet formulas*), which express $\mathfrak{T}'$, $\mathfrak{H}'$, $\mathfrak{B}'$ as linear combinations of $\mathfrak{T}$, $\mathfrak{H}$, $\mathfrak{B}$; merely two functions of s enter in as coefficients, which one calls the *curvature* and *torsion*.

Fundamental theorem of the theory of curves: *A curve is established uniquely, up to a motion, by being given the curvature $k(s) \geq 0$ and torsion $w(s)$ as arbitrary functions of the arc-length s* ([1]).

A *strip* is the structure that is dual to a curve, and thus, a one-parameter set of planes. The strips that are dual to planar curves consists of the contact planes of a cone or the planes of a pencil; the strips that are dual to non-planar curves will be generated by the osculating planes of a non-planar curve, and will have the tangent surfaces to these curves as enveloping surfaces. We briefly call the lines of the enveloping cone (the lines of the enveloping tangent surface, resp.) the *generators* of the strip. The tangent surfaces

([1]) One must make a suitable convention regarding the sign of w.

(cones, resp.) are are also called *developable surfaces;* they are the only surfaces that can be rolled into a plane.

6. Surfaces. Let a *surface* be given in the parametric representation $\mathfrak{X} = \mathfrak{X}(u^1, u^2)$. Two families of curves in the surface in a certain region will define a *net of curves* when two curves of that family do not meet, but two curves of different families have precisely one point of intersection (which is not a point of contact) in common. We assume that the parameter curves u^1 = const., u^2 = const. define a net of curves. One then has $\mathfrak{X}_{u^1} \times \mathfrak{X}_{u^2} \neq 0$, and every point of the surface is associated with the *contact plane:*

$$<\mathfrak{N} - \mathfrak{X},\ \mathfrak{X}_{u^1},\ \mathfrak{X}_{u^2} > = 0$$

that is spanned by $\mathfrak{X}_{u^1}$ and $\mathfrak{X}_{u^2}$. $\mathfrak{X}$ is the position vector of the point of the surface, and $\mathfrak{Y}$ is the position vector of an arbitrary point of the contact plane, while the symbols u^1 and u^2 mean partial differentiation.

A curve $\mathfrak{X} = \mathfrak{X}(u^1(t), u^2(t))$ on the surface will be established by $u^1 = u^1(t)$, $u^2 = u^2(t)$, and a direction of advance will be established by the ratio $\dot{u}^1 : \dot{u}^2$ (in which the dot means differentiation with respect to t), and therefore, a tangent to the surface, as well. The tangents to the surface that are given by the quadratic equation:

$$\alpha \dot{u}^1\dot{u}^1 + 2\beta \dot{u}^1\dot{u}^2 + \gamma \dot{u}^2\dot{u}^2 = 0 \qquad (\beta^2 - \alpha\gamma > 0)$$

will be harmonically separated by every pair I, II of tangents that is given by:

$$\alpha \dot{u}_{\mathrm{I}}^1 \dot{u}_{\mathrm{II}}^1 + \beta(\dot{u}_{\mathrm{I}}^1 \dot{u}_{\mathrm{II}}^2 + \dot{u}_{\mathrm{II}}^1 \dot{u}_{\mathrm{I}}^2) + \gamma \dot{u}_{\mathrm{I}}^2 \dot{u}_{\mathrm{II}}^2 = 0. \tag{5}$$

In particular, the tangent-pair:

$$\alpha \dot{u}^1\dot{u}^1 + \gamma \dot{u}^2\dot{u}^2 = 0$$

is harmonic to the tangent-pair $\dot{u}_{\mathrm{I}}^1 = 0 \mid \dot{u}_{\mathrm{I}}^2 \neq 0$ and $\dot{u}_{\mathrm{II}}^1 \neq 0 \mid \dot{u}_{\mathrm{II}}^2 = 0$ of the parameter curves u^1 = const., u^2 = const.

The contact planes of the surface along a curve κ on the surface define a strip (viz., a *contact strip*), which we will assume does not degenerate to a fixed contact plane. One calls the generators of the contact strip the *surface tangents that are conjugate* to the tangents to the contact curve κ.

A net of curves is called a *conjugate net* when the generators of the contact strip along one family of curves are tangents to the family of curves, and conversely. In other words: *The tangents to the one family of curves along a curve κ of the other family define a cone or the tangent surface to a curve γ* (Fig. 1, on the right). A conjugate net of curves is given by:

$$L\, \dot{u}_{\mathrm{I}}^1 \dot{u}_{\mathrm{II}}^1 + M(\dot{u}_{\mathrm{I}}^1 \dot{u}_{\mathrm{II}}^2 + \dot{u}_{\mathrm{II}}^1 \dot{u}_{\mathrm{I}}^2) + N\, \dot{u}_{\mathrm{I}}^2 \dot{u}_{\mathrm{II}}^2 = 0, \tag{6}$$

with

(7) $$L = <\mathfrak{X}_{u^1}, \mathfrak{X}_{u^2}, \mathfrak{X}_{u^1u^1}>, \quad M = <\mathfrak{X}_{u^1}, \mathfrak{X}_{u^2}, \mathfrak{X}_{u^1u^2}>, \quad N = <\mathfrak{X}_{u^1}, \mathfrak{X}_{u^2}, \mathfrak{X}_{u^2u^2}>.$$

In particular, the parameter net is characterized as a conjugate net by $M = 0$. For developable surfaces (no. **5**), any family of curves will be conjugate to the generators, while in all other cases there will exist precisely one family of curves that is conjugate to any family of curves.

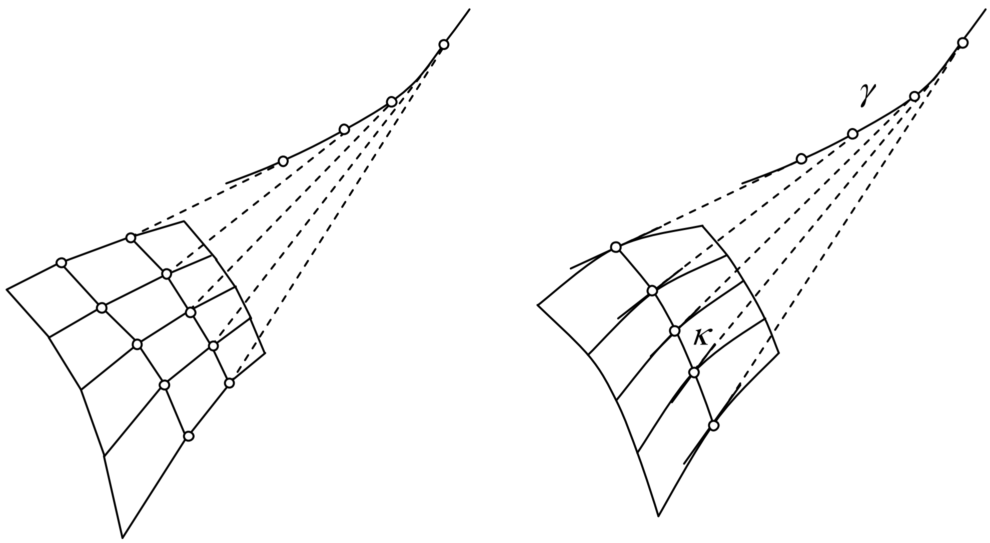

Figure 1.

A contact strip is called a *principal tangent strip* and its contact curve, a *principal tangent curve*, when the generators of the strip coincide with the tangents to the contact curve. In order for that to be true, it is necessary and sufficient that the osculating planes of the contact curve should be, at the same time, contact planes to the surface. We call the tangents to the principal tangent curves the *principal tangents* to the surface. A line that lies on the surface will always be a principal tangent curve.

A net of curves is called a *principal tangent net* when the curves of the two families are principal tangent curves. The principal tangent curves to a surface are given by:

(8) $$L\,\dot{u}^1\dot{u}^1 + 2M\,\dot{u}^1\dot{u}^2 + N\,\dot{u}^2\dot{u}^2 = 0.$$

The parameter curve net is characterized as the principal tangent net by $L = M = 0$. A non-planar surface with $LN - M^2 < 0$ ($= 0$, > 0, resp.) contains 2 (1, 0, resp.) families of principal tangent curves (viz., *negatively curves, developable*, and *positively-curved surfaces,* resp.) For negative curvature, the principal tangent that goes through a point of the surface will be harmonically separated by any pair of conjugate tangents.

One observes that the concepts of conjugate surface tangents and principal tangents are *collinearly invariant*, as well as *correlatively invariant.*

Just like the theory of curves, the theory of surfaces is based upon *differential equations.* Here, two kinds of differences will appear:

1. It is not possible to replace the parameters u^1, u^2 with geometrically-distinguished "natural parameters" that would be analogous to the arc-length s in the theory of curves.

2. The coefficients in the differential equations are not independent of each other, but are coupled by certain *integrability conditions*. We shall content ourselves now with these brief remarks, since will have to come back to the analogous situation in line geometry more thoroughly later on.

7. Models. In order to make things more intuitive, we shall regard *paths of line segments* as discrete-geometric models of curves and *rectangle nets* (curve nets) on surfaces, which are composed of planar or non-planar rectangles as on a chessboard (Fig. 1, on the left).

Analogous concepts:

Discrete geometry	**Differential geometry**
path of line segments	curve
plane through two successive segments	osculating plane
rectangle nets (i.e., two families of line segment paths)	nets of curves (i.e., two families of curves)
rectangle nets with planar vertices: The lines of the one family of line-segment paths cut the second family at a fixed point or intersect successively (Fig. 1, left).	*conjugate nets of curves:* The tangents to one family of curves along a curve of the second family define a cone or a tangent surface (Fig. 1, right).
rectangle nets with planar quadrilaterals: The four segments that emanate from an (internal) node lie in the same plane.	*principal tangent nets:* The osculating planes of the curves are, at the same time, contact planes of the surface.

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CHAPTER I

Basic concepts of algebraic line geometry.

§ 3. Line coordinates.

8. Definition of line coordinates. A line is determined by two distinct points x, x' with the homogeneous point coordinates x_i, x_i' (i = 1, 2, 3, 4). We define the six not-all-vanishing *homogeneous projective line coordinates* ([1]):

$$(9)\qquad \begin{aligned} \sigma p_1 &= x_1' x_4 - x_4' x_1, & \sigma p_2 &= x_2' x_4 - x_4' x_2, & \sigma p_3 &= x_3' x_4 - x_4' x_3, \\ \sigma p_4 &= x_2 x_3' - x_3 x_2', & \sigma p_5 &= x_3 x_1' - x_1 x_3', & \sigma p_6 &= x_1 x_2' - x_2 x_1' \end{aligned}$$

from the matrix ([2]) $\| x, x' \|$ of rank 2. $\sigma \neq 0$ is an arbitrary proportionality factor in this. One will get the identity:

$$(10)\qquad p_1 p_4 + p_2 p_5 + p_3 p_6 = 0$$

from the **Laplace** development of the determinant ([1]) $| x, x', x, x' | \equiv 0$ of the sub-determinants of the first two rows.

The six homogeneous line coordinates are not independent of each other then, but are coupled by the auxiliary condition (10). That corresponds to the fact the lines in projective space define a 4-parameter set.

Homogeneous line coordinates will remain unchanged when one multiplies the x_i (x_i', resp.) by an arbitrary number $\rho \neq 0$ ($\rho' \neq 0$, resp.) or when one replaces the points x, x' with any two distinct points y, y' of the line. It will then follow from:

$$\begin{aligned} y_i &= \alpha x_i + \beta x_i', \\ y_k' &= \alpha' x_i + \beta' x_i', \qquad \text{with} \quad d \equiv \begin{vmatrix} \alpha & \alpha' \\ \beta & \beta' \end{vmatrix} \neq 0, \end{aligned}$$

that:

$$y_i' y_k - y_k' y_i = d\,(x_i' x_k - x_k' x_i)\,.$$

Line coordinates of the edges of the coordinate tetrahedron: In Fig. 2, the vertices $w^i = 0$ of the coordinate tetrahedron are denoted by i; those of the coordinate plane that is opposite to the vertex i will have the equation $x_i = 0$. The edges of the tetrahedron will each have five vanishing line coordinates; the sixth non-vanishing line coordinate is given in the figure.

([1]) Line coordinates were introduced by **J. Plücker**.

([2]) The matrix (determinant, resp.) whose rows are the coordinates of the points x, y, … will be denoted by $\| x, y, \dots \|$ ($| x, y, \dots |$, resp.).

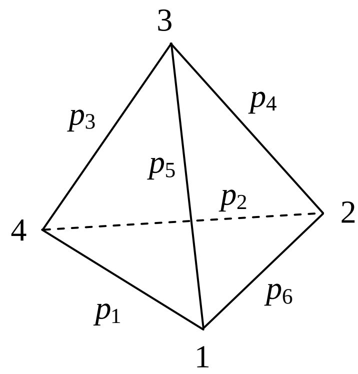

Figure 2.

9. Arrow coordinates. We interpret the x_i and x'_k as homogeneous rectangular coordinates. If x, x' are real points (i.e., $x_4 \neq 0$, $x'_4 \neq 0$) with the position vectors $\mathfrak{X}$, $\mathfrak{X}'$ (Fig. 3) then, from (9), one will have:

$$\left.\begin{array}{l} \sigma p_1,\ \sigma p_2,\ \sigma p_3 \\ \sigma p_4,\ \sigma p_5,\ \sigma p_6 \end{array}\right\} = \text{coordinates of the vector} \left\{\begin{array}{l} \mathfrak{P} = \mathfrak{X}' - \mathfrak{X}, \\ \bar{\mathfrak{P}} = \mathfrak{X} \times \mathfrak{X}' = \mathfrak{X} \times \mathfrak{P}. \end{array}\right.$$

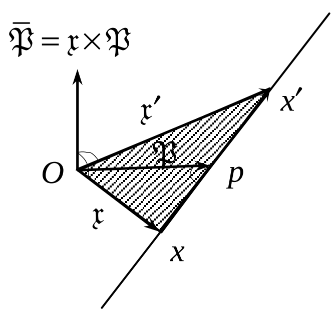

Figure 3.

Equation (10) will then go to the condition for perpendicularity:

(11) $$\mathfrak{P}\bar{\mathfrak{P}} = 0.$$

We refer to the perpendicular vector-pair $\mathfrak{P}$, $\bar{\mathfrak{P}}$ ($\mathfrak{P}\bar{\mathfrak{P}} = 0$, $\mathfrak{P} \neq 0$) as an *arrow* (viz., a *real arrow*) and the six rectangular coordinates of $\mathfrak{P}$, $\bar{\mathfrak{P}}$ as (inhomogeneous) *arrow coordinates.* An oriented, real point-pair x, x' determines the arrow $\mathfrak{P}$, $\bar{\mathfrak{P}}$ uniquely; two oriented point-pairs determine the same arrow if and only if they lie on the same line and can be made to coincide with each other by displacement. Any arrow $\mathfrak{P}$, $\bar{\mathfrak{P}}$ can be represented by oriented point-pairs x, x':

$$\mathfrak{P} = \text{vector } \overrightarrow{xx'},$$

$$\mathfrak{B} = \frac{\mathfrak{P}\times\bar{\mathfrak{P}}}{\mathfrak{P}\bar{\mathfrak{P}}} = \text{altitude vector from } O \text{ to the line } xx'.$$

(See Fig. 3) Since $\mathfrak{B}$ is independent of s, proportional arrows $\mathfrak{P}$, $\bar{\mathfrak{P}}$ and $\sigma\mathfrak{P}$, $\sigma\bar{\mathfrak{P}}$ will produce point-pairs on the same line, and conversely.

For an imaginary line, from (9), one will have $p_1 = p_2 = p_3 = 0$; p_4, p_5, p_6 will then be the coordinates of a vector $\bar{\mathfrak{P}}$ that that is perpendicular to the parallel planes that go through the imaginary line. Thus, an imaginary line will be established by any "*imaginary*" *arrow* $\mathfrak{P} = 0$, $\bar{\mathfrak{P}} \neq 0$; two imaginary arrows will determine the same imaginary line if and only if their vectors $\bar{\mathfrak{P}}$ are proportional.

10. Establishing a line by line coordinates. In no. **8**, we showed that:

A given (real or imaginary) line determines a sextuple of line coordinates p_ρ *that do not all vanish and satisfy equation* (10) *uniquely, up to an arbitrary proportionality factor* $\sigma \neq 0$ ([1]).

We shall now prove the converse:

Any sextuple of numbers p_ρ *that do not all vanish and satisfy equation* (10) *will determine a line with the line coordinates* p_ρ *uniquely.*

Proof: From no. **9**, we first interpret the p_ρ as arrow coordinates. A real ($\mathfrak{P} \neq 0$) or imaginary line ($\mathfrak{P} = 0$) is then established uniquely by $\mathfrak{P}$ ($\mathfrak{P}\bar{\mathfrak{P}} = 0$). We can then once more regard the p_ρ as projective line coordinates relative to an arbitrary coordinate tetrahedron.

11. Representation of line coordinates by plane coordinates. A line will also be determined by two planes w, w' that contain it and have the homogeneous projective coordinates w^i, w'^i (i = 1, 2, 3, 4). The line coordinates that were defined in (9) can be expressed as follows by the plane coordinates w^i, w'^i :

$$(12)\qquad \begin{aligned} \sigma' p_1 &= w^2 w'^3 - w^3 w'^2, &\quad \sigma' p_2 &= w^3 w'^1 - w^1 w'^3, &\quad \sigma' p_3 &= w^1 w'^2 - w^2 w'^1, \\ \sigma' p_4 &= w'^1 w^4 - w'^4 w^1, &\quad \sigma' p_5 &= w'^2 w^4 - w'^2 w^4, &\quad \sigma' p_6 &= w'^3 w^4 - w'^4 w^3. \end{aligned}$$

Once more, $\sigma' \neq 0$ is an arbitrary proportionality factor.

([1]) We shall denote the indices of the point and plane coordinates, which run from 1 to 4, by Latin characters and the indices of the line and complex coordinates (no. **21**), which run from 1 to 6, by Greek ones.

Proof: The fact that the right-hand sides of equations (9) and (12) are proportional to each other follows from the equations of united position of the points x, x' on the line with the planes w, w' through the line:

$$w^i x_i = w^i x'_i = w'^i x_i = w'^i x'_i = 0. \tag{13}$$

Proof of that: We interpret the coordinates rectangularly by no. **9** and distinguish three cases:

a) Real line, not through O: We can assume that x, x' are real points, and w, w' are planes that do not go through O. When one goes from x, x' to new points y, y' (cf., no. **8**) or from w, w' to new planes v, v', the right-hand side of (9) [(12), resp.] will change by only a common factor. Equations (13) will then read:

$$\mathfrak{W}\mathfrak{X} + 1 = \mathfrak{W}\mathfrak{X}' + 1 = \mathfrak{W}'\mathfrak{X} + 1 = \mathfrak{W}'\mathfrak{X}' + 1 = 0$$

in vector notation (cf., no. **1**). It follows from this that:

$$\mathfrak{W}\,(\mathfrak{X}' - \mathfrak{X}) = \mathfrak{W}'\,(\mathfrak{X}' - \mathfrak{X}) = \mathfrak{X}\,(\mathfrak{W}' - \mathfrak{W}) = \mathfrak{X}'\,(\mathfrak{W}' - \mathfrak{W}) = 0,$$

so

$$\mathfrak{X}' - \mathfrak{X} = \lambda\, \mathfrak{W} \times \mathfrak{W}', \quad \mathfrak{X}' \times \mathfrak{X} = \nu\,(\mathfrak{W}' - \mathfrak{W}).$$

From (1), when one substitutes the first of these equations into the second one, one will get:

$$\nu\,(\mathfrak{W}' - \mathfrak{W}) = \lambda\, \mathfrak{X} \times (\mathfrak{W} \times \mathfrak{W}') = -\lambda\,(\mathfrak{W}\mathfrak{X})\,\mathfrak{W}' + \lambda\,(\mathfrak{W}'\mathfrak{X})\,\mathfrak{W} = \lambda\,(\mathfrak{W}' - \mathfrak{W}),$$

so $\nu = \lambda$; i.e., the 2×3 coordinates of $\mathfrak{X}' - \mathfrak{X}$, $\mathfrak{X} \times \mathfrak{X}'$ and $\mathfrak{W} \times \mathfrak{W}'$, $\mathfrak{W}' - \mathfrak{W}$ will be proportional to each other.

b) Real lines through O: Due to the fact that $w^4 = w'^4 = 0$, the fourth terms in equations (13) will vanish, and one will straightaway get:

$$x_1 : x_2 : x_3 = (w^2\, w'^3 - w^3\, w'^2) : (w^3\, w'^1 - w^1\, w'^3) : (w^1\, w'^2 - w^2\, w'^1)\ .$$

In addition:

$$x_1 : x_2 : x_3 = x'_1 : x'_2 : x'_3 = (x'_1 x_4 - x'_4 x_1) : (x'_2 x_4 - x'_4 x_2) : (x'_3 x_4 - x'_4 x_3)\,,$$

so the expressions for p_1, p_2, p_3 in (9) and (12) will be proportional to each other. Finally, $p_4 = p_5 = p_6 = 0$ in (9), since $x_1 : x_2 : x_3 = x'_1 : x'_2 : x'_3$, and in (12), since $w^4 = w'^4$.

c) Imaginary line: Since $x_4 = x'_4 = 0$, the proof is dual to that of *b*).

12. Provisional concept of a six-vector. We combine the six line coordinates p_ρ into the *six-vector* $\mathfrak{p}$, and likewise combine the sextuple of numbers $\lambda\, p_\rho + \mu\, q_\rho$ that is defined by the line coordinates p_ρ, q_ρ into the six-vector $\lambda\,\mathfrak{p} + \mu\,\mathfrak{q}$.

Furthermore, we define the *scalar product* by:

$$\mathfrak{pq} = p_1\, q_4 + p_2\, q_5 + p_3\, q_6 + p_4\, q_1 + p_5\, q_2 + p_6\, q_3\ ;$$

in particular:

$$\mathfrak{pp} = 2\,(p_1\, p_4 + p_2\, p_5 + p_3\, p_6).$$

One then has the rules of calculation:

$$\mathfrak{p} + \mathfrak{q} = \mathfrak{q} + \mathfrak{p}, \qquad \mathfrak{pq} = \mathfrak{qp},$$
$$\lambda\,(\mathfrak{p} + \mathfrak{q}) = \lambda\,\mathfrak{p} + \lambda\,\mathfrak{q}, \qquad \mathfrak{p}\,(\mathfrak{q} + \mathfrak{r}) = \mathfrak{pq} + \mathfrak{pr}.$$

For the six-vector $\mathfrak{p}$ that is defined by the line coordinates p_ρ, one has:

$$\mathfrak{pp} = 0.$$

Later (no. **15**), we will generalize the concept of six-vector that we have introduced provisionally here, and learn about the *singular* six-vectors ($\mathfrak{pp} = 0$), as well as the *non-singular* six-vectors ($\mathfrak{pp} \neq 0$); we denote the *null six-vector* by 0. *The linear dependency* of six-vectors is defined just as it is for ordinary vectors (no. **1**).

Seven six-vectors are always linearly-dependent; any six-vector $\mathfrak{q}$ *can be a linear combination of six linearly-independent six-vectors* $\mathfrak{p}^{\mathrm{I}}, \mathfrak{p}^{\mathrm{II}}, \ldots, \mathfrak{p}^{\mathrm{VI}}$.

Proof: The six homogeneous linear equations for the λ_i that are summarized by:

$$\lambda_1\mathfrak{p}^{\mathrm{I}} + \lambda_2\mathfrak{p}^{\mathrm{II}} + \ldots + \lambda_7\,\mathfrak{p}^{\mathrm{VII}} = 0$$

always possess non-trivial systems of solutions. If $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{VI}}$ are linearly-independent then one must have $\lambda_7 \neq 0$, and $\mathfrak{p}^{\mathrm{VII}}$ will then take the from of a linear combination of the $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{VI}}$.

The contents of no. **10** can now be reformulated when one speaks of singular six-vectors, instead of coordinate sextuples: *Any singular six-vector* $\mathfrak{p} \neq 0$ *determines a line. Two singular six-vectors* $\mathfrak{p} \neq 0$, $\mathfrak{q} \neq 0$ *determine the same line if and only if they are linearly dependent.* We will call the line whose line coordinates define the six-vector $\mathfrak{p}$ "the line $\mathfrak{p}$," for brevity.

Upon establishing a rectangular coordinate system (no. **9**), we shall also denote the six-vector that is defined by $\mathfrak{P}$, $\overline{\mathfrak{P}}$ with:

$$\mathfrak{p} = \{\mathfrak{P} \mid \bar{\mathfrak{P}}\}.$$

13. United position of lines, points, and planes.

Line $\mathfrak{p}$ and point z:

The line $\mathfrak{p}$ contains the point z if and only if the four equations are satisfied:

$$(14)\quad \begin{aligned} 0 &= \ * \qquad - z_1 p_4 - z_2 p_5 - z_3 p_6 , \\ 0 &= z_1 p_4 \quad * \quad - z_2 p_3 + z_3 p_2 , \\ 0 &= z_4 p_5 + z_1 p_3 \quad * \quad - z_3 p_1 , \\ 0 &= z_4 p_6 - z_1 p_2 + z_2 p_1 \quad * \quad . \end{aligned}$$

Proof: The four coordinates of the plane that goes through the three points x, x', z are proportional to the four three-rowed determinants of the matrix $\| x, x', z \|$. They will all be zero if and only if the plane is undetermined; i.e., if z lies in the line xx'. (14) follows by setting the four determinants equal to zero and substituting in (9).

The coefficient matrix of the system of equations (14) for z_i has rank 2.

Line $\mathfrak{p}$ and plane w:

Dual to (14), in order for the line $\mathfrak{p}$ and the plane w to be in united position, it is necessary and sufficient that:

$$(15)\quad \begin{aligned} 0 &= \ * \qquad - w^1 p_1 - w^2 p_2 - w^3 p_3 , \\ 0 &= w^4 p_1 \quad * \quad - w^2 p_6 + w^3 p_5 , \\ 0 &= w^4 p_2 + w^1 p_6 \quad * \quad - w^3 p_4 , \\ 0 &= w^4 p_3 - w^1 p_5 + w^2 p_4 \quad * \quad . \end{aligned}$$

The coefficient of this system of equations for w^k again has rank 2.

Line $\mathfrak{p}$ and line $\mathfrak{q}$:

Two distinct lines $\mathfrak{p}$, $\mathfrak{q}$ will cut if and only if:

$$(16)\qquad \mathfrak{p}\mathfrak{q} = 0$$

is fulfilled.

Proof: Let x, x' (y, y', resp.) be two points on the line $\mathfrak{p}$ ($\mathfrak{q}$, resp.), so $| x, x', y, y' | = 0$ is necessary and sufficient for the intersection of $\mathfrak{p}$ and $\mathfrak{q}$. (16) follows by developing the determinant in the two-rowed sub-determinants of the first two rows.

From (14), one finds that the *point of intersection* x of the two lines $\mathfrak{p}$, $\mathfrak{q}$ is:

$$(17)\qquad \begin{aligned} &x_1 : x_2 : x_3 : x_4 \\ &\quad = (q_6\, p_5 - q_5\, p_6) : (q_4\, p_6 - q_6\, p_4) : (q_5\, p_4 - q_4\, p_5) : (q_1\, p_4 + q_2\, p_5 + q_3\, p_6), \end{aligned}$$

and from (15), one finds the *connecting plane* w of those lines:

$$(18)\qquad \begin{aligned} &w^1 : w^2 : w^3 : w^4 \\ &\quad = (q_6\, p_2 - q_2\, p_6) : (q_1\, p_3 - q_3\, p_1) : (q_2\, p_1 - q_1\, p_2) : (q_4\, p_1 + q_5\, p_2 + q_6\, p_3)\,. \end{aligned}$$

Other formulas will be obtained when one permutes the vertices 1, 2, 3, 4 of the coordinate tetrahedron in any fashion and then renumbers the p_ρ, q_ρ corresponding to Fig. 2. It can happen that (17) or (18) is undetermined, and one will then be forced to employ one of the other formulas.

Pencil of lines, bundle of lines, line field:

Two independent singular six-vectors $\mathfrak{p}$, $\mathfrak{q}$ with:

$$\mathfrak{p}\mathfrak{q} = 0$$

determine two intersecting lines, and therefore, a *pencil of lines.* All of the lines of the pencil are determined by all of the six-vectors $\lambda_1\mathfrak{p} + \lambda_2\mathfrak{q}$ (except for $\lambda_1 = \lambda_2 = 0$), as one can confirm by reverting to point coordinates. The line condition:

$$(\lambda_1\mathfrak{p} + \lambda_2\mathfrak{q})\,(\lambda_1\mathfrak{p} + \lambda_2\mathfrak{q}) = 0$$

is fulfilled for any λ_1, λ_2.

Three linearly-independent singular six-vectors $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ with:

$$\mathfrak{q}\mathfrak{r} = \mathfrak{r}\mathfrak{p} = \mathfrak{p}\mathfrak{q} = 0$$

determine three mutually-intersecting lines that do not belong to a pencil, and thus, either three lines of a *bundle of lines* or three lines of a *line field;* bundles and fields will be switched under correlative transformations (cf., no. **3**). All of the lines of a bundle (field, resp.) will be given by $\lambda_1\mathfrak{p} + \lambda_2\,\mathfrak{q} + \lambda_3\,\mathfrak{r}$ (except for $\lambda_1 = \lambda_2 = \lambda_3 = 0$).

§ 4. Projective maps.

14. Mapping equations in line coordinates. We would now like to represent the projective maps that were given in point and plane coordinates in (3a, b) in terms of line coordinates. To that end, we replace the $\tilde{x}_i$, $\tilde{x}_k$ ($\tilde{w}^i$, $\tilde{w}^k$, resp.) in the right-hand side of:

$$\sigma \tilde{p}_1 = \tilde{x}_1' \tilde{x}_4 - \tilde{x}_4' \tilde{x}_1, \qquad (\sigma \tilde{p}_1 = \tilde{w}^2 \tilde{w}'^3 - \tilde{w}^3 \tilde{w}'^2 \text{, resp.}), \qquad \text{etc.}$$

with x_i, x_k' by means of (3a) [(3b), resp.], and thus obtain linear forms for the p_ρ from (9), thus:

(19) $$\tilde{p}_\rho = \gamma_\rho^1 p_1 + \gamma_\rho^2 p_2 + \ldots + \gamma_\rho^6 p_6 \equiv \gamma_\rho^\mu p_\mu \qquad (\rho, \mu = 1, 2, \ldots, 6).$$

The coefficient determinant γ does not vanish, since conversely the lines will be transformed uniquely under a projective map.

The 36 coefficients γ_ρ^σ are not independent of each other, but are determined by the 16 coefficients α_i^k of (3a) [β^{ik} of (3b), resp.]:

Defining law for the coefficients γ_ρ^σ in terms of the α_i^k [β^{ik}, resp.] under a:

a) **Collinear map:**

(20a)
$$\begin{array}{llll}
\gamma_1^1 = \alpha_1^1 \alpha_4^4 - \alpha_1^4 \alpha_4^1, & \cdots\ \cdots\ \cdots & \gamma_4^1 = \alpha_3^1 \alpha_2^4 - \alpha_3^4 \alpha_2^1, & \cdots\ \cdots\ \cdots \\
\gamma_1^2 = - \quad - & - \ - \ - & \gamma_4^2 = - \quad - & - \ - \ - \\
\gamma_1^3 = - \quad - & - \ - \ - & \gamma_4^3 = - \quad - & - \ - \ - \\
\gamma_1^4 = \alpha_1^3 \alpha_4^2 - \alpha_1^2 \alpha_4^3, & \cdots\ \cdots\ \cdots & \gamma_4^4 = \alpha_2^2 \alpha_3^3 - \alpha_2^3 \alpha_3^2, & \cdots\ \cdots\ \cdots \\
\gamma_1^5 = - \quad - & - \ - \ - & \gamma_4^5 = - \quad - & - \ - \ - \\
\gamma_1^6 = - \quad - & - \ - \ - & \gamma_4^6 = - \quad - & - \ - \ -
\end{array}$$

b) **Correlative map:**

(20*b*)
$$\begin{array}{llll}
\gamma_1^1 = \beta^{31} \beta^{24} - \beta^{34} \beta^{21}, & \cdots\ \cdots\ \cdots & \gamma_4^1 = \beta^{11} \beta^{44} - \beta^{11} \beta^{41}, & \cdots\ \cdots\ \cdots \\
\gamma_1^2 = - \quad - & - \ - \ - & \gamma_4^2 = - \quad - & - \ - \ - \\
\gamma_1^3 = - \quad - & - \ - \ - & \gamma_4^3 = - \quad - & - \ - \ - \\
\gamma_1^4 = \beta^{22} \beta^{33} - \beta^{23} \beta^{32}, & \cdots\ \cdots\ \cdots & \gamma_4^4 = \beta^{13} \beta^{42} - \beta^{12} \beta^{43}, & \cdots\ \cdots\ \cdots \\
\gamma_1^5 = - \quad - & - \ - \ - & \gamma_4^5 = - \quad - & - \ - \ - \\
\gamma_1^6 = - \quad - & - \ - \ - & \gamma_4^6 = - \quad - & - \ - \ -
\end{array}$$

The expressions that are suggested by ellipses follow from the ones to the left of them when one permutes the lower (first, resp.) index through 1, 2, 3 cyclically. The expressions that are suggested by hyphens follow from the ones above them when one cyclically permutes the upper (second, resp.) index through 1, 2, 3 (while keeping the index 4 fixed).

Special projective maps (cf., no. 3):

a) **Affinity:** It follows from $\alpha_4^1 = \alpha_4^2 = \alpha_4^3 = 0$ that $\gamma_\rho^4 = \gamma_\rho^5 = \gamma_\rho^6 = 0$ ($\rho = 1, 2, 3$), so the $\tilde{p}_1$, $\tilde{p}_2$, $\tilde{p}_3$ will be linear forms in only the p_1, p_2, p_3. As a result, the real lines will be mapped to real lines in a one-to-one correspondence.

b) **Polarity:** The fact that $\tilde{w}^i = x_i$ ($i = 1, 2, 3, 4$) implies that:

$$\tilde{p}_1 = p_4, \qquad \tilde{p}_2 = p_5, \qquad \tilde{p}_3 = p_6, \qquad \tilde{p}_4 = p_1, \qquad \tilde{p}_5 = p_2, \qquad \tilde{p}_6 = p_3.$$

From the interpretation of the p_ρ as rectangular arrow coordinates (no. **9**), the transformation will consists of the permutation of the vectors $\mathfrak{P}$, $\bar{\mathfrak{P}}$; since $\mathfrak{P}\bar{\mathfrak{P}} = 0$, corresponding real lines will then be perpendicular to each other.

15. Ultimate concept of a six-vector. In no. **12**, we introduced the singular six-vectors as sextuples of line coordinates. We now define, more generally:

Any sextuple of numbers p_ρ that transforms by (19) *under projective maps defines a six-vector* $\mathfrak{p}$.

As in no. **12**, we distinguish *singular* six-vectors with $\mathfrak{pp} = 0$ and *non-singular* six-vectors with $\mathfrak{pp} \neq 0$. We will append the geometric interpretation of the non-singular six-vector in § 6.

Since line coordinates will transform into line coordinates under (19), the fact that $\mathfrak{pp} = 0$ must always imply that $\tilde{\mathfrak{p}}\tilde{\mathfrak{p}} = 0$. It follows from this that:

$$\tilde{\mathfrak{p}}\tilde{\mathfrak{p}} = k\mathfrak{pp} \tag{21}$$

for the projective transformations of arbitrary (i.e., singular and non-singular) six-vectors. In this, $k \neq 0$ is a constant that is independent of $\mathfrak{p}$ and is determined by the coefficients γ_ρ^σ; since $\mathfrak{pp}$ and $\tilde{\mathfrak{p}}\tilde{\mathfrak{p}}$ are quadratic forms in the p_ρ, they can then differ by only a constant factor. From (21), we get that the scalar product of any two six-vectors is:

$$\tilde{\mathfrak{p}}\tilde{\mathfrak{q}} = k\mathfrak{pq}; \tag{22}$$

If one then replaces $\mathfrak{p}$ with $\mathfrak{p} + \lambda\mathfrak{q}$ in $\mathfrak{p}$ then one will get:

$$\tilde{\mathfrak{p}}\tilde{\mathfrak{p}} + \lambda^2\tilde{\mathfrak{q}}\tilde{\mathfrak{q}} + 2\lambda\tilde{\mathfrak{p}}\tilde{\mathfrak{q}} = k\,(\mathfrak{p}\mathfrak{p} + \lambda^2\,\mathfrak{q}\mathfrak{q} + 2\lambda\,\mathfrak{p}\mathfrak{q}),$$

from which, (22) will follow immediately, since:

$$\tilde{\mathfrak{p}}\tilde{\mathfrak{p}} = k\mathfrak{p}\mathfrak{p}, \qquad \tilde{\mathfrak{q}}\tilde{\mathfrak{q}} = k\mathfrak{q}\mathfrak{q}\ .$$

Equation (21) is not only a necessary, but also a sufficient, condition for projective maps:

A linear transformation (19) *with a non-vanishing coefficient determinant* γ *will always yield a projective map when* (21) *is fulfilled for all six-vectors; the requirement* (21) *is thus equivalent to the conditions* (20a) *and* (20b).

Proof: Let T be any transformation (19) that fulfills (21) and the inequality $\gamma \neq 0$. We must show that T represents either a collineation or a correlation: T maps a line to a line in a one-to-one correspondence, and since (22) follows from (21), intersecting lines will map to other such lines. The linear dependence (independence, resp.) of six-vectors will remain preserved. However, every pencil of lines must be transformed invertibly into either *a*) a pencil of lines or *b*) a line field.

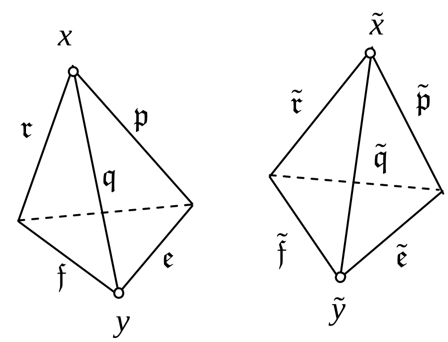

Figure 4.

We next prove the:

Lemma: *If one pencil x is mapped to a pencil* $\tilde{x}$ *under a transformation T then every pencil y will again correspond to another pencil* $\tilde{y}$.

One sees this as follows: Let the three lines $\mathfrak{p}$ (= line xy), $\mathfrak{q}$, $\mathfrak{r}$ through the point x be linearly-independent, and likewise for the three lines $\mathfrak{p}$, $\mathfrak{e}$, $\mathfrak{f}$ that go through a point y that is different from x, of which, $\mathfrak{e}$ should cut the line $\mathfrak{q}$, and $\mathfrak{f}$ should cut the line $\mathfrak{r}$ (Fig. 4). The lines $\tilde{\mathfrak{p}}$, $\tilde{\mathfrak{q}}$, $\tilde{\mathfrak{r}}$ that are transforms under T will then likewise be linearly-independent, and by assumption, they will go through the point $\tilde{x}$. In addition, $\tilde{\mathfrak{p}}$, $\tilde{\mathfrak{e}}$, $\tilde{\mathfrak{f}}$ must be linearly-independent lines of a field or a pencil that is different from $\tilde{x}$. Since $\tilde{\mathfrak{e}}$, $\tilde{\mathfrak{f}}$ must not go through $\tilde{x}$, but must cut the line $\tilde{\mathfrak{q}}$ ($\tilde{\mathfrak{r}}$, resp.), $\tilde{\mathfrak{p}}$, $\tilde{\mathfrak{e}}$, $\tilde{\mathfrak{f}}$ cannot lie in a plane. As a result, $\tilde{\mathfrak{p}}$, $\tilde{\mathfrak{e}}$, $\tilde{\mathfrak{f}}$ are lines of a pencil $\tilde{y}$, so the pencil y will be transformed into the pencil $\tilde{y}$.

Now, two cases are possible for the given transformation T:

a) A pencil x is mapped to a pencil $\tilde{x}$: From the lemma, every pencil (point, resp.) will then go to a pencil (point, resp.), and from the main theorem of projective geometry (cf., no. **3**), the given map T will be a collineation.

b) A pencil x is mapped to a plane field $\tilde{w}$: The resulting transformation $R = UT$, in which T is the given map, and U is any correlation, will then transform the pencil x into a pencil again, so from *a*), it will be a collineation. As a result, the given map $T = U^{-1}R$ (U^{-1} = the correlative map that is inverse to U) will be a correlation.

16. Classification of projective maps. One can distinguish four kinds of projective maps according to the sign of the determinant (19) $\gamma \neq 0$ and the sign of the constant $k \neq 0$ in (21):

a) α) $k > 0, \gamma > 0$, β) $k < 0, \gamma < 0$, *b*) α) $k > 0, \gamma < 0$, α) $k < 0, \gamma > 0$.

The maps a) *are collinear, while the maps b*) *are correlative.*

Proof: *a*) Any collineation can be converted into either α) the identity I or β) the reflection S in a plane by a continuous change of the coefficients γ_ρ^σ, and without γ or k vanishing (hence, while preserving the signs of γ and k). One sees this as follows: A collineation is given by associating the 2×5 points 1, 2, 3, 4, 5 and $\tilde{1}$, $\tilde{2}$, $\tilde{3}$, $\tilde{4}$, $\tilde{5}$, no four of which lie in a plane. One can take 1, 2, 3 and $\tilde{1}$, $\tilde{2}$, $\tilde{3}$ to imaginary points by a continuous change in projective space without any four points moving into a plane, and thus take the given collineation to a affinity. We now think of that affinity as being established by the association of two real tetrahedra $\Delta, \tilde{\Delta}$. We then bring $\tilde{\Delta}$ into coincidence with Δ (the mirror image Δ' of Δ with respect to a plane, resp.) by a continuous, non-degenerate, affine distortion and a continuous motion. In that way, the affinity will go to the identity (reflection in a plane).

We must then calculate k and γ for the maps:

$$I: \tilde{x}_i = x_i \quad (i = 1, 2, 3, 4) \text{ and } \quad S: \tilde{x}_i = -x_1, \quad \tilde{x}_i = x_i \quad (i = 2, 3, 4).$$

For I, one gets:

$$\tilde{p}_\rho = \tilde{\sigma} p_\rho \quad (\rho = 1, 2, \ldots, 6),$$

so

$$k = \tilde{\sigma}^2 > 0, \quad \gamma = \tilde{\sigma}^6 > 0;$$

for S, one gets:

$$\tilde{p}_1 = -\tilde{\sigma} p_1, \quad \tilde{p}_2 = \tilde{\sigma} p_2, \quad \tilde{p}_3 = \tilde{\sigma} p_3, \quad \tilde{p}_4 = \tilde{\sigma} p_4, \quad \tilde{p}_5 = -\tilde{\sigma} p_5, \quad \tilde{p}_6 = -\tilde{\sigma} p_6,$$

so

$$k = -\tilde{\sigma}^2 < 0, \quad \gamma = -\tilde{\sigma}^6 < 0.$$

b) The resulting map $R = PD$ of a correlation D and a polarity P (cf., no. **3**) is a collineation. For P, one has:

$$\tilde{p}_1 = \tilde{\sigma} p_4, \quad \tilde{p}_2 = \tilde{\sigma} p_5, \quad \tilde{p}_3 = \tilde{\sigma} p_6, \quad \tilde{p}_4 = \tilde{\sigma} p_1, \quad \tilde{p}_5 = \tilde{\sigma} p_2, \quad \tilde{p}_6 = \tilde{\sigma} p_3,$$

so

$$k_P = \tilde{\sigma}^2 > 0, \quad \gamma_P = -\tilde{\sigma}^6 < 0.$$

From *a*), for R, one will have either:

$$k_R > 0, \ \gamma_R > 0 \quad \text{or} \quad k_R < 0, \ \gamma_R < 0.$$

Since:

$$k_R = k_P\, k_D, \quad \gamma_R = \gamma_P\, \gamma_D,$$

it will follow that:

$$k_D > 0, \ \gamma_D < 0 \quad \text{or} \quad k_D < 0, \ \gamma_D > 0.$$

We call the projective maps with $k > 0$ *principal projectivities* ($\gamma > 0$: *principal collineations*, $\gamma < 0$: *principal correlations*). The projective maps *a*) β) and *b*) β) are obtained from the principal projectivities by adding a reflection. For that reason, there is no essential restriction if we agree that:

We now understand projective maps to mean principal projectivities exclusively ($k > 0$).

The principal projectivities define a subgroup of the projective maps, and the principal collineations define a subgroup of the principal projectivities.

Proof: For the map $C = BA$ that results from two principal projectivities, since $k_C = k_B \cdot k_A$, it will follow from $k_A > 0$, $k_B > 0$ that one also has $k_C > 0$. For principal collineations, one likewise shows that $\gamma_C > 0$ for resultant maps and $\gamma_B > 0$ for inverse maps.

17. Representation of principal projectivities by unity transformations. The γ_ρ^σ are established only up to an arbitrary proportionality factor τ by representing a projective map as in (19). If one replaces the γ_ρ^σ with $\tau \cdot \gamma_\rho^\sigma$ then the constant $\tau^2 \cdot k$ will enter in place of k. As a result, a principal projectivity ($k > 0$) can always be represented by a linear transformation (19) that is normalized by:

(23) $$\tilde{\mathfrak{p}}\tilde{\mathfrak{p}} = \mathfrak{p}\mathfrak{p}, \qquad \text{hence also} \qquad \tilde{\mathfrak{p}}\tilde{\mathfrak{q}} = \mathfrak{p}\mathfrak{q}$$

by means of $\tau = 1 / \sqrt{k}$. We call the transformations, thus-normalized, *unity transformations.*

Unity transformations define a group. We will essentially reduce the study of line geometry to the investigation of this group in a manner that is similar to the way that one bases metric geometry in rectangular point coordinates on the group of orthogonal transformations.

Relations are true for the coefficient matrix of a unity transformation that are similar to the ones that are true for an orthogonal matrix:

A given 6×6 *matrix* $\| \gamma_\rho^\sigma \|$ *of rank 6 is the coefficient matrix of a unity transformation if and only if the equations:*

(24) $$\gamma_1^\rho \gamma_4^\sigma + \gamma_2^\rho \gamma_5^\sigma + \gamma_3^\rho \gamma_6^\sigma + \gamma_4^\rho \gamma_1^\sigma + \gamma_5^\rho \gamma_2^\sigma + \gamma_6^\rho \gamma_3^\sigma = \begin{cases} 1 & \text{for } \rho - \sigma = \pm 3 \\ 0 & \text{for } \rho - \sigma \neq \pm 3 \end{cases}$$

are fulfilled.

Proof: It follows from (19) and (23) that:

$$\begin{aligned} p_1\, q_4 + p_2\, q_5 + p_3\, q_6 + p_4\, q_1 + p_5\, q_2 + p_6\, q_3 &= \mathfrak{pq} \\ &= \tilde{\mathfrak{p}}\tilde{\mathfrak{q}} = \tilde{p}_1\tilde{q}_4 + \tilde{p}_2\tilde{q}_5 + \tilde{p}_3\tilde{q}_6 + \tilde{p}_4\tilde{q}_1 + \tilde{p}_5\tilde{q}_2 + \tilde{p}_6\tilde{q}_3 \\ &= (\gamma_1^\rho \gamma_4^\sigma + \gamma_2^\rho \gamma_5^\sigma + \gamma_3^\rho \gamma_6^\sigma + \gamma_4^\rho \gamma_1^\sigma + \gamma_5^\rho \gamma_2^\sigma + \gamma_6^\rho \gamma_3^\sigma)\, p_\rho\, q_\sigma ; \end{aligned}$$

one obtains (24) by identifying the coefficients of $p_\rho\, q_\sigma$ in the first and last expressions.

The conditions (24) can be interpreted geometrically as follows:

The 6×6 *coefficients* $\gamma_1^\rho, \ldots, \gamma_6^\rho$ *can be regarded as the line coordinates of the six edges* $\mathfrak{g}^\rho$ *of a tetrahedron; the line coordinates of the pairs of skew tetrahedral edges are partially normalized by* $\mathfrak{g}^{\mathrm{I}}\mathfrak{g}^{\mathrm{IV}} = \mathfrak{g}^{\mathrm{II}}\mathfrak{g}^{\mathrm{V}} = \mathfrak{g}^{\mathrm{III}}\mathfrak{g}^{\mathrm{VI}} = 1$.

The relations (24) immediately yield the *solution to the mapping equations* (19) for the p_ρ:

(25) $$k\, p_\rho = \gamma_4^{\rho\pm3}\tilde{p}_1 + \gamma_5^{\rho\pm3}\tilde{p}_2 + \gamma_6^{\rho\pm3}\tilde{p}_3 + \gamma_1^{\rho\pm3}\tilde{p}_4 + \gamma_2^{\rho\pm3}\tilde{p}_5 + \gamma_3^{\rho\pm3}\tilde{p}_6$$

for arbitrary projective maps ([1]); for unity transformations, one sets $k = 1$. One then gets the coefficients of the unity transformation that is inverse to a unity transformation (19) when one switches the rows and columns of the coefficient matrix $\| \gamma_\rho^\sigma \|$ and replaces the ρ, σ with $\rho \pm 3$, $\sigma \pm 3$. By applying (24) to the inverse transformation (25), it will then follow that:

([1]) One computes the right-hand side with the use of (24).

The necessary and sufficient conditions (24) *are equivalent to the requirements that:*

$$\gamma_\rho^1\gamma_\sigma^4 + \gamma_\rho^2\gamma_\sigma^5 + \gamma_\rho^3\gamma_\sigma^6 + \gamma_\rho^4\gamma_\sigma^1 + \gamma_\rho^5\gamma_\sigma^2 + \gamma_\rho^6\gamma_\sigma^3 = \begin{cases} 1 & \text{for } \rho - \sigma = \pm 3 \\ 0 & \text{for } \rho - \sigma \neq \pm 3 \end{cases} \tag{26}$$

which follow from (24) *by switching the rows and columns in the coefficient matrix.*

§ 5. Projective force transformation and projective kinematic transformation.

We interrupt our geometric investigations here for *an application to the statics and kinematics of rigid bodies* ([1]).

18. Static and kinematical interpretation of arrows. In the *statics* of rigid bodies, the real arrows ($\mathfrak{P} \neq 0$, $\mathfrak{P}\bar{\mathfrak{P}} = 0$) that were introduced in no. **9** can be interpreted as forces (*real forces*), while the imaginary arrows ($\mathfrak{P} = 0$, $\bar{\mathfrak{P}} \neq 0$) can be interpreted as force couples (*imaginary forces*). The vector $\mathfrak{P}$ gives magnitude and direction to the real force, while the vector $\bar{\mathfrak{P}}$ gives the moment of the real force relative to the origin *O* (the moment of the force-couple, resp.); the 2×3 coordinates of $\mathfrak{P}$, $\bar{\mathfrak{P}}$ are line coordinates of the (real or imaginary, resp.) *line of action* of the force.

In the *kinematics* of rigid bodies, we interpret the real arrow as the rotational velocity (*real rotations*), and the imaginary arrow as the translational velocity (*imaginary rotation*). $\mathfrak{P}$ gives the magnitude and sense of the rotational velocity, while $\bar{\mathfrak{P}}$ gives that of translational velocity of the point *O* that is caused by the rotation (for $\mathfrak{P} = 0$, the translational velocity, per se, resp.). The real (imaginary, resp.) *rotational axis* has the 2×3 coordinates of $\mathfrak{P}$, $\bar{\mathfrak{P}}$ as line coordinates.

We call the 2×3 coordinates of $\mathfrak{P}$, $\bar{\mathfrak{P}}$ *force coordinates* (*rotation coordinates*, resp.), and we once more combine them into an (inhomogeneous) six-vector $\mathfrak{p}$. Two linearly-dependent six-vectors now yield forces (rotations, resp.) with the same line of action (rotational axis).

For several forces (rotations, resp.) on a rigid body, one has the necessary and sufficient conditions for static (kinematic, resp.) *equilibrium:*

$$\sum \mathfrak{P} = 0, \qquad \sum \bar{\mathfrak{P}} = 0.$$

These two equations can be combined using the notation of six-vectors into:

$$\sum \mathfrak{p} = 0. \tag{27}$$

([1]) Cf., on this, *Enz. der math. Wiss IV₁*, pp. 128, *et seq.*; furthermore **R. Sauer:** "Proj. Sätze in der Statik des starren Körpers," Math. Ann. **110** (1934).

The sums are extended over all effective forces (rotations, resp.).

19. Projective force transformations and projective kinematical transformations. The static (kinematic, resp.) interpretation of the six-vector $\mathfrak{p}$ will associate forces (rotations, resp.) with forces (rotations, resp.) in a one-to-one correspondence by way of equations (19) with constant coefficients γ_ρ^σ that satisfy condition (21). We refer to that association as a *projective force transformation (kinematic transformation, resp.)*. It has the following properties:

A) *The line of action (rotational axis) will be transformed projectively.*

B) *The equilibrium conditions remain preserved; i.e., every equilibrium system goes to an equilibrium system, and every non-equilibrium system goes to a non-equilibrium system*. Statement A) follows immediately from no. **14**. One gets the proof of B) from (19) and (27); then, from:

$$\sum_{k=1}^{n} p_\rho^{(k)} = 0 \qquad (\rho = 1, 2\ldots, 6),$$

one will obtain from (19):

$$\sum_{k=1}^{n} \tilde{p}_\rho^{(k)} = \gamma_\rho^1 \sum p_1^{(k)} + \ldots + \gamma_\rho^6 \sum p_6^{(k)} = 0 \quad (\rho = 1, 2, \ldots, 6).$$

The upper index k and the summations refer to the n given forces (rotations, resp.)

Special projective force transformations (cf., no. 14):

a) **Affine transformations:** The lines of action are mapped affinely; the real forces always go to real forces again.

Special cases:

1. Change of units for force: $\tilde{\mathfrak{p}} = \tau\,\mathfrak{p}$. The lines of action remain fixed, while all vectors $\mathfrak{P}$, $\bar{\mathfrak{P}}$ are multiplied by the same constant τ.

2. Motion transformations: All lines of action and all vectors $\mathfrak{P}$, $\bar{\mathfrak{P}}$ are subjected to the same transformation. The motion transformations can also be interpreted as *coordinate transformations of the forces.*

b) **Polar force transformations:** The lines of action are polar transformed. The forces of a pencil are then mapped to forces in a plane with conservation of the equilibrium conditions. *One can employ that map to the solution of the problems of* *spatial graphical statics for forces with fixed points of application by constructions in the polar plane of the point of application as the reference plane.*

The same thing can be said for any arbitrary correlative force transformation; e.g., the following map that was given by **R. v. Mises** ([1]):

$$\tilde{p}_1 = p_1, \qquad \tilde{p}_2 = p_2, \qquad \tilde{p}_3 = p_3, \qquad \tilde{p}_4 = p_4, \qquad \tilde{p}_5 = p_5, \qquad \tilde{p}_6 = p_6,$$

which transforms the force $\mathfrak{p}$ that acts at the origin O into the force $\tilde{\mathfrak{p}}$ in the x_1x_2-plane (viz., the base plane). In Fig. 5, the given force that acts at O is drawn with a bold line. The image vectors arises from the base plane vector of the given force by parallel translation, while the projection of the given force into the 3-axis is the moment vector of the image force with respect to O.

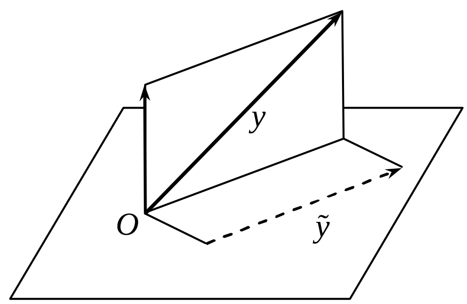

Figure 5.

20. Application to frameworks. We shall now give an application of the projective force transformations to the theory of frameworks.

A *framework* ([2]) consists of rigid connecting pieces, which are idealized as weightless, that can rotate arbitrarily about the nodes (= endpoints of the line segments) with no reaction forces. The framework is called *geometrically indeterminate* when it is not rigid, taken as a whole (e.g., the framework that consists of the twelve edges of a cube). A framework that is rigid as a whole is called *geometrically determinate* when it would be geometrically indeterminate if one omitted any of its line segments (e.g., the framework that consists of the twelve edges of an octahedron), and it is called *geometrically over-determined* when at least one line segment can be omitted without it becoming indeterminate (e.g., the framework with the edges of an octahedron + one diagonal). If external forces act upon the nodes then stress forces will be produced in the line segments, and indeed the stress forces at the endpoints of every segment will be equal and opposite. It is necessary and sufficient for the equilibrium of a framework that the equilibrium equation (27) are fulfilled at every node for the external and stress forces

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([1]) **R. v. Mises:** "Graph. Statik räumlicher Kräftesysteme," Zeit. f. Math. u. Phys. (1916). One will find a collinear force transformation in **W. Prager:** "Formänderungen von Raumfachwerken," Zeit. f. Math. u. Phys. **7** (1927).

([2]) Cf., e.g., *Enz. der math. Wiss. IV$_1$*, pp. 385, *et seq.*

that act at the node. In particular, the system of external forces must satisfy the equilibrium conditions. The equilibrium conditions will be preserved under projective force transformations, so equal and opposite forces on a line will go to other such forces.

If one maps the stresses and external forces that act upon a given framework to a collinear force transformation then any equilibrium state of a framework will again imply an equilibrium state for any collinear framework.

One refers to a framework with the structure of a geometrically-determinate framework as a *statically-exceptional framework* when the following condition is fulfilled: It can be converted into a system of external forces that satisfy the equilibrium conditions in such a way that no equilibrium can come about in the framework by means of finite stress forces. As a trivial example of an exceptional framework, let us mention the edge framework of a tetrahedron that has been collapsed into a plane; external forces whose lines of action do not lie in that plane cannot be equilibrated by finite stress forces. One has the following theorem for exceptional frameworks:

Any framework that is collinear to an exceptional framework will again be exceptional.

Proof: The geometric determinacy of a framework depends upon only the arrangement of the rods, so it will be collinearly invariant. Furthermore, a collinear force transformation will take a force system that cannot be equilibrated by finite stress forces to another such force system in the collinear framework.

The theorem that was just proved might seem surprising when one does not start with projective force transformations, since one will then recognize the affine invariance of the exceptional frameworks, but not the collinear invariance.

§ 6. Linear complex.

21. Definition of a linear complex. We refer to a three-parameter set of lines that is given by:

$$a_4\, p_1 + a_5\, p_2 + a_6\, p_3 + a_1\, p_4 + a_2\, p_5 + a_3\, p_6 = 0$$

as a *linear complex* when not all coefficients a_ρ vanish. Under the linear transformation (19) of the line coordinates p_ρ, it follows from:

$$a_4\, p_1 + \ldots + a_3\, p_6 = \tilde{a}_4 \tilde{p}_1 + \ldots + \tilde{a}_3 \tilde{p}_6$$

that the a_ρ will transform under projective maps just as the line coordinates p_ρ transform under (25); they then define a six-vector $\mathfrak{a}$, and the equation of the linear complex can be written more briefly as:

(28) $$\mathfrak{a}\mathfrak{p} = 0 \qquad (\mathfrak{a} \text{ is not the null vector!}).$$

Any singular or non-singular six-vector determines a linear complex, and linearly-dependent six-vectors represent the same complex; a linear complex is then established by the a_ρ as its *homogeneous, complex coordinates.* We call the complex with the coordinates a_ρ the *complex* $\mathfrak{a}$, for brevity.

From (16), a *singular complex* ($\mathfrak{a}\mathfrak{a} = 0$) consists of the totality of lines that intersect the line $\mathfrak{a}$; the line $\mathfrak{a}$ is called the *axis* of the singular complex. Due to (21), a singular complex will go to another singular complex under any projective map.

Lines (linear complexes, resp.) shall be called *linearly-dependent* when their six-vectors are linearly-dependent. One will then have:

A linear complex $\mathfrak{a}$ *is determined uniquely by five linearly-independent lines that belong to it.* From (28), one will then have, in fact, five linearly-independent homogeneous equations for the a_ρ .

One can map the set of all linear complexes to the *points of five-dimensional projective space* in a one-to-one correspondence when one reinterprets the a_ρ as homogeneous projective point coordinates. The points that correspond to the singular complexes will then define the non-singular second-order hypersurface:

$$a_1\, a_4 + a_2\, a_5 + a_3\, a_6 = 0.$$

22. Null system. Reciprocal lines. In this and the next number, we shall exclude the singular complexes.

If one replaces the p_ρ in (28) with the expressions that are given (9) then one will get:

$$a_4\,(x_1'\,x_4 - x_4'\,x_1) + \ldots + a_1\,(x_2\,x_3' - x_3\,x_2') + \ldots = 0. \tag{29}$$

In that way, any fixed point x will be correlatively assigned to a linear equation in the x_i', and thus a plane w. The plane w will contain the point x, since (29) will be fulfilled for $x_i = x_i'$. The correlation $x \leftrightarrow w$ is called a *null system* (cf., no. **40**), where x is a *null point* and w is a *null plane.* From (29), any line of the complex $\mathfrak{a}$ that goes through the null point x will lie in the null plane w, and conversely, any line in the plane w that goes through the point x will be a line of the complex. Thus:

The lines of the complex $\mathfrak{a}$ *that go through a point x (and lie in a plane w) define a pencil in the null plane w (whose vertex is the null point x).*

Two lines $\mathfrak{p}$, $\mathfrak{q}$, are called *reciprocal* with respect to the non-singular complex $\mathfrak{a}$ when the relation exists:

$$\mathfrak{a} = \lambda\,\mathfrak{p} + \mu\,\mathfrak{q}\,. \tag{30}$$

For every line $\mathfrak{p}$ *that does not belong to the complex* $\mathfrak{a}$ (i.e., $\mathfrak{a}\mathfrak{p} \neq 0$), *there is precisely one reciprocal line* $\mathfrak{q}$; *it does not belong to the complex* $\mathfrak{a}$, *either* ($\mathfrak{a}\mathfrak{q} \neq 0$), *and is skew to* $\mathfrak{p}$ (viz., $\mathfrak{p}\mathfrak{q} \neq 0$).

Proof: It follows from:

$$0 = \mu^2\, \mathfrak{q}\mathfrak{q} = (\mathfrak{a} - \lambda\, \mathfrak{p})\, (\mathfrak{a} - \lambda\, \mathfrak{p}) = \mathfrak{a}\mathfrak{a} - 2\lambda\, \mathfrak{a}\mathfrak{p}\ ,$$

since $\mathfrak{q}$ is determined only up to a factor, that:

$$\mathfrak{q} = 2\, (\mathfrak{a}\mathfrak{p})\, \mathfrak{a} - (\mathfrak{a}\mathfrak{a})\, \mathfrak{p}, \quad \text{with} \quad \mu = \frac{1}{2(\mathfrak{a}\mathfrak{p})},$$

and furthermore:

$$\mathfrak{a}\mathfrak{q} = (\mathfrak{a}\mathfrak{p})\, (\mathfrak{a}\mathfrak{a}) \neq 0, \qquad \mathfrak{p}\mathfrak{q} = 2\, (\mathfrak{a}\mathfrak{p})\, (\mathfrak{a}\mathfrak{p}) \neq 0.$$

(30) cannot be fulfilled by any line $\mathfrak{p}$ of the complex $\mathfrak{a}$ (i.e., $\mathfrak{a}\mathfrak{p} = 0$), since it would follow from $\mathfrak{q}\mathfrak{q} = 0$ that $\mathfrak{a}\mathfrak{a} = 0$, contrary to assumption.

From (30), $\mathfrak{a}\mathfrak{r} = \mathfrak{p}\mathfrak{r} = 0$ always has $\mathfrak{q}\mathfrak{r} = 0$ as a consequence. That is: *Any line* $\mathfrak{r}$ *of the complex* $\mathfrak{a}$ *that meets one of two reciprocal lines* $\mathfrak{p}$, $\mathfrak{q}$ *will intersect the other one.* For that reason, all complex lines through a point x of the line $\mathfrak{p}$ must cut the line $\mathfrak{q}$; i.e., the plane $\{x\mathfrak{q}\}$ will be the null plane of the null point x. We can then characterize the reciprocal lines $\mathfrak{p}$, $\mathfrak{q}$ geometrically as follows:

If a line $\mathfrak{p}$ *that does not belong to the complex* $\mathfrak{a}$ *goes through the point x then the null planes w will define a pencil whose axis is* $\mathfrak{q}$. *The sequence of points* (x) *is perspective to the pencil of planes* (w).

23. Non-singular linear complex in rectangular coordinates. We go to homogeneous, rectangular coordinates by a coordinate transformation (19), and set (no. **9**):

$$\mathfrak{P} = \text{vector}\ (p_1 \mid p_2 \mid p_3), \qquad \mathfrak{A} = \text{vector}\ (a_1 \mid a_2 \mid a_3),$$
$$\bar{\mathfrak{P}} = \text{vector}\ (p_4 \mid p_5 \mid p_6), \qquad \bar{\mathfrak{A}} = \text{vector}\ (a_4 \mid a_5 \mid a_6).$$

Equation (28), (29) can then be written as follows:

(28*) $\bar{\mathfrak{A}}\mathfrak{P} + \mathfrak{A}\bar{\mathfrak{P}} = 0,$ (29*) $\bar{\mathfrak{A}}\,(\mathfrak{X}' - \mathfrak{X}) + \mathfrak{A}\,(\mathfrak{X} \times \mathfrak{X}') = 0.$

Since we have excluded singular complexes, $\mathfrak{A}\bar{\mathfrak{A}} \neq 0$, from which it will follow that $\mathfrak{A}\mathfrak{A} \neq 0$, in particular. Under the coordinate transformation:

$$\mathfrak{X} = \mathfrak{T} + \mathfrak{Y}$$

(viz., parallel translation of the coordinate system), the six-vector $\mathfrak{a} = \{\mathfrak{A} \mid \bar{\mathfrak{A}}\}$ will be transformed into $\mathfrak{b} = \{\mathfrak{B} \mid \bar{\mathfrak{B}}\}$. We show:

The displacement vector $\mathfrak{T}$ can be chosen in such a way that the vectors $\mathfrak{B}, \bar{\mathfrak{B}}$ are linearly-dependent; i.e., parallel to the same line.

Proof: Under the displacement $\mathfrak{X} = \mathfrak{T} + \mathfrak{Y}$, (29*) will go to:

$$0 = \bar{\mathfrak{A}}(\mathfrak{Y}' - \mathfrak{Y}) + \mathfrak{A}\{(\mathfrak{T} + \mathfrak{Y}) \times (\mathfrak{T} + \mathfrak{Y})\} = \bar{\mathfrak{B}}(\mathfrak{Y}' - \mathfrak{Y}) + \mathfrak{B}(\mathfrak{Y} \times \mathfrak{Y}'),$$

with

$$\mathfrak{B} = \mathfrak{A}, \qquad \bar{\mathfrak{B}} = \bar{\mathfrak{A}} + \mathfrak{A} \times \mathfrak{T};$$

so, from no. **1**, one will have:

$$\mathfrak{A}(\mathfrak{T} \times \mathfrak{Y}') + \mathfrak{A}(\mathfrak{Y} \times \mathfrak{T}) = <\mathfrak{A}, \mathfrak{T}, \mathfrak{Y}' - \mathfrak{A}> = (\mathfrak{A} \times \mathfrak{T})(\mathfrak{Y}' - \mathfrak{Y}).$$

The requirement:

$$\bar{\mathfrak{B}} = \rho\, \mathfrak{B} \qquad (\text{so} \qquad \bar{\mathfrak{A}} + \mathfrak{A} \times \mathfrak{T} = \rho\, \mathfrak{A})$$

will give:

$$\mathfrak{T} = \frac{\mathfrak{A} \times \bar{\mathfrak{A}}}{\mathfrak{A}\mathfrak{A}} + \sigma \mathfrak{A}, \qquad \rho = \frac{\mathfrak{A}\bar{\mathfrak{A}}}{\mathfrak{A}\mathfrak{A}} \neq 0$$

under external (inner, resp.) multiplication by $\mathfrak{A}$, in which σ is arbitrary.

The vector equation $\mathfrak{T} = \mathfrak{T}(\sigma)$ establishes a line $\mathfrak{s}$ uniquely, which we will call the *axis of the complex (null system, resp.)*. In the old coordinates, $\mathfrak{s}$ is established by:

$$\mathfrak{s} = \{\mathfrak{A} \mid \mathfrak{B} \times \mathfrak{A}\} \qquad \text{with } \mathfrak{B} = \frac{\mathfrak{A} \times \bar{\mathfrak{A}}}{\mathfrak{A}\mathfrak{A}}; \tag{31}$$

$\mathfrak{A}$ gives the direction of the axis, and $\mathfrak{B}$ is the altitude vector from *O* to the axis of the complex.

(31) *is also true for the axes of singular complexes* (no. **21**); since $\mathfrak{A}\bar{\mathfrak{A}} = 0$, $\mathfrak{B}$ is then the altitude vector from *O* to the line $\mathfrak{s} = \mathfrak{a}$ (no. **9**).

Normal form for a null system:

We refer the null system to a rectangular coordinate system whose three axes coincide with the axes of the complex. The fact that:

$$\bar{\mathfrak{B}}(\mathfrak{X}' - \mathfrak{X}) + \mathfrak{B}(\mathfrak{X} \times \mathfrak{X}'), \qquad \text{with} \quad \mathfrak{B} = 0 \mid 0 \mid b, \quad \bar{\mathfrak{B}} = 0 \mid 0 \mid \bar{b}$$

will then imply that the equation of the null system is:

$$\bar{b}(x_3' x_4 - x_4' x_3) + b(x_1 x_2' - x_2 x_1') = 0. \tag{32}$$

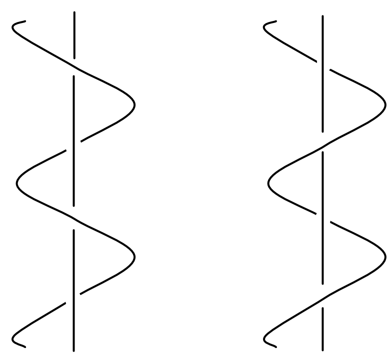
(32) can be interpreted as the equation of the normal plane of the helix that goes through the point x:

$$y_1 = r \cos bu, \qquad y_2 = r \sin bu, \qquad y_3 = bu, \qquad y_4 = 1.$$

One then has:

Any non-singular linear complex determines a screw around a line $\mathfrak{s}$ *(viz., the axis of the complex) with the constant ratio* $b : \bar{b}$ $(b \neq 0,\ \bar{b} \neq 0)$ *of the rotational and translational velocities. Any point thus corresponds to the helix that goes through the point as the null plane to the normal plane.*

The right-hand screws are characterized by $b\bar{b} > 0$, while the left-hand screws are characterized by $b\bar{b} < 0$ (Fig. 6).



Left screw Right screw

Figure 6.

Reciprocal real lines $\mathfrak{p}$, $\mathfrak{q}$ *will project to parallel lines in the image plane under any projection that is parallel to the axis* $\mathfrak{s}$.

Proof: From (30), $\mathfrak{B} = \lambda\,\mathfrak{P} + \mu\,\mathfrak{Q}$; since $\mathfrak{B} = 0 \mid 0 \mid b$, it follows from this that:

$$p_1 : p_2 = q_1 : q_2 .$$

One observes that the concepts that were developed in this number are not of a projective nature. In particular, the axes $\mathfrak{s}$ *of non-singular complexes will correspond only for special projective maps; e.g., motions and reflections.*

Application to the Cremona plane ([1]):

Let two polyhedra Π_1, Π_2 with planar faces be given in space that correspond to each other in the null system (29). Any vertex (*n*-edge) of Π_1 (Π_2, resp.) is associated with a face (planar *n*-vertex) of Π_2 (Π_1, resp.). Corresponding edges of Π_1, Π_2 are reciprocal lines, and thus project as parallel lines under a parallel projection of Π_1, Π_2 that is parallel to the complex axis. The parallel outlines Φ_1, Φ_2 of Π_1, Π_2 in the image plane are called *reciprocal figures* (Cremona, *ibid.*): *Any line segment of* Φ_1 *corresponds to a parallel segment of* Φ_2 *, and conversely; the planar n-edge that emanates from a point of* Φ_1 *corresponds to a closed polygon in* Φ_2*, and conversely.* Fig. 13 gives an example of this. (X) and (Y) are reciprocal figures in it, if one regards them as *planar* structures; parallel line segments that are associated with each other are denoted with the same symbols. Reciprocal figures are employed in graphical statics as the so-called *Cremona plane;* the one figure will be interpreted as a framework, while the other one is the associated *force plane*.

§ 7. Invariants of *n* six-vectors.

Here, we summarize the *invariant-theoretic theorems* ([2]) that will define the analytical foundations for all of the investigations that follow.

24. Semi-invariants, complete invariants, and sign invariants. As we agreed in no. **16**, we shall restrict ourselves to the principal projectivities ($k > 0$). Any linear transformation (19) that represents a principal projectivity can be regarded as the resultant of a *unity transformation* (no. **17**), which is characterized by $k = 1$, and a so-called *renormalization:*

$$\tilde{p}_\rho = \sigma p_\rho \qquad (s \neq 0).$$

A function $\Phi(\mathfrak{p}^{\mathrm{I}}, \mathfrak{p}^{\mathrm{II}}, \ldots, \mathfrak{p}^n)$ of the coordinates p_ρ of n six-vectors is called *semi-invariant* when one has:

$$\Phi(\mathfrak{p}^{\mathrm{I}}, \mathfrak{p}^{\mathrm{II}}, \ldots, \mathfrak{p}^n) = \Phi(\tilde{\mathfrak{p}}^{\mathrm{I}}, \tilde{\mathfrak{p}}^{\mathrm{II}}, \ldots, \tilde{\mathfrak{p}}^n) \tag{33}$$

for all unity transformations. If (33) remains fulfilled, in addition, by all renormalizations for which the normalization factor σ for each of the six-vectors $\mathfrak{p}^{\mathrm{I}}$, $\mathfrak{p}^{\mathrm{II}}$, …, $\mathfrak{p}^n$ can be chosen arbitrarily then one will call the function Φ *completely invariant.* We refer to signs of semi-invariants that remain invariant under all renormalizations as *sign invariants.*

If one represents geometric structures by means of homogeneous six-vectors then the projective-invariant relations will be given by complete invariants and sign invariants; we will give examples of them in § 9. By contrast, if one normalizes the six-vectors inhomogeneously in such a way that they remain normalized under unity transformations

([1]) **L. Cremona**: *Les figures réciproques en statique graphique*, Paris, 1885.
([2]) Cf., **W. Blaschke** and **G. Thomsen:** *Differentialgeometrie III*, § 10.

then the semi-invariants already produce the projective-invariant properties. We will treat families of lines, systems of lines, and line complexes in that way in the later chapters.

If n six-vectors $\mathfrak{p}^{\mathrm{I}}, \mathfrak{p}^{\mathrm{II}}, \ldots, \mathfrak{p}^{n}$ *are given then:*

A) *All scalar products* $\mathfrak{p}^i\mathfrak{p}^k$,

B) *All coefficients of possible linear combinations:*

$$\mathfrak{p}^m = \delta_1^m \mathfrak{p}^{\mathrm{I}} + \ldots + \delta_n^m \mathfrak{p}^n$$

will be semi-invariant.

Proof: A) follows from (23), and one confirms B) by a short calculation. From:

$$p_\rho^m = \delta_1^m p_\rho^{\mathrm{I}} + \ldots + \delta_n^m p_\rho^n,$$

(19) will imply, after multiplying by γ_σ^ρ and summing over ρ from 1 to 6:

$$\tilde{p}_\sigma^m = \delta_1^m \tilde{p}_\sigma^{\mathrm{I}} + \ldots + \delta_n^m \tilde{p}_\sigma^n .$$

We will see that all invariants will revert to types A) and B).

25. Semi-invariants of $n = 6$ **linearly-independent six-vectors.**

2×6 *linearly-independent six-vectors* $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{VI}}$ *and* $\tilde{\mathfrak{p}}^{\mathrm{I}}, \ldots, \tilde{\mathfrak{p}}^{\mathrm{VI}}$ *can be taken to each other by a unity transformation if and only if:*

$$\mathfrak{p}^i\mathfrak{p}^k = \tilde{\mathfrak{p}}^i\tilde{\mathfrak{p}}^k$$

is fulfilled for all i, k = I, …, VI. *As a result, the scalar products* $\mathfrak{p}^i\mathfrak{p}^k$ *define a complete system of semi-invariants of* $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{VI}}$; *i.e., all semi-invariants of the* $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{VI}}$ *can be expressed in terms of the scalar products* $\mathfrak{p}^i\mathfrak{p}^k$.

Proof: There is precisely one transformation:

$$\tilde{\mathfrak{p}}_\rho = \gamma_\rho^\sigma p_\sigma$$

that takes the $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{VI}}$ to $\tilde{\mathfrak{p}}^{\mathrm{I}}, \ldots, \tilde{\mathfrak{p}}^{\mathrm{VI}}$. For any fixed ρ (r = I, …, VI), the six unknowns $\gamma_\rho^1, \ldots, \gamma_\rho^6$ can, in fact, be determined uniquely by the six linear inhomogeneous equations:

$$\tilde{\mathfrak{p}}^{\mathrm{I}}_{\rho} = \gamma^{\sigma}_{\rho}\, p^{\mathrm{I}}_{\sigma}, \quad \tilde{\mathfrak{p}}^{\mathrm{II}}_{\rho} = \gamma^{\sigma}_{\rho}\, p^{\mathrm{II}}_{\sigma}, \quad \ldots, \qquad \tilde{\mathfrak{p}}^{\mathrm{VI}}_{\rho} = \gamma^{\sigma}_{\rho}\, p^{\mathrm{VI}}_{\sigma},$$

since the determinant of the system of equations | $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{VI}}$ | ([1]) does not vanish, due to the linear independence of $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{VI}}$. Since the $\tilde{\mathfrak{p}}^{\mathrm{I}}, \ldots, \tilde{\mathfrak{p}}^{\mathrm{VI}}$ are linearly independent, the determinant of the γ^{σ}_{ρ} is not zero, either.

The transformation thus-found is a unity transformation: From no. **12**, any six-vector $\mathfrak{q}$ can, in fact, be a linear combination of $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{VI}}$:

$$\mathfrak{q} = \delta_1 \mathfrak{p}^{\mathrm{I}} + \ldots + \delta_6 \mathfrak{p}^{\mathrm{VI}};$$

since the $\delta_1, \ldots, \delta_6$ are semi-invariant, from no. **24**, one will also have:

$$\tilde{\mathfrak{q}} = \delta_1 \tilde{\mathfrak{p}}^{\mathrm{I}} + \ldots + \delta_6 \tilde{\mathfrak{p}}^{\mathrm{VI}}$$

then, and finally:

$$\tilde{\mathfrak{q}}\tilde{\mathfrak{q}} = \sum_{i,k=1}^{6} \delta_i \delta_k \tilde{\mathfrak{p}}^i \tilde{\mathfrak{p}}^k = \sum \delta_i \delta_k \mathfrak{p}^i \mathfrak{p}^k = \mathfrak{q}\mathfrak{q}.$$

26. Discriminant of *n* six-vectors. We shall call the semi-invariant expression:

$$\begin{vmatrix} (\mathfrak{p}^{\mathrm{I}}\mathfrak{p}^{\mathrm{I}}) & (\mathfrak{p}^{\mathrm{I}}\mathfrak{p}^{\mathrm{II}}) & \cdots & (\mathfrak{p}^{\mathrm{I}}\mathfrak{p}^{n}) \\ \cdots & \cdots & \cdots & \cdots \\ (\mathfrak{p}^{n}\mathfrak{p}^{\mathrm{I}}) & (\mathfrak{p}^{n}\mathfrak{p}^{\mathrm{II}}) & \cdots & (\mathfrak{p}^{n}\mathfrak{p}^{n}) \end{vmatrix},$$

which we will also write as $D_n(\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{n})$, for brevity, the *discriminant* of $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{n}$; D_n is a symmetric determinant. For the sake of what follows, we shall provide some lemmas regarding the discriminants D_n :

1. *One necessarily has* $D_n = 0$ *for n linearly-dependent six-vectors* $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{n}$.

