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A field theory of dislocations in a COSSERAT continuum

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1. Introduction

In this treatise, a question shall be addressed that leads to a response in the continuum mechanics of dislocations. A moving, deforming continuum with mobile dislocations is characterized by the following kinematical quantities: velocity, state of deformation, its volume element, dislocation density and dislocation current density (number of moving dislocations in a volume element per unit time). Our question reads: What "forces" are these kinematic quantities associated with? One knows that the impulse $\mathbf{p}$ and stress $\boldsymbol{\sigma}$, along with $\mathbf{v}$ and $\boldsymbol{\varepsilon}$, define the power densities $\mathbf{p}\dot{\mathbf{v}}$ and $\boldsymbol{\sigma}\dot{\boldsymbol{\varepsilon}}$. However, what power densities are associated with dislocation densities? Only when this question is clarified can one formulate the next question regarding the material laws (i.e., constitutive equations) that exist between the "forces" and the kinematical quantities.

The field equations of linear dislocation theory admit an analogy with the MAXWELL equations of electromagnetic field [**1**]. This analogy asserts that the constitutive equations of dislocation theory up to now break down. Thus, it also becomes clear that a complete dislocation theory must possess another structure beyond the MAXWELL-LORENTZ theory. In dislocation theory, the dislocation field interacts with the matter field. The electromagnetic field of LORENTZ's theory is indeed analogous to the dislocation field, so there is no energy and impulse exchange with a matter field in an electron. There is only the four-vector of the LORENTZ force density.

Fortunately, in the electrodynamics of MIE [**2**], we possess a theory that is essentially more general than LORENTZ's, makes the electromagnetic field and the matter field in an electron interact, brings clarity to the constitutive equations, and ultimately leads to the LAGRANGE density. This more than 50-year-old theory of MIE – one can find a presentation of it in WEYL [**3**] that is concise, but emphasizes the essentials – is naturally superseded by quantum field theory nowadays. However, since the continuum theory of displacements is a macroscopic theory in the context of classical mechanics, we can follow the train of thought of MIE's theory, and analogously unify the dislocation field and the matter field into a common field, for which an action quantity is defined that remains stationary during an infinitesimal variation of the field state.

If we consider the dislocation theory of a COSSERAT continuum [**4**] here, it is because the continuum theory of dislocations up to now has – consciously or unconsciously – employed the geometric model of an incompatible COSSERAT continuum [**7**]. We shall not discuss the still-debatable question of whether such a macroscopic theory can describe crystal plasticity here.

Let our problem be outlined once more: The present continuum theory of dislocations is incomplete; it lacks constitutive equations. We must investigate how, and in what place, the theory of constitutive equations must be introduced. We will consider our problem as having been solved when we have found a LAGRANGE density that, on the one hand, delivers all of the constitutive equations, and on the other, defines an action integral whose variation leads to a complete system of field equations.

2. Notations

The kinematical and dynamical equations of a COSSERAT continuum can be brought into a lucid form when one employs the following differential operators [**5**], [**6**]:

$$\mathrm{Grad}\begin{bmatrix}\overset{1}{\mathbf{a}}\\ \overset{2}{\mathbf{a}}\end{bmatrix} \equiv \begin{cases}\partial_i \overset{1}{a}_k ,\\ \partial_i \overset{2}{a}_k - \varepsilon_{ikl}\overset{1}{a}_k ,\end{cases} \tag{2.1}$$

$$\mathrm{Rot}\begin{bmatrix}\overset{1}{\mathbf{A}}\\ \overset{2}{\mathbf{A}}\end{bmatrix} \equiv \begin{cases}\varepsilon_{ikl}\partial_k \overset{1}{A}_{lm} ,\\ \varepsilon_{ikl}\partial_k \overset{2}{A}_{lm} + \varepsilon_{mkn}\overset{1}{A}_{\mathrm{ln}} ,\end{cases} \tag{2.2}$$

$$\mathrm{Div}\begin{bmatrix}\overset{1}{\mathbf{R}}\\ \overset{2}{\mathbf{R}}\end{bmatrix} \equiv \begin{cases}\partial_i \overset{1}{R}_{ik} ,\\ \partial_i \overset{2}{R}_{ik} + \varepsilon_{klm}\overset{1}{R}_{lm} .\end{cases} \tag{2.3}$$

This symbolism is justified by the two identities:

$$\mathrm{Rot\,Grad} = 0, \qquad \mathrm{Div\,Rot} = 0, \qquad (2.4), (2.5)$$

All indices range from 1 to 3. The summation convention is in effect. ε_{ikl} is a unit tensor that is alternating in all indices. We use Cartesian coordinates x_1, x_2, x_3 throughout and the abbreviation $\partial_i = \partial / \partial x_i$.

