“Ueber das Gauss’schen Grundsetz der Mechanik,” Zeit. Math. Phys. **3** (1858), 197-223, 261-274.

On Gauss’s fundamental law of mechanics

or

The principle of least constraint, as well as another new basic law of mechanics,
with an excursion into various situations that the mechanical principles apply to

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1. – Development of Gauss's law

In treatise no. 18 in Crelle's Journal for Mathematics, vol. **4**, pp. 232, our great mathematician **Gauss** enriched mechanics with a general fundamental law that should not be missing from any textbook on analytical mechanics, along with **d'Alembert**'s principle and the principle of virtual velocities, since it alone, without the aid of a second fundamental law, suffices completely to determine the motion and equilibrium of any system of bodies, and can thus be taken to be the foundation for all of mechanics. Even though the greater simplicity of Gauss's law, in principle, does not always bring about a greater *practical* simplicity or analytical brevity in the treatment of special problems, since in many cases, **d'Alembert**'s principle, in conjunction with that of virtual velocities, would be easier to implement, nonetheless, there are also cases in which Gauss's law would be more direct and convenient to employ. However, in addition, it reveals a most interesting property of every system of bodies that is found to be in *motion*, as well as every one that is in *equilibrium*, especially because it expresses a criterion for the laws of *motion* and *rest* with equal generality.

The fact that this law has not enjoyed a general acquaintance is perhaps based in the brevity of presentation that the inventor himself gave to it, by which the actual essence of that law and its relationship to the usual general fundamental laws of mechanics might not seem sufficiently clear to many. Therefore, it might be advisable to direct the attention of the mathematical public to that important law with some emphasis, and to that end, to explain the law itself somewhat more thoroughly and illustrate its application in some special cases.

However, in addition, we will take this opportunity to digress a bit further on the basic laws of mechanics and add something new to it.

Gauss defined his law, which one can rightly call ***Gauss's*** *principle*, or from its content, the *principle of least constraint*, as opposed to d'Alembert's principle, in words as follows:

The motion of a system of material points that are coupled to each other in whatever way, and whose motion is likewise constrained by whatever sort of restrictions, will take place at each moment with the greatest possible agreement with the free motion, or the ***least possible constraint****, in which one considers a measure of the constraint that the entire system experiences at each moment in time to be the sum of the products of the squares of the deflections of each point from its free motion with their masses.*

If one then has that (Table II, Fig. 1)(†):

$m, m', m'', \ldots$ are the masses of the material points

$a, a', a'', \ldots$ are their positions at time t,

$b, b', b'', \ldots$ are the locations that they would assume after the infinitely-small time interval dt as a result of the forces (that are *applied* to them)

(†) Translator: I have not been able to find the cited figures and tables.

during that time interval and the velocities and directions that they would attain if they were all free,

$c, c', c'', \ldots$ are the locations that they would actually assume at time interval *dt*

then, from the principle above, of all of the locations that are compatible with the conditions on the system, the actual locations will be the ones for which the expression:

$$m\,(cb)^2 + m'\,(c'b')^2 + m''\,(c''b'')^2 + \ldots \tag{1}$$

is a minimum.

Equilibrium is obviously only a special case of the general law of motion, since in that case, the actual locations $c, c', c'', \ldots$ would coincide with the original ones a, a', a'', …, as long as the equilibrium exists in the *rest* state, so for a system that is found in equilibrium, the expression:

$$m\,(ab)^2 + m'\,(a'b')^2 + m''\,(a''b'')^2 + \ldots \tag{2}$$

must be a minimum. It likewise follows from this that *the persistence of the system in the rest state lies closer to the free motion of the individual points than any possible way that they might emerge from it.*

Gauss's law can be derived from d'Alembert's principle and that of virtual velocities as follows. Let (Table II, Fig. 2):

p be the force that acts upon the material point a, which acts during the time interval *dt*, and if that point were completely free then it would go to b when one considers the velocity and direction that one achieves at time t.

q be the force that acts upon the point a and is produced by the constraint on the system, as a result of which the point would deflect from b to c in as a completely-free mass from the rest state during the time interval *dt*.

r be the resultant of p and q, by whose action, the point a would actually go from a to c as a completely-free mass during the time interval *dt* when one considers the velocity and direction that are achieved at time t; hence, it is the so-called *effective* force on the point a.

Since the point a moves under the action of the force p and the constraints on the system as if it were free and merely affected with the force r, it would follow that if the force r, which acts in the opposite direction to the force p (so the force $-r$), were applied to a, in addition to p (so it would be subjected to the force $-q$ that is composed of p and $-r$, which would lead the completely-free point through the points in space of *cb* during the time interval *dt* under the remaining constraints on the system), then the system would be found in the *equilibrium* state. In fact, the forces $-q, -q', -q''$, … represent the so-called *lost forces*, which must keep the system in equilibrium under the remaining constraints on the system, from **d'Alembert**'s principle.

If we apply the principle of virtual velocities in order to exhibit the condition equation for that equilibrium then we let $\gamma, \gamma', \gamma'', \ldots$ be the locations where the points $a, a', a'', \ldots$ might possibly arrive after the time dt, which are different from $c, c', c'', \ldots$, but compatible with the conditions on the system. Now, obviously $c\gamma, c'\gamma', c''\gamma'', \ldots$ are also the virtual motions that the points $c, c', c'', \ldots$ could assume under the constraints for the system that is found in equilibrium under the forces $-q, -q', -q'', \ldots$

If one drops a perpendicular $\gamma\beta$ from each of the points $\gamma, \gamma', \gamma'', \ldots$ (e.g., from γ) to cb then since the force $-q$ acts parallel to cb, $-q\,(c\beta)$ will be the virtual moment of that force. If one lets $\varphi, \varphi', \varphi'', \ldots$ denote the angles $bc\gamma, b'c'\gamma', b''c''\gamma'', \ldots$ that $c\gamma, c'\gamma', c''\gamma'', \ldots$ make with $cb, c'b', c''b'', \ldots$, resp., then $-q\,(c\gamma)\cos\varphi, -q'\,(c'\gamma')\cos\varphi', -q''\,(c''\gamma'')\cos\varphi'', \ldots$ will be the virtual moments of the forces $-q, -q', -q'', \ldots$

Since the force $-q$ is such that it would push the mass m (which is thought to be completely *free*) from the rest state through the points of cb during the time dt, it will be proportional to the product $m\,(cb)$. If we then set the forces $-q, -q', -q'', \ldots$ equal to the values $m\,(cb), m'\,(c'b'), m''\,(c''b''), \ldots$, resp. (which are proportional to them), then their virtual moments will be:

$$m\,(cb)(c\gamma)\cos\varphi,\quad m'\,(c'b')(c'\gamma')\cos\varphi',\quad m''\,(c''b'')(c''\gamma'')\cos\varphi'', \ldots,$$

respectively.

From the principle of virtual velocities, the sum of those moments must be *equal to zero*. One will then have:

(3) $$\sum m\,(cb)(c\gamma)\cos\varphi = 0.$$

Now, since:

$$(\gamma b)^2 = (cb)^2 + (c\gamma)^2 - 2\,(cb)(c\gamma)\cos\varphi$$

or

(4) $$(cb)^2 = (\gamma b)^2 - (c\gamma)^2 + 2\,(cb)(c\gamma)\cos\varphi,$$

one will have:

$$\sum m\,(cb)^2 = \sum m\,(\gamma b)^2 - \sum m\,(c\gamma)^2 + 2\sum m\,(cb)(c\gamma)\cos\varphi.$$

It will then follow from equation (3) that:

(5) $$\sum m\,(cb)^2 = \sum m\,(\gamma b)^2 - \sum m\,(c\gamma)^2.$$

The length cb is the *actual* deviation of the mass m from the *free motion*, while γb represents *any other possible* deviation. Now, since from equation (5), one has that $\sum m\,(cb)^2$ is always less than $\sum m\,(\gamma b)^2$, in that, one will find the proof of the *principle of least constraint* that was expressed above, namely, that the sum of the products of the actual deflections of the individual points from the free motion of the masses at those point must be a minimum; i.e., it must be smaller than the sum of the products of any other deflections of those masses that are possible under the conditions on the system.

For equilibrium in the rest state, equation (5) will become:

(5a) $$\sum m\,(ab)^2 = \sum m\,(\gamma b)^2 - \sum m\,(a\gamma)^2.$$

2. – Explanation for Gauss's law.

The foregoing law requires some explanation, and for the sake of applying it in certain cases, a transformation of equation (5) or the wording of the principle that it expresses might be absolutely necessary.

When the force p that acts upon a free mass m is capable of endowing that mass with the velocity g in a unit of time, such that g represents the acceleration that the force p gives to the mass p, it is known that the following relationship exist:

(6) $$p = m\,g.$$

The length s of the path that is that is traversed from the rest state at time t is:

(7) $$s = \tfrac{1}{2} g\,t^2 .$$

The path $ds = gt\,dt$ will be traveled in the time interval dt. For the first time interval that follows the rest state (so the one for which one has $t = 0$), that path length will be equal to zero, from the formula itself. However, that zero value for ds at $t = 0$ tells one only that for $t = 0$, the value of ds is no longer an infinitely-small quantity of *degree one* relative to dt, but one of higher degree. In fact, when one either sets t equal to dt directly in equation (7) or when one sets $t = 0$ in the value of the complete increment of s, so in:

$$\Delta s = \frac{ds}{dt}\cdot dt + \frac{1}{1\cdot 2}\frac{d^2 s}{dt^2}\cdot dt^2 + \frac{1}{1\cdot 2\cdot 3}\frac{d^3 s}{dt^3}\cdot dt^3 + \cdots$$

$$= gt\,dt + \tfrac{1}{2} g\,dt^2 ,$$

one will get:

(8) $$\Delta s = \tfrac{1}{2} g\,dt^2$$

for the path that is traversed in the first time interval.

Now, if:

$-q, -q_1, -q_2$ are the forces that would push the free mass m at the material point a from c to γ during the time interval dt.

$f, \quad f_1, \quad f_2$ are the accelerations that the forces $-q$, $-q_1$, $-q_2$, resp., endow the mass m with per unit time

then one will have:

(9) $$-q = m f, \qquad -q_1 = m f_1, \qquad -q_2 = m f_2,$$

(10) $$cb = \tfrac{1}{2} f t^2, \qquad \gamma b = \tfrac{1}{2} f_1 t^2, \qquad c\gamma = \tfrac{1}{2} f_2 t^2.$$

When one substitutes the values in (10), formulas (3) and (5) will assume the form:

(11) $$\sum m f f_2 \cos \varphi = 0,$$

(12) $$\sum m f^2 = \sum m f_1^2 - \sum m f_2^2,$$

and when one introduces the forces q, q_1, q_2 from equation (9), those equations will be converted into:

(13) $$\sum q f \cos \varphi = 0,$$

(14) $$\sum q f = \sum q_1 f_1 - \sum q_2 f_2.$$

If one so wishes, one can also write those formulas as:

(15) $$\sum q\,(c\gamma) \cos \varphi = 0,$$

(16) $$\sum q\,(cb) = \sum q\,(\gamma b) - \sum q\,(c\gamma).$$

In the form of equation (12), the quantities that one treats in the principle of least constraint are freed from the consideration of *infinitely-small* paths. In that form, one deals with only *finite* values, since the *measure of the constraint* for any material point now appears as the *product of its mass with the square of its acceleration due to the deflecting force.*

In the form of equation (14), the *deflecting forces q* themselves are introduced in place of the *masses m*. The *measure of the constraint* is now the *product of the deflecting force with its acceleration.*

In the case where a point has no mass at all, but only represents a geometric position in the system upon which the force p acts, eliminating the mass m of a point a by means of the formulas will take the form of an unacceptable necessity, because one would then have that the mass $m = 0$ for any point of that kind, but a finite force p would assign an infinitely-large acceleration to an infinitely-small mass, so the point b (Fig. 3) would be at an infinite distance. cb and $b\gamma$ (or also f and f_1) would become *infinitely large*, and in that way terms would arise in the formulas above would that take the form $0 \cdot \infty$ or ∞, which would make those formulas unusable. Only equation (13) would still remain useful in those cases.

