THE

PRINCIPLES OF DYNAMICS

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FOREWORD

The content of the present little book is based upon a lecture that I gave as an extension of the general lecture on mechanics in Spring-Summer 1918. My goal was to show how the various principles of dynamics develop from each other and are connected together. In particular, it seemed to me that a simple, but rigorous, explanation of how the various forms of the principle of least action come about would be desirable today since that principle has proved to be a powerful heuristic force in **Einstein**'s theory of gravitation. I have entered into the **Hamilton-Jacobi** theory because it has likewise approached the sphere of interest of physicists by the work of **Schwarzschild**, **Sommerfeld**, and **Epstein**.

I wish to express my heartfelt thanks to **Adolf Kneser** for his interest and advice.

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C. Schaefer.

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As is known, the equations of equilibrium for a system of *n* completely-free mass-points (so one with 3*n* degrees of freedom) read:

$$(1) \qquad \begin{cases} X_\nu = 0, \\ Y_\nu = 0, \\ Z_\nu = 0, \end{cases} \qquad (\nu = 1, 2, \ldots, n),$$

in which X_ν, Y_ν, Z_ν mean the rectangular components of the total force that acts upon the ν^{th} mass-particle. Those three equations admit a simple combination when we multiply each of equations (1) by an arbitrary function, which we would like to denote by δx_ν, δy_ν, δz_ν, respectively, and then add the entire system:

$$(2) \qquad \sum_{\nu=1}^{N} (X_\nu\, \delta x_\nu + Y_\nu\, \delta y_\nu + Z_\nu\, \delta z_\nu) = 0 .$$

Due to the arbitrariness in the choice of the 3*n* functions δx_ν, δy_ν, δz_ν, that equation is *completely equivalent* to the system (1); it says nothing more or less than the latter. That is because we can choose, e.g., $\delta x_1 \neq 0$, and choose all of the remaining δx_ν, as well as all of δy_ν and δz_ν, to be equal to zero. Equation (2) will then produce the result:

$$X_1\, \delta x_1 = 0 ,$$

or since we have $\delta x_1 \neq 0$, by assumption, it will follow that $X_1 = 0$. We will once more obtain all equations in the system (1) in succession in that way.

If one interprets the quantities δx_ν, δy_ν, δz_ν as the components of a displacement that one imagines to be infinitely small, and which all mass-points suffer, then the expression $X_\nu\, \delta x_\nu + Y_\nu\, \delta y_\nu + Z_\nu\, \delta z_\nu$ will represent the work δA_ν that the forces X_ν, Y_ν, Z_ν do under the displacement that one imagines. The left-hand side of (2), which can be written as $\sum_\nu \delta A_\nu = \delta A$, will then represent the total work δA that is done by all forces when each of the *n* mass-points suffers a displacement $\delta \mathfrak{s}_\nu$. We can then formulate the equilibrium condition as follows:

For a completely-free system of mass-points, the work done by an infinitely-small displacement of the system will be equal to zero in equilibrium.

Equation (2) represents the simplest case of the so-called *principle of virtual displacements*. Namely, the imagined displacements that are assigned to the system, which one cares to denote by the symbol δ from the calculus of variations, in contrast to the *actual displacements*, which we will characterize by the symbol "d," will be called by the medieval term *virtual* (i.e., *possible*) displacements, in order to emphasize the fact that they differ from the *actual* displacements. That definition, which is sufficient for the present purposes, must be subjected to suitable restrictions in what follows. Here, it is important to point out that in the case of complete freedom of motion for the mass-system, which we have indeed assumed here, *any conceivable displacement is referred to as virtual*. In particular, the *actual* displacements are also "possible," i.e., *virtual* ones.

In the case of a completely unconstrained system that was just considered, the principle of virtual displacements that is included in equation (2) says nothing more than the system of equations (1) and has no advantage over the latter. Things will be different when we shift our attention to the fact that the freedom of motion might be restricted by certain equations of constraint (there might be m of them with $m < 3n$) that exist between the coordinates x_1, y_1, z_1, x_2, y_2, z_2, …, x_n, y_n, z_n. For instance, we might prescribe that each mass-point might displace only on a certain surface, and we would then ask what equilibrium in the system, thus-restricted, would be.

There would then exist equations of the form:

$$(3) \qquad \varphi_\kappa (x_1, y_1, z_1, \ldots, x_\nu, y_\nu, z_\nu, \ldots, x_n, y_n, z_n) = 0 \qquad (\kappa = 1, 2, \ldots, m).$$

What form will the equilibrium conditions take now?

The fact that this problem is essentially more difficult than the previous one is illuminated when one makes the following clear: Instead of restricting the degrees of freedom of the mass-points by the constraint equations (3), one can also combine the old forces X_ν, Y_ν, Z_ν with suitably-chosen new ones Ξ_ν, H_ν, Z_ν, with the stipulation that the additional forces should be arranged such that they will prevent the mass-points from leaving the prescribed surface. Their ultimate effect should then be to replace the constraint equations (3). One will indeed know the effect of those additional forces then, but not their magnitudes. In order to have a suitable name for those additional forces, we would like to call them the *forces of constraint*.

For the sake of simplicity, we shall first consider a mass-point (mass m_ν, coordinates x_ν, y_ν, z_ν) and a constraint equation:

$$\varphi_\kappa (x_\nu, y_\nu, z_\nu) = 0 .$$

We can always go on to the general case of several mass-points and constraints later by summation then. The simple case that we have taken up here makes it possible for us to give it a geometric interpretation in ordinary three-dimensional space. The constraint equation $\varphi_\kappa = 0$ represents a surface in it upon which the point must remain. If we replace the constraint equation $\varphi_\kappa = 0$ with the constraint forces Ξ_ν, H_ν, Z_ν then *our mass-point will once more be completely free*, except that it will no longer be acted upon by the forces X_ν, Y_ν, Z_ν, but by the forces $X_\nu + \Xi_\nu$, $Y_\nu + \mathrm{H}_\nu$, $Z_\nu + \mathrm{Z}_\nu$. We can then apply the equilibrium condition (1) for a free system and have:

$$(4) \qquad X_\nu + \Xi_\nu = 0 , \quad Y_\nu + \mathrm{H}_\nu = 0 , \quad Z_\nu + \mathrm{Z}_\nu = 0 ,$$

or when we go over to the domain of the principle of virtual displacements, i.e., we subject our mass-point to a displacement $\overline{\delta \mathfrak{s}_\nu}$, then we will find from (2) that:

$$(X_\nu + \Xi_\nu)\,\overline{\delta x_\nu} + (Y_\nu + \mathrm{H}_\nu)\,\overline{\delta y_\nu} + (Z_\nu + \mathrm{Z}_\nu)\,\overline{\delta z_\nu} = 0\,, \tag{5}$$

or also:

$$X_\nu\,\overline{\delta x_\nu} + Y_\nu\,\overline{\delta y_\nu} + Z_\nu\,\overline{\delta z_\nu} = -\,(\Xi_\nu\,\overline{\delta x_\nu} + \mathrm{H}_\nu\,\overline{\delta y_\nu} + \mathrm{Z}_\nu\,\overline{\delta z_\nu})\,, \tag{6}$$

i.e., *the work done by the explicit forces* Ξ_ν, H_ν, Z_ν *under the imagined motion will be equal and opposite to the work done by the forces of constraint* Ξ_ν, H_ν, Z_ν *in equilibrium.* That equation is really of no use, since the Ξ_ν, H_ν, Z_ν are generally unknown. Therefore, in order to go any further so that we can make a definite statement about equilibrium in this case of restricted degrees of freedom, we must add something new. That new thing takes the form of adding a further statement about the nature of the forces of constraint that can be quite plausible in certain concrete cases, but not proved in general, and it can therefore be introduced only as a *hypothesis* in the general case. We then address the following situation: If we observe that the forces of constraint Ξ_ν, H_ν, Z_ν are only supposed to *prevent the mass-point from leaving the surface* then that suggests that we might assume that the forces of constraint are directed normal to the surface $\varphi_\kappa = 0$, i.e., that Ξ_ν, H_ν, Z_ν should be proportional to the direction cosines of the surface:

$$\frac{\dfrac{\partial\varphi_\kappa}{\partial x_\nu}, \dfrac{\partial\varphi_\kappa}{\partial y_\nu}, \dfrac{\partial\varphi_\kappa}{\partial z_\nu}}{\sqrt{\left(\dfrac{\partial\varphi_\kappa}{\partial x_\nu}\right)^2 + \left(\dfrac{\partial\varphi_\kappa}{\partial y_\nu}\right)^2 + \left(\dfrac{\partial\varphi_\kappa}{\partial z_\nu}\right)^2}}\,.$$

If λ_κ means the proportionality factor then from the assumption above, the forces of constraint can be written:

$$\Xi_\nu = \frac{\partial\varphi_\kappa}{\partial x_\nu}\,, \qquad \mathrm{H}_\nu = \frac{\partial\varphi_\kappa}{\partial y_\nu}\,, \qquad \mathrm{Z}_\nu = \frac{\partial\varphi_\kappa}{\partial z_\nu}\,.$$

The equilibrium conditions for our mass-point would then be:

$$\left\{\begin{aligned} X_\nu + \lambda_\kappa \frac{\partial\varphi_\kappa}{\partial x_\nu} &= 0\,, \\ Y_\nu + \lambda_\kappa \frac{\partial\varphi_\kappa}{\partial y_\nu} &= 0\,, \\ Z_\nu + \lambda_\kappa \frac{\partial\varphi_\kappa}{\partial z_\nu} &= 0\,, \end{aligned}\right. \tag{7}$$

or also when we use (6) to go over to the principle of virtual displacements:

$$(8) \qquad X_\nu \overline{\delta x_\nu} + Y_\nu \overline{\delta y_\nu} + Z_\nu \overline{\delta z_\nu} = -\lambda_\kappa \left(\frac{\partial \varphi_\kappa}{\partial x_\nu} \overline{\delta x_\nu} + \frac{\partial \varphi_\kappa}{\partial y_\nu} \overline{\delta y_\nu} + \frac{\partial \varphi_\kappa}{\partial z_\nu} \overline{\delta z_\nu} \right).$$

The $\overline{\delta x_\nu}$, $\overline{\delta y_\nu}$, $\overline{\delta z_\nu}$ in that are *completely independent of each other*, since we have indeed suppressed the equation of constraint. In order to distinguish those completely-free displacements $\overline{\delta x_\nu}$, … from more specialized ones that are likewise introduced, we have provided them with an overbar. The four unknowns x_ν, y_ν, z_ν, λ_κ can be determined from the equilibrium condition (7) or (8), in conjunction with the equation of the surface $\varphi_\kappa = 0$. We then see how the introduction of our hypothesis proves to be practicable.

Up to now, as we have emphasized, the quantities $\overline{\delta x_\nu}$, $\overline{\delta y_\nu}$, $\overline{\delta z_\nu}$ could be chosen to be completely arbitrary and independent of each other. For that reason, the right-hand side of (8) cannot be made to vanish, in general. However, if that right-hand side of (8) can be made to vanish then we would gain a great advantage due to the fact that we would get the same expression for the equilibrium condition in the case of restricted degrees of freedom that we would get in the case of a completely-free system. That is because (8) and (2) would be equal to each other in that case.

Now, how can one succeed in making the right-hand side of (8) equal to zero? Obviously, to that end, the $\overline{\delta x_\nu}$, $\overline{\delta y_\nu}$, $\overline{\delta z_\nu}$ cannot be completely arbitrary, but the components of the displacement must be suitably restricted, and indeed as follows:

Before the displacement, the coordinates x_ν, y_ν, z_ν of the mass-point will obey the condition $\varphi_\kappa = 0$ that was imposed upon them. By contrast, up to now, the coordinates $x_\nu + \overline{\delta x_\nu}$, $y_\nu + \overline{\delta y_\nu}$, $z_\nu + \overline{\delta z_\nu}$ did not need to do that *after* the displacement since I can indeed choose the displacement to be, say, perpendicular to the surface so it points away from the surface. That is because the displacement $\overline{\delta x_\nu}$, … *was chosen to be completely free*. *I shall now restrict my choice of displacement in such a way that I demand that the mass-point should also still lie on the surface after the displacement*, i.e., If I now impose the condition upon the displacement $\overline{\delta x_\nu}$, … that it must be compatible with the equilibrium conditions then we will have:

$$(9) \qquad \begin{cases} \varphi_\kappa(x_\nu, y_\nu, z_\nu) = 0, \\ \varphi_\kappa(x_\nu + \delta x_\nu, y_\nu + \delta y_\nu, z_\nu + \delta z_\nu) = 0, \end{cases}$$

if the displacements, thus-specialized, are denoted by δx_ν, δy_ν, δz_ν (with no overbar). Thus, when we develop the second equation in a **Taylor** series and truncate it after the linear terms, we will get:

$$\varphi_\kappa(x_\nu, y_\nu, z_\nu) + \frac{\partial \varphi_\kappa}{\partial x_\nu} \delta x_\nu + \frac{\partial \varphi_\kappa}{\partial y_\nu} \delta y_\nu + \frac{\partial \varphi_\kappa}{\partial z_\nu} \delta z_\nu = 0 .$$

Finally, upon subtracting the first equation (9) from the one that was just obtained, we will get:

$$\frac{\partial\varphi_\kappa}{\partial x_\nu}\,\delta x_\nu + \frac{\partial\varphi_\kappa}{\partial y_\nu}\,\delta y_\nu + \frac{\partial\varphi_\kappa}{\partial z_\nu}\,\delta z_\nu = 0 \,. \tag{10}$$

We can now say that: *We choose our displacements components* δx_ν, δy_ν, δz_ν *in such a way that they will obey equation* (10). Naturally, the right-hand side of equation (8) would vanish then, as we will get the principle of virtual displacements in the *old* form:

$$X_\nu\,\delta x_\nu + Y_\nu\,\delta y_\nu + Z_\nu\,\delta z_\nu = 0 \,, \tag{11}$$

but it will no longer be true for *arbitrary* displacements $\overline{\delta x_\nu}$, $\overline{\delta y_\nu}$, $\overline{\delta z_\nu}$, but only for ones δx_ν, δy_ν, δz_ν that obey the varied constraint equation $\varphi_\kappa = 0$, i.e., equation (10).

From now on, we shall restrict the term "virtual displacement" in the following way:

We now understand the virtual displacements to be only those imagined displacements that are compatible with the equilibrium conditions. In what follows, we shall always denote them by the symbol δ.

The equilibrium conditions for our mass-point are included in (11), which is valid simultaneously with (10). Naturally, we can now no longer conclude that $X_\nu = Y_\nu = Z_\nu = 0$ from (11), i.e., the principle of virtual displacements, as we could in the case of complete degrees of freedom, since it would indeed also be false, according to equation (7), but rather since the three displacement components are no longer independent of each other, we must proceed as follows: We multiply (10) by an arbitrary factor λ_κ and add it to (11). It would then follow that:

$$\left(X_\nu + \lambda_\kappa\,\frac{\partial\varphi_\kappa}{\partial x_\nu}\right)\delta x_\nu + \left(Y_\nu + \lambda_\kappa\,\frac{\partial\varphi_\kappa}{\partial y_\nu}\right)\delta y_\nu + \left(Z_\nu + \lambda_\kappa\,\frac{\partial\varphi_\kappa}{\partial z_\nu}\right)\delta z_\nu = 0 \,. \tag{12}$$

From equation (10), we can regard one of the three displacements in that as being determined by the other two, say, δx_ν is determined in terms of δy_ν and δz_ν. Therefore, whereas δy_ν and δz_ν can be chosen arbitrarily, δx_ν will be determined by them from (10). We now choose λ_κ, which is still undetermined, but it can still be chosen freely, in such a way that the coefficient of δx_ν in equation (12) will vanish. All that will then remain are the terms with mutually-independent displacements δy_ν and δz_ν, and their coefficients must vanish in their own right due to just that independence of δy_ν and δz_ν. We will then get the following three equations, which are completely symmetric, although the argument that leads to the first one is different from the one that leads to the last two:

$$X_\nu + \lambda_\kappa\,\frac{\partial\varphi_\kappa}{\partial x_\nu} = 0 \,,$$

$$Y_\nu + \lambda_\kappa\,\frac{\partial\varphi_\kappa}{\partial y_\nu} = 0 \,,$$

$$Z_\nu + \lambda_\kappa \frac{\partial \varphi_\kappa}{\partial z_\nu} = 0 \, .$$

They are once more equations (7), which suffice to determine the equilibrium state in conjunction with constraint equation $\varphi_\kappa = 0$, as was mentioned above.

What was carried out for *one* mass-point and *one* constraint equation can be easily generalized to n mass-points and m constraint equations. A geometric representation is also possible then that is completely analogous to the one above, except that we must now go to $3n$-dimensional space, i.e., we must regard the coordinates (x_1, y_1, z_1, …, x_ν, y_ν, z_ν, …, x_n, y_n, z_n) that determine the configuration of the system as the coordinates of *one* point in $3n$-dimensional space. Every constraint equation of the form $\varphi_\kappa (x_1, \ldots, z_n) = 0$ will then represent a *hypersurface* in that space, and when m such constraint equations are present, that will mean that the point (x_1, y_1, z_1, …, x_n, y_n, z_n) that represents the configuration of the system must be found on the common intersection of those m hypersurfaces. Moreover, we can then proceed precisely as we did in the foregoing, and indeed we can even use the same words as before.

Thus, if λ_κ ($\kappa = 1, 2, \ldots, m$) denote the so-called ***Lagrange** multipliers* then we will get:

$$\sum_{\nu=1}^{n} (X_\nu \, \delta x_\nu + Y_\nu \, \delta y_\nu + Z_\nu \, \delta z_\nu) = 0 \tag{13}$$

as the equilibrium condition for those displacements δx_ν, δy_ν, δz_ν that satisfy the following equations:

$$\left\{ \begin{array}{lllllllll} \dfrac{\partial \varphi_1}{\partial x_1}\delta x_1 & + \dfrac{\partial \varphi_1}{\partial y_1}\delta y_1 & + \dfrac{\partial \varphi_1}{\partial z_1}\delta z_1 & + \cdots & + \dfrac{\partial \varphi_1}{\partial x_n}\delta x_n & + \dfrac{\partial \varphi_1}{\partial y_n}\delta y_n & + \dfrac{\partial \varphi_1}{\partial z_n}\delta z_n & = 0 \, , \\ \dfrac{\partial \varphi_\kappa}{\partial x_1}\delta x_1 & + & & & & & + \dfrac{\partial \varphi_\kappa}{\partial z_n}\delta z_n & = 0 \, , \\ \dfrac{\partial \varphi_m}{\partial x_1}\delta x_1 & + & & & & & + \dfrac{\partial \varphi_m}{\partial z_n}\delta z_n & = 0 \, , \end{array} \right. \tag{14}$$

i.e., more briefly: *for all virtual displacements* (in the sense that was just established). If equations (14) were successively extended with the factors λ_1, …, λ_m and added to (13) then that would produce the equation:

$$\sum_{\nu=1}^{n} \left[(X_\nu \, \delta x_\nu + Y_\nu \, \delta y_\nu + Z_\nu \, \delta z_\nu) + \sum_{\kappa=1}^{m} \lambda_\kappa \left(\frac{\partial \varphi_\kappa}{\partial x_\nu} \, \delta x_\nu + \frac{\partial \varphi_\kappa}{\partial y_\nu} \, \delta y_\nu + \frac{\partial \varphi_\kappa}{\partial z_\nu} \, \delta z_\nu \right) \right] = 0 \, ,$$

from which, the same argument that we applied to equation (11) would produce the following system:

$$(15)\qquad \left\{\begin{aligned} X_\nu + \sum_{\kappa=1}^{m} \lambda_\kappa \frac{\partial \varphi_\kappa}{\partial x_\nu} &= 0, \\ Y_\nu + \sum_{\kappa=1}^{m} \lambda_\kappa \frac{\partial \varphi_\kappa}{\partial y_\nu} &= 0, \\ Z_\nu + \sum_{\kappa=1}^{m} \lambda_\kappa \frac{\partial \varphi_\kappa}{\partial z_\nu} &= 0. \end{aligned}\right. \qquad (\nu = 1, 2, \ldots, n)$$

Together with the m constraint equations $\varphi_\kappa = 0$ ($\kappa = 1, 2, \ldots, m$), they will suffice to determine the ($3n + m$) unknowns ($x_1, y_1, z_1, \ldots, x_n, y_n, z_n, \lambda_1, \ldots, \lambda_m$). Equilibrium is then established by the formula for the principle of virtual displacements.

The constraint equations $\varphi_\kappa (x_1, \ldots, z_n) = 0$ that were considered here do not include time explicitly, and for that reason, **Boltzmann** called them *scleronomic* equations, in order to distinguish them from equations of the form $\varphi_\kappa (x_1, \ldots, z_n, t) = 0$, which **Boltzmann** called *rheonomic*, due to the explicit appearance of time in them. Both types of equation have in common that they represent relations between the finite quantities $x_1, \ldots, z_n$, and possibly time t. However, more general forms that represent relations between the differentials $dx_1, \ldots, dz_n$, and possibly the time differential dt are conceivable:

$$(16)\qquad \sum_{\nu=1}^{n} (a_{\kappa\nu}\, dx_\nu + b_{\kappa\nu}\, dy_\nu + c_{\kappa\nu}\, dz_\nu) = 0$$

and

$$(17)\qquad \sum_{\nu=1}^{n} (a'_{\kappa\nu}\, dx_\nu + b'_{\kappa\nu}\, dy_\nu + c'_{\kappa\nu}\, dz_\nu) = 0 .$$

In case those equations are *not integrable*, i.e., the coefficients a, b, c (a', b', c', resp.) do not fulfill the integrability conditions, the latter equations cannot be brought into the form that was considered up to now. One distinguishes the two forms of the constraint equations from each other by the names *holonomic* and *non-holonomic*, which go back to **H. Hertz**. (The terminology goes back to the Greek roots *όλος* = *whole* and *νόμος* = *law*.) Equation (16) will then be referred to as *non-holonomic-scleronomic*, while equation (17) is *non-holonomic-rheonomic*.

Up to now, we have exhibited the principle of virtual displacements only for holonomic-scleronomic constraint equations, and in the following section, we will investigate how it is formulated for non-holonomic and rheonomic ones. That will come down to establishing what we understand virtual displacements to mean in that case.

§ 2.

The principle of virtual displacements for rheonomic and non-holonomic equations of constraint.

We next ask what the physical meaning of a holonomic and rheonomic equation of constraint:

$$\varphi_\kappa (x_1, y_1, z_1, \ldots, x_n, y_n, z_n, t) = 0 \tag{18}$$

might be. Obviously, for every constant value of t, it represents a *hypersurface* in $3n$-dimensional space. If we let t vary then we will get a surface moving in that space. A simple example of that is the following one: Let a mass-point be constrained to move on the surface of a sphere whose center moves with constant velocity c parallel to the x-axis. We will examine that example in more detail later. *Is the principle of virtual displacements also valid here in the form* (13)?

In order to answer that question, we once more start from the fact that the equations of constraint can be replaced with forces of constraint Ξ_ν, H_ν, Z_ν. We again make the hypothesis that the force of constraint is perpendicular to the surface, i.e., that Ξ_ν, H_ν, Z_ν are proportional to the direction cosines of the surface. For one mass-point and one equation of constraint, we will then get the old relations:

$$\Xi_\nu = \lambda_\kappa \frac{\partial \varphi_\kappa}{\partial x_\nu}, \quad \mathrm{H}_\nu = \lambda_\kappa \frac{\partial \varphi_\kappa}{\partial y_\nu}, \quad \mathrm{Z}_\nu = \lambda_\kappa \frac{\partial \varphi_\kappa}{\partial z_\nu}.$$

The equilibrium condition:

$$X_\nu + \lambda_\kappa \frac{\partial \varphi_\kappa}{\partial x_\nu} = 0, \quad Y_\nu + \lambda_\kappa \frac{\partial \varphi_\kappa}{\partial y_\nu} = 0, \qquad Z_\nu + \lambda_\kappa \frac{\partial \varphi_\kappa}{\partial z_\nu} = 0 \tag{19}$$

will then follow in this case, just as before, or when we focus upon a completely-arbitrary displacement $\overline{\delta \mathfrak{s}_\nu}$ with the independent components $\overline{\delta x_\nu}$, $\overline{\delta y_\nu}$, $\overline{\delta z_\nu}$:

$$X_\nu \overline{\delta x_\nu} + Y_\nu \overline{\delta y_\nu} + Z_\nu \overline{\delta z_\nu} = -\lambda_\kappa \left(\frac{\partial \varphi_\kappa}{\partial x_\nu} \overline{\delta x_\nu} + \frac{\partial \varphi_\kappa}{\partial y_\nu} \overline{\delta y_\nu} + \frac{\partial \varphi_\kappa}{\partial z_\nu} \overline{\delta z_\nu} \right). \tag{20}$$

Just as before, we will now return to the old form (13) of the principle of virtual displacements when we restrict the allowable displacements in such a way that they will satisfy the equation:

(21) $$\frac{\partial \varphi_\kappa}{\partial x_\nu}\delta x_\nu + \frac{\partial \varphi_\kappa}{\partial y_\nu}\delta y_\nu + \frac{\partial \varphi_\kappa}{\partial z_\nu}\delta z_\nu = 0 ,$$

in which we now drop the overbar again.

Those constraint equations arise formally in such a way that one varies the constraint equation $\varphi_\kappa(x_1, y_1, z_1, t) = 0$ *while holding time constant*. We then establish that *we understand rheonomic constraints to mean virtual displacements that obey* (21), *i.e., the varied constraint equation* $\varphi_\kappa(x_1, y_1, z_1, t) = 0$ *when time is not varied*. The old equation:

(22) $$X_\nu\,\delta x_\nu + Y_\nu\,\delta y_\nu + Z_\nu\,\delta z_\nu = 0 ,$$

in conjunction with (21), will again produce the equilibrium conditions.

That result generalizes directly from *one* mass-point and *one* constraint to a system of *n* mass-points and *m* constraints in the way that was given in the first section, which does not need to be specified in more detail here. For us, it is enough that we can formulate the result once more as:

For holonomic constraints, whether scleronomic or rheonomic, the principle of virtual displacements is valid in the form (13) *for all virtual displacements that are defined by equations* (14) *in each case.*

On first glance, it might seem remarkable that time does not need to be varied along with everything else, because time will naturally be varied by an actual displacement, which would indeed result in time. If we denote the components of such a thing (in the simple case of *one* mass-point and *one* constraint) by, say, dx_ν, dy_ν, dz_ν then that must naturally correspond to the equation:

(23) $$\frac{\partial \varphi_\kappa}{\partial x_\nu}dx_\nu + \frac{\partial \varphi_\kappa}{\partial y_\nu}dy_\nu + \frac{\partial \varphi_\kappa}{\partial z_\nu}dz_\nu + \frac{\partial \varphi_\kappa}{\partial t}dt = 0$$

since the surface $\varphi_\kappa(x_\nu, y_\nu, z_\nu, t) = 0$ will indeed advance accordingly in the time interval dt. That implies that *here* the actual displacements *do not* belong with the virtual ones, although that was the case for scleronomic constraints. The confusion will be resolved, moreover, when we consider the fact that in the case of rheonomic constraints, the partial derivatives $\frac{\partial \varphi_\kappa}{\partial x_\nu}, \frac{\partial \varphi_\kappa}{\partial y_\nu}, \ldots$ *generally include time as a parameter, so it is by no means true that the system must obey the same constraints in the scleronomic and rheonomic cases*.