As much as is possible, we adopt the notation for physical quantities that is used in the work of KLUGE [**8**].

We will further make use of the scalar product of the motor algebra [**5**]. It is defined by:

$$\begin{bmatrix}\overset{1}{\mathbf{a}}\\ \overset{2}{\mathbf{a}}\end{bmatrix}\circ\begin{bmatrix}\overset{1}{\mathbf{b}}\\ \overset{2}{\mathbf{b}}\end{bmatrix} = \overset{1}{\mathbf{a}}\circ\overset{2}{\mathbf{b}}+\overset{2}{\mathbf{a}}\circ\overset{1}{\mathbf{b}} = \overset{1}{a}_i\circ\overset{2}{b}_i+\overset{2}{a}_i\circ\overset{1}{b}_i \tag{2.6}$$

for vectors and by:

$$\begin{bmatrix} \overset{1}{\mathbf{A}} \\ \overset{2}{\mathbf{A}} \end{bmatrix} \circ\circ \begin{bmatrix} \overset{1}{\mathbf{B}} \\ \overset{2}{\mathbf{B}} \end{bmatrix} = \overset{1}{\mathbf{A}} \circ \overset{2}{\mathbf{B}} + \overset{2}{\mathbf{A}} \circ \overset{1}{\mathbf{B}} = \overset{1}{A}_{ik} \circ \overset{2}{B}_{ik} + \overset{2}{A}_{ik} \circ \overset{1}{B}_{ik} \tag{2.7}$$

for tensors of second rank.

3. Basic dynamical and kinematical equations

The theory of dislocations that lies before us [**8**] consists of the dynamical equations:

$$\mathrm{Div}\begin{bmatrix} \boldsymbol{\sigma} \\ \boldsymbol{\mu} \end{bmatrix} = \begin{bmatrix} \rho\dot{\mathbf{v}} \\ \theta\dot{\mathbf{s}} \end{bmatrix}, \tag{3.1}$$

and the kinematical equations:

$$\mathrm{Rot}\begin{bmatrix} \boldsymbol{\kappa} \\ \boldsymbol{\varepsilon} \end{bmatrix} = \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix}, \tag{3.2}$$

$$\mathrm{Grad}\begin{bmatrix} \mathbf{s} \\ \mathbf{v} \end{bmatrix} - \begin{bmatrix} \dot{\boldsymbol{\kappa}} \\ \dot{\boldsymbol{\varepsilon}} \end{bmatrix} = \begin{bmatrix} \mathbf{S} \\ \mathbf{I} \end{bmatrix}. \tag{3.3}$$

For – e.g. – elastic bodies, a generalized Hooke law couples ($\boldsymbol{\sigma}$, $\boldsymbol{\mu}$) to ($\boldsymbol{\varepsilon}$, $\boldsymbol{\kappa}$).

σ_{ik} and μ_{ik} are the asymmetric tensors of force and moment stresses. $v_k = \dot{u}_k$ is the translation velocity and $s_k = \dot{\alpha}_k$ is the angular velocity of a material point, $\rho\, v_k = p_k$ is the impulse along the path, and $\Theta\, s_k = q_k$ is the proper rotation (i.e., spin), both of which are referred to unit volume. By restricting to linearity – as we do here and in the further reasoning – the dot means the partial derivative with respect to time t.

