"Über das Verhalten eines Elektrons im homogenen elektrischen Feld nach der relativistischen Theorie Diracs," Zeit. f. Phys. **69** (1931), 742-764.

On the behavior of an electron in a homogeneous electric field in Dirac's relativistic theory

By **Fritz Sauter** in Munich

Translated by D. H. Delphenich

With 6 figures. (Received on 21 April 1931)

The solutions of the Dirac equation with the potential $V = vx$ will be obtained and their behavior will be discussed. Along with the region of the function that also appears in the non-relativistic calculations, there is a region in the Dirac theory in which the impulse and velocity of the electron possess opposite signs. In conjunction with that, the probability will be computed for an electron to go from the "positive impulse" region to the "negative impulse" region. This yields the result that transition probability first takes on finite values when the magnitude of the potential ramp over a distance that is equal to the Compton wavelength is comparable to the rest energy of the electron. The large values for the transition probability that were computed by O. Klein for a potential well whose order of magnitude is twice the rest energy are understood to be limiting values in the case of an infinitely steep potential ramp.

Some time ago, an interesting work by O. Klein[*] appeared on the reflection of electrons by a potential well. The computation in terms of Dirac's relativistic theory yielded the following result: If one lets the height P of the potential well increase from 0 then the reflection coefficient R also takes on values from null to 1, which it attains when $P = E - E_0$. (E is the relativistic energy of the electron; E_0 is its rest energy.) With further increases in P, R remains constantly equal to 1, up to the value $P = E + E_0$. If one lets the height of the potential well increase still more then the reflection coefficient goes down again, and in the limiting case of $P = \infty$ it approaches the value $\frac{E - cp}{E + cp}$. (p = impulse of the electron before the transition through the potential well). In Dirac's theory, an electron therefore possesses a finite probability that it might pass on through a very high potential well that is completely reflecting in the classical analysis.

The state that the electron attains after this transition is thus recognized to be one in which its velocity (group velocity) is oppositely directed to its impulse.

The appearance of a "negative impulse" is no longer surprising, since one has already learned to compute with the concept of "negative energy. [**]" The large value that Klein found for the probability of making the transition from a state of positive impulse to one of negative impulse is therefore noteworthy. N. Bohr made the conjecture that this high

[*] O. Klein, ZS. f. Phys. **53**, 157, 1929.

[**] Cf., dispersion theory, in which I. Waller (ZS. f. Phys. **61**, 837, 1930) has shown precisely that the states of negative energy take on a special meaning as intermediate states.

value comes about only as a result of the assumption of a potential *well*, thus, an infinitely steep potential ramp, and that finite transition probabilities are to be expected when and only when the ramp is so steep that the potential at a distance on the order of a Compton wavelength h/mc increases by an amount whose order of magnitude is the rest energy of the electron *.

The goal of the following investigation is to test and verify this opinion of Bohr. To that end, the rectangular potential well $AB'C'D$ (see Fig. 1) that the calculations of O. Klein were based upon will be replaced by a potential ramp $ABCD$ that exists between regions (I and III) of constant potential, between which one finds a region (II) of linearly increasing potential; hence, a region of constant electric field. The question is also posed in this case of what the transition probability would be for an electron to go from region I to region III.

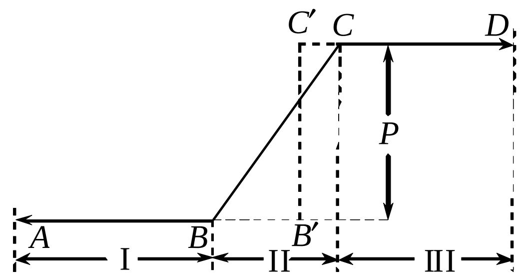

Fig. 1.

In order to respond to this question, it is necessary to solve the Dirac equation for the case of a homogeneous electric field. The first three sections of the following investigation are concerned with arriving at this solution and a discussion of it, while in the fourth section we will treat the problem posed above of calculating the probability for the transition of an electron from positive to negative impulse.

1. *Solution of the Dirac equation.*

The potential V may be put into the form:

$$V = vx; \tag{1}$$

the Dirac equation then reads:

$$\left\{\gamma_1 \frac{\partial}{\partial x} + \gamma_2 \frac{\partial}{\partial y} + \gamma_3 \frac{\partial}{\partial z} + \gamma_4 \left(\frac{1}{ic}\frac{\partial}{\partial t} + \kappa v x\right) + \kappa E_0\right\}\psi = 0, \tag{2}$$

with the abbreviations:

$$E_0 = m\,c^2, \qquad \kappa = \frac{2\pi}{hc}. \tag{3}$$

By means of the Ansatz:

$$\psi = e^{\frac{2\pi i}{h}(y p_y + z p_z - Et)} \cdot \chi(x)\,, \tag{4}$$

(2) goes to:

* I would like to thank Herrn Prof. Heisenberg for the friendly tip about this hypothesis of N. Bohr.

$$\left\{\gamma_1 \frac{d}{dx} + \kappa\gamma_4 \left(vx - E\right) + \kappa(E_0 + ic\gamma_2 p_y + ic\gamma_3 p_z)\right\}\chi = 0\,. \tag{5}$$

This equation may be further converted: One sets:

$$K^2 = E_0^2 + c^2(p_y^2 + p_z^2) \tag{6}$$

and:

$$\gamma_5\, K = \gamma_1(E_0 + ic\ \gamma_2\, p_y + + ic\ \gamma_3\, p_z), \tag{7}$$

in which γ_5 is obviously anticommutative with $\gamma_1\ \gamma_4$, and has the property that $\gamma_5^2 = 1$, such that after left multiplication by γ_1 (5) becomes:

$$\left\{\frac{d}{dx} + \kappa\gamma_1\gamma_4 \left(vx - E\right) + \kappa K\gamma_5\right\}\chi = 0. \tag{5a}$$

It is indicated that we introduce rational units; with:

$$\xi = \sqrt{\frac{\kappa}{v}}\,(vx - E), \tag{8}$$

$$k = \sqrt{\frac{\kappa}{v}}\,K, \tag{9}$$

one obtains from (5a):

$$\left\{\frac{d}{d\xi} + \gamma_1\gamma_4\xi + \gamma_5 k\right\}\chi = 0. \tag{10}$$

One can then integrate this equation when one now introduces four-rowed matrices for the γ_ν and a column of four functions for χ in the usual way, and solve the resulting simultaneous system of four first order differential equations. The integration of (10) then becomes simpler when one does not specialize the γ_ν, but only uses their commutation relations and regards χ as a linear aggregate of the γ_ν *.

It is recommended that one puts *c* into the form:

$$\chi = [f(\xi) + \gamma_5\, g(\xi)] \cdot (1 + i\ \gamma_1\, \gamma_4)\ \Gamma, \tag{11}$$

in which *f* and *g* include no γ_ν, and Γ may mean an arbitrary operator connected with the γ_ν. If one introduces this Ansatz into (10) then, due to the factor $1 + i\ \gamma_1\, \gamma_4$, one obtains:

$$\left\{\left[\frac{df}{d\xi} - i\xi\, f + kg\right] + \gamma_5\left[\frac{dg}{d\xi} + i\xi\, f + kf\right]\right\}(1 + i\ \gamma_1\, \gamma_4)\ \Gamma = 0.$$

* F. Sauter, ZS. f. Phys. **63**, 803, 1930; **64**, 295, 1930.