Proof: One obtains a system of n linear homogeneous equations in the δ_i that do not all vanish from:

$$\delta_1 \mathfrak{p}^{\mathrm{I}} + \ldots + \delta_n \mathfrak{p}^{n} = 0$$

by scalar-multiplying this by $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{n}$; one must then have $D_n = 0$ for the determinant D_n of the system of equations.

2. *Six six-vectors* $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{n}$ *fulfill the identity:*

([1]) | $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{VI}}$ | means the determinant of the 6×6 coordinate of $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{VI}}$.

(34)
$$D_6 = \begin{vmatrix} (\mathfrak{p}^{\mathrm{I}}\mathfrak{p}^{\mathrm{I}}) & \cdots & (\mathfrak{p}^{\mathrm{I}}\mathfrak{p}^{\mathrm{VI}}) \\ \cdots & \cdots & \cdots \\ (\mathfrak{p}^{\mathrm{VI}}\mathfrak{p}^{\mathrm{I}}) & \cdots & (\mathfrak{p}^{\mathrm{VI}}\mathfrak{p}^{\mathrm{VI}}) \end{vmatrix} = - \mid \mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{VI}} \mid^2,$$

in which $\mid \mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{VI}} \mid$ *means the determinant of the* 6×6 *coordinates of* $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{VI}}$.

Proof: It follows from a double application of the multiplication law for determinants that:

$$\begin{vmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{vmatrix} \cdot \mid \mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{VI}} \mid \cdot \mid \mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{VI}} \mid = \begin{vmatrix} (\mathfrak{p}^{\mathrm{I}}\mathfrak{p}^{\mathrm{I}}) & \cdots & (\mathfrak{p}^{\mathrm{I}}\mathfrak{p}^{\mathrm{VI}}) \\ \cdots & \cdots & \cdots \\ (\mathfrak{p}^{\mathrm{VI}}\mathfrak{p}^{\mathrm{I}}) & \cdots & (\mathfrak{p}^{\mathrm{VI}}\mathfrak{p}^{\mathrm{VI}}) \end{vmatrix}.$$

3. Let $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^n$ and $\mathfrak{q}^{\mathrm{I}}, \ldots, \mathfrak{q}^n$ be n linearly-independent six-vectors, and let each $\mathfrak{q}^i$ be a linear combination of the $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^n$:

(35)
$$\mathfrak{q}^i = \delta_1^i \mathfrak{p}^{\mathrm{I}} + \ldots + \delta_n^i \mathfrak{p}^n \qquad (i = 1, 2, \ldots, n).$$

We further set:

$$\mathfrak{p}^i \mathfrak{p}^k = p^{ik}, \qquad \mathfrak{q}^i \mathfrak{q}^k = q^{ik},$$

to abbreviate. One then has: *The quadratic forms* $p^{ik} \lambda_i \lambda_k$ *and* $q^{rs} \mu_r \mu_s$ *are equivalent – i.e., they can go to each other by a non-singular (real) linear transformation, namely:*

(36)
$$\lambda_i = \delta_i^1 \mu_1 + \ldots + \delta_i^n \mu_n .$$

Proof: By substituting (36), one will get:

$$p^{ik} \lambda_i \lambda_k = p^{ik} \ \delta_i^r \delta_k^s \mu_r \mu_s = q^{rs} \mu_r \mu_s ,$$

since, from (35), one has:

$$q^{rs} = \delta_i^r \mathfrak{p}^i \delta_k^s \mathfrak{p}^k = \delta_i^r \delta_k^s \ p^{ik}.$$

From the fundamental theorems on equivalent quadratic forms, which we can assume to be known from algebra ([1]), and which we have already mentioned in no. **4** (to some extent), one will now get:

([1]) Cf., e.g., **O. Perron**, *Algebra I*, volume 8 in this collection, 1932, pps. 116 to 122.

The discriminants $D_n(\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^n)$ *and* $D_n(\mathfrak{q}^{\mathrm{I}}, \ldots, \mathfrak{q}^n)$ *have the same rank. The* δ_k^i *in* (35), *and thus, the linearly-independent six-vectors* $\mathfrak{q}^{\mathrm{I}}, \ldots, \mathfrak{q}^n$, *can be chosen in such a way that* $q^{ik}\,\mu_i\,\mu_k$ *reduces to the pure quadratic form:*

$$q^{11}(\mu_1)^2 + \ldots + q^{rr}(\mu_r)^2,$$

in which the r coefficients q^{ii} *are equal to* ± 1. *The number of positive and negative* q^{ii} *in this cannot change under any real non-singular linear transformation (the law of inertia for quadratic forms; cf., no.* **4**). *For* $r = n$, *the discriminants* $D_n(\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^n)$ *and* $D_n(\mathfrak{q}^{\mathrm{I}}, \ldots, \mathfrak{q}^n)$ *will have the same signs.*

4. *The discriminant* $D_6(\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{IV}})$ *of six linearly-independent six-vectors has rank* $r = 6$. *The form* $p^{ik}\,\lambda_i\,\lambda_k$ *can be transformed into the pure quadratic form:*

$$(\mu_1)^2 + (\mu_2)^2 + (\mu_3)^2 - (\mu_4)^2 - (\mu_5)^2 - (\mu_6)^2$$

(viz., three plus signs and three minus signs!).

Proof: $r = 6$ follows from (34). Since every six-vector $\mathfrak{q}$ is a linear combination of the $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{IV}}$, we can set, e.g.:

$$\sqrt{2}\,\mathfrak{q}^{\mathrm{I}} = 1 \mid 0 \mid 0 \mid 1 \mid 0 \mid 0, \qquad \sqrt{2}\,\mathfrak{q}^{\mathrm{II}} = 0 \mid 1 \mid 0 \mid 0 \mid 1 \mid 0, \qquad \sqrt{2}\,\mathfrak{q}^{\mathrm{III}} = 0 \mid 0 \mid 1 \mid 0 \mid 0 \mid 1,$$
$$\sqrt{2}\,\mathfrak{q}^{\mathrm{IV}} = 1 \mid 0 \mid 0 \mid -1 \mid 0 \mid 0, \qquad \sqrt{2}\,\mathfrak{q}^{\mathrm{V}} = 0 \mid 1 \mid 0 \mid 0 \mid -1 \mid 0, \qquad \sqrt{2}\,\mathfrak{q}^{\mathrm{VI}} = 0 \mid 0 \mid 1 \mid 0 \mid 0 \mid -1,$$

from which, the assertion follows, since:

$$\mathfrak{q}^{\mathrm{I}}\mathfrak{q}^{\mathrm{I}} = \mathfrak{q}^{\mathrm{III}}\mathfrak{q}^{\mathrm{III}} = -\,\mathfrak{q}^{\mathrm{IV}}\mathfrak{q}^{\mathrm{IV}} = -\,\mathfrak{q}^{\mathrm{V}}\mathfrak{q}^{\mathrm{V}} = -\,\mathfrak{q}^{\mathrm{VI}}\mathfrak{q}^{\mathrm{VI}} = 1; \qquad \mathfrak{q}^i\,\mathfrak{q}^k = 0 \qquad (i \neq k).$$

As a consequence of Theorem 1 and Theorem 4, we mention especially: *Six six-vectors are linearly dependent if and only if the discriminant* D_6 *vanishes.*

27. Semi-invariants of *n* arbitrary six-vectors.

Semi-invariants of $n < 6$ *linearly-independent six-vectors:*

Let $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^n$ and $\tilde{\mathfrak{p}}^{\mathrm{I}}, \ldots, \tilde{\mathfrak{p}}^n$ ($n < 6$) be given, linearly-independent six-vectors whose scalar products agree. By adding suitable ([1]) six-vectors $\mathfrak{p}^{n+1}, \ldots, \mathfrak{p}^{\mathrm{VI}}$ and $\tilde{\mathfrak{p}}^{n+1}, \ldots, \tilde{\mathfrak{p}}^{\mathrm{VI}}$, one will obtain 2×6 six-vectors, for which:

([1]) We leave the proof of the existence of this six-vector to the reader.

$$\mathfrak{p}^i \mathfrak{p}^k = \tilde{\mathfrak{p}}^i \tilde{\mathfrak{p}}^k$$

is fulfilled for all i, k from I to VI. From no. **25**, there will then exist a unity transformation that takes $\mathfrak{p}^1, \ldots, \mathfrak{p}^{\mathrm{VI}}$ and $\tilde{\mathfrak{p}}^1, \ldots, \tilde{\mathfrak{p}}^{\mathrm{VI}}$, and one will have the theorem:

The theorem that was proved in no. **25** *for* $n = 6$ *is also true for* $2 \times n$ *linearly-independent unit vectors* $\mathfrak{p}^1, \ldots, \mathfrak{p}^n$ *and* $\tilde{\mathfrak{p}}^1, \ldots, \tilde{\mathfrak{p}}^n$ *for* $n < 6$.

Semi-invariants of n **arbitrary six-vectors:**

Let n six-vectors be given, and let $r \leq n$ be the greatest possible number of linearly-independent six-vectors that are included in them. We then enumerate r linearly-independent six-vectors by $\mathfrak{p}^1, \ldots, \mathfrak{p}^r$ and obtain linear combinations:

$$\mathfrak{p}^m = \delta_1^m \mathfrak{p}^{\mathrm{I}} + \ldots + \delta_r^m \mathfrak{p}^r \qquad (m = r + 1, \ldots, n)$$

for the remaining six-vectors. The δ_i^k fix the $\mathfrak{p}^m$ in terms of the $\mathfrak{p}^1, \ldots, \mathfrak{p}^r$ uniquely, and from no. **24**, they will be semi-invariant. As a result, one will have:

The $2 \times n$ *six-vectors* $\mathfrak{p}^1, \ldots, \mathfrak{p}^n$ *and* $\tilde{\mathfrak{p}}^1, \ldots, \tilde{\mathfrak{p}}^n$, *each of which have linear dependency relations, can be taken to each other by a unity transformation if and only if:*

A) *The scalar products of the linearly-independent six-vectors* $\mathfrak{p}^1, \ldots, \mathfrak{p}^r$ ($\tilde{\mathfrak{p}}^1, \ldots, \tilde{\mathfrak{p}}^r$) *coincide,*

B) *The coefficients of the linear combinations of the linearly-independent six-vectors* $\mathfrak{p}^{r+1}, \ldots, \mathfrak{p}^n$ *and* $\tilde{\mathfrak{p}}^{r+1}, \ldots, \tilde{\mathfrak{p}}^n$ *coincide.*

The scalar products in A) *and the coefficients in* B) *will then define a complete system of semi-invariants of the* $\mathfrak{p}^1, \ldots, \mathfrak{p}^n$.

28. Complete invariants and sign invariants of n **arbitrary six-vectors**. The complete invariants are the semi-invariants that do not change under renormalizations:

$$\tilde{\mathfrak{p}}^{\mathrm{I}} = \sigma^{\mathrm{I}} \mathfrak{p}^{\mathrm{I}}, \qquad \ldots, \; \tilde{\mathfrak{p}}^n = \sigma^n \mathfrak{p}^n$$

(no. **24**). The scalar products transform as:

$$\tilde{\mathfrak{p}}^i \tilde{\mathfrak{p}}^k = \sigma^i \sigma^k \mathfrak{p}^i \mathfrak{p}^k \qquad \text{(not summed!)} \tag{37}$$

under a renormalization.

Moreover, it follows from a linear combination:

$$\mathfrak{p}^m = \delta_1^m \mathfrak{p}^{\mathrm{I}} + \ldots + \delta_n^m \mathfrak{p}^n ,$$

after a renormalization, that:

$$\frac{1}{\sigma^m}\tilde{\mathfrak{p}}^m = \frac{\delta_1^m}{\sigma^1}\tilde{\mathfrak{p}}^{\mathrm{I}} + \ldots + \frac{\delta_n^m}{\sigma^n}\tilde{\mathfrak{p}}^n ,$$

and one will then have:

(38) $$\tilde{\delta}_k^m = \frac{\sigma^m}{\sigma^k}\delta_k^m \qquad \text{(not summed!)}$$

for the transformation of the coefficients. This implies the following prescription for the definition of complete invariants and sign invariants of n given six-vectors $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^n$:

One combines the substitution formulas (37), (38) *that were given in no.* **27** *for the semi-invariants* A) *and* B). *The expressions* $\Phi(\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^n) = \Phi(\tilde{\mathfrak{p}}^{\mathrm{I}}, \ldots, \tilde{\mathfrak{p}}^n)$ *that remain after eliminating the* σ^i *will be complete invariants. At the same time, the process of elimination will produce sign invariants* Ω, *namely, the signs of the expression that are only multiplied by powers of* σ^1 *under renormalization* ([1]).

One has the following theorem for the invariants Φ and Ω that are obtained in that way:

n six-vectors $\tilde{\mathfrak{p}}^{\mathrm{I}}, \ldots, \tilde{\mathfrak{p}}^n$ can be taken to n six-vectors $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^n$ by a principal projectivity, and thus, into $\sigma^1\mathfrak{p}^{\mathrm{I}}, \ldots, \sigma^n\mathfrak{p}^n$, with suitably-chosen $\sigma^i \neq 0$, by a unity transformation if and only if the complete invariants Φ and sign invariants Ω in the $\mathfrak{p}^i$ and $\tilde{\mathfrak{p}}^i$ coincide.

Proof: By the agreement of the complete invariants Φ and the sign invariants Ω, the σ^i are determined to be real from (37) and (38). From no. **27**, there will then exist a unity transformation that takes the $\tilde{\mathfrak{p}}^i$ to the $\sigma^i\mathfrak{p}^i$.

The prescription for determining projective invariants that was given in this paragraph will be carried out for some simple examples in § 9.

§ 8. Manifolds of linear complexes.

29. Notations. We shall call the complex manifold:

$$\mathfrak{a} = \lambda_1\mathfrak{a}^1 + \ldots + \lambda_r\mathfrak{a}^r$$

([1]) It is not necessary that the sign invariants should be independent of each other and the complete invariants.

that is spanned by $r < 6$ linearly-independent complexes $\mathfrak{a}^1, \ldots, \mathfrak{a}^r$, when we exclude $\lambda_1 = \lambda_2 = \ldots = \lambda_r = 0$:

A pencil of complexes	*for* $r = 2$,
A bundle of complexes	“ $r = 3$,
A bush of complexes	“ $r = 4$,
A forest of complexes	“ $r = 5$.

For $r = 6$, one will get the *totality of all complexes* ([1]).

Under the mapping of complexes to five-dimensional projective space (no. **21**), the pencils of complexes will correspond to lines, the bundles of complexes, to planes, etc.

Two complexes $\mathfrak{a}$, $\mathfrak{b}$, whose six-vectors fulfill the equation:

$$\mathfrak{a}\mathfrak{b} = 0$$

shall be called *conjugate*. For example, two singular complexes with intersecting axes will be conjugate, as well as a non-singular complex $\mathfrak{a}$ and a singular complex $\mathfrak{b}$ whose axis belongs to the complex $\mathfrak{a}$. From (22), projective maps will always transform conjugate complexes into conjugate ones.

All complexes that are conjugate to a complex will define a forest, and the complexes that are conjugate to all complexes of a pencil (bundle, bush, resp.) will define a bush (bundle, pencil, resp.). Precisely one complex is conjugate to all complexes of a forest. The following then correspond as conjugates:

individual complex $\leftrightarrow$ *forest, pencil* $\leftrightarrow$ *bush, bundle* $\leftrightarrow$ *bundle.*

Proof: The complexes $\mathfrak{b}$ that are conjugate to the manifold $\lambda_1\mathfrak{a}^1 + \ldots + \lambda_r \mathfrak{a}^r$ are given by:

$$\mathfrak{a}^{\mathrm{I}}\mathfrak{b} = 0, \ldots, \mathfrak{a}^r\mathfrak{b} = 0.$$

All solutions of this system of linear equations of rank r in $b_1, \ldots, b_6$ can be expressed as linear combinations of $(6 - r)$ independent solutions.

30. Classification of the pencils. The singular complexes $\mathfrak{c}$ that are contained in the pencil $\lambda_1\mathfrak{a}^{\mathrm{I}} + \lambda_2\mathfrak{a}^{\mathrm{II}}$ are given by:

$$\mathfrak{c}\mathfrak{c} = \lambda_1\lambda_1(\mathfrak{a}^{\mathrm{I}}\mathfrak{a}^{\mathrm{I}}) + 2\,\lambda_1\lambda_2(\mathfrak{a}^{\mathrm{I}}\mathfrak{a}^{\mathrm{II}}) + \lambda_2\lambda_2(\mathfrak{a}^{\mathrm{II}}\mathfrak{a}^{\mathrm{II}}) = 0. \tag{39}$$

([1]) One will find a thorough synthetic treatment of linear complexes and their linear manifolds in e.g., **Th. Reye**: *Geometrie der Lage II*, Leipzig, 1923.

If one interprets the λ_1, λ_2 as the homogeneous projective point coordinates of a line then the complexes of the pencil will be mapped to points of that line in a one-to-one correspondence. In particular, the singular complexes (39) will correspond to the point-pair:

$$a^{11}\lambda_1\lambda_1 + 2\,a^{12}\lambda_1\lambda_2 + a^{22}\lambda_2\lambda_2 = 0 \tag{39*}$$

in the image line, in which one has set $\mathfrak{a}^i\mathfrak{a}^k = a^{ik}$. We can then derive the following *projectively-invariant classification of the pencils of complexes* from the projective classification (no. **4**) of point-pairs, in which r means the rank of the discriminant $D_2(\mathfrak{a}^{\mathrm{I}}, \mathfrak{a}^{\mathrm{II}})$, and s is the signature of the quadratic form $a^{ik}\lambda_i\lambda_k$:

a) $r = 2$: $\left\{\begin{array}{l} s = 0: \ \ \textit{hyperbolic} \\ s = \pm 2: \textit{elliptic} \end{array}\right\}$ *pencils with* $\left.\begin{array}{c} 2 \\ 0 \end{array}\right\}$ *singular complexes.*

b) $r = 1$: *parabolic pencil with one singular complex.*

c) $r = 0$: *singular pencil with nothing but singular complexes; their axes will define a pencil of lines.*

The two axes of the singular complexes of a hyperbolic pencil are skew, since singular complexes with intersecting axes would imply that $r = 0$, which would then be a singular pencil.

Any pencil of complexes can be represented in terms of the basic complexes:

$$\mathfrak{a}^{\mathrm{I}} = 1 \mid 0 \mid 0 \mid \eta_1 \mid 0 \mid 0, \qquad \mathfrak{a}^{\mathrm{II}} = 0 \mid 1 \mid 0 \mid 0 \mid \eta_1 \mid 0, \tag{40}$$

after a suitable coordinate transformation, in which:

$$a)\ \begin{array}{l} \eta_1 = -\eta_2 = \pm 1, \\ \eta_1 = \eta_2 = \pm 1, \end{array} \qquad b)\quad \eta_1 = \pm 1,\ \ \eta_2 = 0, \qquad c)\quad \eta_1 = \eta_2 = 0.$$

Proof: From no. **26**, paragraph 3, one can always give linearly-independent linear combinations:

$$\mathfrak{a}^{*\mathrm{I}} = \delta_1^1\mathfrak{a}^{\mathrm{I}} + \delta_2^1\mathfrak{a}^{\mathrm{II}}, \qquad \mathfrak{a}^{*\mathrm{II}} = \delta_1^2\mathfrak{a}^{\mathrm{I}} + \delta_2^2\mathfrak{a}^{\mathrm{II}},$$

for which:

$$\mathfrak{a}^{*\mathrm{I}}\mathfrak{a}^{*\mathrm{I}} = 2\eta_1, \quad \mathfrak{a}^{*\mathrm{II}}\mathfrak{a}^{*\mathrm{II}} = 2\eta_2, \quad \mathfrak{a}^{*\mathrm{I}}\mathfrak{a}^{*\mathrm{II}} = 0.$$

Since the six-vectors $\mathfrak{a}^{*\mathrm{I}}$, $\mathfrak{a}^{*\mathrm{II}}$ have only these three scalar products as semi-invariants, from no. **27**, there will exist a unity transformation that will take $\mathfrak{a}^{*\mathrm{I}}$ ($\mathfrak{a}^{*\mathrm{II}}$, resp.) to $1 \mid 0 \mid 0 \mid \eta_1 \mid 0 \mid 0$ ($0 \mid 1 \mid 0 \mid 0 \mid \eta_2 \mid 0$, resp.)

31. Axes of a pencil of complexes. Cylindroid. We shall now determine the axes of the complex of a pencil $\lambda_1\mathfrak{a}^{\mathrm{I}} + \lambda_2\mathfrak{a}^{\mathrm{II}}$ and recall that the axes of non-singular complexes are invariant under only special projectivities, and in particular, motions and reflections:

We employ a rectangular coordinate system whose axis 1 coincides with the complex axis $\mathfrak{s}^{\mathrm{I}}$ of $\mathfrak{a}^{\mathrm{I}}$, and whose axis 3 cuts both complex axes $\mathfrak{s}^{\mathrm{I}}$, $\mathfrak{s}^{\mathrm{II}}$ perpendicularly (Fig. 7). One can then set:

$$\mathfrak{a}^{\mathrm{I}} = \{\mathfrak{A}^{\mathrm{I}} \mid \bar{\mathfrak{A}}^{\mathrm{I}}\} \quad \text{and} \quad \mathfrak{a}^{\mathrm{II}} = \{\mathfrak{A}^{\mathrm{II}} \mid \bar{\mathfrak{A}}^{\mathrm{II}}\}$$

with

$$\mathfrak{A}^{\mathrm{I}} = 1 \mid 0 \mid 0, \qquad \mathfrak{A}^{\mathrm{II}} = \cos\beta \mid \sin\beta \mid 0,$$
$$\bar{\mathfrak{A}}^{\mathrm{I}} = \rho \mid 0 \mid 0, \qquad \bar{\mathfrak{A}}^{\mathrm{II}} = \sigma \mid \tau \mid 0 .$$

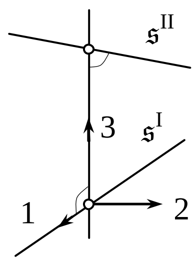


Figure 7.

From (31), $\mathfrak{A}^{\mathrm{I}}$, $\mathfrak{A}^{\mathrm{II}}$ establish the direction of the complex axes, and:

$$\mathfrak{B}^{\mathrm{I}} = \frac{\mathfrak{A}^{\mathrm{I}} \times \bar{\mathfrak{A}}^{\mathrm{I}}}{\mathfrak{A}^{\mathrm{I}}\mathfrak{A}^{\mathrm{I}}} = 0, \qquad \mathfrak{B}^{\mathrm{II}} = \frac{\mathfrak{A}^{\mathrm{II}} \times \bar{\mathfrak{A}}^{\mathrm{II}}}{\mathfrak{A}^{\mathrm{II}}\mathfrak{A}^{\mathrm{II}}} = 0 \mid 0 \mid \tau\cos\beta - \sigma\sin\beta$$

are the altitude vectors from O to the complex axes; in order for $\mathfrak{B}^{\mathrm{II}}$ to be perpendicular to the complex axis $\mathfrak{s}^{\mathrm{II}}$, and therefore parallel to the coordinate axis 3, $\bar{\mathfrak{A}}^{\mathrm{II}}$ must be perpendicular to the coordinate axis 3. Likewise, the axis $\mathfrak{s}$ of an arbitrary complex $\mathfrak{c} = \lambda_1\mathfrak{a}^{\mathrm{I}} + \lambda_2\mathfrak{a}^{\mathrm{II}} = \{\mathfrak{C} \mid \bar{\mathfrak{C}}\}$ with:

$$\mathfrak{C} = \lambda_1\mathfrak{A}^{\mathrm{I}} + \lambda_2\mathfrak{A}^{\mathrm{II}} = \lambda_1 + \lambda_2\cos\beta \mid \lambda_2\sin\beta \mid 0,$$
$$\bar{\mathfrak{C}} = \lambda_1\bar{\mathfrak{A}}^{\mathrm{I}} + \lambda_2\bar{\mathfrak{A}}^{\mathrm{II}} = \lambda_1\rho + \lambda_2\sigma \mid \lambda_2\tau \mid 0,$$

will be established by the direction vector $\mathfrak{C}$ and the altitude vector:

$$\mathfrak{B} = \frac{\mathfrak{C} \times \bar{\mathfrak{C}}}{\mathfrak{C}\mathfrak{C}} = 0 \mid 0 \mid \frac{(\lambda_1 + \lambda_2\cos\beta)\lambda_2\tau - (\lambda_1\rho + \lambda_2\sigma)\lambda_2\cos\beta}{(\lambda_1 + \lambda_2\cos\beta)^2 + (\lambda_2\sin\beta)^2} .$$

We normalize the homogeneous parameters $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ by:

$$(\lambda_1 + \lambda_2 \cos\beta)^2 + (\lambda_2 \sin\beta)^2 = 1,$$

and set:

$$\lambda_1 + \lambda_2 \cos\beta = \cos\varphi, \qquad \lambda_2 \sin\beta = \sin\varphi.$$

We will then get:

$$\mathfrak{C} = \cos\varphi \mid \sin\varphi \mid 0,$$
$$\mathfrak{B} = 0 \mid 0 \mid \lambda_1\lambda_2 (\tau - \rho \sin\beta) + \lambda_2^2 (\tau\cos\beta - \sigma\sin\beta).$$

There are four cases to distinguish now (cf., no. **39**):

1. $\beta = 0$; $(\tau\cos\beta - \sigma\sin\beta)^2 + (\tau - \rho\sin\beta)^2 \neq 0$.

The complex axes define a *pencil of parallels.*

2. $\beta \neq 0$; $(\tau\cos\beta - \sigma\sin\beta)^2 + (\tau - \rho\sin\beta)^2 = 0$.

The complex axes define *pencil of lines with vertex O*.

3. $\beta = 0$; $(\tau\cos\beta - \sigma\sin\beta)^2 + (\tau - \rho\sin\beta)^2 = 0$.

All complexes have the same axis.

4. $\beta \neq 0$; $(\tau\cos\beta - \sigma\sin\beta)^2 + (\tau - \rho\sin\beta)^2 \neq 0$;

hence, one will also have ([1]):

$$(\sigma - \rho\cos\beta)^2 + (\tau - \rho\sin\beta)^2 \neq 0.$$

The Ansatz:

$$\lambda_1\lambda_2 (\tau - \rho\sin\beta) + \lambda_2^2 (\tau\cos\beta - \sigma\sin\beta) = c + h \sin 2(\varphi - \varphi_0)$$

will be satisfied by:

$$4h^2 = \frac{1}{\sin^2\beta}\{(\tau - \rho\sin\beta)^2 + (\sigma - \rho\cos\beta)^2\} \neq 0, \quad \tan 2\varphi_0 = \frac{\rho\cos\beta - \sigma}{\tau - \rho\sin\beta}, \; c = h \sin 2\varphi_0 .$$

One can arrange that $c = \varphi_0 = 0$ by rotating and translating the coordinate system, and finally have:

$$\mathfrak{C} = \cos\varphi \mid \sin\varphi \mid 0, \quad \mathfrak{B} = 0 \mid 0 \mid h \sin 2\varphi.$$

The complex axes will then generate the third-order surface:

([1]) It would follow from $(\sigma - \rho\cos\beta)^2 + (\tau - \rho\sin\beta)^2 = 0$ that $\rho : \sigma : \tau = 1 : \cos\beta : \sin\beta$, and thus, also that $\tau\cos\beta - \sigma\sin\beta = 0$.

$$\frac{x_3}{x_4} = h \sin 2\varphi = 2h \sin \varphi \cos \varphi = 2h \,\frac{x_1 x_2}{x_1^2 + x_2^2}$$

in homogeneous, rectangular coordinates, or:

$$(x_1^2 + x_2^2)\; x_3 = 2h\, x_1\, x_2\, x_3 \qquad (h \neq 0), \tag{41}$$

which one calls a *cylindroid*. We will get more closely acquainted with this surface in no. **67** (cf., Fig. 18).

32. Classification of the bushes. The classification of the bushes $\mu_1\mathfrak{b}^{\mathrm{I}} + \ldots + \mu_{1V}\mathfrak{b}^{\mathrm{IV}}$ can be reduced to the classification of the pencils of complexes $\lambda_1\mathfrak{a}^{\mathrm{I}} + \lambda_2\mathfrak{a}^{\mathrm{II}}$ that are conjugate to the bushes: We appeal to the special representation (40) for the conjugate pencil, and can represent the bush in terms of the basic complexes:

$$\begin{aligned} \mathfrak{b}^{\mathrm{I}} &= -1 \,|\, 0 \,|\, 0 \,|\, \eta_1 \,|\, 0 \,|\, 0, \qquad & \mathfrak{b}^{\mathrm{II}} &= 0 \,|\, -1 \,|\, 0 \,|\, 0 \,|\, \eta_2 \,|\, 0, \\ \mathfrak{b}^{\mathrm{III}} &= 0 \,|\, 0 \,|\, 0 \,|\, 0 \,|\, 0 \,|\, 1, & \mathfrak{b}^{\mathrm{IV}} &= 0 \,|\, 0 \,|\, 1 \,|\, 0 \,|\, 0 \,|\, 0. \end{aligned} \tag{42}$$

In fact, these $\mathfrak{b}^i$ are linearly independent and fulfill the conjugacy requirement $\mathfrak{a}^k\mathfrak{b}^i = 0$.

If r is the rank of the discriminant D_4 ($\mathfrak{b}^{\mathrm{I}}$, $\mathfrak{b}^{\mathrm{II}}$, $\mathfrak{b}^{\mathrm{III}}$, $\mathfrak{b}^{\mathrm{IV}}$) of the bush then one will get the *projective-invariant classification* of the bushes that are conjugate to the pencils that were summarized in no. **30** on the basis of (42):

$$(a)\ r = 4\text{:}\ \left.\begin{cases} \alpha)\ D_4 > 0\text{: } \textit{hyperbolic} \\ \beta)\ D_4 < 0\text{: } \textit{elliptic} \end{cases}\right\}\ \textit{bush}; \qquad \begin{aligned} &b)\ r = 3\text{: } \textit{parabolic bush}, \\ &c)\ r = 2\text{: } \textit{singular bush}. \end{aligned}$$

From paragraph 3 of no. **26**, these case distinctions will also be true when one represents the bush by any four linearly-independent complexes.

The lines that are common to all complexes of the bush $\mu_1\mathfrak{b}^{\mathrm{I}} + \ldots + \mu_4\mathfrak{b}^{\mathrm{IV}}$ are the axes of the singular complexes of the conjugate pencils $\lambda_1\mathfrak{a}^{\mathrm{I}} + \lambda_2\mathfrak{a}^{\mathrm{II}}$. Therefore, a bush has either:

$$a)\ \left.\begin{cases} \alpha) & 2 \\ \beta) & 0 \end{cases}\right\}\ \text{common lines,}$$

b) One common line,

c) One common pencil of lines.

33. Linear systems of lines. The axes of all singular complexes $\mathfrak{b}$ that are contained in $\mu_1\mathfrak{b}^{\mathrm{I}} + \ldots + \mu_4\mathfrak{b}^{\mathrm{IV}}$ define a set of lines that we shall refer to as a *linear system of lines* ([1]). It consists of the lines that are common to all complexes of the conjugate pencil $\lambda_1\mathfrak{a}^{\mathrm{I}} + \lambda_2\mathfrak{a}^{\mathrm{II}}$. In particular, all lines $\mathfrak{d}$ of the system of lines will cut the axes of the singular complex $\mathfrak{c}$ that are contained in the conjugate pencil; these axes will be called the *focal lines* of the system of lines. Since four linearly-independent linear complexes establish a bush uniquely, one will have:

A system of lines is determined uniquely by four linearly-independent lines that belong to it.

We shall distinguish *hyperbolic, elliptic, parabolic, and singular systems of lines* according to the type of bush $\mu_1\mathfrak{b}^{\mathrm{I}} + \ldots + \mu_4\mathfrak{b}^{\mathrm{IV}}$. The following theorem will then be true:

A hyperbolic system of lines has any skew lines as focal lines and consists of the common lines of intersection of these two focal lines; the focal lines do not belong to the system of lines.

A singular system of lines possesses a pencil of lines (vertex x, plane w) of focal lines and decomposes into the bundle of lines x and the line field w.

The proof follows immediately from no. **30**:

A parabolic system of lines possesses a single focal line and consists of a family of line pencils (vertex x, plane w). The vertices x are the points of the focal lines, and the planes w are the planes through the focal lines; the focal line is then itself a line of the system of lines. The sequence of points x and the pencil of planes w are projectively-related.

Proof: We represent the conjugate pencil of complexes $\lambda_1\mathfrak{a}^{\mathrm{I}} + \lambda_2\mathfrak{a}^{\mathrm{II}}$ in terms of the basic complexes (40*b*). One then has $\mathfrak{a}^{\mathrm{I}}\mathfrak{a}^{\mathrm{I}} \neq 0$, $\mathfrak{a}^{\mathrm{II}}\mathfrak{a}^{\mathrm{II}} = 0$ and $\mathfrak{a}^{\mathrm{I}}\mathfrak{a}^{\mathrm{II}} = 0$. $\mathfrak{a}^{\mathrm{II}}$ is then a line of the non-singular complex $\mathfrak{a}^{\mathrm{I}}$, and the lines $\mathfrak{p}$ of the linear system of lines $\mathfrak{a}^{\mathrm{I}}\mathfrak{p} = \mathfrak{a}^{\mathrm{II}}\mathfrak{p} = 0$ are the lines of $\mathfrak{a}^{\mathrm{I}}$ that cut the fixed complex line $\mathfrak{a}^{\mathrm{II}}$. From no. **22**, however, the lines of the complex $\mathfrak{a}^{\mathrm{I}}$ that go through a fixed point x of $\mathfrak{a}^{\mathrm{II}}$ will lie in a plane w through $\mathfrak{a}^{\mathrm{II}}$, and the sequence of points x will be proportional to the pencil of planes w.

Any elliptical system of lines can be transformed projectively into a rotationally-symmetric system of lines. It arises in the following way: Let two parallel planes ε, ε^* be mapped to each other congruently by common altitudes. If one twists the planes ε, ε^* with respect to each other about one of those altitudes then the bundle of parallels of the connecting line between corresponding points ε, ε^* will be twisted into a rotationally-

([1]) In the literature, one also often finds the term "ray net."

symmetric system of lines; in addition, the imaginary lines of ε and ε^* will belong to the system of lines.

Proof: From the special Ansatz (40), one will get:

$$p_1 + p_4 = 0, \qquad p_2 + p_5 = 0$$

for an elliptic system. With consideration given to the fact that $\mathfrak{pp} = 0$, the parametric representation (parameters u^1, u^2):

$$p_1 = u^1, \quad p_2 = u^2, \quad p_3 = 1, \quad p_4 = -u^1, \quad p_5 = -u^2, \quad p_6 = (u^1)^2 + (u^2)^2$$

will follow from this, in which only the line $x_3 = x_4 = 0$ with the line coordinates 0 | 0 | 0 | 0 | 0 | 1 remains excluded. One gets the points of intersection of the line $\mathfrak{p}$ of the system of lines with:

the plane $x_3 = 0$: $\quad x_1 = u^2, \quad x_2 = -u^1, \quad x_4 = 1,$
the plane $x_4 = 0$: $\quad x_1 = u^2, \quad x_2 = \;\; u^2, \quad x_3 = 1$

from (14). We transform the coordinate tetrahedron by a projective map in the manner that is suggested by Fig. 8: 1, 2 are imaginary points, 34 is perpendicular to the plane ε (124) and the plane ε^* (123), 14 is perpendicular to 24, and therefore 13 will be perpendicular to 23. In addition, the unit point will lie on the middle plane that is parallel to ε and ε^*, as well as in the angle-bisecting plane of (134) and (234). The equations $x_1 = u^2$, $x_2 = -u^1$, $x_4 = 1$, and $x_1 = u^1$, $x_2 = u^2$, $x_3 = 1$ will then represent a congruence map of the parallel planes for which the points 4 and 3 will correspond to each other.

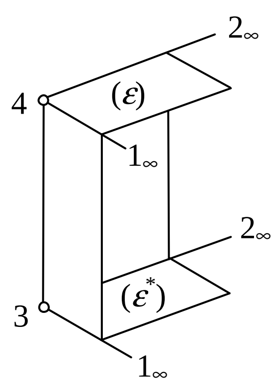


Figure 8.

In *algebraic geometry*, a system of lines is said to have *order n* (*class n*, resp.) when n lines go through every non-special point (n lines lie in every non-special plane, resp.), in the algebraic sense. The linear systems of lines that are treated here are then of order 1 and class 1; every point that does not lie on a focal line and every plane that does not go through a focal lines is incident with precisely one line of the system, which one can obtain by linear operations.

34. Classification of bundles. The singular complexes $\mathfrak{c}$ that are contained in the bundle $\lambda_1\,\mathfrak{a}^{\mathrm{I}} + \lambda_2\,\mathfrak{a}^{\mathrm{II}} + \lambda_3\,\mathfrak{a}^{\mathrm{III}}$ are given by:

$$\mathfrak{c}\mathfrak{c} = \sum_{i,k=1}^{3} \lambda_i \lambda_k\,(\mathfrak{a}^i\,\mathfrak{a}^k) = 0. \tag{43}$$

We interpret the λ_i as homogeneous point coordinates in a plane and map the complexes of the bundle to the points of that plane invertibly. In particular, the singular complexes (43) then correspond to the points of the second-order curve:

$$a^{ik}\lambda_i\lambda_k = 0 \tag{43*}$$

in the image plane, in which we have once more set $a^{ik} = \mathfrak{a}^i\,\mathfrak{a}^k$. By the same argument as in no. **30**, we then get the following *projective-invariant classification of the complex bundles*
:

$a)\ \ r = 3: \begin{Bmatrix} \alpha)\ \ s = \pm 1, \\ \beta)\ \ s = \pm 3, \end{Bmatrix} \qquad b)\ \ r = 2: \begin{Bmatrix} \alpha)\ \ s = 0, \\ \beta)\ \ s = \pm 2, \end{Bmatrix}, \qquad c)\ \ r = 1, \qquad d)\ \ r = 0.$

In this, r means the rank of the discriminant $D_3(\mathfrak{a}^{\mathrm{I}}, \mathfrak{a}^{\mathrm{II}}, \mathfrak{a}^{\mathrm{III}}, \mathfrak{a}^{\mathrm{IV}})$, and s is the signature of the quadratic form $a^{ik}\lambda_i\lambda_k$.

Moreover, one shows, in analogy to no. **30**:

Any complex bundle can be represented in terms of the basic complexes:

$$\mathfrak{a}^{\mathrm{I}} = 1\,|\,0\,|\,0\,|\,\eta_1\,|\,0\,|\,0, \qquad \mathfrak{a}^{\mathrm{II}} = 0\,|\,1\,|\,0\,|\,0\,|\,\eta_2\,|\,0, \qquad \mathfrak{a}^{\mathrm{III}} = 0\,|\,0\,|\,1\,|\,0\,|\,0\,|\,\eta_3\,, \tag{44}$$

after a suitable coordinate transformation, in which the η_i satisfy the conditions:

$a)\ \ (\eta_1)^2 = (\eta_2)^2 = (\eta_3)^2 = 1 \qquad \begin{cases} \alpha)\ \ \eta_1, \eta_2, \eta_3 \text{ are partly } +1, \text{ partly } -1, \\ \beta)\ \ \eta_1 = \eta_2 = \eta_3 = \pm 1, \end{cases}$

$b)\ \ \eta_3 = 0 \qquad \begin{cases} \alpha)\ \ \eta_1 = -\eta_2 = \pm 1, \\ \beta)\ \ \eta_1 = \eta_2 = \pm 1, \end{cases}$

$c)\ \ \eta_1 = \pm 1,\ \ \eta_2 = \eta_3 = 0, \qquad d)\ \ \eta_1 = \eta_2 = \eta_3 = 0.$

Two conjugate complex bundles $\lambda_1\,\mathfrak{a}^{\mathrm{I}} + \lambda_2\,\mathfrak{a}^{\mathrm{II}} + \lambda_3\,\mathfrak{a}^{\mathrm{III}}$ and $\mu_1\,\mathfrak{b}^{\mathrm{I}} + \mu_2\,\mathfrak{b}^{\mathrm{II}} + \mu_3\,\mathfrak{b}^{\mathrm{III}}$ always belong to the same type.

Proof: The complex bundle that is conjugate to (44) can be represented in terms of the basic complexes:

$$(45) \quad \mathfrak{b}^{\mathrm{I}} = -1 \mid 0 \mid 0 \mid \eta_1 \mid 0 \mid 0, \qquad \mathfrak{b}^{\mathrm{II}} = 0 \mid -1 \mid 0 \mid 0 \mid \eta_2 \mid 0, \qquad \mathfrak{b}^{\mathrm{III}} = 0 \mid 0 \mid -1 \mid 0 \mid 0 \mid \eta_3,$$

from which, the statement will follow immediately.

35. Quadrics. The axes of the singular complexes (43) that are contained in the bundle of complexes $\lambda_1\, \mathfrak{a}^{\mathrm{I}} + \lambda_2\, \mathfrak{a}^{\mathrm{II}} + \lambda_3\, \mathfrak{a}^{\mathrm{III}}$ define a set of lines to which we shall give the name of *quadric* ([1]). *A quadric is determined uniquely by three linearly-independent lines.* The lines of the quadric are invertibly related to the points of a second-order curve. For the various types of complex bundles, the associated quadric will contain:

$$a) \begin{cases} \alpha) \text{ A one-parameter family of skew lines,} \\ \beta) \text{ no lines ("null quadric"),} \end{cases}$$

$$b) \begin{cases} \alpha) \text{ Two pencils of lines with one common lines, but with different vertices} \\ \qquad \text{and different planes (i.e., } \textit{crossed pencils of lines}\text{),} \\ \beta) \text{ one line,} \end{cases}$$

c) One pencil of lines.

d) All complexes of the complex bundle are singular; their axes define a bundle of lines (dually: a planar line field).

These properties of quadrics are implied immediately from the discussion of second-order curves (43^*) whose coefficients are assigned according to (44): One of the rectilinear point sequences that belong to the second-order curve (43^*) will correspond to a pencil of lines on the quadric, and conversely. When no rectilinear point sequences can be split off from the second-order curve (43^*), any two lines of the quadric will be skew; along with two intersecting lines, all lines of the pencil that they determine will then belong to the quadric.

From no. **24**, two conjugate complex bundles will provide two conjugate quadrics of the same type. Any line of the one quadric will intersect all lines of the conjugate quadric. That will imply the following *geometric relations for conjugate quadrics:*

a) α) Since:

$$\mathfrak{p}\mathfrak{p} = \mathfrak{p}'\mathfrak{p}' = \mathfrak{p}''\mathfrak{p}'' = 0; \qquad \mathfrak{p}'\mathfrak{p}'' \neq 0, \qquad \mathfrak{p}''\mathfrak{p} \neq 0, \qquad \mathfrak{p}\mathfrak{p}' \neq 0,$$

any three skew lines $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{p}''$ will have a non-vanishing discriminant $D_2(\mathfrak{p}, \mathfrak{p}', \mathfrak{p}'')$. From Theorem 1 of no. **26**, they will then be linearly-independent, and as a result, they will determine a quadric I.

([1]) The name should express the idea that the set of lines is given by the quadratic equation (43^*) in the λ_i .

All common lines $\mathfrak{q}$ that meet $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{p}''$ will define the quadric II that is conjugate to a quadric I; all common lines that meet any three lines $\mathfrak{q}$, $\mathfrak{q}'$, $\mathfrak{q}''$ of the quadric II will generate the quadric I and thus cut all remaining lines of the quadric II (Fig. 9).

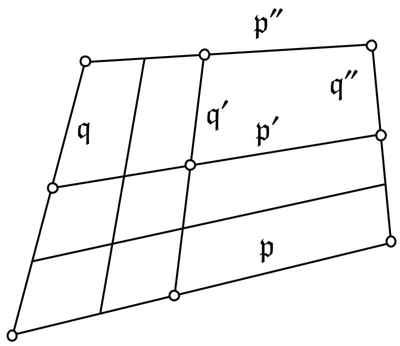


Figure 9.

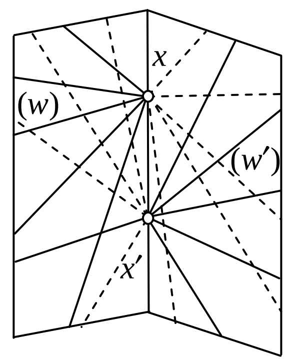


Figure 10.

The lines of the quadric I (II, resp.) cut the lines of the quadric II (I, resp.) along projective sequences of points; the point sequences that are cut out of the lines of the quadric II along two lines of the quadric I – e.g., along $\mathfrak{p}$, $\mathfrak{p}'$ – will then be perspective to the pencil of planes through any third line of the quadric I.

b) α): I consists of the pencils of lines $\{x \mid w\}$ and $\{x' \mid w'\}$, while II consists of the pencils of lines $\{x \mid w'\}$ and $\{x' \mid w\}$ (Fig. 10).

β): I and II degenerate into the same lines.

c): I and II degenerate into the same pencil of lines.

One confirms the statements *b*) and *c*) by juxtaposing (44) and (45).

The surface that is spanned by two conjugate quadrics *a*), α) is a *hyperboloid* (cf., no. **4**); correspondingly, we can the bundle and the quadrics *a*), α) *hyperboloidal*, as well as any four lines of a quadric *a*), *a*).

§ 9. Simplest projective invariants of linear complexes and straight lines.

36. Winding sense of a non-singular complex; three skew lines. A *non-singular complex* $\mathfrak{a}$ yields $\mathfrak{a}\mathfrak{a} \neq 0$ as its only semi-invariant, and therefore, from no. **28**, the sign of $\mathfrak{a}\mathfrak{a}$ is the single sign invariant, but not a complete invariant. As a result, all complexes with $\mathfrak{a}\mathfrak{a} > 0$, and likewise all complexes with $\mathfrak{a}\mathfrak{a} < 0$, will transform into each other by a principal projectivity. We shall call complexes with positive (negative, resp.) sign invariants *positively-wound (negatively-wound*, resp.).

Complexes that are wound the same way will determine screws with the same screw sense. ([1]).

Proof: From no. **23**, two complexes $\mathfrak{a}^{\mathrm{I}}$ and $\mathfrak{a}^{\mathrm{II}}$ can be taken to $0 \mid 0 \mid b^{\mathrm{I}} \mid 0 \mid 0 \mid \bar{b}^{\mathrm{I}}$ and $0 \mid 0 \mid b^{\mathrm{II}} \mid 0 \mid 0 \mid \bar{b}^{\mathrm{II}}$ by motion transformations. Since motions are special principal projectivities, one can represent them by unity transformations, and then have $\mathfrak{a}^{\mathrm{I}}\mathfrak{a}^{\mathrm{I}} = 2\, b^{\mathrm{I}}\bar{b}^{\mathrm{I}}$, $\mathfrak{a}^{\mathrm{II}}\mathfrak{a}^{\mathrm{II}} = 2b^{\mathrm{II}}\bar{b}^{\mathrm{II}}$; however, from no. **23**, the sign of $b\bar{b}$ characterizes the screw sense.

All non-singular complexes of a parabolic pencil are wound the same; one can then classify the parabolic line systems into positively-wound and negatively-wound ones in a manner that is invariant under principal projectivities.

Proof: A parabolic pencil of complex can always be given by:

$$\mathfrak{c} = \lambda_1\, \mathfrak{a}^{\mathrm{I}} + \lambda_2\, \mathfrak{a}^{\mathrm{II}}$$

with

$$\mathfrak{a}^{\mathrm{I}}\mathfrak{a}^{\mathrm{I}} = \mathfrak{a}^{\mathrm{I}}\mathfrak{a}^{\mathrm{II}} = 0, \qquad \mathfrak{a}^{\mathrm{II}}\mathfrak{a}^{\mathrm{II}} \neq 0\,.$$

Since:

$$\mathfrak{c}\mathfrak{c} = (\lambda_2)^2\, \mathfrak{a}^{\mathrm{II}}\mathfrak{a}^{\mathrm{II}},$$

$\mathfrak{c}\mathfrak{c}$ will have the same sign as $\mathfrak{a}^{\mathrm{II}}\mathfrak{a}^{\mathrm{II}}$, for any value of $\lambda_2 \neq 0$.

Two *non-singular non-conjugate complexes* yield the three non-vanishing scalar products $\mathfrak{a}^i\, \mathfrak{a}^k$ as the complete system of semi-invariants. From the prescription of no. **28**, we set:

$$\tilde{\mathfrak{a}}^{\mathrm{I}}\tilde{\mathfrak{a}}^{\mathrm{I}} = \sigma^1\sigma^1\, \mathfrak{a}^{\mathrm{I}}\, \mathfrak{a}^{\mathrm{I}}, \qquad \tilde{\mathfrak{a}}^{\mathrm{I}}\tilde{\mathfrak{a}}^{\mathrm{II}} = \sigma^1\sigma^2\, \mathfrak{a}^{\mathrm{I}}\, \mathfrak{a}^{\mathrm{II}}, \qquad \tilde{\mathfrak{a}}^{\mathrm{II}}\tilde{\mathfrak{a}}^{\mathrm{II}} = \sigma^2\sigma^2\, \mathfrak{a}^{\mathrm{II}}\, \mathfrak{a}^{\mathrm{II}}.$$

One will obtain the single complete invariant:

$$\frac{(\tilde{\mathfrak{a}}^{\mathrm{I}}\, \tilde{\mathfrak{a}}^{\mathrm{I}})(\tilde{\mathfrak{a}}^{\mathrm{II}}\, \tilde{\mathfrak{a}}^{\mathrm{II}})}{(\tilde{\mathfrak{a}}^{\mathrm{I}}\, \tilde{\mathfrak{a}}^{\mathrm{II}})^2} = \frac{(\mathfrak{a}^{\mathrm{I}}\, \mathfrak{a}^{\mathrm{I}})(\mathfrak{a}^{\mathrm{II}}\, \mathfrak{a}^{\mathrm{II}})}{(\mathfrak{a}^{\mathrm{I}}\, \mathfrak{a}^{\mathrm{II}})^2}.$$

Let us skip over the geometric interpretation of this, which is not very simple.

Two skew lines $\mathfrak{p}^{\mathrm{I}}$, $\mathfrak{p}^{\mathrm{II}}$ will yield either a sign invariant or a complete invariant.

Three skew lines $\mathfrak{p}^{\mathrm{I}}$, $\mathfrak{p}^{\mathrm{II}}$, $\mathfrak{p}^{\mathrm{III}}$ are (cf., no. **35**) always linearly-independent. From no. **27** and no. **28**, one will obtain the three non-vanishing semi-invariants $\mathfrak{p}^i\mathfrak{p}^k$ with $i \neq k$, and the sign of $(\mathfrak{p}^{\mathrm{II}}\ \mathfrak{p}^{\mathrm{III}})\ (\mathfrak{p}^{\mathrm{III}}\ \mathfrak{p}^{\mathrm{I}})\ (\mathfrak{p}^{\mathrm{I}}\ \mathfrak{p}^{\mathrm{II}})$ will be the only sign invariant, while there will be no complete invariants. *We shall call triples of lines with the same sign invariants equal-wound.*

([1]) Whether a screw is right-wound or left-wound will depend upon the coordinate system.

Two triples of lines $\mathfrak{p}^{\mathrm{I}}$, $\mathfrak{p}^{\mathrm{II}}$, $\mathfrak{p}^{\mathrm{III}}$ *and* $\mathfrak{q}^{\mathrm{I}}$, $\mathfrak{q}^{\mathrm{II}}$, $\mathfrak{q}^{\mathrm{III}}$ *from two conjugate hyperboloidal quadrics are unequal, so two triples of lines of the same hyperboloidal quadric will be equal-wound; the six six-vectors* $\mathfrak{p}^i$, $\mathfrak{q}^k$ *are linearly-independent.*

Proof: From (34), one has:

$$\begin{aligned} D_6(\mathfrak{p}^{\mathrm{I}}, \mathfrak{p}^{\mathrm{II}}, \mathfrak{p}^{\mathrm{III}}, \mathfrak{q}^{\mathrm{I}}, \mathfrak{q}^{\mathrm{II}}, \mathfrak{q}^{\mathrm{III}}) &= 4\,(\mathfrak{p}^{\mathrm{II}}\,\mathfrak{p}^{\mathrm{III}})\,(\mathfrak{p}^{\mathrm{III}}\,\mathfrak{p}^{\mathrm{I}})\,(\mathfrak{p}^{\mathrm{I}}\,\mathfrak{p}^{\mathrm{II}})\,(\mathfrak{q}^{\mathrm{II}}\,\mathfrak{q}^{\mathrm{III}})\,(\mathfrak{q}^{\mathrm{III}}\,\mathfrak{q}^{\mathrm{I}})\,(\mathfrak{q}^{\mathrm{I}}\,\mathfrak{q}^{\mathrm{III}}) \\ &= -\,|\,\mathfrak{p}^{\mathrm{I}}, \mathfrak{p}^{\mathrm{II}}, \mathfrak{p}^{\mathrm{III}}, \mathfrak{q}^{\mathrm{I}}, \mathfrak{q}^{\mathrm{II}}, \mathfrak{q}^{\mathrm{III}}\,|^2. \end{aligned}$$

The middle expression is non-zero, but the right-hand expression can be negative; as a result, $(\mathfrak{p}^{\mathrm{II}}\,\mathfrak{p}^{\mathrm{III}})\,(\mathfrak{p}^{\mathrm{III}}\,\mathfrak{p}^{\mathrm{I}})\,(\mathfrak{p}^{\mathrm{I}}\,\mathfrak{p}^{\mathrm{II}})$ and $(\mathfrak{q}^{\mathrm{II}}\,\mathfrak{q}^{\mathrm{III}})\,(\mathfrak{q}^{\mathrm{III}}\,\mathfrak{q}^{\mathrm{I}})\,(\mathfrak{q}^{\mathrm{I}}\,\mathfrak{q}^{\mathrm{III}})$ will have unequal signs.

37. Four linearly-independent skew lines. Four linearly-dependent lines. From no. **33**, *four linearly-independent skew lines* $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{IV}}$ are contained in a hyperbolic, elliptic, or parabolic line system, and have:

$$a)\ \begin{cases} \alpha)\ 2 \\ \beta)\ 0 \end{cases}, \qquad b)\ \text{One common line of intersection.}$$

When one eliminates the σ^h from the six equations:

$$(\tilde{\mathfrak{p}}^i\tilde{\mathfrak{p}}^k) = \sigma^i\sigma^k\,(\mathfrak{p}^i\mathfrak{p}^k) \qquad \text{(not summed!)}$$

from no. **28**, the six non-vanishing semi-invariants $\mathfrak{p}^i\mathfrak{p}^k$ with $i \neq k$ will lead to the two complete invariants:

$$G_1 = \frac{(\mathfrak{p}^{\mathrm{I}}\mathfrak{p}^{\mathrm{III}})(\mathfrak{p}^{\mathrm{II}}\mathfrak{p}^{\mathrm{IV}})}{(\mathfrak{p}^{\mathrm{I}}\mathfrak{p}^{\mathrm{IV}})(\mathfrak{p}^{\mathrm{II}}\mathfrak{p}^{\mathrm{III}})}, \qquad G_2 = \frac{(\mathfrak{p}^{\mathrm{I}}\mathfrak{p}^{\mathrm{IV}})(\mathfrak{p}^{\mathrm{II}}\mathfrak{p}^{\mathrm{III}})}{(\mathfrak{p}^{\mathrm{I}}\mathfrak{p}^{\mathrm{II}})(\mathfrak{p}^{\mathrm{III}}\mathfrak{p}^{\mathrm{IV}})}.$$

G_1, G_2 are called *Grassmannian double ratios* of the quadruple of lines. They can be interpreted geometrically for the hyperbolic type as follows:

If $x^{\mathrm{I}}, \ldots, x^{\mathrm{IV}}$ ($x^{\mathrm{I}}, \ldots, x^{\mathrm{IV}}$, resp.) are the points of intersection of the lines $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{IV}}$ with the common line of intersection $\mathfrak{q}^{\mathrm{I}}$ ($\mathfrak{q}^{\mathrm{II}}$, resp.) then between the double ratios:

$$d = (x^{\mathrm{I}}\,x^{\mathrm{II}}\,x^{\mathrm{III}}\,x^{\mathrm{IV}}), \qquad d' = (x'^{\mathrm{I}}\,x'^{\mathrm{II}}\,x'^{\mathrm{III}}\,x'^{\mathrm{IV}})$$

and the Grassmannian double ratios, the relations:

$$G_1 = dd', \qquad \frac{1}{G_2} = 1 + dd' - (d + d') \tag{46}$$

will exist.

Proof: We take x^{I}, x^{II} (x'^{I}, x'^{II}, resp.) to be the vertices 1, 2, 3, 4 of the coordinate tetrahedron, and then have:

$$x^{\mathrm{III}} = 1 \mid \rho \mid 0 \mid 0, \qquad x^{\mathrm{IV}} = 1 \mid \tau \mid 0 \mid 0, \qquad (x'^{\mathrm{III}} = 0 \mid 0 \mid 1 \mid \rho', \qquad x'^{\mathrm{IV}} = 0 \mid 0 \mid 1 \mid \tau', \text{ resp.})$$

for the coordinates of the remaining points.

This implies the line coordinates:

$$\mathfrak{p}^{\mathrm{I}} = 0 \mid 0 \mid 0 \mid 0 \mid 1 \mid 0, \qquad \mathfrak{p}^{\mathrm{II}} = 0 \mid 1 \mid 0 \mid 0 \mid 0 \mid 0,$$
$$\mathfrak{p}^{\mathrm{III}} = \rho' \mid \rho\rho' \mid 0 \mid -\rho \mid 1 \mid 0, \qquad \mathfrak{p}^{\mathrm{IV}} = \tau' \mid \tau\tau' \mid 0 \mid -\tau \mid 1 \mid 0,$$

and from (2):

$$d = \frac{\rho}{\tau}, \qquad d' = \frac{\rho'}{\tau'},$$

moreover, and the assertion follows by a brief calculation.

The relations will become symmetric when one employs the six double ratios of the quadruple of points $x^{\mathrm{I}}\, x^{\mathrm{II}}\, x^{\mathrm{III}}\, x^{\mathrm{IV}}$, namely:

$$d_1 = d, \qquad d_2 = \frac{1}{1-d}, \qquad d_3 = \frac{d-1}{d}, \qquad d_4 = \frac{d}{1-d}, \qquad d_5 = \frac{1}{d}, \qquad d_5 = 1-d,$$

and the corresponding values for the $d_1', \ldots, d_6'$. One then gets:

$$(47)\quad \begin{aligned} G_1 &= d_1 d_1' = \frac{(13)(42)}{(14)(23)}, & G_2 &= d_2 d_2' = \frac{(14)(23)}{(12)(34)}, & G_3 &= d_3 d_3' = \frac{(12)(34)}{(13)(42)}, \\ G_4 &= d_4 d_4' = \frac{(13)(42)}{(12)(34)}, & G_5 &= d_5 d_5' = \frac{(14)(23)}{(13)(43)}, & G_6 &= d_6 d_6' = \frac{(12)(34)}{(14)(23)}. \end{aligned}$$

In this, the brackets (ik) are abbreviations for $\mathfrak{p}^i\,\mathfrak{p}^k$.

In the next chapter, the complete invariant:

$$(48)\qquad I = \frac{D_4(\mathfrak{p}^{\mathrm{I}}, \mathfrak{p}^{\mathrm{II}}, \mathfrak{p}^{\mathrm{III}}, \mathfrak{p}^{\mathrm{IV}})}{\{(\mathfrak{p}^{\mathrm{I}}\mathfrak{p}^{\mathrm{II}})(\mathfrak{p}^{\mathrm{I}}\mathfrak{p}^{\mathrm{III}})(\mathfrak{p}^{\mathrm{I}}\mathfrak{p}^{\mathrm{IV}})(\mathfrak{p}^{\mathrm{II}}\mathfrak{p}^{\mathrm{III}})(\mathfrak{p}^{\mathrm{II}}\mathfrak{p}^{\mathrm{IV}})(\mathfrak{p}^{\mathrm{III}}\mathfrak{p}^{\mathrm{IV}})\}^{2/3}}$$

will also play a role; one obtains the geometric interpretation from (47) by a simple calculation:

$$(49)\qquad I = (G_1G_4)^{2/3} + (G_2G_5)^{2/3} + (G_3G_6)^{2/3} - 2\{(G_1G_6)^{2/3} + (G_2G_4)^{2/3} + (G_3G_5)^{2/3}\}.$$

The first three products contain the two double ratios that are coupled by the substitution $\delta = \frac{d}{d-1}$, while the last three contain the two that are coupled with the substitution $\delta = 1 - d$.

From no. **32**, one has:

$$I \left\{ \begin{array}{lll} > 0 & & \text{hyperbolic} \\ = 0 & \text{in the} & \text{parabolic} \\ < 0 & & \text{elliptic} \end{array} \right\} \text{case;}$$

the numerator of I is then D_4 (no. **32**), while the denominator of I is always positive.

Four linearly-independent, skew lines $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{IV}}$ are hyperboloidal; i.e., they are taken from a hyperboloidal quadric. From no. **35**, three skew lines always determine a hyperboloidal quadric.

From no. **35**, any line $\mathfrak{q}$ of the quadric that is conjugate to $\mathfrak{p}$-quadric will cut the four lines $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{IV}}$ in four points $x^{\mathrm{I}}, \ldots, x^{\mathrm{IV}}$ with the constant double ratio $d = (x^{\mathrm{I}}\, x^{\mathrm{II}}\, x^{\mathrm{III}}\, x^{\mathrm{IV}})$. For this double ratio, it will follow from (46), with $d = d'$, that:

$$2d = 1 + \frac{(\mathfrak{p}^{\mathrm{I}}\mathfrak{p}^{\mathrm{III}})(\mathfrak{p}^{\mathrm{II}}\mathfrak{p}^{\mathrm{IV}}) - (\mathfrak{p}^{\mathrm{I}}\mathfrak{p}^{\mathrm{II}})(\mathfrak{p}^{\mathrm{III}}\mathfrak{p}^{\mathrm{IV}})}{(\mathfrak{p}^{\mathrm{I}}\mathfrak{p}^{\mathrm{IV}})(\mathfrak{p}^{\mathrm{II}}\mathfrak{p}^{\mathrm{III}})}. \tag{50}$$

Four lines of a pencil of lines are always linearly-dependent. For that double ratio of the four lines $\mathfrak{p}^{\mathrm{I}}, \mathfrak{p}^{\mathrm{II}}, \mathfrak{p}^{\mathrm{III}} = \rho\,\mathfrak{p}^{\mathrm{I}} + \tau\,\mathfrak{p}^{\mathrm{II}}, \mathfrak{p}^{\mathrm{IV}} = \rho'\mathfrak{p}^{\mathrm{I}} + \tau'\mathfrak{p}^{\mathrm{II}}$, one will get:

$$d = \frac{\tau\rho'}{\tau'\rho}. \tag{51}$$

Proof: One takes $\mathfrak{p}^{\mathrm{I}}, \mathfrak{p}^{\mathrm{II}}$ to be the sides 14, 24 of the coordinate tetrahedron, so the four lines $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{IV}}$ will cut the side 12 at the points whose coordinates are $x_1 \mid x_2 = 1 \mid 0,\ 0 \mid 1,\ \rho \mid \tau,\ \rho' \mid \tau'$. From (2), the double ratio of these four points is given by (51).

§ 10. Force screws and motion screws.

We conclude Chapter I with some *applications to the mechanics of rigid bodies* ([1]), and thus link them to § 5. As in § 5, we will assume a rectangular coordinate system.

38. Static and kinematic interpretation of six-vectors. One can interpret the six-vector $\mathfrak{a} = \{\mathfrak{A} \mid \bar{\mathfrak{A}}\}$ with the coordinates a_ρ as a *force screw* (*motion screw*, resp.) in the statics (kinematics, resp.) of rigid bodies; the a_ρ are then inhomogeneous *screw*

([1]) Cf., *Enzykl. der math. Wiss IV*$_1$, pp. 128, *et seq.*

coordinates. The force screw $\mathfrak{a}$ is the totality of the real force $\mathfrak{p} = \{\mathfrak{A} \mid 0\}$ with the force vector $\mathfrak{A} = a_1 \mid a_2 \mid a_3$ and the line of action through the origin O, along with the imaginary force $\mathfrak{q} = \{0 \mid \bar{\mathfrak{A}}\}$; i.e., the force-pair with the moment vector $\bar{\mathfrak{A}}$. The motion screw $\mathfrak{a}$ is composed of the real rotation $\mathfrak{p} = \{\mathfrak{A} \mid 0\}$ with the rotation vector $\mathfrak{A}$ and the rotational axis through O, along with the imaginary rotation $\bar{\mathfrak{q}} = \{0 \mid \bar{\mathfrak{A}}\}$; i.e., the displacement velocity $\bar{\mathfrak{A}}$. If the six-vector $\mathfrak{a}$ is singular ($\mathfrak{a}\mathfrak{a} = 0$, so $\mathfrak{A} \perp \bar{\mathfrak{A}}$) then, from no. **18**, the force screw (motion screw, resp.) will specialize to a *singular screw;* i.e., a real or imaginary force (rotation, resp.) whose line of action (rotational axis, resp.) is given by the singular six-vector $\mathfrak{a}$; i.e., the line coordinates a_ρ.

The *conditions of equilibrium* (27) are then true for real (imaginary, resp.) forces (rotations, resp.), as well as for a force screw (motion screw, resp.) that acts upon a given rigid body. From (27), all forces screws (motion screws, resp.) $\mathfrak{a}^{\mathrm{I}}, \ldots, \mathfrak{a}^n$ of a rigid body can always be combined into a single screw $\mathfrak{a} = \mathfrak{a}^{\mathrm{I}} + \ldots + \mathfrak{a}^n$; every "infinitesimal motion" of a rigid body is an "infinitesimal screw."

The projective force (kinematical, resp.) transformation that was treated in § 5 can also be applied with no changes to non-singular force (motion, resp.) screws.

39. Various decompositions of a screw. We shall now translate some of the theorems on linear complexes that we derived before into the theory of screws. For brevity, in each case we shall thus speak of only force (motion, resp.) screws.

1. It follows from no. **23** that:

Any non-singular motion screw $\mathfrak{a} = \{\mathfrak{A} \mid \bar{\mathfrak{A}}\}$ can be decomposed in a unique way into a rotational velocity and a displacement velocity that is parallel to the rotational axis. The complex axis $\mathfrak{s}$ [cf., (31)], which we shall now call the *screw axis,* is the rotational axis, while $\mathfrak{A}$ is the rotational velocity, and $\dfrac{(\mathfrak{A}\bar{\mathfrak{A}})}{(\mathfrak{A}\mathfrak{A})}\mathfrak{A}$ is the displacement velocity.

2. It follows from no. **22** that:

Any non-singular force screw $\mathfrak{a}$ can be decomposed into two forces with skew lines of action (viz., a *force cross*) in infinitely many ways. The lines of action are reciprocal lines in the null system that is given by $\mathfrak{a}$. The force cross is determined uniquely when one gives an arbitrary line that does not belong to the complex $\mathfrak{a}$ as one of the two lines of action.

Theorem 2 is a generalization of Theorem 1, since by Theorem 1, the real rotational axis $\mathfrak{s}$ and the imaginary rotational axis of the displacement velocity are reciprocal with respect to $\mathfrak{a}$.

3. It follows from no. **37** that:

If four forces $\mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{\mathrm{IV}}$ with skew lines of action are in equilibrium then the lines of action will be hyperboloidal.

Under an arbitrary variation of the coefficients λ_1, λ_2, the axes of all screws $\lambda_1\mathfrak{a}^{\mathrm{I}} + \lambda_2 \mathfrak{a}^{\mathrm{II}}$ will define:

1. A pencil of parallels, when $\mathfrak{a}^{\mathrm{I}}$, $\mathfrak{a}^{\mathrm{II}}$ have parallel axes.
2. A pencil of lines with a real vertex, when $\mathfrak{a}^{\mathrm{I}}$, $\mathfrak{a}^{\mathrm{II}}$ have intersecting axes, and the ratios of their rotational and translational velocities coincide.
3. A single line, when $\mathfrak{a}^{\mathrm{I}}$, $\mathfrak{a}^{\mathrm{II}}$ have the same axis.
4. A cylindroid in all remaining cases.

Thus, $\lambda_1\mathfrak{a}^{\mathrm{I}} + \lambda_2 \mathfrak{a}^{\mathrm{II}}$ will be the resultant of the screws $\lambda_1\mathfrak{a}^{\mathrm{I}}$, $\lambda_2 \mathfrak{a}^{\mathrm{II}}$; $\lambda_1\mathfrak{a}^{\mathrm{I}}$ has the same axis as the screw $\mathfrak{a}^{\mathrm{I}}$, but λ_1-times the intensity (i.e., λ_1-times the rotational and translational velocity).

40. Conjugate screws. The scalar product of two six-vectors has a simple mechanical meaning: $L = \mathfrak{a}\mathfrak{b}$ *is the work that is done by the force screw* $\mathfrak{a}$ ($\mathfrak{b}$, resp.) *during the motion screw* $\mathfrak{b}$ ($\mathfrak{a}$, resp.).

Proof: From no. **38**, if one decomposes the force screw $\mathfrak{a}$ into the force $\mathfrak{p} = \{\mathfrak{A} \mid 0\}$ (line of action through O) and the force-couple $\mathfrak{q} = \{0 \mid \bar{\mathfrak{A}}\}$, and further decomposes the motion screw $\mathfrak{b}$ into the real rotation $\mathfrak{u} = \{\mathfrak{B} \mid 0\}$ (rotational axis through O) and the imaginary rotation $\mathfrak{v} = \{0 \mid \bar{\mathfrak{B}}\}$ then

$$\left.\begin{matrix} 0 \\ \mathfrak{A}\bar{\mathfrak{B}} \end{matrix}\right\} \text{ will equal the work that is done by the force } \mathfrak{p} \text{ during the } \left\{\begin{matrix} \text{real rotation } \mathfrak{u} \\ \text{imag. rotation } \mathfrak{v} \end{matrix}\right. ,$$

$$\left.\begin{matrix} \bar{\mathfrak{A}}\mathfrak{B} \\ 0 \end{matrix}\right\} \text{ will equal the work that is done by the force-couple } \mathfrak{q} \text{ during the } \left\{\begin{matrix} \text{real rotation } \mathfrak{u} \\ \text{imag. rotation } \mathfrak{v} \end{matrix}\right. ;$$

hence, the total work done $L = \mathfrak{A}\bar{\mathfrak{B}} + \bar{\mathfrak{A}}\mathfrak{B} = \mathfrak{a}\mathfrak{b}$.

The lines of the non-singular complex $\mathfrak{a}$ are given by $\mathfrak{a}\mathfrak{a} = \mathfrak{a}\mathfrak{b} = 0$. It follows from the foregoing theorem that:

The rotational axes of all rotations $\mathfrak{b}$ *under which the work done* $L = \mathfrak{a}\mathfrak{b}$ *by a given non-singular force screw* $\mathfrak{a}$ *vanishes are the lines of the linear complex* $\mathfrak{a}$.

Due to this connection, one calls the complex lines *null lines* and the correlation (29), a *null system*.

Two screws $\mathfrak{a}$, $\mathfrak{b}$ with vanishing work $\mathfrak{a}\mathfrak{b} = 0$ are called *conjugate screws*, since the complexes $\mathfrak{a}$, $\mathfrak{b}$ are conjugate, from no. **29**. From the principle of virtual work, a rigid body will remain at rest under the influence of a force screw $\mathfrak{a}$ if and only if the work that is done by $\mathfrak{a}$ under the motion screw $\mathfrak{b}$ vanishes for all admissible motions of the rigid body; i.e., when $\mathfrak{a}$ is conjugate to all admissible motion screws.

One says that a rigid body has $n \leq 6$ *degrees of freedom* when the admissible motion screws are linear combinations of n linearly-independent motion screws $\mathfrak{b}^{\mathrm{I}}, \ldots, \mathfrak{b}^{n}$, and thus define an n-dimensional (1) linear manifold $\mu_1\mathfrak{b}^{\mathrm{I}} + \ldots + \mu_n \mathfrak{b}^{n}$.

The force screws under whose influence a body with n degrees of freedom remains at rest define a $(6 - n)$*-dimensional manifold* $\lambda_1\mathfrak{a}^{\mathrm{I}} + \ldots + \lambda_{\mathrm{VI}-n}\, \mathfrak{a}^{\mathrm{VI}-n}$; *the complex manifolds* $\mu_1\mathfrak{b}^{\mathrm{I}} + \ldots + \mu_n \mathfrak{b}^{n}$ *and* $\lambda_1\mathfrak{a}^{\mathrm{I}} + \ldots + \lambda_{\mathrm{VI}-n}\, \mathfrak{a}^{\mathrm{VI}-n}$ *are conjugate* (2). *The interpretation of conjugate screws as force screws and motion screws is interchangeable.*

A fixed-axis rotatable body will serve as an example for $n = 1$: The motion screws are the rotations around the fixed axis. A body that rotates around a fixed point O is an example for $n = 3$: The motion screws are the rotations whose rotational axis goes through O; the force screws can be composed of three forces whose lines of action contain the point O and do no lie in a plane.

§ 11. Unsteady frameworks and rectangle nets.

41. Collinear invariance of the unsteady framework. In no. **20**, we distinguished between geometrically-undetermined, determined, and over-determined frameworks. Among the frameworks whose structure is determined geometrically, there are ones that are indeed rigid under finite wrinkles, but still admit certain "infinitesimal" (3) wrinkles; i.e., one can give small shifts of the nodes:

$$\mathfrak{X}^* = \mathfrak{X} + \varepsilon\overline{\mathfrak{X}}$$

[$\mathfrak{X}$, ($\mathfrak{X}^*$, resp.) = position vector of the original (shifted, resp.) node], in such a way that for the lengths of the rods:

$$l^* = l + \varepsilon^2 \{\ldots\}$$

(1) The corresponding complex manifold $\mu_1\mathfrak{b}^{\mathrm{I}} + \ldots + \mu_n \mathfrak{b}^{n}$ is only $(n - 1)$-dimensional, since the complex coordinates are homogeneous.

(2) **R. S. Ball**, *Theory of Screws*, Dublin, 1876.

(3) In this paragraph, we shall speak of "infinitesimal motions," instead of velocities.

as $\varepsilon \to 0$. One calls such a framework a *geometrically-exceptional framework* or an *unsteady framework*. It can be shown that the unsteady frameworks are, at the same time, statically-exceptional frameworks (no. **20**), and conversely; this is easy to confirm in the trivial example (tetrahedral linkage that has collapsed into a plane) that was given in no. **20**. However, we would like to prove a theorem that was found by **H. Liebmann** ([1]) without referring to that fact:

Any framework that is collinear to an unsteady framework is again unsteady.

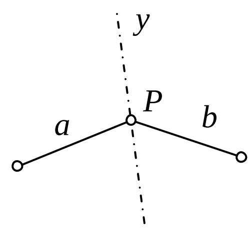


Figure 11.

Proof: One can characterize the "infinitesimal motion" of any rod of an unsteady framework with rods that are rigid to order ε under an infinitesimal wrinkling by a motion screw $\mathfrak{y}$. That motion screw is determined up to an arbitrary additional rotation around the rod as its rotational axis. The relative motion of any two rods a, b with the common node P (Fig. 11) will then be given by $\mathfrak{y}_a - \mathfrak{y}_b$. Now, the condition that the relative motion is a rotation with a rotational axis that goes through P is characteristic of the unsteadiness, so:

$$\mathfrak{p} = \mathfrak{y}_a - \mathfrak{y}_b$$

will be a singular six-vector, and the line $\mathfrak{p}$ will contain the point P. This requirement will be invariant when the motion screw $\mathfrak{y}$ and the singular six-vectors of the rods undergo the same linear transformation (19) with constant coefficients γ_ρ^σ that satisfy the conditions (20a) (viz., a collinear kinematic transformation; cf., no. **38**, conclusion).

42. Face-rigid, unsteady rectangle nets ([2]). A *rectangle net* (cf., no. **7**) is composed of planar or non-planar rectangles that we arrange like the field of a chessboard (Fig. 12). Any vertex that does not lie in the boundary is the vertex of a quadrilateral. We shall call rectangle nets with nothing but planar rectangles (quadrilaterals, resp.) *face-planar* (*vertex-planar*, resp.). We exclude rectangle nets that are both face-planar and vertex-planar. Moreover, the statements will relate to only vertices that do not lie on the boundary.

([1]) **H. Liebmann**: "Ausnahmefachwerkes, etc.," Münchner Berichte **50** (1920); cf., moreover, **W. Blaschke**: "Wackelige Achtflache," Math. Zeit. **6** (1920).

([2]) For this and the following numbers, cf., **R. Sauer:** "Proj. Kinematik, etc." Monatshefte für Math. und Phys. **43** (1936).

A rectangle net can be completely rigid or flexible while keeping the individual rectangles rigid. However, there are also *face-rigid, unsteady, rectangle nets* that admit no finite wrinkling while keeping the individual rectangles rigid, but probably infinitesimal wrinkling, in analogy with unsteady frameworks.

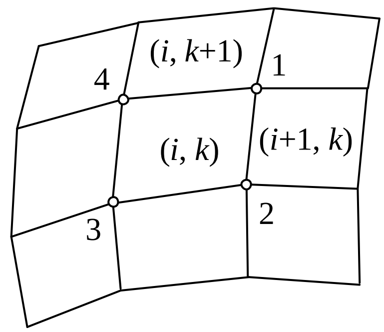


Figure 12.

Characterizing the face-rigid, unsteady, rectangle nets:

Under an infinitesimal wrinkling of a face-rigid, unsteady, rectangle net, every rectangle (i, k) will experience an infinitesimal motion, which we represent by the motion screw:

$$\mathfrak{y}_{ik} = \{\mathfrak{Y}_{ik} \mid \bar{\mathfrak{Y}}_{ik}\}.$$

Therefore, it is necessary and sufficient that the relative motion of two rectangles with a common side should a rotation around that side. One also has:

$$\mathfrak{y}_{i+1,k} - \mathfrak{y}_{ik} = \rho_{ik}\,\mathfrak{q}_{ik}, \qquad \mathfrak{y}_{i,k+1} - \mathfrak{y}_{ik} = \sigma_{ik}\,\mathfrak{p}_{ik} \tag{52}$$

then, in which $\mathfrak{q}_{ik}$ ($\mathfrak{p}_{ik}$, resp.) are singular six-vectors that determine the common sides of the rectangles (i, k), $(i + 1, k)$ [(i, k), $(i, k + 1)$, resp.] (Fig. 12). In ordinary vectors, it will follow from (52) that:

$$\begin{aligned} &\mathfrak{Y}_{i+1,k} - \mathfrak{Y}_{ik} = \rho_{ik}\,\mathfrak{Q}_{ik}, \qquad && \mathfrak{Y}_{i,k+1} - \mathfrak{Y}_{ik} = \sigma_{ik}\,\mathfrak{P}_{ik}, \\ &\bar{\mathfrak{Y}}_{i+1,k} - \bar{\mathfrak{Y}}_{ik} = \rho_{ik}\,\bar{\mathfrak{Q}}_{ik}, \qquad && \bar{\mathfrak{Y}}_{i,k+1} - \bar{\mathfrak{Y}}_{ik} = \sigma_{ik}\,\bar{\mathfrak{P}}_{ik}, \end{aligned} \tag{53}$$

with

$$\mathfrak{Q}_{ik}\,\bar{\mathfrak{Q}}_{ik} = 0, \qquad \mathfrak{P}_{ik}\,\bar{\mathfrak{P}}_{ik} = 0.$$

If one denotes the position vector of the vertices 1, …, 4 of the rectangle (i, k) by $\mathfrak{X}_{ik}^h$ (h = 1, …, 4) then the shift of a vertex h will be given by:

$$\bar{\mathfrak{X}}_{ik}^h = \bar{\mathfrak{Y}}_{ik} + \mathfrak{Y}_{ik} \times \mathfrak{X}_{ik}^h; \tag{54}$$

in this, $\bar{\mathfrak{Y}}_{ik}$ gives the displacement of the entire rectangle (i, k), and $\mathfrak{Y}_{ik} \times \mathfrak{X}_{ik}^h$ gives the displacement of the vertex h that is provoked by the rotation $\mathfrak{Y}_{ik}$ (rotational axis through O). It follows from (54) that:

(55) $$\bar{\mathfrak{X}}_{ik}^2 - \bar{\mathfrak{X}}_{ik}^1 = \bar{\mathfrak{Y}}_{ik} + \mathfrak{Y}_{ik} \times (\mathfrak{X}_{ik}^2 - \mathfrak{X}_{ik}^1).$$

Consequences:

Just as the given face-rigid, unsteady rectangle net (X) is given by the position vectors $\mathfrak{X}_{ik}^h$, the position vectors $\mathfrak{Y}_{ik}$, $\bar{\mathfrak{Y}}_{ik}$, $\bar{\mathfrak{X}}_{ik}^h$ will produce three more rectangle nets (Y), $(\bar{Y})$, $(\bar{X})$; we assume that they do not degenerate. Due to the kinematic meaning of $\mathfrak{Y}_{ik}$ and $\bar{\mathfrak{Y}}_{ik}$, we shall refer to the rectangle net (Y) [$(\bar{Y})$, resp.] as the *rotational crack* (*displacement crack*, resp.] and the rectangle-pair (Y), $(\bar{Y})$ collectively as the *screw crack*. One obtains some theorems from equations (52) to (55):

1. Relations between (X) **and** (Y) (cf., Fig. 13):

Every edge of (X) corresponds to a parallel edge of (Y); every rectangle (quadrilateral, resp.) of (X) is associated with a quadrilateral (rectangle, resp.) of (Y). We have already encountered this kind of association with the Cremona planes for plane figures (no. **23**).

