"Maxwell-Gleichung bewegter Versetzungen im COSSERAT-Kontinuum," Abh. Braunschweig. Wiss. Ges. **21** (1969), 480-486.

MAXWELL's equations for dislocations moving in a COSSERAT continuum (*)

By **Hermann Schaefer**

Translated by D. H. Delphenich

(Received on 21 May 1969)

Summary. – The kinematic and dynamical equations of a **Cosserat** continuum with moving dislocations assume the same form as **Maxwell**'*s* equations when one makes use of the differential operators Grad, Div, and Rot of motor analysis.

1. Introduction. **–** It is known that in a coherent system of units **Maxwell**'s equations for the electromagnetic field can be given a form in which no constants or parameters appear any more that would pertain to the properties of a particular medium. With the conventional notations, one will then have:

$$\mathrm{rot}\,\mathbf{E} + \frac{\partial}{\partial t}\mathbf{B} = 0, \qquad \mathrm{div}\,\mathbf{B} = 0, \tag{1.1}$$

$$\mathrm{rot}\,\mathbf{H} - \frac{\partial}{\partial t}\mathbf{D} = \mathbf{s}, \qquad \mathrm{div}\,\mathbf{D} = \rho\,. \tag{1.2}$$

Furthermore, one has:

$$\boldsymbol{f} = \rho\,\mathbf{E} + \mathbf{s} \times \mathbf{B}, \tag{1.3}$$

$$\lambda = \mathbf{s} \cdot \mathbf{E} \tag{1.4}$$

for the four-vector of the **Lorentz** force $\boldsymbol{f}$ and the power density λ, or in coordinate notation:

$$f_k = \rho\,E_k + e_{klm}\,s_l\,B_m\,, \tag{1.5}$$

$$\lambda = s_l\,E_l\,. \tag{1.6}$$

(*) Dedicated to Herrn Prof. Dr. **Luigi Sobrero**, Trieste, on his 60th birthday.

The impressive elegance of these equations, which encompass the vast complex of macroscopic electromagnetic phenomena, has also defined the yardstick for other field theories of physics, and in particular, it has also led to the explanations for complicated connections in continuum mechanics by drawing upon electromagnetic analogies. One finds a wealth of examples for this in volume 2 of the brilliant **Feynman** lectures on physics. Hence, in the still-young continuum theory of dislocations and internal stresses, one also frequently refers to analogies to electromagnetic phenomena, such as when one explains the laws of vortices in hydrodynamics as examples of **Ampère**'s law between current density and the magnetic field. Our present exposition shall now show that not only are individual analogies between the theory of dislocations and **Maxwell**'s theory present here and there, but that both theories are mathematically isomorphic, moreover. Generally, that isomorphism exists only when one considers the theory of dislocations in a **Cosserat** continuum. In it, any mass "point" has the degrees of freedom of a rigid body, so it possesses orbital momentum and spin. Along with force stresses, moment stresses can appear. The deformations of the continuum are described by two asymmetric tensors [**6**]. However, the kinematic and dynamical equations of **Cosserat** continuum are consistently clearer than those of the classical continuum. There are cases in which the degenerate equations of the classical continuum first become understandable when one considers the corresponding equations of the **Cosserat** continuum.

§ 2. Notations. – All indices run from 1 to 3. The summation convention is observed. ε_{ikl} is the unit tensor that is alternating in all indices. We employ Cartesian coordinates x_1, x_2, x_3 and the abbreviation $\partial_i = \partial / \partial x_i$ throughout.