If one multiplies this equation on the left by $1 + i\,\gamma_1\,\gamma_4$ then one recognizes that the expressions in the two square brackets must vanish identically:

$$\left.\begin{aligned}\left(\frac{d}{d\xi}-i\xi\right)f+kg=0,\\ \left(\frac{d}{d\xi}+i\xi\right)g+kf=0.\end{aligned}\right\} \tag{12}$$

These equations may be easily solved. If one understands $F(\alpha, \gamma, x)$ to mean a degenerate hypergeometric function, which is defined in the entire complex plane by the convergent series:

$$F(\alpha,\ \gamma,\ x)=\sum_0^\infty\frac{\Gamma(\alpha+\nu)}{\Gamma(\alpha)}\frac{\Gamma(\gamma)}{\Gamma(\gamma+\nu)}\frac{x^\nu}{\nu!}, \tag{13}$$

then one can write down the two solutions of (12) in the form:

$$\left.\begin{aligned}f_1 &= e^{\frac{i\xi^2}{2}}F\left(-\frac{k^2}{4i},\ \frac{1}{2},-i\xi^2\right),\\ g_1 &= -k\xi e^{\frac{i\xi^2}{2}}F\left(-\frac{k^2}{4i}+1,\frac{3}{2},-i\xi^2\right),\end{aligned}\right\} \tag{14a}$$

$$\left.\begin{aligned}f_2 &= -k\xi e^{\frac{i\xi^2}{2}}F\left(-\frac{k^2}{4i}+\frac{1}{2},\frac{3}{2},-i\xi^2\right),\\ g_2 &= e^{\frac{i\xi^2}{2}}F\left(-\frac{k^2}{4i}+\frac{1}{2},\frac{1}{2},-i\xi^2\right).\end{aligned}\right\} \tag{14b}$$

On the basis of the relation:

$$F(\alpha,\ \gamma,\ x) = e^x\,F(\gamma - \alpha,\ \gamma,\ -x),$$

one easily verifies that:

$$f_1 = g_2^*, \qquad f_2 = g_1^*, \tag{15}$$

in which the star refers to the complex conjugate value.

In the following, along with the series development that is given by (14), an integral representation for f and g in higher weights will also find application. We therefore derive such an integral representation for the function *:

$$\varphi = e^{\frac{i\xi^2}{2}}\,F(\alpha,\ \gamma,\ -i\xi^2). \tag{16}$$

* The integral representation that W. Gordon gave [Ann. d. Phys. (5) **2**, 1031, 1929] is not convenient for the following computations.

Due to (13), one has the development for φ:

$$\varphi = e^{\frac{i\xi^2}{2}} \sum_0^\infty \binom{-\alpha}{\nu} \frac{\Gamma(\gamma)}{\Gamma(\gamma+\nu)} (i\xi^2)^\nu .$$

For $\frac{1}{\Gamma(\gamma+\nu)}$, the well-known integral representation for the gamma function:

$$\frac{1}{\Gamma(\gamma+\nu)} = \frac{e^{i\pi n}}{2i\pi} \int \frac{e^{-t}\,dt}{t^{n+1}}$$

may be introduced; the path of integration in it is a loop around $t = 0$ that goes to infinity in the direction of the positive real axis. The summation may then be carried out, and yields:

$$\varphi = \frac{\Gamma(\gamma)}{2\pi} e^{i\pi(\gamma-3/2)} e^{\frac{i\xi^2}{2}} \int e^{-t} t^{\alpha-\gamma} (t - i\xi^2)^{-\alpha}\, dt .$$

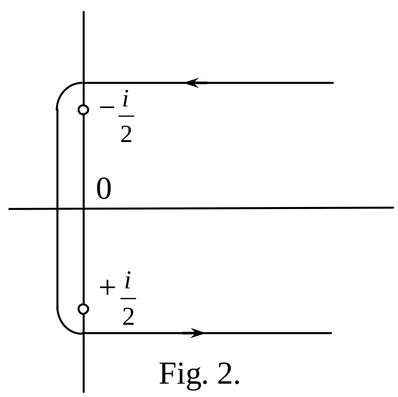

Fig. 2.

One obtains a symmetric form by means of the transformation:

$$t = \xi^2 \left(s + \frac{i}{2} \right);$$

this gives:

$$\varphi = \frac{\Gamma(\gamma)}{2\pi} e^{i\pi(\gamma-3/2)} \,|\, \xi \,|^{2-2\gamma} \int e^{-\xi^2 s} \left(s + \frac{i}{2} \right)^{\alpha-\gamma} \left(s - \frac{i}{2} \right)^{-\alpha} ds . \tag{16a}$$

The path of integration is depicted in Fig. 2. The arguments of $s \pm i/2$ run from 0 to 2π.

If one introduces this expression into (14) the one obtains the integral representations for the solutions:

$$\left.\begin{aligned} f_1 &= -\frac{1}{2\sqrt{\pi}}|\xi|\int e^{-\xi^2 s}\left(s+\frac{i}{2}\right)^{-\frac{k^2}{4i}-\frac{1}{2}}\left(s-\frac{i}{2}\right)^{\frac{k^2}{4i}} ds, \\ g_1 &= -\frac{1}{2\sqrt{\pi}}\frac{k|\xi|}{2\xi}\int e^{-\xi^2 s}\left(s+\frac{i}{2}\right)^{-\frac{k^2}{4i}-\frac{1}{2}}\left(s-\frac{i}{2}\right)^{\frac{k^2}{4i}-1} ds, \end{aligned}\right\} \quad (17a)$$

$$\left.\begin{aligned} f_2 &= -\frac{1}{2\sqrt{\pi}}\frac{k|\xi|}{2\xi}\int e^{-\xi^2 s}\left(s+\frac{i}{2}\right)^{-\frac{k^2}{4i}-1}\left(s-\frac{i}{2}\right)^{\frac{k^2}{4i}-\frac{1}{2}} ds, \\ g_2 &= -\frac{1}{2\sqrt{\pi}}|\xi|\int e^{-\xi^2 s}\left(s+\frac{i}{2}\right)^{-\frac{k^2}{4i}}\left(s-\frac{i}{2}\right)^{\frac{k^2}{4i}-\frac{1}{2}} ds. \end{aligned}\right\} \quad (17b)$$

One easily verifies that these expressions satisfy equations (12). The absolute value sign on ξ is necessary in order to guarantee the symmetry of the functions f and g at the location $\xi = 0$ that is given by (14), along with a continuous path to this place *.

2. *Series development for f and g for large k.*

For physical applications, it is necessary to have some knowledge of the way that f and g depend upon the independent variables. The series development that one obtains from (14), by means of (13), converges for very small values of ξ and k so well that one can reduce it to the first pair of terms. For larger values of ξ and k this development is not suitable in practical calculations.

It is indicated that we make the order of magnitude of k clear; by definition, [cf. (6) and (9)], it is:

$$k = \sqrt{\frac{2\pi}{hc}\cdot\frac{E_0^2 + c^2(p_y^2 + p_z^2)}{v}}.$$

If the components of the impulse in the y and z directions are negligibly small compared to E_0/c then k depends only upon the magnitude of the potential ramp. If one introduces the numerical values and expresses v/e in volt/cm then this gives:

$$k = \frac{1.15\cdot 10^8}{\sqrt{v/e}}.$$

One recognizes that for the highest attainable electrostatic fields of several million volt/cm k is still several powers of ten higher than 1. The first time that k becomes comparable to 1 is for extremely high, practically unrealizable, field strengths of 10^{16} volt/cm. In the following, we will therefore only deal with the case $k \gg 1$ throughout.