As far as the remaining formulas are concerned, it is clear that it is not at all necessary to take the sum $\sum$ over *all* points of the system at once. One could also first take the sum S over a certain complex of points and then take the sum $\mathfrak{S}$ over the remaining points, such that one would then have $\sum = \text{S} + \mathfrak{S}$. In that way, equation (3) would then become:

$$\text{(17)} \qquad \text{S}\, m\,(cb)\,(c\gamma)\cos\varphi + \mathfrak{S}\, m\,(cb)\,(c\gamma)\cos\varphi = 0,$$

and when one splits $\sum m\,(cb)^2$ into $\text{S}\, m\,(cb)^2 + \mathfrak{S}\, m\,(cb)^2$ in equation (5) and then substitutes the values from equation (4) that correspond to the partial sum $\mathfrak{S}\, m\,(cb)^2$ in it, equation (5) will become:

$$\text{(18)} \qquad \text{S}\, m\,(cb)^2 + \mathfrak{S}\, m\,(cb)\,(c\gamma)\cos\varphi = \text{S}\, m\,(cb)^2 - \text{S}\, m\,(c\gamma)^2.$$

If one would like to eliminate the sum $\mathfrak{S}$ from equation (18) with the help of equation (17) then one would indeed obtain an equation with only the summation sign S, which would then refer to only an arbitrary part of the masses of the system. However, one would easily find that this equation was only a result of equation (4), so it is only a *geometric* relationship between the masses that enter into it, but not a *mechanical* one.

If one substitutes the value $m(cb) = \frac{1}{2}(-q)\,dt^2$ for the product $m\,(cb)$ in the summation sign $\mathfrak{S}$, by means of the relations (9) and (10), then that will yield:

$$\text{(19)} \qquad \text{S}\, m\,(cb)\,(c\gamma)\cos\varphi + \tfrac{1}{2}(-q)\,dt^2\,\mathfrak{S}\,(-q)\,(c\gamma)\cos\varphi = 0,$$

$$\text{(20)} \qquad \text{S}\, m\,(cb)^2 + dt^2\,\mathfrak{S}\,(-q)\,(c\gamma)\cos\varphi = \text{S}\, m\,(\gamma b)^2 - \text{S}\, m\,(c\gamma)^2,$$

instead of (17) and (18), resp.

Should the sign $\mathfrak{S}$ refer to only those points of the system that have no mass then one would have to observe that for each such point *cb* that is parallel to *ab*, the magnitude and direction of the *lost* force $-q$ would have to be precisely equal to the *applied* force *p* that acts upon the point *a* (Fig. 2). One would then have:

$$\text{(21)} \qquad \text{S}\, m\,(cb)\,(c\gamma)\cos\varphi + \tfrac{1}{2}dt^2\,\mathfrak{S}\,p\,(c\gamma)\cos\varphi = 0,$$

$$\text{(22)} \qquad \text{S}\, m\,(cb)^2 + dt^2\,\mathfrak{S}\,p\,(c\gamma)\cos\varphi = \text{S}\, m\,(\gamma b)^2 - \text{S}\, m\,(c\gamma)^2,$$

under that assumption, and infinitely-large or indeterminate quantities would no longer enter in those formulas.

If one would also like to let the lost force $-q$ appear under the S sign in place of the mass *m* [since that is true of equations (15) and (16)], then one would get:

(23) $\mathrm{S}\,(-q)\,(cb)\,(c\gamma)\cos\varphi + \mathfrak{S}\,p\,(c\gamma)\cos\varphi = 0,$

(24) $\mathrm{S}\,(-q)(cb) + 2\,\mathfrak{S}\,p\,(c\gamma)\cos\varphi = \mathrm{S}\,(-q)(\gamma b) - \mathrm{S}\,(-q)(c\gamma)\,.$

In order to fix the directions and the angle φ precisely, one must once more point out that $-q$ is the *lost* force, which acts in the direction *cb*, so the *deflecting* force *q* would act in the directly *opposite* direction *bc*, but the angle $\varphi = bc\gamma$ lies between the direction *cb* of the *lost* force $-q$ and the direction $c\gamma$, in which γ refers to any other displaced position of the point *a* that is possible under the constraints on the system.

As far as the case in which the system is found to be in *equilibrium* and *at rest* is concerned, the deflecting force *q* will be equal to $-p$ or the lost force $-q = p$. In that case, equations (21) and (22) will assume the form:

(25) $\mathrm{S}\,m\,(ab)\,(a\gamma)\cos\varphi + \tfrac{1}{2}\,dt^2\,\mathfrak{S}\,p\,(a\gamma)\cos\varphi = 0,$

(26) $\mathrm{S}\,m\,(ab)^2 + dt^2\,\mathfrak{S}\,p\,(a\gamma)\cos\varphi = \mathrm{S}\,m\,(\gamma b)^2 - \mathrm{S}\,m\,(a\gamma)^2,$

resp., and equations (23) and (24) will assume the form:

(23) $\sum p\,(cb)\,(a\gamma)\cos\varphi = 0,$

(24) $\mathrm{S}\,p\,(ab) + 2\,\mathfrak{S}\,p\,(a\gamma)\cos\varphi = \mathrm{S}\,p\,(\gamma b) - \mathrm{S}\,p\,(a\gamma)\,.$

If the system were not precisely in the rest state, but in *uniform motion*, when it is in *equilibrium* then the point *c* would not in fact fall upon *a*, but at some other point a_1 to which the point *a* would be led during the time interval *dt* by the velocity, once it has been achieved. However, one would always have $q = -p$, and the foregoing equations (25) to (28) would remain valid under the assumption that *ab* represents the path that the point *a* would traverse in the time *dt* as a result of only the force *p*, but with no consideration given to the previously-achieved velocity, and that *g* refers to another location for the point *a* that likewise does not consider that velocity.

3. – Relationship between Gauss's law and d'Alembert's principle and the principle of virtual velocities.

Equation (4) expresses only a *geometric* relationship that prevails in the system, while equation (3) expresses a *mechanical* relationship. Now, since formula (5) is the result of simply combining (3) and (4), it will follow that in a strict mechanical sense, formula (5) is *equivalent* to formula (3).

However, formula (3) will represent **d'Alembert**'s principle immediately (under which, the system will be in equilibrium with the lost forces $-q$) once one applies the *principle of virtual velocities*, whereas formula (5) is the complete expression for **Gauss**'s principle, in that it does not merely tell one that $\sum m\,(cb)^2$ is smaller than any possible $\sum m\,(\gamma b)^2$, and is thus a *minimum*, but at the same time, it shows *by how much* the former sum is smaller than the latter.

One sees from this that Gauss's principle includes d'Alembert's, in conjunction with the principle of virtual velocities, which are the *two* fundamental laws that define the basis for *statics* and *dynamics* in its usual presentation, and when they are taken together, one can deem that to be a *general* or *higher* principle of mechanics.

Gauss himself said, in the aforementioned treatise:

> "The special character of the principle of virtual velocities consists of the fact that it is a general formula for solving all static problems, and is therefore a replacement for all other principles without, however, immediately taking credit for the fact that it already seems plausible by itself, to the extent that it was only expressed. In *that* regard, the principle that I will present here seems to have the advantage. However, it also has a *second* one, namely, that it encompasses the law of motion and rest in exactly the same way in greatest generality."

The second advantage, namely, that Gauss's principle characterizes the state of motion and rest at once is sufficiently clear from the foregoing. However, the first advantage, namely, that this principle appears to be a *fundamental law* of mechanics, requires more explanation.

The general wording of Gauss's principle, namely, that the motion of a system at any moment proceeds with the greatest possible agreement with the free motion or *with the smallest possible constraint*, generally seems to be entirely plausible, and a proof would not be required. However, what is *constraint* in the strictly-scientific sense? How does one define the mathematical expression for that general concept? Obviously, for any material point *a* (Fig. 3), the constraint that leads it from the location *b* of its *free* motion to the location *c* of its *actual* motion in the infinitely-small time *dt*, must initially be proportional to the force *q* that pushed it away from the point *b*, and in addition, to the length of the path *bc* through which that mass was pushed, so it should be proportional to the product *q* (*bc*), which represents the *work* done by the deflecting force. One can then take that product itself to be the constraint in question that the mass *m* of the point *a* experiences. Now, should the sum of the constraints that are exerted over the entire system be as small as possible then one would be led immediately to the condition that $\sum q\,(bc)$ must be a minimum.

If *m* denotes the mass of the point *a* then since the force *q* is proportional to the product *m* (*cb*), from equations (9) and (10) one can also impose the demand that $\sum m\,(cb)^2$ must be a minimum, which constitutes the mathematical expression for Gauss's principle.

When one places Gauss's principle at the pinnacle of mechanics in that way, that will imply the remaining fundamental law – namely, the principle of virtual velocities – by the following argument:

Due to equation (4), on purely-geometric grounds, one has:

$$\sum m\,(cb)^2 = \sum m\,(\gamma b)^2 - \sum m\,(c\gamma)^2 + 2\sum m\,(cb)\,(c\gamma)\cos\varphi\,.$$

Now, since $\sum m\,(cb)^2$ is a minimum, so it is always smaller than $\sum m\,(\gamma b)^2$, the quantity $2\sum m\,(cb)\,(c\gamma)\cos\varphi$ will either be negative or smaller than $\sum m\,(c\gamma)^2$, when it is positive, which is a position that the point γ might also assume.

Now let $c\gamma$ (Table II, Fig. 4) be any infinitely-small motion that the point c is in a state to assume according to the constraints on the system, and let $c\gamma_1$ be the advance under the backwards motion that this point would adopt under the return from γ to c. That shows that any point γ_2 that lies between γ and γ_1 can be regarded as the endpoint of the motion $c\gamma_2$. Now, if $X\,dx$ is the analytical expression for $c\,g$, in which X is the function of any quantity x that has one and the same meaning for *all* points of the entire system, such that $X\,dx$, $X'\,dx$, $X''\,dx$, … refer to the material points a, a', a'', …, resp., or c, c', c'', …, resp., then $X\,dx$ will obviously represent *any* infinitely-small motion like $c\gamma_2$ that lies between γ and γ_1 when one merely replaces dx with the corresponding value, and it will be clear that the motions of *all* points of the system inside the infinitely-close limits γ and γ_1 *increase and decrease proportionally, as well as change signs simultaneously.*

Now, since:

$$2\sum m\,(cb)\,(c\gamma)\cos\varphi = 2\,dx\sum m\,(cb)\,X\cos\varphi$$

and

$$\sum m\,(c\gamma)^2 = dx^2\sum m\,X^2$$

from that relation, one sees immediately that for a suitable choice of dx, when the quantity $2\sum m\,(cb)\,(c\gamma)\cos\varphi$ is multiplied by the first power of dx, it will always be *positive*, and also always greater than the quantity $\sum m\,(c\gamma)^2$ times the second power of dx might be if it possessed any nonzero-values at all.

It follows from this, in general, that one must have:

$$\sum m\,(cb)\,(c\gamma)\cos\varphi = 0,$$

which implies equation (3).