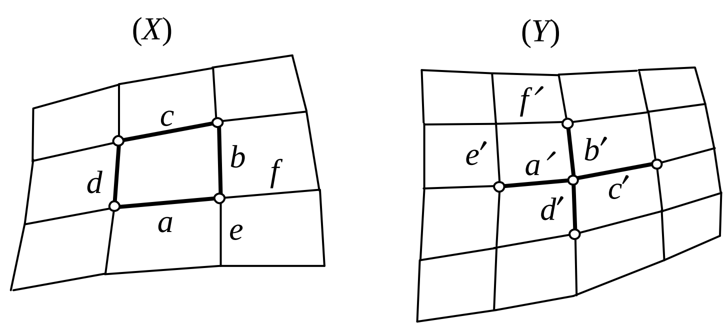


Figure 13.

2. Relations between (X) **and** $(\bar{X})$ [(Y) **and** $(\bar{Y})$, **resp.**]:

Every edge of (X) [(Y), resp.] corresponds to a perpendicular edge of $(\bar{X})$ [$(\bar{Y})$, resp.]. From (55), the four edges of a quadrilateral of (Y) [(X), resp.] with the vertex $\mathfrak{Y}_{ik}$ ($\mathfrak{X}_{ik}^h$, resp.) correspond to four lines in $(\bar{X})$ [$(\bar{Y})$, resp.] that are perpendicular to $\mathfrak{Y}_{ik}$ ($\mathfrak{X}_{ik}^h$, resp.) as the sides of a rectangle. For that reason, the rectangle nets $(\bar{X})$ and $(\bar{Y})$ will be planar.

3. Collinear invariance.

Any rectangle net that is collinear to face-rigid, unsteady rectangle net (X) is also face-rigid, unsteady. The screw crack (Y), ($\bar{Y}$) of (X) implies a screw crack in the rectangle net that is collinear to (X) when one map maps the screws $\mathfrak{y}_{ik} = \{\mathfrak{Y}_{ik} \mid \bar{\mathfrak{Y}}_{ik}\}$ by a collinear kinematic transformation.

Proof: The conditions (52) are invariant under collinear kinematic transformations.

4. Interchangeability of the screw cracks (X), ($\bar{X}$) and (Y), ($\bar{Y}$).

With the rectangle net (X), the rectangle net (Y) is also face-rigid and unsteady. Just as (Y), ($\bar{Y}$) is a screw crack of (X), (X), ($\bar{X}$) will be a screw crack of (Y).

Proof: We combine the position vectors $\mathfrak{X}_{ik}^h$ and $\bar{\mathfrak{X}}_{ik}^h$ of corresponding vertices of (X) and ($\bar{X}$) into a six-vector:

$$\mathfrak{x}_{ik}^h = \{\mathfrak{X}_{ik}^h \mid \bar{\mathfrak{X}}_{ik}^h\}$$

and associate it with the rectangle of (Y) that corresponds to the vertex $\mathfrak{X}_{ik}^h$ of the rectangle net (X), from Theorem 1. For two neighboring vertices of (X), one then has, with the use of (55), e.g.:

$$\mathfrak{x}_{ik}^2 - \mathfrak{x}_{ik}^1 = \{(\mathfrak{X}_{ik}^2 - \mathfrak{X}_{ik}^1) \mid \mathfrak{Y}_{ik} \times (\mathfrak{X}_{ik}^2 - \mathfrak{X}_{ik}^1)\}. \tag{56}$$

The six-vector $\mathfrak{x}_{ik}^2 - \mathfrak{x}_{ik}^1$ is then singular and represents the line of the rectangle net (Y) that is, from Theorem 1, associated with the connecting line of the vertices $\mathfrak{X}_{ik}^1$ and $\mathfrak{X}_{ik}^2$ as a parallel. As a result, (56) characterizes the unsteadiness of (Y) just as (52) characterized the unsteadiness of (X).

Static reinterpretation:

We interpret the six-vectors $\mathfrak{y}_{ik}$ as force screws, and thus interpret the differences $\mathfrak{y}_{i+1,k} - \mathfrak{y}_{ik}$ and $\mathfrak{y}_{i,k+1} - \mathfrak{y}_{ik}$ as forces and consider the edges of the rectangle net (X) as strings under tension that are knotted together at the net points. This net of strings will then be in equilibrium without the addition of external forces when one makes the string tensions proportional to corresponding rectangle sides of (Y).

43. Vertex-rigid, unsteady, rectangle nets. From now on, it will not be the individual rectangles that remain rigid, but the quadrilaterals; let any two rectangles with a common side rotate around it without slipping. The rectangle net is then either completely rigid or *vertex-rigid unsteady;* i.e., infinitesimal wrinkles with quadrilaterals that remain rigid are possible ([1]).

([1]) It can be shown that finite wrinkles are impossible in a vertex-rigid rectangle net.

The characterization of face-rigid, unsteady rectangle nets (no. **42**) can be carried over by analogy to the vertex-rigid, unsteady rectangle nets, with the difference that here the screws $\mathfrak{y}_{ik}$ are not associated with the rectangles, but with the quadrilaterals of (*X*). As a result, the rectangles (quadrilaterals, resp.) of (*X*) will always, in turn, correspond to rectangles (quadrilaterals, resp.) in the rectangle nets (*Y*), $(\bar{Y})$, $(\bar{X})$. Moreover, Theorems 1-4 of no. **42** are also valid, *mutatis mutandis*, for the vertex-rigid, unsteady rectangle nets.

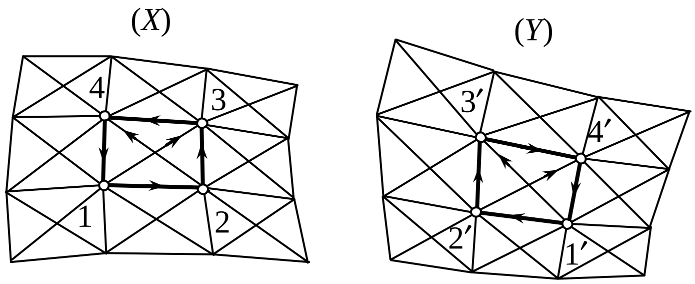


Figure 14.

The vertex-rigid, unsteady rectangle nets (in contrast, to the face-rigid, unsteady rectangle nets) are always planar. Plane rectangles of (*X*) and (*Y*) that correspond to each other will have anti-parallel sides and diagonals in the sense that is characterized by Fig. 14 (the arrows denote parallelism).

Proof: The shift in the vertex 3 is determined by establishing the quadrilateral of (*X*) with the vertex 1 in two ways, either by rotating the quadrilateral 2 (rotational axis 12) or by rotating the quadrilateral 4 (rotational axis 14). For that reason, one has:

$$(\mathfrak{Y}^2 - \mathfrak{Y}^1) \times (\mathfrak{X}^2 - \mathfrak{X}^1) = (\mathfrak{Y}^4 - \mathfrak{Y}^1) \times (\mathfrak{X}^3 - \mathfrak{X}^4). \tag{57}$$

Since corresponding sides of (*X*) and (*Y*) are parallel, in addition, the vectors $\mathfrak{Y}^2 - \mathfrak{Y}^1$, $\mathfrak{X}^2 - \mathfrak{X}^1$ ($\mathfrak{Y}^4 - \mathfrak{Y}^1$, $\mathfrak{X}^3 - \mathfrak{X}^4$, resp.) are linearly dependent, and the planarity of the rectangles 1234 and 1′2′3′4′ will follow from (57). If one now makes 1′4′ = 14 by a similarity transformation of the rectangle net (*Y*) and joins the rectangles 1234 and 1′2′3′4′, as in Fig. 15 then, from (57), the triangles 31′2′ and 341 will be equal as faces, so the diagonals 13 and 2′4′ will be parallel. One proves the parallelism of the diagonals 24 and 1′3′ in the same way.

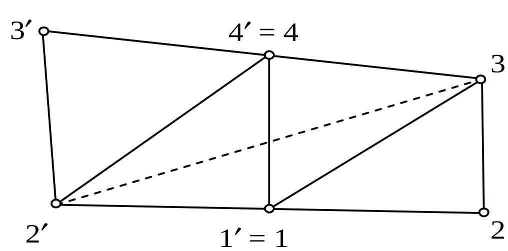


Figure 15.

44. Correlative, unsteady rectangle nets. A planar rectangle net will be mapped into another planar rectangle net by a correlation; the planar rectangles will transform into non-planar quadrilaterals, and conversely. The conditions (52) that characterize the unsteadiness are correlative, as well as collinear invariant. However, the screws $\mathfrak{y}_{ik}$ that are associated with the rectangles are transformed correlatively into screws of the quadrilaterals, or conversely. From no. **43**, the conditions of face-rigid unsteadiness and vertex-rigid unsteadiness are then switched, and one will get:

The rectangle nets that are correlative to a planar, face-rigid (vertex-rigid, resp.), unsteady rectangle net are again planar, but vertex-rigid (face-rigid, resp.) and unsteady.

That theorem is also true, in particular, for the polarity (no. **14**). In that case, $\mathfrak{y} = \{\mathfrak{Y} \mid \bar{\mathfrak{Y}}\}$ will be transformed into $\{\bar{\mathfrak{Y}} \mid \mathfrak{Y}\}$; i.e., the screw crack (Y), $(\bar{Y})$ goes to $(\bar{Y})$, (Y), so the rotational crack and displacement crack are switched.

Finally, let us prove the following theorem:

If (X) *and* (Y) *are vertex-rigid unsteady (hence, also face-planar) then* $(\bar{X})$ *and* $(\bar{Y})$ *will be vertex-planar. The connecting lines between corresponding vertices* 1 *and* 1′ *of* $(\bar{X})$ *and* $(\bar{Y})$ *lie in the planes of the two planar quadrilaterals with the vertices* 1 *and* 1′.

Proof: From (54), the vector $\bar{\mathfrak{X}} - \bar{\mathfrak{Y}}$ (hence, the connecting line 11′) is perpendicular to $\mathfrak{Y}$. From (55), the four quadrilateral sides $\bar{\mathfrak{X}}^2 - \bar{\mathfrak{X}}^1$, etc., that emanate from the vertex 1 will also be perpendicular to $\mathfrak{Y}$. Thus, the quadrilateral 1 will be planar and the line 11′ will lie in its plane. The proof for the quadrilateral 1′ is analogous.

There are rectangle nets that are face-rigid, unsteady for one wrinkle and vertex-rigid, unsteady for another wrinkle. The diagonals of a rectangle net that is defined by the generators of a hyperboloid produce such a face-rigid, as well as vertex-rigid, unsteady net; we shall leave the proof of that to the reader.

CHAPTER II

Families of lines

§ 12. Definition of a family of lines.

45. Parametric representation. A *family of lines* is given by:

$$\mathfrak{p} = \mathfrak{p}(u) \;;$$

i.e., the line coordinates p_ρ are regular functions of the parameter u in a domain $u_a \leq u \leq u_e$. Due to the homogeneity of line coordinates, the six-vector $\mathfrak{p}(u)$ is determined only up to an arbitrary function $\sigma(u) \neq 0$ as a factor. One can generate families of lines in such a way that one lays a line through every point of a given curve.

The derivatives $\dfrac{dp_\rho}{du}$ $\left(\dfrac{d^2 p_\rho}{du^2}, \text{etc., resp.}\right)$ transform like the p_ρ ($\rho = 1, \ldots, 6$) under projective maps (19), and thus define six-vectors $\dfrac{d\mathfrak{p}}{du}$ $\left(\dfrac{d^2\mathfrak{p}}{du^2}, \text{etc., resp.}\right)$. We denote the derivatives by dots and demand that for every location u in the domain of definition:

$\mathfrak{p}$, $\dot{\mathfrak{p}}$ *are linearly independent; in particular, either* $\mathfrak{p}$ *or* $\dot{\mathfrak{p}}$ *will therefore be a null vector.*

This yields the following *map of the family of lines onto the points of an image line:*

Every u' in the domain $u_a < u < u_e$ can be assigned a finite sub-domain for which the values of u are associated with the lines $\mathfrak{p}(u)$ in a one-to-one correspondence ([1]). The lines of the family of lines will then be mapped to the points of an image line segment in a one-to-one correspondence by $u = \xi$ (ξ = abscissa of an image line) in the sub-domain.

One obtains the identities:

$$(58) \qquad \mathfrak{p}\mathfrak{p} \equiv 0, \qquad \mathfrak{p}\dot{\mathfrak{p}} \equiv 0, \qquad \mathfrak{p}\ddot{\mathfrak{p}} + \dot{\mathfrak{p}}\dot{\mathfrak{p}} \equiv 0, \quad \mathfrak{p}\dddot{\mathfrak{p}} + 3\dot{\mathfrak{p}}\ddot{\mathfrak{p}} \equiv 0, \text{ etc.,}$$

from $\mathfrak{p}(u)\ \mathfrak{p}(u) \equiv 0$ by repeated differentiation.

46. Ruled families and torses. A family of lines is called a *ruled family* (*torse*, resp.) when the requirement $\dot{\mathfrak{p}}\dot{\mathfrak{p}} \neq 0$ ($\dot{\mathfrak{p}}\dot{\mathfrak{p}} = 0$, resp.) is fulfilled for all values of u. From § 7, this distinction is projectively invariant. We exclude discrete lines with $\dot{\mathfrak{p}}\dot{\mathfrak{p}} = 0$. We

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([1]) For a sufficiently small absolute value of δ, the six-vectors $\mathfrak{p}(u)$, $\mathfrak{p}(u + \delta) = \mathfrak{p}(u) + \delta\dot{\mathfrak{p}}(u) + \ldots$ will be linearly independent, so they will determine distinct lines.

have already encountered special ruled families in no. **35** in the form of hyperboloidal quadrics.

For torses, since:

$$\mathfrak{p}\mathfrak{p} = \dot{\mathfrak{p}}\dot{\mathfrak{p}} = \mathfrak{p}\dot{\mathfrak{p}} = 0, \qquad \dot{\mathfrak{p}} \neq \sigma\,\mathfrak{p},$$

the singular six-vectors $\mathfrak{p}$, $\dot{\mathfrak{p}}$ represent two intersecting lines. The point of intersection x is called the *point of regression*, and the connecting plane w is called the *plane of regression* of the line $\mathfrak{p}$ of the torse.

Classification of the torses:

1. *The torses with* $\ddot{\mathfrak{p}}\ddot{\mathfrak{p}} = 0$ *and linearly-dependent* ([1]) $\mathfrak{p}$, $\dot{\mathfrak{p}}$, $\ddot{\mathfrak{p}}$ *are pencils of lines (fixed point of regression x = vertex of the pencil, fixed plane of regression w = plane of the pencil).*

Proof: If the torse is a pencil of lines then any three lines $\mathfrak{p}(u)$, $\mathfrak{p}(u + \varepsilon)$, $\mathfrak{p}(u + 2\varepsilon)$ will be linearly dependent. The linear dependency of $\mathfrak{p}(u)$, $\dot{\mathfrak{p}}(u)$, $\ddot{\mathfrak{p}}(u)$ will follow upon developing these expressions in powers of ε. Conversely, if:

$$\lambda_1(u)\ \mathfrak{p}(u) + \lambda_2(u)\,\dot{\mathfrak{p}}(u) + \lambda_3(u)\ \ddot{\mathfrak{p}}(u) \equiv 0 \qquad \text{(and thus, also } \ddot{\mathfrak{p}}\ddot{\mathfrak{p}} \equiv 0)$$

is given then all of the derivatives $\mathfrak{p}^{(m)}(u)$ with $m > 1$ can be represented as linear combinations of $\mathfrak{p}(u)$, $\dot{\mathfrak{p}}(u)$ by repeated differentiation of this identity, and as a result, $\mathfrak{p}(u + \varepsilon)$ can also be represented in that way for all values of ε, so $\mathfrak{p}(u + \varepsilon)$ will provide a pencil of lines.

2. *The torses with* $\ddot{\mathfrak{p}}\ddot{\mathfrak{p}} = 0$ *and linearly-independent* $\mathfrak{p}$, $\dot{\mathfrak{p}}$, $\ddot{\mathfrak{p}}$ *are:*

a) *The tangent families of plane curves, which are then called curves of regression (point of regression x = vertex of a curve, plane of regression w = plane of the curve).*

b) *The family of generators of a cone, which is then called a cone of regression (fixed point of regression x = vertex of a cone, plane of regression w = contact plane of the cone).*

Proof:

α) Let the tangent family of a planar curve be given. We take the plane of the curve to be the plane $x_4 = 0$ and then have $p_1 = p_2 = p_3 = 0$, so $\mathfrak{p}^{(m)}\,\mathfrak{p}^{(m)} = 0$ for any arbitrarily

---

([1]) Due to (58), the linear-dependency of $\mathfrak{p}$, $\dot{\mathfrak{p}}$, $\ddot{\mathfrak{p}}$ will have $\mathfrak{p}^{(m)}\,\mathfrak{p}^{(m)} = 0$ as a consequence for every arbitrarily-high m^{th} derivative $\mathfrak{p}^{(m)}$.

high derivative $\mathfrak{p}^{(m)}$. One will reach an analogous conclusion for the generating family of a cone.

β) Let a torse be given with $\ddot{\mathfrak{p}}\ddot{\mathfrak{p}} = 0$ and linearly-independent $\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}$. With the help of the identities (58), one obtains, from (34):

$$| \mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}, \mathfrak{a}, \mathfrak{b} |^2 = - D_6(\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}, \mathfrak{a}, \mathfrak{b}) = 0.$$

As a result, $\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}, \mathfrak{a}, \mathfrak{b}$ will be linearly dependent, and since $\mathfrak{a}, \mathfrak{b}$ are entirely arbitrary six-vectors, $\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}$ will also be linearly independent, in their own right. By repeated differentiation of:

$$A\mathfrak{p} + B\dot{\mathfrak{p}} + C\ddot{\mathfrak{p}} + D\dddot{\mathfrak{p}} = 0,$$

one can then obtain any arbitrarily-high derivative of $\mathfrak{p}^{(m)}$ as a linear combination of $\mathfrak{p}(u)$, $\dot{\mathfrak{p}}(u)$, $\ddot{\mathfrak{p}}(u)$, and therefore obtain $\mathfrak{p}(u + \varepsilon)$ for all values of ε, as well. Any line $\mathfrak{p}(u + \varepsilon)$ of the torse will then cut the three fixed lines $\mathfrak{p}(u)$, $\dot{\mathfrak{p}}(u)$, $\ddot{\mathfrak{p}}(u)$, which will then span the line field w (line bundle x, resp.), since:

$$\mathfrak{p}\mathfrak{p} = \dot{\mathfrak{p}}\dot{\mathfrak{p}} = \dot{\mathfrak{p}}\ddot{\mathfrak{p}} = \ddot{\mathfrak{p}}\mathfrak{p} = \mathfrak{p}\dot{\mathfrak{p}} = 0.$$

All lines of the torse will then lie in the fixed plane of regression w (go through the fixed point of regression x, resp.).

γ) For the connecting line $\mathfrak{g}$ of the curve points $x(u)$ and $x(u + \varepsilon)$, one has:

$$\mathfrak{g} = \lambda(u)\, \mathfrak{p}(u) + \mu(u)\, \dot{\mathfrak{p}}(u) = \lambda\mathfrak{p} + \mu\dot{\mathfrak{p}} + \varepsilon\, \{\dot{\lambda}\mathfrak{p} + (\lambda + \dot{\mu})\dot{\mathfrak{p}} + \mu\ddot{\mathfrak{p}}\} + \ldots$$

Due to the linear independence of $\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}$, as $\varepsilon \to 0$, one will get:

$$\dot{\lambda} = \lambda + \dot{\mu} = \mu = 0, \qquad \text{thus,} \quad \mathfrak{g} = \rho\,\mathfrak{p}\,;$$

i.e., the lines $\mathfrak{p}$ of the torse will be the tangents to the curve of regression.

3. *The torses with* $\ddot{\mathfrak{p}}\ddot{\mathfrak{p}} \neq 0$ *are the tangent families of non-planar curves, which can then be called curves of regression (point of regression* x *= point of curve, plane of regression* w *= osculating plane).*

Proof:

α) Let the tangent family to a non-planar curve be given. If $\mathfrak{X}(u)$ is the position vector of a point of the curve then one will have:

$$\mathfrak{p} = \{\dot{\mathfrak{X}} \mid \mathfrak{X}\times\dot{\mathfrak{X}}\}, \qquad \dot{\mathfrak{p}} = \{\ddot{\mathfrak{X}} \mid \mathfrak{X}\times\ddot{\mathfrak{X}}\}, \qquad \ddot{\mathfrak{p}} = \{\dddot{\mathfrak{X}} \mid \dot{\mathfrak{X}}\times\ddot{\mathfrak{X}} + \mathfrak{X}\times\dddot{\mathfrak{X}}\},$$

so:

$$\dot{\mathfrak{p}}\dot{\mathfrak{p}} = 0, \qquad \ddot{\mathfrak{p}}\ddot{\mathfrak{p}} = 2 < \dot{\mathfrak{X}}, \ddot{\mathfrak{X}}, \dddot{\mathfrak{X}} > \;\not\equiv 0\,;$$

$< \dot{\mathfrak{X}}, \ddot{\mathfrak{X}}, \dddot{\mathfrak{X}} > \equiv 0$ is characteristic of plane curves.

β) Let a torse be given with $\ddot{\mathfrak{p}}\ddot{\mathfrak{p}} \neq 0$. The points of regression generate a space curve, since otherwise $\ddot{\mathfrak{p}}\ddot{\mathfrak{p}} = 0$ would follow from 2*b*), which would be contrary to assumption.

γ) The connecting line $\mathfrak{g}$ of the curve points $x(u)$ and $x(u + \varepsilon)$ satisfies the conditions:

$$\mathfrak{g}\mathfrak{p}(u) = \mathfrak{g}\dot{\mathfrak{p}}(u) = \mathfrak{g}\mathfrak{p}(u + \varepsilon) = \mathfrak{g}\dot{\mathfrak{p}}(u + \varepsilon) = 0.$$

As $\varepsilon \to 0$, one will get:

$$\mathfrak{g}\mathfrak{p} = \mathfrak{g}\dot{\mathfrak{p}} = \mathfrak{g}\ddot{\mathfrak{p}} = \mathfrak{g}\dddot{\mathfrak{p}} = 0.$$

Since the discriminant $D_4(\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}})$ has rank $r = 3$, these equations determine only the single six-vector $\mathfrak{g} = \mathfrak{p}$ (no. **32**); the lines $\mathfrak{p}$ are then the tangents of the curve that is generated by x, and from 2*a*), that curve will not be planar.

Dually, this likewise implies that the plane w defines a strip whose generators are the lines $\mathfrak{p}$; the planes w are then the osculating planes of the curve of regression.

In what follows, we shall always assume that $\ddot{\mathfrak{p}}\ddot{\mathfrak{p}} \neq 0$ *for the study of torses; i.e., pencils of lines, tangent families of plane curves, and the generating families of cones will remain excluded from consideration.*

The surfaces that are spanned by a family of lines (ruled family, resp.) are called *line surfaces (ruled surfaces, resp.).* The surfaces that are generated by torses are the *developable surfaces* (no. **5**).

47. Models with discrete lines; *ε*-models, torsal models. In order to make the differential-geometric relationships more intuitive, we compare the *models* for line families that are defined by n discrete lines $\mathfrak{p}^0, \mathfrak{p}^{\mathrm{I}}, \ldots, \mathfrak{p}^{n-1}$.

a) *ε-models* (Fig. 16):

The lines of the model are chosen from the given family of lines, namely:

$$\mathfrak{p}^0 = \mathfrak{p}(u),\ \mathfrak{p}^{\mathrm{I}} = \mathfrak{p}(u + \varepsilon),\ \ldots,\ \mathfrak{p}^{n-1} = \mathfrak{p}(u + (n-1)\,\varepsilon).$$

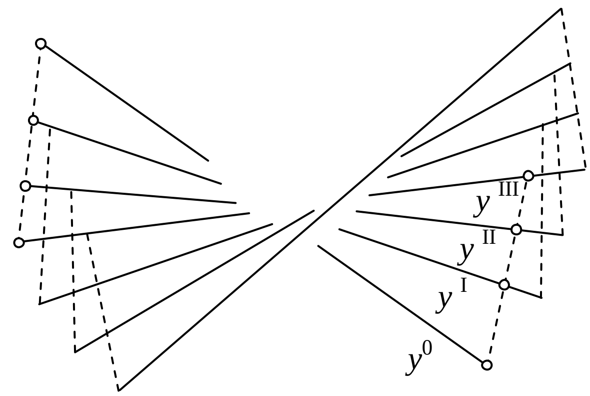

Figure 16.

ε is a small number in this. Under the passage to the limit $\varepsilon \to 0$, all lines of the ε-model will converge to the fixed line $\mathfrak{p}^0$, and the properties of the ε-model that are given by the six-vectors $\mathfrak{p}^0$, $\mathfrak{p}^{\mathrm{I}}$, $\mathfrak{p}^{\mathrm{II}}$, etc., will go to the corresponding differential-geometric properties of the line family that are representable by $\mathfrak{p}$, $\dot{\mathfrak{p}}$, $\ddot{\mathfrak{p}}$, etc. The ε-models depend upon the choice of parameter u; under a substitution $u = u(\bar{u})$, the family of lines will give an ε-model with other lines.

Two successive lines $\mathfrak{p}^0$, $\mathfrak{p}^{\mathrm{I}}$ of the ε-model yield the scalar product:

$$\mathfrak{p}^0\, \mathfrak{p}^{\mathrm{I}} = \mathfrak{p}(u)\, \mathfrak{p}(u+\varepsilon) = -\tfrac{1}{2}\varepsilon^2\, \dot{\mathfrak{p}}\dot{\mathfrak{p}} - \tfrac{1}{2}\varepsilon^3\, \dot{\mathfrak{p}}\ddot{\mathfrak{p}} - \tfrac{1}{24}\varepsilon^4\, (3\dot{\mathfrak{p}}\ddot{\mathfrak{p}} + 4\dot{\mathfrak{p}}\dddot{\mathfrak{p}}) + \dots$$

with the use of the identities (58).

The coefficient of ε^2 is non-zero for ruled families; the coefficients of ε^2 and ε^3 vanish for torses, while the coefficient of ε^4 will be equal to $\dfrac{1}{24}\ddot{\mathfrak{p}}\ddot{\mathfrak{p}}$, and thus, non-zero.

The intersection condition (16) *will then be fulfilled for* $\mathfrak{p}^0$ *and* $\mathfrak{p}^{\mathrm{I}}$ *under the passage to the limit* $\varepsilon \to 0$ *for torses in higher order, as well as ruled families.* Occasionally, one appeals to the vague way of saying this that takes the form: "Neighboring lines are skew for ruled families, while they will intersect for torses."

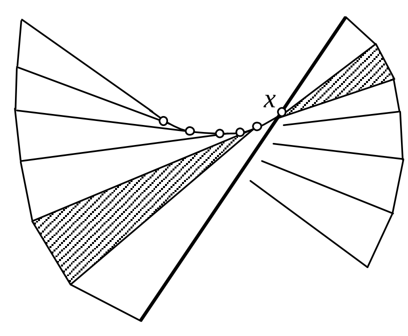

Figure 17.

For a sufficiently small ε, the first non-vanishing term of the power series will outweigh the entire series that follows it. Consequently, if ε is not too large then any two successive lines of an ε-model will always be skew.

b) **Torsal models** (Fig. 17). In order to make the differential-geometric properties of the torses more intuitive, we will often employ so-called "torsal" models instead of ε-models. The lines of the torsal model will not be chosen from the given family of lines, but will be arbitrary, up to the following coupling requirements:

Any two successive lines shall intersect, while no three successive lines shall have a common point or lie in a plane (Fig. 17).

§ 13. Contact structures.

In a similar way to how we introduced tangents and osculating planes in the theory of curves (no. **5**), we would now like to define *contact structures* for a given family of lines, namely, quadrics, linear systems of lines, and the like, that are determined by $\mathfrak{p}$, $\dot{\mathfrak{p}}$, $\ddot{\mathfrak{p}}$, etc., for a fixed value of *u*.

48. Tangent systems and contact correlations. We start from pencils of complexes $\lambda_1\mathfrak{p} + \lambda_2\dot{\mathfrak{p}}$. The lines $\mathfrak{t}$ that are common to all of the complexes of this pencil are determined by:

$$\mathfrak{t}\mathfrak{p} = 0, \qquad \mathfrak{t}\dot{\mathfrak{p}} = 0,$$

and define a linear system of lines (no. **33**) that we would like to call a *tangent system.*

Ruled families: Since $\dot{\mathfrak{p}}\dot{\mathfrak{p}} \neq 0$, the discriminant $D_2(\mathfrak{p}, \dot{\mathfrak{p}})$ will have rank $r = 1$. For that reason, the tangent system is parabolic (no. **33**) and can be decomposed into a series of pencils of lines; the vertices *y* of the pencils lie on $\mathfrak{p}$, while the planes *v* of the pencil will intersect in $\mathfrak{p}$. The points *y* are projectively-related to the planes *v* (*contact correlation*). Since $\mathfrak{t}\mathfrak{p} = 0$, $\mathfrak{t}\,(\mathfrak{p} + \varepsilon\dot{\mathfrak{p}}) = 0$, the lines $\mathfrak{t}$ of the tangent system will be the tangents, and therefore the planes *v* of the contact planes of the ruled surface along $\mathfrak{p}$ that is spanned by the ruled family.

For $\dot{\mathfrak{p}}\dot{\mathfrak{p}} > 0$ (< 0, resp.), the tangent system is positively-wound (negatively-wound, resp.) (no. **36**); this distinction is invariant under parameter substitutions and principal projectivities. We agree that:

In what follows, it shall always be assumed that $\dot{\mathfrak{p}}\dot{\mathfrak{p}} > 0$.

This implies no essential restriction, since one can convert the given ruled family with $\dot{\mathfrak{p}}\dot{\mathfrak{p}} < 0$ into one with $\dot{\mathfrak{p}}\dot{\mathfrak{p}} > 0$ by a reflection.

Torses: $D_2(\mathfrak{p}, \dot{\mathfrak{p}})$ has rank $r = 0$. The tangent system is singular; it decomposes into the line field of the plane of regression w and the bundle of lines with the point of regression x as its vertex. The developable surface has the fixed contact plane w along $\mathfrak{p}$.

49. Osculating quadrics. Principal tangents. We start with the bundle of complexes $\lambda_1\mathfrak{p} + \lambda_2\dot{\mathfrak{p}} + \lambda_3\ddot{\mathfrak{p}}$ ([1]). The axes of the singular complexes that are contained in the bundle of complexes define a quadric that we shall call the *first osculating quadric*. We refer to the conjugate quadric as the *second osculating quadric*. It consists of lines $\mathfrak{h}$ that are common to all complexes of the bundle $\lambda_1\mathfrak{p} + \lambda_2\dot{\mathfrak{p}} + \lambda_3\ddot{\mathfrak{p}}$. The lines $\mathfrak{h}$ are contained in the tangent system and fulfill the requirements:

$$\mathfrak{h}\mathfrak{p} = \mathfrak{h}\dot{\mathfrak{p}} = \mathfrak{h}\ddot{\mathfrak{p}} = 0. \tag{59}$$

Ruled families: $D_3(\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}})$ has rank $r = 3$. From no. **35**, the two osculating quadrics are hyperboloidal ([2]), and thus span a hyperboloid (viz., the *osculating hyperboloid*).

The lines $\mathfrak{h}$ *of the second osculating quadric are principal tangents of the ruled surface.*

Proof: The lines $\mathfrak{h}$ are principal tangents to the osculating hyperboloids. We thus have only to show that the given ruled surface and the osculating hyperboloid coincide in the principal tangents along $\mathfrak{p}$. We assume that the first osculating quadric that is associated with $\mathfrak{p}(0)$ is:

$$\mathfrak{p}^* = \mathfrak{p}(0) + u\,\dot{\mathfrak{p}}(0) + \frac{u^2}{2}\ddot{\mathfrak{p}}(0)\,;$$

for $u = 0$, one then has:

$$\mathfrak{p} = \mathfrak{p}^*, \qquad \dot{\mathfrak{p}} = \dot{\mathfrak{p}}^*, \qquad \ddot{\mathfrak{p}} = \ddot{\mathfrak{p}}^*.$$

However, the principal tangents along $\mathfrak{p}(0)$ are determined by $\mathfrak{p}(0), \dot{\mathfrak{p}}(0), \ddot{\mathfrak{p}}(0)$: The ruled surface is represented by:

$$\mathfrak{X} = \frac{\mathfrak{P}\times\bar{\mathfrak{P}}}{\mathfrak{P}\mathfrak{P}} + v\,\mathfrak{P}.$$

The first and second derivatives of $\mathfrak{X}$ with respect to u and v are established by $\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}$, and thus, from (8), the principal tangents.

A ruled surface has two families of principal tangent curves:

([1]) On the linear independence of $\mathfrak{p}$, $\dot{\mathfrak{p}}$, $\ddot{\mathfrak{p}}$, cf., no. **54**.

([2]) The bundle of complexes $\lambda_1\mathfrak{p} + \lambda_2\dot{\mathfrak{p}} + \lambda_3\ddot{\mathfrak{p}}$ cannot be of type a) β) (no. **34**); there would then exist no quadric for ones of type a) β), but $\mathfrak{p}$ is a line of the quadric.

1. The lines $\mathfrak{p}$ of the ruled family.

2. The integral curves of the direction field that is established by the principal tangents $\mathfrak{h}$ to the ruled surface.

The lines of the ruled family will be cut from the second family of principal tangent curves along projective sequences of points.

Proof: Let $d(u)$ be the double ratio of the point of intersection of the four principal tangent curves of the second family with the lines $\mathfrak{p}(u)$ of the ruled family. Since these principal tangent curves contact the lines $\mathfrak{h}$, and the lines $\mathfrak{h}$ cut the lines of the first osculating quadric along the projective sequences of points (no. **35**), $\dot{d}(u) = 0$, so d = const.

Torses: $D_3(\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}})$ has rank $r = 1$. The osculating quadrics are of type *c*) (no. **35**) and both of them degenerate into a pencil of lines with the point of regression x as its vertex and the plane of regression w as the plane of the pencil.

50. Osculating system. Nodal tangents. We consider the bush of complexes $\lambda_1\mathfrak{p} + \ldots + \lambda_4\dddot{\mathfrak{p}}$ ([1]). The linear system of lines that is defined by the axes of the singular complex of the bush is called the *osculating system*.

Ruled families: $D_4(\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}})$ has rank $r = 4$ or $r = 3$. We then have the case distinction (no. **32**):

$$\left.\begin{array}{ll} r = 4 & \left\{\begin{array}{ll} D_4 > 0: & \text{hyperbolic} \\ D_4 < 0: & \text{elliptic} \end{array}\right. \\ r = 3: & \qquad\qquad \text{parabolic} \end{array}\right\} \text{osculating system.}$$

We then speak of *hyperbolic, elliptic, or parabolic ruled families.* The focal lines of the osculating system – i.e., lines that cut all lines of the osculating system – are determined by:

$$\mathfrak{k}\mathfrak{p} = \mathfrak{k}\dot{\mathfrak{p}} = \mathfrak{k}\ddot{\mathfrak{p}} = \mathfrak{k}\dddot{\mathfrak{p}} = 0, \tag{60}$$

and shall be called *nodal tangents* $\mathfrak{k}$ of the ruled family. From (59), the nodal tangents will be special principal tangents.

A hyperbolic (elliptic, parabolic, resp.) ruled family has 2 (0, 1, resp.) *nodal tangents for any line* $\mathfrak{p}$. *In the first case, the* 2 *nodal tangents are mutually skew (cf., no.* **33**).

([1]) $\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}$ are assumed to be linearly-independent in this (cf., no. **54**).

Torses: $D_4(\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}})$ has rank $r = 3$. The osculating system is parabolic, so it will have only *one* focal line (nodal tangent). That nodal tangent is the line $\mathfrak{p}$ itself; the conditions (60) will be fulfilled by $\mathfrak{k} = \mathfrak{p}$, since the identities that follow from $\mathfrak{p}\mathfrak{p} = \dot{\mathfrak{p}}\dot{\mathfrak{p}} = 0$ by repeated differentiation.

51. Nodal families of a ruled family. Here, we shall speak of only hyperbolic and parabolic ruled families, and introduce the following terms:

Nodal point = point of intersection of the line $\mathfrak{p}$ with the nodal tangent.
Nodal family = families of lines of the nodal tangents.
Nodal curves = geometric locus of the nodal points.

If a nodal curve is rectilinear then the associated nodal family will degenerate into that line.

If a nodal family does not degenerate into a line then it will never be a torse, but always a ruled family.

Proof: For a non-rectlinear node, one has $\dot{\mathfrak{k}} \neq \sigma\mathfrak{k}$ ([1]). Now if $\dot{\mathfrak{k}}\dot{\mathfrak{k}} = 0$ were true then $\lambda_1\mathfrak{k} + \lambda_2\dot{\mathfrak{k}}$ would be a pencil of lines, and from (60) and the equations that follow from (60), namely:

$$\dot{\mathfrak{k}}\mathfrak{p} = \dot{\mathfrak{k}}\dot{\mathfrak{p}} = \dot{\mathfrak{k}}\ddot{\mathfrak{p}} = 0,$$

any line of this pencil would cut all lines of the first osculating quadric. That is impossible, since osculating quadrics are hyperboloidal.

If the ruled family II is the nodal family of the ruled family I then the ruled family I will also be the nodal family of the ruled family II.

Proof: It follows from (60) that:

$$\mathfrak{k}\mathfrak{p} = \dot{\mathfrak{k}}\mathfrak{p} = \ddot{\mathfrak{k}}\mathfrak{p} = \dddot{\mathfrak{k}}\mathfrak{p} = 0.$$

If a ruled family is parabolic then one will have:

$$\mathfrak{k}\ddddot{\mathfrak{p}} = 0, \tag{61}$$

along with (60).

([1]) We exclude discrete points with $\dot{\mathfrak{k}} = \sigma\mathfrak{k}$.

Proof: For a parabolic osculating system, the nodal tangent $\mathfrak{k}$ itself belongs to the osculating system. $\mathfrak{k}$ can then be a linear combination of $\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}$, which we assume to be linearly-independent (no. **54**), so:

$$\mathfrak{k} = A\mathfrak{p} + B\dot{\mathfrak{p}} + C\ddot{\mathfrak{p}} + D\dddot{\mathfrak{p}} .$$

Since the nodal tangents do not belong to the first osculating quadric, $D \neq 0$, and:

$$0 = \frac{1}{D}\mathfrak{k}\dot{\mathfrak{k}} = \frac{1}{D}\mathfrak{k}(\dot{A}\mathfrak{p} + \cdots + (C + \dot{D})\dddot{\mathfrak{p}} + D\ddddot{\mathfrak{p}}) = \mathfrak{k}\ddddot{\mathfrak{p}} .$$

Conversely, one has:

If (61) *is true identically for u, along with* (60), *then the ruled family will be parabolic or the nodal curve that is associated with* $\mathfrak{k}$ *will be rectilinear.*

Proof: It follows from (60) and (61) that:

$$\mathfrak{p}\mathfrak{k} = \dot{\mathfrak{p}}\mathfrak{k} = \ddot{\mathfrak{p}}\mathfrak{k} = \dddot{\mathfrak{p}}\mathfrak{k} = 0 \qquad \text{and} \qquad \mathfrak{p}\dot{\mathfrak{k}} = \dot{\mathfrak{p}}\dot{\mathfrak{k}} = \ddot{\mathfrak{p}}\dot{\mathfrak{k}} = \dddot{\mathfrak{p}}\dot{\mathfrak{k}} = 0.$$

Thus, either $\dot{\mathfrak{k}} = \sigma\,\mathfrak{k}$ (rectilinear nodal curve) or the bush of complexes $\lambda_1\mathfrak{p} + \ldots + \lambda_4\dddot{\mathfrak{p}}$ will be conjugate to the pencil of complexes $\mu_1\mathfrak{k} + \mu_2\dot{\mathfrak{k}}$, and thus parabolic, like them.

52. Osculating complex. Projective and parameter invariance of the contact structure. We consider the forest of complexes $\lambda_1\mathfrak{p} + \ldots + \lambda_5\overset{\cdot\cdot\cdot\cdot\cdot}{\mathfrak{p}}$, under the assumption that $\mathfrak{p}, \dot{\mathfrak{p}}, \ldots, \overset{\cdot\cdot\cdot\cdot\cdot}{\mathfrak{p}}$ are linearly independent, and refer to the conjugate complex as the *osculating complex* $\mathfrak{s}$. It contains the axes of all singular complexes that are contained in the forest of complexes and satisfies the conditions:

(62) $$\mathfrak{p}\mathfrak{s} = \dot{\mathfrak{p}}\mathfrak{s} = \ldots = \overset{\cdot\cdot\cdot\cdot\cdot}{\mathfrak{p}}\mathfrak{s} = 0.$$

All of the contact structures that were introduced in this paragraph are not only projectively invariant, but also invariant under parameter substitutions:

(63) $$u = u(\bar{u}), \qquad \text{with} \qquad \frac{du}{d\bar{u}} \neq 0$$

and renormalizations:

(64) $$\hat{\mathfrak{p}} = \sigma(u)\mathfrak{p}, \qquad \sigma \neq 0.$$

In fact, it will follow from:

$$\frac{d\hat{\mathfrak{p}}}{d\bar{u}} = (\sigma\dot{\mathfrak{p}} + \dot{\sigma}\mathfrak{p})\frac{du}{d\bar{u}}$$

that the pencil of complexes $\lambda_1\hat{\mathfrak{p}} + \lambda_2 \frac{du}{d\bar{u}}$ is identical with the pencil of complexes $\lambda_1\mathfrak{p}$ + $\lambda_2\dot{\mathfrak{p}}$, and the same thing will be true for the bundle of complexes $\lambda_1\mathfrak{p}$ + $\lambda_2\dot{\mathfrak{p}} + \lambda_3\ddot{\mathfrak{p}}$, etc.

53. Explanation for the model. We make the contact structures more intuitive by the following *comparison with the ε-model* (Fig. 16):

ε-model	**Family of lines**
System of lines with the focal lines $\mathfrak{p}^0$, $\mathfrak{p}^{\mathrm{I}}$	Tangent system
Quadric with the skew lines $\mathfrak{p}^0$, $\mathfrak{p}^{\mathrm{I}}$, $\mathfrak{p}^{\mathrm{II}}$ and the conjugate quadric of the lines that meet it.	First and second osculating quadric.
Lines that meet four successive lines $\mathfrak{p}^0$, $\mathfrak{p}^{\mathrm{I}}$, $\mathfrak{p}^{\mathrm{II}}$, $\mathfrak{p}^{\mathrm{III}}$.	Nodal tangents
System of lines with the lines $\mathfrak{p}^0$, $\mathfrak{p}^{\mathrm{I}}$, $\mathfrak{p}^{\mathrm{II}}$, $\mathfrak{p}^{\mathrm{III}}$.	Osculating system
Line complex with the lines $\mathfrak{p}^0$, $\mathfrak{p}^{\mathrm{I}}$, …, $\mathfrak{p}^{\mathrm{IV}}$.	Osculating complex

Under passage to the limit $\varepsilon \to 0$, the line structures of the model will converge to the contact structures of the family of lines.

If one extends the ε-model by the addition of the lines that meet any four successive lines of the model (dashed lines in Fig. 16) then one will have the following further analogy with ruled families with non-rectilinear nodal curves:

ε-**model**	**Family of lines**
Any four successive lines of intersection will cut one line of the mode; i.e., the relationship between the model lines and the lines of intersection is reciprocal.	The relationship between ruled families and nodal families is reciprocal (no. **51**).

The properties of the contact structures of torses will become clearer (cf., no. **47**) by a *comparison of the torsal models* (Fig. 17):

Torsal model	**Torse**
The lines of the model envelope a non-planar segmented path.	The lines of the torse are tangents to a non-planar curve (curve of regression).
Point of intersection x (connecting plane w, resp.) of successive lines.	Points (osculating planes, resp.) of the curve of regression.
Line system with successive lines as focal lines: breaks down into the plane w and the bundle of lines with vertex x (Fig. 17) in the line field.	Tangent system: breaks down into the plane of regression and the bundle of lines with the point of regression as its vertex in the line field.

54. Families of lines with fixed contact structures. In no. **46**, we showed:

$\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}$ *are linearly independent for all u if and only the line family is a pencil of lines.*

One likewise proves:

$\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}$ ($\mathfrak{p}, \dot{\mathfrak{p}}, \ldots, \ddddot{\mathfrak{p}}$ or $\mathfrak{p}, \dot{\mathfrak{p}}, \ldots, \overset{\cdot\cdot\cdot\cdot\cdot}{\mathfrak{p}}$, *resp.) are linearly independent for all u if and only if the families of lines is a quadric (contained in a linear system of lines, contained in a linear complex of lines, resp.). That quadric (that system of lines, that complex of lines, resp.) is then the first osculating quadric (osculating system, osculating complex, resp.) for lines of the family of lines.*

For *ruled families*, the linear independence of $\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}$ follows from the first of the aforementioned theorems. We further deduce that:

$\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}$ *are linearly independent. Therefore, quadrics will be excluded from further consideration, from now on.* All of the contact structures that were introduced in this paragraph will then be determined uniquely, with the exception of the osculating complexes; they are established uniquely when $\mathfrak{p}, \dot{\mathfrak{p}}, \ldots, \ddddot{\mathfrak{p}}$ are also linearly independent.

For *torses*, the linear independent of $\mathfrak{p}, \dot{\mathfrak{p}}, \ldots, \ddddot{\mathfrak{p}}$ follows from:

$$D_5(\mathfrak{p}, \dot{\mathfrak{p}}, \ldots, \overset{\cdot\cdot\cdot\cdot\cdot}{\mathfrak{p}}) = (\ddot{\mathfrak{p}}\,\ddot{\mathfrak{p}})^5 \neq 0.$$

All contact structures, up to the osculating complexes inclusively, are established uniquely now.

§ 14. Invariants of a family of lines.

55. Reduction to semi-invariants. The projective differential geometry of the families of lines that we would now like to develop treats the projective-invariant properties "in the neighborhood" of a line $\mathfrak{p}$ of the family of lines, in which, as agreed (no. **16**), we restrict ourselves to the principal projectivities ($k > 0$). Analytically, this means: *We have to ascertain the complete invariants* Φ *and the sign invariants of the six-vectors* $\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}$, *etc., up to arbitrarily-high derivatives, using the prescriptions of* § 7. In no. **24**, each of the given discrete six-vectors could be multiplied by an arbitrary renormalization factor σ; here, one can renormalize $\hat{\mathfrak{p}} = \sigma\,\mathfrak{p}$ with the arbitrary function $\sigma(u)$ and then have $\dot{\hat{\mathfrak{p}}} = \dot{\sigma}\mathfrak{p} + \sigma\dot{\mathfrak{p}}$, etc. Equations (37), (38) are altered correspondingly.

The invariants thus-determined are, in general, still independent of the choice of parameter u. However, since we seek geometric properties of the families of lines that are not qualified by a special choice of u, we now come to the additional requirement:

The complete invariants Φ *and sign invariants shall remain unchanged under arbitrary parameter substitutions* (63); one must then have:

$$\Phi\left(\mathfrak{p}, \frac{d\mathfrak{p}}{du}, \frac{d^2\mathfrak{p}}{du^2}, \ldots\right) = \Phi\left(\mathfrak{p}, \frac{d\mathfrak{p}}{d\bar{u}}, \frac{d^2\mathfrak{p}}{d\bar{u}^2}, \ldots\right).$$

Now, instead of next defining semi-invariants and complete invariants, as well as sign invariants (and from them, parameter-invariant expressions), as in § 7, we shall embark upon the following preferable path:

1. A distinguished parameter $s = s(u)$ is defined that is a complete invariant and is invariant under the parameter substitutions (63), analogous to the motion-invariant arc-length in the metric theory of curves. Based upon that analogy, we call the parameter s the *arc-length*; we will denote derivatives with respect to s by primes; e.g., $\mathfrak{p}' = d\mathfrak{p} / ds$.

2. The six-vector $\mathfrak{p}$, and therefore the derivatives $\mathfrak{p}'$, $\mathfrak{p}''$, etc., as well, will be *inhomogeneously normalized* in such a way that the normalized six-vector is semi-invariant; i.e., it will remain normalized under any unity transformation (no. **17**).

3. Since, from 1 and 2, parameter substitutions and renormalizations no longer come into question, merely the semi-invariants of the six-vectors $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{p}''$, etc., will be determined.

56. Fundamental system and differential equations. The determination of the semi-invariants of $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{p}''$, etc., to which our problem has been reduced in no. **55**, proceeds as follows:

A) We define the scalar products of the largest-possible number $n \leq 6$ of the linearly-independent six-vectors $\mathfrak{p}, \mathfrak{p}', \ldots, \mathfrak{p}^{(n-1)}$.

B) We determine the coefficients δ_i^n of the linear combination:

$$\mathfrak{p}^n = \delta_0^n \, \mathfrak{p} + \delta_1^n \, \mathfrak{p}' + \ldots + \delta_{n-1}^n \, \mathfrak{p}^{(n-1)}.$$

From no. **26**, all semi-invariants of $\mathfrak{p}, \mathfrak{p}', \ldots, \mathfrak{p}^{(n)}$ can be expressed in terms of the scalar products of *A*) and the coefficients δ_i^n of *B*). However, one even has, moreover:

All semi-invariants of $\mathfrak{p}, \mathfrak{p}', \ldots, \mathfrak{p}^{(m)}$ (*m arbitrarily large) are functions of the invariants A*) *and B*), *and correspond to higher derivatives of the invariants B*) *with respect to the arc-length s.*

Proof: By repeated derivation of the linear combination *B*) with respect to s and elimination of the $\mathfrak{p}^{(n)}$, $\mathfrak{p}^{(n+1)}$, etc., that appear on the right-hand side, we will get the six-vector $\mathfrak{p}^{(m)}$ as a linear combination of $\mathfrak{p}, \mathfrak{p}', \ldots, \mathfrak{p}^{(n-1)}$ for any arbitrarily high m. The coefficients are functions of the δ_i^n and correspondingly higher derivatives of the δ_i^n with respect to s. From no. **27**, all invariants of $\mathfrak{p}, \mathfrak{p}', \ldots, \mathfrak{p}^{(m)}$ can be expressed in terms of these coefficients and the scalar products *A*).

Two families of lines are projective to each other and relate to each other projectively by the same s-values line-wise if and only if they have the same functions of the arc-length s for the invariants A) *and B*). *All projectively-invariant properties of a family of lines are then characterized by the invariants A*) *and B*).

Proof: For each of the two families of lines, we develop $\mathfrak{p}(s + \varepsilon)$ in powers of ε and replace the $\mathfrak{p}^{(m)}(s)$ with linear combinations of $\mathfrak{p}, \mathfrak{p}', \ldots, \mathfrak{p}^{(n-1)}$ for $m > n - 1$. In that way,

$\mathfrak{p}(s + \varepsilon)$ itself will be represented as a linear combination of $\mathfrak{p}$, $\mathfrak{p}'$, …, $\mathfrak{p}^{(n-1)}$. Due to the agreement of the invariants *B*), one will obtain the same linear combination for both families of lines. Moreover, due to the agreement in the invariants *A*), from no. **27**, the six-vectors $\mathfrak{p}(s)$, $\mathfrak{p}'(s)$, …, $\mathfrak{p}^{(n-1)}(s)$, and $\mathfrak{p}(s + \varepsilon)$ of the one family of lines transform into the corresponding six-vectors of the other family of lines under a unity transformation that is independent of ε, so the two families of lines are projective. Conversely, projective families of lines will yield the same invariants *A*) and *B*) in a natural way.

We shall refer to the *n* linearly-independent six-vectors $\mathfrak{p}$, $\mathfrak{p}'$, …, $\mathfrak{p}^{(n-1)}$ as the *fundamental system* and the linear combination *B*) as the *differential equation.* In place of the $\mathfrak{p}$, $\mathfrak{p}'$, …, $\mathfrak{p}^{(n-1)}$, one can also employ any other *n* linearly-independent, inhomogeneously-normalized linear combinations of them – say, $\mathfrak{r}^{\mathrm{I}}$, $\mathfrak{r}^{\mathrm{II}}$, …, $\mathfrak{r}^{n}$ – as the fundamental system. The linear combinations for $\mathfrak{r}'^{\mathrm{I}}$, $\mathfrak{r}'^{\mathrm{II}}$, …, $\mathfrak{r}'^{n}$ then appear in place of the linear combination *B*) as the differential equations. One will choose the fundamental system $\mathfrak{r}^{\mathrm{I}}$, $\mathfrak{r}^{\mathrm{II}}$, …, $\mathfrak{r}^{n}$ in such a way that as many scalar products as possible will vanish, or at least remain constant.

In the metric theory of curves, the fundamental system $\mathfrak{r}^{\mathrm{I}}$, $\mathfrak{r}^{\mathrm{II}}$, …, $\mathfrak{r}^{n}$ corresponds to the triad of unit vectors of the tangent, principal normal, and binormal, while the differential equations are the *Frenet formulas* (no. **5**).

We will now carry out the program that was presented here for hyperbolic and parabolic ruled families and for torses, in turn. We shall leave the treatment of the elliptic ruled families, which can result from the same process, to the reader.

§ 15. Hyperbolic ruled families.

57. Arc-length *s* and normalization of $\mathfrak{p}$.

Arc-length s: Following the process of **G. Thomsen** (1926), we define the arc-length *s* by:

$$\dot{s}^4 = \frac{D_4(\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}})}{(\dot{\mathfrak{p}}\dot{\mathfrak{p}})^4}. \tag{65}$$

s will be established to be real by (65) since the numerator is positive for hyperbolic ruled families, from no. **50**, and the denominator is positive anyway. As for the metric arc-length of the curves, an additive constant (e,g,, the origin of the *s*-numbers) and the sign (e.g., positive sense of the *s*-numbers) will remain arbitrary.

The arc-length s = s(*u*) *is completely invariant and invariant under the parameter substitutions* (63).

Proof: The right-hand side of (65) is semi-invariant as a function of the scalar products of $\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}$. However, it is also completely invariant, so under the renormalization $\hat{\mathfrak{p}} = \sigma\mathfrak{p}$, one will obtain:

$$(\dot{\hat{\mathfrak{p}}}\dot{\hat{\mathfrak{p}}})^4 = \sigma^8(\dot{\mathfrak{p}}\dot{\mathfrak{p}})^4, \qquad D_4(\dot{\hat{\mathfrak{p}}}, \ddot{\hat{\mathfrak{p}}}, \dddot{\hat{\mathfrak{p}}}, \hat{\mathfrak{p}}) = \sigma^8 D_4(\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}).$$

With the parameter substitution (63), one will get, after a brief reduction:

$$\left(\frac{ds}{d\bar{u}}\right)^4 = \dot{s}^4\left(\frac{du}{d\bar{u}}\right)^4, \qquad \left(\frac{d\mathfrak{p}}{d\bar{u}}\frac{d\mathfrak{p}}{d\bar{u}}\right)^4 = (\dot{\mathfrak{p}}\dot{\mathfrak{p}})^4\left(\frac{du}{d\bar{u}}\right)^8,$$
$$D_4\left(\mathfrak{p}, \frac{d\mathfrak{p}}{d\bar{u}}, \frac{d^2\mathfrak{p}}{d\bar{u}^2}, \frac{d^3\mathfrak{p}}{d\bar{u}^3}\right) = D_4(\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}})\left(\frac{du}{d\bar{u}}\right)^{12};$$

one gets the parameter invariance of equation (65) by substituting these expressions.

Interpretation of the arc-length s in terms of the ε-model:

For any four lines $\mathfrak{p}^0$, $\mathfrak{p}^{\mathrm{I}}$, $\mathfrak{p}^{\mathrm{II}}$, $\mathfrak{p}^{\mathrm{III}}$; $\mathfrak{p}^{\mathrm{I}}$, $\mathfrak{p}^{\mathrm{II}}$, $\mathfrak{p}^{\mathrm{III}}$, $\mathfrak{p}^{\mathrm{IV}}$; $\mathfrak{p}^{\mathrm{II}}$, $\mathfrak{p}^{\mathrm{III}}$, $\mathfrak{p}^{\mathrm{IV}}$, $\mathfrak{p}^{\mathrm{V}}$, etc., we form the invariant I that is described by (48) and (49), and sum the absolute values of the fourth roots $\sqrt[4]{I}$. Upon passing to the limit $\varepsilon \to 0$, the sum:

$$s_\varepsilon = \frac{1}{\sqrt[3]{18}}\sum\left|\sqrt[4]{I}\right|$$

will converge to the arc-length s when one fixes the initial line $\mathfrak{p}^0 = \mathfrak{p}(0)$ and the final line $\mathfrak{p}^n = \mathfrak{p}(n\varepsilon)$, and the number of intermediate lines will increase without bound as $n \to \infty$, $\varepsilon = 1 / n \to 0$.

Proof: It follows from:

$$\mathfrak{p}^0 = \mathfrak{p},\ \mathfrak{p}^{\mathrm{I}} = \mathfrak{p} + \varepsilon\dot{\mathfrak{p}} + \frac{\varepsilon^2}{2}\ddot{\mathfrak{p}} + \ldots, \qquad \mathfrak{p}^{\mathrm{II}} = \mathfrak{p} + 2\varepsilon\dot{\mathfrak{p}} + 2\varepsilon^2\ddot{\mathfrak{p}} + \ldots,$$
$$\mathfrak{p}^{\mathrm{III}} = \mathfrak{p} + 3\varepsilon\dot{\mathfrak{p}} + \frac{9}{2}\varepsilon^2\ddot{\mathfrak{p}} + \frac{9}{2}\varepsilon^3\dddot{\mathfrak{p}} + \ldots$$

that:

$$(\mathfrak{p}^0\mathfrak{p}^{\mathrm{I}})\,(\mathfrak{p}^0\mathfrak{p}^{\mathrm{II}})\,(\mathfrak{p}^0\mathfrak{p}^{\mathrm{III}})\,(\mathfrak{p}^{\mathrm{I}}\mathfrak{p}^{\mathrm{II}})\,(\mathfrak{p}^{\mathrm{I}}\mathfrak{p}^{\mathrm{III}})\,(\mathfrak{p}^{\mathrm{II}}\mathfrak{p}^{\mathrm{III}}) = \frac{9}{4}\varepsilon^{12}(\dot{\mathfrak{p}}\dot{\mathfrak{p}})^6 + \ldots,$$

$$D_4(\mathfrak{p}^0, \mathfrak{p}^{\mathrm{I}}, \mathfrak{p}^{\mathrm{II}}, \mathfrak{p}^{\mathrm{III}}) = D_4(\mathfrak{p}, \mathfrak{p} + \varepsilon\dot{\mathfrak{p}}, \mathfrak{p} + 2\varepsilon\dot{\mathfrak{p}} + 2\varepsilon^2\ddot{\mathfrak{p}}, \mathfrak{p} + 3\varepsilon\dot{\mathfrak{p}} + \frac{9}{2}\varepsilon^2\ddot{\mathfrak{p}} + \frac{9}{2}\varepsilon^3\dddot{\mathfrak{p}}) + \ldots$$
$$= 81\,\varepsilon^{12}\,D_4(\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}) + \ldots;$$

hence, from (48):

$$I = 18^{4/3}\,\varepsilon^4\,\frac{D_4(\mathfrak{p},\dot{\mathfrak{p}},\ddot{\mathfrak{p}},\dddot{\mathfrak{p}})}{(\dot{\mathfrak{p}}\dot{\mathfrak{p}})^4} + \ldots$$

Normalization of $\mathfrak{p}$: After introducing the arc-length s, we normalize $\mathfrak{p}$ by the requirement that:

(66) $$\mathfrak{p}'\,\mathfrak{p}' = 1.$$

The six-vector $\mathfrak{p}(s)$ is established up to sign in that way. The requirement (66) can always be fulfilled by real six-vectors, due to our assumption that $\dot{\mathfrak{p}}\dot{\mathfrak{p}} > 0$ (no. **48**). As we deduced in no. **55**, the normalization condition (66) is semi-invariant; a normalized $\mathfrak{p}$ thus remains normalized under any unity transformation.

One deduces the normalized six-vector $\hat{\mathfrak{p}} = \sigma(s)\,\mathfrak{p}(s)$ from an unnormalized one $\mathfrak{p}(s)$ by setting:

$$\sigma^2 = \frac{1}{(\mathfrak{p}'\mathfrak{p}')}.$$

For a normalized $\mathfrak{p}(s)$, one gets from (65) and (66) that:

(67) $$D_4\,(\mathfrak{p}, \mathfrak{p}', \mathfrak{p}'', \mathfrak{p}''') = 1.$$

58. Fundamental system and differential equations. We take the *fundamental system* to be the six-vectors $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{p}''$, $\mathfrak{p}'''$, $\mathfrak{k}$, $\overline{\mathfrak{k}}$, in which $\mathfrak{k}$, $\overline{\mathfrak{k}}$ are the six-vectors of the two skew nodal tangents that belong to the line $\mathfrak{p}(s)$. The scalar products of the fundamental system are summarized in the following table:

	$\mathfrak{p}$	$\mathfrak{p}'$	$\mathfrak{p}''$	$\mathfrak{p}'''$	$\mathfrak{k}$	$\overline{\mathfrak{k}}$
$\mathfrak{p}$	0	0	-1	0	0	0
$\mathfrak{p}'$	0	1	0	$-2a$	0	0
$\mathfrak{p}''$	-1	0	$2a$	a'	0	0
$\mathfrak{p}'''$	0	$-2a$	a'	$4a^2 - 1$	0	0
$\mathfrak{k}$	0	0	0	0	0	b
$\overline{\mathfrak{k}}$	0	0	0	0	b	0

In this, we have set:

(68) $$2a(s) = \mathfrak{p}''\,\mathfrak{p}'', \qquad b(s) = \mathfrak{k}\,\overline{\mathfrak{k}} \neq 0.$$

The remaining scalar products follow from (66), (67), and (68) with the use of (58). Thus, e.g., $\mathfrak{p}'\mathfrak{p}'$ will imply:

$$\mathfrak{p}' \, \mathfrak{p}'' = 0 \qquad \text{and} \qquad \mathfrak{p}' \, \mathfrak{p}''' = - \mathfrak{p}'' \, \mathfrak{p}''' = - 2a$$

by differentiating twice.

From (34), the table of products will imply that:

$$| \, \mathfrak{p}, \mathfrak{p}', \mathfrak{p}'', \mathfrak{p}''', \mathfrak{k}, \bar{\mathfrak{k}} \, |^2 = - D_6(\mathfrak{p}, \mathfrak{p}', \mathfrak{p}'', \mathfrak{p}''', \mathfrak{k}, \bar{\mathfrak{k}}) = b^2 > 0;$$

the six six-vectors of our fundamental system are then, in fact, linearly independent, and can therefore be employed in plane of $\mathfrak{p}, \mathfrak{p}', \ldots, \mathfrak{p}^{(5)}$.

If $\mathfrak{p}, \dot{\mathfrak{p}}, \ldots, \overset{....}{\mathfrak{p}}$ ($\mathfrak{p}, \dot{\mathfrak{p}}, \ldots, \overset{.....}{\mathfrak{p}}$, resp.) are linearly dependent then the ruled family will be contained in a linear system of lines (a linear complex, resp.), as we can also employ a simple fundamental system with only four (five, resp.) six-vectors (no. **56**) in place of our fundamental system. We will come back to it in no. **60**.

Normalization of $\mathfrak{k}, \bar{\mathfrak{k}}$: $\mathfrak{k}$ and $\bar{\mathfrak{k}}$ must now be normalized, just like $\mathfrak{p}$ before. That shall come about from the demands that:

$$\mathfrak{p}'''' \, \mathfrak{k} = 1, \qquad \mathfrak{p}'''' \, \bar{\mathfrak{k}} = 1 \,. \tag{69}$$

$\mathfrak{k}, \bar{\mathfrak{k}}$ will then be determined uniquely and semi-invariant. The condition (69) breaks down for:

$$\overset{....}{\mathfrak{p}} \, \mathfrak{k} = 0 \qquad (\overset{.....}{\mathfrak{p}} \, \bar{\mathfrak{k}} = 0, \text{ resp.}).$$

From no. **51**, these equations are fulfilled identically in u only for parabolic ruled families or for hyperbolic ruled families with one or two rectilinear nodal curves. The former will be treated in § 16, while the latter will be treated in no. **60**, and in this paragraph we will assume that the nodal curves are non-rectilinear. We extend the foregoing table by scalar products of the derivatives $\mathfrak{p}''''$, $\mathfrak{k}, \bar{\mathfrak{k}}$ with six-vectors of the fundamental system:

	$\mathfrak{p}$	$\mathfrak{p}'$	$\mathfrak{p}''$	$\mathfrak{p}'''$	$\mathfrak{k}$	$\bar{\mathfrak{k}}$
$\mathfrak{p}''''$	$2a$	$-3a'$	$a'' - 4a^2 + 1$	$4aa'$	0	0
$\mathfrak{k}'$	0	0	0	-1	0	c
$\bar{\mathfrak{k}}'$	0	0	0	-1	$b' - c$	0

In this, we have set:

$$c(s) = \mathfrak{k}' \, \bar{\mathfrak{k}} \,. \tag{70}$$

The *differential equations* then read:

$$\mathfrak{p}'''' \quad = -(a'' + 1) \, \mathfrak{p} - 3a'\mathfrak{p}' - 2a\mathfrak{p}'' \qquad * \qquad + \frac{1}{b}\mathfrak{k} + \frac{1}{b}\bar{\mathfrak{k}} \,,$$

(71)
$$\mathfrak{k}' = +a'\mathfrak{p} + 2a\mathfrak{p}' \quad * \quad + \mathfrak{p}''' + \frac{c}{b}\mathfrak{k} \quad *,$$
$$\bar{\mathfrak{k}}' = +a'\mathfrak{p} + 2a\mathfrak{p}' \quad * \quad + \mathfrak{p}''' \quad * \quad + \frac{b'-c}{b}\bar{\mathfrak{k}}.$$

Proof: In order to prove, e.g., (71_1), we first set:

$$\mathfrak{p}'''' = A\mathfrak{p} + B\mathfrak{p}' + C\mathfrak{p}'' + D\mathfrak{p}''' + E\mathfrak{k} + F\bar{\mathfrak{k}}$$

with undetermined coefficients, and define the scalar product with $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{p}''$, $\mathfrak{p}'''$, $\mathfrak{k}$, $\bar{\mathfrak{k}}$, namely:

$$\begin{aligned}
\mathfrak{p}\,\mathfrak{p}'''' &= 2a && = -C,\\
\mathfrak{p}'\,\mathfrak{p}'''' &= -3a' && = B - 2aD,\\
\mathfrak{p}''\,\mathfrak{p}'''' &= a'' - 4a^2 + 1 && = -A + 2aC + a'D,\\
\mathfrak{p}'''\,\mathfrak{p}'''' &= 4aa' && = -2AB + a'C + (4a^2 - 1)\,D,\\
\mathfrak{k}\,\mathfrak{p}'''' &= 1 && = bF,\\
\bar{\mathfrak{k}}\,\mathfrak{p}'''' &= 1 && = bE.
\end{aligned}$$

The solution of this linear system of equations yields (71_1).
From no. **56**, it follows from (71) that:

The hyperbolic ruled family with non-rectilinear nodal curves are characterized projectively by three functions $a(s)$, $b(s) \neq 0$, $c(s)$ *of the arc-length s; i.e., they are projective to each other if and only if they agree in the invariants* $a(s)$, $b(s)$, $\neq 0$, $c(s)$.

A curve will be established in an analogous way in the metric theory of curves by two invariants (viz., curvature and torsion), up to motions (no. **5**).

59. Consequences of the differential equations.

1. *The tangent system of the family of nodes and the given ruled family are differently-wound* (no. **48**).

Proof: From (66), $\mathfrak{p}'\mathfrak{p}' = +1$, so it will follow from (71) that $\mathfrak{k}'\mathfrak{k}' = \bar{\mathfrak{k}}'\bar{\mathfrak{k}}' = -1$.

2. *The osculating complex* $\mathfrak{s}$ *is determined uniquely by:*

(72)
$$\mathfrak{s} = \mathfrak{k} - \dot{\mathfrak{k}},$$
and since:
$$\mathfrak{s}\mathfrak{s} = -2b \neq 0,$$
it is non-singular.

Proof: From (60) and (62), the osculating complex $\mathfrak{s}$ is contained in the pencil of complexes $\lambda_1\mathfrak{k} + \lambda_2\overline{\mathfrak{k}}$; $\lambda_1 : \lambda_2$ is determined by the demand that:

$$0 = \mathfrak{p}'''' \mathfrak{s} = \mathfrak{p}'''' (\lambda_1\mathfrak{k} + \lambda_2\overline{\mathfrak{k}}) = \lambda_1 + \lambda_2 .$$

3. *The ruled families with fixed non-singular osculating complexes are characterized by:*

$$2c - b' = 0.$$

Proof: From (71_2) and (71_3):

$$\mathfrak{s} = \mathfrak{k} - \overline{\mathfrak{k}}' = \sigma\mathfrak{s} = \sigma(\mathfrak{k} - \overline{\mathfrak{k}})$$

is necessary and sufficient for $2c - b' = 0$.

4. *For* $b > 0$, *there are precisely two lines* $\mathfrak{q}$, $\overline{\mathfrak{q}}$ *that belong to the second osculating quadric, as well as the osculating complex* s. *These lines* $\mathfrak{q}$, $\overline{\mathfrak{q}}$, *and the nodal tangents* $\mathfrak{k}$, $\overline{\mathfrak{k}}$ *define a hyperboloidal quadruple of lines with the double ratio* $d = -1$.

Proof: If one starts with:

$$\mathfrak{q} = A\mathfrak{p} + B\mathfrak{p}' + C\mathfrak{p}'' + D\mathfrak{p}''' + E\mathfrak{k} + F\overline{\mathfrak{k}},$$

with undetermined coefficients, scalar multiplies it by $\mathfrak{p}$, $\mathfrak{p}'$, etc., and employs the fact that:

$$\mathfrak{q}\mathfrak{p} = \mathfrak{q}\mathfrak{p}' = \mathfrak{q}\mathfrak{p}'' = \mathfrak{q}\mathfrak{s} = 0,$$

which comes from the table in no. **58**, then one will get:

$$\mathfrak{q} = a'\mathfrak{p} + 2a\mathfrak{p}' + \mathfrak{p}''' + \lambda(\mathfrak{k} + \overline{\mathfrak{k}}).$$

It will then follow from $\mathfrak{q}\mathfrak{q} = 0$ that:

$$\lambda = \pm\frac{1}{\sqrt{2b}},$$

and the assertion will follow from (50) with $\mathfrak{p}^{\mathrm{I}} = \mathfrak{k}$, $\mathfrak{p}^{\mathrm{II}} = \overline{\mathfrak{k}}$, $\mathfrak{p}^{\mathrm{III}} = \mathfrak{q}$, $\mathfrak{p}^{\mathrm{IV}} = \overline{\mathfrak{q}}$.

60. Hyperbolic ruled families with one or two rectilinear nodal curves. If a *nodal curve is rectilinear* then one will have:

(73) $$\mathfrak{p}''''\mathfrak{k} = 0, \qquad \mathfrak{p}''''\overline{\mathfrak{k}} = 1,$$

instead of (69), so $\mathfrak{k}$ will be a constant, singular six-vector that is established up to an arbitrary numerical factor. All lines of the ruled family will be contained in the *fixed singular osculating complex* $\mathfrak{s} = \mathfrak{k}$. Instead of the differential equation (71_1), one will have:

$$\mathfrak{p}'''' = -(a'' + 1)\,\mathfrak{p} - 3a'\mathfrak{p}' - 2a\,\mathfrak{p}'' + \mathfrak{k}. \tag{74}$$

All derivatives of $\mathfrak{p}$ can be expressed as linear combinations of $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{p}''$, $\mathfrak{p}'''$, $\mathfrak{k}$ with the help of this differential equation. As a result, we can already establish the five six-vectors $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{p}''$, $\mathfrak{p}'''$, $\mathfrak{k}$ as a *fundamental system* (no. **56**). It then follows from (74) that:

The hyperbolic ruled families with precisely one rectilinear nodal curve are characterized as projectively-invariant by the two invariants $a(s)$, $b(s) \neq 0$.

If *both nodal curves are rectilinear* then the ruled family will be contained in the hyperbolic system of lines with $\mathfrak{k}$, $\overline{\mathfrak{k}}$ as the focal lines (*fixed osculating system*). Equation (69) must be replaced with:

$$\mathfrak{p}''''\mathfrak{k} = \mathfrak{p}''''\,\overline{\mathfrak{k}} = 0, \tag{75}$$

and the differential equation (71_1) with:

$$\mathfrak{p}'''' = -(a'' + 1)\,\mathfrak{p} - 3a'\mathfrak{p}' - 2a\mathfrak{p}''. \tag{76}$$

Thus, the four six-vectors $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{p}''$, $\mathfrak{p}'''$ already define a *fundamental system*, and one will have:

The hyperbolic ruled families with two rectilinear nodal curves are characterized as projectively-invariant by the single invariant $a(s)$.

Equations (74) [(76), resp.] are derived in the same way as (71).

In order to make things more intuitive, we juxtapose the hyperbolic ruled families with one or two rectilinear nodal tangents and the ε-models.

ε-model	**Ruled family**
All lines of the model have one common line of intersection $\mathfrak{k}$, and thus belong to the singular linear complex $\mathfrak{k}$; $\mathfrak{p}^0$, $\mathfrak{p}^{\mathrm{I}}$, $\mathfrak{p}^{\mathrm{II}}$, $\mathfrak{p}^{\mathrm{III}}$, $\mathfrak{k}$, and any other line $\mathfrak{p}^m$ of the model, will be linearly dependent.	The ruled family has one rectilinear nodal curve $\mathfrak{k}$, and thus belongs to the singular linear complex $\mathfrak{k}$; $\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}$, $\mathfrak{k}$, and any other derivative $\mathfrak{p}^{(m)}$, will be linearly dependent.

All lines of the model have two common lines of intersection, and thus belong to a hyperbolic linear system of lines; $\mathfrak{p}^0$, $\mathfrak{p}^{\mathrm{I}}$, $\mathfrak{p}^{\mathrm{II}}$, $\mathfrak{p}^{\mathrm{III}}$, and any other line $\mathfrak{p}^m$ of the model will be linearly dependent.

The ruled family has two rectilinear nodal curves, and thus belongs to a hyperbolic, linear system; $\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}$, and any other derivative $\mathfrak{p}^{(m)}$ will be linearly dependent.

§ 16. Parabolic ruled families.

61. Parabolic ruled families with non-rectilinear nodal curves. In order to define the *arc-length s* and *normalization of the six-vectors* $\mathfrak{p}$, $\mathfrak{k}$, we set:

$$\mathfrak{p}'\mathfrak{p}' = 1, \qquad \mathfrak{k}'\,\mathfrak{k}' = -1, \qquad \mathfrak{p}'''\,\mathfrak{k}''' = 1. \tag{77}$$

Starting with the unnnormalized $\mathfrak{p}$, $\mathfrak{k}$, and an arbitrary parameter u, one will deduce the normalized $\hat{\mathfrak{p}} = \sigma\mathfrak{p}$, $\hat{\mathfrak{k}} = \tau\mathfrak{k}$, and the arc-length $s(u)$ by means of the equations:

$$1 = \hat{\mathfrak{p}}'\hat{\mathfrak{p}}' = \frac{\sigma^2}{\dot{s}^2}\dot{\mathfrak{p}}\dot{\mathfrak{p}}, \qquad -1 = \hat{\mathfrak{k}}'\hat{\mathfrak{k}}' = \frac{\tau^2}{\dot{s}^2}\dot{\mathfrak{k}}\dot{\mathfrak{k}}, \qquad 1 = \hat{\mathfrak{p}}'''\hat{\mathfrak{k}}'' = \frac{\sigma\tau}{\dot{s}^2}\dddot{\mathfrak{p}}\ddot{\mathfrak{k}}. \tag{78}$$

As in no. **57**, s is completely invariant and invariant under the parameter substitutions (63), while the normalization of $\mathfrak{p}$ and $\mathfrak{k}$ is semi-invariant. The sign and additive constant of s will remain undetermined.

The first two requirements in (77) are always fulfilled by real six-vectors: From no. 48, $\dot{\mathfrak{p}}\dot{\mathfrak{p}} > 0$, while from no. **51**, $\dot{\mathfrak{k}}\dot{\mathfrak{k}} \neq 0$ for non-rectilinear nodal curves. From (34), it then follows from this that:

$$\left| \mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \mathfrak{k}, \dot{\mathfrak{k}}, \ddot{\mathfrak{k}} \right|^2 = -\,D_6(\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \mathfrak{k}, \dot{\mathfrak{k}}, \ddot{\mathfrak{k}}) = -\,(\dot{\mathfrak{p}}\dot{\mathfrak{p}})^3(\dot{\mathfrak{k}}\dot{\mathfrak{k}})^3 > 0,$$

and thus $\dot{\mathfrak{k}}\dot{\mathfrak{k}} < 0$.

The third equation in (77) can likewise be satisfied by real six-vectors under the assumption of a non-rectilinear nodal curve, since the quadrics were excluded (no. **54**); one will then have:

If $\dddot{\mathfrak{p}}\ddot{\mathfrak{k}} = 0$ is valid identically in u then either the ruled family is a quadric or one is dealing with a rectilinear nodal curve.

Proof: From (60), (61), and for $\dddot{\mathfrak{p}}\ddot{\mathfrak{k}} = 0$, one will have:

$$\begin{aligned}
\mathfrak{p}\mathfrak{k} &= \dot{\mathfrak{p}}\mathfrak{k} = \ddot{\mathfrak{p}}\mathfrak{k} = \dddot{\mathfrak{p}}\mathfrak{k} = 0, \\
\mathfrak{p}\dot{\mathfrak{k}} &= \dot{\mathfrak{p}}\dot{\mathfrak{k}} = \ddot{\mathfrak{p}}\dot{\mathfrak{k}} = \dddot{\mathfrak{p}}\dot{\mathfrak{k}} = 0, \\
\mathfrak{p}\ddot{\mathfrak{k}} &= \dot{\mathfrak{p}}\ddot{\mathfrak{k}} = \ddot{\mathfrak{p}}\ddot{\mathfrak{k}} = \dddot{\mathfrak{p}}\ddot{\mathfrak{k}} = 0\,.
\end{aligned}$$

Either $\mathfrak{k}, \dot{\mathfrak{k}}$ are linearly dependent (viz., a rectilinear nodal curve) or $\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}$ are conjugate to the bundle of complexes $\lambda_1 \mathfrak{k} + \lambda_2 \dot{\mathfrak{k}} + \lambda_3 \ddot{\mathfrak{k}}$, and are thus themselves linearly dependent (viz., a quadric).