* One observes that the point $\xi = 0$ represents a singularity for the integral representation (16a).

In order to achieve of rapid convergence, it thus seems promising to develop *f* and *g* in descending powers of *k*, which can come about on the basis of Debye's saddle point method. The functions (17) that are to be developed possess the form:

$$F(\xi) = \int e^{-\xi^2 s} \left(\frac{s+\frac{i}{2}}{s-\frac{i}{2}} \right)^{\frac{k^2 t}{4}} G(s)\, ds\,, \tag{18}$$

in which the function $G(s)$ can be regarded as slowly varying compared to the first two factors of the integrand. With the abbreviation:

$$h(s) = \xi^2 s - \frac{k^2 i}{4} \log \frac{s+\frac{i}{2}}{s-\frac{i}{2}}\,,$$

F assumes the form:

$$F = \int e^{-h(s)}\, G(s)\, ds. \tag{18a}$$

This integral will be evaluated in such a manner that one seeks a (saddle-) point, at which the integral possesses a possible sharp maximum; if one then directs the path of integration around this point and develops the integrands at this place then one will obtains a series that falls off quickly enough that one can, at least in our case, truncate it after the first term.

As the position of the saddle point, one obtains from:

$$\frac{dh(s)}{ds} = 0$$

the two points:

$$s_{1,\,2} = \pm \frac{1}{2} \sqrt{\frac{k^2}{\xi^2} - 1}\,. \tag{19}$$

They lie on the real or imaginary axis according to whether $|\,\xi\,|$ is smaller than or greater than *k*, respectively.

The points $\xi = \pm\, k$ thus take on a special position mathematically. One easily arrives at it, since they are also physically distinguished. Due to (6), (8), and (9) it is the point for which one has:

$$(E - vx)^2 = E_0^2 + c^2 (p_y^2 + p_z^2)\,;$$

i.e., at this point, classically speaking, the impulse component p_x in the field direction vanishes. It therefore represents the antipodal point to the classical path, which lies in the region $|\,\xi\,| > k$, while $|\,\xi\,| < k$ represents the classically forbidden region.

For the computation of (18a) we must still clarify the nature of the integration path (I.P.) around the saddle points. As is well known, one chooses it advantageously in such a way that the real part of the exponent – $R(h)$ increases as quickly as possible, while the imaginary part of h remains constant. If one writes s in the form:

$$s = \sigma + i\tau,$$

then the latter will be given by:

$$J(h(s)) = \xi^2\tau - \frac{k^2}{8}\log\frac{\sigma^2 + (\tau + \frac{1}{2})^2}{\sigma^2 + (\tau - \frac{1}{2})^2} = J(h(s_0)).$$

We would now like to treat the two cases $|\xi| < k$ and $|\xi| > k$ separately:

1. $|\xi| < k$. In this case, the saddle points lie on the real axis. The I.P. will, from the above, be given by:

$$\xi^2\tau - \frac{k^2}{8}\log\frac{\sigma^2 + (\tau + \frac{1}{2})^2}{\sigma^2 + (\tau - \frac{1}{2})^2} = 0.$$

Its definition can be gathered from Fig. 3. (The real axis $\sigma = 0$ is also a branch of the I.P.) The arrows refer to the removal of the real part of h, thus to an increase in the integrands. One obtains a useful I.P., which may be continuously deformed into the I.P. of Fig. 2, when one goes rectilinearly from + ∞ to s_2, from there along the indicated curve over to s_1 in the positive sense around the two branching points $\pm i/2$, and again rectilinearly from s_2 back to + (cf., the dashed curve in Fig. 3, which, for the sake of clarity, is indicated near the correct integration path. The rectilinear parts of the I.P. lie on different Riemann surfaces.). The integrand assumes its maximal value at the location *:

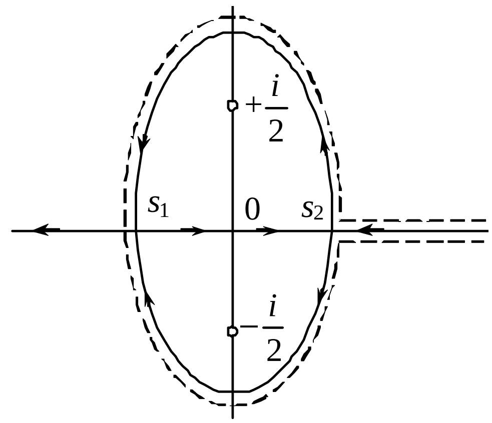

Fig. 3.

$$s_1 = -\frac{1}{2}\sqrt{\frac{k^2}{\xi^2} - 1}^{\,|}\;;$$

the development of $h(s)$ at this saddle point reads like:

$$h(s) = h(s_1) - (s - s_1)^2 \cdot \frac{2\xi^2}{k^2}\sqrt{\frac{k^2}{\xi^2} - 1}^{\,|} + \ldots$$

with:

* As usual, the stroke | on the root $\sqrt{}^{\,|}$ shall imply that its positive value is to be taken.

$$h(s_1) = \frac{\xi^2}{2}\sqrt{\frac{k^2}{\xi^2}-1} - \frac{k^2}{2}\arcsin\frac{|\xi|}{k}.$$

One has, in fact, due to the assumption that was made above on the arguments of $s \pm i/2$:

$$s_1 + \frac{i}{2} = \frac{1}{2}\frac{k}{|\xi|}e^{i\left(\frac{\pi}{2}+\arccos\frac{|\xi|}{k}\right)},$$

$$s_1 - \frac{i}{2} = \frac{1}{2}\frac{k}{|\xi|}e^{i\left(\frac{\pi}{2}-\arccos\frac{|\xi|}{k}\right)},$$

where arcsin and arccos refer to the principal values of the cyclometric functions (i.e., the ones between 0 and $\pi/2$).

For the computation of F, along with $e^{-h(s)}$, one must also develop $G(s)$ in powers of $s - s_1$ and integrate termwise. We would then like to restrict ourselves to just the first term of the series. (The series goes in increasing powers of $1/\xi^2$, and therefore falls off for sufficiently large $\xi^2 \gg 1$ very rapidly). $G(s)$ can then be treated as constant for the integral and one has:

$$F = G(s_1)\,e^{-h(s_1)}\int e^{(s-s_1)^2\cdot\frac{2\xi^4}{k^2}\sqrt{\frac{k^2}{\xi^2}-1}}\,ds + \dots$$

In the same approximation, the I. P. can be replaced by its tangent at the saddle point; one then integrates rectilinearly from $s_1 + i\infty$ to $s_1 - i\infty$. One obtains:

$$F = G(s_1)\,e^{-h(s_1)-\frac{i\pi}{2}}\frac{k\sqrt{\frac{\pi}{2}}}{\xi^2\sqrt[4]{\frac{k^2}{\xi^2}-1}} + \dots \tag{20a}$$

This result is valid as long as $\sqrt{\frac{k^2}{\xi^2}-1}$ is greater than 1 or at least comparable to it, since in the development (20a) this root appears in the denominator. The case $|\xi| \sim k$ must be treated specially. In that case, both saddle points move to the coordinate origin and there is some advantage in developing from the position $|\xi| = k$ outward. Since we will not need this development in what follows, its derivation may be suppressed, for the sake of brevity. The result reads like:

$$F = G(0)\,e^{\frac{k^2\pi}{4}-\frac{i\pi}{2}}\cdot\frac{\sqrt{3}}{(6k)^{2/3}}\left(\Gamma\left(\frac{1}{3}\right) - \frac{\Gamma\left(-\frac{1}{3}\right)}{(6k)^{2/3}}\left[\left(\frac{d\log G}{ds}\right)_{s=0} + k^2 - \xi^2\right]\right) + \dots, \tag{20b}$$

which is valid for $|\xi^2 - k^2| \ll 1$.

