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The stress functions of the three-dimensional continuum and elastic bodies (*)

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The stress function tensor, whose divergence is prescribed by the equilibrium conditions and whose rotation is prescribed by the compatibility conditions, will be determined with the help of a tensor potential, namely, the stress function tensor. For the covariant calculations, close analogies with **Einstein**'s theory of gravitation are indicated. In the special case of isotropic, elastic bodies, one will obtain the **Boussinesq-Neuber** representation of stresses and displacements in terms of potential functions.

1. Notations.

In the following discussion, rectangular parallel coordinates x_i (i = 1, 2, 3) will be employed throughout. To abbreviate, we write $\partial / \partial x_i = \partial_i$. Summation signs will be omitted. The operator Δ means $\partial_r \partial_r = \sum_{r=1}^{3} \frac{\partial^2}{\partial x_r^2}$, as usual.

2. Problem statement.

As is known, the basic equations of the classical theory of elasticity can be formulated in terms of the stresses alone. Except for the boundary conditions and in the absence of volume forces, the symmetric stress tensor σ_{ik} has to satisfy the three equilibrium conditions:

$$\partial_k \sigma_{ik} = 0 \tag{2.1}$$

and the six compatibility conditions (**Beltrami** conditions):

$$\Delta\sigma_{ik} + \frac{m}{m+1} \partial_i \partial_k \sigma = 0 \tag{2.2}$$

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After my presentation, **H. Richter**, Haltingen, made me aware of the work of **V. I. Bloch**: "Stress functions in the theory of elasticity," Priklad. Mat. Mech. Moscow **14** (1950), 415-422 [Russian]. **Richter**'s discussion of that work in the Zentralblatt für Mathematik **39** (1951) had escaped me. In the meantime, **Richter** was so kind as to give some glimpse of **Bloch**'s treatise. With the same objective, **Bloch**'s formulation and representation (viz., the nabla symbolism) of the problem was so fundamentally different from my own that I do not feel that my publication is superfluous. **Neuber**'s representation of stresses, to which one can simplify **Bloch**'s results, seems to be unknown in the Russian literature.

(m = **Poisson** transverse contraction number; $\sigma = \sigma_{rr}$) [**1**].

These nine equations for the six components of the stress tensor are not independent of each other, since three differential identities must exist between them, as well. We shall return to this situation at a later point.

As is known, if the stress state is independent of some coordinate – say, x_3 – then one will arrive at a clear formulation of the problem of integration by the introduction of stress functions – viz., the **Airy** and torsion functions [**2**].

That raises the question of whether one can proceed correspondingly in the general case of a three-dimensional stress state [**3**]. With such an integration process, one would initially treat the equilibrium conditions (2.1) of a continuum that are satisfied identically with the help of stress functions. It is plausible that one will require at least three stress functions for that. (With a terminology that **G. Prange** used, the equilibrium problem is "threefold functionally undetermined.") One finds the Ansätze of **Maxwell** and **Morera** [**4**] in the literature of the classical theory of elasticity, which are complementary, in a sense that will be clarified later. In both Ansätze, the stresses are represented by sums of second derivatives of just three stress functions, and one can show that any stress state in equilibrium can be ascertained by means of those Ansätze.

Now, once the three equilibrium conditions (2.1) are satisfied in that way, one introduces one of those three-stress functions Ansätze into the six compatibility conditions (2.2). One then obtains six fourth-order differential equations for the three stress functions that are once more not independent of each other. Those equations were presented for the **Maxwell** case, as well as the **Morera** case [**5**]. However, they seem to be so complicated that any attempt to find a general method of integration for them would seem hopeless from the outset.

The question of whether the equilibrium conditions can be fulfilled by stress functions was addressed by **C. Weber** [**6**]. He could show that the combination of the two Ansätze of **Maxwell** and **Morera** (hence, an Ansatz that contains six stress functions) preserves its form under an orthogonal transformation, as long as one demands that the six stress functions transform like the components of a symmetric tensor of rank two. One concludes from this that neither of the two Ansätze of **Maxwell** and **Morera** are covariant by themselves, which might make their lack of utility for a general integration procedure more understandable.