We employ the six normalized linearly-independent six-vectors $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{p}''$, $\mathfrak{k}$, $\mathfrak{k}'$, $\mathfrak{k}''$ as the *fundamental system.* They imply the table of scalar products:

	$\mathfrak{p}$	$\mathfrak{p}'$	$\mathfrak{p}''$	$\mathfrak{k}$	$\mathfrak{k}'$	$\mathfrak{k}''$
$\mathfrak{p}$	0	0	–1	0	0	0
$\mathfrak{p}'$	0	1	0	0	0	0
$\mathfrak{p}''$	– 1	0	$2a$	0	0	0
$\mathfrak{k}$	0	0	0	0	0	1
$\mathfrak{k}'$	0	0	0	0	–1	0
$\mathfrak{k}''$	0	0	0	1	0	$2c$

with:

$$| \mathfrak{p}, \mathfrak{p}', \mathfrak{p}'', \mathfrak{k}, \mathfrak{k}', \mathfrak{k}'' |^2 = - D_6(\mathfrak{p}, \mathfrak{p}', \mathfrak{p}'', \mathfrak{k}, \mathfrak{k}', \mathfrak{k}'') = 1.$$

In this, one sets:

(79) $$2a(s) = \mathfrak{p}''\mathfrak{p}'', \qquad 2c(s) = \mathfrak{k}''\,\mathfrak{k}''.$$

We then get:

(80) $$\begin{aligned} \mathfrak{p}''' &= - a'\mathfrak{p} - 2a\,\mathfrak{p}' + \mathfrak{k}, \\ \mathfrak{k}''' &= \quad c'\mathfrak{k} + 2c\,\mathfrak{k}' + \mathfrak{p} \end{aligned}$$

as the differential equations. Thus:

The parabolic ruled families with non-rectilinear curves are characterized in a projectively-invariant way by the two invariants $a(s)$ and $c(s)$.

From (80_2), $D_4(\mathfrak{k}, \mathfrak{k}', \mathfrak{k}'', \mathfrak{k}''') = 0$. This gives the theorem:

The nodal family of a parabolic ruled family is again a parabolic ruled family; both ruled families have the same nodal curve.

It follows from this indirectly, and upon referring to no. **51**, that:

The nodal families of a hyperbolic ruled family are again hyperbolic ruled families; they each have one nodal curve with the given ruled family in common.

The interchangeability of the parabolic ruled family $\mathfrak{p}(s)$ and the nodal family $\mathfrak{k}(s)$ is clearly expressed in the table of scalar products and in the same construction for equations (80_1), (80_2).

It follows from (60), (61), (62) that $\mathfrak{s} = \mathfrak{k}$; i.e., *the osculating complex is singular and has the nodal tangent as its axis.*

62. Parabolic ruled families with rectilinear nodal curves. Since the osculating system is parabolic, as in no. **61**, it will contain the focal line $\mathfrak{k}$ as a line of the system, and $\mathfrak{k}, \mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}$ are therefore linearly dependent. One then has:

$$\dddot{\mathfrak{p}} = A\mathfrak{p} + B\dot{\mathfrak{p}} + C\ddot{\mathfrak{p}} + D\,\mathfrak{k} \qquad (D \neq 0). \tag{81}$$

In this, the six-vector $\mathfrak{k}$ is constant, in contrast to no. **61**. As a result, one will obtain all derivatives $\mathfrak{p}^{(m)}$ as linear combinations of $\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \mathfrak{k}$ by repeated derivation of (81). We can then define the four linearly-independent six-vectors $\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \mathfrak{k}$ to be a *fundamental system*, and then have (81) as the single differential equation. Due to the constancy of $\mathfrak{k}$, the linear dependence of $\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}, \ddddot{\mathfrak{p}}$ follows from (81). Thus, we will have (no. **54**):

The ruled family is contained in a parabolic linear system of lines (fixed osculating system) with the focal line $\mathfrak{k}$.

We use the prescription:

$$\mathfrak{p}'\mathfrak{p}' = 1, \qquad D = 1 \tag{82}$$

for the definition of the arc-length and the normalization of $\mathfrak{p}$. In place of (78), one will then have:

$$1 = \hat{\mathfrak{p}}'\hat{\mathfrak{p}}' = \frac{\sigma^2}{\dot{s}^2}\dot{\mathfrak{p}}\dot{\mathfrak{p}}, \qquad 1 = \hat{D} = \frac{\sigma}{\dot{s}^3}D, \tag{83}$$

which makes s completely invariant again and invariant under (63) once more, and $\mathfrak{p}$ is shown to be semi-invariant. The constant singular six-vector $\mathfrak{k}$, and thus also the function $D(u)$ that was used in (83) to normalize, is determined only up to an arbitrary numerical factor. For that reason, the arc-length is established only up to an arbitrary constant factor (along with the arbitrary additive constant).

Analogous to what we did in no. **61**, we will get the differential equation:

$$\mathfrak{p}''' = -\,a'\mathfrak{p} - 2\,a\,\mathfrak{p}' + \mathfrak{k}, \tag{84}$$

which agrees with (80_1), with the invariant $a(s)$ that is defined by (79). We then obtain the theorem:

The parabolic ruled families with rectilinear nodal curves are characterized in a projectively-invariant way by the single invariant $a(s)$.

§ 17. Torses.

63. Fundamental system and classification. As we agreed in no. **46**, we assume that $\ddot{\mathfrak{p}}\ddot{\mathfrak{p}} \neq 0$; i.e., pencils of lines, tangent families of planar curves, and the cone will remain excluded. Torses will then be the tangent families of non-planar curves, and *the theory of torses is identical with the projective differential geometry of the space curves.*

The osculating complex $\mathfrak{s}$ is always determined uniquely (no. **54**). In addition, we have: *The osculating complex is non-singular.*

Proof: The bundle of complexes $\lambda_1\mathfrak{p} + \lambda_2\dot{\mathfrak{p}} + \lambda_3\ddot{\mathfrak{p}}$ has type *c*) (nos. **34** and **49**), and the two conjugate quadrics will coincide in the pencil of lines $\mu_1\mathfrak{p} + \mu_2\dot{\mathfrak{p}}$. Any singular six-vector $\mathfrak{s}$ that satisfies the first three equations of (62):

$$\mathfrak{p}\mathfrak{s} = \dot{\mathfrak{p}}\mathfrak{s} = \ddot{\mathfrak{p}}\mathfrak{s} = 0$$

must then be representable by $\mathfrak{s} = \mu_1\mathfrak{p} + \mu_2\dot{\mathfrak{p}}$. The last two equations of (62):

$$\dddot{\mathfrak{p}}\mathfrak{s} = \ddddot{\mathfrak{p}}\mathfrak{s} = 0$$

then yield:

$$\mu_2(\dot{\mathfrak{p}}\dddot{\mathfrak{p}}) = -\mu_2(\ddot{\mathfrak{p}}\ddot{\mathfrak{p}}) = 0, \qquad \mu_1(\mathfrak{p}\ddddot{\mathfrak{p}}) = \mu_1(\ddot{\mathfrak{p}}\ddot{\mathfrak{p}}) = 0,$$

so $\mu_1 = \mu_2 = 0$. (62) will not be satisfied by any singular six-vector apart from the null six-vector.

The osculating complex $\mathfrak{s}(u)$ contains the four pencils of lines whose vertices are four "successive" points of the curve of regression, and which lie in four "successive" osculating planes; i.e., $\mathfrak{s}(u)$ fulfills the conditions:

$$\mathfrak{s}(u)\,\mathfrak{p}(u+\varepsilon) = \mathfrak{s}(u)\,\dot{\mathfrak{p}}(u+\varepsilon) = 0,$$

up to order ε^3 inclusive.

We take the six-vectors $\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}, \ddddot{\mathfrak{p}}, \mathfrak{s}$ to be a *fundamental system*, subject to the definition of arc-length and the normalization of $\mathfrak{p}$ and $\mathfrak{s}$. Due to the fact that:

$$(85) \qquad |\,\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}, \ddddot{\mathfrak{p}}, \mathfrak{s}\,|^2 = -D_6(\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}, \ddddot{\mathfrak{p}}, \mathfrak{s}) = -(\ddot{\mathfrak{p}}\ddot{\mathfrak{p}})^5(\mathfrak{s}\mathfrak{s}) > 0,$$

they will be linearly-independent.

In what follows, we distinguish:

a) $\overset{\cdot\cdot\cdot\cdot\cdot}{\mathfrak{p}}\,\mathfrak{s} \neq 0$: *Torses with varying osculating complex,*

b) $\overset{\cdot\cdot\cdot\cdot\cdot}{\mathfrak{p}}\,\mathfrak{s} = 0$: *Torses with a fixed osculating.*

In fact, $\overset{\cdot\cdot\cdot\cdot\cdot}{\mathfrak{p}}\,\mathfrak{s} = 0$, together with the relations (62), which are true in any case, is necessary and sufficient for the linear dependency of $\mathfrak{p}, \dot{\mathfrak{p}}, \ldots, \overset{\cdot\cdot\cdot\cdot\cdot}{\mathfrak{p}}$, and thus for the constancy of the osculating complex $\mathfrak{s}$ (no. **54**). We omit discrete values u with $\overset{\cdot\cdot\cdot\cdot\cdot}{\mathfrak{p}}\,\mathfrak{s} = 0$.

64. Differential equations.

a) *Torses with varying osculating complexes.*

In order to define the *arc-length s and the normalization of* $\mathfrak{p}$ *and* $\mathfrak{s}$, *we set:*

$$\mathfrak{p}''\mathfrak{p}'' = 1, \qquad \mathfrak{s}\mathfrak{s} = -1, \qquad \mathfrak{p}^{(5)}\mathfrak{s} = \eta = 1. \tag{86}$$

In order for the first demand to be fulfillable, the torses with $\ddot{\mathfrak{p}}\ddot{\mathfrak{p}} < 0$ must be converted into torses with $\ddot{\mathfrak{p}}\ddot{\mathfrak{p}} > 0$ by reflection. From (85), it is then necessary that $\mathfrak{s}\mathfrak{s} < 0$, so the second condition of (86) can be likewise satisfied. The last demand is admissible, since $\overset{\cdot\cdot\cdot\cdot\cdot}{\mathfrak{p}}\,\mathfrak{s} \neq 0$. Starting with unnormalized $\mathfrak{p}$, $\mathfrak{s}$, and an arbitrary parameter u, one calculates $s(u)$ and the normalized $\hat{\mathfrak{p}} = \rho\,\mathfrak{p}$, $\hat{\mathfrak{s}} = \sigma\,\mathfrak{s}$, by using:

$$1 = \rho^2\,\frac{\ddot{\mathfrak{p}}\ddot{\mathfrak{p}}}{\dot{s}^4}, \qquad -1 = \sigma^2\,\mathfrak{s}\mathfrak{s}, \qquad 1 = \rho\sigma\,\frac{\overset{\cdot\cdot\cdot\cdot\cdot}{\mathfrak{p}}\,\mathfrak{s}}{\dot{s}^5}.$$

From the table of scalar products:

	$\mathfrak{p}$	$\mathfrak{p}'$	$\mathfrak{p}''$	$\mathfrak{p}'''$	$\mathfrak{p}''''$	$\mathfrak{s}$
$\mathfrak{p}$	0	0	0	0	1	0
$\mathfrak{p}'$	0	0	0	-1	0	0
$\mathfrak{p}''$	0	0	1	0	$-2a$	0
$\mathfrak{p}'''$	0	-1	0	$2a$	a'	0
$\mathfrak{p}''''$	1	0	$-2a$	a'	$2c$	0
$\mathfrak{s}$	0	0	0	0	0	-1

with

$$(87) \qquad 2a(s) = \mathfrak{p}''' \, \mathfrak{p}''', \qquad 2c(s) = \mathfrak{p}'''' \, \mathfrak{p}'''',$$

one will obtain the differential equations:

$$(88) \qquad \mathfrak{p}^{(5)} = (c' - 4aa')\,\mathfrak{p} - (4a^2 + a'' - 2c)\,\mathfrak{p}' - 3a'\mathfrak{p}'' - 2a\mathfrak{p}''' - \eta\mathfrak{s}, \qquad \mathfrak{s}' = -\,\eta\,\mathfrak{s}.$$

One will then have:

The torses with varying osculating complexes are characterized in a projectively-invariant way by the two invariants $a(s)$, $c(s)$.

b) *Torses with fixed osculating complexes.*

Since $\mathfrak{p}, \dot{\mathfrak{p}}, \ldots, \overset{\cdot\cdot\cdot\cdot\cdot}{\mathfrak{p}}$ are linearly dependent, the five six-vectors $\mathfrak{p}, \dot{\mathfrak{p}}, \ldots, \ddddot{\mathfrak{p}}$ already define a fundamental system (no. **56**). As we did in a), we will then obtain (88_1) again as a differential equation, but with $\eta = 0$. In this, one must observe that the two functions $a(s)$, $c(s)$ can be subjected to a normalization condition; since $\eta = 0$, equation (86_3) is distinguished as a normalization condition. We will come back to this in no. **68**.

§ 18. Establishing a family of lines by invariants. Self-projective families of lines.

65. Existence theorem. Up to now, we have shown that a given family of lines is determined as a function of the projective arc-length s, up to principal projectivities, by invariant scalar products of the fundamental system and the invariant coefficients of the differential equations. Conversely, we now start with the invariants $a(s)$, etc., being given and then prove the following theorem, which is analogous to the main theorem of the theory of curves (no. **5**):

There always exists a family of lines in nos. **58**, **60**, **61**, **62** *that always has arbitrarily-given functions* $a(s)$, $b(s) \neq 0$, *etc. of the arc-length s as invariants.*

Proof: We restrict ourselves to proving this for parabolic ruled families with non-rectilinear nodal curves (no. **61**). The other cases can be resolved analogously.

α) From known theorems about the existence of solutions of ordinary differential equations, one has: For arbitrary, given, regular functions $a(s)$, $c(s)$, the differential equations ($80_{1,2}$) will possess precisely one solution $\mathfrak{p}(s)$, $\mathfrak{k}(s)$ with the arbitrary initial conditions:

$$\mathfrak{p}(0),\ \mathfrak{p}'(0),\ \mathfrak{p}''(0); \qquad \mathfrak{k}(0),\ \mathfrak{k}'(0),\ \mathfrak{k}''(0).$$

β) For every system of solutions $\mathfrak{p}(s)$, $\mathfrak{k}(s)$, one has the identities:

$$2a\, p_\rho p_\rho - p'_\rho p'_\sigma + p_\rho p''_\sigma + p''_\rho p_\sigma + 2ck_\rho k_\sigma + k'_\rho k'_\sigma - k_\rho k''_\sigma - k''_\rho k_\sigma = \text{const.}, \tag{89}$$

with ρ, σ = 1, …, 6; the derivative of the expression on the left-hand side equal sign with respect to s will then vanish identically on the basis of the differential equations ($80_{1,2}$).

γ) The initial conditions – which, from a), are arbitrary – will be chosen as follows:

$$\begin{aligned}
\mathfrak{p}(0) &= 1 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0,\\
\mathfrak{p}'(0) &= 0 \left|\frac{1}{\sqrt{2}}\right| 0 \mid 0 \left|\frac{1}{\sqrt{2}}\right| 0,\\
\mathfrak{p}''(0) &= -a(0) \mid 0 \mid 0 \mid -1 \mid 0 \mid 0,\\
\mathfrak{k}(0) &= 0 \mid 0 \mid 1 \mid 0 \mid 0 \mid 0,\\
\mathfrak{k}'(0) &= 0 \left|\frac{1}{\sqrt{2}}\right| 0 \mid 0 \left|-\frac{1}{\sqrt{2}}\right| 0,\\
\mathfrak{k}''(0) &= 0 \mid 0 \mid c(0) \mid 0 \mid 0 \mid 0.
\end{aligned} \tag{90}$$

They obviously fulfill the table of scalar products that was prescribed in no. **61**.

δ) We choose the system of solutions that are determined from the initial conditions (90) in the following form:

$$\begin{aligned}
p_\rho(s) &= \gamma_\rho^1, & k_\sigma(s) &= \gamma_\rho^3,\\
p'_\rho(s) &= \frac{1}{\sqrt{2}}(\gamma_\rho^2 + \gamma_\rho^5), & k'_\rho(s) &= \frac{1}{\sqrt{2}}(\gamma_\rho^2 - \gamma_\rho^5),\\
p''_\rho(s) &= -a(s)\,\gamma_\rho^1 - \gamma_\rho^4, & k''_\rho(s) &= c(s)\,\gamma_\rho^3 + \gamma_\rho^6.
\end{aligned} \tag{91}$$

This Ansatz is admissible, since the 36 linear equations (91) can be solved for the 36 unknown functions γ_ρ^σ. If one substitutes, first the initial conditions (90). and then the expressions (91) in (89) then that will give:

$$\gamma_\rho^1\gamma_\rho^4 + \gamma_\rho^2\gamma_\rho^5 + \gamma_\rho^3\gamma_\rho^6 + \gamma_\rho^4\gamma_\rho^1 + \gamma_\rho^5\gamma_\rho^2 + \gamma_\rho^6\gamma_\rho^3 = \begin{cases} 1 & \rho - \sigma = \pm 3 \\ 0 & \rho - \sigma \neq \pm 3 \end{cases}.$$

From (26), the γ_ρ^σ are the coefficients of a unity transformation for every fixed value of s, and thus also fulfill equations (24). With the help of (24), however, it can easily be shown that the expressions (91) fulfill the table of scalar products of no. **61** identically in s. That means: The family of lines that is defined by $\mathfrak{p}(s)$ is a parabolic ruled family with arc-length s, normalized six-vector $\mathfrak{p}(s)$, and invariants $a(s)$, $c(s)$.

66. Self-projective ruled families. As an example, we now treat the ruled families that are characterized by the constancy of the invariants, by the main theorem in no. **65**. Due to the constancy of the invariants, these so-called *self-projective ruled families* are transformed into themselves by:

$$s = \tilde{s} + k,$$

with the arbitrary constant k; i.e., at least one *one-parameter, continuous, projective group* exists under which the ruled family remains fixed as a whole. All maps are collineations, as they would be for any continuous, projective group. If we restrict ourselves to a suitable s-domain then there will always be a collineation of the ruled family into itself that transforms any two lines of the ruled family to each other.

Just like the self-projective ruled families, their *ε-models* will also be transformed into themselves by certain collineations. These collineations are given by:

$$s = \tilde{s} + m\varepsilon \qquad (m = \text{whole number})$$

and define a discontinuous group.

The self-projective ruled families define the line-geometric counterpart of the *W-curves*, which were investigated by **S. Lie** and **F. Klein** especially, which we would prefer to call *self-projective curves*. Naturally, the self-projective ruled families will remain nodal curves as a whole under all collineations, so those curves will be self-projective curves.

The explicit representation of the self-projective ruled families is obtained by integrating differential equations with constant coefficients. Since the differential equations are linear differential equations, the integral $\mathfrak{p}(s)$ can be represented as elementary functions of the arc-length. We would like to restrict ourselves to determining the *self-projective ruled families with rectilinear nodal curves:*

a) *Hyperbolic self-projective ruled families with two rectilinear nodal curves.*

From (76), we have to integrate the differential equation:

$$\mathfrak{p}'''' + 2a\mathfrak{p}'' + \mathfrak{p} = 0$$

and choose the initial conditions in such a way that the table of scalar products in no. **58** is fulfilled. By elementary calculation, one obtains:

$a > 1$:

$$\sqrt{2\kappa\lambda}\,\mathfrak{p} = \sin\kappa s\cos\lambda s \mid \sin\kappa s\sin\lambda s \mid 0 \mid \cos\kappa s\sin\lambda s \mid -\cos\kappa s\cos\lambda s \mid 0,$$

$$\text{with } \kappa = \left|\sqrt{\frac{a+1}{2}}\right|, \quad \lambda = \left|\sqrt{\frac{a-1}{2}}\right|;$$

$a = 1$:

$$\sqrt{2}\,\mathfrak{p} = \cos s \mid \sin s \mid 0 \mid -s \sin s \mid s \cos s \mid 0,$$

$1 > a > -1$:

$$\sqrt{4\kappa\lambda}\,\mathfrak{p} = e^{-\kappa s} \cos \lambda s \mid e^{-\kappa s} \sin \lambda s \mid 0 \mid -e^{\kappa s} \sin \lambda s \mid -e^{\kappa s} \cos \lambda s \mid 0,$$

$$\text{with } \kappa = \left|\sqrt{\frac{1-a}{2}}\right|, \quad \lambda = \left|\sqrt{\frac{1+a}{2}}\right|;$$

$a = -1$:

$$\sqrt{2}\,\mathfrak{p} = \operatorname{Cos} s \mid \operatorname{Sin} s \mid 0 \mid -s \operatorname{Sin} s \mid s \operatorname{Cos} s \mid 0,$$

$a < -1$:

$$\sqrt{2\kappa\lambda}\,\mathfrak{p} = \operatorname{Sin} \kappa s \operatorname{Cos} \lambda s \mid \operatorname{Sin} \kappa s \operatorname{Sin} \lambda s \mid 0 \mid \operatorname{Cos} \kappa s \operatorname{Sin} \lambda s \mid -\operatorname{Cos} \kappa s \operatorname{Cos} \lambda s \mid 0,$$

$$\text{with } \kappa = \left|\sqrt{\frac{-a+1}{2}}\right|, \quad \lambda = \left|\sqrt{\frac{-a-1}{2}}\right|.$$

For the ruled surfaces that are spanned by the ruled family, one gets, for:

$$(92) \qquad \left.\begin{array}{lllll} a>1: & x_1 = v\cos\lambda s, & x_2 = v\sin\lambda s, & x_3 = -\cot\kappa s, \\ a=1: & x_1 = v\cos s, & x_2 = v\sin s, & x_3 = s, \\ 1>a>-1: & x_1 = v\cos\lambda s, & x_2 = v\sin\lambda s, & x_3 = e^{2\kappa s}, \\ a=-1: & x_1 = v\operatorname{Cos} s, & x_2 = v\operatorname{Sin} s, & x_3 = s, \\ a<-1: & x_1 = v\operatorname{Cos}\lambda s, & x_2 = v\operatorname{Sin}\lambda s, & x_3 = -\cot\kappa s, \end{array}\right\} \quad x_4 = 1,$$

with the parameters s, v. The lines $x_1 = x_2 = 0$, and $x_3 = x_4 = 0$ are the two nodal curves. If one interprets the x_i as homogeneous, rectangular coordinates then one can easily discuss the forms of the surfaces. For $a = 1$, one will obtain the *spiral screw surface*, and for $1 > a > -1$, one will get *Lie's spiral surfaces.* The group of projective maps that transform the ruled family into itself consists of screws (rotational stretchings, resp.)

b) *Parabolic self-projective ruled families with rectilinear nodal curves.*

From (84), one must integrate the differential equation:

$$\mathfrak{p}'''' + 2a\,\mathfrak{p}'' = 0$$

with the initial conditions that correspond to the scalar products of no. **61**. For $a = 0$, one will get:

$$\sqrt{2}\,\mathfrak{p} = s \mid 1 \mid 0 \mid -s \mid s^2 \mid s^3.$$

The ruled surface that is spanned by the ruled family is given by:

$$(93) \qquad x_1 = vs, \qquad x_2 = v - s^2, \qquad x_3 = s, \qquad x_4 = 1.$$

Performing the calculation for $a \neq 0$ will be left to the reader.

67. Ruled surface of degree 3. Among the self-projective ruled surfaces that were treated in no. **66**, one will find the *ruled surfaces of degree 3* ([1]), namely, the ruled surfaces (92) with $a = \mp 5/3$, and thus $\kappa = 2\lambda$, and the ruled surfaces (93). After eliminating the parameters s, v, one will obtain the equations:

$$(94) \qquad \begin{aligned} (x_1^2 + x_2^2)\,x_3 + 2x_1x_2x_4 &= 0, \\ (x_1^2 - x_2^2)\,x_3 + 2x_1x_2x_4 &= 0, \\ (x_2x_4 + x_3^2)\,x_3 - x_1x_4^2 &= 0 \end{aligned}$$

from (92), with $a = \mp 5/3$, and from (93), after switching x_3, x_4, resp.

Discussion of the forms:

a) (94_1) [(94_2), resp.] (Fig. 18):

We interpret the x_i as homogeneous rectilinear coordinates. The generators will then intersect the 3-axis perpendicularly. The 3-axis and the imaginary line in the plane $x_3 = 0$ are the nodal curves. We introduce the angle ω between the generators and the plane $x_2 = 0$. One then gets:

$$\frac{x_3}{x_4} = \begin{cases} -\sin 2\omega & \text{for } (94_1), \\ -\tan 2\omega & \text{for } (94_2). \end{cases}$$

It follows from this that: The curve of intersection of the cylinder of rotation $x_1^2 + x_2^2 = x_4^2$ with the ruled surface (94_1) [(94_2), resp.] goes to a sinusoid (four periods of a tangent line, resp.) under the planar development of the cylinder of rotation into two periods.

([1]) In algebraic line geometry, it will be shown that there are no other ruled surfaces of degree three besides the surfaces that are treated here.

We have encountered the ruled surface (94_1) under the name of *cylindroid* in no. **31**: 0 (2, resp.) generators go through the points of the 3-axis with $x_3^2 > x_4^2$ ($x_3^2 < x_4^2$, resp.), and only one generator will go through the points A, B ($x_3^2 = x_4^2$). These generators are *torsal*; i.e., they are discrete lines with $\dot{\mathfrak{p}}\dot{\mathfrak{p}} = 0$. They will be fixed or permuted with each other under the collineations of the cylindroid into itself.

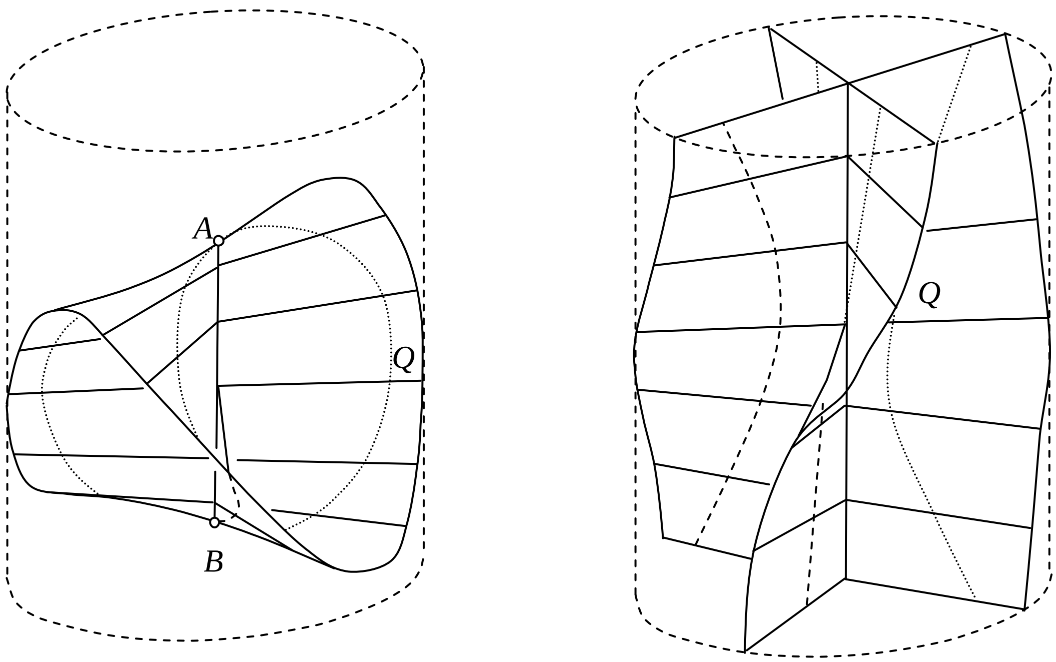

Figure 18.

From (94_2), two generators will go through any point of the 3-axis; the surface is cut out in Fig. 18 by two planes parallel to the plane $x_3 = 0$, and likewise, the cylindroid is cut through a cylinder of rotation.

b) (94_3) *Cayley surface* (Fig. 19):

The nodal curve is the line of intersection g of the planes $x_3 = 0$ and $x_4 = 0$. The plane $x_1 = 0$ cuts the surface along the non-singular conic section κ with the equation $x_2\,x_4 + x_3^2 = 0$; κ and g have the point P in common. Since the generators belong to a parabolic system of lines with the axis g, the sequence of intersection points of the generators with g will be projective to the pencil of planes through g and the generators. As a result, the generators cut the conic section k and the line g along a projective sequence of points, for which the point of intersection P of k and g does not correspond to itself. In Fig. 19, all four vertices of the coordinate tetrahedron are assumed to be real points; a section of the surface that is bounded by the nodal line g, the conic section k, and two generators is represented.

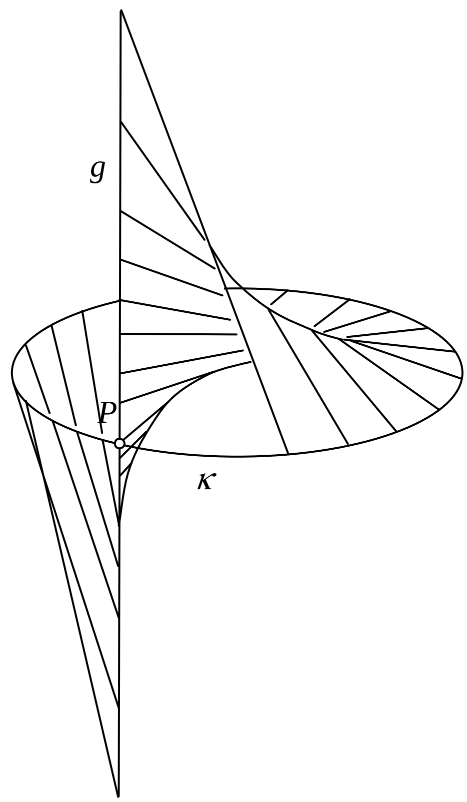

Figure 19.

Projective generator of the ruled family of degree 3 (Fig. 20):

We have just now defined the ruled family (94_3) to be the set of connecting lines between points of a line g and a projectively-related conic section κ, in which g and κ have the point P in common, which does not correspond to itself.

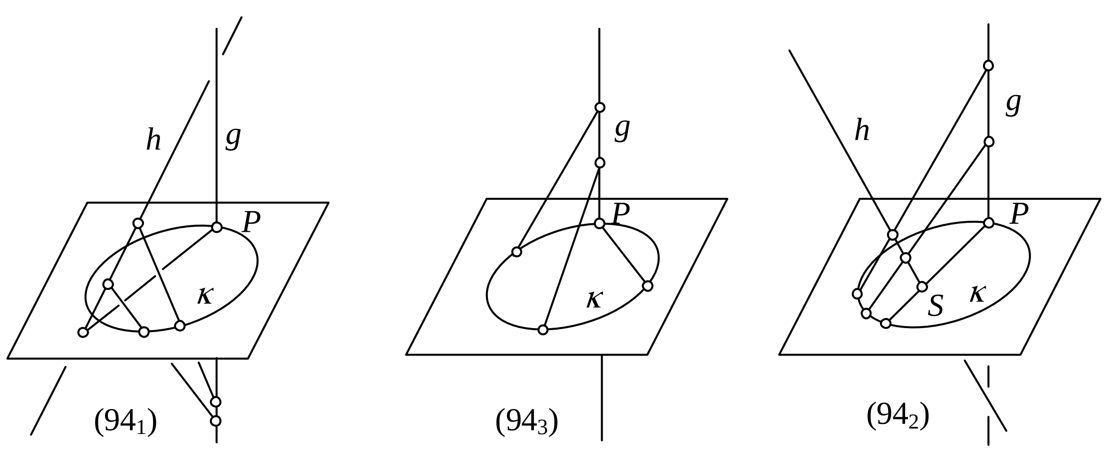

Figure 20.

The ruled families (94_1) [(94_2), resp.] are obtained as sets of common lines that are met by two skew lines g, h, and a non-singular conic section k whose plane will be cut by g at the curve point P and by h at the external (internal, resp.) point S.

Proof: We transform the configuration for (94_1) that was specified in Fig. 20 by a projective map in such a way that g will be the 3-axis of a rectangular coordinate system, h will be the imaginary line of the plane $x_3 = 0$, and κ will be an ellipse with a circular base projection. As an elementary calculation will show, the lines that meet them will then define a cylindroid. One proves the assertion for (94_2) analogously: in place of the ellipse k with circular base projection, one will have a conic section whose base projection is an equilateral hyperboloid.

The generating family of the Cayley surface (94_3) *will be transformed into itself by not only* $s = \hat{s} + k$, *but also the two-parameter continuous group:*

$$s = k_1\hat{s} + k_2 ,$$

(k_1, k_2 = *arbitrary constants)*. That is based upon the fact that in no. **62** the arc-length was established only up to an arbitrary constant factor and an arbitrary additive constant.

The transformations that were considered up to now are superimposed for all ruled surfaces of degree 3 by a one-parameter group of collineations that leaves the individual generators fixed, namely:

$$x_1 = \rho\,\tilde{x}_1, \qquad x_2 = \rho\,\tilde{x}_2, \qquad x_3 = \rho\,\tilde{x}_3, \qquad x_4 = \tilde{x}_4$$

for (94_1) and (94_2) and:

$$x_1 = \tilde{x}_1 + \rho\,\tilde{x}_3, \qquad x_2 = \tilde{x}_2 + \rho\,\tilde{x}_4, \qquad x_3 = \tilde{x}_3, \qquad x_4 = \tilde{x}_4$$

for (94_3). *With the addition of these collineations, the surfaces* (94_1) *and* (94_2) *will be invariant under a two-parameter group of collineations and the surface* (94_3), *under a three-parameter group of collineations.*

68. Self-projective torses (self-projective curves. Just like the self-projective ruled families, the *self-projective torses* will yield invariants by integrating the differential equations with constants. The curves of regression will remain fixed under the collineations that transform the self-projective torses into themselves, and will thus be *self-projective curves* (no. **66**).

We thus satisfy ourselves by determining the self-projective torses *that are contained in a linear complex (= fixed osculating complex)*. From (88_1) with $\eta = 0$, one obtains them by integrating the differential equation:

$$\mathfrak{p}^{(5)} + 2a\mathfrak{p}''' + (4a^2 - 2c)\,\mathfrak{p}' = 0 \tag{95}$$

with initial conditions from the table of scalar products in no. **64**. In that, we recall that we still have to dispose of a normalization condition for $\mathfrak{p}$ and the definition of s. The first normalization condition (86) will remain fulfilled under a substitution:

$$\tilde{\mathfrak{p}} = \sigma^2 \mathfrak{p}, \qquad \tilde{s} = \sigma s \qquad (\sigma = \text{const.}) \tag{96}$$

since

$$\mathfrak{p}'' = \frac{d^2\tilde{\mathfrak{p}}}{d\tilde{s}^2}.$$

The invariants remain constant and transform as:

$$2\tilde{a} = \frac{1}{\sigma^2} 2a, \quad 2\tilde{c} = \frac{1}{\sigma^4} 2c,$$

so

$$4\tilde{a}^2 - 2\tilde{c} = \frac{1}{\sigma^4}(2a^2 - 2c).$$

We can then transform equation (95) into:

$$\mathfrak{p}^{(5)} + 2a\mathfrak{p}''' \pm \mathfrak{p}' = 0 \tag{97}$$

or

$$\mathfrak{p}^{(5)} + 2a\mathfrak{p}''' = 0$$

by a substitution (96).

If (97_1), we restrict ourselves to the plus sign, and from elementary calculations we will get the torses with the following curves of regression:

$$\begin{array}{rllll} a > 1: & x_1 = \cos\lambda s, & x_2 = \sin\lambda s, & x_3 = -\cos\kappa s, & x_4 = \sin\kappa s, \\ a = 1: & x_1 = \cos s, & x_2 = \sin s, & x_3 = s, & x_4 = 1, \\ 1 > a > -1: & x_1 = \cos\lambda s, & x_2 = \sin\lambda s, & x_3 = e^{\kappa s}, & x_4 = e^{-\kappa s}, \\ a = -1: & x_1 = \operatorname{Cos} s, & x_2 = \operatorname{Sin} s, & x_3 = s, & x_4 = 1, \\ a < -1: & x_1 = \operatorname{Cos}\lambda s, & x_2 = \operatorname{Sin}\lambda s, & x_3 = -\operatorname{Cos}\kappa s, & x_4 = \operatorname{Sin}\kappa s. \end{array} \tag{98}$$

In them, κ and λ are the same positive numbers as in no. **66**.

By means of (8), one will get:

The self-projective curves (98) *are identical, up to collineations, to the second family of principal tangent curves of the corresponding ruled surface* (92).

In homogeneous rectangular coordinates x_i, (98) will yield the *helices* for $a = 1$ and the *spatial logarithmic spirals* (= curves on paraboloids of rotation with logarithmic spirals as their base projections) for $1 > a > -1$.

In (97_2), we restrict ourselves to $a = 0$. That will give the torse with the curve of regression:

$$x_1 = s^3, \qquad x_2 = s^2, \qquad x_3 = s, \qquad x_4 = 1. \tag{99}$$

It then follows from (8) that:

The curve (99) *is identical, up to collineations, with the principal tangent curves of the second family of the Cayley surface* (93).

69. Self-projective spatial curves of order 4 and type 2. The curves that are contained in the curves (98) with $a = \mp 5/3$ (and thus, $\kappa = 2\lambda$) are special *space curves of order 4 and type 2* ([1]). By the substitutions:

$$\tan\frac{\lambda s}{2} = u \qquad \left(\tan\frac{\lambda s}{2} = u, \text{ resp.}\right),$$

one will obtain the equations of these curves from (98), upon switching x_3 and x_4:

$$(100)\quad \begin{aligned} &x_1 = 1 - u^4, \quad x_2 = 2u\,(1-u^2), \quad x_3 = 4u\,(1+u^2), \quad x_4 = -\,4u^2 - (1+u^2)^2, \\ &x_1 = 1 - u^4, \quad x_2 = 2u\,(1+u^2), \quad x_3 = 4u\,(1-u^2), \quad x_4 = +\,4u^2 - (1-u^2)^2, \end{aligned}$$

resp. From no. **68**, the curves (100_1) [(100_2), resp.] lie on ruled surfaces of order 3. They define a subset of the intersection of these surfaces with the surfaces of order 2:

$$x_1^2 - x_2^2 + x_3^2 - x_4^2 = 0 \qquad [\,x_1^2 + x_2^2 - x_3^2 - x_4^2 = 0, \text{ resp.}]$$

The rest of the intersection is the real ("imaginary," resp.) pair of lines:

$$x_3 + x_4 = x_1 - x_2 = 0, \qquad x_3 - x_4 = x_1 + x_2 = 0,$$

or

$$x_3 + ix_4 = x_1 + ix_2 = 0, \qquad x_3 - ix_4 = x_1 - ix_2 = 0,$$

resp.

Discussion of forms in homogeneous, rectangular coordinates (Fig. 21): One will obtain:

$$\left.\begin{aligned} &x_1 = \sqrt{\cos 2\varphi}\cos\varphi, \quad x_2 = \sqrt{\cos 2\varphi}\sin\varphi, \quad x_3 = -\sin 2\varphi, \\ &x_1 = \frac{\cos\varphi}{\cos 2\varphi}, \quad x_2 = \frac{\sin\varphi}{\cos 2\varphi}, \quad x_3 = -\tan 2\varphi \ \text{(resp.)}, \end{aligned}\right\} x_4 = 1$$

from (100_1) [(100_2), resp.] after introducing a new parameter φ.

The curves are dashed in Fig. 18, while their base projections are represented in Fig. 21. The points Q in Figs. 18 and 21 correspond to each other.

([1]) One will find more details on space curves of order 4 and type 2 in, e.g., **E. Pascal:** Repert. der höh. Math. II_2, Leipzig-Berlin 1922, Chapter 29. The space curves of order 4 and type 1 are known, since one obtains them as the intersections of two second-order surfaces.

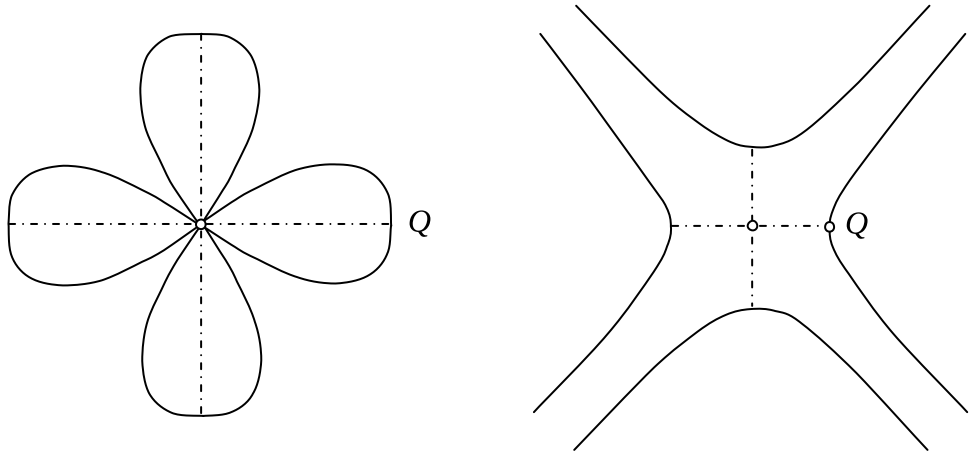

Figure 21.

70. Cubic space curves. The curve (99) and the curves that are collinear to it are the simplest algebraic, non-planar curves, and are called *cubic space curves* ([1]). They are also obtained from (98) with $a = \pm 5 / 4$, so $\kappa = 3\lambda$.

The cubic space curves will be transformed to themselves under not only a one-parameter group of collineations, but also a three-parameter group of collineations. Those collineations are determined from:

$$s = \frac{k_1 \tilde{s} + k_2}{k_3 \tilde{s} + k_4} \qquad (k_1\, k_4 - k_2\, k_3 \neq 0). \tag{101}$$

Proof: By means of (101), (99) will again yield a cubic parameter representation in $\tilde{s}$. From (101), that map of the curve (99) into itself will then give a linear transformation of the point coordinates x_i, and can thus be realized by a collineation.

Discussion of the form (Fig. 22):

The curve (99) is partly the intersection of the cones of order 2:

$$x_3^2 - x_2 x_4 = 0, \quad x_2^2 - x_1 x_3 = 0,$$

which have the line e ($x_2 = x_3 = 0$) in common. Let one cone be transformed into a cylinder by a suitable collineation; it is dashed in Fig. 22. A section of another cone (vertex S) that is bounded by a cubic space curve and two generators is represented. The generator a of the cylinder is an asymptote.

([1]) Cf., **Th. Reye:** *Geometrie der Lage II*, Leipzig, 1923, pp. 163, *et seq.*; furthermore, **E. Pascal:** Repert. der höh. Math. II_2, Leipzig-Berlin, 1922, Chapter 29.

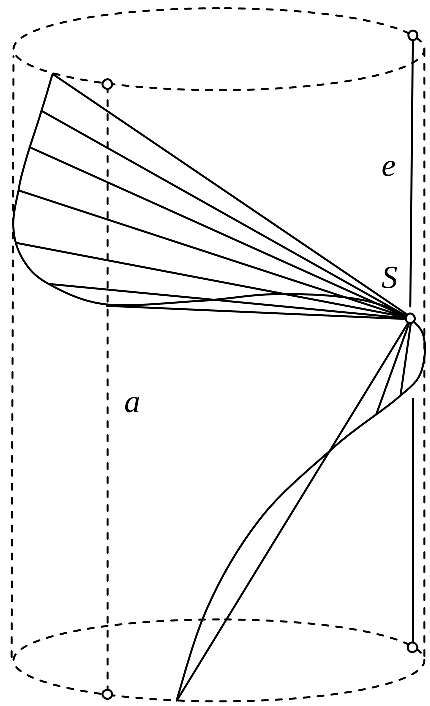

Figure 22.

Chapter III

Line systems

§ 19. Definition of a line system.

71. Parametric representation. A *line system* ([1]) is given by:

$$\mathfrak{p} = \mathfrak{p}\,(u^1, u^2)$$

with the independent parameters u^1, u^2, which vary in the region $u_a^1 \leq u^1 \leq u_b^1$, $u_a^2 \leq u^2 \leq u_b^2$; the six-vector $\mathfrak{p}\,(u^1, u^2)$ is determined only up to an arbitrary factor $\sigma\,(u^1, u^2) \neq 0$. One can then generate the line system in such a way that one line will go through every point of a given plane.

The partial derivatives $\frac{\partial p_\rho}{\partial u^i}$ ($\frac{\partial^2 p_\rho}{\partial u^i\,\partial u^k}$, resp., etc.) transform just like the p_ρ ($\rho = 1, \ldots, 6$) under projective maps by (19) and thus define the six-vectors $\mathfrak{p}_i$ ($\frac{\partial^2 p_\rho}{\partial u^i\,\partial u^k}$, resp., etc.). We denote the first ([2]) partial derivatives by lower indices 1, 2, and demand that for every location u^1, u^2 in the domain of definition:

$\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$ *are linearly-independent; in particular, either* $\mathfrak{p}$ *or its first derivative* $\mathfrak{p}_i$ *will be a zero vector.*

That will give the following *map of the line system to the points of an image plane:*

At any point u'^1, u'^2 in the domain of definition, there exists a sub-domain $u'^1 - \varepsilon \leq u^1 \leq u'^1 + \varepsilon$, $u'^2 - \varepsilon \leq u^2 \leq u'^2 + \varepsilon$ for which the number-pair u^1, u^2 is associated with the line $\mathfrak{p}(u^1, u^2)$ in a uniquely-invertible way ([3]). If one interprets the u^i as inhomogeneous, rectangular point-coordinates in an image plane then the lines of the line system in the sub-domain will be mapped to the points of a quadratic domain in the image plane. Any curve $u^i = u^i\,(t)$ of the image domain will be in one-to-one correspondence with a family of lines that is contained in the line system, and any direction of advance $\dot{u}^1 : \dot{u}^2$ in the

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([1]) One also finds the terminology "ray congruence" used quite often.

([2]) We shall not denote the second partial derivatives by $\mathfrak{p}_{ik}$, in order to express the fact that they do not define a tensor (cf., no. **75**).

([3]) We consider the three parameter-pairs u^1, u^2; $u^1 + a^1$, $u^2 + a^2$; $u^1 + b^1$, $u^2 + b^2$. For $a^1\,b^2 - a^2\,b^1 \neq 0$ and sufficiently small absolute values of a^i, b^i, any two of the three six-vectors:

$$\mathfrak{p}(u^1, u^2);\ \mathfrak{p}(u^1 + a^1, u^2 + a^2) = \mathfrak{p} + a^1\,\mathfrak{p}_1 + a^2\,\mathfrak{p}_1 + \ldots;\ \mathfrak{p}(u^1 + b^1, u^2 + b^2) = \mathfrak{p} + b^1\,\mathfrak{p}_1 + b^2\,\mathfrak{p}_1 + \ldots$$

will be linearly-independent, so they will determine distinct lines.

image domain will correspond to a "direction of advance" in the line system. Here, the dot means the derivative with respect to the parameter t.

By repeated differentiation of $\mathfrak{p}\mathfrak{p} \equiv 0$ with respect to i, k = 1 or 2, one will get the identities:

(102) $$\mathfrak{p}\mathfrak{p} \equiv 0, \qquad \mathfrak{p}\;\mathfrak{p}_i \equiv 0, \qquad \mathfrak{p}\,\frac{\partial^2 \mathfrak{p}}{\partial u^i\,\partial u^k} + \mathfrak{p}_i\,\mathfrak{p}_k \equiv 0, \quad \text{etc.}$$

72. Classification of line systems. Torsal model. We now consider the family of lines in the line system:

$$\mathfrak{p} = \mathfrak{p}\,(u^1(t),\, u^2(t))$$

that is given by $u^i = u^i(t)$. From:

$$\dot{\mathfrak{p}} = \mathfrak{p}_1\,\dot{u}^1 + \mathfrak{p}_2\,\dot{u}^2 \equiv \mathfrak{p}_i\,\dot{u}^i, \qquad \text{so} \qquad \dot{\mathfrak{p}}\dot{\mathfrak{p}} = \mathfrak{p}_i\,\mathfrak{p}_k\,\dot{u}^i\dot{u}^k,$$

one will get from no. **46**:

(103) $$g_{ik}\,\dot{u}^i\dot{u}^k = 0,$$

with

$$g_{ik} = \mathfrak{p}_i\,\mathfrak{p}_k = -\,\mathfrak{p}\,\frac{\partial^2 \mathfrak{p}}{\partial u^i\,\partial u^k} = g_{ki}$$

as the characteristic condition for the given family of lines to be a torse. We have omitted the summation sign $\sum_{i,k=1}^{2}$ from (103), corresponding to our previous convention.

We let r denote the rank and let g denote the value of the determinant of the g_{ik}, and we will then have the following *projectively-invariant and parameter-invariant* (§ 20) *case distinction:*

$$a)\quad r = 2 \left\{ \begin{array}{l} \alpha)\ \ g < 0:\ \textit{hyperbolic} \\ \beta)\ \ g > 0:\ \textit{elliptic} \end{array} \right\} \textit{line system},$$

b) $r = 1$: *parabolic line system,*

c) $r = 0$: *singular line system,*

$$\text{with} \quad \left\{ \begin{array}{ll} a) & \begin{array}{l} \alpha)\ \textit{Two families of torses,} \\ \beta)\ \textit{No torse,} \end{array} \\ b) & \quad \textit{One family of torses,} \\ c) & \quad \textit{Just one torse.} \end{array} \right.$$

The singular line systems are the bundles of lines and the line fields.

Proof: The fact that the bundles of lines and line fields are singular line systems is obvious. Conversely, one has: For $r = 0$, one will have the identity:

$$\left| \mathfrak{p},\ \mathfrak{p}_1,\ \mathfrak{p}_2,\ \frac{\partial^2 \mathfrak{p}}{\partial u^i\,\partial u^k},\ \mathfrak{a},\ \mathfrak{b} \right| \equiv 0$$

for any choice of i, k = 1 or 2, and arbitrary six-vectors $\mathfrak{a}$, $\mathfrak{b}$. From no. **26**, the six given six-vectors (and since $\mathfrak{a}$, $\mathfrak{b}$ can be chosen quite arbitrarily, the four six-vectors $\mathfrak{p}$, $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\frac{\partial^2 \mathfrak{p}}{\partial u^i\,\partial u^k}$) are already linearly dependent. On the other hand, $\mathfrak{p}$, $\mathfrak{p}_1$, $\mathfrak{p}_2$, are linearly independent, by assumption (no. **71**). One will then have:

$$\frac{\partial^2 \mathfrak{p}}{\partial u^i\,\partial u^k} = A\mathfrak{p} + B\mathfrak{p}_1 + C\mathfrak{p}_2,$$

from which, it will follow that:

$$\frac{\partial^2 \mathfrak{p}}{\partial u^i\,\partial u^k}\frac{\partial^2 \mathfrak{p}}{\partial u^m\,\partial u^n} = 0,$$

since $r = 0$ for an arbitrary choice of i, k, m, n = 1 or 2. One will then have $\dot{\mathfrak{p}}\dot{\mathfrak{p}} = 0$, as well as $\ddot{\mathfrak{p}}\ddot{\mathfrak{p}} = 0$, for any family of lines in the line system – i.e., any family of lines is a pencil of lines, which is the family of tangents to a planar curve or the family of generators of a cone (no. **46**). Therefore, any two lines of the system will intersect, since they can be coupled by a family of lines of the line system, and any of these families of lines will go through a fixed point or lie in a fixed plane. However, all lines of the system must then belong to a bundle (line field, resp.).

In what follows, we shall exclude the singular line systems from examination as trivial, and likewise the isolated line with rank $r = 0$.

Torsal model: We shall employ the so-called *torsal models* in order to make the properties of line systems that relate to torses (§ 22) more intuitive.

For $\left\{\begin{matrix}\text{hyperbolic}\\ \text{parabolic}\end{matrix}\right\}$ line systems, the torsal model consists of the lies of a family of segments in a rectangular net with planar $\left\{\begin{matrix}\text{rectangles}\\ \text{quadrilateral}\end{matrix}\right\}$ (cf., no. **7** and nos. **80**, **87**).

Moreover, the lines of a torsal model are arbitrary.

§ 20. Tensors.

73. Definition of tensors. For families of lines, we can eliminate the influence of an arbitrary parameter substitution $u = u(\bar{u})$ by the introduction of a "natural parameter," namely, the arc length s. However, we have two independent parameters u^1, u^2 for line systems. Hence, the parameter substitutions:

$$u^1 = u^1(\bar{u}^1, \bar{u}^2), \qquad u^2 = u^2(\bar{u}^1, \bar{u}^2), \qquad \text{with} \quad \Delta \equiv \begin{vmatrix} \dfrac{\partial u^1}{\partial \bar{u}^1} & \dfrac{\partial u^1}{\partial \bar{u}^2} \\ \dfrac{\partial u^2}{\partial \bar{u}^1} & \dfrac{\partial u^2}{\partial \bar{u}^2} \end{vmatrix} \neq 0 \tag{104}$$

cannot be eliminated in such a simple way as they are for the families of lines. In order to arrive at parameter-invariant expressions, we would like to introduce a suitable analytical tool in the form of the concept of *tensor* in this paragraph. The considerations that were carried out for two parameters u^1, u^2, can be carried over to n parameters with no further asumptions, and will be used in Chapter VI in the study of line complexes for $n = 3$.

One comes to the concept of tensor in the following way:

We consider the $\bar{u}^1$, $\bar{u}^2$ to be functions of t and get from (104), by differentiating with respect to t:

$$\dot{u}^i = \delta^i_k \, \dot{u}^k ,$$

in which the summation sign $\sum_{k=1}^{2}$ has once more been omitted, and we have set:

$$\delta^i_k = \frac{\partial u^i}{\partial \bar{u}^k} ,$$

to abbreviate. Any system of quantities a^1, a^2 that is transformed like the $\dot{u}^1$, $\dot{u}^2$ under the parameter substitution (104), and thus, like:

$$a^i = \delta^i_k \, \bar{a}^k , \tag{105}$$

is called a *contravariant tensor of rank 1* (upper index!). By contrast, any system of quantities b_1, b_2 that is transformed by:

$$\bar{b}_i = \delta^k_i \, b_k \tag{106}$$

is called a *covariant tensor of rank 1* (lower index!).

We now also understand belatedly why we gave point coordinates x_i lower indices and plane coordinates w^i upper ones in no. **2**: We shall speak of collineations, instead of parameter substitutions, for the moment; the x_i then transform analogously to (106) under (3a) and the w^i transform analogously to (105) under (4a).

Tensors of higher rank can be defined with the help of tensors of rank 1: For example, one understands a third-rank tensor a_{ik}^l that is covariant in i and k and contravariant in l to mean a system of quantities:

$$a_{11}^1,\ a_{11}^2,\ a_{12}^1,\ a_{12}^2,\ a_{21}^1,\ a_{21}^2,\ a_{22}^1,\ a_{22}^2,$$

such that the system of quantities transforms like $p_i\, q_k\, r^l$ under any parameter substitution (104), in which p_i, q_k, r^l are tensors of rank 1.

One will get:

$$a_i\, b^i = \bar{a}_k\, \bar{b}^k$$

from (105) and (106) for two arbitrary tensors a_i and b^k of rank one (a_i covariant, b^k contravariant!); i.e., *the sum* $a_1\, b^1 + a_2\, b^2$ *is invariant under arbitrary parameter substitutions.* It likewise follows that for higher-order tensors, one will have, e.g.:

$$a_{ik}^l\, p^i q^k r_l = \bar{a}_{ik}^l\, \bar{p}^i\, \bar{q}^k\, \bar{r}_l,$$

in which p^i, q^k, r_l are tensors of rank 1.

Example: In (103), $g_{ik} = g_{ki}$ is a symmetric, covariant tensor of rank 2, and $g_{ik}\dot{u}^i\,\dot{u}^k$ is a parameter invariant. A special mixed tensor of rank 2 will be given by:

$$g_i^k = \begin{cases} 1 & \text{for } i = k, \\ 0 & \text{for } i \neq k. \end{cases} \tag{107}$$

All components of this tensor are individually parameter-invariant. It follows from:

$$g_i^k\, p_k\, q_i = p_1\, q^1 + p_2\, q^2 = p_i\, q^i = \bar{p}_i\, \bar{q}^i = \bar{g}_i^k\, \bar{p}_k\, \bar{q}^i,$$

in which p_i, q^k are arbitrary tensors, that g_i^k transforms like $a_i\, b^k$, so it is, in fact, a tensor of rank 2.

74. Rules of calculation. One has the following rules of calculation:

a) **Addition:** One will again get a tensor upon adding the corresponding components of equal-rank tensors; e.g.:

$$a_{ik}^l + b_{ik}^l = c_{ik}^l.$$

b) **Multiplication:** A new tensor will result upon multiplying all components of one tensor by all components of a second tensor; e.g.:

$$a_{ik}\, b_{rs}^{\ \ t} = c_{ikrs}^{\ \ \ \ t}.$$

c) **Contraction:** If one sets one upper and one lower indices in a tensor equal to each other and sums over that index then one will get a new tensor of lower rank; e.g.:

$$a_{ikr}^{\ \ \ r} = b_{ik}\,.$$

In particular, in no. **73**, we got the invariant $a_i\, b^i$ (= tensor of rank zero) by contracting $a_i\, b^k$, and similarly, we got the invariant $a_{ik}^{\ \ l}\, p^i\, q^k\, r_l$ from $a_{ik}^{\ \ l}\, p^h\, q^s\, r_t$ by triple contraction.

Proof: *a*) and *b*) follow immediately from the definition of the tensor. In order to assert *c*), we must prove only that the tensor:

$$a_{ikr}^{\ \ \ s}\, p^i\, q^k = c_r^s \qquad (p^i,\ q^k \text{ are tensors of rank 1})$$

has the sum c_s^s as an invariant; the $a_{iks}^{\ \ \ s}$ transform like $m_i\, m_k$, so they define a covariant tensor of rank 2. However, with the help of the tensor g_i^k that was defined in (107), one will have:

$$c_s^s = c_r^s g_s^r;$$

i.e., c_s^s will transform like $p_r\, q^s\, a_s\, b^r$, so from no. **73**, it will be, in fact, invariant.

The determinant a of the components a_{ik} of a covariant tensor of rank 2 transforms like:

(108)
$$\bar{a} = a\, \Delta^2$$

under parameter substitutions (104).

Proof: By applying the law of multiplication for determinants twice, one will get:

$$\bar{a} = \begin{vmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{vmatrix} = \begin{vmatrix} \delta_1^i \delta_1^k a_{ik} & \delta_1^i \delta_2^k a_{ik} \\ \delta_2^i \delta_1^k a_{ik} & \delta_2^i \delta_2^k a_{ik} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} \delta_1^1 & \delta_1^2 \\ \delta_2^1 & \delta_2^2 \end{vmatrix}^2.$$

75. Covariant derivation.

The first partial derivatives φ_1, φ_2 *an arbitrary function* $\varphi\,(u^1, u^2)$ *define a covariant tensor of rank 1 (covariant derivative).*

Proof: It follows from:

$$\varphi\,(u^1, u^2) = \varphi(u^1(\bar{u}^1, \bar{u}^2), u^2(\bar{u}^1, \bar{u}^2)) = \bar{\varphi}(\bar{u}^1, \bar{u}^2)$$

by differentiation and summation that:

$$\varphi_i \dot{u}^i = \bar{\varphi}_i \dot{\bar{u}}^i .$$

By contrast, one observes that the higher partial derivatives do not define a tensor; for example:

$$\sum \frac{\partial^2 \varphi}{\partial u^i \, \partial u^k} \dot{u}^i \dot{u}^k \quad \text{and} \quad \sum \frac{\partial^2 \bar{\varphi}}{\partial \bar{u}^i \, \partial \bar{u}^k} \dot{\bar{u}}^r \, \dot{\bar{u}}^s$$

will not go to each other under arbitrary parameter substitutions, but only for special ones.

Whereas we have previously investigated the behavior of the complex (line, resp.) coordinates and their derivatives under projective transformations, we would now like to establish the effect of parameter substitutions:

Any individual complex coordinate $p_\rho (u^1, u^2)$ with fixed ρ ($\rho = 1, \dots, 6$) is invariant under (104), and thus a tensor of rank zero. For that reason, the partial derivatives $(p_\rho)_1$, $(p_\rho)_2$ will define a covariant tensor of rank 1 for any fixed ρ. On the other hand, the six derivatives $(p_1)_i, \dots, (p_6)_i$ define a six-vector $\mathfrak{p}_i$ for fixed i ($i = 1, 2$) (no. **71**).

The scalar product of the first partial derivatives of two six-vectors $\mathfrak{p}(u^1, u^2)$, $\mathfrak{q}(u^1, u^2)$ defines a covariant tensor of rank 2:

$$a_{ik} = \mathfrak{p}_i \ \mathfrak{q}_k .$$

§ 21. Invariants of a line system.

76. Invariant derivatives. Just as we did in no. **55** for the families of lines, our problem for line systems is to find the properties "in the neighborhood" of a system line $\mathfrak{p}$ that are invariant under principal projectivities; i.e., to determine the complete invariants and sign invariants of the six-vectors $\mathfrak{p}$, $\mathfrak{p}_1$, $\mathfrak{p}_2$, etc., up to partial derivatives of arbitrarily higher order. As in no. **55**, we will again normalize the six-vectors in such a way that they remain normalized for arbitrary unity transformations; instead of the complete invariants, we will then have to ascertain only the semi-invariants.

However, similarly to the situation with families of lines, we also now come to the fact that only expressions that are invariant under parameter substitutions (104) have a geometric meaning. In the exhibition of such parameter-invariant expressions in § 20, we introduced the concept of a tensor, and we now proceed as follows:

Let m^i and m'^k be two contravariant tensors of rank 1 with $m^1 \, m'^2 - m^2 \, m'^1 \neq 0$; let the components m^i and m'^k be semi-invariants. From any semi-invariant $\varphi (u^1, u^2)$, we will then get the functions:

(109) $$\varphi_1 = m^i \varphi_i , \qquad \varphi_2 = m'^k \varphi_k ,$$

which are semi-invariant, as well as parameter-invariant, and refer to φ_1, φ_2 as the *invariant derivatives* of φ. The new parameter-invariant six-vectors:

$$\mathfrak{q}_1 = m^i\,\mathfrak{q}_i\,, \qquad\qquad \mathfrak{q}_2 = m'^{\,k}\,\mathfrak{q}_k$$

follow from a six-vector $\mathfrak{q}\,(u^1, u^2)$ by invariant differentiation.

Solving (109) for φ_i will yield:

(110) $$\varphi_i = m_i\,\varphi_1 + m'_i\,\varphi_2\,,$$

with the covariant tensors m_i, m'_i, which are determined by:

(111) $$m_i\,m^i = 1, \qquad m_i\,m'^i = 0, \qquad m'_i\,m^i = 0, \qquad m'_i\,m'^i = 1.$$

The fact that m_i, m'_i are tensors follows by substituting:

$$m^i = \delta^i_k\,\bar{m}^k\,, \qquad m'^i = \delta^i_k\,\bar{m}'^k$$

into (111), which will yield:

$$m_i\,\delta^i_k = \bar{m}_k\,, \qquad m'_i\,\delta^i_k = \bar{m}'_k\,.$$

By repeated invariant derivation, one comes to the semi-invariants with partial derivatives of arbitrarily high order; e.g.:

(112) $$\begin{aligned}\mathfrak{q}_{12} &= m'^k\,(m^i\,\mathfrak{q}_i)_k = m'^k\,m^i\,\frac{\partial^2\mathfrak{q}}{\partial u^i\,\partial u^k} + m'^k\,\frac{\partial m^i}{\partial u^k}\,\mathfrak{q}_i\,,\\ \mathfrak{q}_{21} &= m^i\,(m'^k\,\mathfrak{q}_k)_i = m^i\,m'^k\,\frac{\partial^2\mathfrak{q}}{\partial u^i\,\partial u^k} + m^k\,\frac{\partial m'^k}{\partial u^i}\,\mathfrak{q}_k\,,\end{aligned}$$

in which one sums over i, k.

The same *rules of calculation* are true for invariant differentiation as for ordinary differentiation:

$$\begin{aligned}(\varphi + \psi)_1 &= \varphi_1 + \psi_1\,, & (\varphi + \psi)_2 &= \varphi_2 + \psi_2\,,\\ (\varphi\cdot\psi)_1 &= \varphi_1\cdot\psi + \varphi\cdot\psi_1\,, & (\varphi\cdot\psi)_2 &= \varphi_2\cdot\psi + \varphi\cdot\psi_2\,.\end{aligned}$$

By contrast, in place of:

$$\frac{\partial^2\varphi}{\partial u^i\,\partial u^k} = \frac{\partial^2\varphi}{\partial u^k\,\partial u^i}\,,$$

one now has the complicated *integrability condition:*

(113) $$\varphi_{12} + q\varphi_1 = \varphi_{21} + q\,\varphi_2\,,$$

with

(114) $$q = (m^1\,m'^2 - m^2\,m'^1)\left(\frac{\partial m_1}{\partial u^2} - \frac{\partial m_2}{\partial u^1}\right), \qquad q' = (m'^1\,m^2 - m'^2\,m^1)\left(\frac{\partial m'_1}{\partial u^2} - \frac{\partial m'_2}{\partial u^1}\right).$$

Proof: From (109) and (110), one has:

$$(\varphi_1)_i = m_i\,\varphi_{12} + m_i'\,\varphi_{12}\,, \qquad (\varphi_2)_i = m_i\,\varphi_{21} + m_i'\,\varphi_{22}\,,$$

$$\frac{\partial^2\varphi}{\partial u^i\,\partial u^k} = (m_i\,\varphi_1 + m_i'\,\varphi_2)_k = \frac{\partial m^i}{\partial u^k}\,\varphi_1 + \frac{\partial m'^i}{\partial u^k}\,\varphi_2 + m_i\,m_k\,\varphi_{11} + m_i\,m_k'\,\varphi_{12}$$
$$+\ m_i'\,m_k\,\varphi_{11} + m_i'\,m_k'\,\varphi_{22}\,.$$

The integrability condition:

$$\frac{\partial^2\varphi}{\partial u^i\,\partial u^k} = \frac{\partial^2\varphi}{\partial u^k\,\partial u^i}$$

can also be written in the form:

$$\sum (m^i\,m'^k - m^k\,m'^i)\,\frac{\partial^2\varphi}{\partial u^i\,\partial u^k} = 0;$$

one will obtain the equation (113) that was to be proved by substituting the expression above and with the help of (111).

From (113), the sequence of invariant derivatives *does not commute*, in general.

The expressions q, q' that were defined in (113) are not only semi-invariant, like the m^i, m'^k, but also parameter invariant.

Proof: If one chooses φ such that $\varphi_2 \equiv 0$, $\varphi_1 \neq 0$, (the other way around, resp.) then (111) will give $q = -\,\varphi_{12} / \varphi_1$ ($q' = -\,\varphi_{12} / \varphi_1$, resp.). q is then parameter-invariant, just like φ_1, φ_{12} .

77. Differential equations and integrability conditions. Just as we did for families of lines, we associate any system line $\mathfrak{p}(u^1, u^2)$ with a projectively and parametrically-invariant fundamental system of $n \leq 6$ normalized linearly-independent six-vectors $\mathfrak{p}$, $\mathfrak{q}$, … We refer to the representation of the $2\times n$ invariant derivatives $\mathfrak{p}^1$, $\mathfrak{p}^2$, $\mathfrak{q}^1$, $\mathfrak{q}^2$, … as linear combinations of the six-vectors $\mathfrak{p}$, $\mathfrak{q}$, … of the fundamental system as a *differential equation.* One then has that:

A) The scalar products of the fundamental system and

B) The coefficients of the differential equations

are semi-invariant with respect to projective maps and parameter-invariant. One difference in comparison to the families of lines consists of the fact that the coefficients of the differential equation are mutually-independent; moreover, they must fulfill the *integrability conditions* that follow from:

$$\mathfrak{p}_{12} + q\,\mathfrak{p}_1 = \mathfrak{p}_{21} + q'\,\mathfrak{p}_2\,, \qquad \text{etc.}$$

The line system is characterized in a projectively-invariant way by the m^i, m'^k, and the invariants *A*), *B*); one then has the main theorem:

Two line systems are projectively related to each other and line-wise projectively related to each other for equal u^1, u^2*-value-pairs if and only if they agree in their invariants A*), *B*), *and the functions* m^i, m'^k, *and thus also the invariants q, q′.*

Proof: From (110), the $\mathfrak{p}_i$, and as one sees upon further differentiation, all partial derivatives of $\mathfrak{p}$ of arbitrarily high order, are linear combinations of the six-vectors of the fundamental system, on the basis of the differential equations. The coefficients of these linear combinations are determined by m^i, m'^k, and the invariants *A*) and *B*). Moreover, one then concludes as one did in no. **56**.

78. Line systems with constant invariants. Later on, we shall deal with line systems with *constant invariants*, in particular. For these line systems, we shall prove the theorem:

Two line systems with constant invariants A), *B*), *and q, q′ are projective to each other if and only if they agree in those invariants.*

Proof: We assume that:

$$m^2 \equiv 0, \qquad m'^1 \equiv 0,$$

which can always be arranged by a parameter substitution (104); we will get to know about these parameters as "torse parameters" (no. **81**) ["principal tangent parameters" no **88**), resp.] for the hyperbolic (parabolic, resp.) line systems. There are three cases to distinguish:

1) $q \neq 0$, $q' \neq 0$: It follows from (111) and (114) that:

$$\frac{\partial m_1}{\partial u^2} = q\, m_1\, m_2', \qquad \frac{\partial m_2'}{\partial u^1} = q'\, m_1\, m_2',$$

so, due to the constancy of q and q':

$$\frac{\partial}{\partial u^2}(q' m_1) = \frac{\partial}{\partial u^1}(q m_2') .$$

We can then set:

$$q'\, m_1 = \frac{\partial \varphi}{\partial u^1}, \quad q m_2' = \frac{\partial \varphi}{\partial u^1} ;$$

from (114), the functions $\varphi(u^1, u^2)$ must then satisfy the equation:

$$\frac{\partial^2\varphi}{\partial u^1\,\partial u^2} - \frac{\partial\varphi}{\partial u^1}\frac{\partial\varphi}{\partial u^2} = 0.$$

By the substitution:

$$\varphi = -\ln\,\psi,$$

one will obtain the differential equation:

$$\frac{\partial^2\varphi}{\partial u^1\,\partial u^2} = 0.$$

From the general solution of this differential equation:

$$\psi = f(u^1) + g(u^2), \qquad \text{so} \qquad \varphi = -\ln[f(u^1) + g(u^2)],$$

one gets:

$$m_1 = -\frac{\dot f}{q'(f+g)}, \qquad m_2' = -\frac{\dot g}{q(f+g)},$$

in which the dot means the differentiation symbol.

Finally, it follows from the parameter substitution:

$$\bar u^1 = f(u^1), \qquad \bar u^2 = g(u^2)$$

that:

(115)
$$\bar m_1 = -\frac{1}{q'(\bar u^1 + \bar u^2)}, \qquad \bar m_2 = -\frac{1}{q(\bar u^1 + \bar u^2)}.$$

One thus also gets the same functions $\bar m^i$, $\bar m'^k$, for all line systems with the same constant invariants; the line systems are then projective to each other, from the main theorem of no. **77**.

2) $q = 0$, $q \neq 0$: It follows from:

$$\frac{\partial m_1}{\partial u^2} = 0, \qquad \frac{\partial m_2'}{\partial u^1} = q' m_1 m_2'$$

that:

$$m_1 = \dot f(u^1), \quad m_2' = \dot g(u^2)\cdot e^{q' f(u^1)},$$

and from the parameter substitution:

$$\bar u^1 = f(u^1), \qquad \bar u^2 = g(u^2),$$

so one will get:

(116)
$$\bar m_1 = 1, \qquad \bar m_2' = e^{q'\bar u^1}.$$

3) $q = q' = 0$: Here, one finds that:

$$\bar{m}_1 = 1, \qquad \bar{m}'_2 = 1. \tag{117}$$

In the following paragraph, we will treat the hyperbolic and parabolic line systems, in turn. The elliptic line systems can be considered formally to be hyperbolic line systems with "imaginary" focal surfaces or focal curves (no. **79**).

§ 22. Contact structures for hyperbolic line systems.

79. Focal surfaces or focal curves. Since $g < 0$, any line of the system will yield the two real, distinct directions of advance:

$$\dot{u}^i = m^i(t) \qquad \text{and} \qquad \dot{u}^i = m'^i(t), \qquad m^1\, m'^2 - m^2\, m'^1 \neq 0, \tag{118}$$

from (103). By integrating them, one will get the two families of torses that are contained in the line system:

$$u^i = \omega^i\,(t, c) \quad \text{and} \quad u^i = \omega'^i\,(t, c)$$

with the integration constants c. A torse of the first family and one of the second will intersect at any line $\mathfrak{p}$ of the system. The points of regression x, x' of those torses that lie along $\mathfrak{p}$ are called *focal points*, and the planes of regression that go through w, w' are called *focal planes.*

The focal points x, x' and the focal planes w, w' determine two restricted pencils of lines $\{x \mid w\}$ and $\{x' \mid w'\}$ with the system line $\mathfrak{p}$ as the common line (Fig. 10). From no. **46**, since:

$$\dot{\mathfrak{p}} = m^1\,\mathfrak{p}_1 + m^2\,\mathfrak{p}_2 \qquad (\dot{\mathfrak{p}} = m'^1\,\mathfrak{p}_1 + m'^2\,\mathfrak{p}_2\,, \text{ resp.}),$$

that line pencil can be represented by:

$$\{x \mid w\} : \rho\,\mathfrak{p} + \sigma(m'^1\,\mathfrak{p}_1 + m'^2\,\mathfrak{p}_2) \qquad [\{x' \mid w'\}: \rho\,\mathfrak{p} + \sigma(m^1\,\mathfrak{p}_1 + m^2\,\mathfrak{p}_2), \text{ resp.}].$$

Therefore, the focal points x, x', as well as the focal planes w, w', will be distinct. The pencils of lines $\{x \mid w\}$ and $\{x' \mid w'\}$ define the quadric of the bundle of complexes $\lambda^0\,\mathfrak{p} + \lambda^1\,\mathfrak{p}_1 + \lambda^2\,\mathfrak{p}_2$, which has type *b*), α) (no. **34**), since $g < 0$, and since the discriminant $D_3\,(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ has rank $r = 2$. We will encounter two complexes that are conjugate to the bundle $\lambda^0\,\mathfrak{p} + \lambda^1\,\mathfrak{p}_1 + \lambda^2\,\mathfrak{p}_2$ in no. **83** in the form of the so-called *contact complexes.*

The surfaces (curves, resp.) that are generated by the focal points x (x', resp.) are called the *focal surfaces* Φ (*focal curves* Φ', resp.) of the line system.

80. Classification of hyperbolic line systems. If the torses of a family are cones or pencils of lines then the vertices of the cones will define a focal curve; by contrast, if the torses are families of tangents then the curves of regression will define a focal surface. One then has the following *case distinction:*

1) *Line systems with two focal surfaces* Φ, Φ′:

The curves of regression of the first (second, resp.) family of torses define a family of curves on the focal surface Φ (Φ′, resp.). We call these families of curves *families of longitudes,* the families of curves that are conjugate to them, *lateral curve families,* and the curve nets that are generated by the longitudinal and lateral curve families, *contact nets.* Since the longitudinal and lateral curve families are conjugate, the lines of the system along a curve of Φ (Φ′, resp.) will define a torse of the second (first, resp.) family; i.e., a torse with a longitudinal curve of Φ′ (Φ, resp.). As a result, the torses of the first (second, resp.) family will contact the focal surface Φ′ (Φ, resp.) along lateral curves. The contact planes of the focal surface Φ (Φ′, resp.) are then identical with the osculating planes of the longitudinal curves of Φ′ (Φ, resp.); i.e., with the focal planes w' (w, resp.). One then has: *The focal surfaces* Φ′ (Φ, *resp.*) *can also be defined to be the envelopes of the focal planes* w' (w, *resp.*). The relationship between Φ and Φ′ is illustrated in Fig. 23.

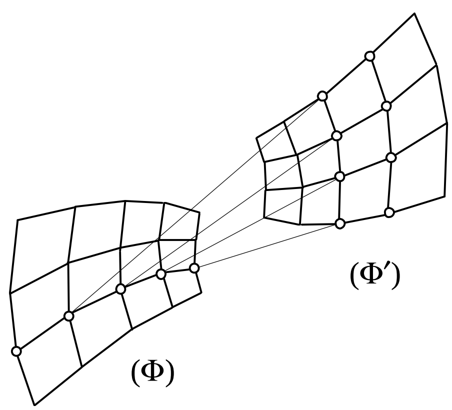

Figure 24.

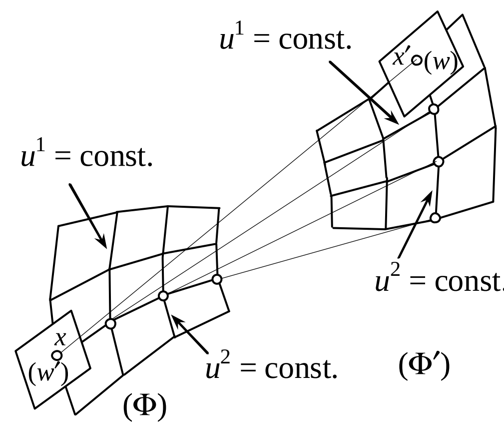

Figure 23.

Either the longitudinal or the lateral curves can be principal tangent curves; since the principal tangent curves are self-conjugate, the longitudinal and lateral curves would coincide, and there would exist only one family of torses, which would contradict the fact that $g < 0$.

If a focal surface Φ is developable then, from no. **6**, the family of generators can belong to any family of longitudes of Φ as the conjugate to the family of lateral curves. The contact planes of Φ along the generators will then be constant and cut out longitudinal curves from the focal surfaces Φ′; the longitudinal curves of Φ′ will then be planar curves. For two developable focal surfaces, both contact nets will consist of the generators and a family of planar curves.

II) *Line systems with a focal surface* Φ *and a focal curve* Φ′:

The contact net of the longitude and lateral curves is defined on the focal surface Φ as in I). Any torse of the first family has a longitudinal curve of Φ as a curve of regression and goes through the focal curve Φ′. Any torse of the second family is a cone or pencil of lines whose vertex lies on the focal curve Φ′ and contacts the focal surface Φ along a lateral curve.

If the focal surface Φ is developable then all torses of the second family will be pencils of lines with their vertices on the focal curve Φ′.

III) *Line systems with two focal curves* Φ, Φ′:

Both families of torses consist of cones or pencils of lines. The cones of the first (second, resp.) family have their vertices on Φ (Φ′, resp.) and go through the focal curve Φ′ (Φ, resp.).

If the one focal curve Φ′ is rectilinear then the torses of the first family will be pencils of lines whose vertices run through the other focal curves Φ. The line systems with two skew, rectilinear, focal curves will be the linear hyperbolic line systems (no. **33**).

Under *correlative maps,* there is a correspondence between:

non-developable focal surfaces	↔	non-developable focal surfaces
developable focal surface	↔	focal curves,

so, in particular:

line systems with two developable focal surfaces	↔	line systems with two focal curves

The torses with the longitude curves of Φ as their curves of regression will be correlatively transformed into torses with curves of regression on Φ′, and conversely. The tangents to a longitudinal curve on Φ will then go to the system lines along a lateral curve of Φ under a correlation (duality of the tangents of a curve and the generators of the correlative strip).

A hyperbolic line system I) or II) can be given by any surface Φ as its focal surface and a conjugate net of Φ as its contact net. *Our theory of hyperbolic line systems is then, at the same time, a projective differential geometry of the conjugate curve nets.*

We elucidate the hyperbolic line systems with two focal surfaces by *comparing them to torsal models:*

Torsal model (Fig. 24)	Line system (Fig. 23)
Two rectangle nets Φ, Φ′ with planar quadrilaterals. Longitudinal and lateral polygons.	Two focal surfaces Φ, Φ′ with conjugate contact nets. Longitudinal and lateral curves.