The follow presentation will be based upon a **Cosserat** continuum. It has been shown that the kinematical and dynamical equations of that continuum can be written quite clearly when one appeals to the following symbolism [**1**, **2**, **3**, **4**]:

$$\text{Grad}\begin{bmatrix} \overset{1}{\mathbf{a}} \\ \overset{2}{\mathbf{a}} \end{bmatrix} \equiv \begin{cases} \partial_i \overset{1}{a}_k , \\ \partial_i \overset{2}{a}_k - \varepsilon_{ikl} \overset{1}{a}_k , \end{cases} \tag{2.1}$$

$$\text{Rot}\begin{bmatrix} \overset{1}{\mathbf{A}} \\ \overset{2}{\mathbf{A}} \end{bmatrix} \equiv \begin{cases} \varepsilon_{ikl} \partial_k \overset{1}{A}_{kl} , \\ \varepsilon_{ikl} (\partial_k \overset{2}{A}_{lm} + \varepsilon_{mkn} \overset{1}{A}_{ln} , \end{cases} \tag{2.2}$$

$$\text{Div}\begin{bmatrix} \overset{1}{\mathbf{R}} \\ \overset{2}{\mathbf{R}} \end{bmatrix} \equiv \begin{cases} \partial_i \overset{1}{R}_{ik} , \\ \partial_i \overset{2}{R}_{ik} + \varepsilon_{klm} \overset{1}{R}_{lm} , \end{cases} \tag{2.3}$$

and as one can confirm by a simple calculation, the two identities:

$$\text{Div Rot} \equiv 0, \quad \text{Rot Grad} \equiv 0 \qquad (2.4), (2.5)$$

exist between those three differential operators.

§ 3. Basic dynamical equations and stress functions. – The equations of the continuum read:

$$\text{Div}\begin{bmatrix}\boldsymbol{\sigma}\\ \boldsymbol{\mu}\end{bmatrix}=\begin{bmatrix}\rho\,\dot{\mathbf{v}}\\ \theta\,\dot{\mathbf{s}}\end{bmatrix}. \tag{3.1}$$

σ_{ik} and μ_{ik} are the asymmetric tensors of the force and moment stresses. On the right-hand side of (3.1), v_k and s_k mean the velocity and angular velocity, resp., of a mass element whose density is $\rho\,(x_1, x_2, x_3)$ and whose spin is $\theta\,(x_1, x_2, x_3)\,\mathbf{s}$. If we restrict ourselves to linearity (a nonlinear theory of the **Cosserat** continuum still does not exist) then we will have to replace the total derivatives $\dot{\mathbf{v}}$ and $\dot{\mathbf{s}}$ with $\partial\mathbf{v} / \partial t$ and $\partial\mathbf{s} / \partial t$, resp.

From **W. Günther** [**5**], the condition for static equilibrium:

$$\text{Div}\begin{bmatrix}\boldsymbol{\sigma}\\ \boldsymbol{\mu}\end{bmatrix}=0 \tag{3.2}$$

can be fulfilled identically in the eighteen stress functions φ_{ik} , Φ_{ik} by the Ansatz:

$$\begin{bmatrix}\boldsymbol{\sigma}\\ \boldsymbol{\mu}\end{bmatrix}=\text{Rot}\begin{bmatrix}\boldsymbol{\varphi}\\ \boldsymbol{\Phi}\end{bmatrix}. \tag{3.3}$$

In order to fulfill the inhomogeneous equations (3.1), one now sets:

$$\begin{bmatrix}\boldsymbol{\sigma}\\ \boldsymbol{\mu}\end{bmatrix}=\text{Rot}\begin{bmatrix}\boldsymbol{\varphi}\\ \boldsymbol{\Phi}\end{bmatrix}-\begin{bmatrix}\dot{\boldsymbol{\psi}}\\ \dot{\boldsymbol{\chi}}\end{bmatrix} \tag{3.4}$$

so the tensor-pair ψ_{ik} , χ_{ik} must then satisfy the equations:

$$\text{Div}\begin{bmatrix}\boldsymbol{\psi}\\ \boldsymbol{\chi}\end{bmatrix}=-\begin{bmatrix}\rho\,\mathbf{v}\\ \theta\,\mathbf{s}\end{bmatrix}. \tag{3.5}$$

The dots in (3.4) again mean partial derivatives with respect to time. One already sees the analogy between (3.4) and (3.5) and the inhomogeneous **Maxwell** equations (1.2). The continuity equation of the charge density:

$$\frac{\partial\rho}{\partial t}+\text{div}\,\mathbf{s}=0 \tag{3.6}$$

in them corresponds to (3.1) here.