Since $m\,(cb)$ is thought to be proportional to the deflecting force $-\,q$, or also in the opposite direction (which one calls the lost force), the foregoing equation will go to:

$$\sum(-q)\,(c\gamma)\cos\varphi = 0 \qquad \text{or} \qquad \sum(-q)\,(c\beta) = 0.$$

However, equilibrium must exist under those lost forces $-\,q$, as **d'Alembert**'s principle itself would say, and proof would not be required. The foregoing formula then expresses a fundamental law of the forces that are found to be in equilibrium, and recognizes the *principle of virtual velocities* in it, which gets its foundation from Gauss's principle in that way.

4. – Simpler proof of the principle of virtual velocities.

In the above, Gauss's principle was derived from d'Alembert's and the principle of virtual velocities. However, it can also be shown how the principle of virtual velocities will follow when one assumes Gauss's principle.

As a rule, one would probably prefer the first path of development, partly because d'Alembert's principle and the principle of virtual velocities are implied by elementary intuitions and admit proofs that are free from objections, but partly because in most cases the last two principles admit an immediate and simpler application to given cases.

In regard to the latter, one must, in fact, observe that since $a, a', a'', \ldots$, as well as $c, c', c'', \ldots$, are positions that the masses $m, m', m'', \ldots$, resp., can actually assume according to the constraints on the system (namely, the former, at the beginning, and the latter, at the end, of the time interval dt), when one applies d'Alembert's principle and the principle of virtual velocities, one can start immediately from the given positions $a, a', a'', \ldots$ of those masses at time t as the point of application of the lost forces $(-q)$, $(-q')$, $(-q'')$, …, whereas the application of Gauss's principle requires the consideration of the fictitious positions $b, b', b'', \ldots$ at which those masses would arrive at time interval dt if they were entirely free, as well as the positions $c, c', c'', \ldots$ at which they would actually arrive at that time, as well as the conditions under which $\sum m\,(cb)^2$ would become a minimum, or from equation (5), equal to $\sum m\,(\gamma b)^2 - \sum m\,(c\gamma)^2$, which are, as a rule, more cumbersome to develop than the conditions under which the sum of the virtual moments of the lost forces that act upon $a, a', a'', \ldots$ would be equal to zero.

However, one will always be able to derive *some* relationships from Gauss's principle more simply and directly than from the other two principles with a suitable handling of the formulas in question.

Meanwhile, if one starts with those other two principles then the simplest-possible proof of the principle of virtual velocities would be desirable. I shall then allow myself to communicate such a thing here.

The principle in question reads:

If the point of application of a system of forces that is found to be in equilibrium is displaced infinitely little, and indeed in a way that would be permitted by the constraints on the system, then sum of the products of the forces and the lengths of the paths that are traversed, which are parallel to the directions of those forces, will be equal to zero.

Therefore, if the point of application of any of the forces P describes the path δp under that motion in a direction that subtends the angle φ with the forward direction of the force P then:

$$\sum P\,\delta p\cos\varphi = 0. \tag{29}$$

We next consider a *rigid* system of points; i.e., one in which all points are *rigidly* coupled to each other. At those points, let:

X, Y, Z be the components of the force P that acts upon the point a that are parallel to the three rectangular coordinate axes,

α, β, γ be the angles of inclination of the forward direction of P with respect to those axes,

$\alpha_1, \beta_1, \gamma_1$ be the angles of inclination of the forward direction along which the infinitely-small displacement δp of the point of application of P will result with respect to those axes, whereas

φ represents the angle of inclination of P with respect to δp,

x, y, z are the coordinates of the point of application a, and

$\delta x, \delta y, \delta z$ are the displacements of that point relative to the three axes or the projections of δp .

Any motion of a rigid body consists of a rectilinear *advance* and a *rotation* around any axis, as an entirely simple argument that requires no calculation will show (cf., my *Situationskalkul*, pp. 191). That advance can be resolved into three advancing motions that are parallel to three *given rectilinear axes*, and thus, parallel to our coordinate axes, and the aforementioned rotation can be resolved into three rotations about those coordinate axes. Therefore, the equilibrium of the system will require that the forces P have no ambition to move in the direction of any axis, so:

$$\sum X = 0, \qquad \sum Y = 0, \qquad \sum Z = 0. \tag{30}$$

In addition, equilibrium will require that there is also no ambition to rotate about an axis, so the moment equations:

$$\sum (xY - yX) = 0, \quad \sum (yZ - zY) = 0, \quad \sum (zX - xZ) = 0 \tag{31}$$

must be valid.

If we focus our attention on a motion under which only a rotation about the z-axis takes place [so one for which the first of equations (31) is true], and we denote the infinitely-small positive angle of rotation from right to left by γ then, as we can easily infer from Fig. 5, the ordinate x will change by the quantity $an = -\varphi y$, and the ordinate y will change by the quantity $nm = \varphi x$. For a similar rotation around the x-axis through the angle ψ, the ordinate y will change by $-\psi x$ and the ordinate z, by ψy. Likewise, under a rotation about the y-axis through an angle of χ, the ordinate z will change by $-\chi x$ and the ordinate x will change by χz . As a result of all those three rotations:

the ordinate x will change by $\delta x = -\varphi y + \chi x$,
″ ″ y ″ ″ $\delta y = -\psi z + \varphi x$,
″ ″ z ″ ″ $\delta z = -\chi x + \psi y$.

If one now multiplies the first, second, and third of equations (31) by φ, ψ, χ, resp., and adds all three then that will give:

$$\sum [(-\varphi y + \chi z) X + (-\psi z + \varphi x) Y + (-\chi x + \psi y) Z] = 0$$

or

$$(32) \qquad \sum (X\,\delta x + Y\,\delta y + Z\,\delta z) = 0.$$

It is important to point out that this *one* equation does not merely replace the *three foregoing ones*, which refer to the *rotation*, completely, but also the *three* equations (30), which refer to the advance, because depending upon whether one sets δz or δy or δx equal to zero, one will get three equations for the relevant rotation about an axis, and depending upon whether one sets δz and δy or δx and δz or δx and δy equal to zero, one will get three equations for the relevant advance.

Equation (32) is then *necessary and sufficient* for equilibrium.

If one now sets:

$$X = P \cos \alpha, \qquad Y = P \cos \beta, \qquad Z = P \cos \gamma,$$

$$\delta x = \delta p \cos \alpha_1, \quad \delta x = \delta p \cos \beta_1, \quad \delta z = \delta p \cos \gamma_1$$

then one will have:

$$X\,\delta x + Y\,\delta y + Z\,\delta z = P\,\delta p\,(\cos \alpha \cos \alpha_1 + \cos \beta \cos \beta_1 + \cos \gamma \cos \gamma_1)$$

$$= P\,\delta p \cos \varphi.$$

With that, equations (32), which express the principle of virtual velocities (which was to be proved), will assume the simplest form:

$$(33) \qquad \sum P\,\delta p \cos \varphi = 0.$$

If one now connects that rigid system to a second, likewise rigid, system in such a way that it is not a rigid coupling that exists at the contact point, but a moving one, then the motion of the former system will be restricted by that in a certain way; i.e., certain displacements that were previously possible will now become impossible.

Obviously, the first system must be in equilibrium under all forces that act upon it when one counts among those forces not merely the P that are applied to it, but also those Q that the second system that is coupled to it will exert upon it at the contact point.

When one considers all forces P and Q, equation (33) will then be true for any remaining displacement of the first system, so obviously, also for any *possible* one. One will then have:

$$\sum P\,\delta p \cos \varphi + \sum Q\,\delta q \cos \psi = 0.$$

The same thing will be true for the second system, for which one will have:

$$\sum P' \delta p' \cos\varphi' + \sum Q' \delta q' \cos\psi' = 0$$

when one puts primes on all of its forces, for clarity. If one now observes that at each contact point between the two systems, the pressure on the one is equal to the counter-pressure on the other, so $Q' = Q$, and that for any possible displacement, the virtual motion of the point of application of Q' will be opposite to that of Q, so $\delta q' \cos \psi' = - \delta q \cos \psi$, then that will imply that $\sum Q' \delta q' \cos \psi' = - \sum Q \, \delta q \cos \psi$. If one then adds the two foregoing equations then all terms in Q will vanish, and only the form of equation (33) will remain, which P will now refer to all forces that are applied to the total moving system that is composed of the two individual ones, and the displacements are restricted to the ones that are still *possible* under the constraints on both systems.

In the same way, one can combine a third and fourth rigid system with previous ones by a constraint that is moving, but based upon immediate contact, without the principle of virtual velocities ceasing to be valid.

Finally, if one ponders the fact that from the foregoing consideration, it is irrelevant whether one or more of the systems considered have finite or infinitely-small dimensions (so the system would reduce to a material point in the latter case) then it would follow that the principle in question will remain applicable to any system at all, as well as couplings by rigid, flexible, extensible, compressible bodies, etc., because every non-rigid, finite body can be decomposed into infinitely-small parts that one can consider to be rigid.

At this point, I must remark that the proof of the principle of virtual velocities that **Moseley** gave in the book *The mechanical principles of engineering and architecture*, and which I also adapted in my own book that appeared with the title *Die mechanischen Principen der Ingenieurkunst und Architectur*, § **121**, pp. 170, is *incorrect*. Namely, that proof starts from the assumption that the components X, as well as Y and Z, that are parallel to the system are in equilibrium by themselves, so $\sum X \, \delta x = 0$, $\sum Y \, \delta y = 0$, $\sum Z \, \delta z = 0$, from which it would generally follow very simply that $\sum (X \, \delta x + Y \, \delta y + Z \, \delta z) = 0$. However, that assumption is inadmissible, since indeed the sum of the forces in each of the three parallel systems is equal to zero, so $\sum X = 0$, $\sum Y = 0$, $\sum Z = 0$, but by no means is the sum of the *moments* of them about any axis always equal to zero, much less will each of the three systems reduce to a *force-couple*, which *does not* represent *equilibrium*.

5. – Special remark by Gauss on the principle of virtual velocities.

In the oft-mentioned treatise, **Gauss** made the following remarks about the principle of virtual velocities:

> "…it is more correct to say that the sum of the virtual moments can *never be positive*, while one ordinarily says that it must be *equal to zero*, because the ordinary expression tacitly assumes that the opposite to any possible motion is likewise possible [or that the opposite of any impossible motion is likewise impossible], such as when a point is required to remain on a

certain surface, when the distance between two points is required to be invariant, and the like. By itself, that is an unnecessary restriction. The outer surface of an impermeable body does not constrain a material point that is found on it to remain on it, but merely prohibits it from appearing on one side. A tensed inextensible, but flexible, string between two points makes only an increase in distance impossible, but not a decrease, etc. Why then would one not wish to express the law of virtual velocities in such a way that it would encompass all cases right from the outset?"

I believe that one could perhaps respond to that remark and question as follows: In general, fixed points, lines, and surfaces are often prohibited from executing *certain* motions, but will admit the direct *opposite ones.* Now, as long as one actually takes advantage of the fixed nature of such points, so an ambition to move in the impossible direction will prevail, there will exist forces on those points that take the form of the *resistance* of those points, which are quite necessary for the equilibrium of the system. However, whenever a displacement takes place in a possible direction that makes the resistance of a point inactive, the forces on the system that represent that resistance will *vanish*, which will generally have nothing but *positive* virtual moments for such a motion, when it persists, and will thus leave a *negative* sum for the moments of the remaining forces by its vanishing. However, the system will no longer remain the same under the vanishing of part of the original forces, and will then leave the equilibrium state. Such motions will also never be suitable to determine the resistance of the fixed points, which plays the role of *external applied forces* entirely, and is necessary for equilibrium, and thus for developing the conditions for equilibrium *completely.* Accordingly, motions that lie directly opposite to the impossible ones will be likewise kept inadmissible if they have a change in the given system of forces as a consequence. The principle of virtual velocities will always require the *vanishing* of the sum of the virtual moments then when the *forces* and *resistances* that are necessary for the *complete* determination of equilibrium are applied to that principle.