The aforementioned example will make the situation clearer, in which one asks what equilibrium would be for a *point* that is supposed to remain on a sphere of radius *R* about the point (x_0, y_0, z_0) when gravity acts upon it at the same time. Let the sphere be initially at rest, so the constraint is scleronomic. It the *z*-axis points vertically *upwards* then the explicit forces will be:

(24) $$X = Y = 0\,, \qquad Z = -\,m\,g\,,$$

and the constraint will further read:

(25) $$\varphi\,(x, y, z) \equiv (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 - R^2 = 0\,.$$

The virtual displacements are then subject to the following constraint here:

(26) $$(x - x_0)\;\delta x + (y - y_0)\;\delta y + (z - z_0)\;\delta z = 0\,.$$

Along with a factor λ, in conjunction with the principle of virtual displacements that was formulated in (13), that will yield the equilibrium conditions:

(27) $$\begin{cases} 0 + \lambda\,(x - x_0) = 0, \\ 0 + \lambda\,(y - y_0) = 0, \\ -m\,g + \lambda\,(z - z_0) = 0. \end{cases}$$

The equilibrium position $(\bar{x}, \bar{y}, \bar{z})$ is calculated from that as follows: The first two equations in (27) imply that:

(28.a) $$\begin{cases} \bar{x} = x_0\,, \\ \bar{y} = y_0\,. \end{cases}$$

Thus, from the constraint equation (25), one further has:

(28.b) $$\bar{z} - z_0 = \pm\,R\,,$$

i.e., the point is at either highest or lowest point in the sphere at rest, which is explained immediately. (Naturally, the equilibrium is labile in the first case, which is established by a special examination in each case.) With those values of $\bar{x}$, $\bar{y}$, $\bar{z}$, one finally finds the value of λ from the third equation in (27):

(28.c) $$\lambda = \pm\,\frac{m\,g}{R}\,,$$

in which the upper or lower sign will be true according to whether the mass-point is at the highest or lowest point, resp.

The center of the sphere might now move along the positive x-axis with a velocity of c. The constraint will then be rheonomic, and indeed one will have:

(29) $$\varphi\,(x, y, z, t) \equiv (x - x_0 - c\,t)^2 + (y - y_0)^2 + (z - z_0)^2 - R^2 = 0\,.$$

From our convention, the virtual displacements are subject to the following constraint here:

$$(x - x_0 - c\,t)\,\delta x + (y - y_0)\,\delta y + (z - z_0)\,\delta z = 0\,, \tag{30}$$

and although time is not also varied, we see that this says something totally different from the constraint (26) in the scleronomic case. Together with the principle of virtual displacements that was formulated in (13), along with a factor λ, we will get the following equilibrium equations here:

$$\left\{ \begin{aligned} 0 + \lambda\,(x - x_0 - ct) &= 0, \\ 0 + \lambda\,(y - y_0) &= 0, \\ -m\,g + \lambda\,(y - y_0) &= 0, \end{aligned} \right. \tag{31}$$

which correspondingly differ from (27).

The equilibrium position $(\bar{x}, \bar{y}, \bar{z})$ and the factor λ follow in the same way as before:

$$\left\{ \begin{aligned} \bar{x} &= x_0 + ct\,, \\ \bar{y} &= y_0\,, \\ \bar{z} - z_0 &= \pm R\,, \\ \lambda &= \pm \frac{m\,g}{R}\,, \end{aligned} \right. \tag{32}$$

i.e., the point can once more be at the highest or lowest point of the moving sphere in equilibrium.

We shall now move on to the non-holonomic (scleronomic or rheonomic) constraints. It is easy to see now that the principle of virtual displacements will again be valid in the form (13) when we define the virtual displacements as follows: They must obey the constraints:

(33) for scleronomic constraints: $\sum_\nu (a_{\kappa\nu}\,\delta x_\nu + b_{\kappa\nu}\,\delta y_\nu + c_{\kappa\nu}\,\delta z_\nu) = 0\,,$

(34) ″ rheonomic ″ $\sum_\nu (a'_{\kappa\nu}\,\delta x_\nu + b'_{\kappa\nu}\,\delta y_\nu + c'_{\kappa\nu}\,\delta z_\nu) = 0\,.$

Those equations will be obtained from equations (16) and (17), which typically represent non-holonomic constraints, when one switches the symbol "*d*" with "δ" and sets *dt* equal to zero, i.e., time is not varied. Namely, that will follow directly from the fact that in our previous explanation of the equations of constraint, we did not actually use them in their "holonomic" form $\varphi_\kappa(x_1, \ldots, z_n, t) = 0$, but only in their *varied* form, in which they represented differential relations between the differentials (variations, resp.). The fact that they fulfilled the integrability conditions was not used at all.

At the same time, let it be remarked here that such non-holonomic constraints always appear when one deals with the rolling of spheres and similar bodies on a certain surface, e.g., a plane. If the surface on which the sphere should roll moves in space then we would simultaneously have a non-holonomic-rheonomic constraint.

As the final result, we can then establish that:

The principle of virtual displacements is true for every type of constraint – holonomic, non-holonomic, rheonomic, scleronomic – in the form (13), *in which virtual displacements are understood in the most general case to mean that they satisfy equations of the form* (34).

That is because the non-holonomic, rheonomic equations of constraint (17) reduce to scleronomic ones in the special case in which time does not appear explicitly and to holonomic ones when the integrability conditions are fulfilled.

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§ 3.

D'Alembert's principle.

As is known from **Newton**'s second axiom, the equations of motion for a system of n completely-free mass-points (so $3n$ degrees of freedom) read:

$$\text{(35)} \qquad \left\{ \begin{aligned} X_\nu - m_\nu\,\ddot{x}_\nu &= 0, \\ Y_\nu - m_\nu\,\ddot{y}_\nu &= 0, \\ Z_\nu - m_\nu\,\ddot{z}_\nu &= 0. \end{aligned} \right. \qquad (\nu = 1, 2, \ldots, n)$$

If we denote the left-hand sides briefly by X'_ν, Y'_ν, Z'_ν then we can write those equations as:

$$\text{(36)} \qquad \left\{ \begin{aligned} X'_\nu &= 0, \\ Y'_\nu &= 0, \\ Z'_\nu &= 0. \end{aligned} \right. \qquad (\nu = 1, 2, \ldots, n)$$

They then take the same form as equations (1) for equilibrium in a free system of n mass-points, as we considered it in § **1**.

The fruitful interpretation of the equations of motion, for which we have **d'Alembert** to thank, is based upon this formal agreement between equations (36) and (1), and is what goes by the name of ***d'Alembert's** principle*.

Obviously, one can formulate (36) as:

Every mass-point that exhibits an accelerated motion under the influence of the explicit forces X_ν, Y_ν, Z_ν will be brought into equilibrium under the common influence of the forces $X_\nu - m_\nu\,\ddot{x}_\nu = X'_\nu$, $Y_\nu - m_\nu\,\ddot{y}_\nu = Y'_\nu$, $Z_\nu - m_\nu\,\ddot{z}_\nu = Z'_\nu$,

or:

If one adds the so-called d'Alembertian inertial forces $-m_\nu\,\ddot{x}_\nu$, $-m_\nu\,\ddot{y}_\nu$, $-m_\nu\,\ddot{z}_\nu$ to the impressed forces X_ν, Y_ν, Z_ν then those forces taken together, i.e., the forces $X_\nu - m_\nu\,\ddot{x}_\nu$, $Y_\nu - m_\nu\,\ddot{y}_\nu$, $Z_\nu - m_\nu\,\ddot{z}_\nu$, will keep the mass-point in equilibrium.

D'Alembert's principle then reduces the problems of dynamics to problems of equilibrium.

Now, in his *Mécanique analytique*, **Lagrange** took the further step by saying that **d'Alembert**'s principle, as the simplest expression that we can consider, viz., (35) [(36), resp.], should be coupled with the principle of virtual displacements. In fact, if we multiply the $3n$ equations (36) by the arbitrary functions δx_ν, δy_ν, δz_ν and add them then it will follow that:

$$\sum_{\nu=1}^{n}(X'_\nu\,\delta x_\nu + Y'_\nu\,\delta y_\nu + Z'_\nu\,\delta z_\nu) = 0\,,$$

or, from (35):

$$\sum_{\nu=1}^{n}[(X_\nu - m_\nu\,\ddot{x}_\nu)\,\delta x_\nu + (Y_\nu - m_\nu\,\ddot{y}_\nu)\,\delta y_\nu + (Z_\nu - m_\nu\,\ddot{z}_\nu)\,\delta z_\nu] = 0\,. \tag{37}$$

In analytical mechanics, that particular equation is called **d'Alembert**'s principle. We will adopt that terminology here and always understand that principle to mean equation (37).

Equation (37) is precisely equivalent to equations (35) for a completely-free mass-system, because the quantities δx_ν, δy_ν, δz_ν, which we naturally once more consider to be infinitely-small displacements $\delta \mathfrak{s}_\nu$, are mutually independent. However, if certain constraints are present, say, m ($< 3n$) of them, that we will assume to be non-holonomic and rheonomic here, for the sake of generality:

$$\sum_{\nu=1}^{n}(a'_{\kappa\nu}\,dx_\nu + b'_{\kappa\nu}\,dy_\nu + c'_{\kappa\nu}\,dz_\nu) + a'_\nu\,dt = 0 \qquad (\kappa = 1, 2, \ldots, m), \tag{38}$$

then it will follow in analogy to the foregoing section that equation (37) will retain its validity when we understand δx_ν, δy_ν, δz_ν to mean *virtual* displacements, i.e., displacements that obey the following conditions:

$$\sum_{\nu=1}^{n}(a'_{\kappa\nu}\,\delta x_\nu + b'_{\kappa\nu}\,\delta y_\nu + c'_{\kappa\nu}\,\delta z_\nu) = 0 \qquad (\kappa = 1, 2, \ldots, m). \tag{39}$$

With the help of m **Lagrange** multipliers, upon combining equations (37) for **d'Alembert**'s principle and the m equations (38), one will get the equation:

$$\sum_{\nu=1}^{n}[(X_\nu - m_\nu\,\ddot{x}_\nu + \sum_{\kappa=1}^{m}\lambda_\kappa\,a'_{\kappa\nu})\,\delta x_\nu + (Y_\nu - m_\nu\,\ddot{y}_\nu + \sum_{\kappa=1}^{m}\lambda_\kappa\,b'_{\kappa\nu})\,\delta y_\nu + (Z_\nu - m_\nu\,\ddot{z}_\nu + \sum_{\kappa=1}^{m}\lambda_\kappa\,c'_{\kappa\nu})\,\delta z_\nu] = 0\,, \tag{40}$$

which will then give the following system of ***Lagrange*** *equations of motion of the first kind*, in just the same way as what followed from equation (12) in § **1**:

$$\left\{\begin{aligned} X_\nu - m_\nu\,\ddot{x}_\nu + \sum_{\kappa=1}^{m}\lambda_\kappa\,a'_{\kappa\nu} &= 0\,, \\ Y_\nu - m_\nu\,\ddot{y}_\nu + \sum_{\kappa=1}^{m}\lambda_\kappa\,b'_{\kappa\nu} &= 0\,, \\ Z_\nu - m_\nu\,\ddot{z}_\nu + \sum_{\kappa=1}^{m}\lambda_\kappa\,c'_{\kappa\nu} &= 0\,. \end{aligned}\right. \tag{41}$$

Those $3n$ equations, together with the m equations (38), will suffice to determine the $(3n + m)$ unknowns $x_1, y_1, z_1, \ldots, x_n, y_n, z_n, \lambda_1, \lambda_2, \ldots, \lambda_m$.

As we have repeatedly emphasized, virtual displacements are determined in the same way here as they were in the principle of virtual displacements. In particular, *time is not varied* for rheonomic constraints, as we will see, e.g., when we go from (38) to (39).

Naturally, equations (41) also subsume the case of scleronomic and holonomic constraints. If the constraints are initially *scleronomic* then the coefficients $a'_{\kappa\nu}$, $b'_{\kappa\nu}$, $c'_{\kappa\nu}$ will not include time explicitly, which we would like to suggest by dropping the prime, as with our previous notation. If the equations are holonomic, in addition, then the coefficients $a_{\lambda\nu}$, $b_{\lambda\nu}$, $c_{\lambda\nu}$ will be equal to the partial derivatives $\frac{\partial\varphi_\kappa}{\partial x_\nu}$, $\frac{\partial\varphi_\kappa}{\partial y_\nu}$, $\frac{\partial\varphi_\kappa}{\partial z_\nu}$, resp., of a certain function φ_κ .

§ 4.

The energy principle for scleronomic and rheonomic equations of constraint.

This is the place for us to establish the circumstances under which the energy principle will be included in the equations of motion of mechanics (41). It will be shown that this is the case only when the equations of constraint are scleronomic, i.e., they do not include time explicitly, whereas holonomity or non-holonomity is irrelevant to that.

We take the most-general case of non-holonomic-rheonomic constraints, whose equations of motion were exhibited in (41). Thus, perhaps *m* equations of constraint of the form (38) might exist:

$$\sum_{\nu=1}^{n} (a'_{\kappa\nu}\, dx_\nu + b'_{\kappa\nu}\, dy_\nu + c'_{\kappa\nu}\, dz_\nu) + a'_\nu\, dt = 0 \qquad (\kappa = 1, 2, \ldots, m),$$

from which, dividing by *dt* will produce the following relations for the velocity components $\dot{x}_\nu$, $\dot{y}_\nu$, $\dot{z}_\nu$:

$$\sum_{\nu=1}^{n} (a'_{\kappa\nu}\, \dot{x}_\nu + b'_{\kappa\nu}\, y'_\nu + c'_{\kappa\nu}\, \ddot{z}_\nu) + a'_\nu = 0 \qquad (\kappa = 1, 2, \ldots, m).$$

If we successively extend the equations of motion by the velocity components $\dot{x}_\nu$, $\dot{y}_\nu$, $\dot{z}_\nu$, add them, and arrange them then we will get:

$$\sum_{\nu=1}^{n} (X_\nu\, \dot{x}_\nu + Y_\nu\, \dot{y}_\nu + Z_\nu\, \dot{z}_\nu) - \sum_{\nu=1}^{n} m_\nu (\ddot{x}_\nu\, \dot{x}_\nu + \ddot{y}_\nu\, \dot{y}_\nu + \ddot{z}_\nu\, \dot{z}_\nu) + \sum_{\kappa=1}^{m} \lambda_\kappa (a'_{\kappa\nu}\, \dot{x}_\nu + b'_{\kappa\nu}\, \dot{y}_\nu + c'_{\kappa\nu}\, \dot{z}_\nu) = 0 . \tag{43}$$

The individual summands in that equation have simple meanings. The first sum represents the work done by the impressed forces X_ν, Y_ν, Z_ν per unit time; we would like to denote it by dA'/dt. The differential symbol is given a prime in order to suggest that $d'A$ is does not necessarily need to be an exact differential of a pure function of the coordinates, which will be true only in special cases that are generally very important. The second summand can be written:

$$\sum_{\nu=1}^{n} m_\nu (\ddot{x}_\nu\, \dot{x}_\nu + \ddot{y}_\nu\, \dot{y}_\nu + \ddot{z}_\nu\, \dot{z}_\nu) = \tfrac{1}{2} \sum_{\nu=1}^{n} m_\nu (\dot{x}_\nu^2 + \dot{y}_\nu^2 + \dot{z}_\nu^2) = \frac{d}{dt} \sum_{\nu=1}^{n} L_\nu = \frac{dL}{dt} . \tag{44}$$

L_ν means the kinetic energy of the ν^{th} mass-particle, and $\sum L_\nu = L$ means that of the total system. Finally, based upon equations (42), the third sum can be written:

(45) $$\sum_{\kappa=1}^{m} \lambda_\kappa (a'_{\kappa\nu}\, \dot{x}_\nu + b'_{\kappa\nu}\, \dot{y}_\nu + c'_{\kappa\nu}\, \dot{z}_\nu) = -\sum_{\kappa=1}^{m} \lambda_\kappa\, a'_\kappa .$$

Along with (44) and (45), equation (43) will become the following one:

(46) $$\frac{dL}{dt} = \frac{d'A}{dt} - \sum_{\kappa=1}^{m} \lambda_\kappa\, a'_\kappa ,$$

while the energy principle is known to demand that the time derivative of the kinetic energy dL/dt should be equal to the work done per unit time dA'/dt.

We then see that the energy principle is not true in the case of rheonomic constraints, which are indeed characterized by the appearance of the coefficients a'_κ, and in fact it is irrelevant whether the rheonomic constraints are holonomic or non-holonomic. That is because if we assume that the constraint (42) can be converted into a holonomic one (due to the integrability conditions being valid) then we will indeed have:

$$a'_{\kappa\nu} = \frac{\partial\varphi_\kappa}{\partial x_\nu}, \qquad b'_{\kappa\nu} = \frac{\partial\varphi_\kappa}{\partial y_\nu}, \qquad c'_{\kappa\nu} = \frac{\partial\varphi_\kappa}{\partial z_\nu}, \qquad a'_\kappa = \frac{\partial\varphi_\kappa}{\partial t},$$

and equation (46) would go to:

(46.a) $$\frac{dL}{dt} = \frac{d'A}{dt} - \sum_{\kappa=1}^{m} \lambda_\kappa \frac{\partial\varphi_\kappa}{\partial t},$$

which proves the assertion above.

By contrast, if the constraints are scleronomic, i.e., the coefficients a'_κ vanish (in the holonomic case, $\partial\varphi_\kappa / \partial t = 0$, resp.), then the energy principle will, in fact, follow:

(47) $$\frac{dL}{dt} = \frac{d'A}{dt}.$$

The energy principle will then be true for only scleronomic (holonomic or non-holonomic) constraint equations.

In special cases, the forces can be represented as the negatives of partial derivatives of a single function Φ of the coordinates $x_1, y_1, z_1, \ldots, x_n, y_n, z_n$:

(48) $$X_\nu = -\frac{\partial\Phi}{\partial x_\nu}, \quad Y_\nu = -\frac{\partial\Phi}{\partial y_\nu}, \quad Z_\nu = -\frac{\partial\Phi}{\partial z_\nu} \qquad (\nu = 1, 2, \ldots, n),$$

and the expression:

$$\frac{d'A}{dt} = \sum_{\nu=1}^{n} (X_\nu \, \dot{x}_\nu + Y_\nu \, \dot{y}_\nu + Z_\nu \, \dot{z}_\nu)$$

will go to:

(49)
$$\frac{d'A}{dt} = -\sum_{\nu=1}^{n} \left(\frac{\partial \Phi}{\partial x_\nu} \dot{x}_\nu + \frac{\partial \Phi}{\partial y_\nu} \dot{y}_\nu + \frac{\partial \Phi}{\partial z_\nu} \dot{z}_\nu \right) = -\frac{d\Phi}{dt},$$

which will make the energy principle (47) assume the simple form:

(50)
$$\begin{cases} \quad \dfrac{d}{dt}(L + \Phi) = 0, \\ \text{or} \quad L + \Phi = h, \end{cases}$$

in which h means a constant. One calls Φ the *potential energy*, while $h = L + \Phi$ means the *total energy*.

In the latter case, i.e., when a potential energy Φ exists, we would like to refer to equation (50) as the *energy integral*, which we would like to characterize as the *narrower* form of the law of energy, to distinguish it from its broader form (47).

The result that was obtained that the energy principle is not valid for rheonomic constraints is important in the investigation of the least action principle (§ **12**), among other things.

§ 5.

True, transitional, and varied motion.

The fundamental difference between holonomic and non-holonomic constraints can be highlighted in a different way, which we would now like to explain.

We first consider the *true* motion of *one* mass-point. (Everything that is essential here will become clear when we restrict ourselves to this simplest system. Moreover, the statements that will be made here can be adapted *verbatim* to a system of n mass-points when we go to $3n$-dimensional space.) We can think of it as being represented in, say, the form:

$$\left\{\begin{aligned} x_\nu &= x_\nu(t), \\ y_\nu &= y_\nu(t), \\ z_\nu &= z_\nu(t). \end{aligned}\right. \tag{51}$$

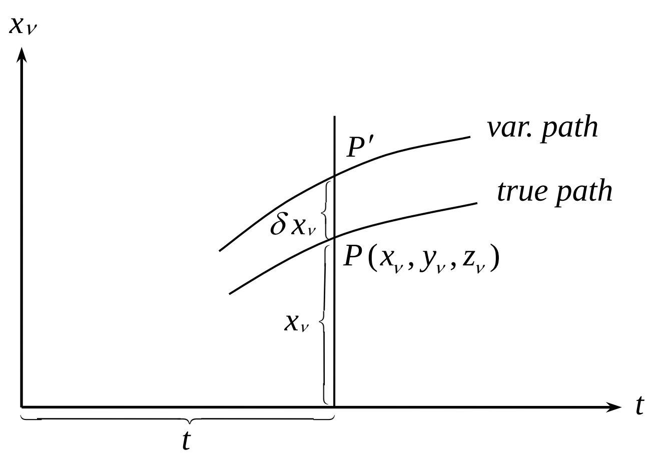

Figure 1.

We associate each point x_ν, y_ν, z_ν with of the *true* path at time t with another neighboring point ($x_\nu + \delta x_\nu$, $y_\nu + \delta y_\nu$, $z_\nu + \delta z_\nu$) at the same time t by means of the *virtual displacements* δx_ν, δy_ν, δz_ν. The set of all those latter points will likewise define a continuous path, at least when δx_ν, δy_ν, δz_ν are continuous functions of time, which we assume. We shall call that path the *varied* path. Naturally, we can associate the true path with a varied path in very different ways, such as when we let the point x_ν, y_ν, z_ν at time t on the *true* path correspond to the point ($x_\nu + \Delta x_\nu$, $y_\nu + \Delta y_\nu$, $z_\nu + \Delta z_\nu$) at time $t + \Delta t$ on the *varied* path. However, the displacements Δx_ν, Δy_ν, Δz_ν *would not be virtual* displacements then, in general, which is why they are also not denoted by the symbol δ. Nonetheless, we shall only deal with virtual displacements, so for our considerations, the varied path will emerge from the true path in the first way that was given.

We can perhaps depict that graphically by considering only the $x_\nu t$-plane (Fig. 1). Naturally, we would then get only the projection of the true and varied paths onto the x-axis.

As is shown in the figure, a point P on the *true* path is associated with a point P' that corresponds to the same value of time t. We call $\overline{PP'}$ the *transitional path*. The components of $\overline{PP'}$ are δx_ν, δy_ν, δz_ν.

Now let a holonomic constraint exist that we would initially like to assume is scleronomic, for simplicity:

$$\varphi(x_\nu, y_\nu, z_\nu) = c_1 , \tag{52}$$

in which the constant c_1 can naturally also have the value 0, but it does not need to. For the sake of clarity, we prefer that the value of the constant should remain undetermined.

We shall now address the issue of what sort of constraints the true, the transitional, and the varied paths should be subject to.

For the *true* path, according to the problem that was posed, equation (52) must be fulfilled for every position of the material point along it, e.g., for the point (x_ν, y_ν, z_ν), as well as for its neighboring point $(x_\nu + d\,x_\nu, y_\nu + d\,y_\nu, z_\nu + d\,z_\nu)$ (which it will assume at time $t + dt$, although that will not be an issue here). Thus, along with (52), one will also have the equation:

$$\varphi(x_\nu + d\,x_\nu, y_\nu + d\,y_\nu, z_\nu + d\,z_\nu) = 0$$

for the neighboring point, or:

$$\varphi(x_\nu, y_\nu, z_\nu) + \frac{\partial\varphi}{\partial x_\nu} dx_\nu + \frac{\partial\varphi}{\partial y_\nu} dy_\nu + \frac{\partial\varphi}{\partial z_\nu} dz_\nu = c_1 ,$$

or after subtracting equation (52):

$$\frac{\partial\varphi}{\partial x_\nu} dx_\nu + \frac{\partial\varphi}{\partial y_\nu} dy_\nu + \frac{\partial\varphi}{\partial z_\nu} dz_\nu \equiv d\varphi = 0 . \tag{53}$$

The components of the true displacement $d\,x_\nu$, $d\,y_\nu$, $d\,z_\nu$ then obey the relation (53), which says just that the value c_1 of φ does not change along the true motion.

How does that work for the *transitional motion?*

During it, which is indeed produced by virtual displacements $\delta\,x_\nu$, $\delta\,y_\nu$, $\delta\,z_\nu$ as a result of **d'Alembert**'s principle, the latter displacements must obey the equation:

$$\frac{\partial\varphi}{\partial x_\nu} \delta x_\nu + \frac{\partial\varphi}{\partial y_\nu} \delta y_\nu + \frac{\partial\varphi}{\partial z_\nu} \delta z_\nu \equiv \delta\,\varphi = 0 , \tag{54}$$

as was shown before, i.e., the change in the value c_1 of φ will also be zero during the transitional motion, that is, *the transitional motion will also obey the constraint equation* (52).

A statement about the behavior of the *varied* path will now follow directly from that.

φ has the value c_1 at the point P on the true path. φ will not change during the transitional motion that leads to the point P' ($\delta\varphi = 0$). *Therefore, the point P' on the varied path will also be assigned the same value $\varphi = c_1$, i.e., the varied path will also obey the constraint* (52) *under our assumptions.*

We have then arrived at the following result:

For holonomic constraints, the true, the transitional, and the varied path will obey the equations of constraint.

As we had already anticipated in the formulation above, that will also be true when the holonomic constraint equations are *rheonomic* since t is treated as a constant parameter during the transitional motion.

In the above, we concluded the behavior of the *varied* path from that of the *true* and the *transitional* paths. Had we demanded from the outset that the varied path should also satisfy the holonomic constraints, along with the true path, then we could have concluded in the same way that the transitional path would have to do so, as well, i.e., that the displacements δx_ν, δy_ν, δz_ν that are thus constituted would have to be virtual ones.

That behavior will change when we now move on to non-holonomic equations.

Therefore, let a non-holonomic (but scleronomic) constraint be given now:

$$d'\varphi \equiv a_{\kappa\nu}\, dx_\nu + b_{\kappa\nu}\, dy_\nu + c_{\kappa\nu}\, dz_\nu = 0 \,, \tag{55}$$

in which the $a_{\kappa\nu}$, … are functions of x_ν, y_ν, z_ν that do not fulfill the integrability conditions. We have denoted the left-hand side of the constraint by $d'\varphi$, in which the prime suggests that it should not be treated as an exact differential of a function φ.

We will now first present the conditions for the *transitional path* to obey the constraint (55), i.e., for the displacements to be virtual. *Previously*, that followed for *holonomic* constraints from the fact that the varied path also obeyed the equations of constraint. Secondly, we will formulate the condition for the *varied* path to obey equation (55) and see whether those two determinations are also compatible with each other now ([1]).

We shall first present the equations that have to be valid along the transitional path!

The virtual displacements δx_ν, δy_ν, δz_ν that characterize that motion have to obey the constraint that is given by (55) when we switch d with δ, i.e., the equation:

$$\delta'\varphi \equiv a_{\kappa\nu}\, \delta x_\nu + b_{\kappa\nu}\, \delta y_\nu + c_{\kappa\nu}\, \delta z_\nu = 0 \,, \tag{56}$$

whose left-hand side we logically denote by $\delta'\varphi$. Now, that equation will be as valid when we complete the transition to the varied path from the point (x_ν, y_ν, z_ν) on the true path as it is when

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([1]) On this, cf., **O. Hölder**, "Über die Prinzipien von Hamilton und Maupertuis," Sitz.-Ber. d. Ges. d. Wiss. zu Göttingen (1896), pp. 122, *et seq.*

we go to the neighboring point ($x_\nu + d\,x_\nu$, $y_\nu + d\,y_\nu$, $z_\nu + d\,z_\nu$) on the true path, i.e., along with (56) [i.e., along with $\delta'\varphi(x_\nu, y_\nu, z_\nu) = 0$], it must also be true that:

$$\delta'\varphi(x_\nu + d\,x_\nu, y_\nu + d\,y_\nu, z_\nu + d\,z_\nu) = 0 \,,$$

or after developing that in a **Taylor** series and subtracting equation (56):

(57) $$d\,(\delta'\varphi) \equiv d\,(a_{\kappa\nu}\,\delta x_\nu + b_{\kappa\nu}\,\delta y_\nu + c_{\kappa\nu}\,\delta z_\nu) = 0 \,.$$

Let us make note here of the special case that will emerge from (57) for a holonomic constraint equation:

(57.a) $$d\,(\delta\varphi) \equiv d\left(\frac{\partial\varphi}{\partial x_\nu}\,\delta x_\nu + \frac{\partial\varphi}{\partial y_\nu}\,\delta y_\nu + \frac{\partial\varphi}{\partial z_\nu}\,\delta z_\nu\right) = 0 \,.$$

(57) is the condition that is implied by the demand that the *transitional path* should obey the prescribed equation.