2. $|\xi| > k$. Now, the saddle points lie on the imaginary axis. Since $\sigma_{1,2} = 0$, $\tau_{1,2} = \pm\frac{1}{2}\sqrt{1-\frac{k^2}{\xi^2}}$ one obtains the I. P. from:

$$\xi^2\tau - \frac{k^2}{8}\log\frac{\sigma^2+(\tau+\frac{1}{2})^2}{\sigma^2+(\tau-\frac{1}{2})^2} = \xi^2\tau_{1,2} - \frac{k^2}{4}\log\frac{\tau_{1,2}+\frac{1}{2}}{\tau_{1,2}-\frac{1}{2}}$$

(see Fig. 4; the arrows have the same meaning as in Fig. 3.) One obtains a useful I. P. in the following way: From $+\infty$, one comes in to the branch point $+i/2$ and goes around it in the positive sense and crosses over the saddle point s_1, and goes back to $+\infty$. One then comes in to the second saddle point s_2, around the branch point $-i/2$, again in the positive sense, and the goes back to $+\infty$. (See the dashed curve in Fig. 4.) This I. P. may obviously be continuously deformed into the one in Fig. 2 when the path segments $s_1 \to \infty$ and $\infty \to s_2$ lie on the same Riemann surface.

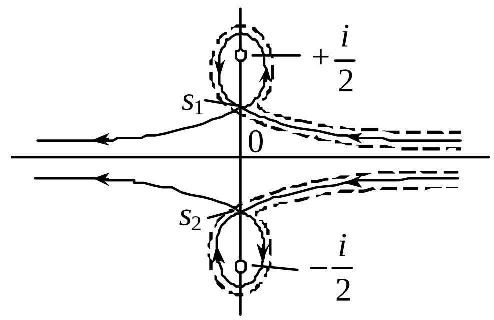

Fig. 4.

For the determination of F one must add the contributions of the integrals in the neighborhoods of both saddle points. The aforementioned assignment yields:

$$\left.\begin{aligned} s_\nu + \frac{1}{2} &= \frac{e^{\frac{i\pi}{2}}}{2}\left(1 \pm \sqrt{1-\frac{k^2}{\xi^2}}\right), \\ s_\nu + \frac{1}{2} &= \frac{e^{\frac{3i\pi}{2}}}{2}\left(1 \mp \sqrt{1-\frac{h^2}{\xi^2}}\right), \end{aligned}\right\} \qquad (\nu = 1, 2).$$

(The upper sign refers to s_1 and the lower one, to s_2.) For $h(s)$ one obtains the development:

$$h(s) = h(s_\nu) \pm (s - s_\nu)2\frac{2\xi^4 i}{k^2}\sqrt{1-\frac{k^2}{\xi^2}} + \ldots$$

with:

$$h(s_\nu) = \pm\frac{i}{2}\xi^2\sqrt{1-\frac{k^2}{\xi^2}} \mp \frac{k^4 i}{4}\log\frac{1+\sqrt{1-\frac{h^2}{\xi^2}}}{1-\sqrt{1-\frac{h^2}{\xi^2}}}.$$

The I. P. intersects the imaginary axis at an angle of 45° at the saddle points; in the neighborhood of the first saddle point, if we set:

$$s = s_1 + te^{-\frac{i\pi}{4}},$$

and for the second one set:

$$s = s_2 + te^{-\frac{3i\pi}{4}}$$

then the integrations can be extended in the approximation considered rectilinearly from $t = -\infty$ to $t = +\infty$. One thus obtains:

$$\left.\begin{aligned} F &= \left\{ e^{-h(s_1)-\frac{i\pi}{4}} G(s_1) + e^{-h(s_2)-\frac{3i\pi}{4}} G(s_2) \right\} \cdot \int_0^\infty e^{-\frac{2\xi^4}{k^2}\sqrt{1-\frac{k}{\xi^2}}\,t^2} dt + \cdots \\ &= \frac{k\sqrt{\frac{\pi}{2}}}{\xi^2\sqrt{1-\frac{k}{\xi^2}}} \left\{ e^{-h(s_1)-\frac{i\pi}{4}} G(s_1) + e^{-h(s_2)-\frac{3i\pi}{4}} G(s_2) \right\} + \cdots \end{aligned}\right\} \tag{20c}$$

This development progresses in powers of $1/k^2$ and thus converges for arbitrary values of ξ sufficiently strongly that one can satisfy oneself with the first approximation. As in the first case, it loses its validity when one approaches $|\xi|$ at the location k, in which limiting case, the development (20b) is to be employed.

If one substitutes the expressions (20a) and (20c) into the functions (17) then one obtains for the first system of solutions:

$$|\xi| < k: \quad \left.\begin{aligned} f_1 &= \frac{1}{2\sqrt[4]{1-\frac{\xi^2}{k^2}}} e^{\frac{k|\xi|}{2}\sqrt{k^2-\xi^2} + \frac{k^2+i}{2}\arcsin\frac{|\xi|}{k}}, \\ g_1 &= \frac{-1}{2\sqrt[4]{1-\frac{\xi^2}{k^2}}} \frac{|\xi|}{\xi} e^{\frac{k|\xi|}{2}\sqrt{k^2-\xi^2} + \frac{k^2-i}{2}\arcsin\frac{|\xi|}{k}}; \end{aligned}\right\} \tag{21}$$

$$|\xi| > k: \quad \left.\begin{aligned} f_1 &= \frac{ke^{\frac{k^2\pi}{4}}}{2|\xi|\sqrt{w}} \left\{ \frac{ie^{-\frac{i\xi^2 w}{2}+\frac{k^2 i}{4}\log\frac{1+w}{1-w}}}{\sqrt{1+w}} + \frac{ie^{-\frac{i\xi^2 w}{2}-\frac{k^2 i}{4}\log\frac{1+w}{1-w}}}{\sqrt{1-w}} \right\}, \\ g_1 &= +\frac{ke^{\frac{k^2\pi}{4}}}{2|\xi|\sqrt{w}} \left\{ \frac{ie^{-\frac{i\xi^2 w}{2}+\frac{k^2 i}{4}\log\frac{1+w}{1-w}}}{\sqrt{1-w}} + \frac{ie^{-\frac{i\xi^2 w}{2}-\frac{k^2 i}{4}\log\frac{1+w}{1-w}}}{\sqrt{1+w}} \right\}, \end{aligned}\right\} \tag{22}$$

in which w has been used as an abbreviation for $\sqrt{1-\frac{k^2}{\xi^2}}$. One obtains the corresponding development for the second system of solutions from this one on the basis of the relations (15):

$$f_2 = g_1^*, \qquad g_2 = f_1^*.$$

3. *Discussion of the solutions.*

First, we must examine the behavior of the functions f and g. As we are to gather from (14), they are symmetric (anti-symmetric, resp.) about the point $\xi = 0$, and furthermore one of the functions is even when the other one is odd. One can deduce their behavior from the developments (21) and (22). For $\xi < -k$ and $\xi > +k$, they represent an oscillation with variable frequency and amplitude, as is depicted schematically in Fig. 5. (The functions are essentially complex, so Fig. 5 therefore serves only to give a rough idea of the functional behavior.) In the intermediate region, their absolute values decay like a power of e to 1 or 0 [cf. (14)] as one approaches $\xi = -k$ and then they again increase exponentially.