In the kinematical equations, the asymmetric tensors κ_{ik} and ε_{ik} describe the state of deformation of the COSSERAT continuum [**4**]. B_{ik} and D_{ik} are the asymmetric tensors of dislocation density – or more precisely, disclination and dislocation density, resp. In a geometrically compatible continuum, the deformations are defined by:

$$\begin{bmatrix} \boldsymbol{\kappa} \\ \boldsymbol{\varepsilon} \end{bmatrix} = \mathrm{Grad}\begin{bmatrix} \boldsymbol{\alpha} \\ \mathbf{u} \end{bmatrix}, \tag{3.4}$$

in which α_k and u_k are the vectors of the infinitesimal rotation and displacement of a material point. In a compatible, dislocation-free continuum, (3.4) vanishes as a result of the right-hand sides in (3.2) and (3.3). One recognizes that the asymmetric tensors S_{ik} and I_{ik} in (3.3) are the measure of how many disclinations and dislocations are found in a unit volume element per unit time or leave it. One calls them the *dislocation flux densities.*

One succeeds in eliminating ($\boldsymbol{\kappa}$; $\boldsymbol{\varepsilon}$), ($\mathbf{s}$, $\mathbf{v}$) from (3.2) and (3.3) with the help of the identities (2.4), (2.5), and this leads to the equations:

$$\mathrm{Rot}\begin{bmatrix}\mathbf{S}\\ \mathbf{I}\end{bmatrix}+\begin{bmatrix}\dot{\mathbf{B}}\\ \dot{\mathbf{D}}\end{bmatrix}=0;\qquad \mathrm{Div}\begin{bmatrix}\mathbf{B}\\ \mathbf{D}\end{bmatrix}=0, \qquad (3.5),\ (3.6)$$

which are analogous to the homogeneous MAXWELL equations.

We consider the integration theory in the example of an elastic body. For this, we summarize the equations that are obtained by differentiation with respect to time; we call them the *developed equations.*

$$\text{Ia)}\quad -\begin{bmatrix}\dot{\mathbf{p}}\\ \dot{\mathbf{q}}\end{bmatrix}+\mathrm{Div}\begin{bmatrix}\boldsymbol{\sigma}\\ \boldsymbol{\mu}\end{bmatrix}=0, \qquad \begin{bmatrix}\mathbf{s}\\ \mathbf{v}\end{bmatrix}$$

$$\text{Ib)}\quad -\begin{bmatrix}\dot{\boldsymbol{\kappa}}\\ \dot{\boldsymbol{\varepsilon}}\end{bmatrix}+\mathrm{Grad}\begin{bmatrix}\mathbf{s}\\ \mathbf{v}\end{bmatrix}=0, \qquad \begin{bmatrix}\boldsymbol{\sigma}\\ \boldsymbol{\mu}\end{bmatrix}$$

$$\text{Ic)}\quad \begin{bmatrix}\dot{\mathbf{B}}\\ \dot{\mathbf{D}}\end{bmatrix}+\mathrm{Rot}\begin{bmatrix}\mathbf{S}\\ \mathbf{I}\end{bmatrix}=0. \qquad \begin{bmatrix}\boldsymbol{\varphi}\\ \boldsymbol{\Phi}\end{bmatrix}$$

They will be accompanied by equations that do not include the time derivatives, namely:

$$\text{IIa)}\quad \begin{bmatrix}\mathbf{B}\\ \mathbf{D}\end{bmatrix}-\mathrm{Rot}\begin{bmatrix}\boldsymbol{\kappa}\\ \boldsymbol{\varepsilon}\end{bmatrix}=0;\qquad \text{IIb)}\quad \mathrm{Div}\begin{bmatrix}\mathbf{B}\\ \mathbf{D}\end{bmatrix}=0.$$

To this, one adds the constitutive equations:

$$\text{IIIa)}\quad p_k=\rho\, v_k;\qquad q_k=\theta\, s_k,$$

and for elastic bodies:

$$\text{IIIb)}\quad \sigma_{ik}=\overset{1}{C}_{iklm}\,\varepsilon_{lm};\qquad \mu_{ik}=\overset{2}{C}_{iklm}\,\kappa_{lm}.$$

It now follows from Ic), and further from Ib) and Ic), that:

$$\text{IVa)}\quad \frac{\partial}{\partial t}\mathrm{Div}\begin{bmatrix}\mathbf{B}\\ \mathbf{D}\end{bmatrix}=0;\qquad \text{IVb)}\quad \frac{\partial}{\partial t}\left\{\begin{bmatrix}\mathbf{B}\\ \mathbf{D}\end{bmatrix}-\mathrm{Rot}\begin{bmatrix}\boldsymbol{\kappa}\\ \boldsymbol{\varepsilon}\end{bmatrix}\right\}=0.$$