We shall now formulate the other demand that the *varied* path should also do that. However, that obviously means that whereas the equation $\delta'\varphi(x_\nu, y_\nu, z_\nu) = 0$ is fulfilled for the true motion, the same *equation* should be valid for the varied coordinates ($x_\nu + \delta x_\nu$, $y_\nu + \delta y_\nu$, $z_\nu + \delta z_\nu$), so:

$$d'\varphi\ (x_\nu + \delta x_\nu, y_\nu + \delta y_\nu, z_\nu + \delta z_\nu) = 0 \,,$$

or upon developing that it in a **Taylor** series and subtracting equation (55):

(58) $$\delta(d'\varphi) \equiv \delta(a_{\kappa\nu}\,\delta x_\nu + b_{\kappa\nu}\,\delta y_\nu + c_{\kappa\nu}\,\delta z_\nu) = 0 \,,$$

which will again specialize to:

(58.a) $$\delta(d\,\varphi) \equiv \delta\left(\frac{\partial\varphi}{\partial x_\nu}\,dx_\nu + \frac{\partial\varphi}{\partial y_\nu}\,dy_\nu + \frac{\partial\varphi}{\partial z_\nu}\,dz_\nu\right) = 0$$

for a holonomic constraint. *We now ask whether the two conditions that were thus obtained* (which we already know to be identical in the case of holonomity, which will also be shown again soon) *are compatible with each other for non-holonomic equations*.

By performing the differentiations that are suggested in (57), it will follow that:

(59) $$\left(\frac{\partial a_{\kappa\nu}}{\partial x_\nu}\,dx_\nu + \cdots\right)\delta x_\nu + \left(\frac{\partial b_{\kappa\nu}}{\partial x_\nu}\,dx_\nu + \cdots\right)\delta y_\nu + \left(\frac{\partial c_{\kappa\nu}}{\partial x_\nu}\,dx_\nu + \cdots\right)\delta z_\nu + (a_{\kappa\nu}\;d\;\delta x_\nu + \cdots) = 0 \,,$$

and likewise from (58):

$$(60)\qquad \left(\frac{\partial a_{\kappa\nu}}{\partial x_\nu}\delta x_\nu + \cdots\right) dx_\nu + \left(\frac{\partial b_{\kappa\nu}}{\partial x_\nu}\delta x_\nu + \cdots\right) dy_\nu + \left(\frac{\partial c_{\kappa\nu}}{\partial x_\nu}\delta x_\nu + \cdots\right) dz_\nu + (a_{\kappa\nu}\, d\,\delta x_\nu + \cdots) = 0\,.$$

Subtraction will give the following relation that must exist between the two equations (57) and (58) in the case of compatibility:

$$(61)\qquad \left\{ \begin{aligned} &\left(\frac{\partial a_{\kappa\nu}}{\partial y_\nu} - \frac{\partial b_{\kappa\nu}}{\partial x_\nu}\right)(dy_\nu\,\delta x_\nu - dx_\nu\,\delta y_\nu) + \left(\frac{\partial b_{\kappa\nu}}{\partial z_\nu} - \frac{\partial c_{\kappa\nu}}{\partial y_\nu}\right)(dz_\nu\,\delta y_\nu - dy_\nu\,\delta z_\nu) \\ &\qquad + \left(\frac{\partial c_{\kappa\nu}}{\partial x_\nu} - \frac{\partial a_{\kappa\nu}}{\partial x_\nu}\right)(dx_\nu\,\delta z_\nu - dz_\nu\,\delta x_\nu) = 0. \end{aligned} \right.$$

On the other hand, the following proportion can be inferred from (55) and (56):

$$(62)\qquad a_{\kappa\nu} : b_{\kappa\nu} : c_{\kappa\nu} = (dz_\nu\,\delta y_\nu - dy_\nu\,\delta z_\nu) : (dx_\nu\,\delta z_\nu - dz_\nu\,\delta x_\nu) : (dx_\nu\,\delta y_\nu - dy_\nu\,\delta x_\nu)\,,$$

or when we appeal to a proportionality factor A :

$$(63)\qquad \begin{aligned} a_{\kappa\nu} &= A\cdot(dx_\nu\,\delta y_\nu - dy_\nu\,\delta x_\nu)\,, \\ b_{\kappa\nu} &= A\cdot(dx_\nu\,\delta z_\nu - dz_\nu\,\delta x_\nu)\,, \\ c_{\kappa\nu} &= A\cdot(dy_\nu\,\delta x_\nu - dx_\nu\,\delta y_\nu)\,, \end{aligned}$$

with which (61) can be written:

$$(64)\qquad \left(\frac{\partial a_{\kappa\nu}}{\partial y_\nu} - \frac{\partial b_{\kappa\nu}}{\partial x_\nu}\right) c_{\kappa\nu} + \left(\frac{\partial b_{\kappa\nu}}{\partial z_\nu} - \frac{\partial c_{\kappa\nu}}{\partial y_\nu}\right) a_{\kappa\nu} + \left(\frac{\partial c_{\kappa\nu}}{\partial x_\nu} - \frac{\partial a_{\kappa\nu}}{\partial x_\nu}\right) b_{\kappa\nu} = 0\,.$$

For a holonomic constraint equation, that relation will specialize to:

$$(64.a)\qquad \left(\frac{\partial^2\varphi}{\partial x_\nu\,\partial y_\nu} - \frac{\partial^2\varphi}{\partial y_\nu\,\partial x_\nu}\right)\frac{\partial\varphi}{\partial z_\nu} + (\cdots)\frac{\partial\varphi}{\partial x_\nu} + (\cdots)\frac{\partial\varphi}{\partial y_\nu} = 0\,,$$

and we will see that:

This is always fulfilled identically, i.e., in the case of holonomic constraints, the demands that, on the one hand, the varied path and on the other, the transitional path should satisfy the constraints on the motion will be identical to each other, as we know already since the one demand inevitably implies the other.

However, (64) *cannot be fulfilled, in general*, since the coefficients $a_{\kappa\nu}$, $b_{\kappa\nu}$, $c_{\kappa\nu}$ do not vanish simultaneously, and likewise, the expressions in the parentheses cannot be simultaneously zero due to non-holonomity. That is because the vanishing of those quantities would mean that the integrability conditions are fulfilled, which is contrary to the assumption of non-holonomity.

That then implies this result:

For non-holonomic equations of constraint, the true and the transitional motions indeed fulfill the equations of constraint, but not the varied path, in general.

Since, on the one hand, mechanical principles (e.g., **d'Alembert**'s principle) demand that the transitional motion will be composed of virtual displacements, so the transitional motion must always fulfill the equations of constraint, if one would like to determine the displacements $\overline{\delta x_\nu}$, $\overline{\delta y_\nu}$, $\overline{\delta z_\nu}$ from that principle *in such a way that the varied path fulfills the constraints on the motion then one would arrive at false equations of motion, in general*. Both determinations will be equivalent only in the case of holonomic constraints. That distinction is not always stressed clearly enough, e.g., **Hertz** ([1]) had violated it in a known example in his theory of mechanics.

The state of affairs might be explained once more with that **Hertzian** example. A ball *rolls without slipping* on a horizontal plane under the action of arbitrary forces. As we have already mentioned before, the equations of constraint in this problem that express rolling without slipping are non-holonomic. Two neighboring points (x_ν, y_ν, z_ν) and ($x_\nu + d\,x_\nu$, $y_\nu + d\,y_\nu$, $z_\nu + dz_\nu$) along the true path naturally emerge from each other by a pure rolling motion.

The principles of mechanics further demand that the transitional motions that start from the point (x_ν, y_ν, z_ν), as well as the point ($x_\nu + d\,x_\nu$, $y_\nu + d\,y_\nu$, $z_\nu + d\,z_\nu$), should lead to the respective final positions ($x_\nu + \delta\,x_\nu$, $y_\nu + \delta\,y_\nu$, $z_\nu + \delta\,z_\nu$) and ($x_\nu + \delta\,x_\nu + d\,(x_\nu + \delta\,x_\nu)$, …), i.e., that the displacements should be virtual. If that were the case then *the points* ($x_\nu + \delta\,x_\nu$, $y_\nu + \delta\,y_\nu$, $z_\nu + \delta z_\nu$) *and* ($x_\nu + \delta\,x_\nu + dx_\nu + d\,\delta\,x_\nu$, $y_\nu + \delta\,y_\nu + dy_\nu + d\,\delta\,y_\nu$, …) *on the varied path would no longer emerge from each other by a pure rolling motion, but a slide would necessarily have to enter in*.

Hölder ([2]) has carried out the calculations in this example thoroughly and explained the actual situation.

([1]) Cf., *infra*, pp. 36.
([2]) **Hölder**, *loc. cit.*

§ 6.

More general variations (variation of time).

In the foregoing sections, we have already occasionally remarked that we can go from the true path to a neighboring varied one by way of more general types of displacements δx_ν, δy_ν, δz than the virtual ones, i.e., ones that are performed at constant time. We would now like to consider that more general type in more detail. We would now like to associate a space-time point (x_ν, y_ν, z_ν, t), as we would like to say briefly, on the true path with the space-time point $(x_\nu + \Delta x_\nu, y_\nu + \Delta y_\nu, z_\nu + \Delta z_\nu, t + \Delta t)$. The set of all of the latter points defines the varied path under our new rule for variation and for that we would like to expressly reserve the notation Δ. We would again like to clarify the association of the two paths in this graphically by examining the relationship in the $x_\nu t$-plane, i.e., we shall consider the projection of the paths onto the x-axis.

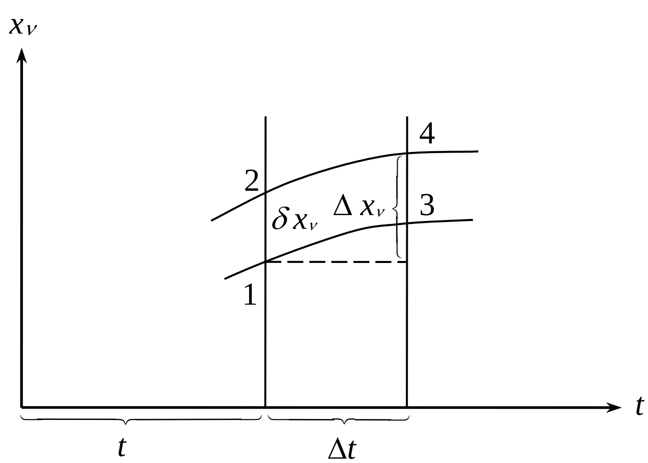

Figure 2.

In Fig. 2, the point 1 means the point on the true path with the coordinates x_ν, y_ν, z_ν at time t. Up to now, we went from a point 1 to a point 2 (with the same value of t, i.e., by means of the operation δ) that has the coordinates $(x_\nu + \delta x_\nu, y_\nu + \delta y_\nu, z_\nu + \delta z_\nu)$. The line segment (1, 2) is accordingly equal to δx_ν. However, we would now like to associate the point 1 with the point 4 that has time $t + \Delta t$ and coordinates $x_\nu + \Delta x_\nu$, $y_\nu + \Delta y_\nu$, $z_\nu + \Delta z_\nu$.

In addition, the point 3 on the *true* path is indicated in the figure, which is attained by the material point at time $t + \Delta t$. If we denote its velocity components by $\dot{x}_\nu$, $\dot{y}_\nu$, $\dot{z}_\nu$ then the point 3 will have the coordinates $x_\nu + \dot{x}_\nu \Delta t$, $y_\nu + \dot{y}_\nu \Delta t$, $z_\nu + \dot{z}_\nu \Delta t$. We will then get the following table for our four points:

Point	Coordinates	Time
1	x_ν, y_ν, z_ν	t
2	$x_\nu + \delta x_\nu, y_\nu + \delta y_\nu, z_\nu + \delta z_\nu$	t
3	$x_\nu + \dot{x}_\nu \Delta t, y_\nu + \dot{y}_\nu \Delta t, z_\nu + \dot{z}_\nu \Delta t$.	$t + \Delta t$
4	$x_\nu + \Delta x_\nu, y_\nu + \Delta y_\nu, z_\nu + \Delta z_\nu$	$t + \Delta t$

The difference between the ordinates of 2 and 1 is accordingly δx_ν, and between 4 and 1, it will be Δx_ν .

We would like to exhibit the relationships between Δx_ν, δx_ν, and Δt. To that end, we observe that the point 4 emerges from 3 under the δ process, and likewise 2 from 1. We can then express the x-coordinate of 4 as follows: $(x_\nu + \dot{x}_\nu \Delta t) + \delta (x_\nu + \dot{x}_\nu \Delta t)$. In general, we will then find the following values for the coordinates of 4:

$$x_\nu + \delta x_\nu + \dot{x}_\nu \Delta t, \qquad y_\nu + \delta y_\nu + \dot{y}_\nu \Delta t, \qquad z_\nu + \delta z_\nu + \dot{z}_\nu \Delta t,$$

when we neglect terms of higher order.

On the other hand, from the table above, its coordinates will be the following ones:

$$x_\nu + \Delta x_\nu, \qquad y_\nu + \Delta y_\nu, \qquad z_\nu + \Delta z_\nu .$$

A comparison will then give the following relations between Δx_ν, δx_ν, and Δt :

$$\left\{ \begin{aligned} \Delta x_\nu &= \delta x_\nu + \dot{x}_\nu \Delta t, \\ \Delta y_\nu &= \delta y_\nu + \dot{y}_\nu \Delta t, \\ \Delta z_\nu &= \delta z_\nu + \dot{z}_\nu \Delta t. \end{aligned} \right. \tag{65}$$

We would further like to apply the operations Δ and δ to a function $\varphi(x, y, z, t)$ and likewise exhibit the relations between $\Delta\varphi$ and $\delta\varphi$. From the definitions of the symbols, we have:

$$\Delta\varphi = \frac{\partial\varphi}{\partial x}\Delta x + \frac{\partial\varphi}{\partial y}\Delta y + \frac{\partial\varphi}{\partial z}\Delta z + \frac{\partial\varphi}{\partial t}\Delta t, \tag{66.a}$$

$$\delta x = \frac{\partial\varphi}{\partial x}\delta x + \frac{\partial\varphi}{\partial y}\delta y + \frac{\partial\varphi}{\partial z}\delta z . \tag{66.b}$$

If we add the time derivative to that:

$$\frac{d\varphi}{dt} = \dot{\varphi} = \frac{\partial\varphi}{\partial x}\dot{x} + \frac{\partial\varphi}{\partial y}\dot{y} + \frac{\partial\varphi}{\partial z}\dot{z} + \frac{\partial\varphi}{\partial t}, \tag{66.c}$$

extend that equation by Δt, and subtract from (66.a) then we will have:

$$\Delta\varphi - \dot{\varphi}\,\Delta t = \frac{\partial\varphi}{\partial x}(\Delta x - \dot{x}\,\Delta t) + \frac{\partial\varphi}{\partial y}(\Delta y - \dot{y}\,\Delta t) + \frac{\partial\varphi}{\partial z}(\Delta z - \dot{z}\,\Delta t)\ ,$$

and from (65), the right-hand side of that is equal to:

$$\frac{\partial\varphi}{\partial x}\delta x + \frac{\partial\varphi}{\partial y}\delta y + \frac{\partial\varphi}{\partial z}\delta z\ ,$$

so from (66.b), it will be equal to $\delta\varphi$. We will then get:

(67) $$\Delta\varphi = \delta\varphi + \dot{\varphi}\,\Delta t\ ,$$

analogous to equation (65).

§ 7.

General coordinates. True and non-holonomic coordinates.

The use of Cartesian coordinates is generally unsuitable, and for that reason one introduces so-called "general coordinates," i.e., quantities q_k that are in one-to-one correspondence with the x_ν, y_ν, z_ν. If holonomic equations of constraint, say m of them, exist between the $3n$ Cartesian coordinates x_ν, y_ν, z_ν then $(3n - m) = N$ quantities q_k can always be chosen such that *no further equations of constraint will exist between them*. If the equations of constraint were scleronomic then the relations that couple the x_ν, y_ν, z_ν to the q_k would not include time t explicitly, and one would also call the q_k "scleronomic" coordinates, in a terminology that is easy to understand. By contrast, if the constraints that were originally present were rheonomic then time would also enter explicitly into the relations between the Cartesian coordinates and the q_k, which would then be called "rheonomic," as well. In both cases, we have, in any event, equations of the following type:

(68.a) $$q_k = q_k\,(x_1, y_1, z_1, \ldots, x_n, y_n, z_n)$$

or $$(k = 1, 2, \ldots, N)$$

(68.b) $$q_k = q_k\,(x_1, y_1, z_1, \ldots, x_n, y_n, z_n, t)\ .$$

In what follows, for the sake of simplicity, we will always restrict ourselves to scleronomic coordinates. Due to the form of the relation (68), the q_k are also called *holonomic* or *true* coordinates. However, in some situations, it is preferable to introduce certain linear differential expressions in the true coordinates that we would like to abbreviate by $d'\pi_r$:

(68) $$d'\pi_r = \sum_{k=1}^{N} \alpha_{kr}\, dq_k \qquad (r = 1, 2, \ldots, N),$$

in which the α_{kr} are functions of the q_k.

If those coefficients α_{kr} satisfy the integrability conditions:

(69) $$\frac{\partial \alpha_{kr}}{\partial q_\lambda} - \frac{\partial \alpha_{\lambda r}}{\partial q_k} = 0 \qquad (r = 1, 2, \ldots, N)$$

then the $d'\pi_r$ will be *exact* differentials, and the quantities π_r can be obtained as functions of the q_k from (68) by integration. In that case, the π_r will be merely new general coordinates of the same character as the q_k . Naturally, the prime on the differential symbol in (68) can then be omitted in that case. By contrast, if the integrability conditions (69) are not fulfilled then there will exist no such quantities as "π_r"; rather, the $d'\pi_r$ will just be *defined* by (68). If one divides

(68) by dt then the quantities $dq_k / dt = \dot{q}_k$ will appear on the right-hand side, namely, the *generalized velocities*. We would not like to denote the quantities $d'\pi_r / dt$ on the left-hand side by $\dot{\pi}_r$, in order to not create the impression that we are dealing with time derivatives of the quantities "π_r". We shall denote the aforementioned quotients by $\overset{*}{\pi}_r$, to distinguish them. They obey the equations:

$$(70) \qquad \overset{*}{\pi}_r = \sum_{k=1}^{N} \alpha_{kr}\, \dot{q}_k \qquad (r = 1, \ldots, N),$$

so they are linear combinations of the generalized velocity components and can even be employed as such generalized velocities. However, they differ from the $\dot{q}_k$ by the fact that the latter are derivatives of quantities q_k, while the former are not. Naturally, for that reason, one cannot employ the "π_r" as generalized coordinates either. Only the q_k and the $\dot{q}_k$ will appear together in the **Lagrange** equations of motion that will be discussed later. The difference between the $\dot{q}_k$ and the $\overset{*}{\pi}_r$ that was explained here makes it plausible from the outset that the quantities $\overset{*}{\pi}_r$ *cannot* be used as velocity components in the **Lagrange** equations since the associated quantities π_r do not even exist. However, as we will explain later, the **Lagrange** equations can be extended in such a way that they will also remain useful in that case. One calls the quantities $d'\pi_r$ (although it is not entirely appropriate) *differentials of non-holonomic coordinates* or *quasi-coordinates* ([1]). An example of those quasi-coordinates is the following one: As is known, one can determine the position of a rigid body that is fixed at a point by the so-called **Euler** angles φ, ψ, ϑ, which are obviously true coordinates. φ is called the angle of precession, ψ is the angle of proper rotation, and ϑ is the pendulum angle. The quantities $\dot{\varphi}$, $\dot{\psi}$, $\dot{\vartheta}$ are correspondingly called velocity components. On the other hand, one can decompose the angular velocity of the rigid body around the instantaneous rotational axis into three components $\overset{*}{\pi}$, $\overset{*}{\chi}$, along three axes that are fixed in the body. They are naturally functions of the $\dot{\varphi}$, $\dot{\psi}$, $\dot{\vartheta}$. Indeed, the mechanics of rigid bodies tells us that the following relations will exist:

$$(71) \qquad \begin{cases} \overset{*}{\pi} = \sin\psi \sin\vartheta \cdot \dot{\varphi} + \cos\psi \cdot \dot{\vartheta} + 0 \cdot \dot{\psi}, \\ \overset{*}{\chi} = \cos\psi \sin\vartheta \cdot \dot{\varphi} - \sin\psi \cdot \dot{\vartheta} + 0 \cdot \dot{\psi}, \\ \overset{*}{\rho} = \qquad \cos\vartheta \cdot \dot{\varphi} + 0 \qquad \cdot \dot{\vartheta} + 1 \cdot \dot{\psi}. \end{cases}$$

([1]) The recognition of this fact, as well as the term "non-holonomic coordinates," goes back to **Boltzmann** [Wien. Sitz.-Ber. **111** (1902), pp. 1603].

In this case, the α_{kr} have the following values:

$$(72)\qquad \begin{cases} \alpha_{11} = \sin\psi\sin\vartheta, & \alpha_{21} = \cos\psi, & \alpha_{31} = 0, \\ \alpha_{12} = \cos\psi\sin\vartheta, & \alpha_{22} = -\sin\psi, & \alpha_{32} = 0, \\ \alpha_{13} = \cos\vartheta, & \alpha_{23} = 0, & \alpha_{33} = 1, \end{cases}$$

and they do not fulfill the integrability conditions (69). For example, one has:

$$\frac{\partial\alpha_{11}}{\partial\vartheta} - \frac{\partial\alpha_{21}}{\partial\varphi} = \frac{\partial}{\partial\vartheta}(\sin\psi\sin\vartheta) - \frac{\partial}{\partial\varphi}(\cos\psi) = \sin\psi\cos\vartheta \neq 0 ,$$

If we multiply equations (71) by dt then the quantities $\overset{*}{\pi}\, dt$, $\overset{*}{\chi}\, dt$, $\overset{*}{\rho}\, dt$ will be *inexact differentials* that we would like to denote by $d'\pi$, $d'\chi$, $d'\rho$:

$$(73)\qquad \begin{cases} d'\pi = \sin\psi\sin\vartheta\cdot d\varphi + \cos\psi\cdot d\vartheta + 0\cdot d\psi, \\ d'\chi = \cos\psi\sin\vartheta\cdot d\varphi - \sin\psi\cdot d\vartheta + 0\cdot d\psi, \\ d'\rho = \cos\vartheta\cdot d\varphi + 0\cdot d\vartheta + 1\cdot d\psi. \end{cases}$$

The quantities $d'\pi$, … have precisely the property that we have just demanded of the differentials of quasi-coordinates. We shall return to this example in more detail later.

In what follows, we shall concern ourselves with an important property of the differentials of non-holonomic coordinates upon which precisely the aforementioned inapplicability of the ordinary **Lagrange** equations is based, in the final analysis.

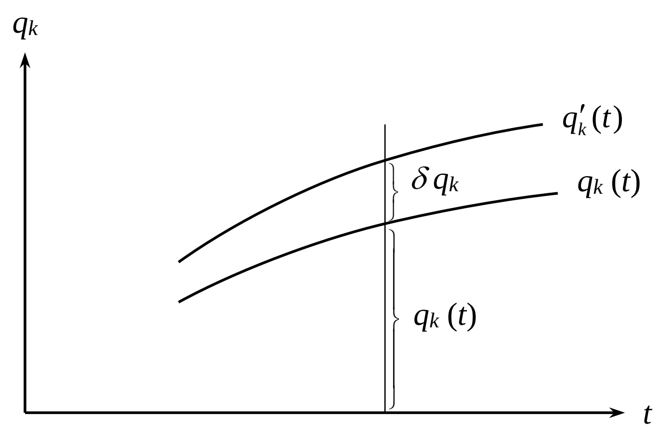

Figure 3.a

Let q_k be a true coordinate and let $\dot{q}_k$ be the associated velocity. We now treat the relationship between the expressions $\delta\left(\frac{dq_k}{dt}\right)$ and $\frac{d}{dt}(\delta q_k)$. We draw the curve $q_k(t)$ in the q_k t-

plane. At the same time, we denote a neighboring curve by $q'_k(t)$. If we associate both curves with each other in such a way that we let each point q_k corresponds to the point q'_k that belongs to the same value of *t*, as is suggested in Fig. 3.a, then by the definition of the δ symbol, we will obviously have $q'_k - q_k = \delta q_k$. Hence:

$$\frac{d}{dt}(\delta q_k) = \frac{dq'_k}{dt} - \frac{dq_k}{dt} = \dot{q}'_k - \dot{q}_k \,. \tag{74}$$

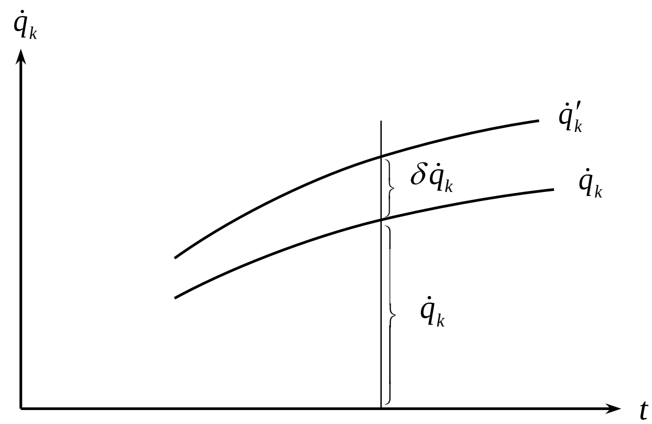

Figure 3.b

If we now draw the curves $\dot{q}_k(t)$ and $\dot{q}'_k(t)$ in the $\dot{q}_k\,t$-plane then from the definition of the δ symbol, we will once more have:

$$\delta\dot{q}_k = \delta\left(\frac{dq_k}{dt}\right) = \dot{q}'_k - \dot{q}_k \,, \tag{75}$$

as we read off directly from Fig. 3.b, and a comparison of (74) and (75) will yield the fundamental relation:

$$\frac{d}{dt}(\delta q_k) = \delta\left(\frac{dq_k}{dt}\right), \tag{76}$$

i.e., *the operations δ and d / dt will commute for holonomic coordinates.* One sees that this is essentially based upon the fact that *t*, the independent variable, remains unvaried. The commutability will not be true for the symbols Δ and d / dt, as one sees immediately.

Let non-holonomic coordinates be defined by the equations:

$$\delta \pi_r = \sum_{k=1}^{N} \alpha_{kr}\, \delta q_k \qquad (r = 1, \ldots, N)\,, \tag{77}$$

and the associated velocities by:

$$\overset{*}{\pi}_r = \sum_{k=1}^{N} \alpha_{kr}\, \dot{q}_k \,, \tag{78}$$

respectively.

If we now take the time derivative of (77) and apply the operation δ to (78) then we will find that:

$$\frac{d}{dt}(\delta\pi_r) = \sum_{k=1}^{N} \alpha_{kr} \quad \frac{d\,\delta q_k}{dt} + \sum_{k=1}^{N} \frac{d\alpha_{kr}}{dt}\delta q_k \ ,$$

$$\delta \overset{*}{\pi}_r = \sum_{k=1}^{N} \alpha_{kr}\, \delta\left(\frac{dq_k}{dt}\right) + \sum_{k=1}^{N} \delta\alpha_{kr} \cdot \frac{dq_k}{dt} \ .$$

Since one has $\frac{d\,\delta q_k}{dt} = \delta\frac{dq_k}{dt}$, from (76), subtraction will yield:

(79) $$\delta \overset{*}{\pi}_r - \frac{d}{dt}(\delta\pi_r) = \sum_k \{\delta\alpha_{kr}\,\dot{q}_k - \dot{\alpha}_{kr}\,\delta q_k\} \ .$$

As calculation will show, the right-hand side of that will be non-zero for non-holonomic coordinates, i.e., the two operations in question do not commute. In order to make that obvious, we write:

$$\dot{\alpha}_{kr} = \sum_\lambda \frac{\partial\alpha_{kr}}{\partial q_\lambda}\dot{q}_\lambda \ ,$$

and when that is substituted in (79), that will produce:

$$\delta \overset{*}{\pi}_r - \frac{d}{dt}(\delta\pi_r) = \sum_k \delta\alpha_{kr}\,\dot{q}_k - \sum_k\sum_\lambda \frac{\partial\alpha_{kr}}{\partial q_\lambda}\delta q_k\,\dot{q}_\lambda \ ,$$

or when we permute the summation indices k and λ in the double sum (which is obviously permissible):

$$\delta \overset{*}{\pi}_r - \frac{d}{dt}(\delta\pi_r) = \sum_k \dot{q}_k \left(\delta\alpha_{kr} - \sum_k\sum_\lambda \frac{\partial\alpha_{kr}}{\partial q_\lambda}\delta q_k\right) \ .$$

Likewise, we have:

$$\delta\,\alpha_{kr} = \sum_\lambda \frac{\partial\alpha_{kr}}{\partial q_\lambda}\delta q_\lambda \ ,$$

so when that is substituted, we will ultimately have:

(80) $$\delta \overset{*}{\pi}_r - \frac{d\,\delta\pi_r}{dt} = \sum_k\sum_\lambda \left\{\frac{\partial\alpha_{kr}}{\partial q_\lambda} - \frac{\partial\alpha_{\lambda r}}{\partial q_k}\right\}\dot{q}_k\,\delta q_k$$

The expression in brackets $\left\{\frac{\partial\alpha_{kr}}{\partial q_\lambda}-\frac{\partial\alpha_{\lambda r}}{\partial q_k}\right\}$ that appears on the right is nothing but the left-hand side of the integrability equation (69), and since that is not fulfilled for non-holonomic coordinates, we will in fact have $\delta\overset{*}{\pi}_r \neq \frac{d\,\delta\pi_r}{dt}$. However, it is precisely the commutability of δ and d / dt that is required in the derivation of the **Lagrange** equations, as we will explain more precisely later.