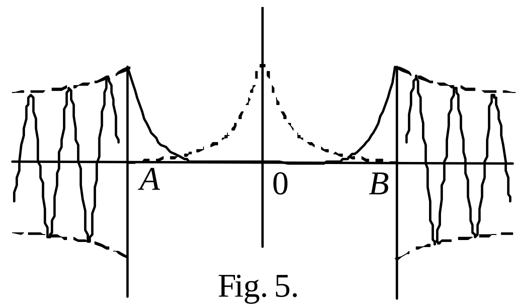

Fig. 5.

It must be pointed out that this exponential decay of the functions around the point $\xi = 0$ is achieved only by the special choice of our system of solutions. If one were to use an I. P. for which the second saddle point:

$$s_2 = \frac{1}{2}\sqrt{\frac{k^2}{\xi^2}-1}$$

in Fig. 3 gives the essential contribution to the integrands then one would obtain an exponential growth of the functions around the null point. Such an I. P. will be given by, e.g., a loop that comes in from positive infinity, as in Fig. 2, and goes around only one of the two branch points. A special linear combination of our two systems of solutions must then give the desired growth. The dotted curve in Fig. 5 depicts this possibility schematically.

It might not be inappropriate to compare this behavior with that of non-relativistic wave mechanics. The Schrödinger equation for the one-dimensional problem reads *:

* In the sequel, we shall set $p_y = p_z = 0$. Due to (6) and (9), one has:

$$k = \sqrt{\frac{\kappa}{\nu}} \cdot E_0 = \sqrt{\frac{2\pi}{hc\nu}} \cdot mc^2.$$

$$\left\{\frac{\partial^2}{\partial x^2} - \frac{8\pi^2 m}{h^2}\left(\frac{h}{2\pi i}\frac{\partial}{\partial t} + vx\right)\right\}\psi = 0.$$

By means of the Ansatz ($\bar{E}$ = non-relativistic energy):

$$\psi = e^{-\frac{2\pi i}{h}\bar{E}t}\chi(x),$$

one obtains, when one further introduces the new variable:

$$\bar{\xi} = \sqrt{\frac{\kappa}{v}}(vx - \bar{E})$$

in place of x, the differential equation for χ:

$$\left\{\frac{d^2}{d\bar{\xi}^2} - 2k\bar{\xi}\right\}\chi(\bar{\xi}) = 0.$$

As is well known, its solution reads:

$$\chi = \sqrt{\bar{\xi}} \cdot Z_{1/3}\left(\frac{2\sqrt{-2k}}{3}\bar{\xi}^{3/2}\right),$$

when $Z_p(x)$ refers to an arbitrary solution of Bessel's differential equation:

$$\left\{\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1 - \frac{p^2}{x^2}\right\}Z_p(x) = 0 .$$

The behavior is represented schematically in Fig. 6: For negative values of $\bar{\xi}$ the argument of $Z_{1/3}$ is real, and the function is thus periodic; for positive $\bar{\xi}$ the argument is complex, so χ represents the superposition of an exponentially increasing branch with an exponentially decreasing one (viz., the decomposition of $Z_{1/3}$ into the two Hankel functions $H_{1/3}^{(1)}$ and $H_{1/3}^{(2)}$).

In order to carry out the transition from the relativistic case to the non-relativistic one, we assume that:

$$E = \bar{E} + E_0,$$

so:

$$\xi = \bar{\xi} - k.$$

The coordinate system in Fig. 5 is therefore displaced by a distance k compared to the one in Fig. 6, so the point $\bar{\xi} = 0$ of Fig. 6 corresponds to the point A ($\xi = -k$) in Fig. 5.

The transition to the non-relativistic case amounts to the passage to the limit lim $k \to \infty$, hence, lim $E_0 \to \infty$, and therefore also lim $k \to \infty$. Under this transition, the points O and B in Fig. 5, which have the abscissas k and $2k$ relative to the non-relativistic system with A as its origin, go to positive infinity. The right half of Fig. 5 therefore goes away, and the left half will be stretched along the entire distance from $-\infty$ to $+\infty$, in which the functional behavior of Fig. 5 obviously takes place.

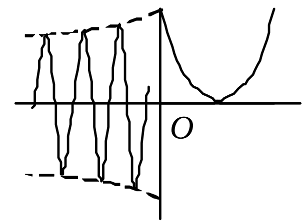

Fig. 6.

Whereas the region $\xi < 0$ therefore corresponds to the domain of validity of the Schrödinger equation, the region $\xi > 0$ possesses no non-relativistic analogue. This region is, as will be shown, characterized by the fact that the wave-mechanical impulse vector is oppositely directed to the velocity.

As is well known, in the Dirac theory the three-dimensional velocity u_ν corresponds to the operator $ic\gamma_\nu$ ($\nu = 1, 2, 3$). One obtains the wave mechanical expectation value of this operator in the form:

$$[u_\nu] = ic\bar{\psi}\,\gamma_\nu\,\psi. \tag{23}$$

In this, $\bar{\psi}$ refers to the adjoint wave function, which satisfies the equation:

$$\bar{\psi}\left\{-\gamma_1\frac{\partial}{\partial x} - \gamma_2\frac{\partial}{\partial y} - \gamma_3\frac{\partial}{\partial z} + \gamma_4\left(-\frac{1}{ic}\frac{\partial}{\partial t} + \kappa v x\right) + \kappa E_0\right\} = 0. \tag{2a}$$

In order to arrive at a relation between the quantities $[u_x]$ that are given by (23) and the expectation value $[p_x]$ of the impulse in the field direction, we multiply equation (2) on the left by $\bar{\psi}\,\gamma_4\gamma_1$ and equation (2a) on the right by $\gamma_1\gamma_4\,\psi$, and then add them together; this yields:

$$\left(\bar{\psi}\gamma_4\frac{\partial}{\partial x}\psi - \frac{\partial\bar{\psi}}{\partial x}\gamma_4\psi\right) + \frac{\partial}{\partial x}(\bar{\psi}\gamma_4\gamma_1\gamma_2\psi) + \frac{\partial}{\partial y}(\bar{\psi}\gamma_4\gamma_1\gamma_3\psi)$$
$$-\bar{\psi}\gamma_1\left(\frac{1}{ic}\frac{\partial}{\partial t} + \kappa v x\right)\psi + \left(\frac{1}{ic}\frac{\partial\bar{\psi}}{\partial t} - \kappa v x\bar{\psi}\right)\gamma_1\psi = 0.$$