However, not even **C. Weber**'s covariant Ansatz has a form that would make its further use promising. For that reason, we have to look for a rational, covariant representation.

3. The compatibility conditions for the infinitesimal state of deformation.

On grounds that will soon become clear, we shall anticipate our consideration of the compatibility conditions.

The symmetric deformation tensor ε_{ik} is defined in terms of the three components u_i of the infinitesimal displacement vector $\mathfrak{u}$ by way of:

$$\varepsilon_{ik} = \tfrac{1}{2}(\partial_i u_k + \partial_k u_i). \tag{3.1}$$

We shall call it the *symmetric gradient* tensor of the vector $\mathfrak{u}$, and write it symbolically as:

$$(\varepsilon_{ik}) = \text{Grad}\, \mathfrak{u}.$$

Obviously, not just any symmetric second-rank tensor can be a deformation tensor. Moreover, it has to satisfy the compatibility conditions, which is a system of second-order field equations that one will obtain most simply by eliminating the components of the displacement vector from (3.1).

We shall deviate from the usual procedure in the literature and carry out that elimination in the following way: We first define:

$$e_{ik} \equiv \varepsilon_{ik} - \tfrac{1}{2}\,\delta_{ik}\,\varepsilon = \tfrac{1}{2}(\partial_i\, u_k + \partial_k\, u_i) - \tfrac{1}{2}\,\delta_{ik}\,\partial_l\, u_l \tag{3.2}$$

(δ_{ik} = **Kronecker** symbol; $\varepsilon = \varepsilon_{ll}$) as a linear combination of equations (3.1), from which we will obtain the divergence:

$$\partial_k\, e_{ik} = \tfrac{1}{2}\Delta\, u_i\,. \tag{3.3}$$

The application of the operator Δ to (3.2) yields:

$$-\Delta\, e_{ik} + \tfrac{1}{2}(\partial_i\, \Delta u_k + \partial_k\, \Delta\, u_i) - \tfrac{1}{2}\,\delta_{ik}\,\partial_l\,\Delta u_l = 0, \tag{3.4}$$

and after one introduces (3.3) into (3.4), the elimination will be complete. The compatibility equations will then take on the form:

$$A_{ik} \equiv -\Delta\, e_{ik} + \partial_i\,\partial_l\, e_{kl} + \partial_k\,\partial_l\, e_{il} - \delta_{ik}\,\partial_l\,\partial_m\, e_{lm} = 0. \tag{3.5}$$

The left-hand side of this defines a symmetric second-rank tensor that we shall refer to as A_{ik}. Those six field equations for the e_{ik} (or also for the ε_{ik}) are not independent of each other. A simple calculation will confirm that:

$$\partial_k\, A_{ik} \equiv 0, \tag{3.6}$$

so the tensor A_{ik} is divergence-free.

In the literature, the compatibility conditions are given in the form:

$$R_{ik,\,lm} \equiv \partial_i\,\partial_l\,\varepsilon_{km} + \partial_k\,\partial_m\,\varepsilon_{il} - \partial_k\,\partial_l\,\varepsilon_{im} - \partial_i\,\partial_m\,\varepsilon_{lm} = 0. \tag{3.7}$$

In that representation, the left-hand side defines a fourth-rank tensor $R_{ik,\,lm}$, which nonetheless possesses only six algebraically-independent components in three dimensions, due to its symmetries:

$$R_{ik,\,lm} = -R_{ki,\,lm}\,, \qquad R_{ik,\,lm} = -R_{ik,\,ml}\,, \qquad R_{ik,\,lm} = R_{lm,\,ik}\,. \tag{3.8}$$

(In a two-dimensional continuum, only one equation will remain, namely, $R_{12,12} = 0$.)

In a deformed continuum, the arc length indeed has the form:

$$ds^2 = (\delta_{ik} + 2e_{ik})\, dx_i\, dx_k\,, \tag{3.9}$$

but the metric remains Euclidian. The **Riemann-Christoffel** tensor $R_{ik,lm}$ of (3.9) must vanish, and that will lead to (3.7) for infinitesimal deformations.