We can modify equation (80) somewhat. Namely, if we solve equation (78) for $\dot{q}_k$ and (77) for δq_k then we can replace $\dot{q}_k$ and δq_λ with $\overset{*}{\pi}_r$ and $\delta\pi_r$, resp., on the right-hand side of (80). Let the solution of the aforementioned equations be the following:

$$\left\{\begin{aligned} \delta q_\lambda &= \sum_\rho \beta_{\rho\lambda}\,\delta\pi_\rho, \\ \dot{q}_k &= \sum_\sigma \beta_{\sigma\lambda}\,\overset{*}{\pi}_\sigma, \end{aligned}\right. \tag{81}$$

in which ρ and σ are two summation indices. (80) will then become:

$$\delta(\overset{*}{\pi}_r) = \frac{d}{dt}(\delta\pi_r)+\sum_k\sum_\lambda\sum_\rho\sum_\sigma\left\{\frac{\partial\alpha_{kr}}{\partial q_\lambda}-\frac{\partial\alpha_{\lambda r}}{\partial q_k}\right\}\beta_{\rho\lambda}\,\beta_{\sigma k}\,\overset{*}{\pi}_\sigma\,\delta\pi_\rho, \tag{82}$$

or when one permutes ρ and σ:

$$\delta(\overset{*}{\pi}_r) = \frac{d}{dt}(\delta\pi_r)+\sum_\sigma\sum_\rho\left[\sum_k\sum_\lambda \beta_{\rho k}\,\beta_{\sigma\lambda}\left\{\frac{\partial\alpha_{kr}}{\partial q_\lambda}-\frac{\partial\alpha_{\lambda r}}{\partial q_k}\right\}\right]\overset{*}{\pi}_\rho\cdot\delta\pi_\sigma\,. \tag{82.a}$$

One can make a slight alteration to (82.a).

The square bracket depends upon the indices k, r, λ, ρ, σ, of which those two symbols k and λ in the double summation that is to be performed are *summation indices*, while the indices r, ρ, σ are running variables. For that reason, we would like to combine them into the notation $(-\gamma_{r\rho\sigma})$ and then get:

$$\delta(\overset{*}{\pi}_r) = \frac{d}{dt}(\delta\pi_r)+\sum_\sigma\sum_\rho \gamma_{r\rho\sigma}\,\overset{*}{\pi}_\rho\,\delta\pi_\sigma\,. \tag{83}$$

The summarizing of the square bracket into $-\gamma_{r\rho\sigma}$ is justified by the fact that $\gamma_{r\rho\sigma}$ depends upon only the mutual couplings between the $\delta\pi_r$ and the δq_k.

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§ 8.

Hamilton's principle of stationary action and its equivalence with d'Alembert's principle.

The left-hand side of equation (37) includes **d' Alembert**'s principle in the form:

$$\sum_{\nu=1}^{n} [(X_\nu - m_\nu \ddot{x}_\nu)\delta x_\nu + (Y_\nu - m_\nu \ddot{y}_\nu)\delta y_\nu + (Z_\nu - m_\nu \ddot{z}_\nu)\delta z_\nu] \ .$$

That expression will vanish when δx_ν, δy_ν, δz_ν are *virtual* displacements. Naturally, it will be non-zero when that is not the case, in general. In what follows, we would like to leave it undecided whether the δx_ν, δy_ν, δz_ν are or are not virtual displacements. For our initial considerations, it will not matter whether the expression above vanishes, but only when in combination. We convert it into the following form:

$$\sum_\nu (X_\nu\, \delta x_\nu + Y_\nu\, \delta y_\nu + Z_\nu\, \delta z_\nu) - \sum_\nu m_\nu (\ddot{x}_\nu\, \delta x_\nu + \ddot{y}_\nu\, \delta y_\nu + \ddot{z}_\nu\, \delta z_\nu) \ . \tag{84}$$

As is known, the first summand:

$$\sum_\nu (X_\nu\, \delta x_\nu + Y_\nu\, \delta y_\nu + Z_\nu\, \delta z_\nu) = \delta' A \tag{85}$$

represents the work done by the forces X_ν, Y_ν, Z_ν under the arbitrary displacement δx_ν, ... The second sum can be converted term-by-term in the following way:

$$\begin{aligned} m_\nu\, \ddot{x}_\nu\, \delta x_\nu &= \frac{d}{dt}(m_\nu\, \dot{x}_\nu\, \delta x_\nu) - m_\nu\, \dot{x}_\nu \frac{d\,\delta x_\nu}{dt} \\ &= \frac{d}{dt}(m_\nu\, \dot{x}_\nu\, \delta x_\nu) - m_\nu\, \dot{x}_\nu\, \delta \dot{x}_\nu \\ &= \frac{d}{dt}(m_\nu\, \dot{x}_\nu\, \delta x_\nu) - \delta\left(\tfrac{1}{2} m_\nu\, \dot{x}_\nu^2\right) \ . \end{aligned}$$

We will then have:

$$m_\nu (\ddot{x}_\nu\, \delta x_\nu + \ddot{y}_\nu\, \delta y_\nu + \ddot{z}_\nu\, \delta z_\nu) = \frac{d}{dt}[m_\nu (\dot{x}_\nu\, \delta x_\nu + \dot{y}_\nu\, \delta y_\nu + \dot{z}_\nu\, \delta z_\nu)] - \delta \frac{m_\nu}{2}(\dot{x}_\nu^2 + \dot{y}_\nu^2 + \dot{z}_\nu^2) \ .$$

The last term is the variation of the kinetic energy L_ν of the ν^{th} mass-point then. Therefore, upon summing over all mass-points of the system, one will ultimately have:

(86) $$m_\nu(\ddot{x}_\nu\,\delta x_\nu + \ddot{y}_\nu\,\delta y_\nu + \ddot{z}_\nu\,\delta z_\nu) = -\,\delta L + \frac{d}{dt}\left[\sum_\nu m_\nu(\dot{x}_\nu\,\delta x_\nu + \dot{y}_\nu\,\delta y_\nu + \dot{z}_\nu\,\delta z_\nu)\right].$$

Subtracting equation (86) from (85) will then produce the following identity:

(87) $$\sum_{\nu=1}^{n}[(X_\nu - m_\nu\,\ddot{x}_\nu)\,\delta x_\nu + \cdots] = \delta L + \delta' A - \frac{d}{dt}\left[\sum_\nu m_\nu(\dot{x}_\nu\,\delta x_\nu + \dot{y}_\nu\,\delta y_\nu + \dot{z}_\nu\,\delta z_\nu)\right].$$

Here, the **d'Alembert** expression on the left-hand side is transformed with no concern for whether the δx_ν, … are or are not virtual displacements. Since a differential quotient with respect to time is found on the right-hand side, that suggests performing an integration over t between two fixed limits t_0 and t_1. That immediately yields:

$$\int_{t_0}^{t_1}\sum_{\nu=1}^{n}[(X_\nu - m_\nu\,\ddot{x}_\nu)\,\delta x_\nu + \cdots]\,dt = \int_{t_0}^{t_1}(\delta L + \delta' A)\,dt - \left[\sum_\nu m_\nu(\dot{x}_\nu\,\delta x_\nu + \dot{y}_\nu\,\delta y_\nu + \dot{z}_\nu\,\delta z_\nu)\right]_{t_0}^{t_1}.$$

The term outside of the integral sign can now be made to vanish by a suitable assumption on the quantities δx_ν, … at the times t_0 and t_1. Namely, if we assume from now on that the δx_ν, δy_ν, δz_ν are *equal to zero* at the two times t_0 and t_1 (the physical meaning of this will be discussed later) then the expression in question will vanish at the two limits, and will get simply:

(88) $$\int_{t_0}^{t_1}\sum_{\nu=1}^{n}[(X_\nu - m_\nu\,\ddot{x}_\nu)\,\delta x_\nu + \cdots]\,dt = \int_{t_0}^{t_1}(\delta L + \delta' A)\,dt\;.$$

Now, if displacements that were assumed to be arbitrary up to now are virtual then the left-hand side will vanish on the grounds of ***d'Alembert****'s principle, so the right*-*hand side will, as well*, and the statement that:

(89) $$\int_{t_0}^{t_1}(\delta L + \delta' A)\,dt = 0$$

will be completely equivalent in its content to **d'Alembert**'s principle and the equations of motion. Equations (89) is called *Hamilton's principle of stationary action*.

In the special case where the forces can be derived from a potential ($X_\nu = -\dfrac{\partial\Phi(x_1,\ldots,x_n)}{\partial x_\nu}$, …), one has $\delta' A = -\,\delta\Phi$, and when one switches the order of variation δ and integration over t, one can write:

$$\delta \int_{t_0}^{t_1} (L - \Phi)\, dt = \delta \int_{t_0}^{t_1} H\, dt = 0 . \tag{90}$$

According to **Helmholtz**, $H = L - \Phi$ is called the *kinetic potential*.

In the form (90), **Hamilton**'s principle of stationary action will admit a simple formulation: namely, except for an irrelevant constant factor, $\int_{t_0}^{t_1} H\, dt$ is the temporal mean of the kinetic potential. We can then summarize the equations of motion in the simple statement:

The temporal mean of the kinetic potential is stationary for the actual motion, i.e., a maximum, a minimum, or a saddle point value.

Therefore, the actual motion (we shall again restrict our notation to the projection onto the *xt*-plane) will be compared to those neighboring (i.e., varied) paths that have the same initial and final points at times t_0 and t_1 as the actual one, since the δx_ν, … should indeed vanish at those two times. The transition from the actual to a varied path must always take place by way of a *virtual* displacement. If non-holonomic constraints are present then from the foregoing the varied motions that one compares to the actual motion will represent *impossible* motions since they will then contradict the equations of motion.

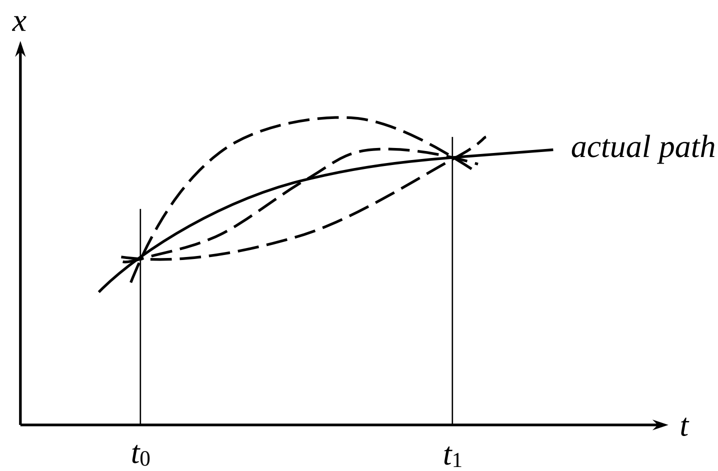

Figure 4.

If we were to demand (but this is not based upon facts) that the *varied* paths should also obey the non-holonomic constraints then that would exclude the use of **Hamilton**'s principle since it would then produce false equations of motion for them. **H. Hertz** ([1]) was of that opinion, although he incorrectly believed that **Hamilton**'s principle should be restricted to holonomic constraints, simply because demanded that the *varied* path should be compatible with the constraints, whereas the principles of mechanics indicate that this should be necessary for the

([1]) **H. Hertz**, *Prinzipien der Mechanik*, pp. 22, *et seq.*

transitional path. In reality, as our derivation shows, **Hamilton**'s principle is true for arbitrary constraints, precisely as **d'Alembert** was.

If one would like to employ **Hamilton**'s principle in order to arrive at the equations of motion then one must merely follow the sequence of equations (89) to (84) in reverse order. In the next section, we will treat the derivation of the **Lagrange** equations from the principle of stationary action in that way.

§ 9.

Lagrange's equations for holonomic and non-holonomic coordinates.

If we introduce general holonomic coordinates q_k and $\dot{q}_k$ into L and the expression $\delta' A$, instead of Cartesian coordinates (which is how they were expressed up to now), then, as is known, L will become a homogeneous quadratic form in the $\dot{q}_k$ whose coefficients are functions of the q_k. One can then say in full generality that L depends upon the q_k and $\dot{q}_k$ then:

$$L = L(q_k, \dot{q}_k),$$

and variation will then yield:

$$\delta L = \sum_{k=1}^{N} \frac{\partial L}{\partial q_k} \delta q_k + \sum_{k=1}^{N} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k . \tag{91}$$

The δx_ν, δy_ν, δz_ν enter into the expression $\delta' A$, which are linear combinations of the δq_k that might take the forms:

$$\delta x_\nu = \sum_k \alpha_k^\nu \, \delta q_k ,$$

$$\delta y_\nu = \sum_k \beta_k^\nu \, \delta q_k$$

$$\delta z_\nu = \sum_k \gamma_k^\nu \, \delta q_k .$$

$X_\nu \, \delta x_\nu + Y_\nu \, \delta y_\nu + Z_\nu \, \delta z_\nu$ will then go to:

$$X_\nu \, \delta x_\nu + Y_\nu \, \delta y_\nu + Z_\nu \, \delta z_\nu = X_\nu \sum_k \alpha_k^\nu \, \delta q_k + Y_\nu \sum_k \beta_k^\nu \, \delta q_k + Z_\nu \sum_k \gamma_k^\nu \, \delta q_k ,$$

so ultimately $\delta' A = \sum_{\nu=1}^{n} (X_\nu \, \delta x_\nu + \cdots)$ will go to:

$$\delta' A = \sum_{\nu=1}^{n} \sum_{k=1}^{N} (X_\nu \, \alpha_k^\nu + Y_\nu \, \beta_k^\nu + Z_\nu \, \gamma_k^\nu) \delta q_k = \sum_{k=1}^{N} \delta q_k \left\{ \sum_{\nu=1}^{n} (X_\nu \, \alpha_k^\nu + Y_\nu \, \beta_k^\nu + Z_\nu \, \gamma_k^\nu) \right\} = \sum_{k=1}^{N} Q_k \, \delta q_k .$$

The Q_k in that is an abbreviation for $\sum_{\nu=1}^{n} (X_\nu \, \alpha_k^\nu + Y_\nu \, \beta_k^\nu + Z_\nu \, \gamma_k^\nu)$.

Its physical meaning is deduced from the fact that when it is multiplied by δq_k, it will have the dimension of work. Accordingly, Q_k is a parallel to X_ν, which also indeed produces a work when multiplied by δx_ν, and for that reason, Q_k will be referred to as the *generalized force* *component* that is produced by the variation δq_k of a coordinate q_k, while the remaining q_k are held constant. One can therefore always write:

(92)
$$\delta' A = \sum_{k=1}^{N} Q_k \, \delta q_k \,,$$

so in the special case where a potential Φ exists, one will naturally have $Q_k = -\frac{\partial \Phi}{\partial q_k}$, so $\delta' A = -\delta\Phi = -\sum_k \frac{\partial \Phi}{\partial q_k} \delta q_k$.

If we substitute (91) and (92) in the principle of stationary action [equation (89)] then we will get:

(93)
$$\int_{t_0}^{t_1} dt \sum_k \frac{\partial L}{\partial q_k} \delta q_k + \int_{t_0}^{t_1} dt \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k + \int_{t_0}^{t_1} dt \sum_k Q_k \, \delta q_k = 0 \,.$$

The second term in that can be converted by partial integration, in which we must make use of the commutability of δ and d / dt, which is true for only holonomic coordinates. That term will become, in turn (we drop the summation sign in the intermediate calculations, for simplicity):

$$\int dt \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k = \int \frac{\partial L}{\partial \dot{q}_k} \frac{d\, \delta q_k}{dt} dt = \left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right]_{t_0}^{t_1} - \int \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k \, dt \,.$$

Since the δx_ν, δy_ν, δz_ν vanish for t_0 and t_1, the same will be true for the δq_k, so the term outside of the integral sign will vanish, and equation (93) will then become:

(94)
$$\int_{t_0}^{t_1} dt \sum_{k=1}^{N} \left\{ \frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) + Q_k \right\} \delta q_k = 0 \,.$$

Since the times t_0 and t_1 are completely arbitrary, but equation (94) is valid in full generality, the integral can vanish only when the integrand is annulled, and due to the complete independence of the N quantities δq_k, it will decompose into N independent equations:

(95)
$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) = Q_k \qquad (k = 1, 2, \ldots, N).$$

Those are the known ***Lagrange** equations of the second kind,* whose validity, as one sees, depends essentially upon the commutability of δ and d / dt, i.e., upon the use of holonomic coordinates.

We would now like to carry out the same calculation ([1]), but under the assumption that we now express the δq_k and $\dot{q}_k$ as non-holonomic coordinate differentials $\delta \pi_r$ and the associated velocities $\overset{*}{\pi}_r$, resp. In that way, $L(q_k, \dot{q}_k)$ might go to $L'(q_k, \overset{*}{\pi}_r)$. If one replaces the δq_k in the given way in the expression $\sum_k Q_k \, \delta q_k$ then $\delta' A$ can obviously be put into the form:

$$\delta' A = \sum_{r=1}^{N} \Pi_r \, \delta\pi_r \tag{96}$$

then, in which the Π_r is the generalized force component that generates the displacement $\delta \pi_r$, while all other $\delta \pi$ are equal to zero.

In **Hamilton**'s principle, we will then have:

$$\int_{t_0}^{t_1} dt \, \delta L' + \int_{t_0}^{t_1} dt \sum_{r=1}^{N} \Pi_r \, \delta\pi_r = 0 ,$$

or, since we have $\delta L' = \sum \frac{\partial L'}{\partial q_k} \delta q_k + \sum_r \frac{\partial L'}{\partial \overset{*}{\pi}_r} \delta \overset{*}{\pi}_r$:

$$\int_{t_0}^{t_1} dt \sum_k \frac{\partial L'}{\partial q_k} \, \delta q_k + \int_{t_0}^{t_1} dt \sum_r \frac{\partial L'}{\partial \overset{*}{\pi}_r} \, \delta \overset{*}{\pi}_r + \int_{t_0}^{t_1} dt \sum_{r=1}^{N} \Pi_r \, \delta\pi_r = 0 . \tag{97}$$

We convert all terms in such a way that they will contain the factor $\delta \pi_r$. We start with the first one. From equation (81), one has (while changing only the summation sign):

$$\delta q_k = \sum_{r=1}^{N} \beta_{rk} \, \delta\pi_r . \tag{98}$$

As a result, the first term can be written:

$$\int_{t_0}^{t_1} dt \sum_{r=1}^{N} \left\{ \sum_{k=1}^{N} \frac{\partial L'}{\partial q_k} \beta_{rk} \right\} \delta\pi_r .$$

Now, if the "π_r" were holonomic coordinates, so from (98), the β_{rk} would be the partial derivatives $\partial q_k / \partial \pi_r$, then one would have:

$$\sum_k \frac{\partial L'}{\partial q_k} \beta_{rk} = \sum_k \frac{\partial L'}{\partial q_k} \frac{\partial q_k}{\partial \pi_r} = \frac{\partial L'}{\partial \pi_r} .$$

([1]) For this, cf., **C. Schaefer**, Phys. Zeit. **19** (1918), pp. 406.

We cannot use this notation here (or only in a *figurative sense*), so we must then make it known that this is an improper notation by enclosing it in parentheses. Thus, we will have:

$$\left(\frac{\partial L'}{\partial \pi_r}\right) \qquad \text{as an abbreviation for:} \qquad \sum_k \frac{\partial L'}{\partial q_k}\beta_{rk}\,.$$

With that, we can write the first term as:

$$\int_{t_0}^{t_1} dt \sum_{k=1}^{N} \frac{\partial L'}{\partial q_k}\delta q_k = \int_{t_0}^{t_1} dt \sum_{r=1}^{N} \left(\frac{\partial L'}{\partial \pi_r}\right)\delta \pi_r\,. \tag{99}$$

In the second term, since the commutation of δ and d / dt is no longer allowed, we must employ the general equation (83):

$$\int_{t_0}^{t_1} dt \sum_r \frac{\partial L'}{\partial \overset{*}{\pi}_r}\,\delta \overset{*}{\pi}_r = \int dt \sum_r \frac{\partial L'}{\partial \overset{*}{\pi}_r}\frac{d\,\delta\pi_r}{dt} - \int dt \sum_r \frac{\partial L'}{\partial \overset{*}{\pi}_r}\sum_\rho\sum_\sigma \gamma_{r\rho\sigma}\,\overset{*}{\pi}_\rho\,\delta\pi_\sigma\,.$$

Since the $\delta\pi_r$ will vanish at the limits of the integral, and because it is true for the δq_k, the first term on the right in that will produce the value $-\int dt \sum_r \frac{d}{dt}\left(\frac{\partial L'}{\partial \overset{*}{\pi}_r}\right)\delta\pi_r$ by ordinary partial integration. We commute the summation signs for r and σ in the second term on the right, in order to get $\delta\pi_r$, instead of $\delta\pi_\sigma$, as in the other expressions. That will then yield:

$$\int dt \sum_r \frac{\partial L'}{\partial \overset{*}{\pi}_r}\,\delta \overset{*}{\pi}_r = -\int dt \sum_r \frac{d}{dt}\left(\frac{\partial L'}{\partial \overset{*}{\pi}_r}\right)\delta\pi_r - \int dt \sum_r\sum_\rho\sum_\sigma \frac{\partial L'}{\partial \overset{*}{\pi}_r}\gamma_{r\rho\sigma}\,\overset{*}{\pi}_r\,\delta\pi_r\,. \tag{100}$$

If one substitutes (99) and (100) in the starting equation (97) then it will follow that:

$$\int_{t_0}^{t_1}\sum_{r=1}^{N}\left[\left(\frac{\partial L'}{\partial \pi_r}\right) - \frac{d}{dt}\left(\frac{\partial L'}{\partial \overset{*}{\pi}_r}\right) - \sum_\rho\sum_\sigma \gamma_{r\rho\sigma}\frac{\partial L'}{\partial \overset{*}{\pi}_\sigma}\overset{*}{\pi}_\sigma + \Pi_r\right]\delta\pi_r = 0\,,$$

from which it will follow by the usual argument that was also used above (viz., the $\delta\pi_r$ are mutually independent, just like the δq_k) that ([1]):

([1]) This equation was first proved by **L. Boltzmann** [Wien. Sitz.-Ber. **111** (1902), pp. 1603; also *Ges. Abh.* III, pp. 682]. Later, **G. Hamel** [Math. Ann. **59** (1904), pp. 416, *et seq.*] devoted an extended investigation to that question.

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \overset{*}{\pi}_r}\right) - \left(\frac{\partial L'}{\partial \pi_r}\right) + \sum_\rho \sum_\sigma \gamma_{r\rho\sigma} \frac{\partial L'}{\partial \overset{*}{\pi}_\sigma} \overset{*}{\pi}_\sigma = \Pi_r \qquad (r = 1, 2, \ldots, N). \tag{101}$$

Since the $\gamma_{r\rho\sigma}$ are proportional to the left-hand sides of the integrability conditions (69), the equation above will reduce immediately to the ordinary **Lagrange** one when the coordinates are holonomic, as it must. Equation (101) is the form of it that is valid for non-holonomic coordinates. In what follows, we will refer to it by the name of the "extended **Lagrange** equation."

§ 10.

Applying the extended Lagrange equations to the Euler equations of a rigid body.

We would like to take the **Euler** equations of a rigid body as an example of the application of the generalized **Lagrange** equations.

If we express the kinetic energy of a rigid body that is fixed at a point in terms of the components $\overset{*}{\pi}$, $\overset{*}{\chi}$, $\overset{*}{\rho}$ of the angular velocities around three axes that are fixed in the body then, as is known, that will give:

$$L' = \tfrac{1}{2} A \overset{*}{\pi}{}^2 + \tfrac{1}{2} B \overset{*}{\chi}{}^2 + \tfrac{1}{2} C \overset{*}{\rho}{}^2 , \tag{102}$$

in which A, B, C are the principal moments of inertia for the fixed point. The quantity of work $\delta' A$ can then be expressed in terms of the infinitesimal rotation angles $\delta\pi$, $\delta\chi$, $\delta\rho$ as:

$$\delta' A = \Pi\, \delta\pi + \mathrm{X}\, \delta\chi + \mathrm{P}\, \delta\rho , \tag{103}$$

in which Π, X, P are the moments of the external forces about the principal axes of inertia.

In equation (71), we already gave the relations between the velocities $\overset{*}{\pi}$, $\overset{*}{\chi}$, $\overset{*}{\rho}$, and those of the **Euler** angles $\dot{\varphi}$, $\dot{\chi}$, $\dot{\vartheta}$:

$$\left\{\begin{aligned} \overset{*}{\pi} &= \sin\psi \sin\vartheta \cdot \dot{\varphi} + \cos\psi \cdot \dot{\vartheta} + 0 \cdot \dot{\psi}, \\ \overset{*}{\chi} &= \cos\psi \sin\vartheta \cdot \dot{\varphi} - \sin\psi \cdot \dot{\vartheta} + 0 \cdot \dot{\psi}, \\ \overset{*}{\rho} &= \qquad \cos\vartheta \cdot \dot{\varphi} + 0 \quad \cdot \dot{\vartheta} + 1 \cdot \dot{\psi}. \end{aligned}\right. \tag{104}$$

With our previous notation, the coefficients α_{kr} will then have the values:

$$\left\{\begin{array}{lll} \alpha_{11} = \sin\psi \sin\vartheta, & \alpha_{21} = \cos\psi, & \alpha_{31} = 0, \\ \alpha_{12} = \cos\psi \sin\vartheta, & \alpha_{22} = -\sin\psi, & \alpha_{32} = 0, \\ \alpha_{13} = \cos\vartheta, & \alpha_{23} = 0, & \alpha_{33} = 1. \end{array}\right. \tag{105}$$

If one solves equations (104) then one will get:

$$
(106) \qquad \begin{cases}
\dot{\varphi} = \dfrac{\sin\psi}{\sin\vartheta}\cdot\overset{*}{\pi} + \dfrac{\cos\psi}{\sin\vartheta}\cdot\overset{*}{\chi} + 0\cdot\overset{*}{\rho}\,, \\
\dot{\vartheta} = \cos\psi\,\overset{*}{\pi} - \sin\psi\cdot\overset{*}{\chi} + 0\cdot\overset{*}{\rho}\,, \\
\dot{\psi} = -\dfrac{\cos\vartheta\sin\psi}{\sin\vartheta}\cdot\overset{*}{\pi} - \dfrac{\cos\vartheta\cos\psi}{\sin\vartheta}\cdot\overset{*}{\chi} + 1\cdot\overset{*}{\rho}\,,
\end{cases}
$$

with the coefficients $\beta_{r\kappa}$:

$$
(107) \qquad \begin{cases}
\beta_{11} = \dfrac{\sin\psi}{\sin\vartheta}, & \beta_{21} = \dfrac{\cos\psi}{\sin\vartheta}, & \beta_{31} = 0, \\
\beta_{12} = \cos\psi, & \beta_{22} = -\sin\psi, & \beta_{32} = 0, \\
\beta_{13} = -\dfrac{\cos\vartheta\sin\psi}{\sin\vartheta}, & \beta_{23} = -\dfrac{\cos\vartheta\cos\psi}{\sin\vartheta}, & \beta_{33} = 1.
\end{cases}
$$

The quantities α_{kr} obviously do not fulfill the integrability conditions (69), so $d\pi$, $d\chi$, $d\rho$ are the differentials of non-holonomic coordinates, i.e., in our previous notation, they would be denoted by $d\pi_1$, $d\pi_2$, $d\pi_3$.

If one would then like to employ the expression (102) for the kinetic energy, as well as (103) for the elementary work $\delta' A$, in order to derive the equations of motion for the rigid body from **Lagrange**’s then one must apply the *extended* ***Lagrange*** equations (101), not the simple ones (95).