For a particular infinitely steep impulse in the y and z direction, the terms with $\partial/\partial y$ and $\partial/\partial z$ [which, due to the dependency on y and z that is given in (4), fall out of the quadratic expressions $\bar{\psi}\,\gamma_4\gamma_1\gamma_{2,3}\psi$] vanish. Since we are considering a stationary state of the energy E, we can carry out the time differentiation, and we obtain:

$$\frac{1}{2}\left(\bar{\psi}\gamma_4\frac{\partial}{\partial x}\psi - \frac{\partial\bar{\psi}}{\partial x}\gamma_4\psi\right) + \kappa(E - vx)\bar{\psi}\gamma_1\psi = 0. \tag{24}$$

Due to (23), the second term may be written in the form:

$$-\frac{2\pi i}{h}\frac{E-vx}{c^2}[u_x].$$

From the meaning of the first expression we consider that the impulse operator in Dirac's theory is given by $\frac{h}{2\pi i}\gamma_4\frac{\partial}{\partial x_\nu}$; the first term of (24) thus represents the symmetrized expectation value of the impulse $[p_x]$, multiplied by the factor $\frac{2\pi i}{h}$. (24) may then be written in the form:

$$[p_x] = \frac{E-vx}{c^2}[u_x]. \tag{25}$$

This equation thus says that for $E > vx$ [or, due to (8), for $\xi < 0$] impulse and velocity carry the same sign, whereas for $E < vx$ (or $\xi > 0$) they are oppositely directed, which was to be proved. One observes, moreover, that (25) represents the wave mechanical translation of the relativistic connection between impulse and velocity:

$$p_x = \frac{mu_x}{\sqrt{1-\frac{u^2}{c^2}}},$$

since:

$$E - vx = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}}.$$

For the application in the next section, the dependency of the velocity $[u_x]$ on the constituent functions f and g can be worked out here. For this, we introduce the expression for ψ that is obtained from (4) and (11):

$$\psi = [f + \gamma_5 g] \cdot e^{\frac{2\pi i}{h}(yp_y+zp_z-Et)}(1 + i\,\gamma_1\,\gamma_4)\,\Gamma; \tag{26a}$$

one easily convinces oneself in the same way as in the first section that the corresponding expression for $\bar{\psi}$ will be given by:

$$\bar{\psi} = \bar{\Gamma}(1-i\gamma_1\gamma_4)[f^* + \bar{\gamma}_5 g^*]e^{-\frac{2\pi i}{h}(yp_y+zp_z-Et)}. \tag{26b}$$

$\bar{\gamma}_5$ refers to the adjoint quantity to γ_5 [see (7)]:

$$\bar{\gamma}_5 = -(E_0 + ic\gamma_2\,p_y + ic\gamma_2\,p_z)\,\gamma_1 = -\,\gamma_1\,\gamma_5\,\gamma_1.$$

If one now constructs the density $\bar{\psi}\,\gamma_4\psi$ then one obtains:

$$\bar{\psi}\,\gamma_4\psi = \bar{\Gamma}(1-i\gamma_1\gamma_4)(f^* - \gamma_1\gamma_5\gamma_1 g^*)\gamma_4(f+\gamma 5g)(1+i\gamma_1\gamma_4)\Gamma = (ff^* + gg^*)\,\Delta, \tag{27}$$

in which:

$$\Delta = 2\,\bar{\Gamma}\,\gamma_4(1 + i\,\gamma_1\gamma_4)\,\Gamma$$

means a constant operator [†]. Analogously, one gets:

$$[u_x] = ic\bar{\psi}\,\gamma_1\psi = -\,c(ff^* - gg^*)\Delta, \tag{28}$$

so $[u_x]$ is also spatially constant for the stationary state under consideration, as one can easily confirm on the basis of (12). Indeed, this also already follows from the divergence condition for the current $S_\nu = e[u_x]$, since we consider a stationary state for which the current components in the z and z directions are constant (viz., a plane wave!).

Because of this constancy, one computes the value of (28) most easily for the location $\xi = 0$. If one assembles the general functions ψ from linear combinations of the constituent functions ψ_1 and ψ_2, hence:

$$\psi = \alpha\psi_1 + \beta\psi_2\,, \tag{29}$$

and therefore:

$$f = \alpha f_1 + \beta f_2\,, \qquad g = \alpha g_1 + \beta g_2\,, \tag{29a}$$

then, from (14), one easily confirms that:

$$ff^* - gg^* = \alpha\alpha^* - \beta\beta^*, \tag{30}$$

since $f_1(0) = g_2(0) = 1$, $f_2(0) = g_1(0) = 0$. The current that results from the two partial solutions ψ_1 and ψ_2 is therefore assembled with no interference into the total current in the x direction.

4. *Transition of an electron into an antifield.*

We will now address the problem that was posed in the introduction of calculating the probability that an electron will go from the region of positive impulse to the region of negative impulse [*]. To that end, we consider the potential function that is represented by the line *ABCD* in Fig. 1:

[†] Cf., F. Sauter, ZS. f. Phys. **64**, 295, 1930; there, the operator Δ was referred to as the "normalization operator."

[*] By "positive" ("negative", resp.) impulse it shall be understood that the impulse and the velocity have the same (opposite, resp.) directions. In the former case, the kinetic energy is positive; in the latter, it is negative.

$$\left.\begin{array}{lll} \text{I.} & -\infty < x < x_1 & \text{or} \quad \xi < \xi_1 < -k \cdots V = vx_1, \\ \text{II.} & x_1 < x < x_2 & \text{"} \quad \xi_1 < \xi < \xi_2 \cdots V = vx, \\ \text{III.} & x_2 < x < +\infty & \text{"} \quad k < \xi_2 < \xi \cdots V = vx_2. \end{array}\right\} \tag{31}$$

We must now solve the Dirac equation for the three cases separately and then match up the three wave functions continuously at the points x_1 and x_2 . For all three cases, the integration can be achieved by the same method, from which the matching will be made essentially easier.

We make the Ansatz:

$$\psi = [f + \gamma_5 g](1 + i\,\gamma_1\gamma_4)\Gamma\, e^{\frac{2\pi i}{h}(yp_y + zp_z - Et)}, \tag{32}$$

where γ_5 is given by (6) and (7). If one now introduces, as above, the new variable:

$$x = \sqrt{\frac{\kappa}{v}}\,(vx - E) \tag{8}$$

then one obtains the equations for f and g precisely as in the first section:

$$\left.\begin{array}{ll} \text{I.} & \left(\frac{d}{d\xi} - i\xi_1\right) f + kg = 0, \quad \left(\frac{d}{d\xi} + i\xi_1\right) g + kf = 0, \\ \text{II.} & \left(\frac{d}{d\xi} - i\xi\right) f + kg = 0, \quad \left(\frac{d}{d\xi} + i\xi\right) g + kf = 0, \\ \text{III.} & \left(\frac{d}{d\xi} - i\xi_2\right) f + kg = 0, \quad \left(\frac{d}{d\xi} + i\xi_2\right) g + kf = 0. \end{array}\right\} \tag{33}$$

k is defined by (9), while ξ_1 and ξ_2 are obtained from (8) by introducing the values x_1 and x_2 at the location x. We write the solutions of the second pair of equations (33) [which is identical with (12)] by the use of the two constants a and b in the form (29a), where f_1, g_1, and f_2 , g_2 are given by (14). For the first and third regions one naturally obtains a plane wave of the form $e^{\pm iq_{1,2}\xi}$, where q_1 and q_2 are determined by means of the equations †:

$$q_\nu^2 = \xi_\nu^2 - k^2, \qquad (\nu = 1, 2). \tag{34}$$

If one computes the coefficient ratios by means of (33) then, with the arbitrary constants a_ν and b_ν, this yields the solutions:

---

† q_ν is connected with the ordinary impulse p_ν by the relation $q_\nu = \sqrt{\frac{\kappa}{v}}\,cp_\nu$. Since $|\,\xi_\nu\,| > k$, q_ν is real and we assume that $q_\nu > 0$.

$$\left.\begin{aligned} f &= a_\nu e^{iq_\nu \xi} + b_\nu e^{-iq_\nu \xi}, \\ g &= \frac{i}{k}[a_\nu(\xi_\nu - q_\nu)e^{iq_\nu \xi} + b_\nu(\xi_\nu + q_\nu)e^{-iq_\nu \xi}]; \end{aligned}\right\} \tag{35}$$

$\nu = 1$ is valid for region I and $\nu = 2$ for region III.