§ 4. The kinematic equations of the incompatible continuum. – The deformations of a **Cosserat** continuum are described by two asymmetric ε_{ik} , κ_{ik} . In a compatible continuum, they are defined by:

$$\begin{bmatrix} \boldsymbol{\kappa} \\ \boldsymbol{\varepsilon} \end{bmatrix} = \text{Grad} \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{u} \end{bmatrix}, \tag{4.1}$$

such that, from (2.6), one will have:

$$\text{Rot} \begin{bmatrix} \boldsymbol{\kappa} \\ \boldsymbol{\varepsilon} \end{bmatrix} = 0. \tag{4.2}$$

In (4.1), u_k and ω_k mean the vector fields of the infinitesimal displacement and rotation of the volume element, and one will have:

$$\begin{bmatrix} \dot{\boldsymbol{\kappa}} \\ \dot{\boldsymbol{\varepsilon}} \end{bmatrix} = \text{Grad} \begin{bmatrix} \mathbf{s} \\ \mathbf{v} \end{bmatrix}. \tag{4.3}$$

One will find an intuitive interpretation of the tensors ε_{ik} and κ_{ik} in [**6**].

An incompatible continuum is characterized by the fact that (4.1), (4.2), and (4.3) are no longer true. In place of (4.2), one now has [**5**]:

$$\text{Rot} \begin{bmatrix} \boldsymbol{\kappa} \\ \boldsymbol{\varepsilon} \end{bmatrix} = \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix}, \qquad \text{Div} \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} = 0, \qquad (4.4), (4.5)$$

in which (4.5) follows from (4.4), due to the fact that Div Rot ≡ 0. **W. Günther** [**5**] has called B_{ik} and D_{ik} *incompatibility tensors.* They are the roots of the fact that no unique field of rotation and translation vectors exist in such a continuum. In the continuum theory of displacements, B_{ik} and D_{ik} are the dislocation densities that one calls *disclinations* and *dislocations* [**7**]. B_{ik} describe rotational dislocations (torsional and bending dislocations), while D_{ik} describe translation dislocations (screw and edge dislocations).

(4.5) is analogous to $(1.1)_2$. It is known that the two homogeneous **Maxwell** equations (1.1) can be fulfilled by the four-potential A_0 , **A**:

$$\mathbf{B} = \text{rot}\, \mathbf{A}, \qquad \mathbf{E} = \text{grad}\, A_0 - \frac{\partial}{\partial t}\mathbf{A}. \qquad (4.6)(4.7)$$

One knows that (4.4) corresponds to the relation (4.6). From the analogies that have been established up to now, we can extend equation (4.3) to the incompatible continuum. If one recalls (4.7) then that will come about by way of:

$$\begin{bmatrix}\mathbf{S}\\ \mathbf{I}\end{bmatrix} = \text{Grad}\begin{bmatrix}\mathbf{s}\\ \mathbf{v}\end{bmatrix} - \begin{bmatrix}\dot{\boldsymbol{\kappa}}\\ \dot{\boldsymbol{\varepsilon}}\end{bmatrix}. \tag{4.8}$$

Due to the fact that Rot Grad ≡ 0, it will follow from (4.8) and (4.4) that:

$$\text{Rot}\begin{bmatrix}\mathbf{S}\\ \mathbf{I}\end{bmatrix} + \begin{bmatrix}\dot{\mathbf{B}}\\ \dot{\mathbf{D}}\end{bmatrix} = 0, \tag{4.9}$$

in analogy with $(1.1)_1$. We can then speak of the "**Maxwell** equations" for the theory of dislocations. The tensors S_{ik} and I_{ik} in (4.8) will be called *dislocation current densities* [**8**, **9**], which is, I believe, an unfortunate choice of terminology. A body possesses dislocation densities B_{ik} and D_{ik} in the unloaded state. If a large enough external load is applied then the dislocations will be put into motion and will then generate new dislocations; the body will deform plastically. That process will be described B_{ik} and D_{ik} . From (4.8) or (4.9), the tensors S_{ik} and I_{ik} are a measure of how many dislocations enter or exit a volume element per unit time.