For example, if a weight P (Table II, Fig. 6) lies on a fixed surface then it must feel a certain force P' of resistance. Above all, the whole criterion for a fixed body in a mechanical context consists of saying that it must be capable of experiencing the resistance that was just required. The impossibility of displacement is, in itself, a corollary to that, and in the spirit of the principle of virtual velocities, one can ignore it completely when one substitutes an external force for the resistance of a fixed barrier.

However, an *essential* condition for the foregoing system is that contact between both bodies must be preserved by the virtual displacement. When one performs a common displacement upwards or downwards through the path δp, that will lead to the formula $P\,\delta p - P'\,\delta p' = 0$, so $P = P'$. By contrast, if one would like to perform a *one-sided* motion of the weight upwards from a plane with no displacement in the plane then one would indeed get the negative value $-P\,\delta p$ for $\sum P\,\delta p$, as Gauss correctly remarked. Only the resistance P' of the plane would vanish then, while the entire system would change, and no more formulas would exist from which one could determine the forces that would be required for equilibrium.

A similar case occurs when a weight P hangs from a fixed point A on a flexible string, as in Fig. 7. That point must respond with the force of resistance $P' = P$. If one merely raises the fixed point then without displacing the fixed point then that resistance P' would vanish, and the forces that are required for the equilibrium of the system would no longer be present.

Therefore, in such cases, when one takes the *fixed nature* of certain points to be absolute, as was done here, and one would like to ignore the forces of resistance that those points produce completely, the principle of virtual velocities would give a likewise incomplete answer, which would consist of saying that the weight P can have *any arbitrary value*, which is indeed correct in itself, but one cannot recognize the essential fact that the fixed point must experience a resistance to the force of *that weight*.

Such a purely-extrinsic view of the concept of *constraints* on a system of material points with no rigorous consideration of the *resistance* of points that are fixed or restricted in their motions can, in turn, easily lead to entirely erroneous arguments. For instance, the case of *elastic* constraints belongs to them. When considered absolutely, such constraints allow arbitrary stretching, compression, and bending, from their mechanical properties. One cannot therefore deny that every motion of a point that is coupled with the remaining system by an elastic band will correspond to one of the conditions on the system, so it will be virtual. However, if one overlooks the intrinsic resistance that appears in that way and the other necessary changes in the forces that are applied to the system then one will get false results.

Having assumed that, let the weight P in Fig. 7 hang from the fixed point A by means of an elastic string, so a stretching of it is very probable, and thus, a motion of the weight P directly downwards, without the fixed point A simultaneously moving with it. However, such a virtual motion will produce a virtual moment $P\,\delta p$ that would be either equal to *zero*, *negative*, or rather decidedly *positive*. For that reason, the result is nonetheless false, because of the fact that stretching of the string cannot happen without overcoming the intrinsic elastic forces, and strictly speaking, without an increase in the weight P.

All of those considerations will lead us to the following rules that must be observed when one applies the principle of virtual velocities.

6. – Consideration of the variability of forces and the intrinsic resistance of a body to a virtual motion.

From the above, one can, with no further discussion, regard a virtual motion of the system to be one that is possible under the constraints on the system; i.e., under which, those constraints will not change, except for the geometrically-allowable absolute and relative motions, so ones under which no other effort is expended than the one that corresponds to the virtual moments of the system of externally-applied forces when the virtual moments of the internal resistance between two contact points of the system do not mutually cancel, and so, from equation (33), vanish by themselves.

That case will always occur when the constraints on the system are *independent* of the forces that are applied to it, such as, e.g., for a system that is composed of nothing but rigid bodies that can rotate about certain points or displace on their outer surfaces.

However, when, in the opposite case, the constraints depend upon the forces that act upon the system, or when the intended displacement of the system is possible at all only under the expenditure of certain quantities of work that are generated in the bonds of the system or when the system must be subjected to special external forces, that displacement can still be regarded as a virtual one *when one takes into account the requisite virtual moments that do not cancel by themselves.*

Hence, if (e.g., in Fig. 7) a weight P hangs from a fixed point A by means of an elastic string, and one would like to displacement the weight through δp downwards then one would find that a quantity of work would be required to rotate it that has the value $W\,dp$, as it also would be from the law of elasticity for the string.

If the length p of the entire string increases by δp then the length of each element dx of the string will increase by $(\delta p / p)\,dx$. The expenditure of work that would be required by that increase in the length of the element dx (which is the differential of the work done by the tension P that acts upon the lower end of that element) will then have the value $-P(\delta p / p)$. The sum of those works over all elements of the string will then be:

$$-\int_0^p P \frac{\delta p}{p} dx .$$

Now, since P depends upon the length p, that tension will always be equal for all elements, so it will be independent of x, and one will then have:

$$-\int_0^p P \frac{\delta p}{p} dx = - P \frac{\delta p}{p} \int_0^p dx = - P \; \delta p .$$

The weight P will produce the virtual moment $P\;\delta p$ under the motion that we speak of. It will then realize equation (33) in the form of $P\;\delta p - P\;\delta p = 0$.

Something similar will occur for the system that is represented in Fig. 6 when one would like to displace the weight P horizontally through δx on the fixed surface, but one makes the assumption that *friction* exists between the weight and the surface, which has the magnitude $f\,P$. In that case, the displacement is, in fact, possible and allowable. However, it can be regarded as only a virtual one when one observes that it requires a force $f\,P$ in the horizontal direction that has not been given up to now, and which produces the positive virtual moment $f\,P\;\delta x$, and at the same time, that the resisting friction of the fixed surface must be overcome, which would, however, yield the opposite moment $-f\,P\;\delta x$, such that equation (33) will now be fulfilled in the form $f\,P\;\delta x - f\,P\;\delta x = 0$.

7. – Correct interpretation of the infinitely-small quantities in the principle of virtual velocities.

In the formula for the principle of virtual velocity, δp is an *infinitely-small* quantity, and therefore a quantity that continually strives to assume the value zero. Since equation (33) will first achieve complete validity for the limiting values of that quantity, but those limiting values are all zero, one can see from that fact itself that the ratios of all those infinitely-small quantities δp to one and the same independent infinitely-small quantity (which we would like to denote by δs) must be determined in such a way that one must get the values $\frac{\delta p}{\delta s}\delta s$, $\frac{\delta p'}{\delta s}\delta s$, $\frac{\delta p''}{\delta s}\delta s$, … for the δp, $\delta p'$, $\delta p''$, …, resp. for equation (33), so once one has divided all terms by the common factor δp, the formula:

(34) $$\sum P \frac{\delta p}{\delta s} \cos\varphi = 0,$$

which includes only *finite* quantities $\delta p / \delta s$.

If, according to the nature of the system, the quantities δp did not all depend upon one and the same basic quantity δs, but upon several such basic quantities in groups, then equation (33) would obviously already decompose into just as many special equations in that way, each term of which would contain the terms of one and the same group.

The aforementioned finite quantities $\delta p / \delta s$ will reduce to the *first differential coefficients* of the function p with respect to the independent variable s under the passage to their limiting values. It would then be entirely superfluous to determine the increment δp more precisely than its first differential, which has the form $A\,ds$, or to determine the quotient $\delta p / \delta s$ more precisely than the first finite term A, since the lower terms of second, third, and higher order are infinitely-small in comparison to them, and from the expression:

$$\frac{\delta p}{\delta s} = A + B\,ds + C\,ds^2 + \ldots,$$

they would all vanish when one passes to the limiting value.

That remark is especially important for those cases in which the first differential coefficient A is coincidentally *equal to zero* precisely, while the higher differential coefficients keep finite values. In such cases, if might seem, on first glance, as if the introduction of the next non-vanishing terms (so, e.g., the substitution $\delta p / \delta s = B\,ds$) would be necessary in order to determine the actual virtual moment of the force in question. That exchange becomes irrelevant by the foregoing remark that the term $B\,ds$ and all higher terms will be effectively equal to zero under the passage to the limiting values, for which only equation (33) will be true.

A practical case of the latter kind is represented, for example, in Fig. 8. In it, one assumes that a weight P is *affixed* to the deepest point of a circular hoop that can *roll* on a horizontal plane. For every small enough motion of the hoop, the weight P generally seems to always accomplish a certain amount of work, because it *lifts* somewhat, while the work done by the resistance P of the plane remains precisely equal to zero, such that equation (33) does not appear to be fulfilled here.

That error can be explained when one observes that for a rolling motion through the infinitely-small angle $\delta\alpha$, the vertical rise of the weight is $\delta p = r - r\cos\delta\alpha$, so, up to second-order terms, $\delta p = \frac{1}{2}(\delta\alpha)^2$, and as a result $\delta p / \delta s = \frac{1}{2}\delta\alpha$. When one passes to the limiting value, one would therefore not merely have that the first differential coefficient $dp / d\alpha$ is equal to zero, but also that the entire expression $\delta p / \delta\alpha$ is equal to zero.

8. – Correct interpretation of the infinitely-small quantities in Gauss's principle.

The infinitely-small quantities that enter into formulas (1), (5), etc., that relate to Gauss's principle of least constraint require much deeper attention. As one would learn from equations (10), the lines *cb*, γb, $c\gamma$ (Fig. 5) are infinitely-small quantities of order *two*, since they are multiplied by dt^2. Nonetheless, *in relation to each other*, they generally behave like *finite* quantities. However, in conjunction with the lines *ab*, *ac*, which are themselves *first*-order quantities, they are infinitely small. Nevertheless, with no further discussion, they can only be first neglected in comparison to those quantities in the end results, where only their ratios to relatively infinitely-large values should be considered. However, it can frequently happen that in the intermediate operations, relatively infinitely-large terms can cancel each other under addition and subtraction and that another relationship between the infinitely-small quantities might remain as a result, such as when one already neglects part of the latter terms prematurely in comparison to infinitely-larger ones.

In regard to that, we point out the following: In order to construct the point *b* to which the mass *m* of the point *a* would move in the time interval *dt* if it were completely free, and the point *c* to which it would actually move, let (Fig. 9, Table II):

v	be the velocity of the mass *m* at the point *a* at time *t* in the direction $a\alpha$, so when one takes $a\alpha = v\,dt$, where α is the point at which the mass *m* would arrive without the influence of any force, merely as a result of the velocity that is achieved during the time element *dt*,
f, g, h	be the velocities that the *applied* force *p*, the *deflecting* force *q*, and the *actual* force *r*, resp., are in a position to impart upon the mass *m* during a *unit of time*,
φ, ψ	be the angles *pav* and *rav*, resp., that the *applied* for *p* and the *actual* force, resp., make with the direction of the velocity *v* or the path of the mass *m* at time *t*, where the angle is thought to be positive or negative according to whether those directions lie on one side or the other of the direction of *v*,
χ	be the angle *par* between the applied and actual forces (which is then $\psi - \varphi$).

If one now makes *ab* parallel to *p* and equal to $\frac{1}{2} f\,dt^2$ then *b* will be the point at which the mass *m* would arrive during the time interval *dt* if it were completely free.

If one takes *bc* to be parallel to *q* and equal to $\frac{1}{2} g\,dt^2$ then *c* will be the point at which the mass actually arrives during that time.