We achieve a continuous connection between the wave functions at the locations x_1 and x_2 (ξ_1 and ξ_2, resp.) when we connect the functions f and g with each other at those locations. This leads to the equations:

$$\left.\begin{aligned} a_1 e^{iq_1\xi_1} + b_1 e^{-iq_1\xi_1} &= \alpha F_1 + \beta G_1^*, \\ \frac{i}{k}[a_1(\xi_1 - q_1)e^{iq_1\xi_1} + b_1(\xi_1 + q_1)e^{-iq_1\xi_1} &= \alpha G_1 + \beta F_1^*; \end{aligned}\right\} \tag{36a}$$

$$\left.\begin{aligned} a_2 e^{iq_2\xi_2} + b_2 e^{-iq_2\xi_2} &= \alpha F_2 + \beta G_2^*, \\ \frac{i}{k}[a_2(\xi_2 - q_2)e^{iq_2\xi_2} + b_2(\xi_2 + q_2)e^{-iq_2\xi_2} &= \alpha G_2 + \beta F_2^*. \end{aligned}\right\} \tag{36b}$$

In this, due to (14), to abbreviate we have set:

$$\left.\begin{aligned} F_\nu &= e^{\frac{i\xi_\nu^2}{2}} F\left(-\frac{k^2}{4i}, \frac{1}{2}, -i\xi_\nu^2\right), \\ G_\nu &= -k\xi_\nu e^{\frac{i\xi_\nu^2}{2}} F\left(-\frac{k^2}{4i}+1, \frac{3}{2}, -i\xi_\nu^2\right), \end{aligned}\right\} \tag{37}$$

and used (15). The system of equations (36) for the determination of the six quantities a_ν, b_ν, α, β is still undetermined; to the mathematical continuity conditions one must also add a physical condition that establishes the behavior of the wave function at infinity.

We shall now, as we have already stressed on many occasions, ascertain the probability that an electron in region I with positive impulse will go over into region III with negative impulse. We must therefore assume an incident particle current that comes in from the left, partly goes through region II and continues on into region III (from left to right), and is partly reflected by the separating surface between x_1 and x_2, which represents a reverse current in region I. The solution must therefore be chosen in such a way that in III only one current flows in the left-to-right direction and none flows in the opposite direction. The current component in the x direction for a wavefunction that was represented by the expression (32) was determined in the third section [equation (28)]. When applied to the plane wave that represented by (35) this produces:

$$S_x = -iec\bar{\psi}\gamma_1\psi = \frac{2ecq_\nu}{k^2}[a_\nu^* a_\nu(-\xi_\nu + q_\nu) + b_\nu^* b_\nu(\xi_\nu + q_\nu)]\Delta,$$

while the particle density will be given by way of (27).

For $\nu = 1$, one thus obtains, since $\xi_1 < -q_1$, an incident current of strength:

$$S_e = \frac{2ecq_1}{k^2}(-\xi_1 + q_1)a_1^* a_1 \Delta$$

and a reflected current:

$$S_r = -\frac{2ecq_1}{k^2}(-\xi_1 - q_1)b_1^* b_1 \Delta \,;$$

for $\nu = 2$, since $\xi_2 > q_2$:

$$S_a = \frac{2ecq_2}{k^2}(\xi_2 + q_2)b_2^* b_2 \Delta$$

represents the required outgoing current, while the incoming current in this region must vanish. One thus obtains $a_2 = 0$ as the physical condition.

One calculates the transmission coefficient:

$$D = \frac{S_a}{S_e} = \frac{q_2}{q_1} \cdot \frac{\xi_2 + q_2}{-\xi_1 + q_1} \frac{b_2^* b_2}{a_1^* a_1}; \tag{38}$$

the corresponding reflection coefficient is given by:

$$R = \frac{-S_r}{S_e} = \frac{-\xi_1 - q_1}{-\xi_1 + q_1} \frac{b_1^* b_1}{a_1^* a_1}, \tag{39}$$

where naturally $R + D = 1$.

If one sets $a_2 = 0$ in (36b) then, since:

$$F_2 F_2^* - G_2 G_2^* = 1$$

the solution to these two equations in α and β reads [see (37) and (13)]:

$$\alpha = b_2 e^{-iq_2\xi_2}\left[\; F_2^* - G_2^* \cdot \frac{i}{k}(\xi_2 + q_2)\right],$$
$$\beta = b_2 e^{-iq_2\xi_2}\left[-G_2 + F_2 \cdot \frac{i}{k}(\xi_2 + q_2)\right].$$

If one substitutes this expression in (36a) and solves it for a_1 and b_1 then one obtains:

$$-\frac{2iq_1}{k} a_1 e^{iq_1\xi_1} = b_2 e^{-iq_2\xi_2}\left[-\frac{i}{k}(\xi_1 + q_1)A + \frac{i}{k}(\xi_2 + q_2)A^* + \frac{(\xi_1 + q_1)(\xi_2 + q_2)}{k^2}B + B^*\right],$$
$$\frac{2iq_1}{k} a_1 e^{-iq_1\xi_1} = b_2 e^{-iq_2\xi_2}\left[-\frac{i}{k}(\xi_1 - q_1)A + \frac{i}{k}(\xi_2 + q_2)A^* + \frac{(\xi_1 - q_1)(\xi_2 + q_2)}{k^2}B + B^*\right],$$

where:

$$A = F_1 F_2^* - G_1^* G_2 ,$$
$$B = F_2 G_1^* - F_1 G_2^* .$$

For the sake of simplicity, we would like to assume symmetric behavior about the point $\xi = 0$:

$$-\xi_1 = \xi_2 = \xi_0 > 0. \tag{40}$$

One will then have:

$$q_1 = q_2 = q_0 ,$$
$$F_1 = F_2 = F_0 , \quad -G_1 = G_2 = G_0 ,$$
$$A = A^* = F_0 F_0^* + G_0 G_0^* , \qquad B_x = -F_0 G_0^* , \quad B^* = -2F_0^* G_0 .$$

One then obtains:

$$a_1 = b_2 \qquad \left[-\frac{\xi_0}{q_0}(F_0F_0^* + G_0G_0^*) + \frac{ki}{q_0}(F_0G_0^* - F_0^*G_0)\right],$$
$$b_1 = b_2 e^{-2iq_0\xi_0}\left[\frac{\xi_0 + q_0}{q_0}(F_0F_0^* + G_0G_0^*) - \frac{i(\xi_0 + q_0)^2}{kq_0}F_0G_0^* + \frac{ki}{q_0}F_0^*G_0\right].$$