One easily confirms by calculation that the components (3.5) are connected by:

$$A_{\alpha\beta} = R_{\alpha+1,\ \alpha+2;\ \beta+1,\ \beta+2}\ . \tag{3.10}$$

(Any index that is greater than three shall be replaced with its residue modulo 3.)

The symmetric curvature tensor:

$$R_{kl} = R_{ik,li} \tag{3.11}$$

arises from $R_{ik,lm}$ by contraction, and a further contraction will give the curvature scalar:

$$R = R_{ll}\ . \tag{3.12}$$

That will then, in turn, imply that:

$$A_{ik} = R_{ik} - \tfrac{1}{2}\,\delta_{ik}\, R, \tag{3.13}$$

which can be shown most simply by exhibiting the individual components, and that means that A_{ik} will be the **Einstein** tensor of the arc length (3.9). The **Riemann-Christoffel** tensor can then be replaced with the **Einstein** tensor in three-dimensional space. One already finds the introduction of the e_{ik} into (3.2) and (3.5), in place of ε_{ik}, in **Einstein**'s theory for weak gravitational fields [**7**].

The representation of the tensor A_{ik} as a matrix product:

$$(A_{ik}) = \begin{bmatrix} 0 & \partial_3 & -\partial_2 \\ -\partial_3 & 0 & \partial_1 \\ \partial_2 & -\partial_1 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix} \tag{3.14}$$

is instructive. The matrix of the deformation tensor is multiplied on the right by the operator matrix of the rotation and on the left by its inverse. For that reason, we would like to call the tensor A_{ik} the *symmetric rotation* of the tensor ε_{ik} and write symbolically:

$$(A_{ik}) = \text{Rot}\ (\varepsilon_{ik}). \tag{3.15}$$

We shall express the fact that the vectorial divergence of A_{ik} vanishes from (3.6) with the notation:

$$\text{Div Rot}\ (\varepsilon_{ik}) \equiv 0. \tag{3.16}$$

The compatibility equations (3.5), which we can now write as:

$$\text{Rot}\ (\varepsilon_{ik}) = 0, \tag{3.17}$$

are the necessary, and also sufficient, integrability conditions for the system of equations (3.1). A symmetric tensor ε_{ik} is the symmetric gradient tensor of a displacement vector field $\mathfrak{u}$ if and only if its symmetric rotation tensor vanishes. The identity:

$$\text{Rot Grad } \mathfrak{u} \equiv 0 \tag{3.18}$$

is then satisfied, as well as the theorem that any symmetric tensor that satisfies (3.17) can be represented as the symmetric gradient tensor of a vector $\mathfrak{u}$.

4. – Covariant Ansätze for the fulfillment of the equilibrium conditions by stress functions.

The equilibrium conditions (2.1) of the continuum require the vanishing of the divergence of the stress tensor. In analogy to (3.16), they can be fulfilled identically in the components F_{ik} of the symmetric tensor of the stress functions by the Ansatz:

$$(\sigma_{ik}) = \text{Rot } (F_{ik}), \tag{4.1}$$

or from (3.5), when one introduces the stress functions:

$$\Phi_{ik} = F_{ik} - \tfrac{1}{2}\, \delta_{ik}\, F \tag{4.2}$$

($F = F_{ll}$), by:

$$\sigma_{ik} = -\,\Delta\Phi_{ik} + \partial_i\, \partial_l\, \Phi_{kl} + \partial_k\, \partial_l\, \Phi_{il} - \delta_{ik}\, \partial_l\, \partial_m\, \Phi_{lm}\,. \tag{4.3}$$

It follows from (3.18) that for a given equilibrium state, the stress functions F_{ik} are determined up to an arbitrary symmetric gradient tensor Grad $\mathfrak{v}$. One can make three of the six F_{ik} (Φ_{ik} , resp.) equal to zero by a suitable choice of the vector $\mathfrak{v}$, but not arbitrarily, in general. One will then be led to, e.g., the non-covariant Ansätze of **Maxwell** ($F_{12} = F_{13} = F_{23} = 0$) and **Morera** ($F_{11} = F_{22} = F_{33} = 0$).