§ 5. Discussion of the analogy. – On the grounds of clarity, we shall write down all of the equations for the linear theory of dislocations one more time:

$$\text{Rot}\begin{bmatrix}\mathbf{S}\\ \mathbf{I}\end{bmatrix} + \frac{\partial}{\partial t}\begin{bmatrix}\mathbf{B}\\ \mathbf{D}\end{bmatrix} = 0, \qquad \text{Div}\begin{bmatrix}\mathbf{B}\\ \mathbf{D}\end{bmatrix} = 0, \qquad (4.9), (4.5)$$

$$\text{Rot}\begin{bmatrix}\boldsymbol{\varphi}\\ \boldsymbol{\Phi}\end{bmatrix} - \frac{\partial}{\partial t}\begin{bmatrix}\boldsymbol{\psi}\\ \boldsymbol{\chi}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\sigma}\\ \boldsymbol{\mu}\end{bmatrix}, \qquad \text{Div}\begin{bmatrix}\boldsymbol{\psi}\\ \boldsymbol{\chi}\end{bmatrix} = -\begin{bmatrix}\rho\,\mathbf{v}\\ \theta\,\mathbf{s}\end{bmatrix}. \qquad (3.4), (3.5)$$

A **Lorentz** gauge for the four potential, as one would find in the field theory of electromagnetism, is lacking here, since the gradient in (4.8) describes the state of deformation velocity of the continuum that actually exists. s_k and v_k must then coincide in (4.8) and (3.5).

The differential operators that were introduced in (2.1), (2.2), (2.3) stem from the motor calculus [2]. In that calculus, the scalar product of the two motors **a** and **b** is defined by:

$$\mathbf{a}\cdot\mathbf{b} = \begin{bmatrix}\overset{1}{\mathbf{a}}\\ \overset{2}{\mathbf{a}}\end{bmatrix}\cdot\begin{bmatrix}\overset{1}{\mathbf{b}}\\ \overset{2}{\mathbf{b}}\end{bmatrix} = \overset{1}{\mathbf{a}}\cdot\overset{2}{\mathbf{b}} + \overset{2}{\mathbf{a}}\cdot\overset{1}{\mathbf{b}} = \overset{1}{a}_k\,\overset{2}{b}_k + \overset{2}{a}_k\,\overset{1}{b}_k\,. \tag{5.1}$$

For that reason, the analogue to the **Lorentz** force (1.5) can be found as follows:

$$F_k = -\begin{bmatrix}\rho\,v_i\\ \theta\,s_i\end{bmatrix}\cdot\begin{bmatrix}S_{kj}\\ I_{kj}\end{bmatrix} + \varepsilon_{klm}\begin{bmatrix}\sigma_{li}\\ \mu_{li}\end{bmatrix}\cdot\begin{bmatrix}B_{mj}\\ D_{mj}\end{bmatrix}$$

$$= -\rho\, v_i\, I_{ki} - \theta\, s_i\, S_{ki} + \varepsilon_{klm}\,(\sigma_{li}\, D_{mi} + \mu_{li}\, B_{mi})\ . \tag{5.2}$$

The expression $\varepsilon_{klm}\ \sigma_{li}\ D_{mi}$ is known as the **Peach-Koehler** force, which acts upon the dislocation D_{ik} , while $\varepsilon_{klm}\ \mu_{li}\ B_{mi}$ is the force that acts upon the disclination B_{ik} . The first two summands in (5.2) are the forces on the dislocation current densities **[9]**. One finds the analogue to (1.6), namely, the power density, in an entirely corresponding way [**9**]:

$$\Lambda = \begin{bmatrix} \boldsymbol{\sigma} \\ \boldsymbol{\mu} \end{bmatrix} \cdot\cdot \begin{bmatrix} \mathbf{S} \\ \mathbf{I} \end{bmatrix} = \sigma_{ik}\, I_{ik} + \mu_{ik}\, S_{ik}\ . \tag{5.3}$$

G. Kluge [**9**] has shown that $\sigma_{ik}\ I_{ik}$ is precisely the work that is done in a moving singular dislocation line by the **Peach-Koehler** force.