The lines of longitudinal polygon of Φ contain the rectangle sides of Φ′ along a lateral polygon (“lateral sequence”).	The tangents to a longitudinal curve of Φ contact Φ′ along a lateral curve.
The planes of any two successive sides of a longitudinal polygon of Φ are rectangle planes of Φ′.	The osculating planes of the longitudinal curves of Φ are contact planes for Φ′.
The lines of a longitudinal polygon on Φ will go to a “lateral sequence” on Φ under a correlation, and thus, to the lines of a longitudinal polygon on Φ′.	The tangents to a longitudinal curve of Φ will go to the lines of a system along a lateral curve on Φ under a correlation, and thus to the tangents to a longitudinal curve on Φ′.

81. Torse parameters and invariant derivatives. Sectional tangents. We refer to the parameters that yield the $\left\{\begin{matrix}\text{first}\\ \text{second}\end{matrix}\right.$ family of torses for $\left\{\begin{matrix}u^2 = \text{const.}\\ u^1 = \text{const.}\end{matrix}\right.$ as *torse parameters*. Using torse parameters, one then has:

$$(119) \qquad g_{11} = g_{22} = 0, \quad g_{12} \neq 0; \qquad m^1 \neq 0, m^2 = 0 \quad \text{and} \quad m'^1 = 0, m'^2 \neq 0$$

in (103) and (118). The lateral curves on Φ (Φ′, resp.) and the longitudinal curves on Φ′ (Φ, resp.) are given by u^1 = const. (u^2 = const. resp.) (Fig. 23). One observes that the notations u^1 and u^2 are switched by correlations.

Subject to the normalization of $\mathfrak{p}(u^1, u^2)$ and the contravariant tensors m^i (m'^i, resp.) that are determined by (118), but only up to an arbitrary proportionality factor $\sigma(u^1, u^2)$ [$\sigma'(u^1, u^2)$, resp.], we define the invariant derivatives by:

$$\varphi_1 = m^i \varphi_i\,, \qquad \varphi_2 = m'^i \varphi_i\,,$$

as in no. **76**. Due to (119), one will then have:

$$(120) \qquad \varphi_1 = m^1 \varphi_1\,, \qquad \varphi_2 = m'^2 \varphi_2\,,$$

in particular, for the torse parameters. Due to the parameter invariance of $m^i \varphi_i$ and $m'^i \varphi_i$, it also follows from that fact that for arbitrary parameters u^1, u^2: *The invariant derivative* 1 (2, *resp.*) *means a differentiation along a torse of the first (second, resp.) family.*

We refer to the tangents $\mathfrak{c}$ ($\mathfrak{c}'$, resp.) to the lateral curves of Φ (Φ', resp.) as *lateral tangents;* if a focal surface degenerates to a focal curve then the tangents to the focal curve will be the lateral tangents. The lateral tangents $\mathfrak{c}$, $\mathfrak{c}'$, together with $\mathfrak{p}$, span the focal planes w, w', so they will be skew to each other. The singular six-vectors $\mathfrak{c}$ and $\mathfrak{c}'$ satisfy the equations:

(121) $$\mathfrak{c}\mathfrak{p} = \mathfrak{c}\mathfrak{p}_1 = \mathfrak{c}\mathfrak{p}_2 = \mathfrak{c}\mathfrak{p}_{12} = 0.$$

Proof: We think of the line system as being referred to torse parameters. For the connecting line $\mathfrak{c}_g$ between the focal points:

$$\left.\begin{matrix} x \\ x_g \end{matrix}\right\} = \text{point of intersection of } \left\{\begin{matrix} \mathfrak{p}(u^1,u^2) & \text{and} & \mathfrak{p}_1(u^1,u^2), \\ \mathfrak{p}(u^1,u^2+\varepsilon) & \text{and} & \mathfrak{p}_1(u^1,u^2+\varepsilon), \end{matrix}\right.$$

one obtains the conditions:

$$\mathfrak{c}_g\mathfrak{p} = \mathfrak{c}_g\mathfrak{p}_1 = \mathfrak{c}_g\,(\mathfrak{p} + \varepsilon\,\mathfrak{p}_1 + \ldots) = \mathfrak{c}_g\left(\mathfrak{p}_1 + \varepsilon\frac{\partial^2\mathfrak{p}}{\partial u^1\partial u^2} + \cdots\right) = 0.$$

Equations (121) follow from this as $\varepsilon \to 0$, with consideration given to (120), and the connecting line $\mathfrak{c}_g$ will converge to the lateral tangent $\mathfrak{c}$.

The bush of complexes $\lambda\,\mathfrak{p} + \mu\,\mathfrak{p}_1 + \nu\,\mathfrak{p}_2 + \rho\,\mathfrak{p}_{12}$ is hyperbolic (no. **32**), since:

$$D_4\,(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_{12}) = (\mathfrak{p}_1\mathfrak{p}_2)^2\,(\mathfrak{p}\mathfrak{p}_{12})^2 = (\mathfrak{p}_1\mathfrak{p}_2)^4 > 0;$$

from (119), one will then have:

$$\mathfrak{p}_1\mathfrak{p}_2 = m^1\,m'^2\,g_{12} \neq 0$$

in torse parameters. From (121), $\mathfrak{c}$, $\mathfrak{c}'$ are the axes of the two singular complexes that are conjugate to the bush of complexes $\lambda\,\mathfrak{p} + \mu\,\mathfrak{p}_1 + \nu\,\mathfrak{p}_2 + \rho\,\mathfrak{p}_{12}$, and thus, uniquely determined by (121) (no. **32**).

We define the symmetric, covariant tensor of rank 2 with the help of the six-vectors $\mathfrak{c}$, $\mathfrak{c}'$:

(122) $$\mathfrak{c}_{ik} = \mathfrak{c}_i\,\mathfrak{p}_k = -\,\mathfrak{c}\frac{\partial^2\mathfrak{p}}{\partial u^i\partial u^k} = \mathfrak{c}_k\,\mathfrak{p}_i = \mathfrak{c}_{ki}, \qquad c'_{ik} = \mathfrak{c}'_i\,\mathfrak{p}_k = -\,\mathfrak{c}'\frac{\partial^2\mathfrak{p}}{\partial u^i\partial u^k} = \mathfrak{c}'_k\,\mathfrak{p}_i = c'_{ki}.$$

It follows from the last equation in (121) that:

(123) $$c_{12} = c'_{12} = 0$$

for the torse parameters.

82. Principal tangent curves and harmonic nets of focal surfaces. In the event that the focal surfaces Φ (Φ', resp.) are not positively-curved, one will have the equations of the *principal tangent curves:*

$$(124)\qquad \begin{vmatrix} g_{11}\dot{u}^1 + g_{12}\dot{u}^2 & c_{11}\dot{u}^1 + c_{12}\dot{u}^2 \\ g_{21}\dot{u}^1 + g_{22}\dot{u}^2 & c_{21}\dot{u}^1 + c_{22}\dot{u}^2 \end{vmatrix} = 0, \qquad \begin{vmatrix} g_{11}\dot{u}^1 + g_{12}\dot{u}^2 & c'_{11}\dot{u}^1 + c'_{12}\dot{u}^2 \\ g_{21}\dot{u}^1 + g_{22}\dot{u}^2 & c'_{21}\dot{u}^1 + c'_{22}\dot{u}^2 \end{vmatrix} = 0, \text{ resp.}$$

Proof: Let the tangents to a principal tangent curve $u^i = u^i(t)$ be represented by $\mathfrak{h} = \lambda\,\mathfrak{p} + \mu\,\mathfrak{c}$. The lines $\mathfrak{h}$ and $\dot{\mathfrak{h}}$ will then span the osculating planes of the principal tangent curves. Since these osculating planes are, at the same time, contact planes to the surface Φ (no. 6), $\mathfrak{h}, \dot{\mathfrak{h}}, \mathfrak{p}, \mathfrak{c}$ must be linearly dependent; it follows from this that:

$$(\lambda\,\mathfrak{p}_i + \mu\,\mathfrak{c}_i) = \rho\,\mathfrak{p} + \sigma\,\mathfrak{c}.$$

Upon scalar-multiplying this by $\mathfrak{p}_1$ and $\mathfrak{p}_2$, one will get:

$$(125)\qquad \begin{aligned} \lambda\,(g_{11}\dot{u}^1 + g_{12}\dot{u}^2) + \mu\,(c_{11}\dot{u}^1 + c_{12}\dot{u}^2) &= 0, \\ \lambda\,(g_{21}\dot{u}^1 + g_{22}\dot{u}^2) + \mu\,(c_{21}\dot{u}^1 + c_{22}\dot{u}^2) &= 0. \end{aligned}$$

Since (125) possesses a non-trivial solution λ, μ, (124) will be true.

In torse parameters, (124) specializes to:

$$(126)\qquad c_{11}(\dot{u}^1)^2 - c_{22}(\dot{u}^2)^2 = 0, \qquad c'_{11}(\dot{u}^1)^2 - c'_{22}(\dot{u}^2)^2 = 0.$$

We denote the determinants of the tensors c_{ik} (c'_{ik}, resp.) by c (c', resp.). Now, from (126), and on the basis of (8), one gets *the projectively-invariant case distinction:*

$$c\ (c',\text{ resp.}) \begin{cases} > 0: \ \textit{negatively} - \\ < 0: \ \textit{positively} - \end{cases}\!\!\Bigg\}\ \textit{curved focal surface}\ \Phi(\Phi', \textit{resp.}), \\ \quad = 0: \textit{developable focal surface or focal curve}\ \Phi(\Phi', \textit{resp.}),$$

at first, with torse parameters, and then, due to (108), for arbitrary parameters, as well. A real curve net is defined on a positively-curved focal surface Φ (Φ', resp.) by:

$$(127)\qquad c_{ik}\dot{u}^i\dot{u}^k = 0 \qquad (c'_{ik}\dot{u}^i\dot{u}^k = 0, \text{ resp.}),$$

which we would like to call a *harmonic net.*

The harmonic net is a conjugate net, and its tangents will be separated harmonically by the tangents to the contact net (no. **80***).*

Proof: In torse parameters, (127) specializes to:

$$c_{11}(\dot{u}^1)^2 + c_{22}(\dot{u}^2)^2 = 0 \qquad (c'_{11}(\dot{u}^1)^2 + c'_{22}(\dot{u}^2)^2 = 0, \text{ resp.}) \tag{128}$$

Since the juxtaposition of (8) and (126) yields:

$$L : M : N = c_{11} : 0 : -c_{22} \qquad (c'_{11} : 0 : -c'_{22}, \text{ resp.}),$$

conjugate nets are given by this from (6). Moreover, from (5), the directions of advance (128) are harmonic to the directions of advance $\dot{u}^1 = 0$ and $\dot{u}^2 = 0$ of the contact net.

The harmonic net of – say – the focal surface Φ can also interpreted as follows: *The lines* $\mathfrak{p}$ *of a system that cut a fixed lateral tangent* $\mathfrak{c}$ *contact* Φ *along two curves that will be contacted at the focal point x of the curves of the harmonic net.*

Proof: It follows from:

$$\begin{aligned} 0 = \mathfrak{c}\mathfrak{p}\,[u^i\,(t+\varepsilon)] &= \mathfrak{c}\left(\mathfrak{p} + \varepsilon\dot{\mathfrak{p}} + \frac{\varepsilon^2}{2}\ddot{\mathfrak{p}} + \cdots\right) \\ &= \mathfrak{c}\left\{\mathfrak{p} + \varepsilon\dot{\mathfrak{p}}_i\dot{u}^i + \frac{\varepsilon^2}{2}\left(\dot{\mathfrak{p}}_i\,\ddot{u}^i + \sum\frac{\partial^2\mathfrak{p}}{\partial u^i\,\partial u^k}\dot{u}^i\,\dot{u}^k\right) + \cdots\right\} \end{aligned}$$

that:

$$0 = \mathfrak{c}\sum\frac{\partial^2\mathfrak{p}}{\partial u^i\,\partial u^k}\dot{u}^i\,\dot{u}^k = -\,c_{ik}\dot{u}^i\dot{u}^k\,.$$

§ 23. Differential equations of hyperbolic line systems.

83. Normalization of the six-vectors $\mathfrak{p}$, $\mathfrak{c}$, $\mathfrak{c}'$ and the tensors m^i, m'^k. According to the plan that was developed in § 21, we must now normalize the six-vectors $\mathfrak{p}$, $\mathfrak{c}$, $\mathfrak{c}'$ by a prescription that is invariant under unity transformations. We thus restrict ourselves to the *line systems with two negatively-curved focal surfaces* and leave the analogous treatment of the other cases to the reader.

The *normalization of* $\mathfrak{p}$, $\mathfrak{c}$, $\mathfrak{c}'$ results from the demands that:

$$g = -\,c = -\,c', \qquad \mathfrak{c}\mathfrak{c}' = 1. \tag{129}$$

One gets the normalized expressions $\hat{\mathfrak{p}} = \rho\,\mathfrak{p}$, $\hat{\mathfrak{c}} = \sigma\,\mathfrak{c}$, $\hat{\mathfrak{c}}' = \sigma'\,\mathfrak{c}'$ from the un-normalized ones on the basis of (129) with:

$$\rho^4\,g = -\,\rho^2\,\sigma^2\,c = -\,\rho^2\,\sigma'^2\,c', \qquad \sigma\sigma'\,(\mathfrak{c}\mathfrak{c}') = 1.$$

ρ, σ, σ' can be determined to be real since $c > 0$, $c' > 0$, $g < 0$; the signs of ρ and σ remain arbitrary, although the sign of σ' will then stay fixed. Due to (108), the normalization conditions are also parameter-invariant.

Normalization of m^i, m'^k:

From (118) and (103), the tensors m^i, m'^k that were employed for the definition of the invariant derivations will be given:

$$g_{ik}\, m^i\, m^k = 0 \quad \text{and} \quad g_{ik}\, m'^i\, m'^k = 0;$$

from the normalization of $\mathfrak{p}$, $\mathfrak{c}$, $\mathfrak{c}'$, they are still determined only up to arbitrary factors. Those factors will now be established, up to a sign, for the tensors m^i and m'^i by the additional requirements that:

$$(130) \qquad c_{ik}\, m^i\, m^k = -1, \qquad c'_{ik} m'^i m'^k = \mp 1,$$

which are invariant under unity transformations. (130) can always be fulfilled by real points. In fact, reverting to the torse parameters, one will get:

$$c_{ik}\, m^i\, m^k = c_{11}\, m^1\, m^1 \neq 0, \qquad c'_{ik} m'^i m'^k = c'_{22} m'^2 m'^2 \neq 0;$$

moreover, one can always arrive at $c_{ik}\, m^i\, m^k < 0$ by a suitable choice of the sign in the normalization of $\mathfrak{p}$ and $\mathfrak{c}$, which still remains arbitrary.

For normalized $\mathfrak{p}$, $\mathfrak{c}$, $\mathfrak{c}'$, the *principal tangents* of Φ (Φ', resp.) are given by:

$$(131) \qquad \mathfrak{h} = \mathfrak{p} + \mathfrak{c}, \qquad \mathfrak{k} = \mathfrak{p} - \mathfrak{c}, \qquad (\mathfrak{h}' = \mathfrak{p} + \mathfrak{c}', \quad \mathfrak{k}' = \mathfrak{p} - \mathfrak{c}', \text{ resp.}).$$

Proof: In torse parameters, (125) yields:

$$\lambda\, g_{12}\, \dot{u}^2 + \mu\, c_{11}\, \dot{u}^1 = 0, \qquad \lambda\, g_{21}\, \dot{u}^2 + \mu\, c_{22}\, \dot{u}^1 = 0.$$

By eliminating $\dot{u}^1$, $\dot{u}^2$ (which do not both vanish), one will get:

$$\lambda^2\, g + \mu^2\, c = 0,$$

so, due to (129):

$$\frac{\lambda}{\mu} = \pm 1.$$

The so-called *contact complexes* $\mathfrak{c} \pm \mathfrak{c}'$ are characterized as follows: They contain the restricted pencils of lines $\{x \mid w\}$ and $\{x' \mid w'\}$ (no. **79**); moreover, $\mathfrak{h}$, $\mathfrak{h}'$, as well as $\mathfrak{k}$, $\mathfrak{k}'$, are reciprocal (no. **22**) relative to $\mathfrak{c} - \mathfrak{c}'$, and $\mathfrak{h}$, $\mathfrak{k}'$, as well as $\mathfrak{h}'$, $\mathfrak{k}$, are reciprocal relative to $\mathfrak{c} + \mathfrak{c}'$.

Proof: The bundles of complexes $\lambda^0\mathfrak{p} + \lambda^1\mathfrak{p}_1 + \lambda^2\mathfrak{p}_2$ and $\mu^0\,\mathfrak{p} + \mu^1\mathfrak{c} + \mu^2\,\mathfrak{c}'$ are conjugate, and determine the restricted pencils of lines $\{x \mid w\}$, $\{x' \mid w'\}$ ($\{x \mid w'\}$, $\{x' \mid w\}$, resp.) as conjugate quadrics. Thus, each of the complexes that contain the pencils of lines $\{x \mid w\}$, $\{x' \mid w'\}$ will be representable by $\mathfrak{a} = \mu^0\,\mathfrak{p} + \mu^1\mathfrak{c} + \mu^2\,\mathfrak{c}'$. From (30), the line $\mathfrak{r}$ that is reciprocal to $\mathfrak{h} = \mathfrak{p} + \mathfrak{c}$ relative to $\mu^0\,\mathfrak{p} + \mu^1\mathfrak{c} + \mu^2\,\mathfrak{c}'$ will be given by

$$\mathfrak{r} = 2(\mathfrak{a}\mathfrak{h})\,\mathfrak{a} - (\mathfrak{a}\mathfrak{a})\,\mathfrak{h} = 2\mu^2\,\mathfrak{a} - 2\mu^1\mu^2\,\mathfrak{h} = 2\mu^2\,\{(\mu^0 - \mu^1)\,\mathfrak{p} + \mu^2\,\mathfrak{c}')\},$$

and the line that is reciprocal to $\mathfrak{k} = \mathfrak{p} - \mathfrak{c}$ will be given by:

$$\mathfrak{s} = -\,2\mu^2\,\{(\mu^0 + \mu^1)\,\mathfrak{p} + \mu^2\,\mathfrak{c}')\}.$$

The demand that $\mathfrak{r} = \text{const} \cdot \mathfrak{h}'$, $\mathfrak{s} = \text{const} \cdot \mathfrak{k}'$ yields $\mu^0 = 0$, $\mu^1 = -\,\mu^2$, so $\mathfrak{a} = \mathfrak{c} - \mathfrak{c}'$.

84. Fundamental system. We denote the lines of intersection of the osculating planes of the longitudinal curves (= focal planes w, w') with the osculating planes of the lateral curves by $\mathfrak{q}$, $\mathfrak{q}'$, so:

$$\left.\begin{matrix}\mathfrak{q}\\ \mathfrak{q}'\end{matrix}\right\} = \text{line of intersection of } \left\{\begin{matrix}w\\ w'\end{matrix}\right. \text{ with the lateral curve osculating plane of } \left\{\begin{matrix}\Phi\\ \Phi'\end{matrix}\right. .$$

Since the focal planes w, w' are, at the same time, contact planes of Φ', Φ, the line $\mathfrak{q}$ ($\mathfrak{q}'$, resp.) will cut the lateral tangent $\mathfrak{c}'$ ($\mathfrak{c}$, resp.). As a result, $\mathfrak{q}$, $\mathfrak{q}'$ are skew, just like $\mathfrak{c}$, $\mathfrak{c}'$ (Fig. 25). The lines $\mathfrak{p}$, $\mathfrak{c}$, $\mathfrak{c}'$, $\mathfrak{q}$, $\mathfrak{q}'$, together with a sixth line $\mathfrak{z}$, are the edges of a tetrahedron. We take the six singular six-vectors $\mathfrak{p}$, $\mathfrak{c}$, $\mathfrak{c}'$, $\mathfrak{q}$, $\mathfrak{q}'$, $\mathfrak{z}$ to be a *fundamental system*. From the product table:

	$\mathfrak{p}$	$\mathfrak{c}$	$\mathfrak{q}$	$\mathfrak{z}$	$\mathfrak{c}'$	$\mathfrak{q}'$
$\mathfrak{p}$	0	0	0	1	0	0
$\mathfrak{c}$	0	0	0	0	1	0
$\mathfrak{q}$	0	0	0	0	0	1
$\mathfrak{z}$	1	0	0	0	0	0
$\mathfrak{c}'$	0	1	0	0	0	0
$\mathfrak{q}'$	0	0	1	0	0	0

one will get:

$$|\,\mathfrak{p}, \mathfrak{c}, \mathfrak{q}, \mathfrak{z}, \mathfrak{c}', \mathfrak{q}'\,|^2 = 1,$$

so the six-vectors of the fundamental system will be linearly independent.

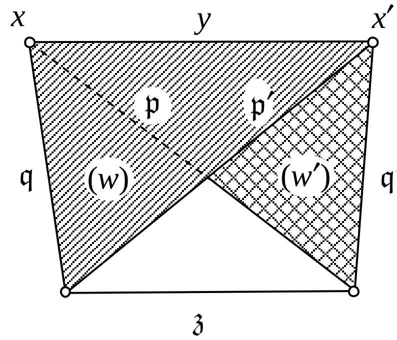

Figure 25.

The normalization of $\mathfrak{z}$, $\mathfrak{q}$, $\mathfrak{q}'$, will be accomplished by the requirements:

$$\mathfrak{p}\mathfrak{z} = 1, \qquad \mathfrak{q}\mathfrak{q}' = 1, \qquad \mathfrak{p}_1\mathfrak{q}' = \pm\, \mathfrak{p}_2\,\mathfrak{q}, \tag{132}$$

the first two of which were already included in the product table. The requirements (132) can be fulfilled, since $\mathfrak{p}$, $\mathfrak{z}$; $\mathfrak{q}$, $\mathfrak{q}'$; $\mathfrak{p}_1$, $\mathfrak{q}$; $\mathfrak{p}_2$, $\mathfrak{q}$ are pairs of skew lines; $\mathfrak{p}_1$ ($\mathfrak{p}_2$, resp.) is a line of the pencil $\{x \mid w\}$ ($\{x' \mid w'\}$, resp.) (Fig. 25). When $(\mathfrak{p}_1\mathfrak{q}')(\mathfrak{p}_2\,\mathfrak{q}) > 0$ (< 0, resp.), one must take the upper (lower, resp.) sign in the last equation of (132). $\mathfrak{z}$ is determined uniquely by (132), while a sign remains arbitrary with $\mathfrak{q}$, but $\mathfrak{q}'$ will again be fixed uniquely.

Determination of the lines $\mathfrak{q}$, $\mathfrak{q}'$: The osculating planes of the longitudinal curves will be spanned by $\mathfrak{p}$, $\mathfrak{p}_1$ ($\mathfrak{p}$, $\mathfrak{p}_2$, resp.), while the osculating planes to the lateral curves will be spanned by $\mathfrak{c}$, $\mathfrak{c}_2$ ($\mathfrak{c}'$, $\mathfrak{c}_1$, resp.). As a result, the lines $\mathfrak{q}$, $\mathfrak{q}'$ can be represented by:

$$\begin{aligned} \mathfrak{q} &= \lambda\mathfrak{p} + \mu\,\mathfrak{p}_1 = \rho\,\mathfrak{c} + \sigma\,\mathfrak{c}_2, \\ \mathfrak{q}' &= \lambda'\mathfrak{p} + \mu'\mathfrak{p}_1 = \rho'\mathfrak{c}' + \sigma'\mathfrak{c}_1', \end{aligned} \qquad (\mu, \mu', \sigma, \sigma' \neq 0). \tag{133}$$

85. Differential equations. In order to calculate the coefficients of the differential equations, one must next determine the scalar products of the six-vectors of the fundamental system with their invariant derivatives:

	$\mathfrak{p}_1$	$\mathfrak{p}_2$	$\mathfrak{c}_1$	$\mathfrak{c}_2$	$\mathfrak{q}_1$	$\mathfrak{q}_2$	$\mathfrak{z}_1$	$\mathfrak{z}_2$	$\mathfrak{c}_1'$	$\mathfrak{c}_2'$	$\mathfrak{q}_1'$	$\mathfrak{q}_2'$
$\mathfrak{p}$	0	0	0	0	0	$-B'$	$-A$	$-A'$	0	0	$-B$	0
$\mathfrak{c}$	0	0	0	0	D	0	$-U$	0	$-N$	N'	0	C'
$\mathfrak{q}$	0	B'	$-D$	0	0	0	$-E$	$-F'$	$-C$	0	R	$-R'$
$\mathfrak{z}$	A	A'	U	0	E	F'	0	0	0	U'	F	E'

$\mathfrak{c}'$	0	0	N	$-N'$	C	0	0	$-U'$	0	0	0	D'
$\mathfrak{q}'$	B	0	0	$-C'$	$-R$	R'	$-F$	$-E'$	0	$-D'$	0	0

Equations (121) and the invariant derivatives of the scalar products of the fundamental systems that are summarized in no. **84** were employed in the construction of this table.

The equations:

$$\mathfrak{c}_2\,\mathfrak{q} = \mathfrak{c}_2\,\mathfrak{z} = \mathfrak{c}'_1\,\mathfrak{q}' = \mathfrak{c}'_1\,\mathfrak{z} = 0$$

are obtained from (133), while the equations:

$$\mathfrak{q}\,\mathfrak{c}'_2 = \mathfrak{q}'\,\mathfrak{c}_1 = 0$$

will be obtained from (133), when one considers the relations:

$$\mathfrak{p}\,\mathfrak{c}'_2 = \mathfrak{p}\,\mathfrak{c}_1 = 0 \qquad\qquad \mathfrak{p}_1\,\mathfrak{c}'_2 = \mathfrak{p}_2\,\mathfrak{c}_1 = 0,$$

which follow from (121). The product tables of no. **84** and no. **85** immediately yield the *differential equations:*

$$(134)\qquad \begin{aligned}
\mathfrak{p}_1 &= A\,\mathfrak{p} + B\,\mathfrak{q}, & \mathfrak{p}_2 &= A'\,\mathfrak{p} + B'\mathfrak{q}',\\
\mathfrak{z}_1 &= -F\mathfrak{q} - E\,\mathfrak{q}' - U\,\mathfrak{c}' - A\,\mathfrak{z}, & \mathfrak{z}_2 &= -F'\,\mathfrak{q}' - E'\,\mathfrak{q} - U'\,\mathfrak{c} - A'\,\mathfrak{z},\\
\mathfrak{c}_1 &= U\mathfrak{p} - D\,\mathfrak{q}' + N\,\mathfrak{c}, & \mathfrak{c}'_2 &= U'\,\mathfrak{p} - D'\,\mathfrak{q} + N'\,\mathfrak{c}',\\
\mathfrak{c}'_1 &= -C\,\mathfrak{q}' - N\,\mathfrak{c}', & \mathfrak{c}_2 &= -C'\,\mathfrak{q} - N'\,\mathfrak{c},\\
\mathfrak{q}_1 &= E\mathfrak{p} - R\,\mathfrak{q} + C\,\mathfrak{c} + B\,\mathfrak{c}', & \mathfrak{q}'_2 &= E'\,\mathfrak{p} - R'\,\mathfrak{q}' + C'\,\mathfrak{c}' + D'\,\mathfrak{c},\\
\mathfrak{q}'_1 &= F\mathfrak{p} + R\,\mathfrak{q}' - B\,\mathfrak{z}, & \mathfrak{q}_2 &= F'\,\mathfrak{p} + R'\,\mathfrak{q} - B'\,\mathfrak{z}\,.
\end{aligned}$$

The auxiliary conditions:

$$(135)\qquad B = \pm B' \neq 0, \qquad BB' = CD' = C'D, \qquad BD = \pm B'D' = 1$$

exist between A, A', B, B', C, C', D, D' in this.

Proof: The first equation follows from (132). The second equations can be derived from (129): In torse parameters, one has:

$$(\mathfrak{p}_1\mathfrak{p}_2)^2 = (\mathfrak{p}_1\mathfrak{c}_1)\,(\mathfrak{p}_2\mathfrak{c}_2) = (\mathfrak{p}_1\mathfrak{c}'_1)(\mathfrak{p}_2\mathfrak{c}'_2)\,,$$

so in invariant derivatives, one will have:

$$(\mathfrak{p}_1\mathfrak{p}_2)^2 = (\mathfrak{p}_1\mathfrak{c}_1)\,(\mathfrak{p}_2\mathfrak{c}_2) = (\mathfrak{p}_1\mathfrak{c}'_1)(\mathfrak{p}_2\mathfrak{c}'_2)\,;$$

the assertion then follows when one considers the differential equations. The last equations are obtained from the requirements (130): They are equivalent to:

$$\mathfrak{p}_1\mathfrak{c}_1 = -1, \qquad \mathfrak{p}_2\mathfrak{c}'_2 = \mp 1,$$

which once more leads to the assertion with the help of the differential equations.

One observes that the sign of the first equation in (135) is chosen according to (132), while the sign of the last equation is chosen according to (130). One further observes that, from (135), none of the six invariants B, B', C, C', D, D' can vanish and that B', C, C', D, D' are determined from B, up to sign.

The invariant derivatives 1 and 2 are switched under *correlations*, and accordingly for the invariant coefficients of the differential equations, in a corresponding way.

86. Integrability conditions. The coefficients of the differential equations (134) are not only coupled by the auxiliary conditions (135), but must also satisfy the following *integrability conditions* (no. **77**):

$$(136)\qquad \begin{aligned}
&B_2 + B\,(R' + q) = A'\,B,\\
&B'_1 + B'(R + q') = A\,B',\\
&A_2 + BF' + Aq = A'_1 + B'F + A'q',\\
&E_2 + E\,(A' + q) - RF' + DU' = F'_1 + F'(A + q') + R'E,\\
&E'_1 + E'(A + q') - R'F + D'U = F_2 + F\,(A' + q) + RE',\\
&U_2 + U\,(N' + A' + q) = E'D - E'C,\\
&U'_1 + U'(N + A + q') = ED' - E'C,\\
&R_2 + R'_1 + BF' + B'F + qR + q'\,R' + CC' + DD' = 0,\\
&D_2 + D\,(N' - R' + q) = B'U,\\
&D'_1 + D'(N - R + q') = BU',\\
&C_2 = C'\,(N + R - q'),\\
&C'_1 = C'\,(N + R - q'),\\
&N_2 + N'_1 + qN + q'N' = DD' - CC'.
\end{aligned}$$

Proof: The equations are obtained by applying the integrability conditions (113) to the six six-vectors of the fundamental system. For example, one has:

$$\begin{aligned}
\mathfrak{c}_{12} &= U\,\mathfrak{p}_2 - D\,\mathfrak{q}'_2 + N\,\mathfrak{c}_2 + U_2\,\mathfrak{p} - D_2\,\mathfrak{q}' + N_2\,\mathfrak{c}\\
&= (A'U - DE' + U_2)\,\mathfrak{p} + (-DD' - NN' + N_2)\,\mathfrak{c} - NC'\,\mathfrak{q} - DC'\,\mathfrak{c}' + (DR' - D_2 + UB')\,\mathfrak{q}',
\end{aligned}$$

$$\begin{aligned}
\mathfrak{c}_{21} &= -C'\,\mathfrak{q}_1 - N'\,\mathfrak{c}_2 - C'_1\,\mathfrak{q} - N'_1\,\mathfrak{c}\\
&= (-EC' - UN')\,\mathfrak{p} - (CC' + NN' + N'_1)\,\mathfrak{c} + (C'R - C'_1)\,\mathfrak{q} - DC'\,\mathfrak{c}' + DN'\,\mathfrak{q}'.
\end{aligned}$$

If one substitutes these expressions, as well as the linear combinations of $\mathfrak{c}_1$ and $\mathfrak{c}_2$ that were given in (134), into (113):

$$\mathfrak{c}_{12} + q\mathfrak{c}_1 - \mathfrak{c}_{12} - q'\mathfrak{c}_2 \equiv 0$$

and arranges the result in terms of the six-vectors of the fundamental system then the coefficients of these six-vectors must vanish; that will yield four of the relations (136).

From the main theorem of no. **77**, one has:

The hyperbolic line systems with two negatively-curved focal surfaces are characterized projectively by the tensors m^i, m'^k, *as well as the invariant coefficients* A, A', *etc., of the differential equations.*

One can calculate all of the remaining invariants for $B^2 \neq 1$ ($B^2 = 1$ characterizes the W-systems that were treated in § 29) as follows with the help of the auxiliary conditions (135) and the integrability conditions (136):

The invariants q, q' are determined from (114) by way of m^i, m'^k; B', C, C', D, D' can be calculated from B, up to sign, using (135), and are non-zero, just like B. The integrability conditions ($136_{11, 12, 9, 10, 1, 2, 3, 8}$) then yield the invariants R, R', U, U', A, A', F, F' as functions of the B, N, N', q, q', and invariant derivatives of the functions. Finally, one solve ($136_{6, 7}$) for E, E' under the assumption that:

$$CC' - DD' \neq 0, \qquad \text{so, from (135),} \qquad B^2 \neq 1.$$

The integrability conditions ($136_{4, 5, 13}$) are not employed in this.

Analogous to no. **65**, we have the *existence theorem* here:

Let the tensors m^i, m'^k, *and the invariants in the table of no.* **85** *be combined in such that way that the auxiliary conditions* (135) *and the integrability conditions* (136) *are fulfilled, but they are otherwise given as arbitrary functions of* u^1, u^2. *There is always a hyperbolic line system with two negatively-curved focal surfaces then for which the invariant derivatives are defined by* m^i, m'^k, *and whose differential equations include the given functions as coefficients.*

The proof proceeds analogously to no. **65**: One must show that $\mathfrak{p}$, $\mathfrak{c}$, $\mathfrak{c}'$, $\mathfrak{q}$, $\mathfrak{q}'$, $\mathfrak{z}$ fulfill the table of scalar products of no. **84**, the normalization requirements (129), (130), (132), and the conditions (121) and (133) for all u^1, u^2. The over-determined system of partial differential equations (134) will appear in place of a system of ordinary differential equations. From known theorems, however, that system will also have precisely one system of solutions $\mathfrak{p}$, $\mathfrak{c}$, $\mathfrak{c}'$, $\mathfrak{q}$, $\mathfrak{q}'$, $\mathfrak{z}$ for the given prescribed initial conditions when the integrability conditions (136) are assumed to be satisfied.

§ 24. Contact structures for parabolic line systems.

87. Focal surface or focal curve. Since $g = 0$, (103) will have a double root $\dot{u}^i = m^i(t)$. The line system will then contain *only one* family of torses, so *one* torse will go through each line of the system $\mathfrak{p}$ (u^1, u^2). *One* focal point x and *one* focal plane w will belong to any line $\mathfrak{p}$ (focal points and focal planes are defined as in no. **79**). The pencil of lines $\{x \mid w\}$ is given by:

$$\lambda\, \mathfrak{p} + \mu\, (m^1\, \mathfrak{p}_1 + m^2\, \mathfrak{p}_2).$$

Depending upon whether the torses are families of tangents of curves or cones (pencils of lines, resp.), the focal points will generate a focal surface or focal curve, respectively.

I) *Line system with focal surface* Φ.

The curves of regression of the torses define a family of curves Σ on the focal surface Φ. These curves must be identical with the one family of principal tangent curves of Φ. Otherwise, there would be a conjugate family of curves that is different from Σ, and the lines $\mathfrak{p}$ of the system along that conjugate family of curves would generate a second family of torses, contrary to assumption. Since the curves of regressions are principal tangent curves, their osculating planes (= focal planes w) will coincide with the contact planes of the focal surface. Thus, as in no. **80**, we will have: *The focal surface* Φ *can also be defined to be the envelope of the focal planes w.*

The focal surface Φ *is always negatively-curved.*

Proof: Since the focal surface Φ possesses real principal tangent curves, it can only be negatively-curved or developable. If Φ were a developable surface then principal tangents would coincide with the generators, and they would define only a one-parameter set, instead of a line system.

II) *Line systems with focal curves* Φ.

All torses of the line system are pencils of lines. Indeed, if they were tangent families of curves that these curves would provide a focal surface, contrary to assumption. However, if they were cones then, by correlation, one would obtain a parabolic line system with a developable focal surface, hence, one with negative curvature, in contradiction to I). Moreover, the planes of the pencils of lines (= focal planes) are contact planes of the focal curve; otherwise, the envelope of the planes of the pencil would be a focal surface, and the line system would be hyperbolic (type II of no. **80**). One then has:

A parabolic line system with a focal curve Φ *consists of the tangents to any surface* Ω *along a curve* γ *of this surface. The points (contact planes, resp.) of* Ω *along* γ *are the focal points (focal planes, resp.), and the pencils of tangents of* Ω *along* γ *are the torses of the line system.*

The tangents to the focal curve γ and the tangents to the surface Ω that are conjugate to them play a distinguished role in this: A second torse goes through each of these lines, apart from a pencil of tangents, namely, the torse of the focal curve tangents (the torse of the conjugate tangents, resp.).

The linear, parabolic, line systems (no. **33**) belong to the parabolic line systems with rectilinear focal curves.

The parabolic line systems with focal surfaces (focal curves, resp.) will be mapped to other such systems *under a correlation*; the tangents to a principal tangent curve of the focal surface again go to the tangents to a principal tangent curve.

Two parabolic line systems of type I are determined by any negatively-curved surface Φ that is not a ruled surface, namely, the two systems of principal tangents of Φ. *Our theory of parabolic line systems is then, at the same time, a projective differential geometry of the negatively-curved surfaces* Φ (cf., § 26.).

We clarify the parabolic line systems with focal surfaces by *juxtaposing them with torsal models:*

Torsal models	**Line systems**
Rectangle nets with planar quadrilaterals.	Negatively-curved focal surfaces with principal tangent nets.
The two sets of lines that are defined by the two families of polygons.	The two parabolic line systems that are defined by the two principal tangents.
The lines of a polygon again go to the lines of a polygon under a correlation.	The tangents along a principal tangent curve again go to the tangents along a principal tangent curve under a correlation.

88. Principal tangent parameter. Invariant derivatives. *From now on, we would like to exclude the parabolic line systems with focal curves from consideration, and assume, in addition, that the focal surface* Φ *is not a ruled surface;* we already treated the principal tangents of the ruled surface in no. **49**. Discrete, rectilinear, principal tangent curves and discrete points with indeterminate osculating planes to the principal tangent curves shall remain unconsidered.

With that restriction, two parabolic line systems $\mathfrak{p}(u^1, u^2)$ and $\mathfrak{p}(u^1, u^2)$ will belong to the focal surface Φ; it proves to be convenient to investigate these two *confocal line systems* at the same time.

The torses of the $\mathfrak{p}$-system ($\mathfrak{p}'$-system, resp.) will be determined by the double roots:

$$\dot{u}^i = m^i \qquad \text{and} \qquad \dot{u}^k = m'^k \qquad \text{with} \quad \begin{vmatrix} m^1 & m^2 \\ m'^1 & m'^2 \end{vmatrix} \neq 0$$

of

(137) $$g_{ik}\dot{u}^i\dot{u}^k = 0 \qquad (g'_{ik}\dot{u}^i\dot{u}^k = 0, \text{ resp.}) \qquad (g = g' = 0).$$

We refer to the parameters that give:

$$\text{the principal tangent curves of the } \begin{cases} \mathfrak{p}-\text{system} \\ \mathfrak{p}'-\text{system} \end{cases} \text{for } \begin{cases} u^2 = \text{const.} \\ u^1 = \text{const.} \end{cases}$$

as the *principal tangent parameters* (Fig. 26). In principal tangent parameters, one has:

(138) $$\begin{aligned} &g_{11} = g_{12} = 0, \qquad &&g_{22} \neq 0, \qquad &&m^1 \neq 0, \qquad &&m^2 = 0, \\ &g'_{12} = g'_{22} = 0, \qquad &&g'_{11} \neq 0, \qquad &&m'^1 = 0, \qquad &&m'^2 \neq 0. \end{aligned}$$

The tensors m^i (m'^k, resp.) are next determined, but only up to a factor $\sigma(u^1, u^2)$ [$\sigma'(u^1, u^2)$, resp.]. We dispense with the indeterminacy, up to a sign, by the *normalization requirements:*

(139) $$g_{ik}\, m'^i\, m'^k = +\,1, \qquad g'_{ik}\, m^i\, m^k = -\,1.$$

The expressions $g_{ik}\, m'^i\, m'^k$ and $g'_{ik}\, m^i\, m^k$ do not vanish. We will prove later on that they must have different signs in no. **89**.

We shall now once more define the *invariant derivatives:*

$$\varphi_1 = m^i\, \varphi_{\,i}, \qquad \varphi_2 = m'^i\, \varphi_{\,i}$$

with the help of the contravariant tensors m^i, m'^k, subject to the normalization of $\mathfrak{p}$ and $\mathfrak{p}'$, and thus for the principal tangent parameters, in particular, one will have:

(140) $$\varphi_1 = m^1\, \varphi_{\,1}, \qquad \varphi_2 = m'^2\, \varphi_{\,2}\,.$$

It follows from this that:

The invariant derivative 1 (2, *resp.*) *means a differentiation along a principal tangent curve of the* $\mathfrak{p}$-*system* ($\mathfrak{p}'$-*system, resp.*).

The covariant tensors m_i, m'_k that are defined by (111) satisfy the relations:

(141) $$g_{ik} = m'_i\, m'_k\,, \qquad g'_{ik} = -\, m_i\, m_k\,,$$

on the basis of (137) and (139).

89. Osculating hyperboloid. One has the identities:

$$(142)\qquad \begin{aligned} \mathfrak{p}\mathfrak{p}_1 &= \mathfrak{p}\mathfrak{p}_2 = \mathfrak{p}'\mathfrak{p}'_1 = \mathfrak{p}'\mathfrak{p}'_2 = 0, \\ \mathfrak{p}\mathfrak{p}'_1 &= \mathfrak{p}\mathfrak{p}'_2 = \mathfrak{p}'\mathfrak{p}_1 = \mathfrak{p}'\mathfrak{p}_2 = 0 \end{aligned}$$

for the six-vectors $\mathfrak{p}$ and $\mathfrak{p}'$, as well as:

$$(143)\qquad \mathfrak{p}_1 = \lambda\,\mathfrak{p} + \mu\,\mathfrak{p}', \qquad \mathfrak{p}'_2 = \lambda'\,\mathfrak{p}' + \mu'\mathfrak{p} \qquad (\mu \neq 0,\ \mu' \neq 0).$$

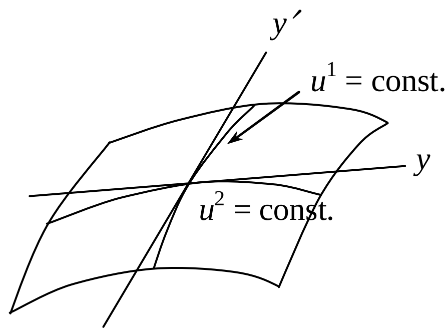

Figure 26.

Proof: In torse parameters, one has:

$$\mathfrak{p}_1 = m^1\mathfrak{p}_1, \qquad \mathfrak{p}_2 = m'^{\,2}\,\mathfrak{p}_2, \qquad \mathfrak{p}'_1 = m^1\,\mathfrak{p}'_1, \qquad \mathfrak{p}'_2 = m'^2\mathfrak{p}'_2 .$$

The osculating planes of the curves u^2 = const. (u^1 = const.) will then be spanned by the lines $\mathfrak{p}$ and $\mathfrak{p}_1$ ($\mathfrak{p}$ and $\mathfrak{p}'_2$, resp.) (Fig. 26). Since these osculating planes are, at the same time, contact planes of the focal surface, the pencil of lines $\rho\,\mathfrak{p} + \sigma\,\mathfrak{p}_1$ ($\rho'\,\mathfrak{p}' + \sigma'\,\mathfrak{p}'_2$, resp.) must also contain the line $\mathfrak{p}$ ($\mathfrak{p}'$, resp.), from which, (143) will follow. $\mu \neq 0$ and $\mu' \neq 0$ in this; for μ = (μ = 0, resp.), $\mathfrak{p}$, $\mathfrak{p}_1$ ($\mathfrak{p}'$, $\mathfrak{p}'_2$, resp.), would be linearly dependent, in contradiction to our discussion in no. **71**.

The first row of (142) is obtained immediately from (102), while the second one can be confirmed as follows: We assume rectangular coordinate throughout and denote the position vector of the focal point by $\mathfrak{X}(u^1, u^2)$. One then has:

$$\mathfrak{p} = \{\mathfrak{X}_1 \mid \mathfrak{X} \times \mathfrak{X}_1\}, \qquad \mathfrak{p}' = \{\mathfrak{X}_2 \mid \mathfrak{X} \times \mathfrak{X}_2\},$$

so, e.g.:

$$\mathfrak{p}_2 = \left\{ \frac{\partial^2 \mathfrak{X}}{\partial u^1 \partial u^2} \,\middle|\, \mathfrak{X}_2 \times \mathfrak{X}_1 + \mathfrak{X} \times \frac{\partial^2 \mathfrak{X}}{\partial u^1 \partial u^2} \right\},$$

and from this:

$$\mathfrak{p}'\mathfrak{p}_2 = \left\langle \frac{\partial^2 \mathfrak{X}}{\partial u^1 \partial u^2}, \mathfrak{X}, \mathfrak{X}_2 \right\rangle + \left\langle \mathfrak{X}, \frac{\partial^2 \mathfrak{X}}{\partial u^1 \partial u^2}, \mathfrak{X}_2 \right\rangle \equiv 0.$$

The relations (142) *are valid for the tangents to any curve net, while* (143) *is characteristic of the principal tangent nets.*

The principal tangents to a family along a principal tangent curve of the other family define ruled families that we would briefly like to call *principal tangent ruled families.* They are given by $\mathfrak{p}(u^1, u^2)$ with u^1 = const. [$\mathfrak{p}'(u^1, u^2)$ with u^2 = const., resp.] in principal tangent parameters; it follows from $\mathfrak{p}_2\mathfrak{p}_2 = g_{22} \neq 0$ ($\mathfrak{p}'_1\mathfrak{p}'_1 = g'_{11} \neq 0$, resp.) [cf., (138)] that these families of lines are not torses. We now state:

The two ruled families of principal tangents that contact a point of the focal surface will have the same osculating quadric there (no. **49***); i.e., the first osculating quadric of the one ruled family of principal tangents is the second osculating quadric for the other ruled family of principal tangents. The hyperboloid that is spanned by both osculating quadrics shall be called the osculating hyperboloid; it was also called that by S. Lie.*

Proof: We merely have to show that the two bundles of complexes:

$$\lambda\,\mathfrak{p} + \mu\,\mathfrak{p}_2 + \nu\,\mathfrak{p}_{22} \qquad \text{and} \qquad \lambda'\,\mathfrak{p}' + \mu'\,\mathfrak{p}'_1 + \nu'\,\mathfrak{p}'_{11}$$

are conjugates, so the scalar products of the $\mathfrak{p}$, $\mathfrak{p}_2$, $\mathfrak{p}_{22}$ with $\mathfrak{p}', \mathfrak{p}'_1, \mathfrak{p}'_{11}$ vanish. That can be confirmed with no difficulty with the help of (142) and (143).

From (34), one has:

$$|\,\mathfrak{p}, \mathfrak{p}_2, \mathfrak{p}_{22}, \mathfrak{p}', \mathfrak{p}'_1, \mathfrak{p}'_{11}\,|^2 = -\,(\mathfrak{p}_2\ \mathfrak{p}_2)^3\ (\mathfrak{p}'_1\,\mathfrak{p}'_1)^3 = -\,(m'^{\,i}\ m'^{\,k}\ \mathfrak{p}_i\ \mathfrak{p}_k)^3\ (m^r\ m^s\ \mathfrak{p}'_r\ \mathfrak{p}'_s)^3;$$

as a result, the non-vanishing expressions:

$$m'^i\ m'^k\ \mathfrak{p}_i\ \mathfrak{p}_k = m'^i\ m'^k\ g_{ik} \qquad \text{and} \qquad m^r\ m^s\ \mathfrak{p}'_r\ \mathfrak{p}'_s = m^r\ m^s\ g'_{rs}$$

have, in fact, different signs [cf., (139)].

In § 25, we will learn about even more contact structures (osculating complexes $\mathfrak{k}$, $\mathfrak{k}'$, etc.).

§ 25. Differential equations of parabolic line systems.

90. Normalization of $\mathfrak{p}$, $\mathfrak{p}'$. Fundamental systems.

Normalization of $\mathfrak{p}$, $\mathfrak{p}'$:

In (143), we assumed that $\mu \neq 0$, $\mu' \neq 0$. We would now like to normalize $\mathfrak{p}$ and $\mathfrak{p}'$ in a way that is invariant under unity transformations by the demand that $\mu = \mu' = 1$; in place of (143), one will then have:

(144) $$\mathfrak{p}_1 = \lambda\,\mathfrak{p} + \mathfrak{p}', \qquad \mathfrak{p}'_2 = \lambda'\,\mathfrak{p}' + \mathfrak{p}.$$

One gets the normalized $\hat{\mathfrak{p}} = \rho\,\mathfrak{p}$, $\hat{\mathfrak{p}}' = \rho'^2\mathfrak{p}'$ from the un-normalized $\mathfrak{p}$, $\mathfrak{p}'$ as follows: From:

$$\hat{g}_{ik} = \hat{\mathfrak{p}}_i\,\hat{\mathfrak{p}}_k = \rho^2\,g_{ik}, \qquad \hat{g}'_{ik} = \rho'^2\,g'_{ik},$$

and according to (139), one will get:

$$\hat{m}^i = \pm\frac{1}{\rho'}m^i, \qquad \hat{m}'^i = \pm\frac{1}{\rho'}m'^i.$$

Due to (143), one then gets:

$$\hat{\mathfrak{p}}_1 = \hat{m}^i\hat{\mathfrak{p}}_i = \pm\frac{\rho}{\rho'}m^i\mathfrak{p}_i \pm\frac{\rho_i}{\rho'}m^i\mathfrak{p} = \pm\frac{\rho}{\rho'}\mathfrak{p}_1 \pm\frac{\rho_i}{\rho'}m^i\mathfrak{p} = (\ldots)\,\hat{\mathfrak{p}} \pm \frac{\rho\mu}{\rho'^2}\hat{\mathfrak{p}}',$$

$$\hat{\mathfrak{p}}_2 = \hat{m}'^i\hat{\mathfrak{p}}'_i = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = (\ldots)\,\hat{\mathfrak{p}}' \pm \frac{\rho'\mu'}{\rho^2}\hat{\mathfrak{p}},$$

and therefore:

$$\frac{\rho\mu}{\rho'^2} = \pm 1, \qquad \frac{\rho'\mu'}{\rho^2} = \pm 1.$$

ρ, ρ' are determined up to sign, in that way.

We take our *fundamental system* ([1]) to be $\mathfrak{p}$, $\mathfrak{p}'$, along with the four normalized six-vectors:

(145)
$$\mathfrak{t} = \mathfrak{p}_2 - 2\lambda'\,\mathfrak{p}, \qquad \mathfrak{q} = \mathfrak{t}_2 + \tfrac{1}{2}(\mathfrak{t}_2\;\mathfrak{t}_2)\,\mathfrak{p},$$
$$\mathfrak{t}' = \mathfrak{p}'_1 - 2\lambda\,\mathfrak{p}', \qquad \mathfrak{q}' = \mathfrak{t}'_1 + \tfrac{1}{2}(\mathfrak{t}'_1\;\mathfrak{t}'_1)\,\mathfrak{p}',$$

([1]) For this and the following numbers, cf., **W. Blaschke** and **G. Thomsen:** *Differentialgeometrie III*, § 90, *et seq.*

in which the invariants λ, λ' are defined by (144).

From the product table:

	$\mathfrak{p}$	$\mathfrak{t}$	$\mathfrak{q}$	$\mathfrak{p}'$	$\mathfrak{t}'$	$\mathfrak{q}'$
$\mathfrak{p}$	0	0	– 1	0	0	0
$\mathfrak{t}$	0	1	0	0	0	0
$\mathfrak{q}$	– 1	0	0	0	0	0
$\mathfrak{p}'$	0	0	0	0	0	1
$\mathfrak{t}'$	0	0	0	0	– 1	0
$\mathfrak{q}'$	0	0	0	1	0	0

and since:

$$|\,\mathfrak{p}, \mathfrak{t}, \mathfrak{q}, \mathfrak{p}', \mathfrak{t}, \mathfrak{q}'\,|^2 = -D_6\,(\mathfrak{p}, \mathfrak{t}, \mathfrak{q}, \mathfrak{p}', \mathfrak{t}, \mathfrak{q}') = 1,$$

the linear independent of the six-vectors of the fundamental system. On the basis of (138) and (139), the scalar products that are summarized in the table yield the following relations:

(146) $\qquad \mathfrak{p}_1\mathfrak{p}_1 = \mathfrak{p}_1\mathfrak{p}_2 = 0, \qquad \mathfrak{p}_1'\,\mathfrak{p}_2' = \mathfrak{p}_2'\,\mathfrak{p}_2' = 0, \qquad \mathfrak{p}_2\mathfrak{p}_2 = 1, \qquad \mathfrak{p}_1'\,\mathfrak{p}_1' = -1,$

along with the conjugacy of the bundles of complexes $\mathfrak{p}$, $\mathfrak{p}_2$, $\mathfrak{p}_{22}$ and $\mathfrak{p}'$, $\mathfrak{p}_1'$, $\mathfrak{p}_{11}'$. One then has, for example:

$$\mathfrak{p}\mathfrak{q} = \mathfrak{p}\mathfrak{t}_2 = \mathfrak{p}\mathfrak{p}_{22} = -\mathfrak{p}_2\,\mathfrak{p}_2 = -1,$$
$$\mathfrak{q}\mathfrak{q} = \mathfrak{t}_2\mathfrak{t}_2 + (\mathfrak{t}_2\mathfrak{t}_2)(\mathfrak{p}\mathfrak{t}_2) = \mathfrak{t}_2\mathfrak{t}_2 - \mathfrak{t}_2\,\mathfrak{t}_2 = 0.$$

We refer to the non-singular, distinctly-wound complexes $\mathfrak{t}$, $\mathfrak{t}'$ as *osculating complexes;* their geometric meaning will be given no. **91**.

From (145), the bundle of complexes that is spanned by $\mathfrak{p}$, $\mathfrak{p}_2$, $\mathfrak{p}_{22}$ ($\mathfrak{p}'$, $\mathfrak{p}_1'$, $\mathfrak{p}_{11}'$, resp.) is identical to the one that is spanned by $\mathfrak{p}$, $\mathfrak{t}$, $\mathfrak{q}$ ($\mathfrak{p}'$, $\mathfrak{t}'$, $\mathfrak{q}'$, resp.). The quadrics of the osculating hyperboloid (no. **89**) are then the axes of the singular complexes that are contained in the bundle:

$$\rho_1\,\mathfrak{p} + \rho_2\,\mathfrak{t} + \rho_3\,\mathfrak{q} \qquad (\rho_1'\,\mathfrak{p}' + \rho_2'\,\mathfrak{t}' + \rho_3'\,\mathfrak{q}', \text{ resp.}).$$

The line $\mathfrak{q}$ ($\mathfrak{q}'$, *resp.*) *can be characterized as the only line of the first (second, resp.) quadric of the osculating hyperboloid that is different from* $\mathfrak{p}$ ($\mathfrak{p}'$, *resp.*) *that simultaneously belong to the osculating complex* $\mathfrak{t}$ ($\mathfrak{t}'$, *resp.*).

Proof: Any line of the first osculating quadric is conjugate to $\mathfrak{p}'$, $\mathfrak{t}'$, $\mathfrak{q}'$, and also conjugate to $\mathfrak{t}$, if it is to belong to the complex $\mathfrak{t}$. Since $D_4\,(\mathfrak{p}', \mathfrak{q}', \mathfrak{t}', \mathfrak{t}) = 1$, and the bush of complexes that is generated by $\mathfrak{p}'$, $\mathfrak{q}'$, $\mathfrak{t}'$, $\mathfrak{t}$ is hyperbolic, it will possess precisely two conjugate lines, and from the product table, they will be the lines $\mathfrak{p}$, $\mathfrak{q}$.

91. Differential equations and integrability conditions.

	$\mathfrak{p}_1$	$\mathfrak{p}_2$	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{q}_1$	$\mathfrak{q}_2$	$\mathfrak{p}'_1$	$\mathfrak{p}'_2$	$\mathfrak{t}'_1$	$\mathfrak{t}'_2$	$\mathfrak{q}'_1$	$\mathfrak{q}'_2$
$\mathfrak{p}$	0	0	0	-1	λ	$2\lambda'$	0	0	0	0	-1	0
$\mathfrak{t}$	0	1	0	0	H	W	0	0	0	0	$-C$	0
$\mathfrak{q}$	$-\lambda$	$-2\lambda'$	$-H$	$-W$	0	0	0	-1	0	C'	$-R$	R'
$\mathfrak{p}'$	0	0	0	0	0	1	0	0	$+1$	0	-2λ	$-\lambda'$
$\mathfrak{t}'$	0	0	0	0	0	$-C'$	-1	0	0	0	$-W'$	$-H'$
$\mathfrak{q}'$	1	0	C	0	R	$-R'$	2λ	λ'	W'	H'	0	0

yield the *differential equations:*

$$
\begin{array}{llll}
\mathfrak{p}_1 &= q'\mathfrak{p} + \mathfrak{p}', & \mathfrak{p}'_2 &= q\mathfrak{p}' + \mathfrak{p}, \\
\mathfrak{p}'_1 &= 2q'\mathfrak{p}' + \mathfrak{t}', & \mathfrak{p}_2 &= 2q\mathfrak{p} + \mathfrak{t}, \\
\mathfrak{t}_1 &= H\,\mathfrak{p}, & \mathfrak{t}'_2 &= H'\mathfrak{p}', \\
\mathfrak{t}'_1 &= W'\mathfrak{p}' + \mathfrak{q}', & \mathfrak{t}_1 &= W\,\mathfrak{p} + \mathfrak{q}, \\
\mathfrak{q}_1 &= H\,\mathfrak{t} - q'\mathfrak{q} - W\mathfrak{p}', & \mathfrak{q}'_2 &= H'\mathfrak{t}' - q\mathfrak{q}' - W'\mathfrak{p}, \\
\mathfrak{q}'_1 &= W'\mathfrak{t}' - 2q'\mathfrak{q}' - W\,\mathfrak{p} + \mathfrak{q}, & \mathfrak{q}_2 &= W\,\mathfrak{t} - 2q\mathfrak{q} - W'\mathfrak{p}' + \mathfrak{q}'.
\end{array}
\tag{147}
$$

The first row of the *integrability conditions* is already employed in this, which can be derived just like in no. **86**:

$$
\begin{array}{l}
C = C' = 0, \quad \lambda = q', \qquad \lambda' = q, \qquad W = -R, \qquad W' = R', \\
H = \mathfrak{q}'_2 - 2q_1 - qq' + 1, \\
H' = q_1 - 2\mathfrak{q}'_2 - qq' + 1, \\
W_1 + 2q'W = H_2 + 3qH, \\
W'_2 + 2q'W = H'_1 + 3q'H', \\
W_2 - W'_1 + 4(qW - q'W') = 0.
\end{array}
\tag{148}
$$

On the basis of the differential equations, one gets the following characterization of the osculating complexes $\mathfrak{t}$, $\mathfrak{t}'$:

The osculating complex $\mathfrak{t}$ ($\mathfrak{t}'$, *resp.*) *contains four "successive" pencils of tangents to the focal surface* ([1]) *along the principal tangent curve* $u^2 = const.$ ($u^1 = const.$) *(cf. Fig, 26); i.e.,* $\mathfrak{t}$ ($\mathfrak{t}'$, *resp.*) *is conjugate to the complexes:*

([1]) Cf., the analogous theorem on the osculating complexes of torses (viz., space curves) in no. **63**.

$$\rho\,\mathfrak{p} + \sigma\,\mathfrak{p}', \;\; (\rho\,\mathfrak{p} + \sigma\,\mathfrak{p}')_1, \ldots, (\rho\,\mathfrak{p} + \sigma\,\mathfrak{p}')_{111}$$
$$[\rho\,\mathfrak{p} + \sigma\,\mathfrak{p}', (\rho\,\mathfrak{p} + \sigma\,\mathfrak{p}')_2, \ldots, (\rho\,\mathfrak{p} + \sigma\,\mathfrak{p}')_{222}, \text{ resp.}].$$

Proof: If follows from (147), after a simple calculation, that:

$$\mathfrak{t}\mathfrak{p} = \mathfrak{t}\mathfrak{p} = \mathfrak{t}\mathfrak{p}_1 = \mathfrak{t}\mathfrak{p}'_1 = \ldots = \mathfrak{t}\mathfrak{p}_{111} = \mathfrak{t}\mathfrak{p}'_{111} = 0.$$

The osculating complex $\mathfrak{t}$ is determined with that; the five complexes:

$$\mathfrak{p}, \mathfrak{p}', \mathfrak{p}_{11} = \mathfrak{t}' + \ldots, \qquad \mathfrak{p}_{111} = \mathfrak{q}' + \ldots, \; \mathfrak{p}'_{111} = \mathfrak{q} + \ldots$$

are then linearly independent.

The invariants H, H' and their invariant derivatives can be calculated with the integrability conditions ($148_{2,3}$) from q, q', and thus, from the tensors m^i, m'^k. One then obtains the following theorem, which is analogous to the one in no. **86**:

The parabolic line systems with focal surfaces that are not ruled surfaces are characterized projectively by the tensors m^i, m'^k, as well as by the invariants W, W'.

Moreover, one has the following *existence theorem*, which is analogous to one in no. **86**:

Let the tensors m^i, m'^k, and the invariants W, W' be such that ($148_{4,5,8}$) *will be fulfilled, but otherwise given as arbitrary functions of u^1, u^2. There is always a parabolic line system with a non-ruled surface as its focal surface then, for which the invariant derivatives are defined in terms of m^i, m'^k, and whose differential equations include W, W' as coefficients.*

The invariants W, W' can be determined uniquely from the tensors m^i, m'^k, except in two cases:

If ($148_{4,5,6}$) are satisfied for W, W' then the equations:

$$w_1 + 2q'\,w = 0, \qquad w'_2 + 2qw' = 0, \qquad w_2 - w'_1 + 4(qw - q'w') = 0 \tag{149}$$

will be true for any further pair of functions $W + w$, $W' + w'$ that likewise fulfills those conditions.

W, W' are thus established uniquely by ($148_{4,5,6}$) if and only if (149) admits only the trivial solution $w = w' = 0$. A non-trivial solution of (149) will exist in the following two *exceptional cases*, and only in them:

a) $w \neq 0$, $w' = 0$ (or vice versa): It follows from:

$$w_1 + 2q'w = 0, \qquad w_2 + 4qw = 0,$$

and the integrability condition:

$$w_{12} + qw_1 = w_{21} + q'w,$$

that:

$$w\,(2q_1 - q_2' + qq') = 0,$$

so, with consideration given to $w \neq 0$ and (142_2):

$$H = 1. \tag{150}$$

b) $w \neq 0$, $w' \neq 0$: In principal tangent parameters, it follows from $(149_{1,2})$ that:

$$m_2' \frac{\partial w}{\partial u^1} + 2w \frac{\partial m_2'}{\partial u^1} = 0, \qquad m_2 \frac{\partial w'}{\partial u^2} + 2w' \frac{\partial m_2}{\partial u^2} = 0,$$

and from this:

$$(m_2')^2\, w = \varphi(u^2), \qquad (m_1)^2\, w' = \psi(u^1).$$

One can arrange that $\varphi \equiv 1$, $\psi \equiv 1$ through a suitable transformation:

$$u^1 = u^1(\bar{u}^1), \quad u^2 = u^2(\bar{u}^2) \qquad \left(\text{so } \bar{m}_2' = \frac{du^2}{d\bar{u}^2} m_2', \bar{m}_1 = \frac{du^1}{d\bar{u}^1} m_1\right)$$

of the principal tangent parameters into themselves, so:

$$w = \frac{1}{(m_2')^2}, \qquad w' = \frac{1}{(m_1)^2}.$$

By substituting this into (149_3), one will then obtain:

$$\frac{\partial}{\partial u^1}\left(\frac{m_2' m_2'}{m_1}\right) - \frac{\partial}{\partial u^2}\left(\frac{m_1 m_1}{m_2'}\right) = 0. \tag{151}$$

We will encounter an example of the exceptional case (150) in no. **97**, while the exceptional case (151) will be characterized geometrically in no. **104**.

§ 26. Projective differential geometry of negatively-curved surfaces.

92. Darboux curves and Segre curves. In no. **87**, we proved that the theory of parabolic line systems can, at the same time, be regarded as the projective differential geometry of negatively-curved surfaces. In this paragraph, we will summarize some fundamental concepts of the *projective theory of surfaces.* As we agreed in no. **88**, we shall exclude the ruled surfaces from consideration.

If the principal tangents along a surface curve define two ruled families with the surface curve as its nodal curve (no. **51**) then will one call that curve a *Darboux curve* of the surface. The definition is obviously projectively-invariant.

Any negatively-curved surface contains a family of Darboux curves; they are given by:

(152) $$(m_i + m_i')\dot{u}^i = 0.$$

Proof: For a Darboux curve $u^i = u^i(t)$, from (60), one has the equations:

$$\mathfrak{k}\ \mathfrak{p} = 0, \qquad \mathfrak{k}(\mathfrak{p}\dot{u}^1 + \mathfrak{p}_2\,\dot{u}^2) = 0,$$
$$\mathfrak{k}\left(\sum \frac{\partial^2 \mathfrak{p}}{\partial u^i\,\partial u^k}\dot{u}^i\,\dot{u}^k + \cdots\right) = 0,$$
$$\mathfrak{t}\left(\sum \frac{\partial^3 \mathfrak{p}}{\partial u^i\,\partial u^k \partial u^l}\dot{u}^i\,\dot{u}^k \dot{u}^l + \cdots\right) = 0,$$

in which $\mathfrak{k}$ is a line of the pencil that is spanned by $\mathfrak{p}$ and $\mathfrak{p}'$. The ellipses suggest sums of the first (second, resp.) partial derivatives of the six-vector $\mathfrak{p}$. Upon going to the principal tangent parameters and finally to invariant derivatives, it will follow that:

$$\mathfrak{k}\mathfrak{p} = 0, \qquad \mathfrak{k}(m_1\mathfrak{p}_1\dot{u}^1 + m_2'\mathfrak{p}_2\dot{u}^2) = 0,$$

$$\mathfrak{k}(m_1m_1\mathfrak{p}_{11}\dot{u}^1\dot{u}^1 + 2m_1m_2'\mathfrak{p}_{12}\dot{u}^1\dot{u}^2 + m_2'm_2'\mathfrak{p}_{22}\dot{u}^2\dot{u}^2 + \cdots) = 0,$$

$$\mathfrak{k}(m_1m_1m_1\mathfrak{p}_{111}\dot{u}^1\dot{u}^1\dot{u}^1 + 3m_1m_1m_2'\mathfrak{p}_{112}\dot{u}^1\dot{u}^1\dot{u}^2 + 3m_1m_2'm_2'\mathfrak{p}_{122}\dot{u}^1\dot{u}^2\dot{u}^2 + m_2'm_2'm_2'\mathfrak{p}_{222}\dot{u}^2\dot{u}^2\dot{u}^2 + \cdots) = 0.$$

On the basis of the differential equations (147), the first two equations will be fulfilled for all lines $\mathfrak{k} = \rho\,\mathfrak{p} + \sigma\,\mathfrak{p}'$, the third equation yields $\mathfrak{k} = \mathfrak{p}'$, and the last one will give:

$$(m_1\dot{u}^1)^3 + (m_2'\dot{u}^2)^3 = 0,$$

after a brief calculation, so

$$m_1\dot{u}^1 + m_2'\dot{u}^2 = 0.$$

In principal tangent parameters, from (138), one will have:

$$m_1\dot{u}^1 + m_2'\dot{u}^2 = (m_i + m_i')\dot{u}^i,$$

from which, the validity of (152) for arbitrary parameters will follow, due to the parameter invariance of $m_i\dot{u}^i$ and $m_i'\dot{u}^i$.

If a surface curve is a nodal curve for one of the two ruled families that are generated by the principal tangents then it must also be a nodal curve for the other ruled family; the relationship between ruled families and nodal families is reciprocal (no. **51**).

The curves that conjugate to the family of Darboux curves are called *Segre curves*. The Darboux tangents will be transformed into Segre tangents under correlations, and conversely. By contrast, the points of a Darboux curve (Segre curves, resp.) will once more go to contact planes along a Darboux curve (Segre curve, resp.). From (152), one will get the differential equation for the Segre curves as:

$$(m_i - m_i')\,\dot{u}^i = 0. \tag{153}$$

From (5), the directions of advance (152) and (153) will separated harmonically by the directions of advance $m_i\dot{u}^i = 0$ and $m_i'\dot{u}^i = 0$ of the principal tangent curves, so they will provide a conjugate curve net.

The *Darboux* and *Segre tangents* are given by the six-vectors:

$$\mathfrak{d}\ (\mathfrak{s}, \text{resp.}) = \mathfrak{p} \pm \mathfrak{p}'. \tag{154}$$

Proof: For the torses of the line system $\mathfrak{p} \pm \mathfrak{p}'$, one has:

$$(\mathfrak{p}_i \pm \mathfrak{p}_i')(\mathfrak{p}_k \pm \mathfrak{p}_k')\,\dot{u}^i\,\dot{u}^k = 0.$$

By reverting to the principal tangent parameters, it will follow that:

$$\mathfrak{p}_2\,\mathfrak{p}_2\dot{u}^2\,\dot{u}^2 + \mathfrak{p}_1'\,\mathfrak{p}_1'\,\dot{u}^1\,\dot{u}^1 = 0,$$

and from that:

$$(m_2'\,\dot{u}^2)^2 - (m_1'\,\dot{u}^1)^2 = 0.$$

However, the Darboux and Segre directions will be determined in principal tangent parameters from (152) and (153) in that way.

Since the demands (60) will be fulfilled for any tangent $\mathfrak{k}$ of the hyperboloid for hyperboloidal quadrics, one will have the theorem:

If the principal tangents along a surface curve generate a hyperboloidal quadric then the surface curve will be a Darboux curve. Thus, for a negatively-curved surface of revolution, the latitudes will be the Darboux curves and the meridians will be the Segre curves.

93. Enveloping structure of the osculating hyperboloid. We state in advance the following general theorem:

Let two line systems or [by the linear dependency of $\mathfrak{g}$, $\mathfrak{g}_1$, $\mathfrak{g}_2$ ($\mathfrak{g}'$, $\mathfrak{g}_1'$, $\mathfrak{g}_2'$, resp.)] families of lines be given by $\mathfrak{g}(u^1, u^2)$, $\mathfrak{g}'(u^1, u^2)$. Any two corresponding lines $\mathfrak{g}(u^1, u^2)$ and $\mathfrak{g}'(u^1, u^2)$ shall intersect and determine a point of intersection $x(u^1, u^2)$ and a connecting plane $w'(u^1, u^2)$. One then has:

$$\mathfrak{g}\mathfrak{g}' = \mathfrak{g}\mathfrak{g}_1' = \mathfrak{g}\mathfrak{g}_2' = \mathfrak{g}'\mathfrak{g}_1 = \mathfrak{g}'\mathfrak{g}_2 = 0 \tag{155}$$

for the necessary and sufficient conditions that the two sets of lines should have the geometric locus Φ of the points x as their common contact structure, i.e.:

Either both sets of lines are tangents to a surface Φ *or they are secants to a curve* Φ (w' *= contact plane of* Φ) *or they all go through a fixed point* Φ.

Proof: We have already proved that (155) is necessary for the contact surface Φ with (142), whose second row is equivalent to (155); the proof proceeds analogously when Φ degenerates to a curve or a fixed point. One recognizes that (155) is sufficient as follows: For any family of lines $\mathfrak{g}(u^1(t), u^2(t))$, one has, from (155):

$$\mathfrak{g}'\mathfrak{g} = \mathfrak{g}'\dot{\mathfrak{g}} = 0;$$

i.e.: $\mathfrak{g}'$ is tangent, and therefore the plane w' of $\mathfrak{g}$ and $\mathfrak{g}'$ will be the contact plane of the line surface that spanned by the $\mathfrak{g}(t)$. As a result, the geometric locus of points x that lie on the line surface will contact the planes w', so those planes will be the contact planes of the set of points x.

With these preparations, we then go on to the determination of the *enveloping structure to the osculating hyperboloid* (no. **89**):

The lines of the two quadric of the osculating hyperboloid can be represented in the homogeneous parameters ρ, τ (ρ', τ', resp.) by:

$$(156) \qquad \mathfrak{g} = \rho\,\mathfrak{p} \pm \sqrt{2\rho\tau}\;\mathfrak{t} + \tau\,\mathfrak{q}, \qquad \mathfrak{g}' = \rho'\mathfrak{p}' \pm \sqrt{2\rho\tau}\;\mathfrak{t}' + \tau'\mathfrak{q}',$$

as would follow from the general Ansatz:

$$\mathfrak{g} = \rho\,\mathfrak{p} + \sigma\,\mathfrak{t} + \tau\,\mathfrak{q}, \qquad \mathfrak{g}' = \rho'\mathfrak{p}' + \sigma'\mathfrak{t}' + \tau'\mathfrak{q}',$$

with $\mathfrak{g}\mathfrak{g} = \mathfrak{g}'\mathfrak{g}' = 0$.

We would now like to separate the sets of lines $\mathfrak{g}(u^1, u^2)$ and $\mathfrak{g}'(u^1, u^2)$ that have a common contact structure from (156) with a suitable choice of ρ, τ and ρ', τ', resp. That contact structure is then, at the same time, the enveloping structure of the system of surfaces that is generated by the osculating hyperboloids:

In order for (156) to represent a set of lines with a common contact structure, it is necessary and sufficient that:

$$(157) \qquad a) \quad \tau = \tau' = 0 \quad \text{or} \quad b) \quad \rho - \tau W = \rho' - \tau' W' = 0.$$

When one replaces the partial derivatives in (156) with invariant ones, with the help of the differential equations (147), it will then follow by an application of (155) to (156) that:

$$\tau(\rho' - \tau' W') = \tau'(\rho - \tau W) = 0,$$

and $\rho = \tau = 0$ or $\rho' = \tau' = 0$ are excluded.

(157a) gives the initial surface that is spanned by $\mathfrak{p}$, $\mathfrak{p}'$ as the trivial enveloping structure of the osculating hyperboloid. In regard to the remaining enveloping structure that is given by (157b), one has the following *projectively-invariant case distinction:*

$$\left.\begin{array}{lr}\alpha)\ W > 0, W' > 0: & 4\\ \beta)\ W = 0, W' > 0, \text{ or vice versa}: & 2\\ \gamma)\ W = W' = 0: & 1\\ \delta)\ W < 0 \text{ or } W' < 0: & 0\end{array}\right\} \begin{array}{l}\textit{enveloping structure of the osculating hyperboloid,}\\ \textit{along with the initial surface.}\end{array}$$

Discussion of the case α): The osculating hyperboloids contact the enveloping structure that is different from the initial surface at four points, namely, the four intersection points of the lines:

$$(158)\qquad \begin{array}{ll}\mathfrak{g} \ = W\mathfrak{p} + \left|\sqrt{2W}\right| \mathfrak{t} + \mathfrak{q}, & \mathfrak{g}' \ = W'\mathfrak{p}' + \left|\sqrt{2W'}\right| \mathfrak{t}' + \mathfrak{q}',\\[2ex] \mathfrak{g}^* = W\mathfrak{p} - \left|\sqrt{2W}\right| \mathfrak{t} + \mathfrak{q}, & \mathfrak{g}^{*\prime} = W'\mathfrak{p}' - \left|\sqrt{2W'}\right| \mathfrak{t}' + \mathfrak{q}'.\end{array}$$

The skew rectangle that is generated by these four lines is called the *Demoulin rectangle*.

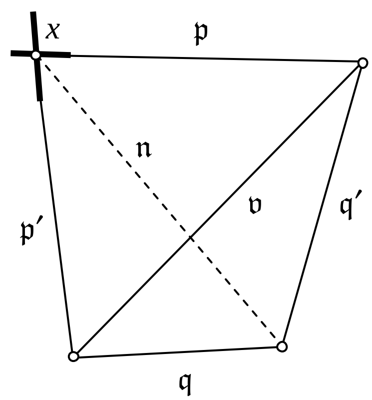

Figure 27.

94. Projective normals and tangents. As the product table of no. **90** shows, the lines $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{q}$, $\mathfrak{q}'$ define a skew rectangle. The line $\mathfrak{n}$ through the point of intersection of $\mathfrak{p}$, $\mathfrak{p}'$ and $\mathfrak{q}$, $\mathfrak{q}'$ shall be called the *projective surface normal*, while the connecting line $\mathfrak{v}$ of the intersection points of $\mathfrak{p}$, $\mathfrak{q}'$ and $\mathfrak{p}'$, $\mathfrak{q}$ shall be called the *projective surface tangent;* $\mathfrak{n}$ goes through the point of the surface and does not lie in the contact plane, while $\mathfrak{v}$ lies in the contact plane and does go through the point of the surface. $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{q}$, $\mathfrak{q}'$, $\mathfrak{n}$, $\mathfrak{v}$ yield the six edges of a tetrahedron (Fig. 27) and can be taken to be a fundamental system in place of $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{q}$, $\mathfrak{q}'$, $\mathfrak{t}$, $\mathfrak{t}'$. Since:

$$(\mathfrak{t} \pm \mathfrak{t}')\,\mathfrak{p} = (\mathfrak{t} \pm \mathfrak{t}')\,\mathfrak{p}' = (\mathfrak{t} \pm \mathfrak{t}')\,\mathfrak{q} = (\mathfrak{t} \pm \mathfrak{t}')\,\mathfrak{q}' = 0,$$

the normal $\mathfrak{n}$ and tangent $\mathfrak{v}$ will be given by:

$$\mathfrak{n}\ (\mathfrak{v}, \text{resp.}) = \mathfrak{t} \pm \mathfrak{t}'. \tag{159}$$

They will be switched with each other under correlations, and for that reason, they cannot be distinguished in (159), *per se.*

In the case α) of no. **93**, $\mathfrak{n}$ and $\mathfrak{v}$ can also be characterized in the following way, as **E. J. Wilczinski** did:

$\mathfrak{n}$ ($\mathfrak{v}$, resp.) *is the line that goes through the point of the surface (lies in the contact plane, resp.) that cuts both diagonals of the Demoulin rectangle.*

Proof: From (158), since:

$$\mathfrak{g}\mathfrak{f} = \mathfrak{g}'\ \mathfrak{f} = \mathfrak{g}^*\mathfrak{f} = \mathfrak{g}^{*\prime}\mathfrak{f} = 0, \qquad \mathfrak{g}\mathfrak{f}' = \mathfrak{g}'\ \mathfrak{f}' = \mathfrak{g}^*\mathfrak{f}' = \mathfrak{g}^{*\prime}\mathfrak{f}' = 0,$$

the diagonals $\mathfrak{f}$, $\mathfrak{f}'$ can be represented by:

$$\mathfrak{f}\ (\mathfrak{f}', \text{resp.}) = W'(W\,\mathfrak{p} - \mathfrak{q}) \pm W\,(W'\mathfrak{p}' - \mathfrak{q}'),$$

from which, the assertion will follow, since:

$$\mathfrak{n}\mathfrak{p} = \mathfrak{n}\mathfrak{p}' = \mathfrak{n}\mathfrak{f} = \mathfrak{n}\mathfrak{f}' = 0, \qquad \mathfrak{v}\mathfrak{p} = \mathfrak{v}\mathfrak{p}' = \mathfrak{v}\mathfrak{f} = \mathfrak{v}\mathfrak{f}' = 0.$$

CHAPTER IV

Special line systems

§ 27. Self-projective line systems.

95. Characterization of self-projective line systems. Just as we considered self-projective families of lines in nos. **66** to **70**, we now investigate *self-projective line systems*, which are characterized by having constant coefficients in the differential equations (134) [(147), resp.].

A self-projective line system will be transformed into itself by at least a *two-parameter, continuous group of collineations*. Upon restricting to a suitable u^1, u^2 domain, any two lines of the system can then be associated with each other collinearly.