The latter would imply the following:

$$
\left(\frac{\partial L'}{\partial \pi}\right) = \left(\frac{\partial L'}{\partial \chi}\right) = \left(\frac{\partial L'}{\partial \rho}\right) = 0\,,
$$

since L' does not depend do not depend upon the coordinates at all, and furthermore:

$$
\frac{\partial L'}{\partial \overset{*}{\pi}} = A\,\overset{*}{\pi}\,, \qquad \frac{\partial L'}{\partial \overset{*}{\chi}} = B\,\overset{*}{\chi}\,, \qquad \frac{\partial L'}{\partial \overset{*}{\rho}} = C\,\overset{*}{\rho}\,,
$$

so one would arrive at the following equations:

$$
(108) \qquad \begin{cases}
A\dfrac{d\,\overset{*}{\pi}}{dt} = \Pi\,, \\
B\dfrac{d\,\overset{*}{\chi}}{dt} = \mathrm{X}\,, \\
C\dfrac{d\,\overset{*}{\rho}}{dt} = \mathrm{P}\,,
\end{cases}
$$

while **Euler**'s read:

$$(109)\qquad \begin{cases} A\,\dfrac{d\overset{*}{\pi}}{dt} + (C-B)\,\overset{*}{\chi}\,\overset{*}{\rho} = \Pi, \\[2ex] B\,\dfrac{d\overset{*}{\chi}}{dt} + (A-C)\,\overset{*}{\rho}\,\overset{*}{\pi} = \mathrm{X}, \\[2ex] C\,\dfrac{d\overset{*}{\rho}}{dt} + (B-A)\,\overset{*}{\pi}\,\overset{*}{\chi} = \mathrm{P}. \end{cases}$$

We must then employ *only* the extended equations, which will then add the term:

$$\sum_{\rho=1}^{3}\sum_{\sigma=1}^{3} \gamma_{r\rho\sigma}\,\frac{\partial L'}{\partial \overset{*}{\pi}_\sigma}\,\overset{*}{\pi}_\sigma$$

to the left-hand sides of equation (108).

One will next find the following values for the $\gamma_{r\rho\sigma}$ by somewhat-tedious, but elementary, calculations:

$$(109.\text{a})\qquad \begin{vmatrix} \gamma_{111}=0 & \gamma_{211}=0 & \gamma_{311}=0 \\ \gamma_{121}=0 & \gamma_{221}=0 & \gamma_{321}=-1 \\ \gamma_{131}=0 & \gamma_{231}=1 & \gamma_{331}=0 \end{vmatrix}, \quad \begin{vmatrix} \gamma_{112}=0 & \gamma_{212}=0 & \gamma_{312}=1 \\ \gamma_{122}=0 & \gamma_{222}=0 & \gamma_{322}=0 \\ \gamma_{132}=-1 & \gamma_{232}=0 & \gamma_{332}=0 \end{vmatrix}, \quad \begin{vmatrix} \gamma_{113}=0 & \gamma_{213}=-1 & \gamma_{313}=0 \\ \gamma_{123}=1 & \gamma_{223}=0 & \gamma_{323}=0 \\ \gamma_{133}=0 & \gamma_{233}=0 & \gamma_{333}=0 \end{vmatrix}.$$

One sees that all quantities $\gamma_{r\rho\sigma}$ with two or three equal indices are equal to zero, while all of the ones with three different indices will be equal to – 1 or + 1. Let us now calculate the nine-term sums:

$$\sum_{\rho=1}^{3}\sum_{\sigma=1}^{3} \gamma_{r\rho\sigma}\,\frac{\partial L'}{\partial \overset{*}{\pi}_\sigma}\,\overset{*}{\pi}_\sigma$$

for each value of r. Here, that will simplify to two terms since from the matrices in (109.a), only two of any nine values of each γ will be non-zero. For $r = 1$, i.e., for $\overset{*}{\pi}_1 = \overset{*}{\pi}$, that will give the following expression:

$$\overset{*}{\pi}_3\,\frac{\partial L'}{\partial \overset{*}{\pi}_2}\,\gamma_{132} + \overset{*}{\pi}_2\,\frac{\partial L'}{\partial \overset{*}{\pi}_3}\,\gamma_{123} = \overset{*}{\pi}_2\,\frac{\partial L'}{\partial \overset{*}{\pi}_3} - \overset{*}{\pi}_3\,\frac{\partial L'}{\partial \overset{*}{\pi}_2} = \overset{*}{\chi}\,\frac{\partial L'}{\partial \overset{*}{\rho}} - \overset{*}{\rho}\,\frac{\partial L'}{\partial \overset{*}{\chi}} \ .$$

If one substitutes those values then one will find that the additional term is:

$$(C-B)\,\overset{*}{\chi}\,\overset{*}{\rho},$$

with which, the incorrect equations (108) will, in fact, go to the correct **Euler** equations (109). Q.E.D.

§ 11.

The Hölder transformation ([1]).

In order to prove the complete equivalence of the principle of stationary action with **d'Alembert**'s, in § **8**, we started from the expression on the left-hand side of **d'Alembert**'s principle:

$$\sum_{\nu=1}^{n} [(X_\nu - m_\nu \ddot{x}_\nu)\delta x_\nu + (Y_\nu - m_\nu \ddot{y}_\nu)\delta y_\nu + (Z_\nu - m_\nu \ddot{z}_\nu)\delta z_\nu] ,$$

in which δx_ν, δy_ν, δz_ν are not initially virtual displacements, so the expression itself needs to be non-zero. The variations that enter into it would all be characterized by the symbol δ, i.e., *time would not be varied along with everything else*.

We would now like to transform that expression, but apply the Δ process everywhere, instead of δ.

That first requires a prefatory remark. Whereas we have proved that $\frac{\delta\, dx}{dt} = \frac{d\,\delta x}{dt}$, the commutability of δ and d / dt, which is based upon the fact that time is not varied, will no longer be true for Δ and d / dt, which we already pointed out at that time. We will then be dealing with the relationship between $\Delta\left(\frac{dx}{dt}\right)$ and $\frac{d\,(\Delta x)}{dt}$. In order to answer that question, it is appropriate to introduce a new independent variable Θ, which we will make depend upon both x and t; it shall also *not* be varied under the Δ process. We shall then reduce the presently-complicated case to the previously-simpler one by that parametric representation of x and t. We have to set: $\frac{dx}{dt} = \frac{x'}{t'}$, when we denote the derivatives with respect to the parameter Θ by a prime $\left(\frac{dx}{d\Theta} = x', \frac{dt}{d\Theta} = t'\right)$.

We find, in succession, that:

$$\Delta\left(\frac{x'}{t'}\right) = \frac{t'\,\Delta x' - x'\,\Delta t'}{t'^2} = \frac{\Delta x'}{t'} - \frac{x'}{t'} \cdot \frac{\Delta t'}{t'}$$

or

([1]) **O. Hölder**, "Über die Prinzipien von **Hamilton** und **Maupertuis**," Nachr. kgl. Ges. Wiss. Göttingen, math.-phys. Klasse (1896), pp. 122, *et seq.*

$$\Delta\left(\frac{x'}{t'}\right) = \frac{\Delta\left(\dfrac{dx}{d\Theta}\right)}{\left(\dfrac{dt}{d\Theta}\right)} - \frac{\left(\dfrac{dx}{d\Theta}\right)}{\left(\dfrac{dt}{d\Theta}\right)} \cdot \frac{\Delta\left(\dfrac{dt}{d\Theta}\right)}{\left(\dfrac{dt}{d\Theta}\right)},$$

or, since $\Delta\left(\dfrac{dx}{d\Theta}\right) = \dfrac{d}{d\Theta}\Delta x$, etc., since Θ is not varied in that, one will have:

$$\Delta\left(\frac{dx}{dt}\right) = \Delta\left(\frac{x'}{t'}\right) = \frac{\dfrac{d\,\Delta x}{d\Theta}}{\dfrac{dt}{d\Theta}} - \left(\frac{dx}{dt}\right) \cdot \frac{\dfrac{d\,\Delta t}{d\Theta}}{\dfrac{dt}{d\Theta}} = \frac{d\,\Delta x}{dt} - \dot{x}\,\frac{d\,\Delta t}{dt}.$$

We ultimately find that:

(110) $$\Delta\left(\frac{dx}{dt}\right) = \frac{d}{dt}(\Delta x) - \dot{x}\,\frac{d\,\Delta t}{dt}.$$

Moreover, one can also derive that equation from (67) directly.

With that preliminary remark, we go on to derive the transformation of the **d'Alembert** expression. Let the kinetic energy of a system of n mass-points be:

$$L = \sum_{\nu=1}^{n} \tfrac{1}{2} m_\nu (\dot{x}_\nu^2 + \dot{y}_\nu^2 + \dot{z}_\nu^2) \;.$$

We then form ΔL :

(111) $$\Delta L = \Delta \sum_{\nu=1}^{n} \tfrac{1}{2} m_\nu (\dot{x}_\nu^2 + \dot{y}_\nu^2 + \dot{z}_\nu^2) = \sum_{\nu=1}^{n} m_\nu\,(\dot{x}_\nu\,\Delta\dot{x}_\nu + \dot{y}_\nu\,\Delta\dot{y}_\nu + \dot{z}_\nu\,\Delta\dot{z}_\nu),$$

or from (110):

$$\begin{aligned}\Delta L &= \sum_{\nu=1}^{n} m_\nu \left[\dot{x}_\nu\left(\frac{d\,\Delta x_\nu}{dt} - \dot{x}_\nu\,\frac{d\,\Delta t}{dt}\right) + \dot{y}_\nu\left(\frac{d\,\Delta y_\nu}{dt} - \dot{y}_\nu\,\frac{d\,\Delta t}{dt}\right) + \dot{z}_\nu\left(\frac{d\,\Delta z_\nu}{dt} - \dot{z}_\nu\,\frac{d\,\Delta t}{dt}\right)\right] \\ &= \sum_{\nu=1}^{n} m_\nu \left(\dot{x}_\nu\,\frac{d\,\Delta x_\nu}{dt} + \dot{y}_\nu\,\frac{d\,\Delta y_\nu}{dt} + \dot{z}_\nu\,\frac{d\,\Delta z_\nu}{dt}\right) - \sum_{\nu=1}^{n} \tfrac{1}{2} m_\nu (\dot{x}_\nu^2 + \dot{y}_\nu^2 + \dot{z}_\nu^2)\,\frac{d\,\Delta t}{dt},\end{aligned}$$

or if one observes the value of L :

(112) $$\Delta L + 2L\,\frac{d\,\Delta t}{dt} = \sum_{\nu=1}^{n} m_\nu \left(\dot{x}_\nu\,\frac{d\,\Delta x_\nu}{dt} + \dot{y}_\nu\,\frac{d\,\Delta y_\nu}{dt} + \dot{z}_\nu\,\frac{d\,\Delta z_\nu}{dt}\right).$$

The right-hand side of that can be further treated – term-by-term – according to the following rule:

$$\dot{x}_\nu \frac{d\,\Delta x_\nu}{dt} = \frac{d}{dt}(\dot{x}_\nu\,\Delta x_\nu) - \ddot{x}_\nu\,\Delta x_\nu\,, \quad \text{etc.}$$

It will then follow that:

$$\Delta L + 2L\frac{d\,\Delta t}{dt} = \frac{d}{dt}\left[\sum_{\nu=1}^{n} m_\nu\left(\dot{x}_\nu\,\Delta x_\nu + \dot{y}_\nu\,\Delta y_\nu + \dot{z}_\nu\,\Delta z_\nu\right)\right] - \sum_{\nu=1}^{n} m_\nu\left(\ddot{x}_\nu\,\Delta x_\nu + \ddot{y}_\nu\,\Delta y_\nu + \ddot{z}_\nu\,\Delta z_\nu\right),$$

or

$$(113)\qquad -\sum_{\nu=1}^{n} m_\nu\left(\ddot{x}_\nu\,\Delta x_\nu + \ddot{y}_\nu\,\Delta y_\nu + \ddot{z}_\nu\,\Delta z_\nu\right) = \Delta L + 2L\frac{d\,\Delta t}{dt} - \frac{d}{dt}\left[\sum_{\nu=1}^{n} m_\nu\left(\dot{x}_\nu\,\Delta x_\nu + \dot{y}_\nu\,\Delta y_\nu + \dot{z}_\nu\,\Delta z_\nu\right)\right].$$

That suggests an integration over time. If we simultaneously determine the Δx_ν, Δy_ν, Δz_ν (as in **Hamilton**'s principle) such that they will vanish at the limits of the integral at the times t_0 and t_1 then the term that emerges from the integral will drop out, and what will remain:

$$(114)\qquad -\int_{t_0}^{t_1} dt \sum_{\nu=1}^{n} m_\nu\left(\ddot{x}_\nu\,\Delta x_\nu + \ddot{y}_\nu\,\Delta y_\nu + \ddot{z}_\nu\,\Delta z_\nu\right) = \int_{t_0}^{t_1}\left(\Delta L + 2L\frac{d\,\Delta t}{dt}\right)dt\,.$$

If we further denote the expression $\sum_{\nu=1}^{n}\left(X_\nu\,\Delta x_\nu + Y_\nu\,\Delta y_\nu + Z_\nu\,\Delta z_\nu\right)$ by $\Delta' A$ and integrate over time from t_0 to t_1 then we will have:

$$(115)\qquad \int_{t_0}^{t_1} dt \sum_{\nu=1}^{n}\left(X_\nu\,\Delta x_\nu + Y_\nu\,\Delta y_\nu + Z_\nu\,\Delta z_\nu\right) = \int_{t_0}^{t_1}\Delta' A\,dt\,.$$

Adding (114) and (115) to the left-hand side of the **d'Alembert** expression, integrating over t, will then yield the desired transformation of the right-hand side:

$$(116)\quad \int_{t_0}^{t_1} dt \sum_{\nu=1}^{n}[(X_\nu - m_\nu\,\ddot{x}_\nu)\Delta x_\nu + (Y_\nu - m_\nu\,\ddot{y}_\nu)\Delta y_\nu + (Z_\nu - m_\nu\,\ddot{z}_\nu)\Delta z_\nu] = \int_{t_0}^{t_1} dt\left(\Delta L + \Delta' A + 2L\frac{d\,\Delta t}{dt}\right).$$

We would like to call this equation the ***Hölder** transformation*. It represents a generalization of equation (88) and differs from it only by the fact that the more general Δ process was applied in it. If we then set $\Delta t = 0$ in equation (116), the Δx_ν, … will become identical to the δx_ν, …, and (116) will go to (88), from which the principle of stationary action will then follow.

The meaning of the **Hölder** transformation lies in the following fact:

Naturally, the variation that appears in (116) is much more general than the previous one, due to the appearance of Δt. We can then prescribe any sort of relation between the quantities Δx_ν, Δy_ν, Δz_ν, and Δt, i.e., *suitably restrict* the most-general variations that enter into (116). For every chosen restriction, we will get a new dynamical principle from (116). The most radical restriction would be $\Delta t = 0$, and that would then imply just **Hamilton**'s principle of stationary action.

In the next section, we shall discuss those constraints that will make (116) lead to the so-called *principle of least action* that was first presented by **Euler** and **Maupertuis** but formulated precisely by **Lagrange**.

§ 12.

The various forms of the principle of least action.

We would next like to treat equation (116) further without introducing any specialization into it.

We first set:

$$\Delta x_\nu = \delta x_\nu + \dot{x}_\nu \, \Delta t\,, \qquad \text{etc.},$$

in the left-hand side of it and obtain:

$$(117)\qquad \left\{ \begin{aligned} &\int_{t_0}^{t_1} dt \sum_{\nu=1}^{n} [(X_\nu - m_\nu\,\ddot{x}_\nu)\,\delta x_\nu + (Y_\nu - m_\nu\,\ddot{y}_\nu)\,\delta y_\nu + (Z_\nu - m_\nu\,\ddot{z}_\nu)\,\delta z_\nu] \\ &+ \int_{t_0}^{t_1} dt \sum_{\nu=1}^{n} [X_\nu\,\delta x_\nu + Y_\nu\,\delta y_\nu + Z_\nu\,\delta z_\nu] \\ &- \int_{t_0}^{t_1} dt \sum_{\nu=1}^{n} m_\nu\,[\ddot{x}_\nu\,\dot{x}_\nu + \ddot{y}_\nu\,\dot{y}_\nu + \ddot{z}_\nu\,\dot{z}_\nu]\,\Delta t = \int_{t_0}^{t_1} dt \left[\Delta L + \Delta' A + 2L\,\frac{d\,\Delta t'}{dt} \right]. \end{aligned} \right.$$

Now, any sort of constraints – say, m of them – might be prescribed, perhaps non-holonomic, rheonomic ones, in order to remain as general as possible:

$$(118)\qquad \sum_{\nu=1}^{n} (a'_{k\nu}\,dx_\nu + b'_{k\nu}\,dy_\nu + c'_{k\nu}\,dz_\nu + a'_\nu\,dt) = 0 \qquad (k = 1, 2, \ldots, m).$$

If we subject the δx_ν, δy_ν, δz_ν, which were completely-free up to now, to the constraints that are valid for *virtual* displacements:

$$\sum_{\nu=1}^{n} (a'_{k\nu}\,\delta x_\nu + b'_{k\nu}\,\delta y_\nu + c'_{k\nu}\,\delta z_\nu) = 0 \qquad (k = 1, 2, \ldots, m)$$

then the first term on the right-hand side of (117) will vanish, according to **d'Alembert**'s principle. Moreover, the square bracket in the second term obviously represents the work done per unit time $d'A/dt$, and the one in the third term represents the change in kinetic energy per unit time dL / dt. If we introduce that into (117) then we will find the following expression for the virtual displacements, which is completely equivalent to **d'Alembert**'s principle, so to the equations of motion:

$$(119)\qquad \int_{t_0}^{t_1} dt \left[\Delta L + \Delta' A + 2L\,\frac{d\,\Delta t}{dt} - \frac{d'A}{dt}\,\Delta t + \frac{dL}{dt}\,\Delta t \right] = 0\,.$$

That equation, in turn, represents a very general principle of dynamics from which the equations of motion can be obtained by purely-formal process.

From what we said before, we can now restrict the variations that enter into (119) in a more suitable way. Every such auxiliary condition will then produce a special dynamical principle.

Now, **Lagrange** ([1]) has shown that one will arrive at the principle of least action that **Euler** first expressed in some special cases when one *demands the validity of the energy principle for the transitional path that is composed of virtual displacements*, i.e., when one demands that one should have:

$$\delta L = \delta' A. \tag{120}$$

That does not say, by any means, that the energy principle should be valid for the true or varied path, which is indeed not the case for rheonomic constraints.

If we now assume, for the sake of simplicity, that the constraints are *scleronomic* (whether holonomic or non-holonomic) then it will follow that the energy equation $\frac{dL}{dt} = \frac{d'A}{dt}$ is, in fact, true for the true motion, and (119) will simplify to:

$$\int_{t_0}^{t_1} dt \left[\Delta L + \Delta' A + 2L \frac{d\,\Delta t}{dt} \right] = 0 . \tag{121}$$

If we introduce the **Lagrange** "transition condition" (120) into that, which we can also write in the following form, in which δ is expressed as Δ [according to equation (67)]:

$$\Delta L - \frac{dL}{dt}\,\Delta t = \Delta' A - \frac{dA}{dt}\,\Delta t , \tag{120.a}$$

then (121) will imply that:

$$\int_{t_0}^{t_1} dt \left[2\,\Delta L + 2L \frac{d\,\Delta t}{dt} \right] = 2\int [\,\Delta L\,dt + L\,d\,\Delta t\,] = 0.$$

However, $\Delta L\; dt + L\; d\;\Delta t$ is equal to $\Delta\,(L\;dt)$, so one can drop the factor of 2 and write:

$$\int_{t_0}^{t_1} \Delta\,(L\,dt) = 0 ,$$

or after switching the integral sign with Δ:

([1]) Cf., also the presentation by **H. von Helmholtz**, "Zur Geschichte des Prinzips der kleinsten Aktion," *Ges. Abhandl.*, Bd. III, pp. 249.

$$\Delta \int_{t_0}^{t_1} L\,dt = 0 . \tag{122}$$

That equation is called the *principle of least action*, and we shall return to its formulation in more detailed words.

Up to now, it was derived under the assumption that the constraints were scleronomic. If we now drop that restricting assumption then the law of energy $\frac{dL}{dt} = \frac{d'A}{dt}$ will no longer be fulfilled for the true motion, and we must once more start from the general equation (119):

$$\int_{t_0}^{t_1} dt \left[\Delta L + \Delta' A + 2L \frac{d\,\Delta t}{dt} - \frac{d'A}{dt}\Delta t + \frac{dL}{dt}\Delta t \right] = 0 , \tag{119}$$

into which we have again introduced the **Lagrange** transition condition in the form (120.a). It will then follow that:

$$\int_{t_0}^{t_1} dt \left[2\,\Delta L - \frac{dL}{dt}\Delta t + \frac{d'A}{dt}\Delta t + 2L \frac{d\,\Delta t}{dt} - \frac{d'A}{dt}\Delta t + \frac{dL}{dt}\Delta t \right] = 0 ,$$

or

$$\int [\,\Delta L\,dt + L\,d\,\Delta t\,] = 0 ,$$

or:

$$\Delta \int_{t_0}^{t_1} L\,dt = 0 , \tag{122}$$

as above, *i.e., the principle of least action, which is now proved under the most general assumptions.*

Except for an irrelevant constant factor, $\int_{t_0}^{t_1} L\,dt$ is the temporal mean value of the kinetic energy. We can then say:

For the true motion, the temporal mean value of the kinetic energy is stationary, i.e., a maximum, minimum, or saddle-point value.

The true path is then compared to those varied paths that have the same starting point and end point as the former has at time t_0 and t_1, and in addition satisfy the condition that the energy principle should be true under the transition from the true to the varied motion.

The first statement would be meaningless without that additional condition.

More precisely, we would like to refer to (122) as the *broader form of the principle of least action*. We once more stress that it is *not* assumed in it that the energy principle is valid for the true (and varied) path, but its validity is required of only the transitional motion.

We will get a *narrower* form for the aforementioned principle when we also assume the energy principle for the true motion, and indeed in the form (as an energy integral):

$$L + \Phi = h \,. \tag{123}$$

According to **Jacobi** ([1]), one can proceed as follows: Time t and the time differential dt are eliminated from equation (122). For the sake of generality, we then set L equal to:

$$2\,L = \sum_{i=1}^{N}\sum_{k=1}^{N} a_{ik}\,\frac{dq_i}{dt}\,\frac{dq_k}{dt} \tag{124}$$

in terms of generalized coordinates q_k and velocities $\dot{q}_k$ (namely, as a homogeneous quadratic form in the latter) then that will imply dt^2 in the form:

$$dt^2 = \frac{\sum_{i,k} a_{ik}\,dq_i\,dq_k}{2L}\,,$$

or from (123):

$$dt = \sqrt{\frac{\sum a_{ik}\,dq_i\,dq_k}{2(h-\Phi)}}\,. \tag{125}$$

Since Φ depends upon only q_k, dt is then expressed in terms of *purely geometrical quantities*.

We likewise replace L with $(h - \Phi)$ in (122) and get:

$$\Delta\int \sqrt{\frac{h-\Phi}{2}}\cdot\sqrt{\sum a_{ik}\,dq_i\,dq_k} = 0\,, \tag{126}$$

instead of (122). Naturally, geometric data is substituted at the limits of the integral at times t_0 and t_1, namely, one gives the configuration of the system at those times.

Equation (126) is the *Jacobi form of the principle of least action*, and from the above, it is valid only when the energy integral is valid for the true path, i.e., only for the so-called *complete systems*. However, it is precisely in mechanics that one must chiefly deal with incomplete systems, and the applicability of (126) will then be restricted. **Jacobi**'s mistake, if one might call it that, consisted of the fact that he believed that this was the only form that was valid, while in reality, (122) is valid much more generally.

Now, the fact that the **Jacobi** principle assumes the energy principle for the true path demands that the energy principle must also be fulfilled for the **Lagrange** transition condition (120). That is because energy has the value $L + \Phi = h$ for each point P of the true path, i.e., it is equal to a constant. If I go to the associated point P' of the varied path then $\delta h = 0$, according to

([1]) **C. G. J. Jacobi**, *Vorlesungen über Dynamik, Werke, Supplementband*, 1884. Lecture 6.

Lagrange, i.e., the point P' will correspond to the same value of energy as P. One can then arrange for that to be true for every point of the true and varied path, and since all points of the true path have the same value of h, that will also be true for the varied path. That is the same line of reasoning by which we previously proved that when a holonomic equation of constraint is fulfilled by the true path, that will also be the case for the varied path since that is the case for the transitional path.

The equations of motion are implied by the broader form (122) when one goes through the line of reasoning that was used to derive it in reverse order. We shall not go into that. Rather, we would like to show that the **Jacobi** form (126) does, in fact, lead to the equations of motion for a complete system. Indeed, we would like to implement that for Cartesian coordinates, which we would like to denote here, not by x_1, y_1, z_1, …, x_n, y_n, z_n, but in general by x_i, which runs through x_1, x_2, x_3, x_4, x_5, …, for the sake of convenience. L will then have the following form:

$$L = \tfrac{1}{2}\sum m_i \left(\frac{dx_i}{dt}\right)^2 ,$$

and it will follow from (126) that:

$$\Delta\int \sqrt{\frac{h-\Phi}{2}}\sqrt{\sum m_i\,dx_i^2} = 0 . \tag{127}$$

Here, it would be convenient to regard the x_i as functions of a parameter Θ that is not affected by the variation. We therefore write (127) as:

$$\Delta\int_{t_0}^{t_1} \sqrt{\frac{h-\Phi}{2}}\sqrt{\sum m_i \left(\frac{dx_i}{d\Theta}\right)^2}\,d\Theta = 0 . \tag{127.a}$$

If we denote the derivatives by a prime, as before, then that will give the general form of the variational principle that is included in (127.a)

$$\Delta\int_{\Theta_0}^{\Theta_1} F(x_i, x_i')\,d\Theta , \tag{128}$$

in which we have set:

$$F(x_i, x_i') = \sqrt{\frac{h-\Phi}{2}}\sqrt{\sum_i m_i\,x_i'^2} . \tag{129}$$

Upon developing (128), it will follow that:

$$\int_{t_0}^{t_1} d\Theta\,\Delta F = \int_{t_0}^{t_1} d\Theta\left[\sum_i \frac{\partial F}{\partial x_i}\Delta x_i + \sum_i \frac{\partial F}{\partial x_i'}\Delta x_i'\right] = 0 . \tag{130}$$

The second term with $\Delta x_i'$ can be converted by partial integration:

$$\int_{t_0}^{t_1} \frac{\partial F}{\partial x_i'}\, \Delta x_i'\, d\Theta = \int_{t_0}^{t_1} \frac{\partial F}{\partial x_i'}\, \frac{d\, \Delta x_i'}{d\Theta} d\Theta = \left[\frac{\partial F}{\partial x_i'} \Delta x_i \right]_{\Theta_0}^{\Theta_1} - \int_{\Theta_0}^{\Theta_1} \frac{d}{d\Theta} \left(\frac{\partial F}{\partial x_i'} \right) \Delta x_i \, d\Theta \, .$$

The term outside the integral drops out since the Δx_i should vanish at the limits, as before. Therefore, (130) will become:

(131) $$\int_{\Theta_0}^{\Theta_1} d\Theta \sum_i \left[\frac{\partial F}{\partial x_i} - \frac{d}{d\Theta} \left(\frac{\partial F}{\partial x_i'} \right) \right] \Delta x_i = 0 \, .$$

If we take the simplest case in which no constraints exist, then it will follow by the argument that has already been applied frequently that:

(132) $$\frac{\partial F}{\partial x_i} - \frac{d}{d\Theta} \left(\frac{\partial F}{\partial x_i'} \right) = 0$$

for all values of *i*.

