"Die Spannungsfunktionen eines Kontinuums mit Momentenspannungen. II," Bull. Acad. Pol. Sci. **15** (1967), 69-73.

The stress functions of a continuum with moment stresses. II (†)

by

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Presented by W. OLSZAK on 15 November 1966.

Translated by D. H. Delphenich

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4. – The second-order stress functions of a continuum with moment stresses.

In order to keep the following discussion clear and concise, we shall introduce a symbolic notation for some differential operators that are completely analogous to the differential operators grad, div, rot of vector analysis.

We first summarize the most important equations of article [**1**] in symbolic notation. We write the equilibrium conditions (1.1) as:

(4.1) $$\text{Div}\begin{pmatrix}\sigma \\ \mu\end{pmatrix} + \begin{pmatrix}X \\ Y\end{pmatrix} = 0 .$$

It follows from (1.2) that:

(4.2) $$\begin{pmatrix}\sigma' \\ \mu'\end{pmatrix} = \text{Rot}\begin{pmatrix}S \\ F\end{pmatrix} ,$$

and (4.2) and (4.1) imply the identity:

(4.3) $$\text{Div Rot} \equiv 0 .$$

(1.3) takes the form:

(4.4) $$\begin{pmatrix}\sigma'' \\ \mu''\end{pmatrix} = \text{Grad}^{*}\begin{pmatrix}\Omega \\ \chi\end{pmatrix} ,$$

so, by analogy with (1.4), while recalling (4.1), one will have:

(†) **Remark**: Since both articles define a whole, the sequence of sections, as well as the sequence of references, is common to both parts.

(4.5) $$\mathrm{Grad}^* \begin{pmatrix} \Omega \\ \chi \end{pmatrix} = \begin{pmatrix} \Delta\Omega \\ \Delta\chi \end{pmatrix} = \Delta \begin{pmatrix} \Omega \\ \chi \end{pmatrix}$$

and

(4.6) $$\Delta \begin{pmatrix} \Omega \\ \chi \end{pmatrix} + \begin{pmatrix} X \\ Y \end{pmatrix} = 0\,.$$

As a rule of calculation, we point out:

(4.7) $$\mathrm{Div}\ \mathrm{Grad}^* \equiv \Delta\,.$$

We now introduce the second-order stress functions Σ_{ik}, Ψ_{ik} by setting:

(4.8) $$\begin{aligned} S_{ik} &= e_{\alpha\beta i}\, \partial_\alpha \Sigma_{\beta k}\,, \\ F_{ik} &= e_{\alpha\beta i}\, (\partial_\alpha \Psi_{\beta k} - e_{k\alpha\mu}\, \Sigma_{\beta\mu})\,, \end{aligned}$$

or, in symbolic notation:

(4.9) $$\begin{pmatrix} S \\ F \end{pmatrix} = \mathrm{Rot}^* \begin{pmatrix} \Sigma \\ \Psi \end{pmatrix}\,.$$

Therefore, from (4.2):

(4.10) $$\begin{pmatrix} \sigma' \\ \mu' \end{pmatrix} = \mathrm{Rot}^*\, \mathrm{Rot}^* \begin{pmatrix} \Sigma \\ \Psi \end{pmatrix}\,.$$

Explicit calculation yields:

(4.11) $$\begin{aligned} \sigma'_{ik} &= -\Delta \Sigma_{ik} + \partial_i\, (\partial_\mu \Sigma_{\mu k})\,, \\ \mu'_{ik} &= -\Delta\, \Psi_{ik} + \partial_i\, (\partial_\mu \Psi_{\mu k} + e_{k\mu\lambda}\, \Sigma_{\mu\lambda}) + e_{ik\alpha}\, (\partial_\mu \Sigma_{\mu\alpha})\,. \end{aligned}$$

In symbolic notation, (4.11) is:

(4.12) $$\begin{pmatrix} \sigma' \\ \mu' \end{pmatrix} = -\Delta \begin{pmatrix} \Sigma \\ \Psi \end{pmatrix} + \mathrm{Grad}^*\, \mathrm{Div} \begin{pmatrix} \Sigma \\ \Psi \end{pmatrix}\,.$$

That implies the rule of calculation:

(4.13) $$\mathrm{Rot}\ \mathrm{Rot}^* \equiv -\Delta + \mathrm{Grad}^*\ \mathrm{Div}\,,$$

which, in turn, can emerge by analogy with vector analysis.