Now we employ the development (22) for F_0 and G_0 † that was given in the second section, which is valid for $\xi^2 > k^2$. Since:

$$w_0 = \sqrt{1 - \frac{k^2}{\xi_0^2}} = \frac{q_0}{\xi_0} ,$$

one has:

$$F_0 = \frac{ke^{\frac{k^2\pi}{4}}}{2\sqrt{\xi_0 q_0}}\left\{\frac{ie^{-\frac{i}{2}\xi_0 q_0 + \frac{k^2 i}{4}\log\frac{\xi_0+q_0}{\xi_0-q_0}}}{\sqrt{1+\frac{q_0}{\xi_0}}} + \frac{e^{\frac{i}{2}\xi_0 q_0 - \frac{k^2 i}{4}\log\frac{\xi_0+q_0}{\xi_0-q_0}}}{\sqrt{1-\frac{q_0}{\xi_0}}}\right\},$$

$$G_0 = \frac{ke^{\frac{k^2\pi}{4}}}{2\sqrt{\xi_0 q_0}}\left\{-\frac{e^{-\frac{i}{2}\xi_0 q_0 + \frac{k^2 i}{4}\log\frac{\xi_0+q_0}{\xi_0-q_0}}}{\sqrt{1-\frac{q_0}{\xi_0}}} + \frac{ie^{\frac{i}{2}\xi_0 q_0 - \frac{k^2 i}{4}\log\frac{\xi_0+q_0}{\xi_0-q_0}}}{\sqrt{1+\frac{q_0}{\xi_0}}}\right\}.$$

† By the application of the development (22), which is valid only for the case where $|\xi|$ does not lie too close to k – for $|\xi| \sim k$ one uses a series of the form (20b) – the validity of the result will be reduced in the case of high velocities for the incoming and outgoing electrons. For small velocities, the problem then loses its interest, since then, from (45) or (46), the "Klein paradox" vanishes in any case.

One thus obtains:

$$F_0 F_0^* + G_0 G_0^* = \quad e^{\frac{k^2\pi}{2}} \left\{ \frac{\xi_0}{q_0} + \frac{k}{q_0} \sin\left(\xi_0 q_0 - \frac{k^2}{2} \log \frac{\xi_0 + q_0}{\xi_0 - q_0} \right) \right\},$$

$$F_0 G_0^* - F_0^* G_0 = -\, i e^{\frac{k^2\pi}{2}} \left\{ \frac{\xi_0}{q_0} + \frac{k}{q_0} \sin\left(\xi_0 q_0 - \frac{k^2}{2} \log \frac{\xi_0 + q_0}{\xi_0 - q_0} \right) \right\},$$

and:

$$a_1 = -b_2 e^{\frac{k^2\pi}{2}},$$

$$b_1 = b_2 e^{\frac{k^2\pi}{2}} \frac{i(\xi_0 + q_0)}{k_0} e^{-i\xi_0 q_0 - \frac{k^2\pi}{2} \log \frac{\xi_0 + q_0}{\xi_0 - q_0}}.$$

From this, on account of (38) and (39), one determines the transmission coefficient:

$$D = e^{-k^2\pi} \tag{41}$$

and reflection coefficient:

$$R = 1. \tag{42}$$

This result is valid up to higher order terms in $1/k^2$ and was derived under the assumption that $k^2 \gg 1$.

This therefore shows that for all electric fields for which $k^2 \gg 1$, hence, for all practically attainable fields (cf., *supra*), the transmission coefficient is vanishingly small; transitions into the region of negative impulse are therefore very rare in this case [†].

For the case of high electron velocity and a symmetric potential function, in the first approximation the value of the transmission coefficient D depends upon only the field strength, hence, upon only the steepness of the potential ramp. This case would (cf., *supra*) correspond to around 10^{16} volt/cm. The location $k^2 \sim 1$ has a special physical meaning. In this case, one has:

$$k^2 = \frac{2\pi}{hc} \frac{(mc^2)^2}{v} \sim 1,$$

or:

$$\frac{vh}{mc} \sim mc^2. \tag{43}$$

This agrees with the conjecture of N. Bohr that was given in the introduction, that one first obtains the finite probability for the transition of an electron into the region of negative impulse when the potential ramp vh/mc over a distance of the Compton wavelength h/mc has the order of magnitude of the rest energy.

It is naturally impossible to experimentally configure fields of this strength. One can possibly imagine that such fields can appear in the interior of an atom in some

[†] This result is naturally independent of the aforementioned assumption of symmetric behavior. In the general case, the final formulas thus become very unclear.

circumstances; in a pure Coulomb field, the critical point that is given by (43) would be at a distance of roughly:

$$r \sim \frac{e}{mc}\sqrt{\frac{h}{c}} \sim 8 \times 10^{-12} \text{ cm}$$

from the center of force; similar high fields in atoms must also exist in Gamow's nuclear model. It will therefore be pointless to draw any conclusions about the behavior of an electron in the immediate neighborhood of nuclei, since Dirac's theory (which is already invalid due to the fact that we are neglecting the nuclear spin) loses its validity at such small distances from the nucleus.

For the sake of completeness, we would still like to consider the case that treated by O. Klein * of a potential well, hence, the limiting case of an infinitely steep potential ramp. The calculation can then be carried out in such a manner that one links the wave functions that are valid in region I and region III to each other directly. v is then to be viewed as an auxiliary mathematical quantity that drops out of the result automatically. In order to use the notation of Klein, one must set:

$$\xi_1 = -\sqrt{\frac{\kappa}{v}}E, \qquad \xi_2 = \sqrt{\frac{\kappa}{v}}(P-E),$$
$$q_1 = \sqrt{\frac{\kappa}{v}}cp, \qquad q_2 = \sqrt{\frac{\kappa}{v}}c\bar{p},$$

where P is then the height of the potential well, E is the energy before the transition to the antifield and p and $\bar{p}$ measure the impulse before and after the transition. One then obtains by simple calculations, which shall be omitted here, in the name of brevity:

$$\left.\begin{aligned} D &= \frac{P-E+c\bar{p}}{E+cp}\cdot\frac{4c^2 p\bar{p}}{[P+c(\bar{p}-p)]^2}, \\ R &= \frac{E-cp}{E+cp}\cdot\left[\frac{P+c(\bar{p}+p)}{P+c(\bar{p}-p)}\right]^2. \end{aligned}\right\} \tag{44}$$

One obtains the formulas that were given in the introduction from these expression in the limiting case of a very high potential well (lim $P \to \infty$):

$$\left.\begin{aligned} D &= \frac{2cp}{E+cp}, \\ R &= \frac{E-cp}{E+cp}. \end{aligned}\right\} \tag{45}$$

---

* O. Klein, loc. cit.

In order to obtain the analogy with the previously-treated case of a symmetric potential function, one must set $\xi_1 = -\xi_2$, and then set $P = 2E$, and thus obtain:

$$\left.\begin{aligned} D &= \frac{c^2 p^2}{E^2} = \beta^2, \\ R &= \frac{m^2 c^4}{E^2} = 1 - \beta^2. \end{aligned} \qquad \left(\beta = \frac{u}{v}\right)\right\} \tag{46}$$

It is a pleasure for me to thank Herrn Professor A. Sommerfeld for his interest in the progress of this investigation.

Moreover, it is fitting for me to thank the Austrian-German Science Fund for granting me a stipend.