We shall discuss a covariant normalization of the Φ_{ik} later.

The close coupling of the stress functions with the compatibility conditions can be clarified by the principle of virtual deformations. We shall content ourselves by sketching out that connection. For the sake of simplicity, the displacements shall be set to zero on the outer surface of the continuum. In the principle of virtual deformations, one can vary the deformations ε_{ik} independently of each other when one equips the compatibility conditions with **Lagrange** multipliers that are included in the integrand of the volume integral:

$$\int (\sigma_{ik}\, \delta\varepsilon_{ik} + \Gamma_{ik,lm}\delta R_{ik,lm})\, dV = 0. \tag{4.4}$$

By definition, the $\Gamma_{ik,lm}$ have to exhibit the same symmetries as $R_{ik,lm}$ [cf., (3.8)]. By partially integrating the second summand in (4.4) twice (the outer surface integrals mutually cancel each other), one will get:

$$\sigma_{km} = \partial_i \, \partial_l \, \Gamma_{ik,lm} \, , \tag{4.5}$$

since the $\delta\varepsilon_{ik}$ are independent of each other, and one convinces oneself that the equilibrium conditions are fulfilled due to the symmetries of the $\Gamma_{ik,lm}$. The six stress functions $\Gamma_{ik,lm}$ define a fourth-rank tensor from which the stress tensor will arise upon taking its divergence twice.

The stress functions Φ_{ik} (F_{ik} , resp.) are connected to the $\Gamma_{ik,lm}$ by way of:

$$\Gamma_{ik,lm} = \delta_{im} \, \Phi_{kl} + \delta_{kl} \, \Phi_{im} - \delta_{il} \, \Phi_{km} - \delta_{km} \, \Phi_{il} \tag{4.6}$$

or

$$F_{ik} = \Gamma_{ik} - \tfrac{1}{2} \, \delta_{im} \, \Gamma, \tag{4.7}$$

resp. (Γ_{ik} is a single contraction of $\Gamma_{ik,lm}$, and Γ is a double contraction of it.) (4.6) and (4.7) are only valid in three dimensions. In the two-dimensional case, only the one component $\Gamma_{12,12}$ will remain, which is the **Airy** stress function.

Any compatibility condition is associated with a stress function. The stress functions are the reactions to the geometric constraint that the metric of the deformation should remain Euclidian under a deformation.

5. – The stress functions of a body that is loaded with volume forces.

If volume forces X_i are present then the equilibrium conditions will read:

$$\partial_k \, \sigma_{ik} + X_i = 0. \tag{5.1}$$

One will get a covariant particular solution by the Ansatz:

$$\sigma_{ik} = \partial_i \, \Phi_k + \partial_i \, \Phi_k - \partial_k \, \Phi_i - \delta_{ik} \, \partial_l \, \Phi_l \, , \tag{5.2}$$

which will yield the determining equation:

$$\Delta\Phi_i + X_i = 0 \tag{5.3}$$

when it is introduced into (5.1). From (4.3), one can define the most general solution of (5.1) with a particular solution Φ_i of this **Poisson** equation:

$$\sigma_{ik} = -\,\Delta\Phi_{ik} + \partial_i \, (\partial_l \, \Phi_{kl} + \Phi_k) + \partial_k \, (\partial_l \, \Phi_{il} + \Phi_i) - \delta_{ik} \, \partial_l \, (\partial_{lm} \, \Phi_{lm} + \Phi_l). \tag{5.2}$$

In the absence of volume forces, the Φ_i are harmonic functions that can be omitted from (5.4), since (5.2) will then be simply the special case $\Delta\Phi_{ik} = 0$ of the Ansatz (4.3).

6. – Analogies with Einstein's general theory of relativity.