Thus, if the equations II) are fulfilled at any initial time t_0 then they remain fulfilled for every later t on the grounds of the developed equations Ib) and Ic). One can thus specify the dislocation densities [$\mathbf{B}$, $\mathbf{D}$] at the time t_0 in the entire body, where IIb) must be fulfilled. By integrating IIa), one then obtains [**5**] the initial values of [$\boldsymbol{\kappa}$; $\boldsymbol{\varepsilon}$]. Now, in order to give some meaning to the developed equations I), one can further prescribe the space-time distribution of the dislocation flux [$\mathbf{S}$, $\mathbf{I}$] for $t \geq t_0$. Ia), Ib), together with the constitutive equation III), then define an integrable system of equations. We have thus presented the current state of dislocation theory.

The presentation of the energy theorem will be useful for later considerations. We scalar multiply the developed equations Ia) and Ib) with the factors that are next to them and add:

$$-\,(\dot{p}_k v_k + \dot{q}_k s_k) - (\dot{\kappa}_{ik}\mu_{ik} + \dot{\varepsilon}_{ik}\sigma_{ik}) + \partial_i (s_k \mu_{ik} + v_k \sigma_{ik}) = \mu_{ik} S_{ik} + \sigma_{ik} I_{ik}\,. \tag{3.7}$$

We can write this as:

$$\frac{\partial w}{\partial t} + \operatorname{div}\mathfrak{s} = l. \tag{3.8}$$

The first two terms in (3.7) are the temporal derivative of the energy w of the matter field, $\mathfrak{s}$ is the vector of the energy flux, and l is the power density that is produced from dislocation fluxes [**S**, **I**] of the matter field. The counterpart to an energy theorem for the dislocation field does not exist. The situation in the MAXWELL-LORENTZ theory is complementary. There, an energy theorem (Poynting) exists for the electromagnetic field (the analogue of the dislocation field), so there is no energy theorem for the material field in the electron. Thus, just as the electron is a foreign body in the MAXWELL-LORENTZ theory, so are the dislocations and dislocation fluxes foreign bodies in the present continuum theory. There, one has the PEACH-KOEHLER force [**9**] in the stress field of the dislocation, like the LORENTZ force in the magnetic field of the electron.

The incompleteness of the dislocation theory becomes clearer when we give the dynamical equations (3.1) the form of the inhomogeneous MAXWELL equations:

$$\mathrm{Rot}\begin{bmatrix}\boldsymbol{\varphi}\\ \boldsymbol{\Phi}\end{bmatrix} - \begin{bmatrix}\dot{\boldsymbol{\psi}}\\ \dot{\boldsymbol{\kappa}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\sigma}\\ \boldsymbol{\mu}\end{bmatrix}; \qquad \mathrm{Div}\begin{bmatrix}\boldsymbol{\psi}\\ \boldsymbol{\kappa}\end{bmatrix} = -\begin{bmatrix}\mathbf{p}\\ \mathbf{q}\end{bmatrix}. \qquad (3.9),\ (3.10)$$

Obviously, (3.1) to (3.9) and (3.10) are fulfilled identically by the stress potentials φ_{ik}, Φ_{ik}, and the impulse potentials ψ_{ik}, χ_{ik}. φ_{ik} and Φ_{ik} are the 18 stress functions that GÜNTHER [**7**] introduced. In the statics of continua with dislocations, they are tools that are often very useful, but still largely superfluous. As a counterpart, but not an analogue, in electrodynamics, one has the LORENTZ four-potential, whose vector and scalar are only mathematical quantities that serve to simplify the integrations. The analogue of the four-potential are, however, the tensors ($\boldsymbol{\kappa}$; $\boldsymbol{\varepsilon}$) and the vectors (**s**, **v**) in the equations (3.2), (3.3) of dislocation theory, which are completely determined physical state quantities here. (3.2) and (3.3) fulfill the homogeneous MAXWELL equations (3.5), (3.6) identically in ($\boldsymbol{\kappa}$; $\boldsymbol{\varepsilon}$) and (**s**, **v**).