One will see that those are actually the equations of motion for a complete system when one performs the differentiations on the *F* in (129). One will initially get:

(133) $$\frac{\partial F}{\partial x_i} = \frac{-\frac{1}{2} \frac{\partial \Phi}{\partial x_i} \sum m_i \, x_i'^{\,2}}{2\sqrt{\frac{h - \Phi}{2}} \cdot \sum m_i \, x_i'^{\,2}} \, .$$

However, one had:

$$2\, L = \sum m_i \left(\frac{dx_i}{d\Theta} \right)^2 \left(\frac{d\Theta}{dt} \right)^2 = \sum m_i \, x_i'^{\,2} \left(\frac{d\Theta}{dt} \right)^2 ;$$

hence:

$$\sum m_i \, x_i'^{\,2} = 2L \left(\frac{dt}{d\Theta} \right)^2 = 2(h - \Phi) \left(\frac{dt}{d\Theta} \right)^2 .$$

When that is substituted in (133), that will give:

(134) $$\frac{\partial F}{\partial x_i} = -\frac{1}{2} \frac{\partial \Phi}{\partial x_i} \left(\frac{dt}{d\Theta} \right).$$

Moreover, from (129):

$$\frac{\partial F}{\partial x_i'} = \frac{\sqrt{\frac{h-\Phi}{2}}\, m_i\, x_i'}{\sqrt{\sum_i m_i\, x_i'^{\,2}}} = \sqrt{\frac{\frac{1}{2}(h-\Phi)}{2\,(h-\Phi)\left(\frac{dt}{d\Theta}\right)^2}}\; m_i\, x_i' = \tfrac{1}{2} m_i \frac{dx_i}{d\Theta}\frac{d\Theta}{dt} = \tfrac{1}{2} m_i \frac{dx_i}{dt} \,,$$

so one will finally have:

$$\frac{d}{d\Theta}\left(\frac{\partial F}{\partial x_i'}\right) = \tfrac{1}{2} m_i \frac{d}{d\Theta}\left(\frac{dx_i}{dt}\right).$$

When (134) and (135) are substituted in (132), that will yield:

$$\frac{\partial \Phi}{\partial x_i}\frac{dt}{d\Theta} + m_i \frac{d}{d\Theta}\left(\frac{dx_i}{dt}\right) = 0 \,,$$

or

$$\frac{\partial \Phi}{\partial x_i} + m_i \frac{d}{d\Theta}\left(\frac{dx_i}{dt}\right)\frac{d\Theta}{dt} = 0 \,,$$

or finally:

$$m_i \frac{d^2 x_i}{dt^2} = -\frac{\partial \Phi}{\partial x_i} \,,$$

which was to be proved.

One can put the principle of least action into yet another form. If one takes the velocity of the i^{th} mass-point to be $v_i = ds_i / dt$, in which ds_i means the path element of that mass-point, then equation (122) can be written:

$$\Delta \int L\, dt = \tfrac{1}{2}\Delta \int \sum m_i\, v_i \frac{ds_i}{dt}\, dt = \tfrac{1}{2}\Delta \int \sum m_i\, v_i\, ds_i = 0 \,,$$

or after cancelling the constant factor:

(136) $$\Delta \int \sum m_i\, v_i\, ds_i = 0 \,.$$

That equation gave the principle its name, since one cares to refer to the expression $m_i\, v_i\, ds_i$ as the "action." Of course, it is by no means the case that the total action $\int \sum m_i\, v_i\, ds_i$ must be a minimum, as the epithet "least" action might suggest. However, such questions, which are interesting in the calculus of variations, play no role in dynamics.

In the special case of *one* mass-point, it will then follow that:

(137) $$\Delta \int_{s_0}^{s_1} m\, v\, ds = 0 \,,$$

and if it is also force-free then since its velocity v would be constant then, it will follow that:

$$\Delta \int_{s_0}^{s_1} ds = 0 . \tag{138}$$

That equation admits a simple physical interpretation:

$\int_{s_0}^{s_1} ds$ is the length of the path that is laid through the force-free particles between the points s_0 and s_1 . From (138), the length shall always have an extremal value, and in the simplest case (which will always occur when the points s_0 and s_1 lie close enough to each other), it will be a minimum. However, the shortest path between two points is a straight line. Thus, (138) immediately implies the statement that a force-free mass-point moves rectilinearly.

In the foregoing, it was tacitly assumed that the force-free mass-point is subject to no kinematical constraints. However, if such a thing is prescribed then, in general, the mass-point will naturally no longer describe a straight line on the surface that it prescribes. However, it is natural that the curve on the surface that is described between two sufficiently-close points s_0 and s_1 will also have the "minimal property" that is expressed by equation (138). Such curves are called *geodetic* or, according to **Hertz**, *straightest*. It is easy to obtain their equation. We regard the x_i in (138) as functions of a parameter Θ in terms of which that equation can be written, and since $ds^2 = dx_1^2 + dx_2^2 + dx_3^2$, we will have:

$$\Delta \int_{\Theta_0}^{\Theta_1} \sqrt{x_1'^2 + x_2'^2 + x_3'^2}\, d\Theta = 0 . \tag{138.a}$$

If we further multiply the prescribed constraint equation $\varphi = 0$ by $d\,\Theta$ and integrate between two limits, as in (138.a), then we will get:

$$\int_{\Theta_0}^{\Theta_1} \varphi\, d\Theta = 0 . \tag{138.b}$$

That equation is fulfilled at the same time as (138.a). We multiply that equation by an unknown factor λ and add it to the integral in (138.a). The variation of the new integral that thus arises must vanish then. That will then give:

$$\Delta \int_{\Theta_0}^{\Theta_1} \left\{ \sqrt{x_1'^2 + x_2'^2 + x_3'^2} + \lambda\, \varphi \right\} d\Theta = \Delta \int_{\Theta_0}^{\Theta_1} F(x_i, x_i')\, d\Theta = 0 .$$

We will then have an integral of the form (128), and performing the variation will produce the equation:

$$\frac{\partial F}{\partial x_i} = \frac{d}{d\Theta}\left(\frac{\partial F}{\partial x_i'}\right).$$

Evaluating that for, e.g., x_1 will then yield:

$$\lambda \frac{\partial \varphi}{\partial x_1} = \frac{d}{d\Theta}\left(\frac{x_1'}{\sqrt{x_1'^2 + x_2'^2 + x_3'^2}}\right).$$

If we now choose the parameter Θ to be the arc-length s then we will obviously have $\sqrt{x_1'^2 + x_2'^2 + x_3'^2} = 1$, and the equation above will go to:

$$\frac{d^2 x}{ds^2} = \lambda \frac{\partial \varphi}{\partial x_1},$$

to which two corresponding ones must be added for x_2 and x_3. They will be the equations for the geodetic line on the surface $\varphi = 0$ that the mass-point describes.

The form (138) can also be obtained for a force-free *system*. To that end, we start from the **Jacobi** equation (127), in which the factor $\sqrt{(h - \Phi)/2}$ will be constant, due to the assumed absence of forces, so it can be dropped. What will remain is:

$$\Delta \int \sqrt{\sum m_i \, dx_i^2} = 0.$$

Now, **Heinrich Hertz** ([1]) defined:

$$ds^2 \sum m_i = \sum m_i \, dx_i^2 \tag{139}$$

in his theory of mechanics.

ds^2 is then a quadratic mean value that **Hertz** referred to as the *square of the path-element of the system*. With that, the **Jacobi** form will again become:

$$\Delta \int ds = 0. \tag{140}$$

Hertz called that formula (140), in conjunction with his definition of ds that is included in (139), the *principle of the straightest path*.

Now, it is interesting that one can also impose that same form when conservative forces are at work. To that end, one must extend **Hertz**'s definition (139) to:

([1]) **H. Hertz**, *Die Prinzipien der Mechanik*, pp. 69.

(141) $$d\sigma^2 \sum m_i = \left(\frac{h-\Phi}{2}\right) \sum m_i\, dx_i^2 \,.$$

Since one interprets $d\sigma$ as the *path-element of the system* (of course, the manner by which it is composed from the dx_i will no longer be the Euclidian form that comes from the Pythagorean theorem), that will obviously give, in turn:

(142) $$\Delta \int d\sigma = 0 \,.$$

One can then reduce the motion of a system in Euclidian space under the influence of conservative forces to the motion of a force-free system in a general Euclidian manifold in a purely-formal way.

That viewpoint has a strong similarity to the appearance of non-Euclidian geometry in **Einstein**'s general theory of relativity.

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§ 13.

Hamilton's canonical equations.

Hamilton transformed the **Lagrange** equations in a convenient way and thus obtained a particularly-symmetric system that has been confirmed in all more-detailed investigations in dynamics, electrodynamics, and statistical mechanics, and in particular, at the hands of **Maxwell**, **Boltzmann**, and **Gibbs**. We shall now move on to the derivation of those equations.

The **Lagrange** equations read:

(95)
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k .$$

One now assumes the existence of a potential Φ, which will make Q_k go to $-\partial\Phi/\partial q_k$. One can then write (95) as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial}{\partial q_k}(L - \Phi) = 0 ,$$

or also, since Φ depends upon only the q_k, *but not* upon the $\dot{q}_k$, when one adds the vanishing term $-\partial\Phi/\partial\dot{q}_k$:

$$\frac{d}{dt}\left(\frac{\partial (L - \Phi)}{\partial \dot{q}_k}\right) - \frac{\partial}{\partial q_k}(L - \Phi) = 0 ,$$

or when one introduces the notation H for the difference $L - \Phi$, which was previously called the *kinetic potential*:

(143)
$$\frac{d}{dt}\left(\frac{\partial H}{\partial \dot{q}_k}\right) - \frac{\partial H}{\partial q_k} = 0 \qquad (k = 1, 2, \ldots, N).$$

The expressions $\partial H/\partial\dot{q}_k$ have a simple physical meaning that we can clarify by a dimensional consideration. H has the dimensions of an energy $[M\,L^2\,T^{-2}]$. If we assume, for the moment, that q_k has the dimension of length, which is indeed the case for, e.g., Cartesian coordinates, then $\dot{q}_k$ will have the dimension of a velocity $[L\,T^{-1}]$, $\partial H/\partial\dot{q}_k$ will have the dimension $[M\,L\,T^{-1}]$, i.e., the dimension of an *impulse*. In fact, for Cartesian coordinates, $\partial H/\partial\dot{q}_k$ will coincide with the impulse components, which one can confirm by calculation. Now, in just the same way that we referred to q_k and $\dot{q}_k$ as generalized coordinates and generalized velocities, even when they did not have the dimensions of length and velocity, and just as we further referred to the quantities Q_k as the generalized force components, we will call the quantities $\partial H/\partial\dot{q}_k$ the *generalized impulse components* and denote them by p_k :

(144) $$\frac{\partial H}{\partial \dot{q}_k} = p_k \qquad (k = 1, 2, \ldots, N).$$

The **Lagrange** equations will then read:

(145) $$\frac{\partial H}{\partial q_k} = \frac{dp_k}{dt}.$$

We now consider a variation δ of $H(q_k, \dot{q}_k)$, which will read:

$$\delta H = \sum_k \frac{\partial H}{\partial q_k} \delta q_k + \sum_k \frac{\partial H}{\partial \dot{q}_k} \delta \dot{q}_k$$

when written out in detail, or from (144) and (145):

(146) $$\delta H = \sum_k \frac{dp_k}{dt} \delta q_k + \sum_k p_k \, \delta \dot{q}_k .$$

If one further forms:

$$\delta \sum_k p_k \, \dot{q}_k = \sum_k p_k \, \delta \dot{q}_k + \sum_k \dot{q}_k \, \delta p_k$$

then it will follow that:

(147) $$\sum_k p_k \, \delta \dot{q}_k = \delta \sum_k p_k \, \dot{q}_k - \sum_k \dot{q}_k \, \delta p_k ,$$

and when that is substituted in (146), that will give:

$$\delta H = \sum_k \frac{dp_k}{dt} \delta q_k + \delta \sum_k p_k \, \dot{q}_k - \sum_k \dot{q}_k \, \delta p_k ,$$

or

(148) $$\delta\left(H - \sum p_k \, \dot{q}_k\right) = \sum_k \frac{dp_k}{dt} \delta q_k - \sum_k \frac{dq_k}{dt} \delta p_k .$$

If we now consider the function $H - \sum p_k \, \dot{q}_k = R$ to depend upon the quantities p_k and q_k then we can further write the variation of R, which is considered to be a function of just those variables, as:

(149) $$\delta R = \sum_k \frac{\partial R}{\partial p_k} \delta p_k + \sum_k \frac{\partial R}{\partial q_k} \delta q_k ,$$

and a comparison of that with (148) will yield the double system of equations:

$$\text{(150)} \qquad \frac{dp_k}{dt} = \frac{\partial R}{\partial q_k}, \qquad -\frac{dq_k}{dt} = \frac{\partial R}{\partial q_k},$$

whose distinctive symmetry is apparent.

Equations (150) are ***Hamilton****'s equations of dynamics*. They are equivalent to the **Lagrange** equations in the form (143). Under some circumstances, one can further simplify **Hamilton**'s equations when the function *R*, viz., the *modified kinetic potential*, takes on a fundamental physical meaning, namely, that of minus the total energy. That will be the case if and only if the kinetic energy *L* is a homogeneous quadratic form in the velocity components $\dot{q}_k$, which does not always need to be the case. One calls the cases in which that requirement is fulfilled "natural" problems. If we now assume that we are dealing with such a thing then, from (144), we will have:

$$\sum p_k \dot{q}_k = \sum \frac{\partial H}{\partial \dot{q}_k} \dot{q}_k ,$$

or, since $H = L - \Phi$, in which Φ does not depend upon $\dot{q}_k$:

$$\sum p_k \dot{q}_k = \sum \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k .$$

Moreover, since *L* is now a homogeneous quadratic form, from **Euler**'s theorem on homogeneous functions:

$$\sum_k \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k = 2 L .$$

Thus, one has $\sum p_k \dot{q}_k = 2 L$, and *R* will become equal to:

$$\text{(151)} \qquad R = H - \sum_k p_k \dot{q}_k = L - \Phi - 2 L = -(L + \Phi) = -E ,$$

in succession, when one lets *E* denote the total energy, which is regarded as a function of the so-called *canonical variables* p_k and q_k . Equations (150) will then become:

$$\text{(152)} \qquad -\frac{dp_k}{dt} = +\frac{\partial E}{\partial q_k}, \qquad \frac{dq_k}{dt} = +\frac{\partial E}{\partial p_k}.$$

That is the form of **Hamilton**'s equations that is ordinarily employed, namely, the so-called *canonical form of the equations of dynamics*. However, the domain of their validity is restricted by the demand that one must be dealing with a "natural" problem.

It would be useful to clarify the meaning of the canonical equations in the example of Cartesian coordinates. We take, say, a mass-point that is pulled back to its rest position by an elastic force. For it, we will have:

$$L = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2), \qquad \Phi = \tfrac{1}{2} k^2 (x^2 + y^2 + z^2).$$

The impulses are:

$$p_1 = m\dot{x}, \qquad p_2 = m\dot{y}, \qquad p_3 = m\dot{z},$$

so E will be a function of the impulses and coordinates:

$$E = \frac{1}{2m}(p^2 + p^2 + p^2) + \frac{k^2}{2}(x^2 + y^2 + z^2).$$

From (152), one will then have:

$$-\frac{dp_1}{dt} = -m\ddot{x} = \frac{\partial E}{\partial x} = k^2 x,$$

etc., and likewise:

$$\frac{dq_1}{dt} = \frac{dx}{dt} = \frac{p_1}{m}.$$

The second equation in (152) then defines the impulse, while the first one yields the actual equations of motion.

§ 14.

General variation of Hamilton’s principal function.

Previously, in the principle of stationary action, one was led to the integral:

$$S = \int_{t_0}^{t_1} (L - \Phi)\, dt = \int_{t_0}^{t_1} H\, dt \tag{153}$$

under the assumption that a potential energy Φ existed. The principle of stationary action, in turn, indeed consisted of saying that δ process would produce the value zero when it is applied to that integral under certain assumptions about the behavior at the limits.

Hamilton had varied the function S, which he called the *principal function*, in a general manner without prescribing any conditions on the limits. At the same time, the variation of time was also permitted. In brief, he had then applied the Δ process to S and achieved some results that were important to dynamics in that way.

We thus form $\Delta S = \Delta \int_{t_0}^{t_1} H\, dt$. It will then follow that:

$$\Delta \int_{t_0}^{t_1} H\, dt = \int_{t_0}^{t_1} \Delta H\, dt + \int_{t_0}^{t_1} H\, d\,\Delta t\,,$$

or when we consider H to be a function of the q_k and $\dot{q}_k$ (we can drop the summation signs up to the conclusion of the calculation, for the sake of clarity):

$$\Delta S = \int \frac{\partial H}{\partial q_k} \Delta q_k\, dt + \int \frac{\partial H}{\partial \dot{q}_k} \Delta \dot{q}_k\, dt + \int H\, d\,\Delta t\,.$$

However, from equation (110), one will have:

$$\Delta \dot{q}_k = \Delta\left(\frac{dq_k}{dt}\right) = \frac{d}{dt}(\Delta q_k) - \dot{q}_k \frac{d\,\Delta t}{dt}$$

since the independent variable t is varied along with the others under the Δ process, and it will further follow that:

$$\Delta S = \int \frac{\partial H}{\partial q_k} \Delta q_k\, dt + \int \frac{\partial H}{\partial \dot{q}_k} \frac{d\,\Delta q_k}{dt}\, dt - \int \frac{\partial H}{\partial \dot{q}_k}\, \dot{q}_k\, d\,\Delta t + \int H\, d\,\Delta t\,,$$

or, after combining the third and fourth terms:

$$(154)\qquad \Delta S = \int \frac{\partial H}{\partial q_k}\Delta q_k\,dt + \int \frac{\partial H}{\partial \dot{q}_k}\frac{d\,\Delta q_k}{dt}\,dt + \int \left(H - \frac{\partial H}{\partial \dot{q}_k}\,\dot{q}_k\right) d\,\Delta t\,.$$

The second, and then the third, integrals are now converted by partial integration:

$$(155)\qquad \int \frac{\partial H}{\partial \dot{q}_k}\frac{d\,\Delta q_k}{dt}\,dt = \left[\frac{\partial H}{\partial \dot{q}_k}\Delta q_k\right]_{t_0}^{t_1} - \int \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{q}_k}\right)\Delta q_k\,dt\,,$$

as well as:

$$(156)\qquad \int \left(H - \frac{\partial H}{\partial \dot{q}_k}\,\dot{q}_k\right) d\,\Delta t = \left[\left(H - \frac{\partial H}{\partial \dot{q}_k}\,\dot{q}_k\right)\Delta t\right]_{t_0}^{t_1} - \int \frac{d}{dt}\left(H - \frac{\partial H}{\partial \dot{q}_k}\,\dot{q}_k\right)\Delta t\,dt\,.$$

If one substitutes both results in equation (154) then one will get:

$$(157)\qquad \left\{\begin{aligned} \Delta S = {} & \int \frac{\partial H}{\partial q_k}\Delta q_k\,dt - \int \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{q}_k}\right)\Delta q_k\,dt - \int \frac{d}{dt}\left(H - \frac{\partial H}{\partial \dot{q}_k}\,\dot{q}_k\right)\Delta t\,dt \\ & + \left[\left(H - \frac{\partial H}{\partial \dot{q}_k}\,\dot{q}_k\right)\Delta t + \frac{\partial H}{\partial \dot{q}_k}\Delta q_k\right]_{t_0}^{t_1}. \end{aligned}\right.$$

The differentiation with respect to t is performed in the third term:

$$(158)\qquad \frac{d}{dt}\left(H - \frac{\partial H}{\partial \dot{q}_k}\,\dot{q}_k\right) = \frac{\partial H}{\partial q_k}\,\dot{q}_k - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{q}_k}\right)\dot{q}_k\,.$$

When combined with (157), that will give:

$$(159)\qquad \left\{\begin{aligned} \Delta S = {} & \int \frac{\partial H}{\partial q_k}\Delta q_k\,dt - \int \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{q}_k}\right)\Delta q_k\,dt - \int \frac{\partial H}{\partial q_k}\,\dot{q}_k\;\Delta t\,dt \\ & + \int \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{q}_k}\right)\dot{q}_k\,\Delta t\,dt + \left[\left(H - \frac{\partial H}{\partial \dot{q}_k}\,\dot{q}_k\right)\Delta t + \frac{\partial H}{\partial \dot{q}_k}\Delta q_k\right]_{t_0}^{t_1}. \end{aligned}\right.$$

When we now reinsert the summation sign everywhere, a suitable combination of that will further imply that:

$$\Delta S = \int \sum_k \left[\frac{\partial H}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{q}_k} \right) \right] \{ \Delta q_k - \dot{q}_k \;\Delta t \} \, dt + \left[\left(H - \sum_k \frac{\partial H}{\partial \dot{q}_k} \, \dot{q}_k \right) \Delta t + \sum_k \frac{\partial H}{\partial \dot{q}_k} \Delta q_k \right]_{t_0}^{t_1} .$$

Now, since $\Delta q_k - \dot{q}_k \;\Delta t = \delta q_k$, that can now be written more simply as:

$$\Delta S = \int \sum_k \left[\frac{\partial H}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{q}_k} \right) \right] \delta q_k \, dt + \left[\left(H - \sum_k \frac{\partial H}{\partial \dot{q}_k} \, \dot{q}_k \right) \Delta t + \sum_k \frac{\partial H}{\partial \dot{q}_k} \Delta q_k \right]_{t_0}^{t_1} . \tag{160}$$

That is **Hamilton**'s fundamental formula, upon which his further conclusions were based.

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§ 15.

Hamilton's differential equation for the principal function and the integrals of the equation of motion.

We would now like to assume that the coordinates q_k are mutually independent. Obviously, that means that the system is *holonomic*, because for such a thing, that can always be achieved by a suitable choice of coordinates. **Lagrange**'s equations are then valid, and the first integral in (160) will vanish. What will remain is:

$$\Delta S = \left[\left(H - \sum_k \frac{\partial H}{\partial \dot{q}_k} \dot{q}_k \right) \Delta t + \sum_k \frac{\partial H}{\partial \dot{q}_k} \Delta q_k \right]_{t_0}^{t_1} . \tag{161}$$

$H - \sum_k \frac{\partial H}{\partial \dot{q}_k} \dot{q}_k = H - \sum_k p_k \dot{q}_k$ is nothing but the function R that was introduced in § **13**. If we then assume that we are dealing with a "natural" problem then $R = -E$, in which E means the total energy. With that, (161) will become:

$$\Delta S = E_0 \, \Delta t_0 - E_1 \, \Delta t_1 - \sum p_k^1 \, \Delta q_k^1 - \sum p_k^0 \, \Delta q_k^0 \tag{162}$$

when written out in detail. In that, we have denoted the values of the impulse components at times t_0 and t_1 by p_k^0 and p_k^1, resp., and the indices on the other quantities mean something analogous.

Naturally, we again associate the true path with a varied path by way of the Δ symbol, which has not been restricted in any way up to now. However, following **Hamilton**, we would now like to establish the following:

1. The neighboring path shall likewise begin at the time-point $t = t_0$, i.e., we shall always set $\Delta t_0 = 0$ in what follows.

2. The neighboring paths shall be generated only by varying the initial conditions q_k^0 and p_k^0 on the true path, i.e., we imagine that a family of varied paths exists that *are subject to the same forces and constraints as the true path, except that the initial values are not* q_k^0 *and* p_k^0*, but* $q_k^0 + \Delta q_k^0$ *and* $p_k^0 + \Delta p_k^0$*, resp.*

When we now drop the upper index of 1, for simplicity's sake, (162) will then become:

(163) $$\Delta S = -E\,\Delta t + \sum_k p_k\,\Delta q_k - \sum p_k^0\,\Delta q_k^0 \;.$$

We would now like to express ΔS in a certain way. The integrals of our system of N degrees of freedom exhibit the coordinates q_k and velocities $\dot{q}_k$ as functions of time t, the initial coordinates q_k^0, and the initial velocities $\dot{q}_k^0$ (the initial impulses p_k^0, resp.):

(164) $$q_k = q_k\,(q_k^0, p_k^0, t)\,,$$

(165) $$\dot{q}_k = \dot{q}_k\,(q_k^0, p_k^0, t)\,.$$

The true motion will be determined completely by that. From the above, the varied motions will emerge from that when we substitute the values $q_k^0 + \Delta q_k^0$ and $p_k^0 + \Delta p_k^0$, in place of q_k^0 and p_k^0, resp. With those equations, we can express the q_k and $\dot{q}_k$ as functions of time, the initial coordinates q_k^0, and the initial impulse p_k^0, and substitute those functions in the principal function, which was naturally considered to be a function of q_k and $\dot{q}_k$ up to now. *By means of the integrals, which are assumed to be known, we will then get S as a function of the initial coordinates and impulses, as well as time t*. Finally, we once more eliminate the initial impulses from S, and likewise by means of the integral (164). Indeed, they represent N relations between the $(3N + 1)$ quantities q_k, q_k^0, p_k^0, t. We can then obtain the N quantities p_k^0 as functions of the quantities q_k, q_k^0, t, i.e., the instantaneous and initial coordinates and time, with the help of those equations, substitute that in S, and ultimately eliminate the p_k^0 from S in that way. *After those operations, S will be represented as something that depends upon the quantities* q_k^0, *the initial coordinates, the* q_k, *the instantaneous coordinates, and time t*. Following **Boltzmann**, we would like to call the function S, when it is represented in that way, the *principal function*, as represented in the "**Hamiltonian** way," and denote it by the index H:

(166) $$S\,(q_k,\ q_k^0,\ t) \equiv S_H\,.$$

E will be represented similarly, and that is why we will also assign it the index H accordingly.

If we now go from the true path to the varied path in the manner that was described above then S will go to $S + \Delta S$, since the q_k^0 will be converted into $q_k^0 + \Delta q_k^0$ and the p_k^0, into $p_k^0 + \Delta p_k^0$, while the upper limit t will be converted into $t + \Delta t$.

From (166), ΔS_H can obviously be written:

(167) $$\Delta S_H = \frac{\partial S_H}{\partial t}\,\Delta t + \sum_k \frac{\partial S_H}{\partial q_k}\,\Delta q_k + \sum_k \frac{\partial S_H}{\partial q_k^0}\,\Delta q_k^0 \;.$$

A comparison of that with (163) will then imply the following three equations:

(168) $$\frac{\partial S_H}{\partial t} + E = 0 \,,$$

(169) $$\frac{\partial S_H}{\partial q_k} = \quad p_k \,,$$

(170) $$\frac{\partial S_H}{\partial q_k^0} = -p_k^0 . \qquad (k = 1, 2, \ldots, N),$$

From the foregoing, the partial derivatives of S_H are understood as follows: $\partial S_H / \partial t$ formed with constant values of q_k and q_k^0. In that case, $(\Delta S_H)_{q_k, q_k^0}$ is non-zero only due to the fact that the upper limit t has been converted into $t + \Delta t$. The same is true of the remaining partial derivatives.

What do equations (168) to (170) mean then?

Equations (168) is a partial differential equation for S_H, namely, the so-called ***Hamilton** differential equation*. It will be satisfied identically by the function S_H that we have constructed from the integrals of the equations of motion, which we have assumed to be known. S_H includes the N arbitrary, non-additive constants q_k^0, and an additive constant can be added to it, in addition, such that we will have $(N + 1)$ independent constants in all, which is exactly as many as the independent variables in (q_k, t). Due to that property, one calls S_H a *complete integral* of the partial differential equation.

The derivatives $\partial S_H / \partial q_k$ and $\partial S / \partial q_k^0$ include *no* differential quotients with respect to time. By contrast, p_k is a linear combination of the general velocity components $\dot{q}_k$, so it will then include the *first* derivatives with respect to t ; the p_k^0 are constants. We can then characterize equations (169) as follows:

They are N equations that express the velocities $\dot{q}_k$ in terms of constants and time: *They are then the N first ("intermediate") integrals of the equations of motion.* Likewise, equations (170) are N relations that express the general coordinates q_k in terms of constants and time: *They are N second integrals of motion*.