In the case of no volume forces, there exists a close analogy between the present developments and **Einstein**'s theory of gravitation in the special case of weak matter fields. The metric:

$$d\mathfrak{s}^2 = (\delta_{ik} + 2F_{ik})\, dx_i\, dx_k \tag{6.1}$$

deviates infinitesimally from the Euclidian one. The stress tensor will then be equal to the **Einstein** tensor of the arc element (6.1):

$$\mathfrak{R}_{ik} - \tfrac{1}{2}\,\delta_{ik}\,\mathfrak{R} = \sigma_{ik}\,. \tag{6.2}$$

$\delta_{ik} + 2F_{ik}$ corresponds to the gravitational potentials g_{ik} , and the stress tensor σ_{ik} is the impulse-energy tensor in **Einstein**'s theory, while our equilibrium conditions are the analogues of the conservation laws for energy and impulse. In the context of this analogy, (6.2) corresponds to the field equations of gravitation. The integration of the field equation, in turn, corresponds to the problem of ascertaining the stress functions Φ_{ik} or F_{ik} for a given equilibrium state. **Einstein**'s method of integration for the linearized field equations, when adapted to this problem statement, is briefly the following one [**8**]: Since the F_{ik} are determined only up to a gradient tensor (and correspondingly for the Φ_{ik}), one can always impose the covariant normalization:

$$\partial_k\, \Phi_{ik} = 0. \tag{6.3}$$

However, (4.3) decomposes into the two systems of equations (6.3) and:

$$\Delta\Phi_{ik} = \sigma_{ik}\,. \tag{6.4}$$

Due to the fact that $\partial_k\, \sigma_{ik} = 0$, it will follow from (6.4) that $\partial_k\, \Phi_{ik}$ is harmonic for any integral of (6.4). One can always fulfill (6.3) by integrating (6.4) with a suitable combination of harmonic functions then.

The spherically-symmetric solution to the field equations of gravitation for empty space that is singular at the center, which can be interpreted as the gravitational field of a mass point, also has its analogy in the context of our problem. Namely, it corresponds to the solution:

$$\Phi_{33} = \ln\sqrt{x_1^2 + x_2^2}\,, \qquad \Phi_{11} = \Phi_{22} = \Phi_{12} = \Phi_{13} = \Phi_{23} = 0$$

to the equations:

$$\Delta\Phi_{ik} = 0, \qquad \partial_k\, \Phi_{ik} = 0,$$

which can be regarded as the stress function tensor of a thin string that is tensed along the x_3-axis. However, in contrast to the four-dimensional theory of relativity, the metric (6.1) is Euclidian in three-dimensional space outside the continuum. For that reason, in this example, the stress functions can be represented as the gradient tensor of an infinitely multi-valued vector that is singular along the x_3-axis.

In the case of a finitely-extended body, the actual stresses that appear will be subject to compatibility conditions. The theory of relativity was first made into an intrinsically-closed theory by such "compatibility conditions" for the impulse-energy tensor, which describes the structure of matter by way of elementary particles.

7. – The elastic body.

We shall now return to our actual problem statement. The tensor field of stresses for an isotropic elastic body can be calculated from its vectorial divergence (equilibrium conditions) and tensorial rotation (compatibility conditions):

$$\mathrm{Div}\,(\sigma_{ik}) = -X_i\,, \tag{7.1}$$
$$\mathrm{Rot}\,(\varepsilon_{ik}) = 0, \tag{7.2}$$

in which σ_{ik} and ε_{ik} are connected by **Hooke**'s law:

$$2G\varepsilon_{ik} = \sigma_{ik} - \frac{1}{m+1}\,\delta_{ik}\,\sigma\,. \tag{7.3}$$

(G = shear modulus)

Recall the known problem in mathematical physics of determining a vector field from its scalar divergence and its vectorial rotation. There, one will be led to the concept of the vector potential. Here, the tensor of stress functions plays the role of a tensor potential.