The analogues of the stress and impulse potentials [$\boldsymbol{\varphi}$, $\boldsymbol{\Phi}$] and [$\boldsymbol{\Psi}$, $\boldsymbol{\chi}$] of dislocation theory in the inhomogeneous MAXWELL equations (3.9), (3.10) are the current potential **H** and the charge potential **D** (both of them have many names in electrodynamics), which are coupled with the vectors **B** and **E** of the magneto-electric field by constitutive equations.

The idea of MIE was to give some physical reality to the LORENTZ four-potential of the electromagnetic field and couple it to a material field of the electron by means of constitutive equations.

We extend the developed equations I) by means of (3.9) to:

$$\text{Id)} \qquad \begin{bmatrix} \dot{\boldsymbol{\psi}} \\ \dot{\boldsymbol{\kappa}} \end{bmatrix} - \text{Rot}\begin{bmatrix} \boldsymbol{\varphi} \\ \boldsymbol{\Phi} \end{bmatrix} = -\begin{bmatrix} \boldsymbol{\sigma} \\ \boldsymbol{\mu} \end{bmatrix} \qquad | \qquad \begin{bmatrix} \mathbf{S} \\ \mathbf{I} \end{bmatrix}.$$

(3.10), which contains no time derivative, comes to II) as:

$$\text{IIc)} \qquad \text{Div}\begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa} \end{bmatrix} + \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} = 0.$$

From Ia) and Id), it follows that:

$$\text{IVc)} \qquad \frac{\partial}{\partial t}\left\{\text{Div}\begin{bmatrix} \boldsymbol{\psi} \\ \boldsymbol{\kappa} \end{bmatrix} + \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix}\right\} = 0.$$

In order to obtain a complete dislocation theory, we must demand that in addition to III) there must exist constitutive equations between [$\boldsymbol{\varphi}$, $\boldsymbol{\Phi}$], [$\boldsymbol{\Psi}$, $\boldsymbol{\chi}$], on the one hand, and [**B**, **D**], [**S**, **I**], on the other – for example, the ones for elastic bodies. Later, we will see that the constitutive equations can be made essentially more general. In any event, one must demand that the still-missing constitutive equations for dislocation theory reduces the number of unknowns in the developed equations Ia to Id) in such a way that a closed systems of equations exists that is soluble when one is given the initial conditions IIa) to IIc).

4. The energy theorem

We now follow MIE's theory. In order to arrive at a starting point for it, like the constitutive equations in the dislocation theory, we demand the validity of an energy theorem of the form:

$$\frac{\partial W}{\partial t} + \text{div}\,\Sigma = 0, \tag{4.1}$$

where W is the energy density and the vector $\boldsymbol{\Sigma}$ is the energy flux. The energy shall then be localizable in space and time.

One then has (3.7). We now multiply the developed equations Ic) and Id) with the scalar factors next to them and then obtain, after an intermediate calculation:

$$\left.\begin{aligned} &(\dot{B}_{ik}\Phi_{ik} + \dot{D}_{ik}\varphi_{ik}) + (\dot{\psi}_{ik}I_{ik} + \dot{\chi}_{ik}S_{ik}) + \partial_i(\varepsilon_{irs}S_{rk}\Phi_{sk} + \varepsilon_{irs}I_{rk}\varphi_{sk}) \\ &\qquad = -(\mu_{ik}S_{ik} + \sigma_{ik}I_{ik}). \end{aligned}\right\} \tag{4.2}$$

Equation (4.2) is obviously the analogue of the POYNTING theorem in MAXWELL's theory. On the right-hand side of (4.2), one finds the power density that the dislocation field delivers to the material field.