One will also get the same point *c* when one takes *ac* to be parallel to *r* and equal to $\frac{1}{2} h\,dt^2$.

Since the mass *m* must describe a continuous curve for forces that act continuously, the smaller that one chooses the time interval *dt* to be, the more that the line *ac* will fall along the tangent *va* to that curve, and its length will become equal to $a\alpha + \alpha c \cdot \cos\,\psi$, such that this line, which represents the increment Δs in the path *s* that has been traversed at time, will then have the value:

$$\Delta s = v\,dt + \tfrac{1}{2} h \cos dt^2, \tag{35}$$

up to terms of dimension two. In that expression, the quantities of both dimensions are carefully kept long enough that one has to compare the line *ac* with similarly-constructed lines, such as *ab*.

It follows form equation (35) that:

$$\frac{\Delta s}{dt} = v + \tfrac{1}{2} h \cos\,\psi\,dt\,.$$

If one passes to limiting values then the second term on the right-hand side will vanish, and one will get the known formula $ds / dt = v$. However, the conclusion would become completely false that because the quantity *v* on the right-hand side of this formula is increased by $\tfrac{1}{2} h \cos\,\psi\,dt$, that increase will probably represent the increase that the velocity suffers during the time interval *dt*, so under the passage of the mass *m* from the point *a* to the point *c*, such one can set:

$$\frac{\Delta s}{dt} = v + dv = v + \tfrac{1}{2} h \cos\,\psi\,dt\,,$$

and therefore:

$$dv = \tfrac{1}{2} h \cos\,\psi\,dt \qquad \text{or} \qquad \frac{dv}{dt} = \tfrac{1}{2} h \cos\,\psi\,.$$

Moreover, the quotient $\Delta s / dt$ expresses nothing but the velocity that the mass would take on during the time *dt* or along the path *ac if it traversed that path with uniform velocity*, and $\tfrac{1}{2} h \cos\,\psi\,dt$ is the excess of that *fictitious* velocity over the one that prevails at the point *a*.

Since the motion of the mass *m* generally accelerates or decelerates, that fictitious velocity, which is, to some extent, the *mean* velocity that exists along the path *ac*, will differ essentially from the one that exists at time interval *dt*, and thus, upon the arrival at the point *c*. The latter velocity is:

$$v + dv = v + h \cos\,\psi\,dt,$$

which is its increase over the one that prevails at time *t*, namely, $dv = h \cos\,\psi\,dt$, so it will be *twice as large* as the aforementioned increase, because one has, in full generality:

$$s + \Delta s = s + \frac{ds}{dt} dt + \tfrac{1}{2} \frac{d^2 s}{dt^2} dt^2 + \ldots,$$

so

$$\Delta s = \frac{ds}{dt}dt + \tfrac{1}{2}\frac{d^2 s}{dt^2}dt^2 + \ldots,$$

or

(36) $$\Delta s = v\,dt + \tfrac{1}{2}\frac{dv}{dt}dt^2 + \ldots,$$

so a comparison of this formula with (35) will give:

$$\frac{dv}{dt} = h \cos \psi.$$

By contrast:

$$v + \Delta v = v + \frac{dv}{dt}dt + \tfrac{1}{2}\frac{d^2 v}{dt^2}dt^2 + \ldots,$$

and therefore:

$$(v + \Delta v)\,dt = v\,dt + \tfrac{1}{2}\frac{dv}{dt}dt^2 + \ldots$$

However, it would be incorrect for one to regard the line *ac*, which is actually equal to Δs, to be $(v + \Delta v)\,dt$, and accordingly, from equation (35), one would have:

$$v\,dt + \frac{dv}{dt}dt^2 = v\,dt + \tfrac{1}{2}h \cos \psi\, dt^2,$$

so one would like to set $dv / dt = \tfrac{1}{2}h \cos \psi$, since that says the same thing as assuming that the mass *m* traverses the path *ac* with the velocity $v + \Delta v$, although $v + \Delta v$ represents the velocity that the mass *s* would achieve at the endpoint *c* of that path.

In reality, from the equality of (36) and (35), the line *ac* has the value:

(38) $$ac = \Delta s = v\,dt + \tfrac{1}{2}\frac{dv}{dt}dt^2 = v\,dt + \tfrac{1}{2}h \cos \psi\, dt^2.$$

Furthermore, from the above and from (27):

(39) $$ac = \tfrac{1}{2}h\,dt^2 = \frac{1}{2\cos\psi}\frac{dv}{dt}dt^2 .$$

The line *ab* is:

(40) $$ab = \tfrac{1}{2} f\,dt^2.$$

Thus, in the triangle $b\alpha c$, in which the angles are $b\alpha c = p\;a\;r = c$, one has that the square of the deflection *cb* is:

$$(cb)^2 = (\alpha b)^2 + (\alpha c)^2 - 2\,(\alpha\, b)\,(\alpha\, c) \cos \chi$$

or

(41) $$(cb)^2 = \tfrac{1}{4} d\,t^4\,(f^2 + h^2 - 2\,f\,h \cos\chi)$$

$$= \tfrac{1}{4} d\,t^4 \left[f^2 + \left(\frac{1}{\cos\psi} \frac{dv}{dt} \right)^2 - 2\,f\,\frac{dv}{dt}\frac{\cos\chi}{\cos\psi} \right].$$

If one would like, then one can also set:

(42) $$(cb)^2 = \tfrac{1}{4} d\,t^4\,[(f \cos\varphi - h \cos\chi)^2 + (f \sin\varphi - h \sin\chi)^2]$$

$$= \tfrac{1}{4} d\,t^2 \left[\left(f \cos\varphi - \frac{dv}{dt} \right)^2 + \left(f \sin\varphi - \frac{dv}{dt} \tan\psi \right)^2 \right],$$

instead of (41).

9. – Transformation of Gauss's formula for the decomposition of the forces along three rectangular axes.

If one decomposes the force that is applied to the mass *m* into its components parallel to the rectangular axes, so if:

f, *g*, *h* are the velocities that those components that might be communicated to the mass *m*, whose coordinates are *x*, *y*, *z*, during a *unit time* at time *t*,

$\tfrac{1}{2} f\,dt^2$, $\tfrac{1}{2} g\,dt^2$, $\tfrac{1}{2} h\,dt^2$

are the distances in space through which the mass *m* in the rest state would be pushed by those forces during the time interval *dt*,

u, *v*, *w* are the velocities parallel to the three axes that the mass *m* will actually possess at time *t*

then:

$$u = \frac{dx}{dt}, \qquad v = \frac{dy}{dt}, \qquad w = \frac{dz}{dt}.$$

When one develops the increments Δx, Δy, Δz up to second-order terms (*ac* in Fig. 10), the *actual* advance of the point *a* in the course of the time interval *dt* in the directions of the three axes will be:

(43)
$$\begin{cases} \Delta x = \dfrac{dx}{dt}dt + \dfrac{1}{2}\dfrac{d^2x}{dt^2}dt^2 = u\,dt + \dfrac{1}{2}\dfrac{du}{dt}dt^2, \\ \Delta y = \dfrac{dy}{dt}dt + \dfrac{1}{2}\dfrac{d^2y}{dt^2}dt^2 = v\,dt + \dfrac{1}{2}\dfrac{dv}{dt}dt^2, \\ \Delta z = \dfrac{dz}{dt}dt + \dfrac{1}{2}\dfrac{d^2z}{dt^2}dt^2 = w\,dt + \dfrac{1}{2}\dfrac{dw}{dt}dt^2. \end{cases}$$

By contrast, the partial distances that the point *a* would traverse in the time *dt* if it were completely free (*ab* in Fig. 10) would be:

(44)
$$\begin{cases} u\,dt + \tfrac{1}{2} f\,dt^2, \\ v\,dt + \tfrac{1}{2} g\,dt^2, \\ w\,dt + \tfrac{1}{2} h\,dt^2. \end{cases}$$

Therefore, the deflections into the direction of the three axes (*cb* = *ab* – *ac* in Fig. 10):

(45)
$$\begin{cases} \tfrac{1}{2} f\,dt^2 - \tfrac{1}{2}\dfrac{du}{dt}dt^2 = \tfrac{1}{2}dt^2\left(f - \dfrac{du}{dt}\right), \\ \tfrac{1}{2} g\,dt^2 - \tfrac{1}{2}\dfrac{dv}{dt}dt^2 = \tfrac{1}{2}dt^2\left(g - \dfrac{dv}{dt}\right), \\ \tfrac{1}{2} h\,dt^2 - \tfrac{1}{2}\dfrac{dw}{dt}dt^2 = \tfrac{1}{2}dt^2\left(h - \dfrac{dw}{dt}\right). \end{cases}$$

Since the square of the actual deflection is equal to the sum of the squares of the deflections along the three axes, Gauss's principle will require that the sum:

(46)
$$\sum m\left(f - \frac{du}{dt}\right)^2 + \sum m\left(g - \frac{dv}{dt}\right)^2 + \sum m\left(h - \frac{dw}{dt}\right)^2$$

should be a minimum.

As far as equation (5) is concerned, when δx, δy, δz denote the projections of any possible *displacement* $c\gamma$ (Fig. 3) of the point *c*, since the projection of any other possible *deflection* γb of the point *b* in the direction of the *x*-axis has the value (*cb*) – δx = $\tfrac{1}{2}dt^2\left(f - \dfrac{du}{dt}\right) - \delta x$, one will then have:

(47)
$$\tfrac{1}{4}dt^4 \sum m\left(f - \frac{du}{dt}\right)^2 + \tfrac{1}{4}dt^4 \sum m\left(g - \frac{dv}{dt}\right)^2 + \tfrac{1}{4}dt^4 \sum m\left(h - \frac{dw}{dt}\right)^2$$

$$= \sum m\left[\tfrac{1}{2}dt^2\left(f-\frac{du}{dt}\right)-\delta x\right]^2+\sum m\left[\tfrac{1}{2}dt^2\left(g-\frac{dv}{dt}\right)-\delta y\right]^2+\sum m\left[\tfrac{1}{2}dt^2\left(h-\frac{dw}{dt}\right)-\delta z\right]^2$$

$$-\sum m(\delta x)^2-\sum m(\delta y)^2-\sum m(\delta z)^2.$$

A development of the squares on the right-hand side leads directly to the known fundamental equation:

(48) $$\sum m\left(f-\frac{du}{dt}\right)\delta x+\sum m\left(g-\frac{dv}{dt}\right)\delta y+\sum m\left(h-\frac{dw}{dt}\right)\delta z = 0,$$

which enters in place of equation (3).

It should be remarked in that regard that when certain forces of the system act, not on *masses m*, but on *massless* points, and one denotes the components of those forces by *X*, *Y*, *Z*, one must convert the Σ sign in equation (47) into an S and add the sum:

$$dt^2\,\mathfrak{S}\,X\,\delta x + dt^2\,\mathfrak{S}\,Y\,\delta y + dt^2\,\mathfrak{S}\,Z\,\delta z$$

to the left-hand side, since the summation sign S refers to the *material* points, and the sign $\mathfrak{S}$ refers to the *massless* points.

10. – Application of Gauss's principle to the motion of a pendulum and the equilibrium of a lever.

In order to make the application of Gauss's principle more intuitive, we would like to consider the motion of two ponderous masses *m*, *m′* (Fig. 11) that are fixed at the endpoints *a*, *a′* of a lever that rotates about *A*. Let:

a, *a′* be the lever arms *Aa*, *Aa′*, resp.,

φ be the angle *BAa* that the lever subtends with the horizontal at time *t*,

v be its angular velocity at that time,

g be the velocity that gravity communicates during the time interval,

p, *p′* = *mg*, *mg′*, resp., be the weights of the masses *m*, *m′*, resp.