We first treat the case of absent volume forces ($X_i = 0$) and write (4.3) as:

$$(\sigma_{ik}) = -\Delta^*(\Phi_{ik}), \tag{7.4}$$

with the introduction of the operator Δ^*. With that operator, the compatibility conditions (3.5) read:

$$-\Delta^*(e_{ik}) = 0, \tag{7.5}$$

and from (7.3), σ_{ik} and e_{ik} are coupled by:

$$2\,G\,e_{ik} = \sigma_{ik} - \frac{1}{2}\delta_{ik}\,\frac{m}{m+1}\sigma\,. \tag{7.6}$$

As one confirms immediately, for an arbitrary symmetric tensor H_{ik}, one will have:

$$\Delta^*(\Delta^*(H_{ik})) = \Delta^*(\Delta H_{ik}). \tag{7.7}$$

We make the Ansatz:

$$\Phi_{ik} = \Psi_{ik} - \tfrac{1}{2}\,\delta_{ik}\,\Omega \tag{7.8}$$

for Φ_{ik}, in which we reserve the right to assign the scalar function Ω at will. It will then follow from (7.4) to (7.8) that:

$$-\Delta^*\left(-\Delta\Psi_{ik} + \tfrac{1}{2}\delta_{ik}\Delta\Omega - \tfrac{1}{2}\delta_{ik}\frac{m}{m+1}\sigma\right) = 0. \tag{7.9}$$

We will now demand of Ω that:

$$\Delta\Omega = \frac{m}{m+1}\sigma, \tag{7.10}$$

and what will remain is:

$$-\Delta^*(\Delta\Psi_{ik}) = 0. \tag{7.11}$$

The $\Delta\Psi_{ik}$ will then satisfy the compatibility conditions, so they can be represented with the help of a vector $\Delta\mathfrak{w}$ by:

$$\Delta\Psi_{ik} = \partial_i \Delta w_k + \partial_k \Delta w_i - \delta_{ik}\,\partial_l \Delta w_l\,. \tag{7.12}$$

Integration yields:

$$\Psi_{ik} = \partial_i w_k + \partial_k w_i - \delta_{ik}\,\partial_l w_l + \Theta_{ik}\,, \tag{7.13}$$

so $\Delta\Theta_{ik} = 0$. The $\Theta_i = \partial_k \Theta_{ik}$ are likewise harmonic functions then. We substitute (7.13) and (7.8) into (7.4), which will make the derivatives of the w_i cancel, and obtain:

$$\sigma_{ik} = -\partial_i\partial_k\Omega + \delta_{ik}\Delta\Omega + \partial_i\Theta_k + \partial_k\Theta_i - \delta_{ik}\,\partial_l\Theta_l\,. \tag{7.14}$$

If volume forces X_i are present then, as a glimpse at (5.2) and (5.3) will show, one can immediately consider the manner by which one can subject the Θ_i in (7.14) to the equations:

$$\Delta\Theta_i + X_i = 0. \tag{7.15}$$

From (7.14), one calculates that:

$$\sigma = 2\,\Delta\Omega + \partial_l\Theta_l\,, \tag{7.16}$$

and with (7.10), one will have:

$$\Delta\Omega = \frac{m}{m+1}\partial_l\Theta_l\,. \tag{7.17}$$

One will get the representation of the displacement vector $\mathfrak{u}$ of the elastic body from (7.14):

$$G\,u_i = \Theta_i - \tfrac{1}{2}\partial_l\,\Omega. \tag{7.18}$$

8. – Concluding remarks.

One finds the results of the last section already in **Boussinesq** [**9**], who arrived at them by starting from the differential equations for the displacements. They were once more discovered by **Neuber** [**10**] by the same method, and he then made them known by way of numerous investigations of spatial stress states.

The fact that our considerations above seem to flow along familiar lines from a chapter in the classical theory of elasticity should not seem miraculous and should have been expected. However, the fact that it would follow precisely from the **Boussinesq-**

Neuber representations of stresses and displacements could not have been suspected from the outset.

One knows that classical mechanics, in particular, the **Hamilton-Jacobi** theory, already carries the skeleton of quantum mechanics in it. The fact that this classical example of a field theory clearly limns out the contours of **Einstein**'s theory of relativity seems to be a non-trivial addition to our knowledge from our present investigations.

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