Proof: The invariants q, q' can be expressed in terms of the constant coefficients in the differential equations with the help of the integrability conditions ($136_{1,2}$) [(148_1), resp.], and are therefore themselves constant. From no. **78**, one can then normalize the torsion parameters (principal tangent parameters, resp.) in such a way that m_1, m_2' become the functions that were given in (115), (116), or (117). Those functions of u^1, u^2 correspond to each under the two-parameter substitution groups of the same functions of the $\bar{u}^1$, $\bar{u}^2$:

$$(160)\qquad \begin{array}{lll} \alpha) & u^1 = a\bar{u}^1 + b, & u^2 = a\bar{u}^2 - b, \quad \text{etc.} \\ \beta) & u^1 = \bar{u}^1 + a, & u^2 = e^{-q'a}\,\bar{u}^2 + b, \quad \text{etc.} \\ \gamma) & u^1 = \bar{u}^1 + a, & u^2 = \bar{u}^2 + b, \end{array}$$

with the arbitrary constants a¸ b.

In that way, a line system with constant coefficients in its differential equations will be transformed projectively into itself by the substitutions $\left.\begin{array}{c}(160\alpha)\\(160\beta)\\(160\gamma)\end{array}\right\}$ for $\left\{\begin{array}{ll} q \neq 0, & q' \neq 0 \\ q = 0, & q' \neq 0 \\ q = q' = 0 & \end{array}\right.$. Since these projective maps define a continuous group, they will be collineations.

Discussion of the group of collineations (160α, β, γ):

For (160γ), there exists a one-parameter subgroup of collineations ($a = 0$, b variable or a variable, $b = 0$) *in each case*, under which, either the torsion (principal tangent, resp.) curves u^1 = const or u^2 = const. will remain fixed individually.

For (160β), there exists such a one-parameter subgroup ($a = 0$, b variable) only for the torsion (principal tangent, resp.) curves u^1 = const., and for (160α) there is no such subgroup.

The focal surfaces of the self-projective line systems are *self-projective surfaces;* they will be transformed into themselves under at least a two-parameter group of collineations, and by restricting to a suitable domain, any point of a surface will go to any other point of the surface. For example, the ruled surfaces that were treated in no. **66** are self-projective surfaces. Any self-projective surface is the focal surface of infinitely-many self-projective line systems.

For (160γ), the longitude curves of both focal surfaces (the principal tangent curves of both families, resp.) are transformed into themselves individually, and will then be *self-projective curves.* For (160β), the longitude curves of a focal surface (the principal tangent curves of a family, resp.) are self-projective. Along with the longitude curves of the one focal surface, the latitude curves of the other one will likewise be self-projective.

96. Self-projective hyperbolic line systems. We restrict ourselves here to (160γ); i.e., to the assumption that $q = q' = 0$. It then follows from (135), and since $N_2 = N_1' = 0$, from (136_{13}), that

$$B = \pm 1, \qquad B' = \pm 1.$$

We choose the plus sign and obtain the relations (135), when taken with the plus signs:

$$B = B' = C = C' = D = D' = 1,$$

and from the integrability conditions ($136_{11,12,9,10,1,2,3,8}$):

$$\begin{array}{llll} R = -N, & R' = -N', & U = 2N', & U' = 2N, \\ A = -N, & A' = -N', & F = F' = -1. & \end{array}$$

Equations ($136_{6,7}$) provide only the one condition:

$$E - E' = 0,$$

while equations ($136_{4,5}$) yield nothing new. With that, in total, the three constants N, N', $E = E' = \rho$ will remain arbitrary.

The special case of $N = N' = 0$:

The differential equations (134) specialize to:

$$(161) \qquad \begin{array}{ll} \mathfrak{p}_1 = \mathfrak{q}, & \mathfrak{p}_2 = \mathfrak{q}', \\ \mathfrak{z}_1 = \mathfrak{q} - \rho\,\mathfrak{q}', & \mathfrak{z}_2 = \mathfrak{q}' - \rho\mathfrak{q}, \\ \mathfrak{c}_1 = -\mathfrak{q}', & \mathfrak{c}_2' = -\mathfrak{q}, \\ \mathfrak{c}_1' = -\mathfrak{q}', & \mathfrak{c}_2 = -\mathfrak{q}, \\ \mathfrak{q}_1 = \rho\,\mathfrak{q} + \mathfrak{c} + \mathfrak{c}', & \mathfrak{q}_2' = \rho\,\mathfrak{p} + \mathfrak{c} + \mathfrak{c}', \\ \mathfrak{q}_1' = -\mathfrak{p} - \mathfrak{z}, & \mathfrak{q}_2 = -\mathfrak{p} - \mathfrak{z}\,. \end{array}$$

When we replace the tensors m^i, m'^k as in (117), the invariant derivatives 1, 2 will go to the usual partial derivatives $\partial / \partial u^1$, $\partial / \partial u^2$ with respect to the torsion parameters. In order to discuss the focal surfaces, we transform the parameters:

$$u^1 = \bar{u}^1 + \bar{u}^2, \qquad u^2 = \bar{u}^1 - \bar{u}^2.$$

It will then follow from (161) for the principal tangents that:

$$\begin{aligned} &\mathfrak{h} = \mathfrak{p} + \mathfrak{c}, \quad &&\mathfrak{h}_{\bar{u}^1} = 0; \quad &&\mathfrak{k} = \mathfrak{p} - \mathfrak{c}, \quad &&\mathfrak{k}_{\bar{u}^2} = 0; \\ &\mathfrak{h}' = \mathfrak{p} + \mathfrak{c}', \quad &&\mathfrak{h}'_{\bar{u}^1} = 0; \quad &&\mathfrak{k}' = \mathfrak{p} - \mathfrak{c}', \quad &&\mathfrak{k}'_{\bar{u}^2} = 0. \end{aligned}$$

The principal tangents $\mathfrak{h}$ ($\mathfrak{h}'$, resp.) of the focal surface Φ (Φ', resp.) are then fixed along the curves $\bar{u}^2$ = const., while the principal tangents $\mathfrak{k}$ ($\mathfrak{k}'$, resp.) are fixed along the curves $\bar{u}^1$ = const.; i.e.:

The focal surfaces Φ, Φ' are hyperboloids.

$$(162) \qquad \begin{aligned} &\mathfrak{h}_{\bar{u}^2\bar{u}^2\bar{u}^2} - 4(\rho+2)\mathfrak{h}_{\bar{u}^2} = 0, \qquad &&\mathfrak{k}_{\bar{u}^1\bar{u}^1\bar{u}^1} - 4(\rho-2)\mathfrak{k}_{\bar{u}^1} = 0, \\ &\mathfrak{h}'_{\bar{u}^2\bar{u}^2\bar{u}^2} - 4(\rho+2)\mathfrak{h}'_{\bar{u}^2} = 0, \qquad &&\mathfrak{k}'_{\bar{u}^1\bar{u}^1\bar{u}^1} - 4(\rho-2)\mathfrak{k}'_{\bar{u}^1} = 0. \end{aligned}$$

With the initial conditions:

$$\begin{aligned} &\mathfrak{p} = 1 \mid 0 \mid 0 \mid 0 \mid 0 \mid 0, \quad &&\mathfrak{q} = 0 \mid 1 \mid 0 \mid 0 \mid 0 \mid 0, \quad &&\mathfrak{c} = 0 \mid 0 \mid 1 \mid 0 \mid 0 \mid 0, \\ &\mathfrak{z} = 0 \mid 0 \mid 0 \mid 1 \mid 0 \mid 0, \quad &&\mathfrak{q}' = 0 \mid 0 \mid 0 \mid 0 \mid 1 \mid 0, \quad &&\mathfrak{c}' = 0 \mid 0 \mid 0 \mid 0 \mid 0 \mid 1, \end{aligned}$$

one will obtain the normalized line coordinates of $\mathfrak{h}$ and $\mathfrak{h}'$ by integrating (162):

$$\begin{aligned} h_1 &= \frac{1}{\rho+2} + \frac{\rho+1}{\rho+2}\cos(\kappa\bar{u}^2), \qquad & h_4 &= -\frac{1}{\rho+2} + \frac{1}{\rho+2}\cos(\kappa\bar{u}^2), \\ h_2 &= \frac{2}{\kappa}\sin(\kappa\bar{u}^2), & h_5 &= -\frac{2}{\kappa}\sin(\kappa\bar{u}^2), \\ h_3 &= \frac{\rho+1}{\rho+2} + \frac{1}{\rho+2}\cos(\kappa\bar{u}^2), & h_6 &= -\frac{1}{\rho+2} + \frac{1}{\rho+2}\cos(\kappa\bar{u}^2), \end{aligned}$$

and

$$\begin{aligned} h'_1 &= \frac{1}{\rho+2} + \frac{\rho+1}{\rho+2}\cos(\kappa\bar{u}^2), \qquad & h_4 &= -\frac{1}{\rho+2} + \frac{1}{\rho+2}\cos(\kappa\bar{u}^2), \\ h'_2 &= \frac{2}{\kappa}\sin(\kappa\bar{u}^2), & h_5 &= -\frac{2}{\kappa}\sin(\kappa\bar{u}^2), \\ h'_3 &= -\frac{1}{\rho+2} + \frac{1}{\rho+2}\cos(\kappa\bar{u}^2), & h_6 &= \frac{\rho+1}{\rho+2} + \frac{1}{\rho+2}\cos(\kappa\bar{u}^2), \end{aligned}$$

in which we have set:

$$\kappa = 2\sqrt{\rho+2}\,,$$

to abbreviate. With that, we will have:

$$(163)\qquad \begin{aligned} &\Phi)\quad h_2 + h_5 = 0, && h_4 - h_6 = 0, && h_1 - h_3 - \rho\, h_4 = 0; \\ &\Phi')\quad h_2' + h_5' = 0, && h_4' - h_3' = 0, && h_1' - h_6' - \rho h_4' = 0. \end{aligned}$$

It follows from these representations that the two focal surfaces Φ, Φ' have the skew quadrilateral of lines:

$$\begin{aligned} \mathfrak{h} &= \rho + 1 \mid \pm\sqrt{\rho+2} \mid 1 \mid \mp\sqrt{\rho+2} \mid 1, \\ \mathfrak{k} &= \rho - 1 \mid \pm\sqrt{\rho-2} \mid -1 \mid \mp\sqrt{\rho-2} \mid 1 \end{aligned}$$

in common when we assume that $\rho > 2$.

The parametric representation:

$$\begin{aligned} \mathfrak{h} &= \tfrac{1}{2}(u^2 + \rho) \mid u \mid \tfrac{1}{2}(u^2 - \rho) \mid 1 \mid -u \mid 1, \\ \mathfrak{h}' &= \tfrac{1}{2}(u^2 + \rho) \mid u \mid \quad 1 \quad \mid 1 \mid -u \mid \tfrac{1}{2}(u^2 - \rho) \end{aligned}$$

can be derived from (163). In point coordinates, that will yield:

$$\begin{aligned} &\Phi)\quad \sigma x_1 = 1 + 2uv, && \sigma x_2 = 2v, && \sigma x_3 = -1, && \sigma x_4 = u + (u^2 - \rho)\, v; \\ &\Phi')\quad \sigma x_1' = u + (u^2 + \rho)\, v, && \sigma x_2' = 1 + 2uv, && \sigma x_3' = 2v, && \sigma x_4' = 1. \end{aligned}$$

By eliminating u, v, one will obtain the equations:

$$\Phi)\quad x_1^2 - x_3^2 - 2x_2 x_4 - \rho x_2^2 = 0, \qquad\qquad \Phi')\quad x_2'^2 - x_4'^2 - 2x_1' x_3' + \rho x_3'^2 = 0\,.$$

They can go to:

$$\Phi)\quad y_1\, y_3 + \lambda\, y_2\, y_4 = 0, \qquad\qquad \Phi')\quad \lambda y_1' y_3' + y_2' y_4' = 0$$

by a suitable choice of coordinate transformation. The two focal surfaces and the two singular second-order surfaces $y_1\, y_3 = 0$, $y_2\, y_4 = 0$, through the skew quadrilateral that is common to the focal surfaces, then yield the double ratio:

$$d = \frac{\lambda_1}{\lambda_2} \qquad \left(\frac{\lambda_2}{\lambda_1} = \frac{\rho \mp \sqrt{\rho^2 - 4}}{\rho \pm \sqrt{\rho^2 - 4}}, \text{resp.}\right) \qquad (d > 0).$$

It then follows that:

Any two hyperboloids that have a quadrilateral in common and produce a positive double ratio d with the two second-order surfaces that are determined by the skew quadrilateral will be focal surfaces of a self-projective line system (161) *with* $\rho > 2$.

97. Self-projective parabolic line systems. From the integrability conditions (148), one has the following possibilities for constant invariants:

$$
\begin{array}{llll}
 & H = H' = \rho < 1, & q = q' = \sqrt{1-\rho}, & W = W' = \tfrac{3}{2}\rho, \\
(164) \qquad \text{or} & H = H' = 1, & q = q' = 0, & W = \rho,\ W' = \rho', \\
\text{or} & H = H' = 0, & q = \rho,\ q' = \dfrac{1}{\rho}, & W = W' = 0.
\end{array}
$$

The special case of $H = H' = 1$, $q = q' = W = W' = 0$:

The differential equations (147) yield:

$$
\begin{array}{llllll}
\mathfrak{p}_1 = \mathfrak{p}', & \mathfrak{p}'_1 = \mathfrak{t}', & \mathfrak{t}_1 = \mathfrak{p}, & \mathfrak{t}'_1 = \mathfrak{q}', & \mathfrak{q}_1 = \mathfrak{t}, & \mathfrak{q}'_1 = \mathfrak{q}, \\
\mathfrak{p}'_2 = \mathfrak{p}, & \mathfrak{p}_2 = \mathfrak{t}, & \mathfrak{t}'_2 = \mathfrak{p}', & \mathfrak{t}_2 = \mathfrak{q}, & \mathfrak{q}'_2 = \mathfrak{t}', & \mathfrak{q}_2 = \mathfrak{q}'.
\end{array}
$$

One more, by the special Ansatz (117), the invariant derivatives 1, 2 will go to the usual partial derivatives $\partial / \partial u^1$, $\partial / \partial u^2$ with the principal tangent parameters. One will then get:

$$
\frac{\partial^6 \mathfrak{p}}{\partial (u^1)^6} = \mathfrak{p}, \qquad \frac{\partial^2 \mathfrak{p}}{\partial u^1\, \partial u^2} = \mathfrak{p},
$$
$$
\frac{\partial^6 \mathfrak{p}'}{\partial (u^2)^6} = \mathfrak{p}', \qquad \frac{\partial^2 \mathfrak{p}'}{\partial u^1\, \partial u^2} = \mathfrak{p}'
$$

from the differential equations. With the parameter substitution:

$$
u^1 = -\,\bar{u}^1 - \bar{u}^2, \qquad u^2 = -\,\bar{u}^1 + \bar{u}^2,
$$

one will get the line coordinates:

$$
\left.\begin{array}{l} \sigma p_1 \\ \sigma p'_1 \end{array}\right\} = e^{\bar{u}^1}\left[\cos(\bar{u}^2\sqrt{3}) \mp \sqrt{3}\sin(\bar{u}^2\sqrt{3})\right],
$$

$$
\left.\begin{array}{l} \sigma p_2 \\ \sigma p'_2 \end{array}\right\} = e^{\bar{u}^1}\left[\sin(\bar{u}^2\sqrt{3}) \pm \sqrt{3}\cos(\bar{u}^2\sqrt{3})\right], \qquad \left.\begin{array}{l} \sigma p_3 \\ \sigma p'_3 \end{array}\right\} = -\,2e^{-2\bar{u}^1},
$$

$$\left.\begin{matrix}\sigma p_4 \\ \sigma p_4'\end{matrix}\right\} = \sqrt{3}\, e^{-\bar{u}^1}\left[-\sqrt{3}\sin(\bar{u}^2\sqrt{3}) \mp \sqrt{3}\cos(\bar{u}^2\sqrt{3})\right],$$

$$\left.\begin{matrix}\sigma p_5 \\ \sigma p_6'\end{matrix}\right\} = \sqrt{3}\, e^{-\bar{u}^1}\left[\sqrt{3}\cos(\bar{u}^2\sqrt{3}) \mp \sqrt{3}\sin(\bar{u}^2\sqrt{3})\right], \qquad \left.\begin{matrix}\sigma p_6 \\ \sigma p_6'\end{matrix}\right\} = \pm\sqrt{3}\, e^{2\bar{u}^1},$$

with a suitable choice of initial conditions.

The focal surface Φ that is given by the $\mathfrak{p}$, $\mathfrak{p}'$ will be represented in point coordinates by:

$$\sigma x_1 = e^{\bar{u}^1}\cos(\bar{u}^2\sqrt{3}), \qquad \sigma x_2 = e^{\bar{u}^1}\sin(\bar{u}^2\sqrt{3}), \qquad \sigma x_3 = e^{-2\bar{u}^1}, \qquad \sigma x_3 = 1.$$

By eliminating $\bar{u}^1$, $\bar{u}^2$, it will follow that the equation of Φ is:

$$(x_1^2 + x_2^2)\, x_3 = x_4^2 .$$

If one interprets the x_i as homogeneous, rectangular coordinates then Φ will be a third-order surface of rotation. As for any surface of rotation (no. **92**), the latitudes $\bar{u}^1$ = const. will be the Darboux curves, while the meridians $\bar{u}^2$ = const. will be Segre curves.

The self-projective transformations are composed of rotations around the axis of the surface of rotation and the affinities:

$$\tilde{x}_1 = \kappa x_1, \qquad \tilde{x}_2 = \kappa x_2, \qquad \tilde{x}_3 = \frac{1}{\kappa^2}\, x_3, \quad \tilde{x}_4 = x_4 .$$

(κ = arbitrary constant). From no. **95**, the principal tangent curves of both families are self-projective.

In plane coordinates, the surface has the equation:

$$27\,[(w^1)^2 + (w^2)^2]\, w^3 + 4\,(w^4)^3 = 0,$$

which is also of class 3.

§ 28. Special parabolic line systems (1).

98. Line systems with $W = W' = 0$. From (157b) and (156), there exist precisely two line systems $\mathfrak{g}(u^1, u^2)$, $\mathfrak{g}'(u^1, u^2)$ for $W = W' = 0$, namely:

$$\mathfrak{g} = \mathfrak{q}, \qquad \mathfrak{g}' = \mathfrak{q}'.$$

(1) Cf., **W. Blaschke and G. Thomsen**, *loc. cit.*, page 119.

As a consequence, the osculating hyperboloids have only a single enveloping structure Ψ, along with the initial surface Φ. The surfaces Φ and Ψ are of equal status, in the following sense:

a) *Just as* $\mathfrak{p}$, $\mathfrak{p}'$ *are the principal tangents to the surface* Φ, *the* $\mathfrak{q}$, $\mathfrak{q}'$ *are principal tangents to the surface* Ψ.

If one associates the points of intersection of corresponding line-pairs $\mathfrak{p}$, $\mathfrak{p}'$ *and* $\mathfrak{q}$, $\mathfrak{q}'$ *with the projective normals* $\mathfrak{n}$ *of* Φ *then the principal tangent curves on* Φ *and* Ψ *will correspond.*

Proof: From (147), one has:

$$\mathfrak{q}_2 = -\,2q\mathfrak{q} + \mathfrak{q}', \qquad \mathfrak{q}'_1 = -\,2q'\mathfrak{q}' + \mathfrak{q}.$$

From (144), this means that $\mathfrak{q}$ and $\mathfrak{q}'$ are principal tangents to the surface Ψ, and the principal tangent parameters of Φ are, at the same time, principal tangent parameters of Ψ. However, one observes that the six-vector $\mathfrak{p}$ ($\mathfrak{p}'$, resp.) corresponds to the six-vector $\mathfrak{q}$ ($\mathfrak{q}'$, resp.) under the comparison with (144). The principal tangent curves of Φ, with the tangents $\mathfrak{p}$ ($\mathfrak{p}'$, resp.), then map to the principal tangent curves of Ψ, with the tangents $\mathfrak{q}$ ($\mathfrak{q}'$, resp.).

b) *The osculating hyperboloids of* Φ *are, at the same time, the osculating hyperboloids of* Ψ.

Proof: it follows from (147):

$$\begin{aligned} \mathfrak{q}_1 &= H\,\mathfrak{t} - q'\mathfrak{q}, &\qquad \mathfrak{q}_{11} &= H^2\,\mathfrak{p} + (H_1 - q'H)\,\mathfrak{t} + (q'^2 - q'_1)\,\mathfrak{q}; \\ \mathfrak{q}'_1 &= H'\,\mathfrak{t}' - q\mathfrak{q}', &\qquad \mathfrak{q}'_{22} &= H'^2\,\mathfrak{p}' + (H'_2 - qH')\,\mathfrak{t}' + (q^2 - q_2)\,\mathfrak{q}'; \end{aligned}$$

As a result of this, the quadrics that are spanned by $\mathfrak{p}$, $\mathfrak{t}$, $\mathfrak{q}$ ($\mathfrak{p}'$, $\mathfrak{t}'$, $\mathfrak{q}'$, resp.) are identical with the ones that are spanned by $\mathfrak{q}$, $\mathfrak{q}_1$, $\mathfrak{q}_{11}$ ($\mathfrak{q}'$, $\mathfrak{q}'_2$, $\mathfrak{q}'_{22}$, resp.).

The special self-projective line system that was treated in no. **97** is, at the same time, an **example** of the line system $W = W' = 0$. Since:

$$\mathfrak{q} = \mathfrak{t}_2 = \mathfrak{p}_{22}\,,$$

after a brief computation, one gets from no. **97** that:

$$q_1 = p'_1, \qquad q_2 = p'_2, \qquad q_3 = p'_3, \qquad q_4 = -\,p'_4, \qquad q_5 = -\,p'_5, \qquad q_6 = -\,p'_6\,.$$

By choosing rectangular coordinates, the line $\mathfrak{p}'$ will go to the $\mathfrak{q}$ under a reflection in the coordinate origin. With that, the surface Ψ that is spanned by the lines $\mathfrak{q}$ will be symmetric to the focal surface Φ:

$$(x_1^2 + x_2^2)\, x_3 - x_4^2 = 0$$

of the line system of the lines $\mathfrak{p}'$, and will have the equation:

$$(x_1^2 + x_2^2)\, x_3 + x_4^2 = 0.$$

99. Line systems with $q = q' = 0$. For any parabolic line system, the tangents to the Darboux curves and the Segre curves of the focal surface Φ define two hyperbolic line systems that have in common the focal surface Φ and the net that is generated by the Darboux curves and Segre curves as the contact net (no. **80**). Along with the common focal surface Φ, the two hyperbolic line systems each have another focal surface or focal curve Φ' (Φ'', resp.).

The line systems with $q = q' = 0$ now have the following special properties:

For $q = q' = 0$ *and* $W \neq W'$, Φ, *as well as* Φ' *and* Φ'', *are negatively-curved surfaces. Under the association of corresponding focal points, the principal tangent curves of* Φ *will be transformed to the principal tangent curves of* Φ' *and* Φ'', *so the line systems of Darboux tangents and Segre tangents are W-systems* (§ 29).

For $q = q' = 0$ *and* $W = W'$, Φ' *and* Φ'' *are focal curves or developable focal surfaces.*

Proof: From (154), the tangents to the Darboux and Segre curves are given by the six-vectors $\mathfrak{p} \pm \mathfrak{p}'$. Since, from (117), when $q = q' = 0$, one can set the tensors m_i, m'_k equal to:

$$m_1 = 1, \qquad m_2 = 0; \qquad m'_1 = 0, \qquad m'_2 = 1,$$

(152) and (153) will give $u^1 \pm u^2$ = const. for the Darboux and Segre curves. As a result, the torsion parameters $\bar{u}^1$, $\bar{u}^2$ of the hyperbolic line system of the Darboux and Segre curves will be coupled to the principal tangent parameters u^1, u^2 of the given parabolic line system by:

(165) $$u^1 = \bar{u}^1 + \bar{u}^2, \qquad u^2 = \bar{u}^1 - \bar{u}^2.$$

We now determine the lateral tangents (no. **81**) $\mathfrak{c}$, $\mathfrak{c}'$ of the line system of Darboux (Segre, resp.) tangents $\mathfrak{p} \pm \mathfrak{p}'$ according to (121). In that way, we can replace the invariant derivatives in (121) with partial derivatives with respect to the torsion parameters $\partial / \partial \bar{u}^1$, $\partial / \partial \bar{u}^2$, and then have, on the basis of (165):

$$\mathfrak{c}\,(\mathfrak{p} \pm \mathfrak{p}') = 0, \qquad \mathfrak{c}\,(\mathfrak{p} \pm \mathfrak{p}')_{u^1} + \mathfrak{c}\,(\mathfrak{p} \pm \mathfrak{p}')_{u^2} = 0, \qquad \mathfrak{c}\,(\mathfrak{p} \pm \mathfrak{p}')_{u^1} - \mathfrak{c}\,(\mathfrak{p} \pm \mathfrak{p}')_{u^2} = 0,$$

$$\mathfrak{c}\,(\mathfrak{p}\pm\mathfrak{p}')_{u^1u^1}-\mathfrak{c}\,(\mathfrak{p}\pm\mathfrak{p}')_{u^2u^2}=0.$$

Since, from (117), the partial derivatives $\partial / \partial u^1$, $\partial / \partial u^2$ are identical with the invariant derivatives of the parabolic line system, one will further get:

$$\mathfrak{c}(\mathfrak{p}\pm\mathfrak{p}') = \mathfrak{c}(\mathfrak{p}_1\pm\mathfrak{p}'_1) = \mathfrak{c}(\mathfrak{p}_2\pm\mathfrak{p}'_2) = \mathfrak{c}(\mathfrak{p}_{11}-\mathfrak{p}_{22}\pm\mathfrak{p}'_{11}\mp\mathfrak{p}'_{22}) = 0.$$

With the help of the differential equations (147), one obtains:

$$(166)\qquad \mathfrak{c}(\mathfrak{p}\pm\mathfrak{p}') = \mathfrak{c}(\mathfrak{p}'\pm\mathfrak{t}') = \mathfrak{c}(\mathfrak{p}\pm\mathfrak{t}) = \mathfrak{c}(\mathfrak{t}'-W\mathfrak{p}-\mathfrak{q}\pm W'\mathfrak{p}'\pm\mathfrak{q}'\mp\mathfrak{t}) = 0.$$

For the general Ansatz:

$$\mathfrak{c} \text{ and } \mathfrak{c}' = \alpha\,\mathfrak{p} + \alpha'\,\mathfrak{p}' + \beta\,\mathfrak{q} + \beta'\,\mathfrak{q}' + \gamma\,\mathfrak{t} + \gamma'\,\mathfrak{t}',$$

it will follow from (166) and $\mathfrak{c}\mathfrak{c} = \mathfrak{c}'\mathfrak{c}' = 0$ that:

$$\beta\mp\beta' = \beta\mp\gamma' = \beta\mp\gamma = -\gamma' + W\beta + \alpha\pm W'\beta'\pm\alpha'\mp\gamma = 0, \qquad \alpha\beta-\alpha'\beta' = 0.$$

That will yield:

$$\mathfrak{c} = \mathfrak{p}\mp\mathfrak{p}' \qquad (\text{with } \alpha = 1,\ \alpha' = \mp 1,\ \beta = \beta' = \gamma = \gamma' = 0),$$

$$\mathfrak{c}' = \left(1-\frac{W+W'}{2}\right)(\mathfrak{p}\pm\mathfrak{p}') + \mathfrak{q} + \mathfrak{q}' \pm \mathfrak{t} + \mathfrak{t}'$$

$$\left(\text{with } \beta = 1, \beta' = \pm 1, \gamma = \pm 1, \gamma' = 1, \alpha = \pm\alpha' = 1-\frac{W+W'}{2}\right).$$

The line system of the Darboux (Segre, resp.) tangents then has (as it should) the Segre (Darboux, resp.) tangents of Φ for its first lateral tangents $\mathfrak{c}$. Furthermore, we now define the tensor c'_{ik} according to (122). In that, one must take $\mathfrak{p}\pm\mathfrak{p}'$ in place of the line $\mathfrak{p}$ of the system in (122), and by referring to the principal tangent parameters of Φ, the partial derivatives can be replaced with the invariant derivatives relative to the parabolic line system. One then has:

$$c'_{11} = (\mathfrak{p}\pm\mathfrak{p}')_1\,\mathfrak{c}'_1 = \frac{W-W'}{2},$$

$$c'_{22} = (\mathfrak{p}\pm\mathfrak{p}')_2\,\mathfrak{c}'_2 = \frac{W-W'}{2},$$

$$c'_{12} = c'_{21} = (\mathfrak{p}\pm\mathfrak{p}')_1\,\mathfrak{c}'_2 = 1 - H' = 0;$$

from (148_3), it will then follow from $q' = 0$ that $H' = 1$.

The determinant c' vanishes when $W = W'$, so Φ', Φ'' will be focal curves or developable focal surfaces (no. **82**). $c' > 0$ when $W \neq W'$. The focal surface Φ' (Φ'', resp.) will then be negatively-curved, and from (124), will have u^1 = const, and u^2 = const. for its principal tangent curves with:

$$\begin{aligned} g_{11} &= (\mathfrak{p} \pm \mathfrak{p}')_1(\mathfrak{p} \pm \mathfrak{p}')_1 &&= -1, \\ g_{22} &= (\mathfrak{p} \pm \mathfrak{p}')_2(\mathfrak{p} \pm \mathfrak{p}')_2 &&= 1, \\ g_{21} = g_{21} &= (\mathfrak{p} \pm \mathfrak{p}')_1(\mathfrak{p} \pm \mathfrak{p}')_2 &&= 0. \end{aligned}$$

The principal tangent parameters of Φ are also principal tangent parameter for Φ' and Φ'' then.

100. Line systems with $H = 0$ **or** $H' = 0$.

$H = 0$ ($H' = 0$) *is the necessary and sufficient condition for the osculating complex* $\mathfrak{t}$ ($\mathfrak{t}'$, *resp.*) *to be fixed along the principal tangent curves* u^2 *= const.* (u^1 *= const.*). *Not only will four "successive" tangents* (no. **91**) *belong to the linear complex* $\mathfrak{t}$ ($\mathfrak{t}'$, *resp.*), *but all pencils of tangents to the focal surface* Φ *along a principal tangent curve* u^2 *= const.* (u^1 *= const.*).

Proof: From (147), for $H = 0$ ($H' = 0$, resp.):

$$\mathfrak{t}_1 = 0 \quad (\mathfrak{t}'_2 = 0);$$

on the other hand, for $H \neq 0$ ($H' \neq 0$, resp.), from (147), $\mathfrak{t}$, $\mathfrak{t}_1$ ($\mathfrak{t}'$, $\mathfrak{t}'_2$, resp.) will be linearly-independent.

In connection with that, let the following theorem be also mentioned:

A surface curve along which all pencils of tangents of the surface are taken from a fixed linear complex $\mathfrak{a}$ *is necessarily a principal tangent curve.*

Proof: The theorem will be trivial with $\mathfrak{a}\mathfrak{a} = 0$ for a rectilinear surface curve or a planar surface curve with fixed contact planes. In the remaining cases, $\mathfrak{a}\mathfrak{a} \neq 0$, and one will then conclude the following: We give the surface curve by $u^2 = 0$ and denote the derivatives with respect to u^1 for $u^2 = 0$ by dots. For the pencil of tangents $\rho\, \mathfrak{r}(u^1) + \sigma\, \mathfrak{s}(u^2)$ along $u^2 = 0$ one has, by assumption:

$$\mathfrak{a}\,(\rho\,\mathfrak{r} + \sigma\,\mathfrak{s}) = 0 \qquad (\mathfrak{a} = \text{const.})$$

for all ρ, σ, so:

$$\mathfrak{a}\mathfrak{r} = \mathfrak{a}\mathfrak{s} = 0 \qquad \text{and thus, also} \quad \mathfrak{a}\dot{\mathfrak{r}} = \mathfrak{a}\dot{\mathfrak{s}} = 0.$$

It follows from this and the identities (155) that:

$$D_{\mathfrak{s}}\,(\mathfrak{r}, \mathfrak{s}, \dot{\mathfrak{r}}, \dot{\mathfrak{s}}, \mathfrak{a}, \mathfrak{w}) = 0.$$

From no. **26**, these six six-vectors (and since $\mathfrak{w}$ means an arbitrary six-vector, the first five six-vectors) are then linearly-independent, so:

$$A\mathfrak{r} + B\mathfrak{s} + C\dot{\mathfrak{r}} + D\dot{\mathfrak{s}} + E\mathfrak{a} = 0;$$

scalar multiplication by $\mathfrak{a}$ will give $E = 0$, since $\mathfrak{a}\mathfrak{a} \neq 0$. Due to the linear independence of $\mathfrak{r}$ and $\mathfrak{s}$, C and D cannot vanish simultaneously. For that reason, one can set, say, $D = -1$, and then have:

(167) $$\dot{\mathfrak{s}} = A\mathfrak{r} + B\mathfrak{s} + C\dot{\mathfrak{r}}.$$

From no. **46**, the tangents to the curve $u^2 = 0$ and the surface tangents that are conjugate to them are given by:

$$0 = (\rho\,\mathfrak{r} + \sigma\mathfrak{s})^{\cdot}\,(\rho\,\mathfrak{r} + \sigma\mathfrak{s})^{\cdot} = \rho^2(\dot{\mathfrak{r}}\dot{\mathfrak{r}}) + \sigma^2(\dot{\mathfrak{s}}\dot{\mathfrak{s}}) + 2\rho\sigma(\dot{\mathfrak{r}}\dot{\mathfrak{s}}).$$

From (167), since:

$$(\dot{\mathfrak{r}}\dot{\mathfrak{r}})(\dot{\mathfrak{s}}\dot{\mathfrak{s}}) - (\dot{\mathfrak{r}}\dot{\mathfrak{s}})^2 = 0,$$

that equation will have a double root $\rho : \sigma$; i.e., the tangents to the curve $u^2 = 0$ are conjugate to themselves, so the curve $u^2 = 0$ will then be a principal tangent curve.

For the line system $H = H' = 0$, the osculating complex $\mathfrak{t}$ is fixed along the principal tangent curves $u^2 =$ const., just as $\mathfrak{t}'$ is fixed along $u^1 =$ const. In addition, one has:

All of the osculating complexes $\mathfrak{t}$ *and* $\mathfrak{t}'$ *of a line system with* $H = H' = 0$ *are contained in two conjugate bundles of complexes.*

Proof: From (147), one has:

$$\begin{aligned}
&\mathfrak{t} = \mathfrak{t}, && \mathfrak{t}_2 = W\mathfrak{p} + \mathfrak{q}, && \mathfrak{t}_{22} = (W_2 + 2W\,q)\,\mathfrak{p} - W'\mathfrak{p}' + 2W\,\mathfrak{t} - 2q\,\mathfrak{q} + \mathfrak{q}', \\
&\mathfrak{t}' = \mathfrak{t}', && \mathfrak{t}'_1 = W'\mathfrak{p}' + \mathfrak{q}', && \mathfrak{t}'_{11} = (W'_1 + 2W'q')\,\mathfrak{p}' - W\,\mathfrak{p} + 2W'\mathfrak{t}' - 2q'\mathfrak{q}' + \mathfrak{q}.
\end{aligned}$$

Thus, $\mathfrak{t}$, $\mathfrak{t}_2$, $\mathfrak{t}_{22}$, and likewise, $\mathfrak{t}', \mathfrak{t}'_1, \mathfrak{t}'_{11}$, will be linearly-independent and span two bundles of complexes. In addition, the scalar products of the $\mathfrak{t}$, $\mathfrak{t}_2$, $\mathfrak{t}_{22}$ with the $\mathfrak{t}', \mathfrak{t}'_1, \mathfrak{t}'_{11}$ will vanish, as a simple calculation will show; i.e., the two bundles of complexes will be conjugate.

In principal tangent parameters, one has $\mathfrak{t} = \mathfrak{t}\,(u^2)$, $\mathfrak{t}' = \mathfrak{t}'(u^1)$. It follows from:

$$\mathfrak{t}(u^2)\,\mathfrak{t}'(u^1) = \mathfrak{t}(u^2)\;\mathfrak{t}'_1(u^1) = \mathfrak{t}(u^2)\;\mathfrak{t}'_{11}(u^1) = 0$$

by repeated differentiation that all arbitrarily higher derivatives $d^n\mathfrak{t} / d(u^2)^n$ are conjugate to the bundle $\mathfrak{t}', \mathfrak{t}'_1, \mathfrak{t}'_{11}$, and are therefore contained in the bundle $\mathfrak{t}$, $\mathfrak{t}_2$, $\mathfrak{t}_{22}$. The same thing will then be true for any complex:

$$\mathfrak{t}(u^2 + \varepsilon) = \mathfrak{t} + \varepsilon\,\frac{d\mathfrak{t}}{du^2} + \frac{\varepsilon^2}{2}\frac{d^2\mathfrak{t}}{d(u^2)^2} + \dots$$

with an arbitrary ε.

101. Line systems with $H = H'$. Two curve nets of a surface shall be called *diagonal* when they are representable by:

$$(168) \qquad u^1 = \text{const.},\ u^2 = \text{const.}, \quad \text{and} \quad u^1 + u^2 = \text{const.},\ \ u^1 - u^2 = \text{const.},$$

after a suitable parameter substitution. Two discrete, arbitrarily closely-meshed, curve nets can be selected from the four families of curves that are diagonally-coupled, as in e.g., Fig. 29. These discrete curve nets will be defined by the curves:

$$u^1 = m\varepsilon, \qquad u^2 = n\varepsilon, \qquad u^1 + u^2 = h\varepsilon, \quad u^1 - u^2 = k\varepsilon,$$

in which ε is an arbitrary, sufficiently-small positive number, and m, n, h, k mean positive and negative whole numbers, as well as zero.

The 2×2 tangents to a diagonal curve net that intersect at a point of the surface are separated harmonically. If the curve net is a principal tangent net then any diagonal curve net will be conjugate and will be called *isothermally conjugate*. These statements follow immediately from (168) and no. **6**. We now further show that:

There are negatively-curved surfaces on which the principal tangent net is diagonal to the net of Darboux and Segre curves. These surfaces can be characterized as focal surfaces of the parabolic line system with $H = H'$.

Proof: It is necessary and sufficient for the diagonality that the differential equations (152) and (153) for the Darboux and Segre curves will go to $\dot{u}^1 \pm \dot{u}^2 = 0$ by a parameter substitution: This is the case in reference to the principal tangent parameters if and only if:

$$m_1 : m'_2 = \varphi(u^1) : \psi(u^2),$$

so

$$(169) \qquad \frac{\partial^2}{\partial u^1 \partial u^2}\ln\left(\frac{m_1}{m'_2}\right) = 0$$

is fulfilled. Since from (114), on the basis of a simple calculation, one will have:

$$q_1 - q_2' = \frac{1}{m_1 m_2'} \frac{\partial^2}{\partial u^1 \partial u^2} \ln\left(\frac{m_1}{m_2'}\right)$$

in principal tangent coordinates, it follows from (169) that:

$$q_1 - q_2' = 0,$$

and from ($148_{2,3}$), one has the assertion that $H = H'$. Conversely, $H = H'$ once more yields (169).

Of the numerous remarkable *properties of the surfaces* ([1]) $H = H'$, let only the following ones be cited:

The torses of the projective normals (no. **94**) *intersect the surface* Φ *along conjugate families of curves if and only if the principal tangent net and the net of Darboux and Segre curves is diagonal to the surface* Φ.

Proof: The torses of the normals are given by:

$$0 = \dot{\mathfrak{n}}\dot{\mathfrak{n}} = \mathfrak{n}_i\,\mathfrak{n}_k\,\dot{u}^i\dot{u}^k = (\mathfrak{n}_1 m_1 \dot{u}^1 + \mathfrak{n}_2 m_2' \dot{u}^2)(\mathfrak{n}_1 m_1 \dot{u}^1 + \mathfrak{n}_2 m_2' \dot{u}^2)$$

in principal tangent parameters. It follows from (159) that:

$$\mathfrak{n}_1 = \mathfrak{t}_1 \pm \mathfrak{t}_1' = H\,\mathfrak{p} \pm W'\mathfrak{p}' \pm \mathfrak{q}',$$
$$\mathfrak{n}_2 = \mathfrak{t}_2' \pm \mathfrak{t}_2' = \pm H'\mathfrak{p}' + W\mathfrak{p} + \mathfrak{q},$$

so

$$0 = \tfrac{1}{2}\dot{\mathfrak{n}}\dot{\mathfrak{n}} = W' m_1 m_2 \dot{u}^1\dot{u}^1 + (H' - H) m_1 m_2' \dot{u}^1\dot{u}^2 - W m_2' m_2' \dot{u}^2\dot{u}^2 .$$

This equation produces conjugate families of curves for $H = H'$, and only in that case.

Special parabolic line systems with $H = H'$**:**

a) The line system $H = H' = 0$ (no. **100**).
b) The line system $q = q' = 0$ (no. **99**), since, from ($148_{2,3}$), it follows this that $H = H'$.
c) The self-projective line system (no. **97**), since for q = const., q' = const, if likewise follows from ($148_{2,3}$) that $H = H'$.
d) The principal tangent system of negatively-curved surfaces of rotation. Due to the rotational symmetry, the net of principal tangent curves will then be diagonal to the net of Darboux and Segre curves (from no. **92**, the latitudes and meridians, resp.).

([1]) **O. Baier** has investigated the analogous surfaces of positive curvature, Diss. Munich, 1931.

§ 29. Special hyperbolic line systems. *W*-systems.

102. Definition of the *W*-systems. A hyperbolic line system with two *negatively-curved focal surfaces* Φ, Φ' is called a *W*-system ([1]) when the principal tangent curves of Φ and Φ' correspond to each other under the association of the focal points x, x'. If one family of principal tangent curves of Φ is mapped to principal tangent curves of Φ' then the same thing will always be true for the second family, and the line system with the focal surfaces Φ, Φ' will be a *W*-system; the contact nets of Φ, Φ' will then correspond without that, and their tangents will be separated harmonically by the principal tangents.

The W-system will be characterized by:

$$DD' - CC' = 0 \tag{170}$$

or the condition:

$$B^2 = 1, \tag{171}$$

which, from (135), *is equivalent to it.*

Proof: In torsion parameters, the differential equations (124) for the principal tangent curves of Φ (Φ', resp.) read:

$$c_{11}(\dot{u}^1)^2 - c_{22}(\dot{u}^2)^2 = 0 \qquad [c'_{11}(\dot{u}^1)^2 - c'_{22}(\dot{u}^2)^2 = 0, \text{ resp.}]$$

A necessary and sufficient condition for the correspondence of the principal tangent curves is then:

$$c_{11} : c_{22} = c'_{11} : c'_{22}\,.$$

It then follows from (170) that:

$$m^1\, m^1\, c_{11} = m^1\, m^1\, \mathfrak{p}_1\, \mathfrak{c}_1 = \mathfrak{p}_1\, \mathfrak{c}_1 = -BD, \qquad m'^2\, m'^2\, c_{22} = \mathfrak{p}_2\, \mathfrak{c}_2 = -B'C',$$
$$m^1\, m^1\, c'_{11} = \mathfrak{p}_1\, \mathfrak{c}'_1 = -BC, \qquad m'^2\, m'^2\, c'_{22} = \mathfrak{p}_2\, \mathfrak{c}'_2 = -B'D'.$$

Since the principal tangents of Φ and Φ' are associated with each other by a *W*-system, any conjugate curve net of the one focal surface must again correspond to a conjugate curve net of the other focal surface. On the other hand, the principal tangents are determined uniquely by two pairs of conjugate directions as the lines that each separate the two given conjugate pairs of tangents harmonically. It follows from this that:

If, along with the contact nets, yet another pair of conjugate nets corresponds under the map of the focal surfaces Φ, Φ' *that associates focal points* x, x' *then the line system will be a W-system.*

([1]) One should recall the name of **J. Weingarten.**

Now, *W*-systems with two *positively-curved focal surfaces* will also be defined by this theorem. They can be characterized, for example, by the demand that the harmonic nets (no. **82**) of Φ and Φ' must correspond. The conditions (170) and (171) are true for positively-curved focal surfaces, as well as for negatively-curved ones.

As *discrete-geometric* models, one can compare the negatively-curved focal surfaces of the *W*-systems of the rectangle nets $(\bar{X})$ and $(\bar{Y})$ that are associated with angle-fixed rectangle nets (no. **43**):

Model	*W*-system
Two rectangle nets $(\bar{X})$, $(\bar{Y})$ with planar quadrilaterals.	Two principal tangent nets of negatively-curved surfaces Φ, Φ'.
The connecting lines of corresponding quadrilateral vertices of $(\bar{X})$ and $(\bar{Y})$ shall each lie in both quadrilateral planes.	The connecting lines of corresponding surface points of Φ, Φ' shall each lie in both contact planes; i.e., they will be common tangents to Φ and Φ'.
Sets of connecting lines.	*W*-systems of connecting lines.

We will come back to this analogy in § 33.

103. Osculating complex. The *W*-systems can be characterized by the demand that:

$$|\,\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_{11}, \mathfrak{p}_{12}, \mathfrak{p}_{22}\,| = 0, \tag{172}$$

instead of (171), or by:

$$\left|\,\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2, \frac{\partial^2\mathfrak{p}}{\partial u^1\partial u^1}, \frac{\partial^2\mathfrak{p}}{\partial u^1\partial u^2}, \frac{\partial^2\mathfrak{p}}{\partial u^2\partial u^2}\,\right| = 0 \tag{173}$$

in ordinary partial derivatives with respect to arbitrary parameters u^1, u^2.

Proof: From (134), one has:

$$\mathfrak{p}_{11} = \alpha\,\mathfrak{p} + \beta\,\mathfrak{q} + B\,(C\mathfrak{c} + D\mathfrak{c}'),$$

$$\mathfrak{p}_{12} = \gamma\,\mathfrak{p} + \delta\,\mathfrak{q} + \varepsilon\,\mathfrak{q}' - BB'\mathfrak{z},$$

$$\mathfrak{p}_{22} = \alpha'\mathfrak{p}' + \beta'\mathfrak{q}' + B'(C'\mathfrak{c}' + D'\mathfrak{c}),$$

so

$$|\,\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_{11}, \mathfrak{p}_{12}, \mathfrak{p}_{22}\,| = |\,\mathfrak{p}, B\mathfrak{q}, B'\mathfrak{q}', B\,(C\mathfrak{c} + D\mathfrak{c}'), -BB'\mathfrak{z}, B'(C'\mathfrak{c}' + D'\mathfrak{c})\,|\,.$$

Since $B \neq 0$, $B' \neq 0$, and due to the fact that $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{q}'$, $\mathfrak{c}$, $\mathfrak{c}'$, $\mathfrak{z}$ are linearly-independent, that determinant will vanish if and only if $C : D = D' : C'$, so (170) will be true. (173) follows from (172) by going from invariant to ordinary partial derivatives, and conversely.

We refer to a linear complex $\mathfrak{s}$ that satisfies the conditions:

(174) $$\mathfrak{s}\mathfrak{p} = \mathfrak{s}\mathfrak{p}_1 = \mathfrak{s}\mathfrak{p}_2 = \mathfrak{s}\,\frac{\partial^2\mathfrak{p}}{\partial u^1\partial u^1} = \mathfrak{s}\,\frac{\partial^2\mathfrak{p}}{\partial u^1\partial u^2} = \mathfrak{s}\,\frac{\partial^2\mathfrak{p}}{\partial u^2\partial u^2} = 0$$

as an *osculating complex* $\mathfrak{s}$. It then contains the first osculating quadric (no. **49**) of all families of lines of the given line system that go through a line $\mathfrak{p}$ of the system. The six linear, homogeneous equations (174) will have a non-trivial solution $\mathfrak{s} \neq 0$ if and only if (173) is true. Thus:

A hyperbolic line system with two focal surfaces possesses an osculating complex for any line of the system if and only if it is a W-system.

104. Confocal *W*-systems. In this and the following numbers, we shall assume that the focal surfaces are negatively-curved. We first pose the problem (cf., no. **116**):

Let a negatively-curved surface Φ be given. Find the W-systems for which Φ is one focal surface.

We start with the confocal, parabolic line systems of the principal tangents $\mathfrak{p}$, $\mathfrak{p}'$ of the given surface Φ. The lines $\mathfrak{r}$ of the desired *W*-system are then given by:

(175) $$\mathfrak{r} = \rho\,\mathfrak{p} + \mathfrak{p}' \qquad (\rho \neq 0),$$

with the auxiliary condition (173):

$$\left|\, \mathfrak{r}, \mathfrak{r}_1, \mathfrak{r}_2, \frac{\partial^2\mathfrak{r}}{\partial u^1\partial u^1}, \frac{\partial^2\mathfrak{r}}{\partial u^1\partial u^2}, \frac{\partial^2\mathfrak{r}}{\partial u^2\partial u^2} \,\right| = 0.$$

By going to the invariant derivatives with respect to the parabolic line system $\mathfrak{p}$, $\mathfrak{p}'$, one again comes to:

$$|\, \mathfrak{r}, \mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_{11}, \mathfrak{r}_{12}, \mathfrak{r}_{22} \,| = 0.$$

This equation differs from (172) by the fact that the invariant derivatives were defined as in no. **81** there, but here they are defined as in no. **88**. By substituting (175) and the use of the differential equations (147), one will get ([1]):

(176) $$\rho\,\rho_{12} - \rho_1\,\rho_2 + \rho^2\,(q'_2 - q_1) + q\,\rho\,\rho_1 - \rho^2\,\rho_2 - \rho_1 - \rho^3\,q + \rho\,q' = 0.$$

([1]) One represents $\mathfrak{r}$, $\mathfrak{r}_1$, $\mathfrak{r}_2$, etc., as linear combinations of $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{q}$, $\mathfrak{q}'$, $\mathfrak{t}$, $\mathfrak{t}'$, and sets the coefficient determinant equal to zero.

When one again expresses the invariant derivatives in terms of ordinary partial derivatives, this will become a second-order partial differential equation for the function $\rho(u^1, u^2)$.

We understand the term *confocal W-systems* to mean two *W*-systems that have a negatively-curved focal surface Φ in common and contact Φ along two conjugate families of curves. The net that is defined by these families of curves will then be a common contact net to the two *W*-systems. Moreover, one has the theorem:

The focal surfaces Φ of confocal W-systems are identical with the focal surfaces of the parabolic line systems that were characterized by (151) *in no.* **91**.

Proof: The focal surfaces of confocal *W*-systems can be characterized by the demand that (176) admits two solutions $+\rho(u^1, u^2)$ and $-\rho(u^1, u^2)$. In fact, the lines $\rho\mathfrak{p} + \mathfrak{p}'$ and $-\rho\mathfrak{p} + \mathfrak{p}'$ of the two *W*-systems will then be separated harmonically by the principal tangents $\mathfrak{p}$ and $\mathfrak{p}'$.

Equation (176) will have $+\rho$ and $-\rho$ for its solutions if and only if the two conditions:

$$\rho^2\rho_2 + \rho_1 + \rho^3 q - \rho q' = 0,$$
$$\rho\rho_{12} - \rho_1\rho_2 + \rho^2(q_2' - q_1) + q\rho\rho_1 = 0 \tag{177}$$

are fulfilled. Now, in principal tangent parameters, one will have:

$$\rho\,\rho_{12} - \rho_1\,\rho_2 + q\,\rho\,\rho_1 = \frac{\rho^2}{m_1 m_2'}\frac{\partial^2}{\partial u^1 \partial u^2}\ln\rho,$$
$$q_1 - q_2' = \frac{\rho^2}{m_1 m_2'}\frac{\partial^2}{\partial u^1 \partial u^2}\ln\frac{m_1}{m_2'}.$$

One gets from (177_2) that:

$$\rho = \frac{m_1}{m_2'}\frac{\varphi(u^2)}{\psi(u^1)},$$

and after the parameter transformation:

$$\bar{u}^1 = \int \psi(u^1)\,du^1, \qquad \bar{u}^2 = \int \varphi(u^2)\,du^2,$$

one will get:

$$\rho = \frac{m_1}{m_2'}.$$

One will get (151) by substituting this expression in (177_1).

105. ***W*-system with a ruled surface for one focal surface.** We assume that the principal tangent curves u^2 = const. are rectilinear, so the focal surface Φ is a ruled surface. One will then have:

The tangents to the second focal surface Φ' *along each principal tangent curve* u^2 = *const. belong to a fixed linear complex* $\mathfrak{t}$. *The principal tangents to the surface* Φ' *then define parabolic line systems with* $H = 0$ (no. **100**).

Proof: We consider the ruled family R of the system lines that contact Φ along one generator e and Φ' along the corresponding principal tangent curve e'. That ruled family R is contained in a linear line system, namely, the parabolic tangent system (no. **48**) that is defined by the tangents to Φ along e. The ruled surface that is spanned by R itself has e and e' for its principal tangent curves. The assertion now follows immediately from the **lemma**:

If a ruled family is implied by a linear line system then the tangents to the ruled surface along each principal tangent curve e' *of the second family will belong to a fixed linear complex.*

Proof of the lemma: For the principal tangents $\mathfrak{h}$ of the principal tangent curves e', one will have:

$$\dot{\mathfrak{h}} = \alpha\,\mathfrak{p} + \beta\,\mathfrak{h}\,;$$

in that way, one expresses the idea that the osculating plane that is spanned by $\mathfrak{h}$ and $\dot{\mathfrak{h}}$ contains the generator $\mathfrak{p}$, and is thus the contact plane to the ruled surface. A linear complex that satisfies the conditions:

$$\mathfrak{t}\mathfrak{p} = \mathfrak{t}\dot{\mathfrak{p}} = \mathfrak{t}\ddot{\mathfrak{p}} = \mathfrak{t}\dddot{\mathfrak{p}} = \mathfrak{t}\mathfrak{h} = 0 \tag{178}$$

will also satisfy the equations:

$$\mathfrak{t}\dot{\mathfrak{h}} = \mathfrak{t}\ddot{\mathfrak{h}} = \mathfrak{t}\dddot{\mathfrak{h}} = \mathfrak{t}\ddddot{\mathfrak{h}} = 0. \tag{179}$$

Now, when the ruled family implies a linear line system, and therefore $\ddddot{\mathfrak{p}}$ is a linear combination $\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}$ (no. **54**), along with (178) and (179), one will also have:

$$\mathfrak{t}\ddddot{\mathfrak{p}} = 0. \tag{180}$$

One further obtains from (178), (179), and (180) that:

$$\dot{\mathfrak{t}}\mathfrak{p} = \dot{\mathfrak{t}}\dot{\mathfrak{p}} = \dot{\mathfrak{t}}\ddot{\mathfrak{p}} = \dot{\mathfrak{t}}\dddot{\mathfrak{p}} = \dot{\mathfrak{t}}\mathfrak{h} = 0\,. \tag{181}$$

In the event that $\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}, \mathfrak{h}$ are linearly-independent, (178) and (181) will yield:

$$\dot{\mathfrak{t}} = \sigma \mathfrak{t} ;$$

i.e., the complex $\mathfrak{t}$ is fixed along the principal tangent curve e'. In case $\mathfrak{p}, \dot{\mathfrak{p}}, \ddot{\mathfrak{p}}, \dddot{\mathfrak{p}}, \mathfrak{h}$ are linearly-dependent, the ruled family will be either *a*) a quadric or *b*) parabolic, and will have $\mathfrak{h}$ as its nodal tangent. For a hyperbolic ruled family, it will then be impossible to have:

$$\mathfrak{h} = \alpha \mathfrak{p} + \beta \dot{\mathfrak{p}} + \gamma \ddot{\mathfrak{p}} + \delta \dddot{\mathfrak{p}} , \tag{182}$$

since otherwise one would have $\mathfrak{h}\mathfrak{k} = \mathfrak{h}\bar{\mathfrak{k}} = 0$, and for parabolic ruled families, the condition (60) $\mathfrak{h}\ddot{\mathfrak{p}} = 0$ for nodal tangents would follow from (182) by scalar multiplying with $\mathfrak{h}$. The case *a*) is trivial, while the assumption *b*) can be fulfilled only for a rectilinear nodal curve, which again makes the lemma trivial.

When the focal surface Φ *is a hyperboloid, the theorem that was stated at the beginning of this number will be true for the principal tangent curves* u^1 *= const., as well as for* u^2 *= const., so the principal tangent to the focal surface* Φ′ *will define a parabolic line system with* $H = H' = 0$ (no. **100**).

We leave it to the reader to prove the contents of this number immediately with the help of the differential equations (134).

106. ***W*-system with two ruled surfaces as focal surfaces.** There is a *W*-system whose focal surfaces Φ, Φ′ are both ruled surfaces with mutually-corresponding generators. The lines of the system that contact Φ and Φ′ along each generator define a hyperboloidal quadric. In analogy with the theorem in no. **49** about the second family of principal tangent curves to a ruled surface, it will then follow that:

The longitudes (no. **80**) *cut the generators of the focal surfaces* Φ, Φ′ *along projective point sequences.*

We now pose the problem:

Determine all W-systems that have two ruled surfaces with mutually-corresponding generators for their focal surfaces Φ, Φ′.

Let the first focal surface Φ be given by the arbitrary ruled family $\mathfrak{q} = \mathfrak{q}(u)$. The second focal surface Φ′ must then be spanned by a ruled family $\mathfrak{q}^* = \mathfrak{q}^*(u)$ with linearly-dependent six-vectors $\mathfrak{q}, \dot{\mathfrak{q}}, \mathfrak{q}^*, \dot{\mathfrak{q}}^*$, since the tangent systems along $\mathfrak{q}(u)$ and $\mathfrak{q}^*(u)$ have a hyperboloidal quadric in common. By a suitable normalization of $\mathfrak{q}^*$, one can arrange that:

$$\dot{\mathfrak{q}}^* = \rho\,\mathfrak{q} + \sigma\dot{\mathfrak{q}}\,,$$

in particular. Since $\mathfrak{q}^*\dot{\mathfrak{q}}^* = 0$, $\mathfrak{q}\mathfrak{q}^* \neq 0$, one will again come to:

$$\rho : \sigma = -(\dot{\mathfrak{q}}\mathfrak{q}^*) : (\mathfrak{q}\mathfrak{q}^*),$$

so

$$\dot{\mathfrak{q}}^* = \tau\{\mathfrak{q}(\dot{\mathfrak{q}}\mathfrak{q}^*) - \dot{\mathfrak{q}}(\mathfrak{q}\mathfrak{q}^*)\}, \qquad \dot{\mathfrak{q}}\dot{\mathfrak{q}}^* = -\tau(\mathfrak{q}\mathfrak{q}^*)(\dot{\mathfrak{q}}\dot{\mathfrak{q}})\,,$$

and since $\dot{\mathfrak{q}}\dot{\mathfrak{q}} \neq 0$, one will finally have:

$$\dot{\mathfrak{q}}^*(\mathfrak{q}\mathfrak{q}^*)(\dot{\mathfrak{q}}\dot{\mathfrak{q}}) + (\dot{\mathfrak{q}}\dot{\mathfrak{q}}^*)\{\mathfrak{q}(\dot{\mathfrak{q}}\mathfrak{q}^*) - \dot{\mathfrak{q}}(\mathfrak{q}\mathfrak{q}^*)\} = 0.$$

If the arbitrarily-given ruled family $\mathfrak{q} = \mathfrak{q}(u)$ is hyperbolic, for example, then we will assume the normalization conditions as in § 15, and for:

$$\mathfrak{q}^* = A\mathfrak{q} + B\mathfrak{q}' + C\mathfrak{q}'' + D\mathfrak{q}''' + E\mathfrak{k} + F\bar{\mathfrak{k}}\,,$$

we will obtain six first-order ordinary differential equations in A', B', C', D', E', F' explicitly with the use of the differential equations (71). One proceeds analogously for a parabolic ruled family $\mathfrak{q}(u)$. One will then have:

Along with the ruled family $\mathfrak{q} = \mathfrak{q}(u)$, *one can also give the line* $\mathfrak{q}^*(0)$ *that is associated with the line* $\mathfrak{q}(0)$ *in the second ruled family essentially arbitrarily. Precisely one W-system is determined in that way.*

107. Examples.

Any self-projective, hyperbolic line system with two focal surfaces and $q = q' = 0$ *is a W-system.*

Proof: One obtains the relation (170) that characterized *W*-systems immediately from the integrability conditions (136_{13}) with N = const., N'= const., $q = q' = 0$.

The first osculating quadric of a hyperbolic ruled family with non-rectilinear nodal curves generates a special W-system: The focal surfaces will be spanned by the two nodal families (no. **51**) *of the given ruled families.*

Proof: For an arbitrary line of a first osculating quadric:

$$\mathfrak{r} = \alpha\,\mathfrak{p} + \beta\dot{\mathfrak{p}} + \gamma\ddot{\mathfrak{p}},$$

and the nodal lines $\mathfrak{k}$, $\overline{\mathfrak{k}}$ follow from (60):

$$\mathfrak{r}\mathfrak{k} = \mathfrak{r}\dot{\mathfrak{k}} = \mathfrak{r}\overline{\mathfrak{k}} = \mathfrak{r}\dot{\overline{\mathfrak{k}}} = 0;$$

i.e., the line $\mathfrak{r}$ contacts the ruled surface that is spanned by the nodal family.

For the sake of completeness, we add that:

The first osculating quadric of a parabolic ruled family with non-rectilinear nodal curves defines a parabolic line system whose focal surface is spanned by the nodal ruled family.

Proof: From (60) and (61), one will have:

$$\mathfrak{r}\mathfrak{k} = \mathfrak{r}\dot{\mathfrak{k}} = \mathfrak{r}\ddot{\mathfrak{k}} = 0,$$

so the line $\mathfrak{r}$ will be the principal tangent to the nodal ruled family.

The line system that was treated in no. **96** is a special *W*-system with two ruled surfaces for its focal surfaces; *both focal surfaces are hyperboloids.*

Discrete-geometric model for the *W*-system with two ruled surfaces for its focal surfaces:

Two discrete line sequences $\mathfrak{q}^0, \mathfrak{q}^{\mathrm{I}}, \mathfrak{q}^{\mathrm{II}}, \ldots$ and $\mathfrak{q}^{*0}, \mathfrak{q}^{*\mathrm{I}}, \mathfrak{q}^{*\mathrm{II}}, \ldots$, with the hyperboloidal line-quadruples $\mathfrak{q}^0, \mathfrak{q}^{\mathrm{I}}, \mathfrak{q}^{*0}, \mathfrak{q}^{*\mathrm{I}}$, as well as $\mathfrak{q}^{\mathrm{I}}, \mathfrak{q}^{\mathrm{II}}, \mathfrak{q}^{*\mathrm{I}}, \mathfrak{q}^{*\mathrm{II}}$, etc., enter in place of the ruled families $\mathfrak{q} = \mathfrak{q}(u)$ and $\mathfrak{q}^* = \mathfrak{q}^*(u)$ (no. **106**). $\mathfrak{q}^{*\mathrm{I}}$ must then lie hyperboloidally with $\mathfrak{q}^0, \mathfrak{q}^{\mathrm{I}}, \mathfrak{q}^{*0}$, and furthermore, $\mathfrak{q}^{*\mathrm{II}}$ must lie hyperboloidally with $\mathfrak{q}^{\mathrm{I}}, \mathfrak{q}^{\mathrm{II}}, \mathfrak{q}^{*\mathrm{I}}$.

The special case for which the *W*-system is constructed from the first osculating quadric of a hyperbolic ruled family corresponds, as a model, to two discrete line sequences $\mathfrak{q}^0, \mathfrak{q}^{\mathrm{I}}, \mathfrak{q}^{\mathrm{II}}, \ldots$ and $\mathfrak{q}^{*0}, \mathfrak{q}^{*\mathrm{I}}, \mathfrak{q}^{*\mathrm{II}}, \ldots$ that possess a common sequence of "nodal lines" $\mathfrak{p}^0, \mathfrak{p}^{\mathrm{I}}, \mathfrak{p}^{\mathrm{II}}, \ldots$, as in Figure 16.

108. Laplacian cycles ([1]) and *W*-systems. Let Φ, Φ' be the focal surfaces of a hyperbolic line system I, and let u^1, u^2 be the torsion parameters, in the terminology of no. **81**. The tangents to the lateral curves u^1 = const. of Φ, and likewise, the tangents to the lateral curves u^2 = const. of Φ', will then define two more hyperbolic line systems II (III, resp.). II has yet another focal surface Φ''', along with Φ; III has the focal surface Φ'', along with Φ'. If one now considers the lateral tangents on Φ'', Φ''', as well, then one

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([1]) Cf., on this, volume 22 of this collection: **E Salkowski:** *Affine Differentialgeometrie*, 1934, pp. 160; furthermore, **G. Darboux**: *Théorie des surfaces II*, Paris, 1925.

will generally obtain two new line systems and two new focal surfaces Φ^{IV}, Φ^{V}, and by continuing with that process, one will get an infinite chain of line systems and focal surfaces. However, it can happen that Φ^{IV}, Φ^{V} coincide with Φ''', Φ'', resp. The chain then "closes" and reduces to the so-called *four-term Laplacian cycle* ([1]) of the four surfaces Φ, Φ', Φ'', Φ'''.

The surfaces of a Laplacian cycle are related to each other point-wise by the line systems I, II, III. The skew rectangle that is defined by the point-quadruple x, x', x'', x''' generates the four surfaces of the cycle as the geometric locus of its vertices, and likewise as the enveloping surfaces of the planes that are spanned by the sides of the rectangle. From no. **80**, paragraph I, the osculating planes of the contact net of a surface Φ of the cycle are simultaneously the contact planes of the "adjacent" surfaces Φ', Φ''', and thus coincide with the osculating planes of the contact net of the "opposite" surface Φ''.

We cannot go deeper into the appealing properties of Laplacian cycles. We mention, in passing, the following theorem that **H. Jonas** ([2]) recently found:

The connecting lines of "opposite" points of a four-term Laplacian cycle define a W-system.

([1]) One will find examples in **H. Jonas**: Berliner Math. Ges. Ber. **29** (1930); Math. Ann. **87** (1922), and Sächs. Akad. Ber. **87** (1935).

([2]) **H. Jonas**, Math. Ann. **114** (1937).

CHAPTER V.

Infinitesimal bending of surfaces

§ 30. Screw cracks.

109. Concept of the infinitesimal bending of a surface. In § 11, we examined infinitesimal wrinkles of nets of rectangles by means of line geometry. In this chapter, we would now like to treat the differential-geometric analogues of these discrete-geometric relationships, namely, the so-called infinitesimal bending of surfaces. As in § 11, we also start here with *the posing of metric problems* and will then eventually arrive at some remarkable *projective-invariant properties* ([1]). In particular, we will soon deduce a close connection with the theory of *W*-systems.

Let a non-developable surface (*X*) be given by:

$$\mathfrak{X} = \mathfrak{X}(u^1, u^2),$$

with the parameter domain $u_a^1 \leq u^1 \leq u_e^1$, $u_a^2 \leq u^2 \leq u_e^2$. We refer to a deformation:

$$\mathfrak{X}^* = \mathfrak{X}(u^1, u^2) + \varepsilon\,\bar{\mathfrak{X}}(u^1, u_2) \qquad (\varepsilon = \text{const.}) \tag{183}$$

as an *infinitesimal bending* of the surface (*X*) when the auxiliary condition:

$$\dot{\mathfrak{X}}^*\dot{\mathfrak{X}}^* = \dot{\mathfrak{X}}\dot{\mathfrak{X}} + \varepsilon^2\dot{\bar{\mathfrak{X}}}\dot{\bar{\mathfrak{X}}} \tag{184}$$

is fulfilled for all $u^1 = u^1(t)$, $u^2 = u^2(t)$. That means: The arc lengths of the curves that lie on the surface (*X*) remain unchanged under the deformation (183) for $\varepsilon \to 0$ in order of magnitude ε. (184) is equivalent to the requirement that:

$$\dot{\mathfrak{X}}\dot{\bar{\mathfrak{X}}} = (\mathfrak{X}_1\dot{u}^1 + \mathfrak{X}_2\dot{u}^2)(\bar{\mathfrak{X}}_1\dot{u}^1 + \bar{\mathfrak{X}}_2\dot{u}^2) = 0; \tag{185}$$

the indices mean partial derivatives with respect to u^1 (u^2, resp.). Since (185) should be fulfilled by the mutually-independent functions $\dot{u}^1$, $\dot{u}^2$, the factors of $\dot{u}^1\dot{u}^1$, $\dot{u}^2\dot{u}^2$, $\dot{u}^1\dot{u}^2$ must vanish individually, so:

$$\mathfrak{X}_1\bar{\mathfrak{X}}_1 = \mathfrak{X}_2\bar{\mathfrak{X}}_2 = \mathfrak{X}_1\bar{\mathfrak{X}}_2 + \mathfrak{X}_2\bar{\mathfrak{X}}_1 = 0. \tag{186}$$

([1]) Cf., for this, volume 22 of this collection: **E. Salkowski**, *Affine Differentialgeometrie*, 1934. In chapter 9, the infinitesimal bending of surfaces is treated in the context of *affine* geometry. Moreover, one should confer the presentation of the bending of surfaces in **L. Bianchi**, *Lezioni di geometria differenziale II*, Bologna, 1923.

The existence ([1]) of a vector $\mathfrak{Y}(u^1, u^2)$, for which:

$$(187) \qquad \dot{\bar{\mathfrak{X}}} = \mathfrak{Y} \times \dot{\mathfrak{X}}_1, \qquad \text{so} \qquad \bar{\mathfrak{X}}_1 = \mathfrak{Y} \times \mathfrak{X}_1, \qquad \bar{\mathfrak{X}}_2 = \mathfrak{Y} \times \mathfrak{X}_2$$

follows from (186); i.e., $\mathfrak{Y} \times d\mathfrak{X}$ is a *complete differential* $d\bar{\mathfrak{X}}$. Since:

$$d\,(\mathfrak{Y} \times \mathfrak{X}) = \mathfrak{Y} \times d\mathfrak{X} - \mathfrak{X} \times d\mathfrak{Y},$$

$\mathfrak{X} \times d\mathfrak{Y}$ is also the complete differential of a function $\bar{\mathfrak{Y}}(u^1, u^2)$ then, and we have:

$$(188) \qquad \dot{\bar{\mathfrak{Y}}} = \mathfrak{X} \times \dot{\mathfrak{Y}}, \qquad \bar{\mathfrak{Y}}_1 = \mathfrak{X} \times \mathfrak{Y}_1, \qquad \bar{\mathfrak{Y}}_2 = \mathfrak{X} \times \mathfrak{Y}_2.$$

The vector $\bar{\mathfrak{Y}}$ is determined from $\mathfrak{X}$ and $\mathfrak{Y}$, up to an additive constant vector, in that way. By a suitable choice of that constant vector, one can always arrange that:

$$(189) \qquad \bar{\mathfrak{Y}} - \bar{\mathfrak{X}} = \mathfrak{X} \times \mathfrak{Y};$$

from (187) and (188), the differentials $d(\bar{\mathfrak{Y}} - \bar{\mathfrak{X}})$ and $d(\mathfrak{X} \times \mathfrak{Y})$ are equal to each other then.

110. Definition of the screw crack. *Any tangent pencil of the surface* (X) *will experience the infinitesimal screw:*

$$(190) \qquad \varepsilon\,\mathfrak{y} = \{\varepsilon\,\mathfrak{Y} \mid \varepsilon\,\bar{\mathfrak{Y}}\}$$

under the infinitesimal bending (183).

Proof: From (189), the infinitesimal displacement $\varepsilon\,\bar{\mathfrak{X}}$ of a point P on the surface can be decomposed into:

$$\varepsilon\,\bar{\mathfrak{X}} = \varepsilon\,\mathfrak{Y} \times \mathfrak{X} + \varepsilon\,\bar{\mathfrak{Y}}.$$

The shift $\varepsilon\,\mathfrak{Y} \times \mathfrak{X}$ can then be produced by a rotation of the surface point P around a rotational axis that goes through the origin O with the rotation vector $\varepsilon\,\mathfrak{Y}$. At the same time, from (187), any surface tangent through the point P will be brought into the direction that is prescribed by (183) under this rotation. By adding the parallel displacement $\varepsilon\,\bar{\mathfrak{Y}}$, the entire tangent pencil with the vertex P will then go to the new position that corresponds to the infinitesimal bending (183).

([1]) Upon excluding $\bar{\mathfrak{X}}_1 \times \bar{\mathfrak{X}}_2 = 0$ (page 163, footnote), due to (186_2), one will also have $\mathfrak{X}_1 \bar{\mathfrak{X}}_2 = -\mathfrak{X}_2 \bar{\mathfrak{X}}_1 \neq 0$. One then gets from (186) that:

$$\mathfrak{Y} = \frac{\bar{\mathfrak{X}}_1 \times \bar{\mathfrak{X}}_2}{\mathfrak{X}_2 \bar{\mathfrak{X}}_1} = \frac{\bar{\mathfrak{X}}_2 \times \bar{\mathfrak{X}}_1}{\mathfrak{X}_2 \bar{\mathfrak{X}}_1}.$$

Three more surfaces (Y), ($\bar{Y}$), ($\bar{X}$) are defined by $\mathfrak{Y}$, $\bar{\mathfrak{Y}}$, $\bar{\mathfrak{X}}$ as position vectors. We assume that these surfaces do not degenerate into curves or points ([1]). Due to the kinematic interpretation of $\mathfrak{Y}$, $\bar{\mathfrak{Y}}$, we refer to the surface (Y) as a *rotational crack,* the surface ($\bar{Y}$), as a *displacement crack*, and the surface-pair (Y), ($\bar{Y}$) as a *screw crack* relative to the infinitesimal bending (183). When the rotational crack (Y) is given, the displacement crack ($\bar{Y}$), as well as the function $\bar{\mathfrak{X}}(u^1, u^2)$ that characterizes the infinitesimal bending (183), can be determined by means of (188) by quadratures and by means of (189). In contrast to the rotational crack, the displacement crack will depend upon the position of the origin O.

As equations (187) and (188) show, the surface-pairs (Y), ($\bar{Y}$) and (X), ($\bar{X}$) are equivalent. *The surface-pair can also be regarded as a screw crack then, namely, with respect to the infinitesimal bending:*

(191) $$\mathfrak{Y}^* = \mathfrak{Y} + \varepsilon\,\bar{\mathfrak{Y}}$$

of the surface (Y).

111. Integrability conditions. From (187), the integrability condition:

$$\frac{\partial^2\bar{\mathfrak{X}}}{\partial u^1\,\partial u^2} = \frac{\partial^2\bar{\mathfrak{X}}}{\partial u^2\,\partial u^1}$$

implies that:

(192) $$\mathfrak{Y}_2 \times \mathfrak{X}_1 = \mathfrak{Y}_1 \times \mathfrak{X}_2\,;$$

i.e., the vectors $\mathfrak{X}_1$, $\mathfrak{X}_2$, $\mathfrak{Y}_1$, $\mathfrak{Y}_2$, are linearly-dependent. As a result, one can replace $\mathfrak{Y}_1$, $\mathfrak{Y}_2$ with linear combinations of $\mathfrak{X}_1$, $\mathfrak{X}_2$:

$$\mathfrak{Y}_1 = \mu\,\mathfrak{X}_1 + \lambda\,\mathfrak{X}_2\,, \qquad \mathfrak{Y}_2 = -\,\nu\,\mathfrak{X}_1 - \mu'\mathfrak{X}_2\,.$$

Upon substituting this into (192), we will get:

$$\mu'\mathfrak{X}_2 \times \mathfrak{X}_1 = \mu\,\mathfrak{X}_2 \times \mathfrak{X}_1, \qquad \text{so} \qquad \mu' = \mu,$$

and thus:

(193) $$\mathfrak{Y}_1 = \mu\,\mathfrak{X}_1 + \lambda\,\mathfrak{X}_2\,, \qquad \mathfrak{Y}_2 = -\,\nu\,\mathfrak{X}_1 - \mu\,\mathfrak{X}_2\,.$$

Due to the linear independence of $\mathfrak{Y}_1$, $\mathfrak{Y}_2$, one will have the auxiliary condition:

$$\lambda\nu - \mu^2 \neq 0.$$

It follows further from (188) that:

([1]) One would then have $\mathfrak{Y}_1 \times \mathfrak{Y}_2 \neq 0$, $\bar{\mathfrak{Y}}_1 \times \bar{\mathfrak{Y}}_2 \neq 0$, $\bar{\mathfrak{X}}_1 \times \bar{\mathfrak{X}}_2 \neq 0$.

(194) $$\bar{\mathfrak{Y}}_1 = \mu\,\mathfrak{X}\times\mathfrak{X}_1 + \lambda\,\mathfrak{X}\times\mathfrak{X}_2, \quad \bar{\mathfrak{Y}}_2 = -\nu\,\mathfrak{X}\times\mathfrak{X}_1 - \mu\,\mathfrak{X}\times\mathfrak{X}_2\,.$$

Let the tangents to the curves u^2 = const. (u^1 = const.) on the surface (X) be given by the singular six-vectors:

(195) $$\mathfrak{p} = \{\mathfrak{X}_1 \mid \mathfrak{X}\times\mathfrak{X}_1\} \qquad (\mathfrak{q} = \{\mathfrak{X}_2 \mid \mathfrak{X}\times\mathfrak{X}_2\}, \text{ resp.}).$$

With the use of these six-vectors $\mathfrak{p}$, $\mathfrak{q}$, and the six-vector $\mathfrak{y}$ that was defined in (190), the two equations (193) and (194) can be combined into:

(196) $$\mathfrak{y}_1 = \mu\,\mathfrak{p} + \lambda\,\mathfrak{q}, \qquad \mathfrak{y}_2 = -\nu\,\mathfrak{p} - \mu\,\mathfrak{q}\,.$$

This implies:

(197) $$\frac{\partial}{\partial u^2}(\mu\,\mathfrak{p} + \lambda\,\mathfrak{q}) + \frac{\partial}{\partial u^1}(\nu\,\mathfrak{p} + \mu\,\mathfrak{q}) = 0$$

as a further integrability condition.

Mapping the surfaces (X), (Y), ($\bar{Y}$), ($\bar{X}$) **to each other:**

We associate the points of the surfaces (X), (Y), ($\bar{Y}$), ($\bar{X}$) with each other when they belong to the same values of the parameters u^1, u^2. It will then follow from (187) [(188), resp.] that:

a) *Any tangent to the surface* (X) [(Y), resp.] *corresponds to a perpendicular tangent to the surface* ($\bar{X}$) [($\bar{Y}$), resp.].

b) *Any contact plane of the surface* ($\bar{X}$) [($\bar{Y}$), resp.] *is perpendicular to the position vector of the corresponding point of the surface* (Y) [(x), resp.].

From (192), one will have:

c) *The contact planes to corresponding points of the surface* (X) *and* (Y) *are parallel to each other.*

Any surface-pair (X), ($\bar{X}$) with the reciprocal relationship *a*) will satisfy equation (187) and thus defines an infinitesimal bending. By contrast, the map of the surfaces (X), (Y) that is given by (192) is still not determined completely by the property *c*). The requirement of the parallelism of corresponding contact planes then brings with it the additional condition $\mu = \mu'$ in (193). We will interpret that condition geometrically in nos. **118** and **119** (Figures 28, 29).

§ 31. Infinitesimal bending of mutually-projective surfaces.