Since the masses can move only along the circular lines in question, the force *r* that acts upon them will fall in the direction of the tangent *r a* to that circle; the angle will then be *rap* = *BAa* = φ. The velocity of the mass *m* is *a v*. If one then takes $(a\alpha) = a\,v\,dt$ then that mass would arrive at α after the time element *dt* by means of its intrinsic velocity. If one makes the vertical $(\alpha b) = \tfrac{1}{2} g\, dt^2$ then *b* will be the point at which that mass would arrive during that time interval if it were completely free. Now, it actually arrives at c, such that the angle will be $\alpha Ac = d\varphi$ and one will have $d\varphi / dt = v$; therefore, let $(\alpha c) = x$.

If one puts primes on the quantities with the same symbols for the mass m' then one will get:

$$(a\,c) \quad = (a\,\alpha) + x \quad = a\,v\,dt + x,$$

$$(a'c') = (a'\alpha') + x' = a'v\,dt + x',$$

so

$$x' = \frac{a'}{a}\,x.$$

Furthermore, in the triangle bca, one has:

$$(bc)^2 = (\alpha b)^2 + (\alpha c)^2 - 2\,(\alpha b)\,(\alpha c)\cos(b\alpha c)$$

$$= \tfrac{1}{4}g^2\,dt^4 + x^2 - g\,x\cos\varphi\,dt^2.$$

Since the angle is $b'a'c' = \pi - \varphi$ here, one will get:

$$(b'c')^2 = \tfrac{1}{4}g^2dt^4\,\frac{a'^2}{a^2}x^2 + \frac{a'}{a}g\,x\cos\varphi\,dt^2$$

for the triangle $b'\alpha'c'$. Thus:

(49) $$\sum m\,(bc)^2 = \tfrac{1}{4}(m + m')\,g^2\,dt^4 + \frac{a^2m + a'^2m'}{a^2}x^2 - \frac{am - a'm'}{a}g\,x\cos\varphi\,dt^2.$$

In order for that sum to be minimum, as in Gauss's principle, we must set its differential with respect to x equal to zero. That will give:

(50) $$x = \frac{am - a'm'}{a^2m + a'^2m'}\,\frac{a\,g\cos\varphi}{2}\,dt^2.$$

Since the velocity of the mass m at time t is equal to av [so $(ac) = x = \tfrac{1}{2}a\dfrac{dv}{dt}dt^2$], when one sets that expression equal to the foregoing one for x, one will get:

(51) $$\frac{dv}{dt} = \frac{am - a'm'}{a^2m + a'^2m'}\,g\cos\varphi,$$

or also, since one has $\dfrac{dv}{dt} = \dfrac{d^2\varphi}{dt^2}$, one will get:

(52) $$\frac{1}{\cos\varphi}\frac{d^2\varphi}{dt^2} = \frac{am - a'm'}{a^2m + a'^2m'}g = \frac{ap - a'p'}{a^2p + a'^2p'}g$$

as the fundamental equation for the pendulum motion to be determined.

If one would like to introduce the angle φ as an independent variable and the angular velocity v as a dependent variable then, since $\frac{dv}{dt} = \frac{dv}{d\varphi}\frac{d\varphi}{dt} = v\frac{dv}{d\varphi}$, equation (51) will give:

$$v\,dv = \frac{ap - a'p'}{a^2p + a'^2p'}g\cos\varphi\,d\varphi,$$

or upon integration, when the angular velocity is $v = v_0$ for $\varphi = 0$:

(53) $$\tfrac{1}{2}(v^2 - v_0^2) = \frac{ap - a'p'}{a^2p + a'^2p'}g\sin\varphi.$$

If one would like to exhibit the conditions for *equilibrium* of the masses m, m' or the weights p, p' on the lever aAa' then, from equation (50), $(ac) = x$ must be equal to *zero*. That will give the known relation:

(54) $$ap = a'p'.$$

11. – Application of Gauss's principle to the motion of a material point on a given surface or line.

The application of Gauss's principle takes an especially simple form for the motion of a material point on a given surface or line. We immediately direct our attention to the most general case of a given *surface*. In Fig. 12, let:

v be the velocity of the point a of the mass m at time t, and

g be the velocity that the force that is applied to that mass (say, gravity) communicates to it in a unit time.

Now, one has $a\alpha = v\,dt$, and the line ab is in the direction of the force that acts and its length equals $\tfrac{1}{2}g\,dt^2$, so b is the location at which the mass m would arrive after the time dt if it were completely free, so c will be the location on the surface at which that mass would actually arrive, and it will then be the base point of the *normal bc that is dropped from b to the surface*, since that would be the *shortest* line that one could draw from b to the surface, and obviously that shortest line will satisfy the condition of Gauss's principle that $\sum m\,(bc)^2 = m\,(bc)^2$ is a minimum.

That property will suffice to develop all conditions for the motion of the given point. Namely, if:

φ denotes the angle $b\alpha n$ that the direction αb of the force makes with the normal αn, which is an angle that is also equal to $\alpha b\, c$, by the infinite smallness of the figure that we speak of, and

ψ denotes the angle $c\,\alpha\, e$ that the intersection $\alpha\, c$ of the normal plane $n\,\alpha\, b$ or $\alpha\, b\, c$ with the tangent plane is inclined from the direction $a\,\alpha\, e$ of the velocity of the mass m at time in question, then:

$$a\,\alpha = v\,dt, \qquad \alpha b = \tfrac{1}{2} g\, t^2, \quad \alpha c = \tfrac{1}{2}\,\alpha b \sin\varphi = \tfrac{1}{2} g \sin\varphi\, dt^2,$$

and

$$ac = a\,\alpha + \alpha\, c \cdot \cos\psi = v\,dt + \tfrac{1}{2} g \sin\varphi\,\cos\psi\, dt^2.$$

Now, since one also has $ac = v\,dt + \frac{1}{2}\frac{dv}{dt}dt^2$, one has the fundamental equation:

$$\frac{dv}{dt} = g \sin\varphi \cos\psi\,.$$

12. – Application of Gauss's principle to the collision of inelastic bodies.

In order to apply Gauss's principle to the collision of inelastic bodies, let the velocity of the two masses m, m', which both move along a straight line, be equal to v, v', resp., before the impact and V after the impact. If no union of the masses occurred at the moment of collision, so there would be no constraint on the motion, then the two masses would move through the distances $v\,dt$, $v'\,dt$, resp., during the time interval dt if they were completely free. Under the conditions on the system (as a compound body), they would actually traverse the distance $V\,dt$. If $v < v'$ then the deflections will amount to $(V - v)\,dt$ and $(v' - V)\,dt$, resp. Therefore, the constraint is:

$$m\,(V - v)^2\,dt^2 + m'(v' - V)^2\,dt^2.$$

In order for that expression to be a minimum according to Gauss's principle, we set its differential with respect to V equal to zero. That will give the known relation:

$$V = \frac{mv + m'v'}{m + m'}\,. \tag{56}$$

(Conclusion in next issue)

13. – Processes that allow one to always consider the *actual* displacement of a system in an infinitely-small time interval to be a *virtual* one.

In Fig. 9 (Table II in the previous issue), α is the position that the material point *a* would occupy at the end of a time interval *dt* due to its intrinsic velocity at time *t if no forces at all acted upon it*. *b* is the position that it would occupy with that velocity under the effect of the force *p* applied to it *if it were completely free*. *c* is the position that it would occupy with that velocity under the action of the applied force *p* and the constraint on the system, so the one that it will *actually occupy* under the under the control of the effective force *r*. In addition, γ denotes any position besides *c* that the point in question *might possible occupy as a result of a virtual displacement of the system*.

One understands a *virtual* displacement to be one that corresponds to the momentary constraint on the system that exists at time *t*. However, in the manner of presentation that is found in all textbooks on mechanics, the constraint itself is always regarded as completely unvarying during the displacement. If that constraint is to also depend upon *time t* then it must still be considered to be constant during the time interval *dt* under the virtual displacement. The infinitely-small path $c\ \gamma$ is therefore only the spatial variation of the point *c* that is allowed by the momentary constraints that exist on the system without one considering those variations that are produced by the way that the constraint might depends upon time *t* . In the determination of those variations, one must then treat time as constant when the law of dependency of the constraint on the system is to be given as a function of time *t*. Obviously, just the same thing is also true of the *forces* that act upon the system, as long as they are supposed to be functions of time *t* or the positions of the masses that they act upon, which are themselves functions of time *t*. Those forces must also be considered to be unvarying under the displacement during the time interval *dt*.

Accordingly, *in general*, the *actual* motion of the system during the time *dt* (hence, the displacement *ca*), cannot be regarded as a *virtual* one. Rather, that can happen only when the constraint on the system is *independent* of time *t* or the *forces* that act upon it. One result of that consideration, among other things, is that the principle of *vis viva* is only valid for those systems whose constraints do not depend upon time.

Obviously, one comes to that restriction of the virtual displacements by the tacitly-made assumption that among the forces *p* that act upon the material part of the system, only the ones that are considered to be *externally-applied*, but *internal*, and which emerge as the *reaction of the couplings* in the system, so ones that are *themselves produced* as a result of the motion that the externally-applied forces bring about in some way, can be disregarded. For the systems with completely *independent* or *unvarying* constraints (e.g., for the ones in which rigid, inelastic materials exist with fixed constraints, rotatable axes, completely-free isolated parts, and similar mechanisms, under which any change in the constraint is, in principle, absolutely *impossible*), the internal reactions within and between the couplings in the system will always be of the sort that for any possible displacement of the system, the *quantity of work* done by all of the reactions *will be equal to zero*. The moments of the internal forces in such system would always vanish then, no matter how one might displace the system. Therefore, that imaginary motion can also be regarded as a virtual one here. By contrast, for the systems with *variable* constraints (e.g., mechanisms with elastic couplings, with compressible or gaseous bodies, and the like, under which certain changes in the constraints are *possible*, as long as the required *forces are applied*), the quantity of work done by internal reactions in the couplings of the system will be *equal to zero* only for those displacements that produce no change in the constraint, but for other displacements under which special forces are developed, that work will have *a finite, positive or negative value*. Therefore, one restricts the field of virtual motions here to the ones that are independent of time *t* or to the ones under which the couplings in the system do not change, because only for those displacements will the moments of the internal forces vanish.

However, such a vanishing of the internal forces in the equation that expresses the principle of virtual velocities would have no particular use whatsoever, because it would be a big mistake to believe that one could avoid considering the internal forces (like elastic forces) completely in that way. That is by no means the case, because when the aforementioned equation is also free from internal forces under the popular restriction on virtual displacements, that will always make their consideration in isolation valid for the complete determination of the motion of the system.

That sheds light upon the fact that when one takes the internal forces (namely, elastic forces) that appear under a certain displacement into account, the concept of the *constraint* on a system can always be extended in such a way that the displacements that are *even possible under the action of forces* will seem to be *allowable* or *virtual*, and in that way, the difference above between the two types of constraints will vanish completely, and in addition, the arbitrary restriction on the virtual displacements will drop out for the latter type of systems. Along with those advantages of the generalization in principle, one also has the convenience of the fact that one will be led directly and necessarily to *all* requisite equations in the presentation of the fundamental equations for the motion of a system, so those former equations do not have to be extended by auxiliary considerations about the internal forces.

Under the latter assumption, one can also regard the *actual* displacement *ac* as a *virtual* one then *in all cases* (which is understood to mean systems with mutually-independent constraints). One will then have to consider the internal stresses to be overcome by the actual motion only in the case of a system with variable constraints.