We now add (3.7) to (4.2) and recognize the vector Σ of energy flux:

$$\Sigma_i = s_k\,\mu_{ik} + v_k\,\sigma_{ik} + \varepsilon_{irs}\,(S_{rk}\,\Phi_{sk} + I_{rk}\,\varphi_{sk}). \tag{4.3}$$

Furthermore, a comparison with our postulated energy theorem (4.1) yields:

$$\frac{\partial W}{\partial t} = (\Phi_{ik}\dot{B}_{ik} + \varphi_{ik}\dot{D}_{ik}) + (I_{ik}\dot{\psi}_{ik} + S_{ik}\dot{\chi}_{ik}) - (v_k\dot{p}_k + s_k\dot{q}_k) - (\mu_{ik}\dot{\mu}_{ik} + \sigma_{ik}\dot{\varepsilon}_{ik})\,. \tag{4.4}$$

Here, in order to make things more organized, it is recommended that one introduce the LAGRANGE density L:

$$W = -\,(p_k\,v_k + q_k\,s_k) + (I_{ik}\,\psi_{ik} + S_{ik}\,\chi_{ik}) - L, \tag{4.5}$$

such that:

$$\frac{\partial L}{\partial t} = -(p_k\dot{v}_k + q_k\dot{s}_k) - (\mu_{ik}\dot{\kappa}_{ik} + \sigma_{ik}\dot{\varepsilon}_{ik}) - (\Phi_{ik}\dot{B}_{ik} + \varphi_{ik}\dot{D}_{ik}) + (\psi_{ik}\dot{I}_{ik} + \chi_{ik}\dot{S}_{ik}) \tag{4.6}$$

contains the kinematical quantities only by way of their temporal derivatives.

The right-hand side of (4.6) must now be the time derivative of a function:

$$L = L(\mathbf{s}, \mathbf{v}, \boldsymbol{\kappa}, \boldsymbol{\varepsilon}, \mathbf{B}, \mathbf{D}, \mathbf{S}, \mathbf{I}), \tag{4.7}$$

where (4.6) must be true as a result of the relations:

$$\left.\begin{aligned} &\frac{\partial L}{\partial v_k} = -p_k, \quad \frac{\partial L}{\partial s_k} = -q_k, \quad \frac{\partial L}{\partial \kappa_{ik}} = \mu_{ik}, \quad \frac{\partial L}{\partial \varepsilon_{ik}} = \sigma_{ik}, \\ &\frac{\partial L}{\partial B_{ik}} = -\Phi_{ik}, \quad \frac{\partial L}{\partial D_{ik}} = -\varphi_{ik}, \quad \frac{\partial L}{\partial S_{ik}} = \chi_{ik}, \quad \frac{\partial L}{\partial I_{ik}} = \psi_{ik}. \end{aligned}\right\} \tag{4.8}$$

These are the constitutive equations.

5. The variational problem for stationary action

We can define the action of the field with (4.7). It is the integral extended over space and time:

$$\int L\, dx_1\, dx_2\, dx_3\, dt\,. \tag{5.1}$$

In (4.7), which includes only geometric field quantities, one must observe that **B**, **D**, **S**, **I** depend upon **s**, **v**, $\boldsymbol{\kappa}$, $\boldsymbol{\varepsilon}$ only by means of equations (3.2) and (3.3). Thus (assuming a knowledge of the material law), L becomes a function of the independent state quantities

s, **v**, $\boldsymbol{\kappa}$, $\boldsymbol{\varepsilon}$, and their spatial and temporal derivatives. The variational problem for the stationary action then has the form:

$$\delta \int L\, dx_1\, dx_2\, dx_3\, dt = \int \delta L\, dx_1\, dx_2\, dx_3\, dt\,, \tag{5.2}$$

with the LAGRANGE density:

$$L = L(s_k\,, v_k\,, \kappa_{ik}\,, \varepsilon_{ik}\,, \partial_i\, s_k\,, \partial_i\, v_k\,, \partial_i\, \kappa_{ik}\,, \partial_i\, \varepsilon_{ik}\,, \dot{\kappa}_{ik}\,, \dot{\varepsilon}_{ik}\,). \tag{5.3}$$

The field equations that our variational problem leads to have already been established with no explicit knowledge of the material law (4.8). One derives the variation of the Lagrange density from (4.6):

$$\begin{aligned} \delta L = &-(p_k\, \delta v_k + q_k\, \delta s_k) + (\mu_{ik}\, \delta\kappa_{ik} + \sigma_{ik}\, \delta\varepsilon_{tk}) \\ &- (\Phi_{ik}\, \delta B_{ik} + \varphi_{ik}\, \delta D_{ik}) + (\psi_{ik}\, \delta I_{ik} + \chi_{ik}\, \delta S_{ik}). \end{aligned} \tag{5.4}$$