The introduction of second-order stress functions by way of (4.9) raises the question of whether one might perhaps select a subset of the set of stress functions S_{ik}, F_{ik} that would lead to an inadmissible restriction on the stress functions σ'_{ik}, μ'_{ik}. The fact that this question can be answered in the negative can be seen from the following argument: The special first-order stress functions:

(4.14) $$S^0_{ik} = \partial_i A_k\,, \qquad F^0_{ik} = \partial_i B_k - e_{ik\alpha} A_\alpha\,,$$

with twice-differentiable, but otherwise arbitrary A_k , B_k are null stress functions, i.e., they imply the stresses $\sigma'_{ik} \equiv 0$, $\mu'_{ik} \equiv 0$. In a symbolic representation, we write (4.14):

$$\begin{pmatrix} S^0 \\ F^0 \end{pmatrix} = \text{Grad} \begin{pmatrix} A \\ B \end{pmatrix}, \tag{4.15}$$

and from (4.2), we have the rule of calculation:

$$\text{Rot Grad} \equiv 0 \,. \tag{4.16}$$

Now, the introduction of second-order stress functions by (4.9) means that we shall impose the following covariant normalization on the first-order stress functions:

$$\partial_\mu S_{\mu e} = 0 \,, \qquad \partial_\mu F_{\mu e} - e_{e\alpha\beta} S_{\alpha\beta} = 0 \,. \tag{4.17}$$

Symbolically:

$$\text{Div}^* \begin{pmatrix} S \\ F \end{pmatrix} = 0 \,. \tag{4.18}$$

That is because one confirms by explicit calculation that:

$$\text{Div}^* \, \text{Rot}^* = 0 \tag{4.19}$$

and furthermore:

$$\text{Div}^* \, \text{Grad} \equiv \Delta \,. \tag{4.20}$$

From (4.2), (4.15), (4.16), $\begin{pmatrix} S \\ F \end{pmatrix}$ and $\begin{pmatrix} S + S^0 \\ F + F^0 \end{pmatrix}$ both lead to the same stress state. In the event that:

$$\text{Div}^* \begin{pmatrix} S \\ F \end{pmatrix} \neq 0 \,,$$

one can always arrive at:

$$\text{Div}^* \begin{pmatrix} S + S^0 \\ F + F^0 \end{pmatrix} = 0 \,. \tag{4.21}$$

That is because from (4.15) and (4.20), (4.21) will read:

(4.22) $$\text{Div}\begin{pmatrix} S \\ F \end{pmatrix} + \Delta\begin{pmatrix} A \\ B \end{pmatrix} = 0\,,$$

and the A_k, B_k can be determined from the known theorems of potential theory in such a way that (4.22) is fulfilled.

5. – The complete representation of the equilibrium state by stress functions in a continuum with moment stresses.

We have to show that for every solution σ_{ik}, μ_{ik} of the equilibrium conditions (4.1):

(5.1) $$\text{Div}\begin{pmatrix} \sigma \\ \mu \end{pmatrix} + \begin{pmatrix} X \\ Y \end{pmatrix} = 0\,.$$

Stress functions Σ_{ik}, Ψ_{ik}, Ω_k, χ_k exist such that from (1.5), (4.4) and (4.12) will be true, so:

(5.2) $$\begin{pmatrix} \sigma \\ \mu \end{pmatrix} = -\,\Delta\begin{pmatrix} \Sigma \\ \Psi \end{pmatrix} + \text{Grad}^*\,\text{Div}\begin{pmatrix} \Sigma \\ \Psi \end{pmatrix} + \text{Grad}^*\begin{pmatrix} \Omega \\ \chi \end{pmatrix},$$

with:

(5.3) $$\Delta\begin{pmatrix} \Omega \\ \chi \end{pmatrix} + \begin{pmatrix} X \\ Y \end{pmatrix} = 0\,.$$

The proof proceeds by analogy with the proof that Gurtin presented in section 2 on the completeness of his generalized Beltrami representation of the symmetric stress tensor.

We next determine Σ_{ik} and Ψ_{ik} from the Poisson equations:

(5.4) $$\sigma_{ik} = -\,\Delta\,\Sigma_{ik}\,, \qquad \mu_{ik} = -\,\Delta\,\Psi_{ik}\,.$$

We will then have:

(5.5) $$\begin{pmatrix} \sigma \\ \mu \end{pmatrix} = -\Delta\begin{pmatrix} \Sigma \\ \Psi \end{pmatrix} + \text{Grad}^*\,\text{Div}\begin{pmatrix} \Sigma \\ \Psi \end{pmatrix} - \text{Grad}^*\,\text{Div}\begin{pmatrix} \Sigma \\ \Psi \end{pmatrix},$$

trivially. If we set:

(5.6) $$-\,\text{Div}\begin{pmatrix} \Sigma \\ \Psi \end{pmatrix} = \begin{pmatrix} \Omega \\ \chi \end{pmatrix}$$

in the last expression then (5.5) will agree formally with (5.2).