However, the displacement $a\alpha$ will also be regarded in that way under *the same* conditions under which the *actual* displacement *ac* will seem to be virtual, since the latter arises from the assumption that the material point *a* will advance uniformly during the time interval *dt* with the velocity that it has gained at time *t*. However, the same state will also be attained due to the fact that one can assign the completely-allowable value of *zero* to the effective force *r* on each material point *a* that arises from all internal stresses under consideration, and in that way, the *actual* displacement *ac* will go directly to the $a\alpha$ that we imagined above.

14. – Explanatory example for the process that was just described.

An example might better explain the foregoing.

From Fig. 13 (Table II in the previous issue), let the material point *a* of mass *m* and weight *n* be coupled to the disc a' of mass m' and weight n' by a weightless string $a\alpha$ of length *c*. The system falls vertically downwards through the air, which makes air resistance act upon the disc a', which can be expressed by $k\,v'^2$, if v' denotes the velocity of the disc at time t, while v is that of the point a.

1) If the string aa' is *inextensible* then one will be dealing with an entirely-unvarying system that can be treated in the usual simple way. Namely, if one denotes the vertical abscissas of the points a and a' from any fixed point by x and x', resp., and the acceleration of gravity by g, and observes that $v' = v$, then one will have that:

$$\text{the lost force of the mass } a \text{ is} \qquad mg - m\frac{dv}{dt},$$

$$\text{〃} \qquad \text{〃} \qquad \text{〃} \quad a' \text{ is} \qquad m'g - k\,v^2 - m'\frac{dv}{dt}.$$

Since those forces must be equilibrium, from d'Alembert's principle, the principle of virtual velocities will imply that:

$$\left(mg - m\frac{dv}{dt}\right)\delta x + \left(m'g - kv^2 - m'\frac{dv}{dt}\right)\delta x' = 0.$$

From the fixed constraint on the system, one has $x = x' + c$, so $\delta x = \delta x'$, and that illuminates the fact that one can also regard the *actual* motion during the time interval dt as a *virtual* one here, or $\delta x = v\,dt$, $\delta x' = v'\,dt = v\,dt$. The foregoing equation will always yield the relation:

$$(m + m')\,g - k\,v^2 - (m + m')\,\frac{dv}{dt} = 0,$$

from which, the law of dependency between v and t can now be found by integration.

2). However, if one assumes that the string aa' is *extensible* then one will be dealing with a system whose constraint depends upon time t. Namely, c will be a function of time t in the equation $x = x' + c$ that represents that constraint.

From the usual prescriptions in the textbooks on mechanics, one would now proceed as follows: From the relation $x = x' + c$, one has:

$$\frac{dx}{dt} = \frac{dx'}{dt} + \frac{dc}{dt}, \qquad \text{i.e.,} \qquad v = v' + \frac{dc}{dt},$$

and furthermore:

$$\frac{dv}{dt} = \frac{dv'}{dt} + \frac{d^2c}{dt^2}.$$

One will then have that the lost forces are:

at a : $$Q = m\,g - m\frac{dv}{dt} = mg - m\frac{dv'}{dt} - m\frac{d^2c}{dt^2},$$

at a': $$Q' = m'g - k\,v'^2 - m'\frac{dv'}{dt}.$$

If one expresses the equilibrium of those forces by the principle of virtual velocities then one will get:

$$Q\,\delta x + Q'\delta x' = 0.$$

In order to determine δx and $\delta x'$ in that equation by the usual procedure, the constraint on the system during the time interval dt must be regarded as unvarying, so the quantity c in the relation $x = x' + c$ must be regarded as constant, and as a result, one must set $\delta x = \delta x'$, from which it will follow that:

$$Q + Q' = 0.$$

That illuminates the fact that in all such situations, it will be impossible for one to set the *virtual* displacements δx, $\delta x'$ equal to the *actual* ones during the time interval dt, since that would imply that $\delta x = v\,dt = v'dt + dc$ and $\delta x' = v'dt$, so:

$$Q\,(v'dt + dc) + Q'v'dt = 0,$$

or

$$Q + Q' = -\frac{1}{v'}\frac{dc}{dt},$$

which is an equation that contradicts the previously-found correct relation $Q + Q' = 0$.

One further sees that this ordinarily-applied process does not merely exclude the assumption that the *actual* motion is a *virtual* one, which seems so natural, but it also leaves the solution of the problem incomplete, along with that, because it will imply only the single equation $Q + Q' = 0$, in addition to the relation $x = x' + c$ for the determination of the three unknown quantities v, v', c, which will assume the form:

$$(m + m')\,g - k\,v'^2 - (m + m')\frac{dv'}{dt} - m\frac{d^2c}{dt^2} = 0$$

under the requisite substitution. In order to get the third equation that is lacking, one must now go further into a consideration of the internal forces on the system (viz., the stresses in and between the links).

To that end, if one defines the law of elasticity for the string aa' then the tension in it will be equal to zero in its original length a and it will increase in proportion to the increase in length under its extension. The length c at time t will require a force of tension that one can set equal to $(c - a)\,q$, in which q is a constant. Since that tension must obviously be equal to the lost force Q of the mass a, one will get the third equation:

$$(c - a)\, q = Q = mg - m\frac{dv'}{dt} - m\frac{d^2c}{dt^2},$$

for which, one can also take the equation:

$$(c - a)\, q = - Q' = - m'g + k\, v'^2 - m'\frac{dv'}{dt},$$

since $Q = - Q'$.

3) However, if one now generalizes the concept of a constraint on the system in the broadest way right from the start, such that each displacement that is possible from the physical nature of the system can be considered to be virtual, in which one considers the requisite internal forces that might appear, then not only will that unnatural restriction on virtual displacements go away, but all equations that are required for the determination of the phenomena of motion will vanish entirely in their own right.

If one then denotes the tension $(c - a)\, q$ in the string when its length is c by E then one will get the equation:

$$E = Q,$$

and the principle of virtual velocities will imply the equation:

$$Q\, \delta x + Q'\, \delta x' - E\, \delta c = 0,$$

when one observes that under the lengthening of the string by the length δc, the elastic forces that must be overcome will have the virtual moment $- E\, \delta c$. If one now sets $\delta x = \delta x' + \delta c$ then since $E = Q$, the foregoing equation will be converted into:

$$Q + Q' = 0,$$

which was also found before by means of the usual procedure.

15. – Magnitude of the constraint that is exerted upon a system.

We once more return to Fig. 9, in which b is the location to which the material point a would move during the time interval dt as a result of the velocity that it had attained at time t if it were completely *free*, c is the location to which it would *actually* move, α is the location to which it would move if it advanced with *uniform velocity* that it has attained with forces acting upon then, and finally, γ is any location that is allowed by the constraints to which that point would be *displaced*.

In no. **13**, we saw that $c\alpha$ can always be considered to be a *virtual* motion. If the constraint on the system is unvarying then no attention at all must be given to the internal forces. However, if the constraint is variable then it will only be necessary for one to consider the requisite internal forces that might appear during that motion and that exist within or between the links in the system; i.e., to treat them like the externally-applied forces p.

Under those assumptions, one can then put α in place of γ in equation (5). That will give the following expression for the total *constraint* $\sum m\,(bc)^2$ that is exerted upon the system:

$$\sum m\,(bc)^2 = \sum m\,(ab)^2 - \sum m\,(\alpha c)^2. \tag{57}$$

Since $m\,(bc)$, $m\,(ab)$, $m\,(\alpha c)$ are proportional to the forces q, p, r, resp., one can also write that equation as:

$$\sum q\,(bc)^2 = \sum p\,(ab)^2 - \sum r\,(\alpha c)^2. \tag{58}$$

That equation teaches us that the effect of the deflecting forces q, or the *constraint* that the system feels as a result of the coupling of its material parts at each moment in time, *is always equal to the difference between the effect that the forces p that are applied to it would provoke if all points were completely free and the effect that the effective forces r would actually provoke*.

Since that constraint is a minimum, from Gauss's principle, it will follow further that *the difference between the effect of the applied and effective forces is always as small as would be possible with the constraints that are given on the system*.

Nonetheless, as equation (57) teaches us, that difference will always be *positive, so as a result of the constraint on the individual material points of a system, there will always be a loss to the internal effect of the applied forces that is capable of being produced when all material points are completely free; however, that loss is always only as small as possible.*

In that, we must once more emphasize that when the constraint on the system is variable, along with the *applied* forces p, one must also include the internal forces – namely, the *elastic forces* – that might appear because of the motion of a to α in the links of the system. When one ignores those elastic forces, one would generally find that the loss that the *remaining external applied forces* suffer in many states of motion where the *overcoming* of the internal forces requires a certain effort will *increase*, but in many other states where the internal forces support the motion, it will *decrease*, and in the latter case there can, in turn, be a *gain* in mechanical work.

In addition, one must also point out here that when forces are present in the system that act upon *massless* points, not *massive* ones, equation (22) must be applied, which assumes the form:

$$\mathrm{S}\; q\,(c\,b)^2 + dt^2\, \mathfrak{S}\; p\,(c\alpha)\cos\varphi = \mathrm{S}\; p\,(\alpha\,b) - \mathrm{S}\; r\,(\alpha\,c)$$

or

$$\mathrm{S}\; q\,(c\,b)^2 = dt^2\, \mathfrak{S}\; p\,(\alpha\,c)\cos\varphi + \mathrm{S}\; p\,(\alpha\,b) - \mathrm{S}\; r\,(\alpha\,c)$$

here, where $c\alpha$ is regarded as a virtual displacement, and φ represents the angle $\alpha c\,b = raq$ between the forward directions of the forces r and q.

16. – A look back at the fundamental law of mechanics above and a comparison of it with the principle of least action of Maupertuis.

In the foregoing, we saw that one can take Gauss's principle to be the starting point of mechanics, as well as the principle of virtual velocities, in conjunction with d'Alembert's principle. Since each of those two foundations possesses the generality that is necessary if one is to develop the entire study of motion and equilibrium mathematically from it, one can already see from the outset, as **Gauss** also remarked in the aforementioned treatise, that when one has expressed the one, there can be no further essentially-new basic principle for mechanics that is not included in the former, according to matter, and would be derived from it. In fact, we have seen how both basic principles imply each other.

However, from **Gauss**'s further remark, it is not at all true that this new principle proves to be worthless due to that situation. Rather, it is always interesting and instructive to arrive at a new and advantageous viewpoint on the laws of nature, if it happens that one can solve this or that problem more easily by means of it or if it reveals a special suitability.

In regard to the latter, we have weighed the two basic principles above against each other many times in the foregoing and found that the principle of virtual velocities, in conjunction with d'Alembert's principle, will permit a simpler or more convenient application in most cases, but that in many special cases, Gauss's principle will allow an immediately employment, and that the latter possesses greater simplicity, in addition, while the former must be composed of *two* laws in a sense, and that ultimately Gauss's principle, from its content, comes closer to the essence of a self-explanatory fundamental law that requires no proof than the principle of virtual velocities.

Up to now, no fundamental law with the same profundity has been expressed besides the foregoing, since the principle of least action that **Maupertuis** proposed carries only the character of a lemma, but can hardly make any claim to the title of a fundamental law. That is because from the statement of that law that was first given correctly by **Lagrange**, the sum over all material points in the system of the integrals of the products of the quantities of motion *mv* and the curve elements *ds* that are described between any two epochs in the motion – so the quantities $\int \sum mv\, ds$ – is a minimum (special cases in which that quantity can also be a maximum must be dealt with). That integral sum is then smaller for the actual motion of the system than it would be if the material points that are pushed by those forces as a result of other constraints that would be necessary to reach the same endpoint of the motion were to follow other paths.