Thus, from (3.2) and (3.3), one has:

$$\begin{bmatrix} \delta\mathbf{B} \\ \delta\mathbf{D} \end{bmatrix} = \mathrm{Rot} \begin{bmatrix} \delta\boldsymbol{\kappa} \\ \delta\boldsymbol{\varepsilon} \end{bmatrix}, \tag{5.5}$$

$$\begin{bmatrix} \delta\mathbf{S} \\ \delta\mathbf{I} \end{bmatrix} = \mathrm{Grad} \begin{bmatrix} \delta\mathbf{s} \\ \delta\mathbf{v} \end{bmatrix} - \frac{\partial}{\partial t} \begin{bmatrix} \delta\boldsymbol{\kappa} \\ \delta\boldsymbol{\varepsilon} \end{bmatrix}. \tag{5.6}$$

Carrying out the variation in (5.2) then delivers the field equations:

$$p_k + \partial_i \psi_{ik} = 0, \qquad (\text{for } \delta v_k), \qquad q_k + (\partial_i \chi_{ik} + \varepsilon_{klm}\, \psi_{lm}) = 0 \qquad (\text{for } \delta s_k),$$

and furthermore:

$$\begin{aligned} &\sigma_{ik} + \psi_{ik} - \varepsilon_{irs}\, \partial_r \varphi_{sk} = 0 && (\text{for } \delta\varepsilon_{tk}), \\ &\mu_{ik} + \dot{\chi}_{ik} - \varepsilon_{irs}(\partial_r \Phi_{sk} + \varepsilon_{krl}\, \varphi_{sl}) = 0 && (\text{for } \delta\kappa_{tk}). \end{aligned}$$

However, these are the inhomogeneous MAXWELL dynamical equations (3.9) and (3.10).

Naturally, one can define the LAGRANGE density L^* that is conjugate to L under a LEGENDRE transformation, and which is then a function of the dynamical field quantities (**p**, **q**), ($\boldsymbol{\sigma}$, $\boldsymbol{\mu}$), ($\boldsymbol{\varphi}$, $\boldsymbol{\Phi}$), ($\boldsymbol{\Psi}$, $\boldsymbol{\chi}$), but from (3.9) and (3.10), this reduces to a function of the impulse potential ($\boldsymbol{\Psi}$, $\boldsymbol{\chi}$), the stress potential ($\boldsymbol{\varphi}$, $\boldsymbol{\Phi}$), and their spatial and temporal derivatives. Carrying out the variations then delivers the kinematical equations (3.2) and (3.3).

In [**4**], I set down how one is to interpret the GÜNTHER stress functions ($\boldsymbol{\varphi}$, $\boldsymbol{\Phi}$) of the COSSERAT continuum intuitively. As of yet, I have still not developed an explanation for how the stress potentials ($\boldsymbol{\varphi}$, $\boldsymbol{\Phi}$) of our theory do work on the moving dislocations as physical quantities.

However, let me be permitted the observation that the tensorial MAXWELL equations of dislocation theory that were presented here are essentially more complicated than the ones for electrodynamics. After all, understanding all of the consequences of the vectorial MAXWELL equations of electrodynamics has already required decades of work.

6. Outlook

The continuum theory of dislocations that was propounded here is the analogue of MIE's electrodynamics, and like it, is a logically closed theory. In general, it is first animated by the material law that couples the kinematical and dynamical field quantities with each other. In order for one to make a series of statements about the general form of these constitutive equations, *a priori*, one would ultimately be led to experiment with the material from which the continuum is composed, as one would be for any macroscopic continuum theory. However, one would then set foot upon an entirely unexplored territory.

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Summary

There is a mathematical analogy between the continuum theory of moving dislocations and electrodynamics. This remarkable analogy points out that the theory of dislocations is incomplete with regard to the missing constitutive equations. The generalized electrodynamical theory of MIE shows a way to complete the theory of moving dislocations that finally leads to the formulation of a Lagrangian density.

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