It might be good for us to explain the type of substitutions and constructions that have occurred here by the simplest-possible example. We deal with the free case. In that case, the integrals are known to read:

(171) $$\begin{cases} \dot{x} = \dot{x}_0 - g\,t, \\ x = x_0 + \dot{x}_0\,t - \tfrac{1}{2}\,g\,t^2. \end{cases}$$

The kinetic and potential energy will then be:

$$L = \tfrac{1}{2} m \dot{x}^2 = \tfrac{1}{2} m (\dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2) \,,$$

$$\Phi = m\,g\,x = m\,g\,(x_0 + \dot{x}_0\,t - \tfrac{1}{2}\,g\,t^2)\,,$$

while the total energy E and the kinetic potential Φ are:

(172)
$$\begin{cases} E = L + \Phi = \tfrac{1}{2}\,m\,\dot{x}_0^2 + m\,g\,x_0, \\ H = L - \Phi = \left(\tfrac{1}{2}\,m\,\dot{x}_0^2 - m\,g\,x_0\right) - 2\,m\,g\,\dot{x}_0\,t + m\,g^2\,t^2. \end{cases}$$

Finally, when we set the lower limit $t_0 = 0$, for simplicity, $S = \int H\,dt$ will become:

(173)
$$S = \left(\tfrac{1}{2}\,m\,\dot{x}_0^2 - m\,g\,x_0\right) t - m\,g\,\dot{x}_0\,t^2 + \tfrac{1}{3}\,m\,g^2\,t^3.$$

However, neither E nor S have been expressed in the ***Hamiltonian*** *way* up to now. To that end, we must represent the $\dot{x}_0$ in terms of x and t by using the second equation in (171):

(174)
$$\dot{x}_0 = \frac{x - x_0 + \tfrac{1}{2}\,g\,t^2}{t}.$$

With that, we will then get:

$$E_H = \tfrac{1}{2}\,m\,\frac{(x - x_0 + \tfrac{1}{2}\,g\,t^2)^2}{t^2} + m\,g\,x_0\,,$$

$$S_H = \frac{m}{2t}\left(x - x_0 + \tfrac{1}{2}\,g\,t^2\right)^2 - m\,g\,x_0\,t - m\,g\,t\left(x - x_0 + \tfrac{1}{2}\,g\,t^2\right) + \tfrac{1}{3}\,m\,g^2\,t^2.$$

Those values of S_H and E_H must fulfill **Hamilton**'s partial differential equation (168). It is:

$$\frac{\partial S}{\partial t} = m\left(x - x_0 + \tfrac{1}{2}\,g\,t^2\right)g - \frac{m}{2t^2}\left(x - x_0 + \tfrac{1}{2}\,g\,t^2\right)^2 - m\,g\,x_0 - m\,g\,(x - x_0) - \tfrac{3}{2}\,m\,g^2 t^2 + m\,g^2 t^2\,.$$

If one now forms equation (168) with the value of E_H above then one will find that:

$$\left\{m\,g\left(x - x_0 + \tfrac{1}{2}\,g\,t^2\right) - \frac{m}{2t^2}\left(x - x_0 + \tfrac{1}{2}\,g\,t^2\right)^2 - m\,g\,x_0 - m\,g\,x + m\,g\,x_0 - \tfrac{1}{2}\,m\,g^2 t^2\right\}$$
$$+ \left\{\frac{m}{2t^2}\left(x - x_0 + \tfrac{1}{2}\,g\,t^2\right)^2 + m\,g\,x_0\right\} = 0\,,$$

and that equation will be fulfilled identically since the terms on the left-hand side cancel pairwise.

Similarly, one will have, e.g., with equation (170): $\dfrac{\partial S_H}{\partial x_0} = -\,m\,\dot{x}_0\,.$

One will find that:

$$\frac{\partial S_H}{\partial x_0} = -\frac{m}{t}\left(x - x_0 + \tfrac{1}{2} g\, t^2\right),$$

and from (174), that is, in fact, equal to $-m\,\dot{x}_0$, which was to be proved.

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§ 16.

Jacobi's converse of Hamilton's theorem on the actual determination of the integrals.

How can one apply those arguments then? If we have only **Hamilton**'s theorem, which is included in equations (168) to (170), then its usefulness would be very limited in terms of obtaining the integrals of the equations of motion, since we had to assume they existed in order to arrive at the theorem.

We would like to see what we can exhibit when we *do not* possess those integrals.

We know the expression for the kinetic energy as a function of the q_k and $\dot{q}_k$, or q_k and p_k, respectively, since we have $\partial L / \partial \dot{q}_k = p_k$; we likewise know Φ as a function of the q_k. We can then construct the total energy $E = L + \Phi$, as well as the kinetic potential $H = L - \Phi$ as a function of the q_k. However, we can eliminate the p_k from E by employing equation (169), i.e., we can set $p_k = \partial S / \partial q_k$. We will then get E as a function of the q_k and $\partial S / \partial q_k$, which we would like to suggest by $E\left(q_k, \dfrac{\partial S}{\partial q_k}\right)$.

Hamilton's partial differential equation (168) for S will then read:

$$\frac{\partial S}{\partial t} + E\left(q_k, \frac{\partial S}{\partial q_k}\right) = 0 . \tag{175}$$

We shall drop the index H from now on.

Up to now, we could calculate only one solution that satisfied that equation identically from the known integrals of the equations of motion, and indeed a complete solution with N independent constants q_k^0. Now, when the integrals are unknown, the same will be true for S, and we must employ the differential equation (175) precisely in order to determine S.

Now, **Jacobi** ([1]) proved the following theorem:

If an *arbitrary* complete integral of equation (175) is known, i.e., a solution with N independent, arbitrary, multiplicative constants $\alpha_1, \alpha_2, \ldots, \alpha_N$ (which do not, by any means, need to be identical to the q_k^0), then one will get the integrals of the motion when one forms equations (169) and (170) with that arbitrary solution, i.e., one determines the impulses p_k by means of the equations:

$$\frac{\partial S}{\partial q_k} = p_k \qquad (k = 1, 2, \ldots, N) \tag{176}$$

([1]) **C. G. J. Jacobi**, *Vorlesungen über Dynamik,* Lecture 20.

(they are the N first or intermediate integrals), and one further sets the derivatives of S with respect to the constants α_k equal to N new arbitrary constants β_k :

(177) $$\frac{\partial S}{\partial \alpha_k} = \beta_k \qquad (k = 1, 2, \ldots, N).$$

Those are the N second integrals of the motion.

In the cases that occur quite frequently in which E does not depend upon time explicitly, one can simplify that result when one "separates" the time variable t from S. In that case, one sets:

$$S = -\alpha_1 t + W(q_k, \alpha_2, \alpha_3, \ldots, \alpha_N),$$

in which α_1 is one of the constants, and W now includes only $(N - 1)$ of the independent constants $\alpha_1, \alpha_2, \ldots, \alpha_N$. One will then have:

$$\frac{\partial S}{\partial t} = -\alpha_1, \qquad \frac{\partial S}{\partial q_k} = \frac{\partial W}{\partial q_k}, \qquad \frac{\partial S}{\partial \alpha_k} = \frac{\partial W}{\partial \alpha_k} \qquad (k = 2, \ldots, N),$$

$$\frac{\partial S}{\partial \alpha_1} = -t + \frac{\partial W}{\partial \alpha_1}.$$

With that, **Hamilton**'s partial differential equation (175) for S will go to the following one for W:

(178) $$E\left(q_k, \frac{\partial W}{\partial q_k}\right) = \alpha_1,$$

and **Jacobi**'s integral equations (176) and (177) will then become:

(179) $$\frac{\partial W}{\partial q_k} = p_k \qquad (N \text{ equations}),$$

(180) $$\frac{\partial W}{\partial \alpha_k} = \beta_k \qquad (N - 1 \text{ equations}),$$

(181) $$\frac{\partial W}{\partial \alpha_1} = t + \beta_1.$$

Before we go on to prove **Jacobi**'s theorem, we would like to clarify its meaning in an example that is as simple as possible.

We shall treat the linear oscillation of a mass-point about a rest position under the influence of an elastic force. We are then given at the outset:

$$L = \tfrac{1}{2} m \dot{x}^2 = \frac{1}{2m} p^2 ,$$
$$\Phi = \tfrac{1}{2} k^2 x^2 ,$$
$$E = L + \Phi = \frac{1}{2m} p^2 + \tfrac{1}{2} k^2 x^2 .$$

Since E does not depend upon t, we can use the function W everywhere, instead of S, and write:

$$E = \frac{1}{2m} p^2 + \tfrac{1}{2} k^2 x^2 = \frac{1}{2m}\left(\frac{dW}{dx}\right)^2 + \tfrac{1}{2} k^2 x^2 .$$

Thus, from equation (178), the differential equation for W will read:

$$\frac{1}{2m}\left(\frac{dW}{dx}\right)^2 + \tfrac{1}{2} k^2 x^2 = \alpha_1 .$$

Here, we need an integral with only *one* arbitrary multiplicative constant, and that constant is already given by α_1 . An additive constant will no longer need to be added then. We will then get:

$$\left(\frac{dW}{dx}\right)^2 = 2 m \alpha_1 \left(1 - \frac{k^2}{2\alpha} x^2\right) ,$$

so

$$\frac{dW}{dx} = \sqrt{2 m \alpha_1} \cdot \sqrt{1 - \frac{k^2}{2\alpha} x^2} .$$

Integration will yield:

$$W = \frac{\alpha_1 \sqrt{m}}{k}\left[\arcsin \frac{k}{\sqrt{2\alpha_1}} + \frac{k\, x}{\sqrt{2\alpha_1}} \cdot \sqrt{1 - \frac{k^2}{2\alpha} x^2}\;\right] .$$

If we correspondingly differentiate (181) with respect to the constant α_1 and set $\partial W / \partial \alpha_1 - t$ equal to a new constant β_1 then it will follow from (181) that:

$$\beta_1 + t = \frac{\sqrt{m}}{k} \arcsin \frac{k\, x}{\sqrt{2\alpha_1}} ,$$

or

$$\frac{k}{\sqrt{m}}(\beta_1 + t) = \arcsin \frac{k}{\sqrt{2\alpha_1}} x ,$$

or

$$x = \frac{\sqrt{2\alpha_1}}{k} \sin \frac{k}{\sqrt{m}} (\beta_1 + t) \ ,$$

which is, in fact, the desired integral.

Naturally, in such a simple case, one would not appeal to the **Hamilton-Jacobi** method in order to find the integrals of the equation. That example should serve only to make it easier to understand the theory.

We now move on to the proof of **Jacobi**'s theorem!

We have to show that the due to the equations:

$$\frac{\partial S}{\partial q_k} = p_k \quad \text{and} \quad \frac{\partial S}{\partial \alpha_k} = \beta_k \qquad (k = 2, \ldots, N),$$

in which S is a complete integral of **Hamilton**'s partial differential equation with N arbitrary independent constants α_k, the equations of motion, which we would like to take in, say, **Hamilton**'s canonical form:

(182) $$\frac{\partial E}{\partial p_k} = \dot{q}_k \,, \qquad - \frac{\partial E}{\partial q_k} = \dot{p}_k \,,$$

will be satisfied.

We start from equation (177): $\frac{\partial S}{\partial \alpha_k} = \beta_k$ and form the complete temporal derivative $\frac{d}{dt}\left(\frac{\partial S}{\partial \alpha_k}\right)$, whose value is zero, for all values of k (= 1, …, N). Since S is a function t, q_k, and α_k, we will then have the following system of equations:

(183) $$\begin{cases} \dfrac{\partial^2 S}{\partial \alpha_1 \, \partial t} + \dfrac{\partial^2 S}{\partial \alpha_1 \, \partial q_1} \dot{q}_1 + \dfrac{\partial^2 S}{\partial \alpha_1 \, \partial q_2} \dot{q}_2 + \cdots + \dfrac{\partial^2 S}{\partial \alpha_1 \, \partial q_N} \dot{q}_N = 0 \,, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial^2 S}{\partial \alpha_N \, \partial t} + \dfrac{\partial^2 S}{\partial \alpha_N \, \partial q_1} \dot{q}_1 + \dfrac{\partial^2 S}{\partial \alpha_N \, \partial q_2} \dot{q}_2 + \cdots + \dfrac{\partial^2 S}{\partial \alpha_N \, \partial q_N} \dot{q}_N = 0 \,. \end{cases}$$

We have to calculate the $\dot{q}_k$ from those equations and substitute that in the first canonical equation in (182): $\frac{\partial E}{\partial p_k} = \dot{q}_k$, which must then be satisfied identically. However, we can also, conversely, substitute the values $\frac{\partial E}{\partial p_k}$ for $\dot{q}_k$ that come from the canonical equation (182) in (183) and show that (183) will be satisfied in that way. We write the system (183) more simply as:

(184) $$\frac{\partial^2 S}{\partial t\,\partial\alpha_k} + \sum_i \frac{\partial^2 S}{\partial\alpha_k\,\partial q_i}\,\dot{q}_i = 0 \qquad (k = 1, 2, \ldots, N),$$

and when we make the stated substitution for $\dot{q}_k$, we will have:

(185) $$\frac{\partial^2 S}{\partial\alpha_k\,\partial t} + \sum_i \frac{\partial^2 S}{\partial\alpha_k\,\partial q_i}\cdot\frac{\partial E}{\partial p_i} = 0 \qquad (k = 1, 2, \ldots, N).$$

Since that equation is an identity, the following can be shown: S is a solution of **Hamilton**'s partial differential equation:

$$\frac{\partial S}{\partial t} + E\left(q_k, \frac{\partial S}{\partial q_k}\right) \equiv 0 \,,$$

which is satisfied identically for arbitrary values of the constants α_k. If we differentiate with respect to α_k then it will follow that:

$$\frac{\partial^2 S}{\partial t\,\partial\alpha_k} + \frac{\partial E}{\partial\alpha_k} \equiv 0 \,.$$

Now, the α_k are found in the expressions $\partial S / \partial q_k$ that are included in E. thus:

$$\frac{\partial E}{\partial\alpha_k} = \sum_i \frac{\partial E}{\partial\left(\dfrac{\partial S}{\partial q_i}\right)}\frac{\partial^2 S}{\partial q_i\,\partial\alpha_k} \equiv 0 \,.$$

However, since one has $\partial S / \partial q_i = p_i$, according to **Jacobi**, that can also be written:

$$\frac{\partial^2 S}{\partial t\,\partial\alpha_k} + \sum_i \frac{\partial^2 S}{\partial q_i\,\partial\alpha_k}\cdot\frac{\partial E}{\partial p_i} \equiv 0 \,,$$

which is satisfied identically and agrees with (185).

One can proceed similarly with the other system of equations. One starts from $\dfrac{\partial S}{\partial q_k} = p_k$ and forms $\dfrac{d}{dt}\left(\dfrac{\partial S}{\partial q_k}\right)$. That will produce a system for the $\dot{p}_k$ that is analogous to equation (183) and can be combined with the second canonical equation (182) in the same way. Those calculations are entirely analogous to the ones that were just represented and will be omitted here.

With that, **Jacobi**'s theorem is proved, and we shall now address the question of how the integration of the partial differential equation for S is to be achieved in practice.

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§ 17.

On the integration of Hamilton's partial differential equation.

The integration of **Hamilton**'s partial differential equation encounters difficulties that can be overcome in only special cases. Namely, under certain conditions, one can *separate all variables*, just as we could previously separate the time variable t, i.e., one can decompose the function W that remains after separating the time variable from S and is a function of all q_k into a sum of functions, each of which depends upon only one variable. The partial differential equation will then go to a series of ordinary ones. Under those suitable conditions, one can then set:

$$W(q_1, q_2, \ldots, q_N) = W(q_1) + W(q_2) + \ldots + W(q_N) . \tag{186}$$

The question of *when* a **Hamiltonian** equation admits a separation of variables as in (186) has not been solved. However, **Stäckel** ([1]), supported by previous work by **Liouville** and **Staude**, has succeeded in solving the problem under the assumption that the function $E\left(q_k, \frac{\partial W}{\partial q_k}\right)$, which is indeed of degree *two* in the $\frac{\partial W}{\partial q_k}$, *includes only the squares of those quantities, but not their double products*. **Stäckel**'s assumption was then that E, which we would like to split into L and Φ for the moment, can be written:

$$E\left(q_k, \frac{\partial W}{\partial q_k}\right) = \tfrac{1}{2}\sum_{k=1}^{N} A_k(q_k)\cdot\left(\frac{\partial W}{\partial q_k}\right)^2 + \Phi(q_k) . \tag{187}$$

As **Stäckel** showed, one can then write out the general solution to the mechanical problem immediately. The problems in which a separation of variables is possible on the basis of **Stäckel**'s theorem have had a fundamental significance in recent times in the so-called quantum theory through the work of **Schwarzschild** ([2]) and **Epstein** ([3]).

For that reason, we would like to discuss at least one example of the separation of variables.

We again deal with a mass-point that performs elastic oscillations about its rest position in the xy-plane. It is therefore a generalization of the problem that we treated in the previous section. Now, for that case, we have:

$$L = \tfrac{1}{2}\, m\dot{x}^2 + \tfrac{1}{2}\, m\dot{y}^2 = \frac{1}{2m}\, p_1^2 + \frac{1}{2m}\, p_2^2 ,$$

([1]) **P. Stäckel**, *Über die Integration der Hamilton-Jacobischen Differentialgleichung mittels Separation der Variabeln*, Habilitationsschrift, Halle, 1891; one might also, cf., the very clear and simple presentation in **Charlier**, *Mechanik des Himmels*, Bd. I, pp. 77, *et seq.*

([2]) **K. Schwarzschild**, Sitz.-Ber. d. Berl. Akad. d. Wiss. (1916), pp. 548.

([3]) **P. S. Epstein**, Phys. Zeit. **17** (1916), pp. 148; Ann. Phys. (Leipzig) **50** (1916), pp. 489; *ibid.*, (1916), pp. 815; *ibid.*, **51** (1916), pp. 168.

$$\Phi = \tfrac{1}{2}\,k^2x^2 + \tfrac{1}{2}\,k^2y^2 .$$

For the total energy, it will follow that:

$$E\,(p_k,\,q_k) = \frac{1}{2m}\;p_1^2 + \frac{1}{2m}\;p_2^2 + \tfrac{1}{2}\,k^2x^2 + \tfrac{1}{2}\,k^2y^2 ,$$

and since that does not include t explicitly, we can use the simplified differential equation for the function W that is in equation (178). For our problem, it will read:

$$(188) \qquad \frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{1}{2m}\left(\frac{\partial W}{\partial y}\right)^2 + \tfrac{1}{2}\,k^2x^2 + \tfrac{1}{2}\,k^2y^2 = \alpha_1 .$$

From the above, the following Ansatz is possible here with no further analysis:

$$(189) \qquad W\,(x,\,y) = W_1\,(x) + W_2\,(y) ,$$

which will make (188) go to:

$$(190) \qquad \frac{1}{2m}\left(\frac{dW_1}{dx}\right)^2 + \frac{1}{2m}\left(\frac{dW_2}{dy}\right)^2 + \tfrac{1}{2}\,k^2x^2 + \tfrac{1}{2}\,k^2y^2 = \alpha_1 ,$$

and we can perhaps set:

$$(191) \qquad \frac{1}{2m}\left(\frac{dW_1}{dx}\right)^2 + \tfrac{1}{2}\,k^2x^2 = \frac{\alpha_2^2}{2m} ,$$

in which α_2 is a new constant. What will then remain is the differential equation for W_2 :

$$(192) \qquad \frac{1}{2m}\left(\frac{dW_1}{dy}\right)^2 + \tfrac{1}{2}\,k^2y^2 = \alpha_1 - \frac{\alpha_2^2}{2m} .$$

We now have only the two *ordinary* differential equations (191) and (192) to deal with then. We find immediately from (191) that:

$$\frac{dW_1}{dx} = \alpha_2\sqrt{1 - \frac{m\,k^2}{\alpha_2^2}\,x^2} ,$$

which can be integrated with no further assumptions:

(193) $$W_1(x) = \frac{\alpha_2^2}{2k\sqrt{m}}\left[\arcsin\frac{k\sqrt{m}}{\alpha_2}x + \frac{k\sqrt{m}}{\alpha_2}x\sqrt{1-\frac{mk^2}{\alpha_2^2}x^2}\right].$$

It likewise follows from (192) that:

$$\frac{dW_2}{dy} = \sqrt{2m\alpha_2 - \alpha_2^2}\cdot\sqrt{1-\frac{mk^2y^2}{2m\alpha_1-\alpha_2^2}},$$

and the integration likewise yields:

(194) $$W_2(x) = \frac{2m\alpha_1-\alpha_2^2}{2k\sqrt{m}}\left[\arcsin\frac{k\sqrt{m}}{\sqrt{2m\alpha_1-\alpha_2^2}}y + \frac{k\sqrt{m}}{\sqrt{2m\alpha_1-\alpha_2^2}}y\sqrt{1-\frac{k^2my^2}{2m\alpha_1-\alpha_2^2}}\right].$$

It will then follow that W is:

(195) $$\left\{\begin{aligned} W(x,y) = {} & \frac{\alpha_2^2}{2k\sqrt{m}}\left[\arcsin\frac{k\sqrt{m}}{\alpha_2}x + \frac{k\sqrt{m}}{\alpha_2}x\sqrt{1-\frac{mk^2}{\alpha_2^2}x^2}\right] \\ & + \frac{2m\alpha_1-\alpha_2^2}{2k\sqrt{m}}\left[\arcsin\frac{k\sqrt{m}}{\sqrt{2m\alpha_1-\alpha_2^2}}y + \frac{k\sqrt{m}}{\sqrt{2m\alpha_1-\alpha_2^2}}y\sqrt{1-\frac{k^2my^2}{2m\alpha_1-\alpha_2^2}}\right].\end{aligned}\right.$$

We will now find the integrals from equations (180) and (181) when we differentiate the constants α_1 and α_2 and set the results equal to $t + \beta_1$ and β_2, resp., where β_1 and β_2 are two new constants. We will then find, by a simple, but somewhat tedious, calculation, that:

$$\frac{\partial W}{\partial \alpha_1} = \beta_1 + t = \frac{\sqrt{m}}{k}\arcsin\frac{k\sqrt{m}}{\sqrt{2m\alpha_1-\alpha_2^2}}y,$$

or

(196) $$y = \frac{\sqrt{2m\alpha_1-\alpha_2^2}}{k\sqrt{m}}\sin\frac{k}{\sqrt{m}}(t+\beta_1),$$

with which one of the integrals has been found, and it actually includes *two* arbitrary constants (viz., an amplitude constant and a phase constant), as it must.

Likewise, one will have:

$$\frac{\partial W}{\partial \alpha_2} = \frac{\alpha_2}{k\sqrt{m}}\left[\arcsin\frac{k\sqrt{m}}{\alpha_2}x - \arcsin\frac{k\sqrt{m}}{\sqrt{2m\alpha_1-\alpha_2^2}}y\right] = \beta_2,$$

from which the following relation between x and y will arise by a simple calculation, which is then the equation of the path:

$$\frac{k^2 m}{\alpha_2^2} x^2 + \frac{2 k^2 m}{\alpha_2 \sqrt{2 m \alpha_1 - \alpha_2^2}} x y + \frac{k^2 m}{2 m \alpha_1 - \alpha_2^2} y^2 = \sin^2 \left(\frac{k \sqrt{m} \beta_2}{\alpha_2} \right) , \tag{197}$$

i.e., the path is an ellipse, as is known. One also finds x as a function of t directly upon combining that with (196), which is the ordinary representation of the second integral:

$$x = \frac{\alpha_2}{k \sqrt{m}} \cdot \sin \frac{k}{\sqrt{m}} \left(t + \beta_1 + \frac{m \beta_2}{\alpha_2} \right) , \tag{198}$$

which likewise includes arbitrary constants. The fact that equation (179) is also fulfilled by the impulses is easy to see.

One finds other examples of the separation of variables in the cited papers by **Epstein**, as well as in **Debye** ([1]) and **Scherrer** ([2]), and they merit special interest due to the fact that they point to the significance of **Hamilton-Jacobi** theory for one of the burning questions of modern physics, as was mentioned before.

([1]) **P. Debye**, Nachr. kgl. Ges. Wiss. zu Göttingen, math.-phys. Klasse (1916), pp. 142.
([2]) **P. Scherrer**, *ibid.* (1916), pp. 154.

§ 18.

Gauss's principle of least constraint.

Gauss ([1]) has added a new principle to dynamics that serves to achieve the same thing as **d'Alembert**'s and the principles that are equivalent to it that we have treated already.

We can characterize the true motion by giving the coordinates x_ν, y_ν, z_ν of the ν^{th} mass-points as functions of time:

$$x_\nu = \psi_\nu(t), \qquad y_\nu = \chi_\nu(t), \qquad z_\nu = \omega_\nu(t).$$

Those expressions constitute the true path. The true velocities and accelerations are:

$$\begin{aligned} \dot{x}_\nu &= \dot{\psi}_\nu(t), & \dot{y}_\nu &= \dot{\chi}_\nu(t), & \dot{z}_\nu &= \dot{\omega}_\nu(t), \\ \ddot{x}_\nu &= \ddot{\psi}_\nu(t), & \ddot{y}_\nu &= \ddot{\chi}_\nu(t), & \ddot{z}_\nu &= \ddot{\omega}_\nu(t). \end{aligned}$$

We now associate the true path with a varied one for which the coordinates and velocities remain unchanged, i.e., we take:

$$\delta x_\nu = \delta y_\nu = \delta z_\nu = \delta \dot{x}_\nu = \delta \dot{y}_\nu = \delta \dot{z}_\nu = 0 .$$

Only the acceleration components should suffer small variations:

$$\delta \ddot{x}_\nu, \qquad \delta \ddot{y}_\nu, \qquad \delta \ddot{z}_\nu \qquad\qquad (\neq 0).$$

We call such a variation that affects only the acceleration a ***Gaussian*** *variation* ([2]) and denote that by putting the index g in the symbol δ when a more precise characterization is desirable.

Now let m constraint equations exist, say non-holonomic-rheonomic ones, to remain as general as possible:

$$\sum_{\nu=1}^{n} (a'_{k\nu}\, dx_\nu + b'_{k\nu}\, dy_\nu + c'_{k\nu}\, dz_\nu + a'_k\, dt) = 0 \qquad (k = 1, 2, \ldots, m), \tag{199}$$

in which the coefficients $a'_{k\nu}$, $b'_{k\nu}$, $c'_{k\nu}$, a'_k are functions of the coordinates and time t. The true velocity components will then obey the equations:

([1]) **C. F. Gauss**, "Über ein neues allgemeines Grundgesetz der Mechanik," J. reine angew. Math. **4** (1829); also *Werke*, Bd. V, pp. 23, *et seq.*, 1877.

([2]) That terminology goes back to **Boltzmann** (*Vorl. über d. Prinzipe der Mechanik*, Bd. I, pp. 209.)

(200) $$\sum_{\nu=1}^{n} (a'_{k\nu}\,\dot{x}_\nu + b'_{k\nu}\,\dot{y}_\nu + c'_{k\nu}\,\dot{z}_\nu + a'_k) = 0 \qquad (k = 1, 2, \ldots, m).$$

Upon differentiating that with respect to t, we will get the relations that the true accelerations must satisfy:

(201) $$\sum_{\nu=1}^{n} (a'_{k\nu}\,\ddot{x}_\nu + b'_{k\nu}\,\ddot{y}_\nu + c'_{k\nu}\,\ddot{z}_\nu) + \Phi_k(x_\nu, y_\nu, z_\nu, \dot{x}_\nu, \dot{y}_\nu, \dot{z}_\nu, t) = 0 \qquad (k = 1, 2, \ldots, m),$$

in which we have combined all of the terms that do not contain second derivatives with respect to time in Φ_k. Φ_k will then depend upon only time, the coordinates, and the velocities.

If we now impose the demand that **Gauss** did, namely, that the varied motion should likewise satisfy equations (201), which are true for the true accelerations, when the quantities $\ddot{x}_\nu$, $\ddot{y}_\nu$, $\ddot{z}_\nu$ are replaced with the varied ones $\ddot{x}_\nu + \delta\ddot{x}_\nu$, $\ddot{y}_\nu + \delta\ddot{y}_\nu$, $\ddot{z}_\nu + \delta\ddot{z}_\nu$:

$$\sum_{\nu=1}^{n} [a'_{k\nu}(\ddot{x}_\nu + \delta\ddot{x}_\nu) + b'_{k\nu}(\ddot{y}_\nu + \delta\ddot{y}_\nu) + c'_{k\nu}(\ddot{z}_\nu + \delta\ddot{z}_\nu) + \Phi_k(x_\nu, y_\nu, z_\nu, \dot{x}_\nu, \dot{y}_\nu, \dot{z}_\nu, t) = 0\,,$$

and we subtract equation (201) from that then it will then follow that the quantities $\delta\ddot{x}_\nu$, $\delta\ddot{y}_\nu$, $\delta\ddot{z}_\nu$ are subject to the constraints:

(202) $$\sum_{\nu=1}^{n} (a'_{k\nu}\,\delta\ddot{x}_\nu + b'_{k\nu}\,\delta\ddot{y}_\nu + c'_{k\nu}\,\delta\ddot{z}_\nu) = 0 \qquad (k = 1, 2, \ldots, m).$$

The quantities $\delta\ddot{x}_\nu$, $\delta\ddot{y}_\nu$, $\delta\ddot{z}_\nu$ then fulfill the constraint equations in the same sense as was previously required of the virtual displacements δx_ν, δy_ν, δz_ν. The conclusions that could be inferred from that property at the time must now be true for the $\delta\ddot{x}_\nu$, $\delta\ddot{y}_\nu$, $\delta\ddot{z}_\nu$ then.