If one also must concede that it is obvious that the motion of a system in the manner that actually results would proceed in the easiest way then it would not be clear, with no further discussion, that the product of the quantity of motion and the path element would be the proper *measure* for that quantity that must be a minimum under such situations. Therefore, the law is very much in need of a proof. However, that law loses the property of a *fundamental law* entirely, since it is not completely general, but rather certain cases remain excluded in which the integral sum above can be a *maximum*.

17. – New fundamental law of mechanics.

From the viewpoint that was presented in the foregoing number, it would not be without interest to become acquainted with *a new, completely-general, fundamental law of mechanics* that takes the place of the other ones completely. Permit me to present it as follows:

From the coupling of material points upon which forces act into a system, those forces will indeed define a certain constraint, such that it will be prevented from performing the maximum of mechanical work that it would be capable of producing if all points were completely free. By itself, it must be regarded as lying in the nature of things or an immediate consequence of the constancy of matter and its forces that *the set of all works that the applied forces actually perform under the motion of the system will also appear completely – i.e., with no loss or gain*, since a loss or a gain in work must have a cause that might reveal itself to be nothing but an equivalent amount of work that appears.

That is what our new fundamental law consists of. It seems that it leaves nothing to be desired in *simplicity* and *evidence*, and that it can be aptly presented without proof as a fundamental law of mechanics, although if one feels that it would be desirable to analyze it and reduce it to the elementary theorems of statics and mechanics then one can also provide it with a special proof, as one will see shortly.

As far as the mathematical expression for that law is concerned, as before, in Fig. 14 (Table II of the previous issue), let α be the location that the material point a of the system would occupy as a result of the intrinsic velocity that it had attained at time t acting over the time interval dt, but with no forces at all acting upon it. Let p be the force *applied* to it, which would lead it from α to b during that time if it were completely free. Let r be the *effective* force, which would actually lead it from α to c, so it would then correspond to the actual motion ac when one recalls the intrinsic velocity that it already has, and finally, let m be the mass of the material point a.

For the sake of brevity, we symbolically let $\mathfrak{A}\ p\ a$ denote the work that a force p performs when its point of application traverses the straight path a, so it will have the expression $pa \cos \alpha$, in which α represents the angle of inclination between the forward direction of the force p and the path a, and the work that is *actually developed* during the motion in Fig. 14 by the *applied* force p during the time interval dt as it traverses the path ac will be equal to $\mathfrak{A}\ p\ (a\ c)$. By contrast, the *apparent* work done on the material point by the effective force is r is $\mathfrak{A}\ r\ (a\ c)$. Thus, from our fundamental law, we must have the equation:

$$\sum \mathfrak{A}\, r\, (a\ c) = \sum \mathfrak{A}\, p\, (a\ c)\,. \tag{59}$$

On simple geometric grounds, the work that is done by a force p when one traverses a broken path whose sides are a_1, a_2, a_3, … will be equal to the work that force does when it traverses the straight lines that connect the endpoints of the broken path; i.e., it is:

$$\mathfrak{A}\ p\ a_1 + \mathfrak{A}\ p\ a_2 + \mathfrak{A}\ p\ a_3 + \ldots = \mathfrak{A}\ p\ a,$$

in which the sum of the projections of the individual line segments a_1, a_2, a_3, … onto the direction of p equals the projection of the line a onto that direction.

It follows from this that the sum of the works done by the force p, as well as the force r, when they traverse the broken path $a\ \alpha c$ is equal to the work done by the force in question when it traverses the diagonal $a\,c$. One can also write:

$$(60)\qquad \sum \mathfrak{A}\, r\,(a\ \alpha) + \sum \mathfrak{A}\, r\,(\alpha c) = \sum \mathfrak{A}\, r\,(a\ \alpha) + \sum \mathfrak{A}\, r\,(\alpha c),$$

instead of equation (59) then.

Now, $a\alpha$ is a motion that the point of the system might exhibit if it had a uniform velocity that was consistent with its constraint, but it was not acted upon by forces. Any *virtual displacement* of the system away from the location a can obviously be regarded as such a motion; i.e., one can think that when the virtual displacements over the time interval dt are replaced with uniform velocities, those velocities can be considered to be ones that the individual points of the system might possibly possess at time t. In this, as in no. **13**, it is generally assumed that should the constraint on the system depend upon time t or be variable then among the applied forces p, the ones that are considered to be required will be the ones by which that variability is required by the constraints.

That further illuminates the fact that, no matter how variable the line $a\alpha$ of the velocities of the points at time t that are consistent with the constraint on the system might be, the line αc will not depend upon those velocities at all, but will merely be required by the applied force p, or if one would prefer, the effective force r, when αc represents the direction that is given to the *deflection* of the material point at time t that is produced by only those forces. In order to make the validity of that assertion clearer, recall that no matter what the law of dependency between the line αc and the forces on the system might be, it can produce no other values for the deflection αc, regardless of whether one determines that deflection from the point a or the point α, because no matter how variable the line $a\alpha$ might also be, that deflection will still be *infinitely small*, which has the consequence that the forces p on the system will have an effect on the point a that differs from the effect on the system at the point *a by only infinitely little;* i.e., when one passes to the limiting state in the sense of differential calculus.

There will be even more evidence for this theorem when one imagines that in the construction of the actual motion along the diagonal ac, it will not be the piece $a\alpha$ that is described with uniform velocity and then the deflection αc, which might give the impression that the *later* component αc can possibly depend upon the *earlier* one $a\alpha$, but, from Fig. 15 (Table II of the previous issue), it would *first* describe the path $a\,c_1 = \alpha c$ that lies in the direction of the effective force r, *so the one that is merely required by the applied force p with no concern for any uniform velocity, and then the path* $c_1\,c = a\ \alpha$, which has a uniform velocity that is *given arbitrarily*.

In order to prevent all misunderstandings, we point out that *in a certain sense* the applied force p, and therefore also the effective force r and the line ac_1 can depend upon the velocity at time t, and therefore on the line $a\alpha$, such as, e.g., for motion in resistant media, in which the resistance of the medium varies with the direction and velocity of the moving mass. By itself, that fact is irrelevant for the present considerations, because we think of the applied forces p as being just the ones that correspond precisely to the *actual* motion at the end of time t. If those forces were not, in fact, also required by the velocities at time t, and therefore functions of that velocity, then we would assume *that they do not change when we substitute any other displacement for the virtual displacement* $a\alpha$.

Under those assumptions, the first terms on the left and right-hand side of equation (60) will appear to be included among the arbitrarily-varying quantities that are given by the laws of constraint on the system, while the second terms are variable; i.e., quantities that are established by the nature of the system and its forces. On that basis already, and also when one considers that the first terms can be equal to zero by the permissible assumption that $a\alpha = 0$ itself, equation (60) will decompose into the following two separate equations:

$$\sum \mathfrak{A}\, r\,(a\;\alpha) = \sum \mathfrak{A}\, p\,(a\;\alpha), \tag{61}$$

$$\sum \mathfrak{A}\, r\,(\alpha\, c) = \sum \mathfrak{A}\, p\,(\alpha\, c)\,. \tag{62}$$

Since $a\alpha$ represents any arbitrary admissible virtual displacement of the point a, and obviously $\alpha\; c = a\; c_1$ is also such a displacement (for $a\alpha = 0$), equation (62) will be contained in equation (61), and thus superfluous.

Equation (61), as the immediate consequence of equation (59), which was given by our fundamental law, can indeed be likewise replaced with equation (59), but it would seem necessary to perform the foregoing derivation and emphasize the remarks that it provoked in order to show more clearly that in the expressions for the *works* done by the forces p and r, the path of the point of application of those forces will remain arbitrary within the limits of the *virtual displacements*, which from equation (59), in which ac denotes the *actual* path of the point a, is no more evident than the arbitrariness in that path in its resolution into *arbitrary* components $\alpha\, a$ and constant components $\alpha\, c$ proves to be, especially since one should not, with no further analysis, overlook that if that were true then any *virtual* motion could be regarded as an *actual* motion *that results from the governing forces p*, although there is no doubt that every virtual motion can be regarded as a motion $a\alpha$ that results *with uniform velocity, but without the action of the forces p*.

From that explanation, if one denotes any virtual displacement of the point a [so the line $a\alpha$ in equation (61) or the line $\alpha\, c$ in equation (59)] by δs then our basic equation will become:

$$\sum \mathfrak{A}\, r\;\delta s = \sum \mathfrak{A}\, p\;\delta s\,. \tag{63}$$

One sees that it can be easily reduced to the formula that represents d'Alembert's principle, with the help of the principle of virtual velocities, because if one denotes a coordinate line that is drawn parallel to the direction of the effective force r by ρ, then denotes the force r by $m\dfrac{d^2\rho}{dt^2}$, and further denotes the angle of inclination of r with respect to the virtual displacement δs of the point a by φ and the angle of inclination of p with respect to δs by ψ then the work done by the force r under that displacement will be equal to $m\frac{d^2\rho}{dt^2}\,\delta s\cos\varphi$, and the work done by the force p will be equal to $p\ \delta s\cos\psi$. In that way, equation (63) will become:

$$\text{(64)}\qquad \sum m\frac{d^2\rho}{dt^2}\,\delta s\cos\varphi = \sum p\ \ \delta s\cos\psi.$$

If one would like to refer all quantities to a rectangular coordinate system, as one usually does, then that work done by any force would split into the works done by its components, and if X, Y, Z are the components of p and δx, δy, δz are the projections of the displacement δs onto the three axes then (from a derivation that was applied before in no. **4**):

$$\sum m\left(\frac{d^2x}{dt^2}\delta x+\frac{d^2y}{dt^2}\delta y+\frac{d^2z}{dt^2}\delta z\right) = \sum(X\,\delta x+Y\,\delta y+Z\,\delta z)\,,$$

which is an equation that is ordinarily presented in the form:

$$\text{(65)}\qquad \sum\left[\left(X-m\frac{d^2x}{dt^2}\right)\delta x+\left(Y-m\frac{d^2y}{dt^2}\right)\delta y+\left(Z-m\frac{d^2z}{dt^2}\right)\delta z\right] = 0,$$

in order to express the equilibrium of the *lost* forces.

It hardly needs to be remarked that our fundamental law encompasses the state of *variable* motion, as well as that of *rest*, or even *equilibrium with uniform motion*, since only the effective forces r need to be set to zero for there to be equilibrium, which will make the entire left-hand side of our fundamental equation reduce to zero.

That further illuminates the fact that this, in itself very plausible, fundamental law possesses the advantage of greater simplicity over d'Alembert's, since the former first requires the assistance of the principle of virtual velocities in order to put the fundamental equations of motion into the form of a mathematical formula, and in addition, requires a detour through the concept of *lost* forces, to which end, certain forces must first be applied to the given system that do not exist in reality and only serve to produce a fictitious system with the so-called lost forces.

In addition, our fundamental law is applicable, with no further analysis, regardless of whether certain forces p in it act upon *material or massless* points, since one only has to set $m = 0$ for the massless points, which one cannot do in Gauss's law, since that would imply *infinite quantities* for massless points, which, as we showed in no. **2**, would make a conversion of the formula necessary, and to some extent the fundamental law itself would be annulled.

If one would like to assign a special name to the new law, for the sake of brevity of reference, then since the motion of the system completely realizes or brings to light the work done by the forces that are applied to it then the terminology of *the principle of the realization of work* might be suitable.