112. Bending of collinear surfaces. Although infinitesimal bending was defined by metric concepts, the following projective-geometric theorem is true ([1]):

Any surface $(\tilde{X})$ *that is collinear to a surface* (X) *will yield a screw crack* $(\tilde{Y})$, $(\bar{\tilde{Y}})$ *from the screw crack* (Y), $(\bar{Y})$ *of* (X) *in such a way that one subjects the screw coordinates of* $\mathfrak{y}$ *and the line coordinates of the surface tangents of* (X) *to the same linear transformation with constant coefficients – i.e., ones that are independent of* u^1, u^2. *One can derive an infinitesimal bending for any collinear surface from a given infinitesimal bending of a surface in that way.*

Proof: The homogeneous, rectangular point coordinates x_i (u^1, u^2) of the points of the surface (X) will be transformed by (3a) into $\tilde{x}_i$ (u^1, u^2), while the line coordinates p_ρ of the surface tangents will be transformed into $\tilde{p}_\rho$ by (19). Thus, the transformation coefficients γ_ρ^σ will be mapped to the α_i^k by (20a), and like them, they should also be constant; i.e., independent of u^1, u^2. We now compute, for example, the first coordinate of the vector $\bar{\mathfrak{X}}_1 = \dfrac{\partial}{\partial u^1}\left(\dfrac{\tilde{x}_1}{\tilde{x}_4}\right)$ and obtain, with consideration given to (195) and (19):

$$\begin{aligned}
\frac{\partial}{\partial u^1}\left(\frac{\tilde{x}_1}{\tilde{x}_4}\right) &= \frac{1}{(\tilde{x}_4)^2}\left(\tilde{x}_4\frac{\partial \tilde{x}_1}{\partial u^1} - \tilde{x}_1\frac{\partial \tilde{x}_4}{\partial u^1}\right) \\
&= \frac{1}{(\alpha_4^1 x_1 + \cdots + \alpha_4^4 x_4)^2}\left\{(\alpha_4^1 x_1 + \cdots + \alpha_4^4 x_4)\left(\alpha_1^1\frac{\partial x_1}{\partial u^1} + \cdots + \alpha_1^4\frac{\partial x_4}{\partial u^1}\right)\right. \\
&\qquad\qquad \left. -\ (\alpha_1^1 x_1 + \cdots + \alpha_1^4 x_4)\left(\alpha_4^1\frac{\partial x_1}{\partial u^1} + \cdots + \alpha_4^4\frac{\partial x_4}{\partial u^1}\right)\right\} \\
&= \frac{(x_4)^2}{(\alpha_4^1 x_1 + \cdots + \alpha_4^4 x_4)^2}\left\{(\alpha_1^1\alpha_4^4 - \alpha_1^4\alpha_4^1)\frac{\partial x_1}{\partial u^1}\left(\frac{x_1}{x_4}\right) + \cdots + (\alpha_1^2\alpha_4^4 - \alpha_1^1\alpha_4^2)\right. \\
&\qquad\qquad \left. - \left(\frac{x_1}{x_4}\frac{\partial}{\partial u^1}\left(\frac{x_2}{x_4}\right) + \cdots + \frac{x_2}{x_4}\frac{\partial}{\partial u^1}\left(\frac{x_1}{x_4}\right)\right)\right\} \\
&= \frac{(x_4)^2}{(\alpha_4^1 x_1 + \cdots + \alpha_4^4 x_4)^2}\{\gamma_1^1 p_1 + \cdots + \gamma_1^6 p_6\} = \frac{(x_4)^2}{(\alpha_4^1 x_1 + \cdots + \alpha_4^4 x_4)^2}\tilde{p}_1 .
\end{aligned}$$

([1]) For this and the following numbers, cf., **R. Sauer**, "Infinitesimale Verbiegungen zueinander projektiver Flächen," Math. Ann. **111** (1935); one already finds the results without the kinematical interpretation (but not in a line-geometric representation) in **G. Darboux**, *Théorie des surfaces IV*, Paris, 1925.

Thus, the first coordinate of the six-vector $\tilde{\mathfrak{p}}$ and the first coordinate of the vector $\tilde{\mathfrak{X}}$ differ by the factor $\frac{(x_4)^2}{(\alpha_4^1 x_1 + \cdots + \alpha_4^4 x_4)^2}$, and since the six-vectors $\tilde{\mathfrak{p}}$ and $\{\tilde{\mathfrak{X}}_1 \mid \tilde{\mathfrak{X}} \times \tilde{\mathfrak{X}}_1\}$ are proportional, one will have:

$$\tilde{\mathfrak{p}} = \omega^2 \{\tilde{\mathfrak{X}}_1 \mid \tilde{\mathfrak{X}} \times \tilde{\mathfrak{X}}_1\} \qquad (\tilde{\mathfrak{q}} = \omega^2 \{\tilde{\mathfrak{X}}_2 \mid \tilde{\mathfrak{X}} \times \tilde{\mathfrak{X}}_2\}), \tag{198}$$

with:

$$\omega = \frac{1}{x_4}(\alpha_4^1 x_1 + \cdots + \alpha_4^4 x_4). \tag{199}$$

Due to the constancy of the transformation coefficients γ_ρ^σ in (19), it follows from (197) that:

$$\frac{\partial}{\partial u^2}(\mu\,\tilde{\mathfrak{p}} + \lambda\,\tilde{\mathfrak{q}}) + \frac{\partial}{\partial u^1}(\nu\,\tilde{\mathfrak{p}} + \mu\,\tilde{\mathfrak{q}}) = 0$$

for the collinear surface $(\tilde{X})$ so, from (198):

$$\frac{\partial}{\partial u^2}(\tilde{\mu}\,\tilde{\mathfrak{X}}_1 + \tilde{\lambda}\,\tilde{\mathfrak{X}}_2) + \frac{\partial}{\partial u^1}(\tilde{\nu}\,\tilde{\mathfrak{X}}_1 + \tilde{\mu}\,\tilde{\mathfrak{X}}_2) = 0,$$

$$\frac{\partial}{\partial u^2}(\tilde{\mu}\,\tilde{\mathfrak{X}} \times \tilde{\mathfrak{X}}_1 + \tilde{\lambda}\,\tilde{\mathfrak{X}} \times \tilde{\mathfrak{X}}_2) + \frac{\partial}{\partial u^1}(\tilde{\nu}\,\tilde{\mathfrak{X}} \times \tilde{\mathfrak{X}}_1 + \tilde{\mu}\,\tilde{\mathfrak{X}} \times \tilde{\mathfrak{X}}_2) = 0,$$

with:

$$\tilde{\lambda} = \lambda\,\omega^2, \qquad \tilde{\mu} = \mu\,\omega^2, \qquad \tilde{\nu} = \nu\,\omega^2. \tag{200}$$

As a result, one can set:

$$\begin{aligned} \tilde{\mathfrak{Y}}_1 &= \tilde{\mu}\,\tilde{\mathfrak{X}}_1 + \tilde{\lambda}\,\tilde{\mathfrak{X}}_2, & \tilde{\mathfrak{Y}}_2 &= -\tilde{\nu}\,\tilde{\mathfrak{X}}_1 - \tilde{\mu}\,\tilde{\mathfrak{X}}_2, \\ \bar{\tilde{\mathfrak{Y}}}_1 &= \tilde{\mu}\,\tilde{\mathfrak{X}} \times \tilde{\mathfrak{X}}_1 + \tilde{\lambda}\,\tilde{\mathfrak{X}} \times \tilde{\mathfrak{X}}_2, & \bar{\tilde{\mathfrak{Y}}}_2 &= -\tilde{\nu}\,\tilde{\mathfrak{X}} \times \tilde{\mathfrak{X}}_1 - \tilde{\mu}\,\tilde{\mathfrak{X}} \times \tilde{\mathfrak{X}}_2, \end{aligned} \tag{201}$$

as well as:

$$\bar{\tilde{\mathfrak{X}}}_1 = \tilde{\mathfrak{Y}} \times \tilde{\mathfrak{X}}_1, \qquad \bar{\tilde{\mathfrak{X}}}_2 = \tilde{\mathfrak{Y}} \times \tilde{\mathfrak{X}}_2; \tag{202}$$

hence, due to (201), the integrability condition:

$$\frac{\partial^2 \bar{\tilde{\mathfrak{X}}}}{\partial u^1\,\partial u^2} = \frac{\partial^2 \bar{\tilde{\mathfrak{X}}}}{\partial u^2\,\partial u^1}$$

will be fulfilled. From (201), $(\tilde{Y})$, $(\bar{\tilde{Y}})$ is the screw crack of $(\tilde{X})$ for the infinitesimal bending:

$$\tilde{\mathfrak{x}}^* = \tilde{\mathfrak{x}} + \varepsilon \bar{\tilde{\mathfrak{x}}}.$$

Due to (198), equations (201) can be combined into:

$$\bar{\mathfrak{y}}_1 = \mu\,\bar{\mathfrak{p}} + \lambda\,\bar{\mathfrak{q}}, \qquad \bar{\mathfrak{y}}_2 = -\nu\,\bar{\mathfrak{p}} - \mu\,\bar{\mathfrak{q}}.$$

The juxtaposition of these relations with (196) shows that the six-vectors $\mathfrak{y}_1$, $\mathfrak{y}_2$ – and thus, up to some inessential constants, the six-vector $\mathfrak{y}$ itself – transform in the same way as the singular six-vectors $\mathfrak{p}$, $\mathfrak{q}$ of the surface tangents.

Geometric interpretation of ω*:* The so-called flight plane:

$$\alpha_4^1 x_1 + \cdots + \alpha_4^4 x_4 = 0$$

is mapped to the virtual plane $\tilde{x}_4 = 0$ under the given collineation (3a). From (199), ω is, up to a constant factor, equal to the distance from the point of the surface (X) from the flight plane.

113. Bending of correlative surfaces.

The theorem that was stated for collineations in no. **112** *is also true for correlations.*

Proof: The homogeneous, rectangular point coordinates x_i (u^1, u^2) of the points of (X) will now be transformed by (3b) into the plane coordinates $\tilde{w}^i$ (u^1, u^2) of the contact planes of the correlative surface $(\tilde{X})$. Moreover, the line coordinates p_ρ (q_ρ, resp.) of the tangents to the curves u^2 = const. (u^1 = const., resp.) of (X), which are normalized by (195), will go to the line coordinates $\tilde{p}_\rho$ ($\tilde{q}_\rho$, resp.) of the tangents to the family of curves on $(\tilde{X})$ that are conjugate to u^2 = const. (u^1 = const., resp.) under (19) [(20b), resp.]; the tangents to a surface curve are then mapped to the conjugate tangents along the corresponding surface curves under a correlation. We will once more assume that the coefficients of the transformation equations (3b) and (19).

One will obtain:

$$\tilde{\mathfrak{p}} = \omega^2 \{\mathfrak{W} \times \tilde{\mathfrak{W}}_1 \mid \tilde{\mathfrak{W}}_1\} \qquad (\tilde{\mathfrak{q}} = \omega^2 \{\mathfrak{W} \times \tilde{\mathfrak{W}}_2 \mid \tilde{\mathfrak{W}}_2\}),$$

by calculations that are analogous to then ones in no. **112**, and furthermore:

$$\text{(203)} \qquad \begin{aligned} \tilde{\mathfrak{Y}}_1 &= \tilde{\mu}\,\tilde{\mathfrak{W}} \times \tilde{\mathfrak{W}}_1 + \tilde{\lambda}\,\tilde{\mathfrak{W}} \times \tilde{\mathfrak{W}}_2, & \tilde{\mathfrak{Y}}_2 &= -\tilde{\nu}\,\tilde{\mathfrak{W}} \times \tilde{\mathfrak{W}}_1 - \tilde{\mu}\,\tilde{\mathfrak{W}} \times \tilde{\mathfrak{W}}_2, \\ \bar{\tilde{\mathfrak{Y}}}_1 &= \tilde{\mu}\,\tilde{\mathfrak{W}}_1 + \tilde{\lambda}\,\tilde{\mathfrak{W}}_2, & \bar{\tilde{\mathfrak{Y}}}_2 &= -\tilde{\nu}\,\tilde{\mathfrak{W}}_1 - \tilde{\mu}\,\tilde{\mathfrak{W}}_2, \end{aligned}$$

with functions ω, $\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{\nu}$ that are defined by:

$$\omega = \frac{1}{x_4}(\beta^{41}x_1 + \beta^{42}x_2 + \beta^{43}x_3 + \beta^{44}x_4).$$

The six-vectors $\{\tilde{\mathfrak{X}}_1 \mid \tilde{\mathfrak{X}}\times\tilde{\mathfrak{X}}_1\}$, $\{\tilde{\mathfrak{X}}_2 \mid \tilde{\mathfrak{X}}\times\tilde{\mathfrak{X}}_2\}$ ($\tilde{\mathfrak{p}}$, $\tilde{\mathfrak{q}}$, resp.) yield the tangents to the curves u^2 = const., u^1 = const. (the tangents to the surface $(\tilde{X})$ that are conjugate to them, resp.). As a result, the following linear combinations exist:

$$\mathfrak{W}\times\tilde{\mathfrak{W}}_1 = \delta_{11}\tilde{\mathfrak{X}}_1 + \delta_{12}\tilde{\mathfrak{X}}_2, \qquad \mathfrak{W}\times\tilde{\mathfrak{W}}_2 = \delta_{21}\tilde{\mathfrak{X}}_1 + \delta_{22}\tilde{\mathfrak{X}}_2,$$
$$\tilde{\mathfrak{W}}_1 = \delta_{11}\tilde{\mathfrak{X}}\times\tilde{\mathfrak{X}}_1 + \delta_{12}\tilde{\mathfrak{X}}\times\tilde{\mathfrak{X}}_2, \qquad \tilde{\mathfrak{W}}_2 = \delta_{21}\tilde{\mathfrak{X}}\times\tilde{\mathfrak{X}}_1 + \delta_{22}\tilde{\mathfrak{X}}\times\tilde{\mathfrak{X}}_2.$$

By substituting this into (203), one will get:

$$\bar{\mathfrak{Y}}_1 = \tilde{M}\tilde{\mathfrak{X}}_1 + \tilde{\Lambda}\tilde{\mathfrak{X}}_2, \qquad \bar{\mathfrak{Y}}_2 = -\tilde{N}\tilde{\mathfrak{X}}_1 - \tilde{M}'\tilde{\mathfrak{X}}_2,$$
$$\bar{\bar{\mathfrak{Y}}}_1 = \tilde{M}\tilde{\mathfrak{X}}\times\tilde{\mathfrak{X}}_1 + \tilde{\Lambda}\tilde{\mathfrak{X}}\times\tilde{\mathfrak{X}}_2, \qquad \bar{\bar{\mathfrak{Y}}}_2 = -\tilde{N}\tilde{\mathfrak{X}}\times\tilde{\mathfrak{X}}_1 - \tilde{M}'\tilde{\mathfrak{X}}\times\tilde{\mathfrak{X}}_2.$$

It follows from:

$$\bar{\bar{\mathfrak{Y}}}_1 = \tilde{\mathfrak{X}}\times\tilde{\mathfrak{Y}}_1, \qquad \bar{\bar{\mathfrak{Y}}}_2 = \tilde{\mathfrak{X}}\times\tilde{\mathfrak{Y}}_2$$

that the integrability condition is:

$$\tilde{\mathfrak{X}}_2\times\tilde{\mathfrak{Y}}_1 = \tilde{\mathfrak{X}}_1\times\tilde{\mathfrak{Y}}_2,$$

and it follows from this that $\tilde{M}' = \tilde{M}$, as in (193). Thus, $(\tilde{Y})$, $(\tilde{\bar{Y}})$ is a screw crack of $(\tilde{X})$. Just as in no. **112**, one shows that $\mathfrak{y}$, $\mathfrak{p}$, $\mathfrak{q}$ go to $\tilde{\mathfrak{y}}$, $\tilde{\mathfrak{p}}$, $\tilde{\mathfrak{q}}$, resp., by the same linear transformation.

Special correlation: polarity.

$\mathfrak{p} = \{\mathfrak{P} \mid \bar{\mathfrak{P}}\}$ will be transformed into $\tilde{\mathfrak{p}} = \{\bar{\mathfrak{P}} \mid \mathfrak{P}\}$ under the polarity (no. **14**):

$$\tilde{p}_\rho = p_{\rho\pm 3},$$

and therefore $\mathfrak{y} = \{\mathfrak{Y} \mid \bar{\mathfrak{Y}}\}$ will also be transformed into $\tilde{\mathfrak{y}} = \{\bar{\mathfrak{Y}} \mid \mathfrak{Y}\}$. Thus, if (Y), $(\bar{Y})$ is a screw crack of the surface (X) then $(\bar{Y})$, (Y) will be a screw crack of the surface $(\tilde{X})$ that is polar to (X); i.e., the rotational crack (Y) and the displacement crack $(\bar{Y})$ will exchange their meanings.

§ 32. Connection with the theory of *W*-systems.

114. Determination of a *W*-system for a given surface bending. *The points of* $(\bar{X})$, $(\bar{Y})$ *that correspond to connecting lines define a system of lines with* $(\bar{X})$, $(\bar{Y})$ *as its focal surfaces.*

Proof: From (189) and (187) [(188), resp.], the vectors $\bar{\mathfrak{Y}} - \bar{\mathfrak{X}}$, $\bar{\mathfrak{X}}_1$, $\bar{\mathfrak{X}}_2$ [$\bar{\mathfrak{Y}} - \bar{\mathfrak{X}}$, $\bar{\mathfrak{Y}}_1$, $\bar{\mathfrak{Y}}_2$, resp.] are perpendicular to the vector $\mathfrak{Y}$ ($\mathfrak{X}$, resp.}, and are therefore linearly dependent; i.e., the points of $(\bar{X})$ and $(\bar{Y})$ that correspond to connecting lines lie in the corresponding contact plane of the surfaces $(\bar{X})$ and $(\bar{Y})$.

Any conjugate net of curves on $(\bar{X})$ *corresponds to a conjugate net of curves on* $(\bar{Y})$, *and conversely. Thus, when one of the two surfaces is negatively curved, its principal tangent curves will correspond to the principal tangent curves of the other surface, and the letter will then be negatively curved, as well.*

Proof: From (8), we must confirm the proportion:

$$< \bar{\mathfrak{X}}_1, \bar{\mathfrak{X}}_2, \bar{\mathfrak{X}}_{u^1u^2} > \,:\, < \bar{\mathfrak{X}}_1, \bar{\mathfrak{X}}_2, \bar{\mathfrak{X}}_{u^1u^2} > \,:\, < \bar{\mathfrak{X}}_1, \bar{\mathfrak{X}}_2, \bar{\mathfrak{X}}_{u^1u^2} >$$
$$= \,< \bar{\mathfrak{Y}}_1, \bar{\mathfrak{Y}}_2, \bar{\mathfrak{Y}}_{u^1u^2} > \,:\, < \bar{\mathfrak{Y}}_1, \bar{\mathfrak{Y}}_2, \bar{\mathfrak{Y}}_{u^1u^2} > \,:\, < \bar{\mathfrak{Y}}_1, \bar{\mathfrak{Y}}_2, \bar{\mathfrak{Y}}_{u^1u^2} > .$$

Now, from (188), (1), and (193), one has, e.g.:

$$< \bar{\mathfrak{Y}}_1, \bar{\mathfrak{Y}}_2, \bar{\mathfrak{Y}}_{u^1u^2} > \,=\, <\mathfrak{X}\times\mathfrak{Y}_1,\ \mathfrak{X}\times\mathfrak{Y}_2,\ \mathfrak{X}_1\times\mathfrak{Y}_1> \,=\, \lambda\,(\lambda\nu - \mu^2)\, <\mathfrak{X}\times\mathfrak{Y}_1,\ \mathfrak{X}\times\mathfrak{Y}_2,\ \mathfrak{X}_1\times\mathfrak{Y}_1>$$
$$= \lambda\,(\lambda\nu - \mu^2)\, <\mathfrak{X},\ \mathfrak{X}_1,\ \mathfrak{X}_2>^2,$$

and likewise, from (187) and (193);

$$< \bar{\mathfrak{X}}_1, \bar{\mathfrak{X}}_2, \bar{\mathfrak{X}}_{u^1u^2} > \,=\, -\frac{\lambda}{(\lambda\nu - \mu^2)^2} <\mathfrak{Y},\ \mathfrak{Y}_1,\ \mathfrak{Y}_2>^2;$$

in this, $\lambda\nu - \mu^2 \neq 0$, and except for some special points, one will also have:

$$<\mathfrak{X},\ \mathfrak{X}_1,\ \mathfrak{X}_2> \neq 0, \qquad <\mathfrak{Y},\ \mathfrak{Y}_1,\ \mathfrak{Y}_2> \neq 0.$$

From these and the corresponding formulas for the other triple products, it will then follow that:

$$< \bar{\mathfrak{X}}_1, \bar{\mathfrak{X}}_2, \bar{\mathfrak{X}}_{u^1u^2} > \,:\, < \bar{\mathfrak{X}}_1, \bar{\mathfrak{X}}_2, \bar{\mathfrak{X}}_{u^1u^2} > \,:\, < \bar{\mathfrak{X}}_1, \bar{\mathfrak{X}}_2, \bar{\mathfrak{X}}_{u^1u^2} > \,=\, \lambda : (-\mu) : \nu,$$
$$< \bar{\mathfrak{Y}}_1, \bar{\mathfrak{Y}}_2, \bar{\mathfrak{Y}}_{u^1u^2} > \,:\, < \bar{\mathfrak{Y}}_1, \bar{\mathfrak{Y}}_2, \bar{\mathfrak{Y}}_{u^1u^2} > \,:\, < \bar{\mathfrak{Y}}_1, \bar{\mathfrak{Y}}_2, \bar{\mathfrak{Y}}_{u^1u^2} > \,=\, \lambda : (-\mu) : \nu.$$

In summary, one obtains from the foregoing two theorems:

The points of $(\bar{X})$, $(\bar{Y})$ *that correspond to connecting lines define a W-system with* $(\bar{X})$, $(\bar{Y})$ *as its focal surfaces.*

115. Determination of a surface bending with a given *W*-system. We shall now show, conversely, how one can determine the surfaces (X), (Y) that are linked by the conditions (187) and (188) to a given arbitrary *W*-system with the focal surfaces $(\bar{X})$, $(\bar{Y})$, and thus obtain an infinitesimal bending of the surface (X).

From (188), $\mathfrak{X}$ is perpendicular to $\bar{\mathfrak{Y}}_1$ and $\bar{\mathfrak{Y}}_2$, so one can set:

$$\mathfrak{X} = \varphi\, \bar{\mathfrak{Y}}_1 \times \bar{\mathfrak{Y}}_2, \tag{204}$$

with the yet-to-be-determined function $\varphi(u^1, u^2)$. It follows from:

$$\mathfrak{X}_1 = \varphi_1\, \bar{\mathfrak{Y}}_1 \times \bar{\mathfrak{Y}}_2 + \varphi \bar{\mathfrak{Y}}_1 \times \bar{\mathfrak{Y}}_{u^1u^2} - \varphi \bar{\mathfrak{Y}}_2 \times \bar{\mathfrak{Y}}_{u^1u^1},$$
$$\mathfrak{X}_2 = \varphi_2\, \bar{\mathfrak{Y}}_1 \times \bar{\mathfrak{Y}}_2 + \varphi \bar{\mathfrak{Y}}_1 \times \bar{\mathfrak{Y}}_{u^2u^2} - \varphi \bar{\mathfrak{Y}}_2 \times \bar{\mathfrak{Y}}_{u^1u^2},$$

and the three requirements (186) that:

$$\begin{aligned}
&\varphi_1 <\bar{\mathfrak{Y}}_1, \bar{\mathfrak{Y}}_2, \bar{\mathfrak{X}}_1> + \varphi\{<\bar{\mathfrak{Y}}_1, \bar{\mathfrak{Y}}_{u^1u^2}, \bar{\mathfrak{X}}_1> - <\bar{\mathfrak{Y}}_2, \bar{\mathfrak{Y}}_{u^1u^1}, \bar{\mathfrak{X}}_1>\} = 0,\\
&\varphi_2 <\bar{\mathfrak{Y}}_1, \bar{\mathfrak{Y}}_2, \bar{\mathfrak{X}}_2> + \varphi\{<\bar{\mathfrak{Y}}_1, \bar{\mathfrak{Y}}_{u^1u^2}, \bar{\mathfrak{X}}_2> - <\bar{\mathfrak{Y}}_2, \bar{\mathfrak{Y}}_{u^1u^1}, \bar{\mathfrak{X}}_2>\} = 0,\\
&\varphi_1 <\bar{\mathfrak{Y}}_1, \bar{\mathfrak{Y}}_2, \bar{\mathfrak{X}}_2> + \varphi_1 <\bar{\mathfrak{Y}}_1, \bar{\mathfrak{Y}}_2, \bar{\mathfrak{X}}_1>\\
&+ \varphi\,\{<\bar{\mathfrak{Y}}_1, \bar{\mathfrak{Y}}_{u^1u^2}, \bar{\mathfrak{X}}_2> - <\bar{\mathfrak{Y}}_2, \bar{\mathfrak{Y}}_{u^1u^1}, \bar{\mathfrak{X}}_2> + <\bar{\mathfrak{Y}}_1, \bar{\mathfrak{Y}}_{u^1u^2}, \bar{\mathfrak{X}}_1> - <\bar{\mathfrak{Y}}_2, \bar{\mathfrak{Y}}_{u^1u^1}, \bar{\mathfrak{X}}_1>\} = 0.
\end{aligned} \tag{205}$$

These are three first-order partial differential equations for the desired function $\varphi(u^1, u^2)$. We shall show that *precisely one solution* $\varphi(u^1, u^2)$ of it exists, up to an arbitrary constant factor:

We first perform a transformation (cf., Figure 23) on the torse parameters of the given *W*-system. We will then have:

$$\begin{aligned}
\bar{\mathfrak{X}} - \bar{\mathfrak{Y}} &= \rho\, \bar{\mathfrak{X}}_1 = \sigma \bar{\mathfrak{Y}}_2 \qquad (\rho,\ \sigma \neq 0),\\
\bar{\mathfrak{X}}_1 &= \bar{\mathfrak{Y}}_1 + \sigma_1 \bar{\mathfrak{Y}}_2 + \sigma \bar{\mathfrak{Y}}_{u^1u^2},\\
\bar{\mathfrak{X}}_2 &= (1 + \sigma_2)\bar{\mathfrak{Y}}_2 + \sigma \bar{\mathfrak{Y}}_{u^1u^2}.
\end{aligned} \tag{206}$$

By substituting this into (205_2), one will get the identity 0 = 0, and from ($205_{2,3}$) one will obtain:

$$\varphi_2\, \sigma = \varphi\,(2 + \sigma_2),$$

(207)
$$\varphi_1 \sigma N = \varphi(\sigma_1 N + \sigma P),$$
with
$$N = < \bar{\mathfrak{Y}}_1, \bar{\mathfrak{Y}}_2, \bar{\mathfrak{Y}}_{u^1u^2} >, \qquad P = < \bar{\mathfrak{Y}}_2, \bar{\mathfrak{Y}}_{u^1u^1}, \bar{\mathfrak{Y}}_{u^2u^2} >.$$

We must now express the idea that the principal tangent curves to $(\bar{X})$ and $(\bar{Y})$ correspond to each other. Since the parameter curves on both surfaces define conjugate nets, the condition that will characterize that is:

$$< \bar{\mathfrak{X}}_1, \bar{\mathfrak{X}}_2, \bar{\mathfrak{X}}_{u^1u^1} >: < \bar{\mathfrak{X}}_1, \bar{\mathfrak{X}}_2, \bar{\mathfrak{X}}_{u^2u^2} > = < \bar{\mathfrak{Y}}_1, \bar{\mathfrak{Y}}_2, \bar{\mathfrak{Y}}_{u^1u^1} >:< \bar{\mathfrak{Y}}_1, \bar{\mathfrak{Y}}_2, \bar{\mathfrak{Y}}_{u^2u^2} >.$$

It follows from this relation, with the help of (206), that:

(208)
$$\frac{\partial}{\partial u^1}\left(\frac{2+\sigma_2}{\sigma}\right) = \frac{\partial}{\partial u^2}\left(\frac{\sigma_1}{\sigma} + \frac{P}{N}\right).$$

(208) is the integrability condition for $(207_{1,\,2})$. The integral is determined up to a constant factor, and can be obtained be quadratures.

Infinitesimal bending of the surface (*X*):

Since (205) possesses precisely *one* solution, up to a numerical factor, precisely *one* surface (*X*) will be determined by (204), up to a similarity transformation. The given surface $(\bar{X})$ will then define an infinitesimal bending of (*X*) by (183). The given surface $(\bar{Y})$ will belong to that bending as the displacement crack; the contact planes of $(\bar{Y})$ will then be established uniquely by the requirement (204) as the planes through the corresponding points of the given surface $(\bar{X})$ that are perpendicular to $\mathfrak{X}$.

116. Summary. Since (186) is formed the same way in regard to $\mathfrak{X}$ and $\bar{\mathfrak{X}}$, not only will the infinitesimal bending $\mathfrak{X}^* = \mathfrak{X} + \varepsilon\bar{\mathfrak{X}}$ of the surface (*X*) be defined by the surface pair (*X*), $(\bar{X})$, but also the infinitesimal bending $\bar{\mathfrak{X}}^* = \bar{\mathfrak{X}} + \varepsilon\,\mathfrak{X}^*$ of the surface $(\bar{X})$. As a result, we can summarize the results of nos. **114** and **115** in the following theorem:

If an infinitesimal bending of a surface $(\bar{X})$ *is given then a W-system that has* $(\bar{X})$ *as a focal surface can be derived from* (187) *and* (189) *by differentiations and algebraic operations. Conversely, if a W-system with* $(\bar{X})$ *as a focal surface is given then one can obtain an infinitesimal bending of the surface* $(\bar{X})$ *from* (205) *and* (204) *by only quadratures. The determination of all infinitesimal bendings of a surface* $(\bar{X})$ *and the determination of all W-systems that have* $(\bar{X})$ *as a focal surface* (204) *are thus equivalent problems.*

Example: We determine the infinitesimal bending that belongs with the *W*-system that was found in no. **96** whose two focal systems are:

$$(\bar{X}): \quad \bar{x}_1\,\bar{x}_3 + \lambda\bar{x}_2\,\bar{x}_4 = 0, \qquad (\bar{Y}): \quad \lambda\bar{y}_1\,\bar{y}_3 + \bar{y}_2\,\bar{y}_4 = 0.$$

The focal surfaces are represented by:

$$\bar{\mathfrak{X}} = -u^1\lambda \mid -u^1\,u^2\lambda \mid -u^2\lambda, \qquad \bar{\mathfrak{Y}} = u^1 \mid u^1\,u^2\lambda \mid -u^2$$

in principal tangent parameters. From (205) and (204), that will imply:

$$\mathfrak{X} = -u^2\lambda \mid 1 \mid u^1\lambda, \qquad \mathfrak{Y} = u^2 \mid -1 \mid u^1,$$

so the infinitesimal bending of the hyperboloid $(\bar{X})$ will be:

$$\bar{\mathfrak{X}}^* = -u^1\lambda - \varepsilon u^2\lambda \mid -u^1\,u^2\lambda + \varepsilon \mid -u^2\lambda + \varepsilon u^1\lambda,$$

and the infinitesimal bending of the plane (X) will be:

$$\mathfrak{X}^* = -u^2\lambda - \varepsilon u^1\lambda \mid 1 - \varepsilon u^1\,u^2\lambda \mid u^1\lambda - \varepsilon u^2\lambda.$$

§ 33. Torsion-fixed and curvature-fixed nets of curves.

117. Definition. In this paragraph, we treat the *differential-geometric analogues of the unsteady net of rectangles.* It is then recommended that the reader should go over § 11 again.

Torsion-fixed nets:

Let a net of curves be given on a surface (X) and an infinitesimal bending (183) of (X). We select two corresponding net-rectangles of (X) and (X^*) and consider the tetrahedron that is defined by the corners of these net-rectangles. If the two net-rectangles contract to points then the quotient of the two tetrahedral volumes will approach the value $1 + \varepsilon^2\{\dots\}$. The net of curves is then called *torsion-fixed* with respect to (183). Being torsion-fixed can also be defined by the requirement that "torsion" in the "transverse ruled surfaces" remains unchanged under the infinitesimal bending (183) to order ε ([1]).

([1]) The transverse ruled surfaces of a net of curves will be spanned by the tangents to one family of curves along the curves of the other family. The torsion of a ruled surface is the limiting value of the quotient $\frac{\text{angle between two generators}}{\text{shortest distance between them}}$. Cf., on this, **R. Sauer**, Math. Ann. **108** (1936), 673-693.

Curvature-fixed net of curves:

A net of curves (X) is called *curvature-fixed* with respect to (183) when the curvature k of the curves of the net remain unchanged to order ε, so one also has $k^* = k + \varepsilon^2\{\dots\}$.

Analogy with the unsteady rectangle net (§ 11):

Rectangle net	**Curve net**
Tetrahedral volume of the rectangle	Tetrahedral volume of the "infinitesimal" net-rectangle
Angle between successive sides of the rectangle	Curvature of the net curves
Face-rigid unsteady: The tetrahedral volume is preserved to order ε.	**Torsion-fixed:** (")
Vertex-rigid unsteady: The angle between successive sides of the rectangle remains unchanged to order ε.	**Curvature-fixed:** The curvature of the net curves remains unchanged to order ε.

The parameter net is torsion-fixed for:

$$(209) \qquad < \mathfrak{X}_1^*, \mathfrak{X}_2^*, \mathfrak{X}_{u^1u^2}^* > \; = \; < \mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_{u^1u^2} > + \, \varepsilon^2\{\dots\}.$$

Proof: We consider the net-curve rectangle whose vertices are:

$$u^1 \mid u^2, \qquad u^1 + \eta^1 \mid u^2, \qquad u^1 \mid u^2 + \eta^2, \qquad u^1 + \eta^1 \mid u^2 + \eta^2.$$

The tetrahedron with these vertices has the spatial volume:

$$V = \tfrac{1}{6} < \mathfrak{X}(u^1 + \eta^1, u^2) - \mathfrak{X}(u^1, u^2), \; \mathfrak{X}(u^1, u^2 + \eta^2) - \mathfrak{X}(u^1, u^2), \; \mathfrak{X}(u^1 + \eta^1, u^2 + \eta^2) - \mathfrak{X}(u^1, u^2) >$$

$$= \tfrac{1}{6} < \mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_{u^1u^2} > (\eta^1)^2 (\eta^2)^2 + \dots,$$

in which the ellipsis means terms of degree higher than four in the η^i. (209) follows from $V^* / V = 1 + \varepsilon^2\{\dots\}$ when $\eta^1 \to 0$, $\eta^2 \to 0$.

In the exceptional case of $< \mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_{u^1u^2} > = 0$, the parameter net is *conjugate.* In the event that it is torsion-fixed, according to (209), it will remain conjugate to order ε under the infinitesimal bending (183).

The parameter net is curvature-fixed for:

(210) $$\mathfrak{X}^*_{u^iu^i}\,\mathfrak{X}^*_{u^iu^i} = \mathfrak{X}_{u^iu^i}\,\mathfrak{X}_{u^iu^i} + \varepsilon^2\{\dots\} \qquad (i = 1, 2).$$

Proof: One has:

$$k^2 = \frac{(\mathfrak{X}_1 \times \mathfrak{X}_{u^1u^1})(\mathfrak{X}_1 \times \mathfrak{X}_{u^1u^1})}{(\mathfrak{X}_1\,\mathfrak{X}_1)^3} = \frac{(\mathfrak{X}_1\,\mathfrak{X}_1)(\mathfrak{X}_{u^1u^1}\,\mathfrak{X}_{u^1u^1}) - (\mathfrak{X}_1\,\mathfrak{X}_{u^1u^1})^2}{(\mathfrak{X}_1\,\mathfrak{X}_1)^3}$$

for the curvature k of, e.g., a curve u^2 = const.

Due to (186), one has:

$$\mathfrak{X}^*_1\,\mathfrak{X}^*_1 = \mathfrak{X}_1\,\mathfrak{X}_1 + \{\dots\}, \qquad \mathfrak{X}^*_1\,\mathfrak{X}^*_{u^1u^1} = \mathfrak{X}_1\,\mathfrak{X}_{u^1u^1} + \varepsilon^2\{\dots\},$$

so:

$$k^{*2} = \frac{(\mathfrak{X}_1\,\mathfrak{X}_1)(\mathfrak{X}^*_{u^1u^1}\,\mathfrak{X}^*_{u^1u^1}) - (\mathfrak{X}_1\,\mathfrak{X}_{u^1u^1})^2}{(\mathfrak{X}_1\,\mathfrak{X}_1)^3} + \varepsilon^2\{\dots\}.$$

(210) will then follow from the fact that $k^{*2} = k^2 + \varepsilon^2\{\dots\}$.

We substitute $\mathfrak{X}^* = \mathfrak{X} + \varepsilon^2\{\dots\}$ into (209) and (210). By setting the coefficients of ε equal to zero, we will get the theorem:

The parameter net of (X) *is characterized as torsion-fixed (curvature-fixed, resp.) by the fact that:*

(211) $$< \bar{\mathfrak{X}}_1, \mathfrak{X}_2, \mathfrak{X}_{u^1u^2} > + < \mathfrak{X}_1, \bar{\mathfrak{X}}_2, \mathfrak{X}_{u^1u^2} > + < \mathfrak{X}_1, \mathfrak{X}_2, \bar{\mathfrak{X}}_{u^1u^2} > = 0$$
$$(\mathfrak{X}_{u^1u^1}\,\bar{\mathfrak{X}}_{u^1u^1} = \mathfrak{X}_{u^2u^2}\,\bar{\mathfrak{X}}_{u^2u^2} = 0, \textit{resp.})$$

118. Properties of the torsion-fixed net. By the replacement of:

$$\bar{\mathfrak{X}}_{u^1u^2} = \frac{\partial}{\partial u^2}(\mathfrak{Y} \times \mathfrak{X}_1) = \mathfrak{Y} \times \mathfrak{X}_1 + \mathfrak{Y} \times \mathfrak{X}_{u^1u^2} = -\mu\,\mathfrak{X}_2 \times \mathfrak{X}_1 + \mathfrak{Y} \times \mathfrak{X}_{u^1u^2}$$

in (211_1), one will obtain:

$$< \bar{\mathfrak{X}}_1, \mathfrak{X}_2, \mathfrak{X}_{u^1u^2} > + < \mathfrak{X}_1, \bar{\mathfrak{X}}_2, \mathfrak{X}_{u^1u^2} > + < \mathfrak{X}_1, \mathfrak{X}_2, \bar{\mathfrak{X}}_{u^1u^2} > = \mu\,(\mathfrak{X}_1 \times \mathfrak{X}_2)\,(\mathfrak{X}_1 \times \mathfrak{X}_2).$$

Therefore, when $\mu = 0$, *one can characterize the parameter net of* (X) *as torsion-fixed.*

The relations (196) then specialize to:

$$\mathfrak{y}_1 = \lambda\,\mathfrak{q}, \qquad \mathfrak{y}_2 = -\lambda\,\mathfrak{p}, \tag{212}$$

which is the differential-geometric counterpart to (52). Moreover, one has the analogues of the corresponding discrete-geometric theorems of no. **43**:

1. Relations between the parameter nets of (X) **and** (Y). From (193), with $\mu = 0$, the tangent to a curve $u^1 = \text{const.}$ ($u^2 = \text{const.}$, resp.) will be parallel to the tangents to the curves $u^2 = \text{const.}$ ($u^1 = \text{const.}$) along the corresponding curve $u^1 = \text{const.}$ ($u^2 = \text{const.}$) in (Y). The bold-faced tangents to (X) and (Y) in Fig. 28 are pair-wise parallel; corresponding points of the surfaces (X) and (Y) are denoted by the same symbols.

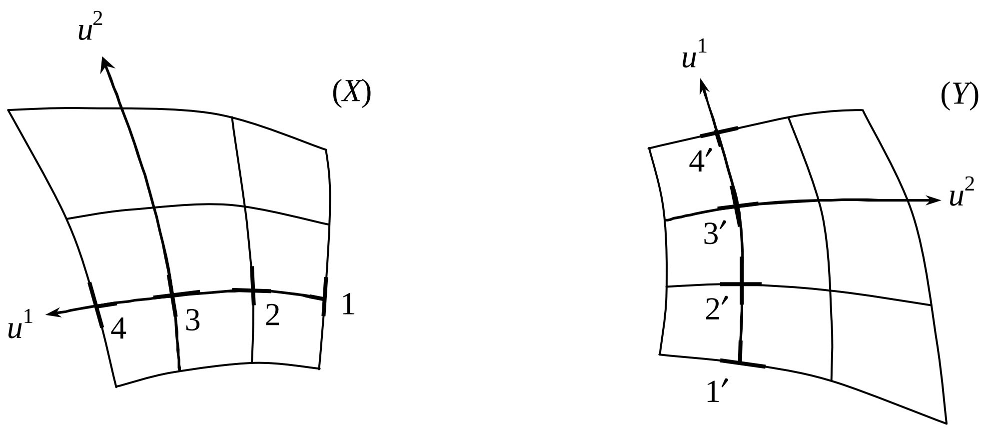

Figure 28.

2. Relations between the parameter nets of (X) **and** $(\bar{X})$, (Y) **and** $(\bar{Y})$. The tangents to the parameter curves of (X) [(Y), resp.] are perpendicular to the corresponding tangents of $(\bar{X})$ [$(\bar{Y})$, resp.] (no. 111, Theorem a). It follows from (187) and (193) that:

$$\begin{aligned} \langle \bar{\mathfrak{X}}_1, \bar{\mathfrak{X}}_2, \bar{\mathfrak{X}}_{u^1u^2} \rangle &= \langle \mathfrak{Y} \times \mathfrak{X}_1, \mathfrak{Y} \times \mathfrak{X}_2, \mathfrak{Y}_1 \times \mathfrak{X}_2 \rangle \\ &= \mu \langle \mathfrak{Y} \times \mathfrak{X}_1, \mathfrak{Y} \times \mathfrak{X}_2, \mathfrak{X}_1 \times \mathfrak{X}_2 \rangle = \mu \langle \mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{Y} \rangle^2, \end{aligned}$$

so $\langle \bar{\mathfrak{X}}_1, \bar{\mathfrak{X}}_2, \bar{\mathfrak{X}}_{u^1u^2} \rangle = 0$ when $\mu = 0$. Conversely, except for special points, $\langle \bar{\mathfrak{X}}_1, \bar{\mathfrak{X}}_2, \bar{\mathfrak{X}}_{u^1u^2} \rangle = 0$ also has the equation $\mu = 0$ as a consequence. One then has:

Any conjugate curve net of $(\bar{X})$ *corresponds to a torsion-fixed curve net on* (X), *and conversely.*

3. Collinear invariance. From (200), it follows from $\mu = 0$ that $\tilde{\mu} = 0$ for any collineation. Thus:

Every net that is collinear to a torsion-fixed curve net is once more torsion-fixed, and in fact, with respect to the bending that was determined in no. **112**.

4. Commutation of (X), $(\bar{X})$ **and** (Y), $(\bar{Y})$. From no. **110**, (X), $(\bar{X})$ can also be regarded as screw cracks, and indeed for the infinitesimal bending (191). From (193), one has:

$$\mathfrak{X}_1 = -\frac{1}{\nu}\mathfrak{Y}_2, \qquad \mathfrak{X}_2 = \mathfrak{Y}_1 . \tag{213}$$

Just as the parameter net of (X) was characterized by (193) with $\mu = 0$, the parameter net of (Y) will be characterized as torsion-fixed. Thus:

If the parameter net of (X) *is torsion-fixed under* (183) *then the parameter net of* (Y) *will be torsion-fixed under* (191).

119. Properties of the curvature-fixed nets. If one substitutes:

$$\bar{\mathfrak{X}}_{u^1u^1} = \lambda\,\mathfrak{X}_2 \times \mathfrak{X}_1 + \mathfrak{Y} \times \mathfrak{X}_{u^1u^1}, \qquad \bar{\mathfrak{X}}_{u^2u^2} = -\nu\,\mathfrak{X}_1 \times \mathfrak{X}_2 + \mathfrak{Y} \times \mathfrak{X}_{u^2u^2}$$

into (211_2) then that will make:

$$0 = \lambda <\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_{u^1u^1}>, \qquad 0 = \nu <\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_{u^2u^2}>.$$

As a result, the principal tangent net is always curvature-fixed. Therefore, we would now like to exclude the principal tangent nets from the curvature-fixed nets as trivial. $\lambda = \nu = 0$ *then characterizes a parameter net as being curvature-fixed.* (196) specializes to:

$$\mathfrak{y}_1 = \mu\,\mathfrak{p}, \qquad \mathfrak{y}_2 = -\mu\,\mathfrak{q}. \tag{214}$$

Once more, this relation corresponds to the discrete-geometric condition (52) of the vertex-rigid rectangle net, and it implies the following theorem, which is analogous to one in no. **43**:

Every net that is collinear to a curvature-fixed curve net is again curvature-fixed, and in fact, with respect to a bending that is determined as in no. **112**.

If the parameter net of (X) *is curvature-fixed under* (183) *then the parameter net of* (Y) *will be curvature-fixed under* (191).

The curvature-fixed nets, in contrast to the torsion-fixed ones, are always conjugate. The tangents to the curves $u^1 = const.$, $u^2 = const.$, $u^1 \pm u^2 = const.$ *of the curvature-fixed* *parameter net of* (X) *are parallel to the tangents to the curves* u^1 = *const.*, u^2 = *const.*, $u^1 \mp u^2$ = *const. of* (Y) (Cf., Fig. 29).

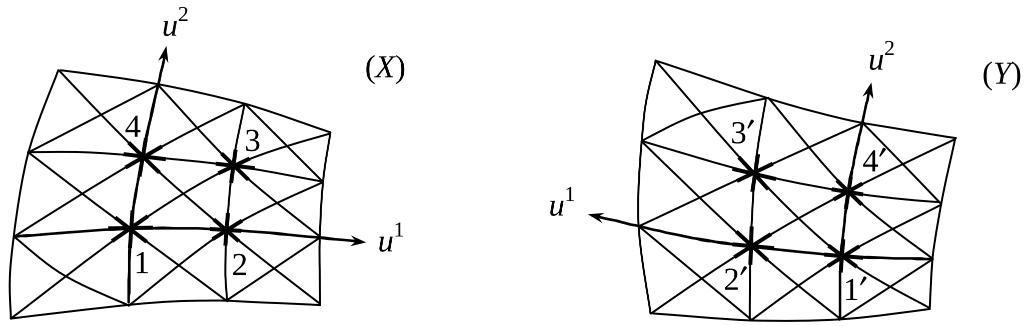

Figure 29.

Proof: (193), with $\lambda = \nu = 0$ implies that:

$$\mathfrak{Y}_1 = \mu\,\mathfrak{X}_1, \qquad \mathfrak{Y}_2 = -\,\mu\,\mathfrak{X}_2,$$

and that will imply:

$$\mathfrak{Y}_1 \pm \mathfrak{Y}_2 = \mu\,(\mathfrak{X}_1 \mp \mathfrak{X}_2),$$

as well as:

$$\mathfrak{Y}_{u^1u^2} = \mu_2\,\mathfrak{X}_1 + \mu\,\mathfrak{X}_{u^1u^2} = -\,\mu_1\,\mathfrak{X}_2 - \mu\,\mathfrak{X}_{u^1u^2}\,.$$

Therefore:

$$<\mathfrak{Y}_1, \mathfrak{Y}_2, \mathfrak{Y}_{u^1u^2}> = -\,\mu^3 <\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_{u^1u^2}> = +\,\mu^3 <\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_{u^1u^2}> = 0,$$

and since $\mu \neq 0$, one will also have:

$$<\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_{u^1u^2}> = 0.$$

The *interpretation* of (192) that was promised in no. **111** is given by the relation between (X) and (Y) that is represented in Figs. 28 and 29. Fig. 28 (29, resp.) is the differential-geometric analogue of Fig. 13 (14, resp.).

If the parameter curves fulfill (214) then the curves $\bar{u}^1 = u^1 + u^2$, $\bar{u}^2 = u^1 - u^2$ will satisfy the condition (212). Thus:

Every net that is diagonal to a curvature-fixed net is torsion-fixed.

Confer Fig. 29 on that; the diagonal net in it has the same relationship to it as the parameter net in Fig. 28.

120. Correlation conjugate torsion-fixed and curvature-fixed curve nets. The following argument is analogous to the one in no. **44**.

The nets that are correlatively-conjugate to a conjugate and torsion-fixed (curvature-fixed, resp.) net are curvature-fixed (torsion-fixed, resp.) with respect to the bending that was determined in **113**.

Proof: For $\mu = 0$ ($\lambda = \nu = 0$, resp.), (203) specializes to:

$$\bar{\mathfrak{Y}}_1 = \tilde{\lambda}\tilde{\mathfrak{W}}\times\tilde{\mathfrak{W}}_2, \qquad \bar{\mathfrak{Y}}_2 = -\tilde{\nu}\tilde{\mathfrak{W}}\times\tilde{\mathfrak{W}}_1,$$

or:

$$\bar{\mathfrak{Y}}_1 = \tilde{\mu}\tilde{\mathfrak{W}}\times\tilde{\mathfrak{W}}_1, \qquad \bar{\mathfrak{Y}}_2 = -\tilde{\mu}\tilde{\mathfrak{W}}\times\tilde{\mathfrak{W}}_2,$$

resp. Thus, the vectors $\tilde{\mathfrak{W}}\times\tilde{\mathfrak{W}}_1$, $\tilde{\mathfrak{W}}\times\tilde{\mathfrak{W}}_2$ will be parallel to the conjugate tangents to the curves u^2 = const., u^1 = const., so, since the parameter net is conjugate, they will be parallel to the tangents to the curves u^1 = const., u^2 = const.; i.e., to $\tilde{\mathfrak{X}}_2$, $\tilde{\mathfrak{X}}_1$, resp.

The principal tangent net of $(\bar{X})$ *corresponds to curvature-fixed net on* (X), *and that is the only curvature-fixed net on* (X) *for the bending* (183). *A curvature-fixed net on* (X) *for the bending* (183) *will then exist if and only if* $(\bar{X})$ *is negatively-curved.*

Proof: (187) and (193) imply that:

$$<\bar{\mathfrak{X}}_1, \bar{\mathfrak{X}}_2, \bar{\mathfrak{X}}_{u^1u^1}> \; = \; <\mathfrak{Y}\times\mathfrak{X}_1, \mathfrak{Y}\times\mathfrak{X}_2, \mathfrak{Y}_1\times\mathfrak{X}_1> \; = \lambda <\mathfrak{Y}\times\mathfrak{X}_1, \mathfrak{Y}\times\mathfrak{X}_2, \mathfrak{X}_2\times\mathfrak{X}_1>$$
$$= -\lambda <\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{Y}>^2.$$

Except for special points, $\lambda = 0$ and $<\bar{\mathfrak{X}}_1, \bar{\mathfrak{X}}_2, \bar{\mathfrak{X}}_{u^1u^1}> \; = 0$ are reciprocal conditions; the same thing will be true for $\nu = 0$ and $<\bar{\mathfrak{X}}_1, \bar{\mathfrak{X}}_2, \bar{\mathfrak{X}}_{u^2u^2}> \; = 0$.

The last part of the last theorem in no. **44** is the discrete-geometric analogue of the fact that was proved in no. **114** that the points of $(\bar{X})$ and $(\bar{Y})$ that correspond to connecting lines will define a W-system with $(\bar{X})$ and $(\bar{Y})$ as its focal surfaces (cf., the discrete-geometric juxtaposition in no. **102**.)

121. Torsion-fixed and curvature-fixed systems of lines. An arbitrary given conjugate net on a surface is, in general, not torsion-fixed or curvature-fixed for any infinitesimal bending of the surface. Therefore, if such a bending does exist then both of the tangents that define the conjugate net shall be called a *torsion-fixed (curvature-fixed, resp.) systems of lines.* A torsion-fixed (curvature-fixed, resp.) system of lines will then be characterized by the collinear-invariant requirement that the contact net (no. **80**) on at least one focal surface is torsion-fixed (curvature-fixed, resp.).

We would now like to characterize the torsion-fixed systems of lines with two focal surfaces by *projective invariants* (no. **85**):

(212) implies the necessary and sufficient collinear-invariant condition for a system of lines $\mathfrak{p}\,(u^1, u^2)$ to be torsion-fixed:

$$\frac{\partial}{\partial u^1}(\rho\,\mathfrak{p})+\frac{\partial}{\partial u^2}(\sigma\,\mathfrak{c}) = 0. \tag{215}$$

As in no. **81**, we have again replaced $\mathfrak{q}$ with $\mathfrak{c}$ in (212) and thought of $\mathfrak{p}$ and $\mathfrak{c}$ as being normalized as in no. **83**, which is why other functions ρ, σ appear in place of λ, μ.

We introduce torsion parameters and obtain from (215), with consideration given to (120):

$$\rho_1\mathfrak{p} + \sigma_2\,\mathfrak{c} + \frac{1}{m^1}\rho\,\mathfrak{p}_1 + \frac{1}{m'^2}\sigma\,\mathfrak{c}_2 = 0,$$

and furthermore, with the help of the derivation equations (134):

$$\mathfrak{p}\left(\rho_1+\frac{1}{m^1}A\rho\right)+\mathfrak{c}\left(\sigma_2-\frac{1}{m'^2}N'\sigma\right)+\mathfrak{q}\left(\frac{1}{m^1}\rho_1 B-\frac{1}{m'^2}\sigma C'\right) = 0.$$

Due to the linear independent of $\mathfrak{p}$, $\mathfrak{c}$, $\mathfrak{q}$, it follows from this that:

$$m^1\rho_1 + A\rho = 0, m'^2\sigma_2 - N'\sigma = 0, \qquad m'^2\rho\, B - m^1\sigma\, C' = 0.$$

The first two equations imply:

$$\rho = \chi\,(u^2)\ e^{-\int\frac{A}{m^1}du^1}, \qquad \sigma = \psi\,(u^1)\ e^{+\int\frac{N'}{m'^2}du^2},$$

with the arbitrary functions $\psi\,(u^1)$, $\chi\,(u^2)$; the last equation yields:

$$\frac{m^1C'}{m'^2B}e^{\int\frac{A}{m^1}du^1+\int\frac{N'}{m'^2}du^2} = \frac{\chi(u^2)}{\psi(u^1)},$$

and thus, after a double logarithmic differentiation with respect to u^1, u^2:

$$\frac{\partial^2}{\partial u^1\partial u^2}\ln\left(\frac{m^1}{m'^2}\right)+\frac{\partial^2}{\partial u^1\partial u^2}\ln\left(\frac{C'}{B}\right)+\frac{\partial}{\partial u^2}\left(\frac{A}{m^1}\right)+\frac{\partial}{\partial u^1}\left(\frac{N'}{m'^2}\right) = 0.$$

Since:

$$\frac{\partial}{\partial u^2}\left(\frac{A}{m^1}\right) = \frac{1}{m^1m'^2}A_2+\frac{\partial}{\partial u^2}\left(\frac{1}{m^1}\right)A = \frac{1}{m^1m'^2}(A_2+qA),$$

$$\frac{\partial}{\partial u^1}\left(\frac{N'}{m'^2}\right) = \frac{1}{m^1m'^2}N'_1+\frac{\partial}{\partial u^1}\left(\frac{1}{m'^2}\right)N' = \frac{1}{m^1m'^2}(N'_1+q'N'),$$

$$\frac{\partial}{\partial u^1}\ln\left(\frac{C'}{B}\right) = \frac{1}{m^1}\left(\ln\frac{C'}{B}\right)_1,$$

$$\frac{\partial^2}{\partial u^1\,\partial u^2}\ln\left(\frac{C'}{B}\right) = \frac{1}{m^1 m'^2}\left(\ln\frac{C'}{B}\right)_{12} + \frac{\partial}{\partial u^2}\left(\frac{1}{m'^1}\right)\left(\ln\frac{C'}{B}\right)_1$$
$$= \frac{1}{m^1 m'^2}\left\{\left(\ln\frac{C'}{B}\right)_{12} + q\left(\ln\frac{C'}{B}\right)_1\right\}$$

$$\frac{\partial^2}{\partial u^1\,\partial u^2}\ln\left(\frac{m^1}{m'^2}\right) = \frac{1}{m^1 m'^2}(q_2' - q_1),$$

one finally comes to:

$$A_2 + q\,A + N_1' + q'N' + q_2' - q_1 + \left(\ln\frac{C'}{B}\right)_{12} + q\left(\ln\frac{C'}{B}\right)_1 = 0. \tag{216}$$

One can characterize the curvature-fixed systems of lines by invariants in an analogous way. We would like to skip over that demonstration, since, from no. **120**, the curvature-fixed systems of lines are correlative to the torsion-fixed ones.

122. Example. From (216), the self-projective hyperbolic systems of lines (constant invariants!) with $q = q' = 0$ are torsion-fixed. Since $q = q' = 0$ and the constancy of the invariants remain valid under correlations, these systems of lines will likewise be *torsion-fixed and curvature-fixed*, and indeed, with respect to the contact net *on both focal surfaces.*

An example of such a system of lines is the self-projective W-system that was described in no. **96**. The likewise curvature-fixed and torsion-fixed contact nets lie on hyperboloids and are diagonal to the generator net of the hyperboloid.

More generally, one has:

Any diagonal net of the generator net of a hyperboloid is torsion-fixed for a bending and curvature-fixed for another bending.

Proof: From the Ansatz:

$$\mathfrak{X} = \varphi(u^1 + u^2) \mid \psi(u^1 - u^2) \mid \varphi(u^1 + u^2)\cdot\psi(u^1 - u^2),$$

with the arbitrary functions φ and ψ, the parameter net is conjugate to the net of generators $u^1 \pm u^2 = \text{const.}$

The function $\mathfrak{Y}(u^1, u^2)$ is determined, up to a constant factor, by:

$$\mathfrak{Y}_1 = \frac{1}{\varphi_1\psi_1}\ \mathfrak{X}_2\,, \qquad\qquad \mathfrak{Y}_2 = -\frac{1}{\varphi_1\psi_1}\ \mathfrak{X}_1\,,$$

(that is:

$$\mathfrak{Y}_1 = \frac{1}{\varphi_1\psi_1}\ \mathfrak{X}_1\,, \qquad\qquad \mathfrak{Y}_2 = -\frac{1}{\varphi_1\psi_1}\ \mathfrak{X}_2\,,$$

resp.), since:

$$\frac{\partial}{\partial u^1}\left(\frac{\mathfrak{X}_1}{\varphi_1\,\psi_1}\right) + \frac{\partial}{\partial u^2}\left(\frac{\mathfrak{X}_2}{\varphi_1\,\psi_1}\right) \equiv 0 \qquad \left[\frac{\partial}{\partial u^1}\left(\frac{\mathfrak{X}_1}{\varphi_1\,\psi_1}\right) + \frac{\partial}{\partial u^2}\left(\frac{\mathfrak{X}_2}{\varphi_1\,\psi_1}\right) \equiv 0, \text{resp.}\right].$$

From no. **118** (no. **119**, resp.), the parameter net of (X) then characterized by being torsion-fixed (curvature-fixed, resp.). The discrete-geometric analogues of these curve nets were mentioned in the conclusion of no. **44**.

§ 34. Stress distributions in membranes.

123. Screw crack of a stress distribution. In no. **42**, we juxtaposed the kinematic picture of the face-rigid, unsteady, rectangular net with a *static interpretation* by replacing the motion screws of the screw crack with force screws. In an analogous way, we would now like to reinterpret the kinematic study of an infinitesimal surface bending statically as results concerning *stress distributions in inextensible membranes* ([1]). We thus assume that the membrane surface is not developable, so planar membranes will be excluded, in particular.

Equilibrium conditions:

Any line element $d\mathfrak{X}$ of the membrane (X) belongs to a stress-force that acts in the contact plane, which we will represent by the six-vector:

(217) $$d\mathfrak{y} = \{d\mathfrak{Y} \mid \mathfrak{X} \times d\mathfrak{Y}\}.$$

If no external forces are present then the stress-force that acts upon the boundary of any simply-connected surface patch of (X) must vanish, so:

(218) $$\oint d\mathfrak{y} = 0\,;$$

i.e., $d\mathfrak{y}$ *will be a complete differential, so the elementary stresses* $d\mathfrak{y}$ *can be derived from a stress function* $\mathfrak{y}(u^1, u^2)$.

([1]) Cf., on this, **W. Blaschke:** "Reziproke Kräftpläne, usw.," Int. Cong. of Math. Cambridge, Proceedings 2, 1913; and furthermore, **M. Lagally**, "Über Spannung und elastische Deformation, usw.," Zeit. f. angew. Math. u. Mech. **4**, 1924.

Screw crack:

When one goes from the six-vector $d\mathfrak{y}$ to the ordinary vectors $d\mathfrak{Y}$, $\mathfrak{X} \times d\mathfrak{Y}$, using (217), it will follow from (218) that $d\mathfrak{Y}$, as well as $\mathfrak{X} \times d\mathfrak{Y}$, are total differentials. There then exist two functions $\mathfrak{Y}(u^1, u^2)$ and $\bar{\mathfrak{Y}}(u^1, u^2)$, which are coupled with the position vector $\mathfrak{X}(u^1, u^2)$ of the point on the membrane by:

$$d\bar{\mathfrak{Y}} = \mathfrak{X} \times d\mathfrak{Y} .$$

This relation is equivalent to the conditions (188) of no. **109**. As a result, the membrane surface (X) and the surfaces (Y) [$(\bar{Y})$, resp.] that are described by the position vector $\mathfrak{Y}$ ($\bar{\mathfrak{Y}}$, resp.) will have the same relationship to each other as the surfaces with the same notations do under the infinitesimal surface bending. *Any infinitesimal bending of* (X) *will then also produce a stress distribution on the membrane* (X) *that will be in equilibrium in the absence of external forces, and conversely.* We call the surface-pair (Y), $(\bar{Y})$ the *screw crack* for the given stress distribution in (X).

124. Juxtaposition of analogous theorems on surface bending and stress distributions. From no. **123**, the theorems on surface bending can be translated immediately into the theory of stresses. For example, one will get:

a) From no. **110**: A given stress distribution in the membrane (X) with the screw crack (Y), $(\bar{Y})$ also determines a stress distribution in the membrane (Y) with the screw crack (X), $(\bar{X})$, for which $(\bar{X})$ is given by (189).

b) From nos. **112** and **113**: A known stress distribution $\mathfrak{y}(u^1, u^2)$ on the membrane (X) will imply a stress distribution in any membrane $(\tilde{X})$ that is collinear or correlative to (X) when one subjects the line coordinates of the tangents of (X) and the coordinates of the screw $\mathfrak{y}$ to the same linear transformation with constant coefficients.

c) From nos. **118** and **119**: Torsion-fixed (curvature-fixed, resp.) nets correspond to *lateral stress nets* (*shear stress nets*, resp.). For those nets, the stress-forces that act along a net curve κ act in the direction of the net curves of the other family (in the direction of the net curve κ, resp.).

One can represent the lateral stress nets by string models with families of intertwined strings in a state of tension. The string stresses yield the lateral stresses in the net. Since, by assumption, the lateral stresses are in equilibrium, the surface that is generated by the strings does not need to be realized by a rigid material.

Any collinear map transforms a lateral stress (shear stress, resp.) into another one.

d) From nos. **119** and **120**: There are conjugate and non-conjugate lateral stress nets, but only conjugate shear stress nets. A correlation will transform a conjugate lateral stress net into a shear stress net, and conversely.

CHAPTER VI

Line complexes

§ 35. Definition of a line complex.

125. Parametric representation. A *line complex* ([1]) is given by:

$$\mathfrak{p} = \mathfrak{p}\,(u^1, u^2, u^3),$$

with the independent parameters u^1, u^2, u^3 that vary in the domain $u_a^i \le u^i \le u_e^i$ (i = 1, 2, 3); the six-vector $\mathfrak{p}\,(u^1, u^2, u^3)$ is determined only up to an arbitrary factor $\sigma(u^1, u^2, u^3) \neq 0$. If one associates any point z of a region in a plane w with a cone or a pencil of lines with the vertex z then those lines will define a line complex in the event that they do not all belong to the plane w. Conversely, any line complex can be generated in that way. Every line of the complex that does not lie in w will then cut w at a well-defined point z, and a one-parameter set of complex lines will go through every non-special point of intersection z.

Just as we did for line systems in no. **71**, here, we introduce the six-vectors $\mathfrak{p}_i = \dfrac{\partial \mathfrak{p}}{\partial u^i}$, as well as $\dfrac{\partial^2 \mathfrak{p}}{\partial u^i \partial u^k}$, etc. Therefore, the identities (102) are once more valid, but this time with i, k = 1, 2, 3. In addition, we demand, as in no. **71**, that:

$\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ *shall be linearly-independent; in particular, either* $\mathfrak{p}$ *or one of the first derivatives* $\mathfrak{p}_i$ *will then be a zero vector.*

One gets the following *map of line complexes to the points of an image space* from that: For every location u'^i in the domain of definition, there exists a finite sub-region $u'^i - \varepsilon \le u^i \le u'^i + \varepsilon$, for which, the triple of numbers u^i is associated with the line $\mathfrak{p}\,(u^1, u^2, u^3)$ in a single-valued, invertible way ([2]). If one interprets the u^i as inhomogeneous, rectangular point coordinates in an image space then the line of the complex in the sub-region will be mapped to the points of a cube in the image space in a one-to-one correspondence. Any curve (surface, resp.) of the image space will correspond to a family of lines (line system, resp.) that is contained in the line complex, while any direction of advance in image space $\dot{u}^1 : \dot{u}^2 : \dot{u}^3$ will correspond to a "direction of advance" in the line complex.

([1]) The metric differential geometry of line complexes has recently been treated thoroughly by **W. Haack**; cf., the citation on page 1.
([2]) Cf., page 100, footnote 3.

Tensors: As we did in § 20 for two variables u_1, u_2, here, we define tensors in three variables that are covariant (contravariant, resp.) tensors with respect to the parameter substitution:

$$(219) \qquad u^i = u^i(\bar{u}^1, \bar{u}^2, \bar{u}^3) \qquad \text{with} \qquad \Delta = \begin{vmatrix} \dfrac{\partial u^1}{\partial \bar{u}^1} & \dfrac{\partial u^1}{\partial \bar{u}^2} & \dfrac{\partial u^1}{\partial \bar{u}^3} \\ \dfrac{\partial u^2}{\partial \bar{u}^1} & \dfrac{\partial u^2}{\partial \bar{u}^2} & \dfrac{\partial u^2}{\partial \bar{u}^3} \\ \dfrac{\partial u^3}{\partial \bar{u}^1} & \dfrac{\partial u^3}{\partial \bar{u}^2} & \dfrac{\partial u^3}{\partial \bar{u}^3} \end{vmatrix} \neq 0.$$

In particular, the coordinates $\dfrac{\partial p_\rho}{\partial \bar{u}^i}$ of the six-vector $\mathfrak{p}_i = \dfrac{\partial \mathfrak{p}}{\partial \bar{u}^i}$ are once more tensors of rank one.

126. Torses in a line complex. The torses $u^i = u^i(t)$ that are contained in a line complex are implied by the condition:

$$(220) \qquad g_{ik} \dot{u}^i \dot{u}^k = 0,$$

which reads the same as (103) formally, in which one finds the symmetric, covariant tensor of rank two:

$$g_{ik} = \mathfrak{p}_i \, \mathfrak{p}_k = - \mathfrak{p} \, \frac{\partial^2 \mathfrak{p}}{\partial u^i \partial u^k} = g_{ki} .$$

Since the indices now run from 1 to 3, (220) now implies a one-parameter set of directions of advance $\dot{u}^1 : \dot{u}^2 : \dot{u}^3$ (no. **128**) for any line $\mathfrak{p}$ of the complex. As a consequence, for the torses that go through a complex line $\mathfrak{p}$, one can prescribe, along with the differential equation (220), an essentially arbitrary relation between two parameters, as well – e.g., $u^2 = \chi(u^1)$; i.e., except for the parameters, the set of torses contains an arbitrary function of variable. Special torses of the complex are the complex lines that go through a fixed point z (viz., a *complex cone*) and dual to that, the complex lines that lie in a fixed plane w (viz., tangents to the *complex curves*). We will learn about other projectively-distinguished torses (no. **133**).

One always has $r > 1$ *for the rank* r *of the matrix* g_{ik} .

Proof: (220) transforms under a parameter transformation (219) as the equation of a second-order curve in homogeneous point coordinates $\dot{u}^i$ does under a coordinate transformation (no. **4**). Thus, for $r \leq 1$, (220) will go to:

$$g_{11}\dot{u}^1\dot{u}^1 = 0, \quad \text{with} \quad g_{11} > 0 \text{ or } < 0.$$

One will then have:

$$\mathfrak{p}_2\,\mathfrak{p}_2 = \mathfrak{p}_3\,\mathfrak{p}_3 = \mathfrak{p}_2\,\mathfrak{p}_3 = \mathfrak{p}_1\,\mathfrak{p}_2 = \mathfrak{p}_1\,\mathfrak{p}_3 = 0.$$

It follows from this, and the fact that:

$$\mathfrak{p}\mathfrak{p} = \mathfrak{p}\mathfrak{p}_1 = \mathfrak{p}\mathfrak{p}_2 = \mathfrak{p}\mathfrak{p}_3 = 0,$$

that $\mathfrak{p}$, $\mathfrak{p}_2$, $\mathfrak{p}_3$ are pair-wise intersecting lines that belong to the linear complex $\mathfrak{p}_1$. For $\mathfrak{p}_1\mathfrak{p}_1 \neq 0$, the three lines $\mathfrak{p}$, $\mathfrak{p}_2$, $\mathfrak{p}_3$ then belong to a pencil, while $\mathfrak{p}_1\mathfrak{p}_1 = 0$, the four lines $\mathfrak{p}$, $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$ lie in a plane or a bundle. In both cases, $\mathfrak{p}$, $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$ will then be linearly-dependent, which contradicts our assumption (no. **125**).

All complex cones and complex curves degenerate into pencils of lines for a *linear complex*. This property is also characteristic of linear complexes. A relation that is linear in the x_i' will then follow from the equation of a complex:

$$F\,(p_1, p_2, \ldots, p_6) = 0$$

upon substituting (9) if and only if F is a linear form in the p_ρ. In that way, we have implicitly given the complex by an equation $F = 0$, instead of an explicit parametric representation of the line coordinates, as an exceptional case.

127. Classification of the line complexes. Since $r > 1$, the following *projectively and parametrically-invariant case distinction follows:*

a) $r = 3$: *non-singular line complexes.*

b) $r = 2$: *singular line complexes.*

The singular complexes are the tangent complexes of the surfaces and the secant complexes of the curves; the latter go to tangent complexes of developable surfaces under correlations.

Proof:

α) Let the tangent complex of a non-developable or developable surface $\mathfrak{X} = \mathfrak{X}(u^1, u^2)$ be given. From $\mathfrak{p} = \{\mathfrak{P} \mid \bar{\mathfrak{P}}\}$, with:

$$\begin{aligned}\mathfrak{P} &= \mathfrak{X}_1 + u^3\,\mathfrak{X}_2,\\ \bar{\mathfrak{P}} &= \mathfrak{X}\times\mathfrak{P} = \mathfrak{X}\times\mathfrak{X}_1 + u^3\,\mathfrak{X}\times\mathfrak{X}_2,\end{aligned}$$

one will get:

$$\begin{aligned}\mathfrak{p}_1\mathfrak{p}_3 &= \mathfrak{P}_1\bar{\mathfrak{P}}_3 + \mathfrak{P}_3\bar{\mathfrak{P}}_1\\ &= (\mathfrak{X}_{u^1u^1} + u^3\mathfrak{X}_{u^2u^2})\,\mathfrak{X}\times\mathfrak{X}_2 + \mathfrak{X}_2\,(\mathfrak{X}\times\mathfrak{X}_{u^1u^1} + u^3\mathfrak{X}\times\mathfrak{X}_{u^2u^2}) = 0,\end{aligned}$$

as well as $\mathfrak{p}_2\mathfrak{p}_3 = 0$ and $\mathfrak{p}_3\mathfrak{p}_3 = 0$, so $r < 3$. The same thing will be true for the secant complexes, since they correspond dually to the tangent complexes of the developable surfaces.

β) Let a line complex with $r = 2$ be given. (220) can then be transformed into:

$$g_{11}\dot{u}^1\dot{u}^1 + 2g_{12}\dot{u}^1\dot{u}^2 + g_{22}\dot{u}^2\dot{u}^2 = 0$$

by a parameter substitution (219). Since $\mathfrak{p}_3\mathfrak{p}_3 = \mathfrak{p}\mathfrak{p}_3 = 0$, $\mathfrak{p}$, $\mathfrak{p}_3$ are two lines that intersect in a well-defined point x and span a well-defined plane w. It follows from:

$$\frac{\partial^2\mathfrak{p}}{\partial u^3\partial u^3}\,\mathfrak{p} = \frac{\partial^2\mathfrak{p}}{\partial u^3\partial u^3}\,\mathfrak{p}_1 = \frac{\partial^2\mathfrak{p}}{\partial u^3\partial u^3}\,\mathfrak{p}_2 = \frac{\partial^2\mathfrak{p}}{\partial u^3\partial u^3}\,\mathfrak{p}_3 = 0$$

and the linear independence of $\mathfrak{p}$, $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$ that:

$$\frac{\partial^2\mathfrak{p}}{\partial u^3\partial u^3} = \lambda_1\mathfrak{p} + \lambda_2\mathfrak{p}_3\,;$$

i.e., the families of lines u^1 = const., u^2 = const. that are contained in the complex are pencils of lines (no. **46**). They have the point x as their vertices and lie in the planes w.

The point x cannot be fixed, since otherwise the complex lines would define a bundle, and only a two-parameter set. If x generates a curve then all complex lines must cut that curve, so the complex would be a secant complex. If x generates a surface then the planes w of the pencils of lines will be the contact planes to the surface, so the complex will be a tangent complex. One sees that as follows: Since:

$$\mathfrak{p}\mathfrak{p}(u^i(t)) = \mathfrak{p}\dot{\mathfrak{p}}(u^i(t)) = \mathfrak{p}_3\mathfrak{p}(u^i(t)) = \mathfrak{p}_3\dot{\mathfrak{p}}(u^i(t)) = 0,$$

$\mathfrak{p}$, $\mathfrak{p}_3$ will be tangents, so w will be the contact plane to a line surface that is spanned by the family of lines of the complex that go through a line $\mathfrak{p}$ of the pencil $\{x \mid w\}$.

In following, we exclude the singular line complexes, as well as the isolated lines with $r = 2$, *from consideration.* For the determinant g of the g_{ik}, we will then always have:

(221) $$g \neq 0.$$

§ 36. Contact structures.

128. Contact complex, contact system, and contact correlation. The bush of complexes $\lambda^0\mathfrak{p} + \lambda^1\mathfrak{p}_1 + \lambda^2\mathfrak{p}_2 + \lambda^3\mathfrak{p}_3$ is established by the four linearly-independent complexes $\mathfrak{p}$, $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$ in a manner that is projectively-invariant, as well as invariant under renormalization and parameter substitutions. The complexes $\mathfrak{b}$ of the pencil of complexes that is conjugate to that bush are determined by:

$$\mathfrak{b}\mathfrak{p} = \mathfrak{b}\mathfrak{p}_1 = \mathfrak{b}\mathfrak{p}_2 = \mathfrak{b}\mathfrak{p}_3 = 0, \tag{222}$$

and are called *contact complexes;* they then contain the complex lines $\mathfrak{p}$ and all complex lines $\mathfrak{p}(u^1 + r\varepsilon, u^2 + s\varepsilon, u^3 + t\varepsilon)$ in the neighborhood of $\mathfrak{p}(u^1, u^2, u^3)$, to order ε.

Since the discriminant $D_4(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)$ has rank $r = 3$, the bush $\lambda^0\mathfrak{p} + \lambda^1\mathfrak{p}_1 + \lambda^2\mathfrak{p}_2 + \lambda^3\mathfrak{p}_3$ will be parabolic (no. **32**), as well as the pencil of the contact complex $\mathfrak{b}$ that is conjugate to it. There is then precisely one singular contact complex, and since (222) will be fulfilled by $\mathfrak{b} = \mathfrak{p}$, it will have the complex line $\mathfrak{p}$ for its axis.

The lines that are common to all contact complexes $\mathfrak{b}$ define a parabolic system of lines that we call the *contact system*. The contact system generates a correlation between the points x of its focal line $\mathfrak{p}$ and planes w through p (no. **33**). This so-called *contact correlation* can be characterized as follows:

The torses of the given complex that go through the line $\mathfrak{p}$ *have* x *for their point of regression* w *for their plane of regression.*

Proof: For any torse $u^i = u^i(t)$, the point of intersection x of $\mathfrak{p}$ and $\dot{\mathfrak{p}} = \mathfrak{p}_i\dot{u}^i$ is the point of regression, and the connecting plane w of $\mathfrak{p}$ and $\dot{\mathfrak{p}}$ is the plane of regression (no. **46**). $\dot{\mathfrak{p}}$ is a singular complex of the bush $\lambda^0\mathfrak{p} + \lambda^1\mathfrak{p}_1 + \lambda^2\mathfrak{p}_2 + \lambda^3\mathfrak{p}_3$, so the line $\dot{\mathfrak{p}}$ will then belong to the contact system, and $x \mid w$ will be an element-pair of the contact correlation. Conversely, since:

$$0 = \mathfrak{s}\mathfrak{s} = \mathfrak{p}_i\mathfrak{p}_k\,\lambda^i\lambda^k = g_{ik}\,\lambda^i\lambda^k,$$

any singular six-vector will produce torses of the given complex with $\dot{u}^1 : \dot{u}^2 : \dot{u}^3 = \lambda^1 : \lambda^2 : \lambda^3$.

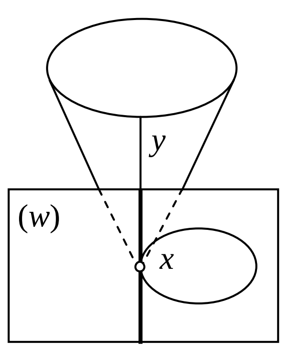

Figure 30.

Some particular torses are the generating families of the complex cone with the vertex x and the tangent family of the complex curves in the plane w (no. **126**). It will then follow, in particular, from the theorem that was proved before that:

If w is the plane, and x is the contact point of a complex curve then w will be, at the same time, the contact plane along $\mathfrak{p}$ *for the complex cone with the vertex x (Fig. 30). That complex cone will contact any torse of the complex through* $\mathfrak{p}$ *whose point of regression is x along* $\mathfrak{p}$.

129. Principal points and principal planes. Let the parabolic pencil of the contact complex be represented by:

$$\mathfrak{b} = \lambda\, \mathfrak{p} + \mu\, \mathfrak{c} \qquad\qquad (\mathfrak{p}\mathfrak{p} = 0,\ \mathfrak{c}\mathfrak{c} \neq 0).$$

With the help of the tensor:

$$c_{ik} = \mathfrak{c}_i\, \mathfrak{p}_k = \mathfrak{c}_k\, \mathfrak{p}_i = -\frac{\partial^2 \mathfrak{p}}{\partial u^i \partial u^k} = c_{ki}\,,$$

when:

(223) $$(\lambda\, g_{ik} + \mu\, c_{ik})\, \dot{u}^i \dot{u}^k = 0,$$

one will get the direction of advance $\dot{u}^1 : \dot{u}^2 : \dot{u}^3$ that contacts the complex $\mathfrak{b}$ to order two; i.e., for which, the condition:

$$0 = \mathfrak{b}\mathfrak{p}\,(u^i(t+\varepsilon)) = \mathfrak{b}\left\{\mathfrak{p} + \varepsilon\dot{\mathfrak{p}} + \frac{\varepsilon^2}{2}\ddot{\mathfrak{p}} + \cdots\right\} = \mathfrak{b}\left\{\mathfrak{p} + \varepsilon\dot{\mathfrak{p}}_i \dot{u}^i + \frac{\varepsilon^2}{2}(\mathfrak{p}_{u^i u^k}\dot{u}^i\dot{u}^k + \mathfrak{p}_i \ddot{u}^i) + \cdots\right\}$$

is fulfilled to order ε^2. Any contact complex is then contacted to order two by a one-parameter set of directions of advance. In particular, by way of (223), with $\mu = 0$, the singular contact complex $\mathfrak{b} = \mathfrak{p}$ will be associated with the directions of advance (220) that determine the contact correlation. The so-called *principal directions of advance* ([1]) that are given by:

(224) $$g_{ik}\dot{u}^i\dot{u}^k = 0, \qquad c_{ik}\dot{u}^i\dot{u}^k = 0$$

are characterized by the fact that *all* contact complexes contact them to order two. The associated points x (planes w, resp.) of the contact correlation (no. **128**) are called the *principal points* (*principal planes,* resp.).

The equation that follows from (223) can be interpreted in a finite way by means of the family of lines of the complex:

([1]) The principal directions of advance were first introduced by **F. Klein**: Math. Ann. **5** (1872).

The first osculating quadrics of all families of lines of the complex through $\mathfrak{p}$ *for a fixed direction of advance* $\dot{u}^i$ *are contained in the contact complex that belongs to* $\dot{u}^i$.