From equation (41), the equations of motion for our system will then read:

(203) $$\begin{cases} X_\nu - m_\nu\,\ddot{x}_\nu + \sum_{k=1}^{m} \lambda_k\,a'_{k\nu} = 0, \\ Y_\nu - m_\nu\,\ddot{y}_\nu + \sum_{k=1}^{m} \lambda_k\,b'_{k\nu} = 0, \\ Z_\nu - m_\nu\,\ddot{z}_\nu + \sum_{k=1}^{m} \lambda_k\,c'_{k\nu} = 0. \end{cases}$$

If we successively extend those equations by $\delta\ddot{x}_\nu$, $\delta\ddot{y}_\nu$, $\delta\ddot{z}_\nu$ and sum over the entire system then we will get:

$$\sum_{\nu=1}^{n} [(m_\nu \ddot{x}_\nu - X_\nu)\,\delta\ddot{x}_\nu + (m_\nu \ddot{y}_\nu - Y_\nu)\,\delta\ddot{y}_\nu + (m_\nu \ddot{z}_\nu - Z_\nu)\,\delta\ddot{z}_\nu]$$
$$= \sum_{k=1}^{m} \lambda_k \left\{ \sum_{\nu=1}^{n} (a'_{k\nu}\,\delta\ddot{x}_\nu + b'_{k\nu}\,\delta\ddot{y}_\nu + c'_{k\nu}\,\delta\ddot{z}_\nu) \right\} .$$

However, since the $\delta\ddot{x}_\nu$, $\delta\ddot{y}_\nu$, $\delta\ddot{z}_\nu$ must obey equations (202), in complete analogy with the virtual displacements, they will annul the right-hand side, and the result will be:

(204) $$\sum_{\nu=1}^{n} [(m_\nu \ddot{x}_\nu - X_\nu)\,\delta\ddot{x}_\nu + (m_\nu \ddot{y}_\nu - Y_\nu)\,\delta\ddot{y}_\nu + (m_\nu \ddot{z}_\nu - Z_\nu)\,\delta\ddot{z}_\nu] = 0 ,$$

which are equivalent to the equations of motion, and obviously d'Alembert's principle, as well.

Now, **Gauss** defined the following expression:

(205) $$\mathsf{Z} = \sum_{\nu=1}^{n} \frac{1}{m_\nu} [(m_\nu \ddot{x}_\nu - X_\nu)^2 + (m_\nu \ddot{y}_\nu - Y_\nu)^2 + (m_\nu \ddot{z}_\nu - Z_\nu)^2]$$

to be the *constraint* (we will explain the rationale for that terminology later), and remarked that applying the operation δ_g to (205) and setting the result equal to zero would produce precisely equation (204), so the equations of motion:

(206) $$\delta_g\,\mathsf{Z} \equiv \sum_{\nu=1}^{n} [(m_\nu \ddot{x}_\nu - X_\nu)^2 + \cdots] = 0 ,$$

such that the statement that $\delta_g\,\mathsf{Z} = 0$ *will be equivalent to the equations of motion.* Indeed, the constraint is always a minimum, since $\delta^2\mathsf{Z} > 0$, which one can convince oneself of by calculation. We then have **Gauss**'s principle:

For the true motion, the constraint is a minimum for a given position and velocity.

Now, how does one come to call the expression (205) the "constraint"? Following **Gauss**, we consider the ν^{th} mass-particle at two successive moments in time t and $t + dt$, and indeed for the time being, we assume that the equations of constraint have been suspended and only the explicit forces X_ν, Y_ν, Z_ν are in effect. We then assume that the mass-

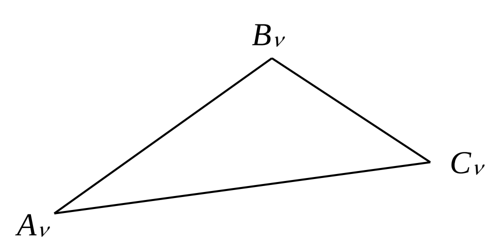

Figure 6.

particle is under the simultaneous effect of the explicit forces and the equations of constraint (which resolve into forces of constraint), i.e., we then consider the true motion of the mass-particle ([1]).

At time t, let the particle in question by at the point A_ν with the coordinates x_ν, y_ν, z_ν in the first way of considering the situation, and at time $t + dt$, let it be at B_ν with the coordinates $x_\nu + \dot{x}_\nu\, dt + \frac{1}{2}\ddot{x}_\nu\, dt^2, \ldots, z_\nu + \dot{z}_\nu\, dt + \frac{1}{2}\ddot{z}_\nu\, dt^2$. That motion will be performed under the common action of the explicit forces and the equations of constraint, and under their combined influence, the true accelerations $\ddot{x}_\nu$, $\ddot{y}_\nu$, $\ddot{z}_\nu$ will occur.

Now, the three line segments $\overrightarrow{A_\nu B_\nu}$, $\overrightarrow{A_\nu C_\nu}$, and $\overrightarrow{B_\nu C_\nu}$ obviously have the following components:

$$\begin{array}{llll} \overrightarrow{A_\nu B_\nu}: & \dot{x}_\nu\, dt + \dfrac{1}{2}\dfrac{X_\nu}{m_\nu}dt^2, & \dot{y}_\nu\, dt + \dfrac{1}{2}\dfrac{Y_\nu}{m_\nu}dt^2, & \dot{z}_\nu\, dt + \dfrac{1}{2}\dfrac{Z_\nu}{m_\nu}dt^2, \\ \overrightarrow{A_\nu C_\nu}: & \dot{x}_\nu\, dt + \frac{1}{2}\ \ddot{x}_\nu\, dt^2, & \dot{y}_\nu\, dt + \frac{1}{2}\ \ddot{y}_\nu\, dt^2, & \dot{z}_\nu\, dt + \frac{1}{2}\ \ddot{z}_\nu\, dt^2, \end{array}$$

so $\overrightarrow{B_\nu C_\nu}$ which is the vectorial difference of the two segments that were just considered, will have the components:

$$\overrightarrow{B_\nu C_\nu}: \qquad \frac{1}{2}\left(\ddot{x}_\nu - \frac{X_\nu}{m_\nu}\right)dt^2, \quad \frac{1}{2}\left(\ddot{y}_\nu - \frac{Y_\nu}{m_\nu}\right)dt^2, \quad \frac{1}{2}\left(\ddot{z}_\nu - \frac{Z_\nu}{m_\nu}\right)dt^2.$$

The magnitude of the segment $\overrightarrow{B_\nu C_\nu}$ then measures the *deviation* from the true motion of the *free* one (as **Gauss** called it), i.e., the one that would be followed in the absence of constraint equations. Now, **Gauss**'s principle demands that the sum $\sum_{\nu=1}^{n} m_\nu \left(\overrightarrow{B_\nu C_\nu}\right)^2$ should be a minimum for the true motion, i.e., that the mass m_ν multiplied by the square of that deviation should be as small as possible. One also sees that it is just that expression that one can use as a measure of the "constraint" (in the sense of the method of least-squares, which obviously guided **Gauss**'s reasoning in presenting the principle). In a certain sense, the mass factors represent the "weight" of the deviation from the "free" motion in each case. If we form $\sum m_\nu \left(\overrightarrow{B_\nu C_\nu}\right)^2$ then will find that:

$$\sum_{\nu=1}^{n} m_\nu \left[\left(\ddot{x}_\nu - \frac{X_\nu}{m_\nu}\right)^2 + \cdots\right] = \sum_{\nu=1}^{n} \frac{1}{m_\nu}[(m_\nu\, \ddot{x}_\nu - X_\nu)^2 + \cdots]$$

([1]) On this, one can confer, say: **Boltzmann**, *Vorlesungen über die Prinzipe der Mechanik*, Bd. I, 1897, pp. 212.

when we omit the constants factors $\frac{1}{4}dt^4$, i.e., we will find just the expression (205), which defines the constraint.

Furthermore, one can easily get back to **Hertz**'s principle of the straightest path from **Gauss**'s principle of least constraint, although we will not go into the details of that here.

§ 19.

The Gibbs-Appell form of the equations of motion.

The analytical formulation of **Gauss**'s principle of least constraint that is included in equation (204) leads to a new form of the equations of motion that goes back to **Gibbs** ([1]) and **Appell** ([2]) and is suited to non-holonomic constraints, unlike the **Lagrange** equations.

Once more, we would like to consider our system of m mass-points whose freedom is, however, restricted by m *holonomic* equations of constraint, such that the number of degrees of freedom will amount to $3n - m = N$.

In any event, we can then express the differentials of the coordinates dx_ν, dy_ν, dz_ν as linear functions of N independent differentials $dq_1, dq_2, \ldots, dq_N$. If the equations are integrable then the coordinates x_ν, y_ν, z_ν are themselves representable as functions of N independent parameters q_1, …, q_N that are then referred to as the *true coordinates*. However, if the integrability conditions are not fulfilled then the N quantities dq_k will represent non-holonomic differentials, which we previously denoted by $d\pi_k$. For the sake of generality, we would like to leave undecided which of the cases we are dealing with, and for that reason, we will pose the relations as follows:

$$(207)\qquad \left\{\begin{aligned} dx_\nu &= a_{1\nu}\,dq_1 + a_{2\nu}\,dq_2 + \cdots + a_{N\nu}\,dq_N\,, \\ dy_\nu &= b_{1\nu}\,dq_1 + b_{2\nu}\,dq_2 + \cdots + b_{N\nu}\,dq_N\,, \\ dz_\nu &= c_{1\nu}\,dq_1 + c_{2\nu}\,dq_2 + \cdots + c_{N\nu}\,dq_N\,. \end{aligned}\right. \qquad (\nu = 1, 2, \ldots, n)$$

We will then obtain the velocities $\dot{x}_\nu$, $\dot{y}_\nu$, $\dot{z}_\nu$ upon dividing that by dt :

$$(208)\qquad \left\{\begin{aligned} \dot{x}_\nu &= a_{1\nu}\,\dot{q}_1 + a_{2\nu}\,\dot{q}_2 + \cdots + a_{N\nu}\,\dot{q}_N\,, \\ \dot{y}_\nu &= b_{1\nu}\,\dot{q}_1 + b_{2\nu}\,\dot{q}_2 + \cdots + b_{N\nu}\,\dot{q}_N\,, \\ \dot{z}_\nu &= c_{1\nu}\,\dot{q}_1 + c_{2\nu}\,\dot{q}_2 + \cdots + c_{N\nu}\,\dot{q}_N\,. \end{aligned}\right. \qquad (\nu = 1, 2, \ldots, n)$$

Upon differentiating that with respect to t, we will ultimately obtain the following relations between the Cartesian components of the acceleration $\ddot{x}_\nu$, $\ddot{y}_\nu$, $\ddot{z}_\nu$ and the general ones $\ddot{q}_k$:

$$(209)\qquad \left\{\begin{aligned} \ddot{x}_\nu &= a_{1\nu}\,\ddot{q}_1 + a_{2\nu}\,\ddot{q}_2 + \cdots + a_{N\nu}\,\ddot{q}_N + \Phi_{1\nu}\,(x_\nu, y_\nu, z_\nu, \dot{x}_\nu, \dot{y}_\nu, \dot{z}_\nu, t), \\ \ddot{y}_\nu &= b_{1\nu}\,\ddot{q}_1 + b_{2\nu}\,\ddot{q}_2 + \cdots + b_{N\nu}\,\ddot{q}_N + \Phi_{2\nu}\,, \\ \ddot{z}_\nu &= c_{1\nu}\,\ddot{q}_1 + c_{2\nu}\,\ddot{q}_2 + \cdots + c_{N\nu}\,\ddot{q}_N + \Phi_{3\nu}\,, \end{aligned}\right.$$

([1]) **J. W. Gibbs**, "On the fundamental formulae of Dynamics," Am. J. Math. **2** (1879), pp. 49, *et seq.*; *Scientific Papers*, vol. II, pp. 1, *et seq.*

([2]) **P. Appell**, "Sur une forme générale des équations de la dynamique," C. R. Acad. Sci. Paris **129** (1899), pp. 317; also *Traité de mécanique rationelle*, 2nd ed., t. II, 1904, pp. 292.

in which we have combined all of the terms that do not include second derivatives with respect to t into the Φ. If we form the **Gauss**ian variation δ_g of the acceleration components, i.e., a variation that leaves x_ν, y_ν, z_ν, $\dot{x}_\nu$, $\dot{y}_\nu$, $\dot{z}_\nu$, t unchanged while affecting only the acceleration, then it will follow that:

(210) $$\left\{\begin{aligned} \delta\ddot{x}_\nu &= a_{1\nu}\,\delta\ddot{q}_1 + a_{2\nu}\,\delta\ddot{q}_2 + \cdots + a_{N\nu}\,\delta\ddot{q}_N, \\ \delta\ddot{y}_\nu &= b_{1\nu}\,\delta\ddot{q}_1 + b_{2\nu}\,\delta\ddot{q}_2 + \cdots + b_{N\nu}\,\delta\ddot{q}_N, \\ \delta\ddot{z}_\nu &= c_{1\nu}\,\delta\ddot{q}_1 + c_{2\nu}\,\delta\ddot{q}_2 + \cdots + c_{N\nu}\,\delta\ddot{q}_N. \end{aligned}\right. \qquad (\nu = 1, 2, \ldots, n)$$

The coefficients $a_{\lambda\nu}$, $b_{\lambda\nu}$, $c_{\lambda\nu}$ will then have the following meanings accordingly:

(211) $$a_{\lambda\nu} = \frac{\partial\ddot{x}_\nu}{\partial\ddot{q}_\lambda}, \qquad b_{\lambda\nu} = \frac{\partial\ddot{y}_\nu}{\partial\ddot{q}_\lambda}, \qquad c_{\lambda\nu} = \frac{\partial\ddot{z}_\nu}{\partial\ddot{q}_\lambda}.$$

We would like to introduce the expressions for $\delta\ddot{x}_\nu$, $\delta\ddot{y}_\nu$, $\delta\ddot{z}_\nu$ into equation (204) for the principle of least constraint. We first split that equation into two parts that correspond to the explicit forces and the inertial forces:

$$\sum_{\nu=1}^{n}[X_\nu\,\delta\ddot{x}_\nu + Y_\nu\,\delta\ddot{y}_\nu + Z_\nu\,\delta\ddot{z}_\nu] = \sum_{\nu=1}^{n} m_\nu\,[\ddot{x}_\nu\,\delta\ddot{x}_\nu + \ddot{y}_\nu\,\delta\ddot{y}_\nu + \ddot{z}_\nu\,\delta\ddot{z}_\nu]\ .$$

If we perform the stated substitution here then we will get:

$$\sum_{\nu=1}^{n}\left[X_\nu\sum_{\lambda=1}^{N}a_{\lambda\nu}\,\delta\ddot{q}_\lambda + Y_\nu\sum_{\lambda=1}^{N}b_{\lambda\nu}\,\delta\ddot{q}_\lambda + Z_\nu\sum_{\lambda=1}^{N}c_{\lambda\nu}\,\delta\ddot{q}_\lambda\right]$$
$$= \sum_{\nu=1}^{n} m_\nu\left[\ddot{x}_\nu\sum_{\lambda=1}^{N}a_{\lambda\nu}\,\delta\ddot{q}_\lambda + \ddot{y}_\nu\sum_{\lambda=1}^{N}b_{\lambda\nu}\,\delta\ddot{q}_\lambda + \ddot{z}_\nu\sum_{\lambda=1}^{N}c_{\lambda\nu}\,\delta\ddot{q}_\lambda\right],$$

or when rearranged:

$$\sum_{\lambda=1}^{N}\left[\sum_{\nu=1}^{n}(X_\nu\,a_{\lambda\nu} + Y_\nu\,b_{\lambda\nu} + Z_\nu\,c_{\lambda\nu})\right]\delta\ddot{q}_\lambda = \sum_{\lambda=1}^{N}\left[\sum_{\nu=1}^{n}m_\nu\,(\ddot{x}_\nu\,a_{\lambda\nu} + \ddot{y}_\nu\,b_{\lambda\nu} + \ddot{z}_\nu\,c_{\lambda\nu})\right]\delta\ddot{q}_\lambda\ .$$

However, due to the independence of the $\delta\ddot{q}_\lambda$, that equation will decompose into N individual equations:

(212) $$\sum_{\nu=1}^{n}(X_\nu\,a_{\lambda\nu} + Y_\nu\,b_{\lambda\nu} + Z_\nu\,c_{\lambda\nu}) = \sum_{\nu=1}^{n}m_\nu\,(\ddot{x}_\nu\,a_{\lambda\nu} + \ddot{y}_\nu\,b_{\lambda\nu} + \ddot{z}_\nu\,c_{\lambda\nu}) \qquad (\lambda = 1, 2, \ldots, N).$$

The expression on the left has a simple mechanical meaning: Namely, it is the general force component Q_λ that strives to vary the coordinate q_λ. One will see that most simply when one introduces the values of δx_ν, δy_ν, δz_ν from (207) into the expression for the work $\sum [X_\nu\,\delta x_\nu +$

$Y_\nu\,\delta y_\nu + Z_\nu\,\delta z_\nu$] and arranges the result in terms of the δq_λ. The factor of δq_λ is then just Q_λ. One likewise introduces the values of the coefficients $a_{\lambda\nu}$, $b_{\lambda\nu}$, $c_{\lambda\nu}$ from (211) into the right-hand side of the last equation. One will then get:

$$Q_\lambda = \sum_{\nu=1}^{n} m_\nu \left(\ddot{x}_\nu \frac{\partial \ddot{x}_\nu}{\partial \ddot{q}_\lambda} + \ddot{y}_\nu \frac{\partial \ddot{y}_\nu}{\partial \ddot{q}_\lambda} + \ddot{z}_\nu \frac{\partial \ddot{z}_\nu}{\partial \ddot{q}_\lambda} \right) \qquad (\lambda = 1, 2, \ldots, N), \tag{213}$$

or finally:

$$Q_\lambda = \frac{\partial}{\partial \ddot{q}_\lambda} \sum_{\nu=1}^{n} \tfrac{1}{2} m_\nu (\ddot{x}_\nu^2 + \ddot{y}_\nu^2 + \ddot{z}_\nu^2) \qquad (\lambda = 1, 2, \ldots, N). \tag{214}$$

Here, the forces Q_λ are expressed as the partial derivatives of a function that is formed in the same way as kinetic energy, except with the components of acceleration, instead of velocity components. We would like to introduce a special terminology for that function: We shall refer to:

$$T = \sum_{\nu=1}^{n} \tfrac{1}{2} m_\nu (\ddot{x}_\nu^2 + \ddot{y}_\nu^2 + \ddot{z}_\nu^2) \tag{215}$$

as the ***Gibbs-Appell** function*. The equations of motion will then read:

$$Q_\lambda = \frac{\partial T}{\partial \ddot{q}_\lambda}. \tag{216}$$

It is clear from the derivation that this will also be true *when the coordinates are non-holonomic.*

We would like to make an application of precisely that formula to the **Euler** equations of a rigid body that is fixed at a point. We have already treated that problem before with the help of the general **Lagrange** equations (§ **10**).

The following equations in the form of (207) are true in this case:

$$\left\{ \begin{aligned} dx_\nu &= z_\nu\, d\chi - y_\nu\, d\rho, \\ dy_\nu &= x_\nu\, d\rho - z_\nu\, d\pi, \\ dz_\nu &= y_\nu\, d\pi - x_\nu\, d\chi. \end{aligned} \right. \tag{217}$$

The x_ν, y_ν, z_ν in that mean the coordinates of a mass-point in the rigid body in a coordinate system that is fixed in the body and whose origin coincides with the fixed point. We will impose further specializations later. Furthermore, $d\pi$, $d\chi$, $d\rho$ are the infinitesimal angles of rotation around the axes of that system. The velocities will follow upon dividing by dt :

$$(218)\qquad \begin{cases} \dot{x}_\nu = z_\nu\,\dot{\chi} - y_\nu\,\dot{\rho}, \\ \dot{y}_\nu = x_\nu\,\dot{\rho} - z_\nu\,\dot{\pi}, \\ \dot{z}_\nu = y_\nu\,\dot{\pi} - x_\nu\,\dot{\chi}. \end{cases}$$

In order to remain true to our previous notations, we shall not actually write $\dot{\pi}$, $\dot{\chi}$, $\dot{\rho}$, but rather $\overset{*}{\pi}$, $\overset{*}{\chi}$, $\overset{*}{\rho}$, since there are no such quantities as π, χ, ρ whose derivatives are the $\dot{\pi}$, $\dot{\chi}$, $\dot{\rho}$. Nonetheless, we would like to preserve the dot in the following calculations, for the sake of simplicity of notation, since hopefully no misunderstanding should arise.

We find that the acceleration components are:

$$\begin{aligned} \ddot{x}_\nu &= z_\nu\,\ddot{\chi} - y_\nu\,\ddot{\rho} + \dot{z}_\nu\,\dot{\chi} - \dot{y}_\nu\,\dot{\rho}, \\ \ddot{y}_\nu &= x_\nu\,\ddot{\rho} - z_\nu\,\ddot{\pi} + \dot{x}_\nu\,\dot{\rho} - \dot{z}_\nu\,\dot{\pi}, \\ \ddot{z}_\nu &= y_\nu\,\ddot{\pi} - x_\nu\,\ddot{\chi} + \dot{y}_\nu\,\dot{\pi} - \dot{x}_\nu\,\dot{\chi}. \end{aligned}$$

If we once more substitute the value of $\dot{x}_\nu$, $\dot{y}_\nu$, $\dot{z}_\nu$ from (218) in that then that will yield:

$$(219)\qquad \begin{cases} \ddot{x}_\nu = z_\nu\,\ddot{\chi} - y_\nu\,\ddot{\rho} + y_\nu\,\dot{\pi}\,\dot{\chi} - x_\nu\,\dot{\chi}^2 - x_\nu\,\dot{\rho}^2 + z_\nu\,\dot{\pi}\,\dot{\rho}, \\ \ddot{y}_\nu = x_\nu\,\ddot{\rho} - z_\nu\,\ddot{\pi} + z_\nu\,\dot{\chi}\,\dot{\rho} - y_\nu\,\dot{\rho}^2 - y_\nu\,\dot{\pi}^2 + x_\nu\,\dot{\chi}\,\dot{\pi}, \\ \ddot{z}_\nu = y_\nu\,\ddot{\pi} - x_\nu\,\ddot{\chi} + x_\nu\,\dot{\rho}\,\dot{\pi} - z_\nu\,\dot{\pi}^2 - z_\nu\,\dot{\chi}^2 + y_\nu\,\dot{\rho}\,\dot{\chi}. \end{cases}$$

If we now form T then it would greatly simplify the calculations if we were to drop all terms in which no double dots appear, because those terms would make no contribution under **Gauss**ian variation. We would then get, e.g., for $\ddot{x}_\nu^2$:

$$\begin{aligned} \ddot{x}_\nu^2 = {} & z_\nu^2\,\ddot{\chi}^2 + y_\nu^2\,\ddot{\rho}^2 - 2z_\nu\,y_\nu\,\ddot{\chi}\,\ddot{\rho} + 2z_\nu\,y_\nu\,\dot{\pi}\,\dot{\chi}\,\ddot{\chi} - 2x_\nu\,z_\nu\,\dot{\chi}^2\,\ddot{\chi} - 2x_\nu\,z_\nu\,\dot{\rho}^2\,\ddot{\chi} \\ & + 2z_\nu^2\,\dot{\pi}\,\dot{\rho}\,\ddot{\chi} - 2y_\nu^2\,\dot{\pi}\,\dot{\chi}\,\ddot{\rho} + 2x_\nu\,y_\nu\,\dot{\chi}^2\,\ddot{\rho} + 2x_\nu\,y_\nu\,\dot{\rho}^2\,\ddot{\rho} - 2y_\nu\,z_\nu\,\dot{\pi}\,\dot{\rho}\,\ddot{\rho} + \cdots \end{aligned}$$

The remaining components of the acceleration follow from that by cyclic permutation of the symbols x, y, z, as well as π, χ, ρ.

If we now form $\sum m_\nu(\ddot{x}_\nu^2 + \ddot{y}_\nu^2 + \ddot{z}_\nu^2)$ then terms with the factors $\sum m_\nu\,x_\nu^2$, $\sum m_\nu\,y_\nu^2$, $\sum m_\nu\,z_\nu^2$, and ones with the factors $\sum m_\nu\,x_\nu\,y_\nu$, $\sum m_\nu\,y_\nu\,z_\nu$, $\sum m_\nu\,z_\nu\,x_\nu$ will appear. However, the latter are known to be the so-called *moments of deviation* about the axes, *and we will succeed in making them drop out when we let the coordinate system that is used in the derivation of the Euler equations coincide with the principal axes of inertia through the fixed point*. We will then have, quite simply:

$$2\,T = \ddot{\pi}^2 \sum_\nu m_\nu\,(y_\nu^2 + z_\nu^2) + \ddot{\chi}^2 \sum_\nu m_\nu\,(z_\nu^2 + x_\nu^2) + \ddot{\rho}^2 \sum_\nu m_\nu\,(y_\nu^2 + x_\nu^2)$$
$$+\; 2\,\dot{\pi}\,\dot{\rho}\,\ddot{\chi} \sum_\nu m_\nu\,(z_\nu^2 - x_\nu^2) + 2\,\dot{\chi}\,\dot{\pi}\,\ddot{\rho} \sum_\nu m_\nu\,(x_\nu^2 - y_\nu^2) + 2\,\dot{\rho}\,\dot{\chi}\,\ddot{\pi} \sum_\nu m_\nu\,(y_\nu^2 - x_\nu^2) + \cdots$$

Now, when the principal moments of inertia are denoted by A, B, C, as usual:

$$A = \sum m_\nu\,(y_\nu^2 + z_\nu^2), \quad B = \sum m_\nu\,(z_\nu^2 + x_\nu^2), \quad C = \sum m_\nu\,(y_\nu^2 + x_\nu^2) ,$$

and one makes suitable subtractions of them:

$$B - A = \sum m_\nu\,(x_\nu^2 - y_\nu^2), \qquad C - B = \sum m_\nu\,(y_\nu^2 - z_\nu^2), \qquad A - C = \sum m_\nu\,(z_\nu^2 - x_\nu^2),$$

the expression for T will become (naturally, except for terms that carry no double dots):

$$T = \tfrac{1}{2} A\ddot{\pi}^2 + \tfrac{1}{2} B\,\ddot{\chi}^2 + \tfrac{1}{2} C\,\ddot{\rho}^2 + (A - C)\,\dot{\pi}\,\dot{\rho}\,\ddot{\chi} + (B - A)\,\dot{\chi}\,\dot{\pi}\,\ddot{\rho} + (C - B)\,\dot{\rho}\,\dot{\chi}\,\ddot{\pi}\,. \tag{220}$$

If we now denote the generalized force components by Π, Χ, Ρ, as before, in the sense that $\Pi\;\delta\pi + \mathrm{X}\;\delta\chi + \mathrm{P}\;\delta\rho$ means the work done under the infinitesimal rotation $\delta\pi$, $\delta\chi$, $\delta\rho$ then we will have from (216) and (220) that:

$$\Pi = \frac{\partial T}{\partial \ddot{\pi}} = A\,\ddot{\pi} + (C - A)\,\dot{\chi}\,\dot{\rho},$$
$$\mathrm{X} = \frac{\partial T}{\partial \ddot{\chi}} = B\,\ddot{\chi} + (A - C)\,\dot{\rho}\,\dot{\pi},$$
$$\mathrm{P} = \frac{\partial T}{\partial \ddot{\rho}} = C\,\ddot{\rho} + (B - A)\,\dot{\pi}\,\dot{\chi},$$

i.e., the **Euler** equations. It is remarkable that the **Gibbs-Appell** form of the equations of motion allows one to achieve that result with significantly less computational work that one expends with the extended **Lagrange** equations for non-holonomic coordinates.