The **Maxwell** equations (1.1), (1.2) of the electromagnetic field must be extended by material laws (viz., constitutive equations), whether one is dealing with empty space or a material medium. As one knows, such material laws can be exceptionally complicated (think of ferromagnetic bodies).

Corresponding statements will be true for the **Maxwell** equations of moving dislocations. We will be entering into virgin territory when we pose the question of what connection exists between, for example, the stress functions and the dislocation densities. Let us hope that the analogy that was discovered here might be useful in the context of the open, and certainly complicated, problems of the theory of dislocations. Perhaps **Mie**'s electrodynamics will show the way [**10**] (*).

The impetus for this work came from the recently-appearing publication of **G. Kluge** [**9**], which has been referred to quite often here, and we have adopted his notations here throughout. Without wishing to diminish the value of **Kluge**'s work, it must be said that his interpretation of the tensor B_{ik} as a measure of the "foreign matter" (following the example of other authors [**8**]) is erroneous. As we said above, B_{ik} represents the density of rotational dislocations – viz., disclinations. Other complications arise from the quite correct, but unnecessarily complicated, stress functions. Moreover, it was not observed that the deformation tensor of the **Cosserat** continuum is asymmetric. However, all of that can be easily repaired, and that should not diminish the meaning of that work as a foray into hitherto-unknown territory. A critical inspection of **Kluge**'s work would lead to the analogy that was presented here.

(*) Added by the editor: The author permits me to refer to his next-appearing papers:

"Maxwell-Gleichungen, Energiesatz und Lagrangedichte in the Kontinuumstheorie der Versetzungen," Acta Mechanica **10** (1970), 59-66.

"Eine Feldtheorie der Versetzungen im COSSERAT-Kontinuum," Zeit. f. Angew. Math u. Phys. **20** (1969), 891-899.

References

1. **H. Schaefer**, "Die Spannungsfunktionen eines Kontinuums mit Momentspannungen," Bull. Acad. Pol. Sc. **15** (1967), 63-67, 69-73.
2. **H. Schaefer**, "Analysis der Motorfelder im Cosserat-Kontinuum," ZAMM **47** (1967), 319-328.
3. **S. Kessel**, "Die Spannungsfunktionen des Cosserat-Kontinuums," ZAMM **47** (1967), 329-336.
4. **S. Kessel**, "Stress Functions and Loading Singularities for the Infinitely-Extended Linear Elastic-Isotropic Cosserat Continuum," *Mechanics of Generalized Continua*, ed., E. Kröner, Springer-Verlag, 1968.
5. **W. Günther**, "Zur Statik und Kinematik des Cosserat-Kontiuum," Abh. d. Braunschweig. Wiss. Ges. **10** (1958), 195-213.
6. **H. Schaefer**, "Das Cosserat-Kontinuum," ZAMM **47** (1967), 485-498.
7. **K. Anthony**, **U. Essmann**, **A. Seeger**, **H. Träuble**, "Dislocations and the Cosserat Continuum with Incompatible Rotations," *Mechanics of Generalized Continua*, ed. E. Kröner, Springer-Verlag, 1968.
8. **H. Günther**, *Zur nichtlinearen Kontinuumstheorie bewegter Versetzungen*, Akademie-Verlag, Berlin, 1967.
9. **G. Kluge**, "Zur Dynamik der allgemeinen Versetzungstheorie bei Berücksichtigung von Momentenspannungen," Int. J. Eng. Sci. **7** (1969), 169-182.
10. **H. Weyl**, *Raum-Zeit-Materie*, 4th ed., Springer-Verlag, Berlin, 1921, pp. 186.