Since σ_{ik}, μ_{ik} fulfill (5.1), it will follow from (5.4) that:

(5.7)
$$\operatorname{Div}\begin{pmatrix}\sigma\\ \mu\end{pmatrix} = -\Delta\operatorname{Div}\begin{pmatrix}\Sigma\\ \Psi\end{pmatrix} = -\begin{pmatrix}X\\ Y\end{pmatrix}.$$

However, (5.3) is also fulfilled with that, along with (5.6), and the proof is complete.

In the case $X_k \equiv 0$, $Y_k \equiv 0$, one has the following theorem, which is the generalization of Gurtin's theorem to the continuum with moment stresses: Every stress field σ_{ik} , μ_{ik} in the equilibrium state is composed of:

$$\sigma_{ik} = \sigma'_{ik} + \sigma''_{ik}, \qquad \mu_{ik} = \mu'_{ik} + \mu''_{ik},$$

in which σ'_{ik} , μ'_{ik} is a stress field in total self-equilibrium, while σ''_{ik} , μ''_{ik} is a harmonic stress state.

6. – Concluding remark.

In order to put the foregoing arguments – in particular, those of the last chapter – in the proper light, we would like to remark that we have proved nothing but the fact that any arbitrary tensor-pair σ_{ik} , μ_{ik} can be represented by:

(6.1)
$$\begin{pmatrix}\sigma\\ \mu\end{pmatrix} = \operatorname{Rot}\begin{pmatrix}S\\ F\end{pmatrix} + \operatorname{Grad}^{*}\begin{pmatrix}\Omega\\ \chi\end{pmatrix}.$$

It would be simple to show, in conjunction with our considerations above, that one also has the dual possibility:

(6.2)
$$\begin{pmatrix}\sigma\\ \mu\end{pmatrix} = \operatorname{Rot}^{*}\begin{pmatrix}S\\ F\end{pmatrix} + \operatorname{Grad}\begin{pmatrix}\Omega\\ \chi\end{pmatrix}.$$

Naturally, the representation (6.2) is worthless for stress tensors, because σ_{ik} , μ_{ik} must satisfy the equilibrium conditions (5.1) with prescribed X_k , Y_k . However, the representation (6.2) allows one to make the transition from the first-order stress functions S_{ik} , F_{ik} to the second-order stress functions Σ_{ik} , Ψ_{ik} in section 4 more precise. From (6.2), one would have to replace (4.9) with:

(6.3)
$$\begin{pmatrix}S\\ F\end{pmatrix} = \operatorname{Rot}^{*}\begin{pmatrix}\Sigma\\ \Psi\end{pmatrix} + \operatorname{Grad}\begin{pmatrix}G\\ H\end{pmatrix},$$

in which the functions Σ_{ik} , Ψ_{ik} and G_k , H_k can be determined for arbitrarily-given S_{ik} , F_{ik} . However. it is clear from (4.15) and (4.16) that (6.3) is in no way more general than (4.9).

A further remark to conclude this discussion is concerned with the stress fields that are found in total self-equilibrium. Günther [**2**] has shown that such stress fields can be subsumed entirely by the Beltrami representation (2.10); his proof was reproduced in Gurtin [**6**]. The representation (1.2) goes back to Günther [**3**] and is complete in the same sense as Beltrami's. From the foregoing considerations of this section, the same thing will apply to our second-order stress functions. For that reason, the representation (4.11) or (4.12) is complete for stress fields in total self-equilibrium. We then subsume the stress states in bodies that are bounded by only a single surface. However, we also subsume the stress states in bodies with cavities (viz., *periphractic* domains), so in bodies that are bounded by multiple surfaces, as long as the resulting dyname of loading vanishes for each surface.

Added in proof. – D. C. Carlson treated the same problem [**7**]. However, his complete representations are more complicated than the ones that were given here. The article contains an error insofar as Gurtin's simpler representation that was derived in eqs. (4.15) to (4.19) was communicated by Schaefer in a letter.

TECHNISCHE HOCHSCHULE, BRAUNSCHWEIG, DEUTSCHES BUNDES REPUBLIK

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