Mapping the directions of advance to the points of an image plane:

Along with the map of complex lines to the points of a spatial domain that was discussed in no. **125**, we now introduce a map ([1]) of the directions of advance $\dot{u}^i$ through a complex line $\mathfrak{p}$ to the points of a plane, in which we interpret the $\dot{u}^i$ as homogeneous, projective point coordinates in a projective plane. Under that map, the directions of advance (226) that contact a well-defined contact complex $\mathfrak{b}$ to order two will be associated with the points of a second-order curve. In particular, the singular contact complex $\mathfrak{b} = \mathfrak{p}$ will correspond to the non-singular second-order curve γ:

$$g_{ik}\dot{u}^i\dot{u}^k = 0.$$

It is non-vacuous. The parabolic bush $\lambda^0\mathfrak{p} + \lambda^i\mathfrak{p}_i$ then contains singular complexes that are not $\mathfrak{p}$, so the equation:

$$0 = (\lambda^0\mathfrak{p} + \lambda^i\mathfrak{p}_i)(\lambda^0\mathfrak{p} + \lambda^k\mathfrak{p}_k) = g_{ik}\lambda^i\lambda^k$$

will admit $\lambda^1, \lambda^2, \lambda^3$ as a real non-vanishing triple of values.

The parameter substitutions (219) of the line complex correspond to coordinate transformations (collineations, resp.) with non-vanishing coefficient determinant Δ:

$$\dot{u}^i = \delta^i_k\,\dot{\bar{u}}^k \qquad \left(\delta^i_k = \frac{\partial u^i}{\partial \bar{u}^k}\right) \tag{225}$$

in the image plane:

130. Case distinction for nonsingular line complexes. A pencil of conics in the image plane is given by (223). The projective classification of these pencils of conics yields a *projectively-invariant and parametrically-invariant case distinction of the non-singular line complex.*

In what follows, we restrict ourselves to the cases in which we assume:

The pencil of conics (223) *shall have precisely four distinct real base points (Fig. 31).*

These four base points then correspond to precisely four principal directions of advance, and thus to four principal planes and four principal points, as well. The quadruple of principal planes is correlative to the quadruple of principal points, so both quadruples will yield the same double ratio.

([1]) This map was first employed by **K. Zindler** and **G. Sannia**.

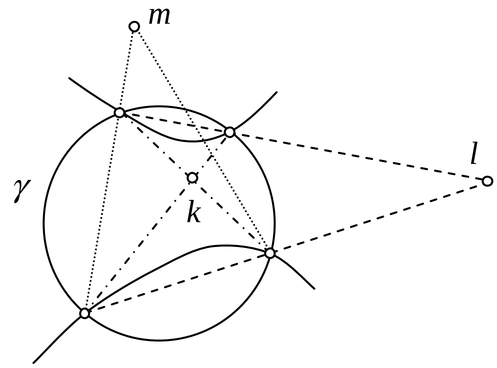

Figure 31.

The *linear complexes*, in particular, will be excluded by the assumption of precisely four distinct real principal directions of advance: The constancy of $\mathfrak{c}$ will imply that $c_{ik} = \mathfrak{c}_i\, \mathfrak{p}_k = 0$, and the pencil of conics will then degenerate to the fixed conic γ, so the base points of the pencil of conics and the principal directions of advance that correspond to them will be undetermined.

In the following numbers, we will learn about other projectively-distinguished contact complexes, in addition to the singular contact complex $\mathfrak{b} = \mathfrak{p}$.

131. Doubly-contacting complexes. We now ask which contact complexes are stationary for certain directions of advance; i.e., a neighborhood contains not only the complex line $\mathfrak{p}(u^i)$, but also the "neighboring line" $\mathfrak{p}(u^i + \varepsilon \dot{u}^i)$, to order ε. These so-called *doubly-contacting complexes* $\mathfrak{b}$ are then defined by the fact that, along with (222), they also fulfill the corresponding conditions with $u^i + \varepsilon \dot{u}^i$:

$$\mathfrak{b}\mathfrak{p}(u^i + \varepsilon \dot{u}^i) = 0, \qquad \mathfrak{b}\mathfrak{p}_k(u^i + \varepsilon \dot{u}^i) = 0 \qquad (k = 1, 2, 3),$$

to order ε. Along with (222), that also yields:

$$0 = \sum_{i=1}^{3} \mathfrak{b} \frac{\partial^2 \mathfrak{p}}{\partial u^i \partial u^k} \dot{u}^i = \sum_{i=1}^{3} (\lambda \mathfrak{p} + \mu \mathfrak{c}) \frac{\partial^2 \mathfrak{p}}{\partial u^i \partial u^k} \dot{u}^i ,$$

so finally:

$$(\lambda\, g_{ik} + \mu\, c_{ik})\, \dot{u}^i = 0 \qquad (k = 1, 2, 3). \tag{226}$$

These three equations, which are linear and homogeneous in $\dot{u}^i$, have a non-trivial system of solutions if and only if the determinant vanishes, so for:

$$|\, \lambda\, g_{ik} + \mu\, c_{ik} \,| = 0. \tag{227}$$

Any root of this third-degree equation in $l : m$ will imply a doubly-contacting complex $\mathfrak{b} = \lambda\,\mathfrak{p} + \mu\,\mathfrak{c}$, and (226) will give the associated direction of advance.

Under the map of the directions of advance to the points of a plane (no. **129**), one will get the singular conics of the pencil of conics (223) by substituting $\lambda : \mu$ in (227); in Fig. 31, they are drawn with dashed (dotted, dash-dotted, resp.) lines. From (226), the directions of advance that belong to the doubly-contacting complexes map to the vertices k, l, m of the polar triangle (= double points of the singular conic) that is common to the pencil of conics (223). It will then follow from the assumption of four real distinct base points that: *There exist precisely three (real) double-contacting complexes.*

We denote the directions of advance for the doubly-contacting complexes that are given by (226) by:

$$\dot{u}^i = k^i, \qquad \dot{u}^i = l^{\,i}, \qquad \dot{u}^i = m^i.$$

Since the vertices of the polar triangle do not lie along a line, k^i, $l^{\,i}$, m^i will be linearly-independent; i.e.:

$$(228) \qquad \begin{vmatrix} k^1 & k^2 & k^3 \\ l^1 & l^2 & l^3 \\ m^1 & m^2 & m^3 \end{vmatrix} \neq 0.$$

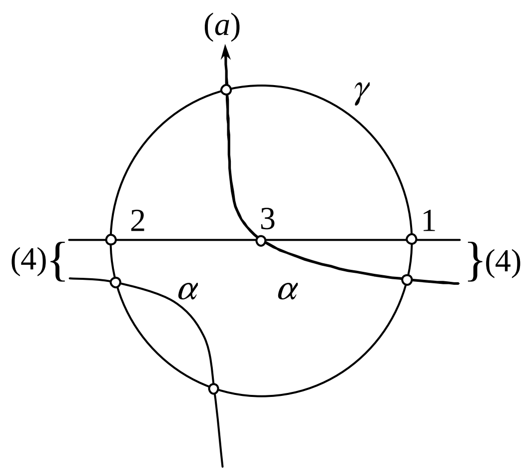

Figure 32.

132. Lemma on apolar conic sections. Apolar complex. In order to find another conic in the pencil (223) that is projectively-distinguished, not in its own right, but relative to γ, we first state the following **lemma** from the projective theory of conic sections:

In a pencil of conics, for every non-singular conic γ there exists a second conic α of the pencil with the following projectively-invariant relations:

a) *The polar of any point a of α relative to the γ intersects γ and α in harmonic point-pairs* 1, 2 *and* 3, 4 *(Fig. 32; a is the imaginary point of the hyperbola α in it).*

b) *Harmonic tangent-pairs to α and γ go through any pole of any tangent to γ relative to α.*

It then follows directly from this that:

c) *There are infinitely many polar triangles of γ whose vertices lie on α.*

d) *There are infinitely many polar triangles of α whose sides contact γ. This relationship between the conic γ and α will be called the **apolarity relationship**; α is called **apolar** to γ.*

Proof of the lemma:

We first introduce the second-rank tensor g^{ik} that is contravariant to the covariant tensor g_{ik} ($g \neq 0$) by the requirements:

(229) $$g_{ik}\, g^{il} = g_k^l = \begin{cases} 1 & \text{for } k = l, \\ 0 & \phantom{\text{for }} k \neq l, \end{cases}$$

with the help of the mixed tensor g_k^l (no. **73**). Since $g \neq 0$, the g^{ik} will be determined uniquely ([1]). They are the subdeterminants of the elements g_{ik} in the matrix g_{ik}, divided by g.

By contracting, one gets from (229) that:

(230) $$g_{ik}\, g^{ik} = 3.$$

Thus, the requirement:

$$g^{ik}\,(\lambda\, g_{ik} + \mu\, c_{ik}) = 0,$$

which is invariant under collineations (225), will imply the ratio $\mu : \lambda = \sigma \neq 0$. With $a_{ik} = g_{ik} + \sigma c_{ik}$, one will have:

(231) $$g^{ik}\, a_{ik} = 0.$$

We shall show that this condition is characteristic for the apolarity of the conics:

$$\gamma)\ g_{ik}\dot{u}^i\dot{u}^k = 0 \text{ and } \qquad \alpha)\ a_{ik}\dot{u}^i\dot{u}^k = 0:$$

a) We take vertex 3 of the coordinate triangle to be a point of α that does not lie on γ, and take the vertices 1, 2 to be two points of the polars of 3 relative to g. We will then have:

([1]) One will then get, e.g., g^{11}, g^{21}, g^{31} from the three linear equations:

$$g_{11}\, g^{11} + g_{21}\, g^{21} + g_{31}\, g^{31} = 1, \qquad g_{12}\, g^{11} + g_{22}\, g^{21} + g_{32}\, g^{31} = 0, \qquad g_{13}\, g^{11} + g_{23}\, g^{21} + g_{33}\, g^{31} = 0,$$

with the non-vanishing coefficient determinant g.

$$a_{33} = 0, \qquad g_{13} = g_{23} = 0, \qquad g^{13} = g^{23} = 0,$$

and (231) will specialize to:

$$g_{33}\,(g_{11}\,a_{22} + g_{22}\,a_{11} - g_{12}\,a_{12}) = 0.$$

Since $g_{33} \neq 0$, the expression in parentheses must vanish. That says that the point-pairs:

$$g_{11}\dot{u}^1\dot{u}^1 + 2g_{12}\dot{u}^1\dot{u}^2 + g_{22}\dot{u}^2\dot{u}^2 = 0, \qquad a_{11}\dot{u}^1\dot{u}^1 + 2a_{12}\dot{u}^1\dot{u}^2 + a_{22}\dot{u}^2\dot{u}^2 = 0,$$

at which the conics γ and α are cut by the polars $\dot{u}^3 = 0$ of vertex 3 relative to γ are separated harmonically. (231) likewise follows conversely from requirement *a*) of the lemma.

b) The proof of statement *b*) of the lemma proceeds dually to *a*).

By basing the polar triangle that is common to the pencil of conics (223) upon the coordinate triangle, γ and α will have the equations:

$$g_{11}(\dot{u}^1)^2 + g_{22}(\dot{u}^2)^2 + g_{33}(\dot{u}^3)^2 = 0, \qquad a_{11}(\dot{u}^1)^2 + a_{22}(\dot{u}^2)^2 + a_{33}(\dot{u}^3)^2 = 0.$$

The apolarity condition (231) specializes to:

$$\frac{a_{11}}{g_{11}} + \frac{a_{22}}{g_{22}} + \frac{a_{33}}{g_{33}} = 0. \tag{232}$$

We now return to the theory of line complexes and define the *apolar complex* $\mathfrak{a}$ to be the contact complex:

$$\mathfrak{a} = \mathfrak{p} + \sigma\,\mathfrak{c},$$

in which $\sigma = \mu : \lambda$ is determined by the apolarity condition:

$$g^{ik}\,(\lambda\, g_{ik} + \mu\, c_{ik}) = 0.$$

Since $\sigma \neq 0$, $\mathfrak{a}$ will be non-singular; $\mathfrak{p}$ is then the single singular complex of the parabolic pencil of contact complexes (no. **128**). The definition of the apolar complex is projectively-invariant and invariant under renormalizations and parameter substitutions. From now on, we will always span the pencil of contact complexes $\mathfrak{b}$ by the singular complex p and the apolar complex $\mathfrak{a}$, so:

$$\mathfrak{b} = \lambda\,\mathfrak{p} + \mu\,\mathfrak{a}\,.$$

From no. **129**, we define the tensor a_{ik} by:

$$a_{ik} = \mathfrak{a}_i\,\mathfrak{p}_k = \mathfrak{a}_k\,\mathfrak{p}_i = -\,\mathfrak{a}\,\frac{\partial^2\mathfrak{p}}{\partial u^i \partial u^k} = a_{ki}\,,$$

and replace c_{ik} with a_{ik} in (223) and (224).

133. Osculating ruled families and osculating torses. Three systems (226) of first-order ordinary differential equations are determined by the three roots $\lambda : \mu$ of (227). Each of these three systems of differential equations will yield a solution:

$$n^i = \varphi^i(t - \bar{t}, \bar{u}^1, \bar{u}^2, \bar{u}^3)\,,$$

with the initial conditions $\bar{t}$, $\bar{u}^1$, $\bar{u}^2$, $\bar{u}^3$. The families of lines $u^i = u^i(t)$ that are given in that way will be ruled families. The solutions $\dot{u}^i$ of (226) will not fulfill the torse condition (220), since the corresponding points $\dot{u}^i = k^i$, $\dot{u}^i = l^i$, $\dot{u}^i = m^i$ of Fig. 31 will not lie on the conic. We call these ruled families the *osculating ruled families* of the line complex. They define three two-parameter sets; three osculating ruled families will go through any complex line. From no. **131**, a doubly-contacting complex will belong to any complex line $\mathfrak{p}$ and any osculating ruled family that goes through $\mathfrak{p}$; i.e., a contact complex that is stationary under advancing in the relevant osculating ruled family.

Just as three two-parameter sets of ruled families are determined by (224), four two-parameter sets of torses will be determined by (224). We call them the *osculating torses* of the line complex; four osculating torses go through any complex line. The osculating torses contact each line $\mathfrak{p}$ of the contact system (no. **128**) to order two; i.e., they fulfill the condition:

$$(\lambda\,\mathfrak{p} + \mu\,\mathfrak{a})\,\mathfrak{p}(u^i(t + \varepsilon)) = 0$$

for every λ and μ to order ε^2 (no. **129**).

Under the map of complex lines $\mathfrak{p}(u^i)$ to the points of space with the rectangular coordinates u^i (no. **125**), the osculating ruled families (osculating torses, resp.) will correspond to three (four, resp.) two-parameter systems of curves.

§ 37. Invariants of a line complex.

134. Definition of the invariant derivatives. In order to find the invariants of a line complex, we proceed essentially as in § 21. Once more, we first define *invariant derivatives.* For that, we employ the three first-rank contravariant tensors k^i, l^i, m^i that were introduced in no. **131**, and which satisfy the inequality (228). They give the directions of advance in the osculating ruled families (no. **133**) of the line complex [the vertices of the polar triangle that is common to the pencil of conics (223), resp.]. As a result, one has the polar relations:

$$g_{ik} l^i m^k = g_{ik} m^i k^k = g_{ik} k^i l^k = 0,$$
$$a_{ik} l^i m^k = a_{ik} m^i k^k = a_{ik} k^i l^k = 0. \tag{233}$$

The tensors g_{ik} and a_{ik} are then dependent upon the normalization of the six-vectors $\mathfrak{p}$, $\mathfrak{a}$ in that way; we shall defer that normalization to no. **136**. Furthermore, the tensors k^i, l^i, m^i are established by (233) only up to an arbitrary proportionality factor. In order to eliminate that indeterminacy, we demand that:

$$g_{ik}\, k^i\, k^k = +1, \qquad g_{ik}\, l^i\, l^k = -1, \qquad g_{ik}\, m^i\, m^k = -1; \tag{234}$$

the three tensors will be determined uniquely, up to a sign, in that way.

The requirements (234) can be fulfilled by real numbers: Since every polar triangle of the conic section γ has one interior point and two exterior ones as its vertices, we can assume that, say, k is interior, and l, m is exterior (Fig. 31). However, one will then have:

$$g_{ik}\, k^i\, k^k > 0 \text{ or } < 0, \qquad g_{ik}\, l^i\, l^k > 0 \text{ or } < 0, \qquad g_{ik}\, m^i\, m^k > 0 \text{ or } < 0 .$$

We restrict ourselves to the left-hand inequality, which can be enforced by a reflection of the given complex, under which, the scalar product $g_{ik} = \mathfrak{p}_i\, \mathfrak{p}_k$ will change sign (no. **16**).

Analogously to (109), we now define the *invariant derivatives* of a semi-invariant function $\varphi(u^1, u^2, u^3)$ by:

$$\varphi_1 = k^i\, \varphi_i\,, \qquad \varphi_2 = l^i\, \varphi_i\,, \qquad \varphi_3 = m^i\, \varphi_i\,; \tag{235}$$

just like the invariant derivatives for line systems, they are parameter invariant under projective maps, as well as semi-invariant.

135. Integrability conditions for the invariant derivatives. By solving equations (235) for φ_i, one obtains:

$$\varphi_i = k_i\, \varphi_1 - l_i\, \varphi_2 - m_i\, \varphi_3\,, \tag{236}$$

with the covariant first-rank tensors:

$$k_i = k^k\, g_{ik}\,, \qquad l_i = l^k\, g_{ik}\,, \qquad m_i = m^k\, g_{ik}\,,$$

for which, from (233_1) and (234), the relations are true:

$$l_i\, m^i = l_i\, k^i = m_i\, k^i = m_i\, l^i = k_i\, l^i = k_i\, m^i = 0,$$
$$k_i\, k^i = +1, \qquad l_i\, l^i = -1, \qquad m_i\, m^i = -1.$$

Proof: The k_i, $(-l_i)$, $(-m_i)$ are equal to the sub-determinants of the elements k^i, l^i, m^i in the determinant $|\, k^i, l^i, m^i \,|$, divided by the determinant $|\, k^i\; l^i\; m^i \,|$. It will then follow from this that:

$$k_i\,k^i - l_i\,l^i - m_i\,m^i = \begin{cases} 1 & \text{for } i = k \\ 0 & \text{for } i \neq k \end{cases}, \tag{237}$$

and (236) will follow from this as the solution of (235).

In place of (113) and (114), one has the *integrability conditions:*

$$\begin{aligned} \varphi_{23} - \varphi_{32} &= q\varphi_1 + q'\varphi_2 + q''\varphi_3, \\ \varphi_{31} - \varphi_{13} &= r\varphi_1 + r'\varphi_2 + r''\varphi_3, \\ \varphi_{12} - \varphi_{21} &= s\varphi_1 + s'\varphi_2 + s''\varphi_3, \end{aligned} \tag{238}$$

with

$$q = -\sum_{i,k}(l^i m^k - l^k m^i)\frac{\partial k_i}{\partial u^k}, \qquad q' = +\sum_{i,k}(l^i m^k - l^k m^i)\frac{\partial l_i}{\partial u^k},$$

$$q'' = +\sum_{i,k}(l^i m^k - l^k m^i)\frac{\partial m_i}{\partial u^k},$$

$$r = -\sum_{i,k}(m^i k^k - m^k k^i)\frac{\partial k_i}{\partial u^k}, \qquad r' = +\sum_{i,k}(m^i k^k - m^k k^i)\frac{\partial l_i}{\partial u^k}, \tag{239}$$

$$r'' = +\sum_{i,k}(m^i k^k - m^k k^i)\frac{\partial m_i}{\partial u^k},$$

$$s = -\sum_{i,k}(k^i l^k - k^k l^i)\frac{\partial k_i}{\partial u^k}, \qquad s' = +\sum_{i,k}(k^i l^k - k^k l^i)\frac{\partial l_i}{\partial u^k},$$

$$s'' = +\sum_{i,k}(k^i l^k - k^k l^i)\frac{\partial m_i}{\partial u^k}.$$

Proof: From (235) and (236), one obtains:

$$(\varphi_1)_i = k_i\,\varphi_{11} - l_i\,\varphi_{12} - m_i\,\varphi_{13}$$

and the second partial derivatives:

$$\begin{aligned} \frac{\partial^2\varphi}{\partial u^i\partial u^k} = {} & \frac{\partial k_i}{\partial u^k}\varphi_1 - \frac{\partial l_i}{\partial u^k}\varphi_2 - \frac{\partial m_i}{\partial u^k}\varphi_3 \\ & + k_i\,k_k\,\varphi_{11} - k_i\,l_k\,\varphi_{12} - k_i\,m_k\,\varphi_{13} \\ & - l_i\,k_k\,\varphi_{21} + l_i\,l_k\,\varphi_{12} + l_i\,m_k\,\varphi_{13} \\ & - m_i\,k_k\,\varphi_{31} + m_i\,l_k\,\varphi_{32} + m_i\,m_k\,\varphi_{33}\,. \end{aligned} \tag{240}$$

The integrability conditions for the partial derivatives:

$$\frac{\partial^2\varphi}{\partial u^i\partial u^k} = \frac{\partial^2\varphi}{\partial u^k\partial u^i}$$

can be replaced with the three requirements:

$$\sum_{i,k}(l^i m^k - l^k m^i)\frac{\partial^2\varphi}{\partial u^i \partial u^k} = 0, \quad \sum_{i,k}(m^i k^k - m^k k^i)\frac{\partial^2\varphi}{\partial u^i \partial u^k} = 0, \quad \sum_{i,k}(k^i l^k - k^k l^i)\frac{\partial^2\varphi}{\partial u^i \partial u^k} = 0.$$

Equations (238) and (239) then follow from this by substitution into (240).

Analogously to no. **76**, one shows that the nine functions q, q', q''; r, r', r''; s, s', s'' are not only projectively-invariant, like the k^i, l^i, m^i, but are also invariant under parameter substitutions.

136. Fundamental system. Normalization of $\mathfrak{p}$. We next take the complexes of *the fundamental system* to be $\mathfrak{p}$, along with the three invariant derivatives of $\mathfrak{p}$:

$$\mathfrak{q} = \mathfrak{p}_1, \qquad \mathfrak{r} = \mathfrak{p}_2, \qquad \mathfrak{s} = \mathfrak{p}_3\,.$$

From (234) and (233_1):

$$\mathfrak{q}\mathfrak{q} = +1, \qquad \mathfrak{r}\mathfrak{r} = \mathfrak{s}\mathfrak{s} = -1,$$
$$\mathfrak{r}\mathfrak{s} = \mathfrak{s}\mathfrak{q} = \mathfrak{q}\mathfrak{r} = 0;$$

i.e., the complexes are non-singular and pair-wise conjugate.

We choose the apolar complex $\mathfrak{a}$ to be the fifth complex. The six-vector $\mathfrak{a}$ is non-singular, and for that reason, it can be normalized by the requirement that:

(241) $$\mathfrak{a}\mathfrak{a} = \pm 1.$$

We will see directly that only the plus sign is possible.

In order to come to the last complex, we start from the bush of complexes $\lambda\,\mathfrak{q} + \mu\,\mathfrak{r} + \nu\,\mathfrak{s} + \rho\,\mathfrak{a}$, which naturally depends upon the normalization of $\mathfrak{p}$ that has yet to be done. Since:

$$D_4(\mathfrak{q}, \mathfrak{r}, \mathfrak{s}, \mathfrak{a}) = (\mathfrak{q}\mathfrak{q})\,(\mathfrak{r}\mathfrak{r})\,(\mathfrak{s}\mathfrak{s})\,(\mathfrak{a}\mathfrak{a}) = (\mathfrak{a}\mathfrak{a}) = \pm 1,$$

that bush will be elliptical or hyperbolic (no. **32**); i.e., there will exist 0 or 2 skew lines that are common to all complexes of the bush. However, since:

$$\mathfrak{p}\mathfrak{q} = \mathfrak{p}\mathfrak{r} = \mathfrak{p}\mathfrak{s} = \mathfrak{p}\mathfrak{a} = 0,$$

the complex line $\mathfrak{p}$ is obviously such a common line. Therefore, the bush is hyperbolic, and there is yet a second common line $\mathfrak{z}$, which is, in fact, skew to $\mathfrak{p}$. It satisfies the conditions:

$$\mathfrak{z}\mathfrak{q} = \mathfrak{z}\mathfrak{r} = \mathfrak{z}\mathfrak{s} = \mathfrak{z}\mathfrak{a} = 0;$$

the six-vector $\mathfrak{z}$ is normalized by:

(242) $$\mathfrak{z}\mathfrak{p} = 1.$$

Since the bush is hyperbolic, $D_4(\mathfrak{q}, \mathfrak{r}, \mathfrak{s}, \mathfrak{a}) > 0$ (no. **32**), so only the plus sign can be used in (241), as was already predicted.

We now have the following table of products for the fundamental system:

	$\mathfrak{p}$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{s}$	$\mathfrak{a}$	$\mathfrak{z}$
$\mathfrak{p}$	0	0	0	0	0	1
$\mathfrak{q}$	0	1	0	0	0	0
$\mathfrak{r}$	0	0	–1	0	0	0
$\mathfrak{s}$	0	0	0	–1	0	0
$\mathfrak{a}$	0	0	0	0	1	0
$\mathfrak{z}$	1	0	0	0	0	0

It follows from this that:

$$|\,\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}, \mathfrak{a}, \mathfrak{z}\,|^2 = -D_6(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}, \mathfrak{a}, \mathfrak{z}) = 1,$$

so the six complexes of the fundamental system are linearly-independent.

We must now attend to the *normalization* of the singular six-vector $\mathfrak{p}$: Along with (242), we demand that:

(243) $$g^2 = z^2,$$

in which g (z, resp.) means the determinant of the $g_{ik} = \mathfrak{p}_i\,\mathfrak{p}_k$ or:

$$z_{ik} = \mathfrak{z}_i\,\mathfrak{z}_k\,,$$

resp. Due to (108), the ratio $g : z$ will be parameter invariant, and therefore, the normalization condition (243), as well. One will get the normalized six-vector $\hat{\mathfrak{p}} = \rho\,\mathfrak{p}$ from an unnormalized one by using (242) and (243):

$$\hat{g} = \rho^6\,g, \qquad \hat{z} = z,$$

from the equation:

$$\rho^{12} = |\,z / g\,|,$$

so $\hat{\mathfrak{p}}$ is determined up to sign. In order for the normalization condition (243) to be fulfilled, we assume from now on that: *The line $\mathfrak{z}$ shall define a non-singular complex, as shall the line $\mathfrak{p}$.* From no. **127**, in fact, one will then have $z \neq 0$, along with $g \neq 0$.

§ 38. Differential equations.

137. Summary of the differential equations. We next give the scalar products of the six-vectors of the fundamental system with the invariant derivatives:

	$\mathfrak{p}$	$\mathfrak{q}$	$\mathfrak{r}$	$\mathfrak{s}$	$\mathfrak{a}$	$\mathfrak{z}$
$\mathfrak{q}_1$	-1	0	s	$-r$	R	D
$\mathfrak{q}_2$	0	0	$-s'$	$\frac{1}{2}(q+r'-s'')$	0	I
$\mathfrak{q}_3$	0	0	$\frac{1}{2}(-q+r'-s'')$	r''	0	H
$\mathfrak{r}_1$	0	$-s$	0	$\frac{1}{2}(q+r'+s'')$	0	I
$\mathfrak{r}_2$	1	s'	0	$-q'$	S	E
$\mathfrak{r}_3$	0	$\frac{1}{2}(q-r'+s'')$	0	$-q''$	0	G
$\mathfrak{s}_1$	0	r	$\frac{1}{2}(-q-r'-s'')$	0	0	H
$\mathfrak{s}_2$	0	$\frac{1}{2}(-q-r'+s'')$	q'	0	0	G
$\mathfrak{s}_3$	1	$-r''$	q''	0	T	F
$\mathfrak{a}_1$	0	$-R$	0	0	0	A
$\mathfrak{a}_2$	0	0	$-S$	0	0	B
$\mathfrak{a}_3$	0	0	0	$-T$	0	C
$\mathfrak{z}_1$	0	$-D$	$-I$	$-H$	$-A$	0
$\mathfrak{z}_2$	0	$-I$	$-E$	$-G$	$-B$	0
$\mathfrak{z}_3$	0	$-H$	$-G$	$-F$	$-C$	0

In this, one must consider that:

a) The conditions (238), when applied to $\mathfrak{p}$, will yield:

$$\begin{aligned} \mathfrak{r}_3 - \mathfrak{s}_2 &= q\,\mathfrak{q} + q'\,\mathfrak{r} + q''\,\mathfrak{s}, \\ \mathfrak{s}_1 - \mathfrak{q}_3 &= r\,\mathfrak{q} + r'\,\mathfrak{r} + r''\,\mathfrak{s}, \\ \mathfrak{q}_2 - \mathfrak{r}_1 &= s\,\mathfrak{q} + s'\,\mathfrak{r} + s''\,\mathfrak{s}. \end{aligned}$$

By scalar multiplying with $\mathfrak{q}$, $\mathfrak{r}$, $\mathfrak{s}$, resp., one will get:

$$\begin{aligned} \mathfrak{q}\mathfrak{r}_3 + \mathfrak{s}\mathfrak{q}_2 &= q, & \mathfrak{r}\mathfrak{s}_2 &= q', & \mathfrak{s}\mathfrak{r}_3 &= -q'', \\ \mathfrak{q}\mathfrak{s}_2 &= r, & \mathfrak{r}\mathfrak{s}_1 + \mathfrak{q}\mathfrak{r}_3 &= -r', & \mathfrak{s}\mathfrak{q}_3 &= r'', \\ \mathfrak{q}\mathfrak{r}_1 &= s, & \mathfrak{r}\mathfrak{q}_2 &= -s', & \mathfrak{s}\mathfrak{q}_2 + \mathfrak{r}\mathfrak{s}_1 &= -s'', \end{aligned}$$

so

$$\mathfrak{q}\mathfrak{r}_1 = \tfrac{1}{2}(q - r' + s''), \quad \mathfrak{s}\mathfrak{q}_2 = \tfrac{1}{2}(q + r' - s''), \quad \mathfrak{r}\mathfrak{s}_1 = \tfrac{1}{2}(-q - r' - s'').$$

b) $$\begin{aligned} G &= \mathfrak{z}\mathfrak{r}_3 = \mathfrak{z}\mathfrak{p}_{23} = \mathfrak{z}\mathfrak{p}_{32} = \mathfrak{z}\mathfrak{s}_2\,, \\ H &= \mathfrak{z}\mathfrak{s}_1 = \mathfrak{z}\mathfrak{p}_{31} = \mathfrak{z}\mathfrak{p}_{13} = \mathfrak{z}\mathfrak{q}_3\,, \\ I &= \mathfrak{z}\mathfrak{q}_2 = \mathfrak{z}\mathfrak{p}_{12} = \mathfrak{z}\mathfrak{p}_{21} = \mathfrak{z}\mathfrak{r}_1\,. \end{aligned}$$

c) $$\mathfrak{a}\mathfrak{q}_2 = -\,\mathfrak{a}_2\,\mathfrak{q} = -\,\mathfrak{a}_2\,\mathfrak{q}_1 = -\,k^i\,l^k\,\mathfrak{p}_i\,\mathfrak{a}_k = -\,k^i\,l^k\,a_{ik} = 0$$

from (233); analogously:

$$\mathfrak{a}\mathfrak{q}_3 = -\,k^i\,m^k\,a_{ik} = 0, \qquad \mathfrak{a}\mathfrak{r}_1 = \mathfrak{a}\mathfrak{r}_3 = \mathfrak{a}\mathfrak{s}_1 = \mathfrak{a}\mathfrak{s}_2 = 0.$$

With the help of the table of products in nos. **136** and **137**, we get the *differential equations:*

$$\begin{array}{llllll}
\mathfrak{p}_1 = \mathfrak{q}, & & \mathfrak{p}_2 = \mathfrak{r}, & \mathfrak{p}_3 = \mathfrak{s}, & \\
\mathfrak{q}_1 = D\mathfrak{p} & & -s\mathfrak{r} & +r\mathfrak{s} & +R\mathfrak{a} - \mathfrak{z}, \\
\mathfrak{q}_2 = I\mathfrak{p} & & +s'\mathfrak{r} & -\tfrac{1}{2}(q+r'-s'')\mathfrak{s}, & \\
\mathfrak{q}_3 = H\mathfrak{p} & & +\tfrac{1}{2}(q-r'+s'')\mathfrak{r} & -r''\mathfrak{s}, & \\
\mathfrak{r}_1 = I\mathfrak{p} & -s\mathfrak{q} & & -\tfrac{1}{2}(q+r'+s'')\mathfrak{s}, & \\
\mathfrak{r}_2 = E\mathfrak{p} & +s'\mathfrak{q} & & +q'\mathfrak{s} & +S\mathfrak{a} + \mathfrak{z}, \\
\mathfrak{r}_3 = G\mathfrak{p} & +\tfrac{1}{2}(q-r'+s'')\mathfrak{q} & & +q''\mathfrak{s}, & \\
\mathfrak{s}_1 = H\mathfrak{p} & +r\mathfrak{q} & +\tfrac{1}{2}(q+r'+s'')\mathfrak{r} & & \\
\mathfrak{s}_2 = G\mathfrak{p} & -\tfrac{1}{2}(q+r'-s'')\mathfrak{q} & -q'\mathfrak{r}, & & \\
\mathfrak{s}_3 = F\mathfrak{p} & -r''\mathfrak{q} & -q''\mathfrak{r} & & +T\mathfrak{a} + \mathfrak{z}, \\
\mathfrak{a}_1 = A\mathfrak{p} & -R\mathfrak{q} & & & \\
\mathfrak{a}_2 = B\mathfrak{p} & & +S\mathfrak{r}, & & \\
\mathfrak{a}_3 = C\mathfrak{p} & & & +T\mathfrak{s}, & \\
\mathfrak{z}_1 = & -D\mathfrak{q} & +I\mathfrak{r} & +H\mathfrak{s} & -A\mathfrak{a}, \\
\mathfrak{z}_2 = & -I\mathfrak{q} & +E\mathfrak{r} & +G\mathfrak{s} & -B\mathfrak{a}, \\
\mathfrak{z}_3 = & -H\mathfrak{q} & +G\mathfrak{r} & +F\mathfrak{s} & -C\mathfrak{a}.
\end{array} \tag{244}$$

138. Integrability conditions and auxiliary conditions. For the sake of brevity, we restrict ourselves in what follows to the special assumption of constant coefficients in the differential equations (244) and vanishing invariants q, q', q''; r, r', r''; s, s', s''. Analogous to what was done in no. **86**, one then obtains from (244) the *integrability conditions:*

$$\begin{aligned} A &= B = C = 0, \\ G &= H = I = 0, \\ D &= -\tfrac{1}{2}(\;\;RS + ST + TR), \\ E &= -\tfrac{1}{2}(-RS + ST + TR), \end{aligned} \tag{245}$$

$$F = -\tfrac{1}{2}(\quad RS + ST - TR).$$

Along with the integrability conditions, one must also consider:

a) The apolarity condition (231).
b) The normalization prescription (243).
c) The requirement of the existence of precisely four principal directions of advance (no. **130**).

That leads to the *auxiliary conditions:*

$$\begin{gathered} R - S - T = 0, \\ D^2 E^2 F^2 = 1, \\ R + S \neq 0, \qquad T - S \neq 0, \qquad R + T \neq 0. \end{gathered} \tag{246}$$

($246_{1,3}$) will then be true in general; i.e., for non-constant invariants, as well.

Proof: From (236), one has:

$$g_{ik} = (k_i\,\mathfrak{p}_1 - l_i\,\mathfrak{p}_2 - m_i\,\mathfrak{p}_3)\,(k_k\,\mathfrak{p}_1 - l_k\,\mathfrak{p}_2 - m_k\,\mathfrak{p}_3) = k_i\,k_k - l_i\,l_k - m_i\,m_k\,,$$

and analogously:

$$a_{ik} = -R\,k_i\,k_k - S\,l_i\,l_k - T\,m_i\,m_k\,,$$

so, from (231) ([1]):

$$0 = g^{ik}\,a_{ik} = -R\,g^{ik}\,k_i\,k_k - S\,g^{ik}\,l_i\,l_k - T\,g^{ik}\,m_i\,m_k = -R + S + T.$$

Moreover, one has:

$$g = D_3(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3) = D_3(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)\begin{vmatrix} k_1 & k_2 & k_3 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix}^2,$$

$$z = D_3(\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3) = D_3(\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3)\begin{vmatrix} k_1 & k_2 & k_3 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix}^2,$$

so, from (243):

$$[D_3(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3)]^2 = [D_3(\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3)]^2;$$

(246_2) will come about upon replacing the differential equations (244) and considering (245).

([1]) It follows from $k^i = g^{ir}\,k_r$ that $k_i = g_{ir}\,k^r$, and therefore $g^{ik}\,k_i\,k_k = k^k\,k_k = 1$, etc.

Upon establishing the common polar triangle to be the coordinate triangle and making a suitable choice of unit point (i.e., a parameter substitution), the pencil of conics (223) will have the equation:

$$\lambda[(\dot{u}^1)^2 - (\dot{u}^2)^2 - (\dot{u}^3)^2] + \mu[R(\dot{u}^1)^2 + S(\dot{u}^2)^2 + T(\dot{u}^3)^2] = 0.$$

The requirement that there be precisely four base points will imply (246_3).

Analogously to what was done in no. **86**, if one excludes $z = 0$ (no. **136**) and assumes that there are precisely four distinct real principal points for every complex line (no. **130**) then:

The non-singular line complexes are characterized projectively by the tensors k^i, l^i, m^i, and the invariants A, B, C, etc.

The functions k^i, l^i, m^i, A, B, C, etc., are not independent of each other. They must satisfy the inequality (228), the integrability conditions, and the auxiliary conditions. They are subject to no other restrictions, so one will obtain the following existence theorem by a development that is similar to the one in no. **65**:

Let the tensors k^i, l^i, m^i, and the functions A, B, C; D, E, F; G, H, I; R, S, T be such that the integrability conditions, as well as the auxiliary conditions, are fulfilled, but are otherwise given as arbitrary functions of the u^1, u^2, u^3. There will then always be a non-singular line complex for which the invariant derivatives with respect to k^i, l^i, m^i are defined and whose differential equations contain the given functions as coefficients.

139. Double ratio of the principal points. From no. **130**, any complex line $\mathfrak{p}$ contains four principal points. *One will get:*

$$d = \frac{R - 2S}{2R - S}$$

for the double ratio d of the principal points.

Proof: For a suitable parameter substitution (cf., no. **138**), the four principal directions of advance will be given by:

$$(\dot{u}^1)^2 - (\dot{u}^2)^2 - (\dot{u}^3)^2 = 0, \qquad R(\dot{u}^1)^2 + S(\dot{u}^2)^2 + T(\dot{u}^3)^2 = 0,$$

so:

$$\dot{u}^1 : \dot{u}^2 : \dot{u}^3 = +\sqrt{-S+T} : \pm\sqrt{R+T} : \pm\sqrt{-R-T}$$

(Four sign combinations!)

For a suitable coordinate transformations, one will have:

$$\mathfrak{p} = 0 \mid \quad 0 \quad \mid 0 \mid 1 \mid \quad 0 \quad \mid 0,$$

$$\mathfrak{q} = 0 \mid \frac{1}{\sqrt{2}} \mid 0 \mid 0 \mid \frac{1}{\sqrt{2}} \mid 0,$$

$$\mathfrak{r} = 0 \mid 0 \mid \frac{1}{\sqrt{2}} \mid 0 \mid 0 \mid \frac{1}{\sqrt{2}},$$

$$\mathfrak{s} = 0 \mid \frac{1}{\sqrt{2}} \mid 0 \mid 0 \mid -\frac{1}{\sqrt{2}} \mid 0 .$$

This Ansatz then satisfies the table of scalar products of no. **136**. Since the principal point is the point of intersection of the lines $\mathfrak{p}$ and $\dot{u}^1\mathfrak{q} + \dot{u}^2\mathfrak{r} + \dot{u}^3\mathfrak{s}$, (17) will then give the coordinates:

$$\begin{aligned} x_1 : x_4 : x_2 : x_3 &= 0 : 0 : +\sqrt{R+T} \;:\; (+\sqrt{-S+T} - \sqrt{-R-S}), \\ &\quad\; 0 : 0 : +\sqrt{R+T} \;:\; (+\sqrt{-S+T} + \sqrt{-R-S}), \text{ resp.}, \\ &\quad\; 0 : 0 : -\sqrt{R+T} \;:\; (+\sqrt{-S+T} + \sqrt{-R-S}), \text{ resp.}, \\ &\quad\; 0 : 0 : -\sqrt{R+T} \;:\; (+\sqrt{-S+T} - \sqrt{-R-S}), \text{ resp.}, \end{aligned}$$

from which, one will get the given double ratio from (2).

§ 39. Tetrahedral complexes.

140. Characterization of the projective invariants. As an example of what was discussed in general in § 38, we now treat the so-called *tetrahedral line complexes*, which are characterized by having constant coefficients in the differential equations and vanishing invariants q, q', q''; r, r', r''; s, s', s''. We have already derived the integrability conditions (245) and the auxiliary conditions (246) for these special complexes.

The tetrahedral line complexes are *self-projective*; i.e., they will be transformed into themselves by a *three-parameter group of collineations*. Upon restricting ourselves to a suitable domain in u^1, u^2, u^3, two arbitrary complex lines can go to each other by a collineation of the group.

Proof: From (239), since $q = r = s = 0$:

$$\frac{\partial k_2}{\partial u^3} = \frac{\partial k_3}{\partial u^2}, \quad \frac{\partial k_3}{\partial u^1} = \frac{\partial k_1}{\partial u^3}, \quad \frac{\partial k_1}{\partial u^2} = \frac{\partial k_2}{\partial u^1},$$

and since $q' = r' = s' = 0$, $q'' = r'' = s'' = 0$, one likewise has:

$$\begin{aligned} \frac{\partial l_2}{\partial u^3} &= \frac{\partial l_3}{\partial u^2}, \quad \frac{\partial l_3}{\partial u^1} = \frac{\partial l_1}{\partial u^3}, \quad \frac{\partial l_1}{\partial u^2} = \frac{\partial l_2}{\partial u^1}, \\ \frac{\partial m_2}{\partial u^3} &= \frac{\partial m_3}{\partial u^2}, \quad \frac{\partial m_3}{\partial u^1} = \frac{\partial m_1}{\partial u^3}, \quad \frac{\partial m_1}{\partial u^2} = \frac{\partial m_2}{\partial u^1}. \end{aligned}$$

One can then set:

$$k_i = \frac{\partial}{\partial u^i}\,\varphi(u^1, u^2, u^3), \qquad l_i = \frac{\partial}{\partial u^i}\,\psi(u^1, u^2, u^3), \qquad m_i = \frac{\partial}{\partial u^i}\,\chi(u^1, u^2, u^3),$$

in which φ, ψ, χ are three arbitrary functions with non-vanishing functional determinants.

With the parameter substitution:

(247) $$\bar{u}^1 = \varphi(u^1, u^2, u^3), \qquad \bar{u}^2 = \psi(u^1, u^2, u^3), \qquad \bar{u}^3 = \chi(u^1, u^2, u^3),$$

one will get:

$$k_i\,\dot{u}^i = \bar{k}_i\,\dot{\bar{u}}^i = \bar{k}_1(k_1\dot{u}^1 + k_2\dot{u}^2 + k_3\dot{u}^3) + \bar{k}_2(l_1\dot{u}^1 + l_2\dot{u}^2 + l_3\dot{u}^3) + \bar{k}_3(m_1\dot{u}^1 + m_2\dot{u}^2 + m_3\dot{u}^3),$$

so:

$$\begin{aligned} k_1 &= \bar{k}_1 k_1 + \bar{k}_2 l_1 + \bar{k}_3 m_1, \\ k_2 &= \bar{k}_1 k_2 + \bar{k}_2 l_2 + \bar{k}_3 m_2, \\ k_3 &= \bar{k}_1 k_3 + \bar{k}_2 l_3 + \bar{k}_3 m_3. \end{aligned}$$

Since, from (228), the coefficient determinant $|\,k_i\;l_i\;m_i\,|$ does not vanish ([1]), these and the corresponding equations for $\bar{l}_i$, $\bar{m}_i$ imply that:

(248) $$\bar{k}_1 = 1,\ \bar{k}_2 = 0,\ \bar{k}_3 = 0; \qquad \bar{l}_1 = 0,\ \bar{l}_2 = 1,\ \bar{l}_3 = 0; \qquad \bar{m}_1 = 0,\ \bar{m}_2 = 0,\ \bar{m}_3 = 1.$$

The assertion then follows on the basis of the transformation of the complex into itself by:

$$u^1 = \bar{u}^1 + a^1, \quad u^2 = \bar{u}^2 + a^2, \quad u^3 = \bar{u}^3 + a^3,$$

with the arbitrary constants a^1, a^2, a^3, in analogy to no. **95**.

In analogy to no. **95**, one further gets:

There exists a one-parameter subgroup of collineations for which every family of lines u^2 = const. u^3 = const. (u^3 = const. u^1 = const., resp., u^1 = const. u^2 = const., resp.) of the tetrahedral complex is transformed into itself alone.

In no. **145**, we will show that these families of lines are the *osculating ruled families.*

([1]) From no. **135**, one has:

$$\begin{vmatrix} k_1 & k_2 & k_3 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix} \cdot \begin{vmatrix} k^1 & k^2 & k^3 \\ l^1 & l^2 & l^3 \\ m^1 & m^2 & m^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 1.$$

141. Explicit representation of the tetrahedral complexes. We choose the parameters that are given by (147) as our reference and then give the special values (248) to k_i, l_i, m_i that make the invariant derivatives go to ordinary partial derivatives. From (245), a tetrahedral complex is characterized projectively by three constants R, S, T. From (246), R, S, T are not mutually independent, but they can be reduced to a *single projectively-invariant constant:*

One the basis of ($246_{1,2}$), one can set:

$$R = \tau(a + 2b), \qquad S = \tau(b + 2a), \qquad T = \tau(b - a),$$
$$(a \neq 0,\ a \neq 0,\ a + b \neq 0;\ \tau \neq 0),$$

in which τ, a, b are arbitrary constants, at first. The integrability conditions (245) then imply that D, E, F are:

$$D = \frac{\tau^2}{2}(a^2 - 5b^2 - 5ab), \qquad E = \frac{\tau^2}{2}(5a^2 - b^2 + 5ab), \qquad F = \frac{\tau^2}{2}(-a^2 - b^2 - 7ab),$$

and (246_2) will give t for any pair of numbers a, b such that the ratio $a : b$ remains arbitrary as a single projectively-invariant constant. In no. **142**, we will interpret that invariant geometrically.

The differential equations (244) yield:

$$\frac{\partial^2 \mathfrak{p}}{\partial u^2 \partial u^3} = 0, \qquad \frac{\partial^2 \mathfrak{p}}{\partial u^3 \partial u^1} = 0, \qquad \frac{\partial^2 \mathfrak{p}}{\partial u^1 \partial u^2} = 0,$$

$$\frac{\partial^3 \mathfrak{p}}{\partial (u^1)^3} = (2D - R^2)\frac{\partial \mathfrak{p}}{\partial u^1}, \quad \frac{\partial^3 \mathfrak{p}}{\partial (u^2)^3} = (2E + S^2)\frac{\partial \mathfrak{p}}{\partial u^2}, \quad \frac{\partial^3 \mathfrak{p}}{\partial (u^3)^3} = (2F + T^2)\frac{\partial \mathfrak{p}}{\partial u^3}.$$

For a suitable choice of initial conditions, the integration will imply:

$$\rho\, \mathfrak{p} = e^{\alpha u^1} \mid e^{\beta u^2} \mid e^{\gamma u^3} \mid \tau a e^{-\alpha u^1} \mid \tau b e^{-\beta u^2} \mid \tau c e^{-\gamma u^3}, \tag{249}$$

with the abbreviations:

$$c = -(a + b),$$

$$\alpha^2 = \tau^2 (2D - R^2) = 9bc\tau^2, \quad \beta^2 = \tau^2 (2E + S^2) = -9ac\tau^2, \tag{250}$$
$$\gamma^2 = \tau^2 (2F + T^2) = -9ab\tau^2.$$

The normalization factor ρ will be established by:

$$1 = \mathfrak{q}\mathfrak{q} = \rho^2\, \mathfrak{p}_1\, \mathfrak{p}_1 = -2\rho^2 \alpha^2 a\tau, \quad 1 = -\mathfrak{q}\mathfrak{q} = -2\rho^2 \beta^2 b\tau, \quad 1 = -\mathfrak{s}\mathfrak{s} = 2\rho^2 \gamma^2 c\tau,$$

namely:

$$\rho^2 = -\frac{1}{18abc\,\tau^3}.$$

The constants α, β, γ, ρ are real for $a < 0$, $b > 0$, $c > 0$; from now on, we shall make that assumption.

It follows further from (249) and (244) that:

$$\begin{aligned}\rho\,\mathfrak{a} = \tau(b-c)\,e^{\alpha u^1} \mid \tau(c-a)\,e^{\beta u^2} \mid \tau(a-b)\,e^{\gamma u^3} \mid \\ \mid \tau^2 a\,(b-c)\,e^{-\alpha u^1} \mid \tau^2 b\,(c-a)\,e^{-\beta u^2} \mid \tau^2 c\,(a-b)\,e^{-\gamma u^3};\end{aligned} \tag{251}$$

$$\rho\,\mathfrak{z} = -De^{\alpha u^1} \mid Ee^{\beta u^2} \mid Fe^{\gamma u^3} \mid -\tau a\;De^{-\alpha u^1} \mid \tau b\;Ee^{-\beta u^2} \mid \tau c\;Fe^{-\gamma u^3}.$$

142. Interpretation of the projective invariant $a : b$. The tetrahedral complex has been the subject of numerous investigations into algebraic line geometry ([1]). Its name is based upon the following property:

All lines of the tetrahedral complex (249) *cut the four planes of the coordinate tetrahedron in quadruples of points that have the constant double ratio:*

$$d = -\frac{a}{b} = \frac{R-2S}{2R-S} > 1. \tag{252}$$

Proof: (14) implies the points of intersection:

$$\begin{array}{lllll}
\text{I)} & x_1 = 0, & x_2 = -\,ce^{\beta u^2}, & x_3 = be^{\gamma u^3}, & x_4 = e^{\alpha u^1+\beta u^2+\gamma u^3}, \\
\text{II)} & x_1 = ce^{\alpha u^1}, & x_2 = 0, & x_3 = -\,ae^{\gamma u^3}, & x_4 = e^{\alpha u^1+\beta u^2+\gamma u^3}, \\
\text{III)} & x_1 = -\,be^{\alpha u^1}, & x_2 = ae^{\beta u^2}, & x_3 = 0, & x_4 = e^{\alpha u^1+\beta u^2+\gamma u^3}, \\
\text{IV)} & x_1 = e^{\alpha u^1}, & x_2 = e^{\beta u^2}, & x_3 = e^{\gamma u^3}, & x_4 = 0,
\end{array}$$

from which the assertion follows by means of (2)

Conversely, one has:

The set of all lines that cut the planes of a tetrahedron with constant double ratio d define a tetrahedral complex (249) *with the projective invariant $d = -a / b$.*

Proof: Among the six values that the double ratio will assume under all possible permutations of the four points, there will always be one of them with $d > 1$. From (252), the ratio $a : b$ is determined from that value of d. We can then set $a < 0$, $b > 0$, $c = -(a + b) > 0$ (no. **141**) and obtain the proportionality factor τ from (246_2). The constants α, β, γ that one obtains from (250) will then be real.

([1]) Cf., e.g., **Th. Reye:** *Geometrie der Lage III,* Leipzig, 1923.

143. Quadratic equation of the tetrahedral complex. It follows from (249) that:

$$c\,p_2\,p_5 - b\,p_3\,p_6 = 0, \quad a\,p_3\,p_6 - b\,p_1\,p_4 = 0, \quad b\,p_1\,p_4 - a\,p_2\,p_5 = 0. \tag{253}$$

Due to the fact that $a + b + c = 0$ and the identity (10), these three equations are independent of each other, and in fact each of them has the other two as a consequence. The tetrahedral complex can then be defined to be the set of all lines whose line coordinates satisfy the condition (253_1). That equation is also satisfied by all lines that go through the four vertices of the coordinate tetrahedron or the four coordinate planes (cf., Fig. 2), so we would not like to count those lines with the tetrahedral complex in what follows. Since (253) has degree two in p_ρ, one calls the tetrahedral complex a *quadratic complex;* however, one observes that conversely, not every quadratic complex is a tetrahedral complex. The theory of general quadratic complexes belongs to the realm of algebraic geometry.

From no. **140**, the tetrahedral complex (253) is self-projective. The three-parameter group of collineations under which it is transformed is given by:

$$\tilde{x}_1 = \sigma_1\,x_1, \qquad \tilde{x}_2 = \sigma_2\,x_2, \qquad \tilde{x}_3 = \sigma_3\,x_3, \qquad \tilde{x}_4 = \sigma_4\,x_4,$$

with arbitrary non-vanishing constants σ_i . It will then consist of all collineations under which the vertices (planes, resp.) of the coordinate tetrahedron are the only fixed points (fixed planes, resp.) (no. **3**). Along with these collineations, there are also correlations that transform the tetrahedral complex into itself. Such a correlation is, e.g., the polarity (no. **14**).

$$\tilde{p}_\rho = p_{\rho\pm 3},$$

under which, the four points of intersection of a complex line with the planes of the tetrahedron go to the four connecting planes of the corresponding complex lines with the vertices of the tetrahedron. Since the double ratio of the quadruple of intersection points is equal to that of the quadruple of the connecting planes, it will follow from no. **142** that:

All lines of the tetrahedral complex (249) *lie with the four vertices of the coordinates in quadruples of planes with the constant double ratio* $d = -a / b$.

One will get the equation of the complex cone of the point x by substituting (9) in (253_1):

$$c(x_2'x_4 - x_4'x_2)(x_3'x_1 - x_1'x_3) - b(x_3'x_4 - x_4'x_3)(x_1'x_2 - x_2'x_1) = 0.$$

It will then follow from this that: *The complex cone of any point that does not lie in a coordinate plane is a non-singular cone of order two.*

The dual theorem reads: *The complex curve of any plane that does not go through a vertex of the coordinate tetrahedron is a non-singular curve of class two.*

We shall now discuss the *complex of the line* $\mathfrak{z}$: From (251_2), it satisfies the equation:

$$E^2\, b\, p_1\, p_4 - E^2\, b\, p_1\, p_4 = 0,$$

so it will again be a tetrahedral complex. Its double ratio δ is coupled with the double ratio d of the tetrahedral complex of the line $\mathfrak{p}$ by:

$$\delta = -\frac{aD^2}{bE^2} = d\,\frac{(d^2+5d-5)^2}{(5d^2-5d-1)^2}.$$

The requirement $d = \delta$ can be fulfilled for only $\delta = d = 2$, since $d > 1$ [cf., (252)]. Thus:

The tetrahedral complex with a harmonic double ratio $\delta = d = 2$ is the only tetrahedral complex for which the $\mathfrak{z}$-complex is identical with the $\mathfrak{p}$-complex.

144. Different ways of generating a tetrahedral complex. The definitions of a tetrahedral complex that were given in nos. **140** and **142** are equivalent to the following way of generating them:

The set of connecting lines of corresponding points x, $\tilde{x}$ under the collineation:

(254) $\quad \tilde{x}_1 = \sigma_1\, x_1,\; \tilde{x}_2 = \sigma_2\, x_2,\; \tilde{x}_3 = \sigma_3\, x_3,\; \tilde{x}_4 = \sigma_4\, x_4 \qquad (\sigma_4 > \sigma_3 > \sigma_1 > \sigma_2\,;\; \sigma_i \neq 0),$

which has the four vertices of the coordinate tetrahedron for its only fixed points, is a tetrahedral complex with the double ratio:

(255) $$d = \frac{(\sigma_3-\sigma_2)(\sigma_4-\sigma_1)}{(\sigma_3-\sigma_1)(\sigma_4-\sigma_2)} > 1.$$

Proof: From (254) and (9), the connecting lines of corresponding points have the line coordinates:

$$\tau\,\mathfrak{p} = (\sigma_4-\sigma_1)\, x_1\, x_4 \mid (\sigma_4-\sigma_2)\, x_2\, x_4 \mid (\sigma_4-\sigma_3)\, x_3\, x_4 \mid (\sigma_2-\sigma_3)\, x_2\, x_3 \mid$$
$$\mid (\sigma_3-\sigma_1)\, x_3\, x_1 \mid (\sigma_1-\sigma_2)\, x_1\, x_2\,.$$

If one sets:

(256) $$x_1 = \frac{1}{\sigma_4-\sigma_1}\, e^{\alpha u^1}, \quad x_2 = \frac{1}{\sigma_4-\sigma_2}\, e^{\beta u^2}, \quad x_3 = \frac{1}{\sigma_4-\sigma_3}\, e^{\gamma u^3}, \quad x_4 = e^{\alpha u^1+\beta u^2+\gamma u^3}$$

then one will get the representation:

$$\rho\,\mathfrak{p} = e^{\alpha u^1} \mid e^{\beta u^2} \mid e^{\gamma u^3} \mid$$

$$\left|\frac{\sigma_2-\sigma_3}{(\sigma_4-\sigma_2)(\sigma_4-\sigma_3)}e^{-\alpha u^1}\right|\frac{\sigma_3-\sigma_1}{(\sigma_4-\sigma_3)(\sigma_4-\sigma_1)}e^{-\beta u^2}\left|\frac{\sigma_1-\sigma_2}{(\sigma_4-\sigma_1)(\sigma_4-\sigma_2)}e^{-\gamma u^3}\right.,$$

which can be identified with (249). In the notation of no. **141**, one has:

$$\tau\, a = \frac{\sigma_2-\sigma_3}{(\sigma_4-\sigma_2)(\sigma_4-\sigma_3)} < 0, \qquad \tau\, b = \frac{\sigma_3-\sigma_1}{(\sigma_4-\sigma_3)(\sigma_4-\sigma_1)} > 0,$$

$$\tau\, c = \frac{\sigma_1-\sigma_2}{(\sigma_4-\sigma_1)(\sigma_4-\sigma_2)} > 0,$$

and (255) will follow from (252); in order to fulfill (246_2), the σ_i must be normalized correspondingly.

Conversely, every tetrahedral complex (249) *can be generated as the set of connecting lines of corresponding points under any collineation of the two-parameter collineation group* (254), *in which the ratios* $\sigma_2 / \sigma_3 < 1$, $\sigma_3 / \sigma_4 < 1$ *are regarded as arbitrary parameters, and* σ_1 / σ_2 *is determined by the double ratio d from* (255).

Since, from no. **143**, the tetrahedral complex will be transformed into itself under the polarity, the connecting line of two points x, $\tilde{x}$ that correspond under (254) must also belong to the line of intersection of the polar planes w, $\tilde{w}$ of the complex. That will then imply a way of generating it that is dual to (254):

The set of lines of intersection of corresponding planes w, $\tilde{w}$ *under a collineation:*

(257) $$\tilde{w}^1 = \sigma_1\, w^1, \quad \tilde{w}^2 = \sigma_2\, w^2, \quad \tilde{w}^3 = \sigma_3\, w^3, \quad \tilde{w}^4 = \sigma_4\, w^4,$$

$$(\sigma_4 > \sigma_3 > \sigma_1 > \sigma_2)$$

define a tetrahedral complex with a double ratio that is determined by (255).

Another manner of generating the tetrahedral complex is the following one:

The set of lines of intersection of corresponding projective pencils of lines with different vertices and in different planes will be a tetrahedral complex when the pencil of lines cuts two projective sequences of points with precisely two fixed points out of the line of intersection.

Proof: Two projective pencils of lines ([1]) are given by:

([1]) The lines $\mathfrak{q}$, $\mathfrak{r}$ have nothing to do with the fundamental complexes $\mathfrak{q}$, $\mathfrak{r}$, $\mathfrak{s}$ that were introduced in no. **136**.

$$\mathfrak{q} = 0 \mid 0 \mid 0 \mid 0 \mid e^{\gamma u^3} \mid -e^{\beta u^2}, \qquad \mathfrak{r} = 0 \mid ce^{-\gamma u^3} \mid -be^{-\beta u^2} \mid 0 \mid 0 \mid 0,$$

$$\mathfrak{q} = 0 \mid 0 \mid 0 \mid -e^{\gamma u^3} \mid 0 \mid e^{\alpha u^1}, \qquad \mathfrak{r} = -ce^{-\gamma u^3} \mid 0 \mid ae^{-\alpha u^1} \mid 0 \mid 0 \mid 0, \text{ resp.,} \tag{258}$$

$$\mathfrak{q} = 0 \mid 0 \mid 0 \mid e^{\beta u^2} \mid -e^{\alpha u^1} \mid 0, \qquad \mathfrak{r} = be^{-\beta u^2} \mid -ae^{-\alpha u^1} \mid 0 \mid 0 \mid 0 \mid 0, \text{ resp.,}$$

with the parameters u^1, u^2, u^3. Its planes intersect along the edge 23 (31, 12, resp.) of the coordinate tetrahedron (Fig. 2) and the projective point-sequences along those edges will have the vertices 2 and 3 (3 and 1, 1 and 2, resp.) for fixed points. The line of intersection $\mathfrak{p}$ of the two skew lines $\mathfrak{q}$, $\mathfrak{r}$ is determined by the conditions:

$$\begin{aligned} &e^{\gamma u^3} p_2 - e^{\beta u^2} p_3 = 0, && ce^{-\gamma u^3} p_5 - be^{-\beta u^2} p_6 = 0,\\ &e^{\alpha u^1} p_3 - e^{\gamma u^3} p_1 = 0, && ae^{-\alpha u^1} p_6 - ce^{-\gamma u^3} p_4 = 0, \text{ resp.,}\\ &e^{\beta u^2} p_1 - e^{\alpha u^1} p_2 = 0, && be^{-\beta u^2} p_4 - ae^{-\alpha u^1} p_5 = 0, \text{ resp.} \end{aligned} \tag{259}$$

The lines of intersection each define a hyperbolic line system with focal lines $\mathfrak{q}$, $\mathfrak{r}$ for $\beta u^2 - \gamma u^3$ ($\gamma u^3 - \alpha u^1$, $\alpha u^1 - \beta u^2$, resp.). Equations (253) will yield the tetrahedral complex upon eliminating u^1, u^2, u^3.

145. Osculating ruled families and osculating torses. In no. **138**, we had:

$$g_{ik} = k_i\, k_k - l_i\, l_k - m_i\, m_k, \qquad a_{ik} = -R\, k_i\, k_k - S\, l_i\, l_k - T\, m_i\, m_k.$$

Due to (248), (227) specializes to:

$$(\lambda - \mu R)\,(\lambda + \mu S)\,(\lambda + \mu T) = 0,$$

and one will obtain the three systems of *osculating ruled families:*

$$u^1 = \text{const.},\ u^2 = \text{const.} \qquad (u^2 = \text{const.},\ u^3 = \text{const.};\ u^3 = \text{const.},\ u^3 = \text{const.}, \text{ resp.})$$

The osculating ruled families are hyperboloidal quadrics. Any quadric of the first system (viz., u^1 = const., u^2 = const.*) contains the opposite edges* 12, 34 *of the coordinate tetrahedron and intersects any two corresponding lines* $\mathfrak{q}$, $\mathfrak{r}$ *of the projective pencil of lines* (258) [*vertex* = 3 (4, resp.), *plane of pencil* = 123 (124, resp.); cf., Fig. 33]. *Conversely, any quadric of that kind is the osculating ruled family of the first system.*

Proof: (249) implies that:

$$e^{\beta u^2} p_1 - e^{\alpha u^1} p_2 = 0, \quad be^{-\beta u^2} p_4 - ae^{-\alpha u^1} p_5 = 0, \qquad \tau ae^{-\alpha u^1} p_1 - e^{\alpha u^1} p_4 = 0. \tag{260}$$

Therefore, any family of lines (S) u^1 = const., u^2 = const. is a quadric, as the intersection of three independent linear complexes, and a hyperbolic quadric, as a ruled family. Since ($260_{1,2}$) agrees with (259_3), any line of (S) will intersect the lines $\mathfrak{q}$, $\mathfrak{r}$ that are given by (259_3). Furthermore, since (260) will be fulfilled by $p_1 = p_2 = p_4 = 0$, (S) will contain the line 12 ($p_6 \neq 0$, $p_\rho = 0$ for $\rho \neq 6$) and 34 ($p_3 \neq 0$, $p_\rho = 0$ for $\rho \neq 3$) of the coordinate tetrahedron (Fig. 2).

From (224), the *osculating torses* are given by:

$$(\dot{u}^1)^2 - (\dot{u}^2)^2 - (\dot{u}^3)^2, \quad R(\dot{u}^1)^2 + S(\dot{u}^2)^2 + T(\dot{u}^3)^2 = 0.$$

As in no. **139**, it follows from this that:

$$\dot{u}^1 : \dot{u}^2 : \dot{u}^3 = \sqrt{-S+T} : \sqrt{R+T} : \sqrt{-R-S} = \sqrt{-a} : \sqrt{b} : \sqrt{c},$$

so

$$u^1 = t\sqrt{-a} + h^1, \qquad u^2 = t\sqrt{b} + h^2, \qquad u^3 = t\sqrt{c} + h^3,$$

with the parameter t and the integration constants h^1, h^2, h^3. The various sign combinations of the roots yield the four systems of osculating torses. By substituting them in (249), one will get, e.g., for the sign combination + + +:

$$\rho\,\mathfrak{p} = \kappa\omega \mid \lambda\omega \mid \mu\omega \left| \frac{\tau a}{\kappa} \right| \frac{\tau b}{\kappa} \left| \frac{\tau c}{\kappa}, \right. \tag{261}$$

with the abbreviations:

$$\omega = e^{\beta\tau\sqrt{-abct}}, \quad \kappa = e^{\alpha h^1}, \qquad \lambda = e^{\beta h^2}, \qquad \mu = e^{\gamma h^3}.$$

The osculating torse (261) is a pencil of lines whose vertex lies in the plane $x_4 = 0$ and whose plane goes through the vertex 4 of the coordinate tetrahedron ([1]). Since the vertices and planes of the coordinate tetrahedron have the same status relative to the tetrahedral complexes, one will then have:

The four osculating torses that go through a complex line $\mathfrak{p}$ are pencils of lines. Their vertices are the four points of intersection of $\mathfrak{p}$ with the planes of the coordinate tetrahedron, while the planes of the pencils contains opposite vertices of the coordinate tetrahedron.

([1]) (261) yields a line in the plane $x_4 = 0$ (through the vertex 4, resp.) for $\omega = 0$ ($1 / \omega = 0$, resp.)

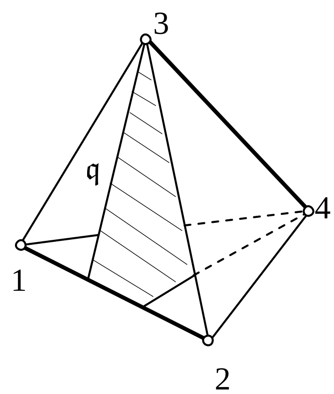

Figure 33.

From no. **149**, the vertices (planes, resp.) of the pencils are the *principal points* (*planes*, resp.) of the complex line $\mathfrak{p}$; the agreement of the formulas for d in nos. **139** and **142** has this result, which one could have expected from the outset.

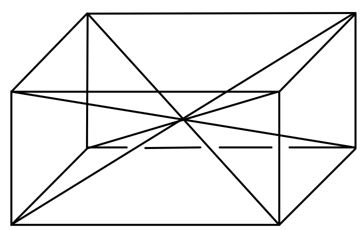

Figure 34.

Under the *map of the line complex to point space* (no. **125**), the three systems of osculating ruled families will correspond to the three bundles that are parallel to the coordinate axes, and the four systems of osculating torses will correspond to the four bundles of parallels that are pair-wise symmetric to the coordinate planes. These seven bundles of parallels can be *linked diagonally* ([1]); i.e., one can give small-celled structures of congruent cuboids in such a way that the edges and diagonals of the cuboid are taken from the 3 + 4 bundles of parallels. A cuboid with four diagonals is represented in Fig. 34. This configuration of cuboid edges and diagonals in image space is associated with an analogous *configuration of osculating ruled families and osculating torses* in the tetrahedral complex: Any three of the osculating ruled families and any four of the osculating torses will collectively have a complex line in common that corresponds to an edge of the cuboid in image space.

([1]) Cf., on this, the footnote in page 216 that refers to **W. Blaschke.**

§ 40. Tetrahedral line systems of class 2 (order 2, resp.).

146. Definition of the tetrahedral line system of class 2 (order 2, resp.)(1). The set of connecting lines of corresponding points of two hyperboloids (H), ($\tilde{H}$) that are associated with precisely four fixed points by an arbitrary collineation (254) is called a *tetrahedral line system of class 2.* The term "tetrahedral" refers to the fact that the line system is contained in the tetrahedral line complex that is determined by (254). Conversely, infinitely many tetrahedral line systems of class 2 can be selected from any tetrahedral complex by varying the hyperboloid (H).

The term "line system of class 2" is justified by the following property (cf., no. **33**):

Precisely two system lines lie (in the algebraic sense) in any non-special plane (i.e., one that contains no vertex of the coordinate tetrahedron).

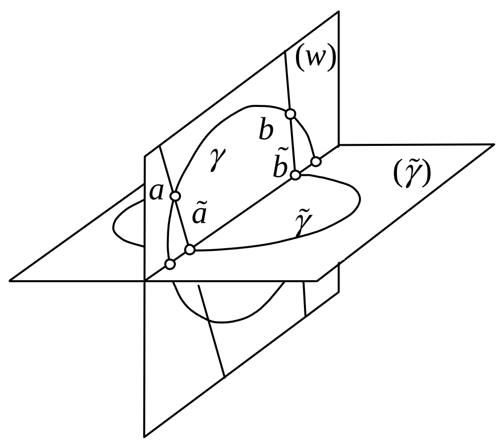

Figure 35.

Proof: Let γ be the second-order curve of intersection of a non-special plane w with the hyperboloid (H), and let $\tilde{\gamma}$ be the curve of intersection of the plane $\tilde{w}$ that corresponds with (254) with ($\tilde{H}$) (Fig. 35). Since the plane does not contain a vertex of the coordinate tetrahedron, and thus contains no fixed point, $\tilde{w}$ will be different from w, and the two points of intersection $\tilde{a}$, $\tilde{b}$ of w with $\tilde{\gamma}$ will be collinearly assigned to other points a, b of γ. The four points $\tilde{a}$, $\tilde{b}$, a, b do not lie on a line, since otherwise that line would contain at least one fixed point, which is contrary to the assumption that w is not special. Thus, precisely the two lines $a\tilde{a}$ and $b\tilde{b}$ of the line lie in the plane w. That count is to be understood in the algebraic sense – i.e., with no concern for multiplicity and reality.

Dually, we define the corresponding *tetrahedral line system of order 2* to be the set of lines of intersection of corresponding contact planes of two hyperboloids that are related collinearly by (254).

(1) Cf., on this, **Th. Reye**: *Geometrie der Lage III*, Leipzig, 1923.

147. Special tetrahedral line systems of class 2 and order 2. The line systems u^1 = const. (u^2 = const., u^3 = const., resp.) that are contained in the tetrahedral complex (249) are special tetrahedral line systems of *order 2, as well as class 2.* With the manner of generation in no. **146**, they will yield corresponding points, as well as corresponding contact planes (by intersection) of two collinear hyperboloids that have the skew quadrilateral 13421 (12341, 14321, resp.) of the coordinate tetrahedron (Fig. 2) in common.

Proof: By eliminating u^2, u^3, it will follow from (256) that the hyperboloid (H) is:

$$e^{2\alpha u^1}(\sigma_4-\sigma_2)(\sigma_4-\sigma_3)\,x_2\,x_3-(\sigma_4-\sigma_1)\,x_1\,x_4=0,$$

and from (254), the corresponding hyperboloid $(\tilde{H})$:

$$e^{2\alpha u^1}(\sigma_4-\sigma_2)(\sigma_4-\sigma_3)\sigma_1\sigma_4\;\tilde{x}_2\tilde{x}_3-(\sigma_4-\sigma_1)\sigma_2\sigma_3\;\tilde{x}_1\tilde{x}_4=0.$$

(H), $(\tilde{H})$ have the skew quadrilateral 13421 in common and generate the line system u^1 = const. according to no. **146** as the line system of class 2 by connecting collinearly-corresponding points of (H), $(\tilde{H})$. Since a line system u^1 = const. will be once more transformed into a line system u^1 = const. by polarity, that line system can also be dually generated by the intersections of associated contact planes of collinear hyperboloids with the common skew quadrilateral 13421.

The line systems u^1 = const. are elliptic. The line systems u^2 = const. and u^3 = const. are hyperbolic and identical with the W-systems that were treated in no. **96** *that had two hyperboloids with common skew quadrilaterals as focal surfaces.*

Proof: From no. **141**, one has:

$$\rho^2\mathfrak{p}_1\mathfrak{p}_1=-2a\alpha^2\tau,\qquad \rho^2\mathfrak{p}_2\mathfrak{p}_2=-2b\beta^2\tau,\qquad \rho^2\mathfrak{p}_3\mathfrak{p}_3=-2c\gamma^2\tau;\qquad \rho^2\mathfrak{p}_i\,\mathfrak{p}_k=0 \text{ for } i\neq k,$$

so one will get:

$$b\beta^2(\dot{u}^2)^2+c\gamma^2(\dot{u}^3)^2=0,\qquad [\,c\gamma^2(\dot{u}^3)^2+a\alpha^2(\dot{u}^1)^2=0,\;\; a\alpha^2(\dot{u}^1)^2+b\beta^2(\dot{u}^2)^2=0,\text{ resp.}]$$

for the differential equation (103) of the torses of the line system u^1 = const. (u^2 = const., u^3 = const., resp.). Since $a < 0$, $b > 0$, $c > 0$ (no. **141**), the first system will have no families of torses, and the second one will have three, but the third system will have two real ones. For example, the torse parameters of the third system will be:

$$\bar{u}^1=\tfrac{1}{2}(\alpha\sqrt{-a}\,u^1+\beta\sqrt{b}\,u^2),\qquad \bar{u}^2=\tfrac{1}{2}(\alpha\sqrt{-a}\,u^1-\beta\sqrt{b}\,u^2).$$

Upon introducing these parameters, (249) will imply that:

$$\rho\,\mathfrak{p} = e^{\frac{1}{\sqrt{-a}}(\bar{u}^1+\bar{u}^2)} \mid e^{\frac{1}{\sqrt{b}}(\bar{u}^1-\bar{u}^2)} \mid e^{\gamma u^3} \mid \tau\, a\; e^{-\frac{1}{\sqrt{-a}}(\bar{u}^1+\bar{u}^2)} \mid \tau\, b\, e^{-\frac{1}{\sqrt{b}}(\bar{u}^1-\bar{u}^2)} \mid \tau\, c\; e^{-\gamma u^3}.$$

One finds the focal surfaces from the geometric loci of the points of intersection x (x', resp.) of the lines $\mathfrak{p}$, $\frac{\partial \mathfrak{p}}{\partial \bar{u}^1}$ ($\mathfrak{p}$, $\frac{\partial \mathfrak{p}}{\partial \bar{u}^2}$, resp.). From (17), one will get:

$$x_1\,x_2 + \lambda\,x_3\,x_4 = 0 \qquad (\,x_1'\,x_2' + \mu\,x_3'\,x_4' = 0, \text{ resp.}),$$

with:

$$\lambda = \frac{\tau\,c^2\,e^{-2\gamma u^3}}{\left(\sqrt{-a}-\sqrt{b}\right)^2}, \quad \mu = \frac{\tau\,c^2\,e^{-2\gamma u^3}}{\left(\sqrt{-a}+\sqrt{b}\right)^2};$$

as in no. **96**, the double ratio λ / μ is positive.

148. The three edge systems of a restricted cubic framework. From no. **144**, any collineation of the one-parameter group of collineations:

$$(262) \quad \tilde{x}_1 = (\sigma_1 + \tau)\,x_1, \qquad \tilde{x}_2 = (\sigma_2 + \tau)\,x_2, \qquad \tilde{x}_3 = (\sigma_3 + \tau)\,x_3, \qquad \tilde{x}_4 = (\sigma_4 + \tau)\,x_4,$$

with the parameter τ generates the tetrahedral complex (249) by connecting corresponding points x, $\tilde{x}$; by varying τ, the point $\tilde{x}(\tau)$ that corresponds to the point x will run through the complex line that is associated with the point x by (254).

Any hyperboloid (H) will be related collinearly by (262) to a one-parameter set of hyperboloids $(\tilde{H}(\tau))$. The points $\tilde{x}(\tau)$ of $(\tilde{H}(\tau))$ that correspond to a point x of (H) will lie on a line, and, from no. **146**, the set of all those lines will be a tetrahedral line system of class 2. That line system and the generating family of the hyperboloid $(\tilde{H}(\tau))$ have *a topologically-remarkable link:*

They span three one-parameter sets of hyperboloids. Upon restricting to a suitable region in projective space, hyperboloids of the same set will have no point in common, while two hyperboloids of different sets will intersect in a generator. The link will become very intuitive when we select three discrete sequences of hyperboloids from the sets. They will then define a spatial arrangement in hexahedra with hyperboloidal faces and rectilinear edges that run through them. (Cf., Fig. 36; for ease of illustration, six cells were removed from the cube.) The configuration is topologically equivalent to a regular spatial arrangement of congruent cubes, and for that reason, it shall be called a *restricted cubic framework* ([1]).

The three edge systems of a restricted cubic framework are equivalent to each other in the following sense: Any system of edges relates the one-parameter set of hyperboloids that are spanned by the other two edge systems to each in a point-wise collinear way, and

([1]) Such topological questions in differential geometry were investigated thoroughly in the last few years by **W. Blaschke** and his students, especially, **G. Bol**.

for that reason, they can be constructed from two of those hyperboloids by connecting corresponding points. If the hyperboloids that are spanned by two of the three edge systems are related to each other by collineations with precisely four fixed points then, from no. **146**, the third edge system will be a tetrahedral line system of class two.

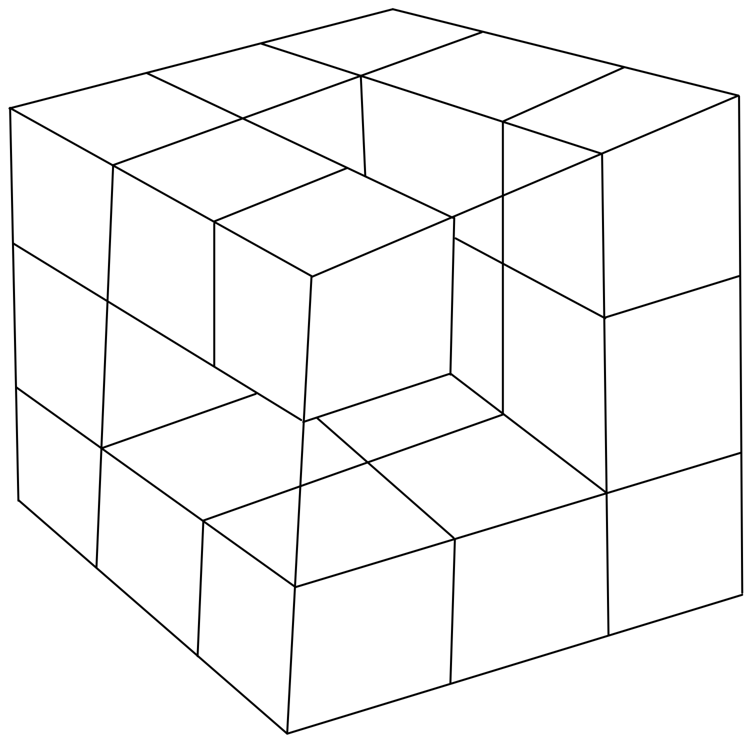

Figure 36.

Specialization: We take the first edge system to the special tetrahedral line system of class two and order two u^1 = const. of no. **147**. The hyperboloids that are spanned by the other edge system will then define a pencil with the skew quadrilateral 13421 as its base curve. As a result, all lines of the second (third, resp.) edge system will intersect two opposite sides of that quadrilateral; the second and third edge systems will then be *hyperbolic linear line systems* with 13, 42 (34, 21, resp.) for their focal lines.