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On an integral form that corresponds to Gauss's principle of least constraint

By

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I. – Statement of the problem.

It is known that the principles of mechanics that **A. Voss** ([1])called principles of the "third kind," can be reduced to ones of a different kind that **Voss** (*loc. cit.*) called the "second kind," which are the ones that involve the equations of motion of mechanics directly. Principles of the "third kind" include, e.g., Hamilton's principle and the principle of least action, while one of the "second kind" include the principle of virtual velocities, d'Alembert's principle, and Gauss's principle of least constraint. Lagrange's equations, in their first form – that is, the one in which the Lagrange multipliers appear – can be counted amongst the "second kind," while the ones of the "third kind" include the ones in which knowledge of the *vis viva* must be assumed (viz., Lagrange's equations in their second form).

For example, Hamilton's principle can be reduced to that of **d'Alembert**, and can be shown to be completely equivalent to it; i.e., if the one were true then the other would be true, as well, and conversely. Now, as is known, the cited principles of the "second kind" are completely equivalent to each other. On the other hand, the principles of the "third kind" are likewise completely equivalent to each other, since they can always be reduced to ones of the "second kind."

([1]) **A. Voß**, "Die Prinzipien der rationellen Mechanik," Encykl. der math. Wiss. IV, 1, pp. 10.

Such considerations prompted Herrn Prof. Dr. **A. Wassmuth** to remark in a seminar on mathematical physics in Graz that there must be a principle that has the form of a time integral between fixed time limits and can be reduced to Gauss's principle of least constraint in a manner that is analogous to the way that **Hamilton**'s principle or the principle of least action reduces to d'Alembert's principle. With the terminology that was given above, a principle of the "third kind" should be given that relates formally to Gauss's principle of least constraint, which belongs to the "second kind," in perhaps the way that Hamilton's principle relates to **d'Alembert**'s. Elaborating upon that formal analogy is the goal of the present article.

II. – Generalities concerning the variational principle and the associated variational conditions.

The proof of the equivalence of a principle of the "third kind" with one of the "second kind" will be accomplished with the help of an identity that has the form:

$$\int_{t_1}^{t_2} (\delta L - A)\, dt = \int_{t_1}^{t_2} \sum_{i=1}^{3n} (X_i - m_i\, \ddot{x}_i)\, \delta \ddot{x}_i \cdot dt\,, \tag{1}$$

in the special case of the equivalence of Hamilton's and d'Alembert's principles.

[In this, L means the *vis viva*, and A means the virtual work done on the system, which is thought of as consisting of n mass-points. (There are therefore $3n$ rectilinear coordinates for a point in it.) $\ddot{x}$ means an acceleration, X means an explicit force, t_1 and t_2 are fixed time limits (viz., the starting point and end point of the motion), and δ is the symbol for a variation. (For more details, see below.)]

In the case in question, in order to prove the equivalence of an integral that is presented with Gauss's principle of least constraint from this, one must start by finding an identity that is expressed analogously to equation (1) and whose right-hand side is obviously composed of the expression:

$$\int_{t_1}^{t_2} \sum_{i=1}^{3n} (X_i - m_i\, \ddot{x}_i)\, \delta \ddot{x}_i \cdot dt\,, \tag{2}$$

since that is the form of Gauss's principle that is analogous to d'Alembert's principle.

If the identity, thus-found, can lead to that conclusion in a manner that is analogous to the way that one concludes the equivalence of Hamilton's principle and d'Alembert's ([1]) from the identity (1) then the left-hand side of that new identity will yield the desired integral form.

The considerations to be made shall be referred to a system of n discrete mass-points, so one will be dealing with $3n$ independent variables in the case of rectilinear coordinates. However, the number of degrees of freedom in the system can be diminished by the condition equations that are imposed on the system. The same

([1]) **Boltzmann**, *Mechanik*, pp. 7, *et seq.*

assumptions shall be made about those condition equations that **Boltzmann** ([1]) drew upon in order to represent the equivalence of Hamilton's principle with d'Alembert's by analogy. The 3*n* rectilinear coordinates will all be denoted by the symbol *x* with indices added [likewise following **Boltzmann** ([2])]. Each of the *n* masses will then enter into only three of the notations in different forms, and indeed m_i, m_{i+1}, m_{i+2} ($i = 1, 4, 7, \ldots, 3n - 2$) will always be regarded as identical. The dots over the symbols x_i will mean derivatives of the coordinate values with respect to time, and indeed, one dot will mean the first derivative, or the velocity in the direction of the coordinate in question, two dots will mean the second derivative (i.e., the acceleration in that direction), and so on. The X_i refer to the explicit forces that correspond to the coordinates with the same indices, that is, the forces that do not arise form constraints that are expressed by the condition equations. The so-called variation of a quantity will be suggested by the operation symbol δ; i.e., a fully-general arbitrary increase in the quantity in question that is "infinitely small," as one says in the customary terminology of mechanics. The demand that is expressed in that way can be posed more precisely by saying that the absolute values of the variations must all lie below a positive quantity that is regarded as arbitrarily-small, but well-defined. Now, those variations are *not* by any means *identical* to the increases in the coordinates or other variables that *actually occur* in the course of motion. The latter increases will be referred to as *differentials* of the quantities in question, in the sense that they should be the changes in the quantities in question that *actually* take place in a time interval *dt*. One will further require that the variations must be *compatible* with the *conditions* on the system. Should the value of a variable – e.g., – always remain smaller than a well-defined quantity, then the "varied" value of that variable could not be greater than or equal to that quantity, either.

If the three coordinates of a mass-point were to take on values that are "varied," with the meaning that was just given to that term, then one would have to refer to the position of the mass-point that is established in that way as a *varied position*. One must once more note in that regard that the varied values of the coordinates must also satisfy the condition equations. For example, if a mass-point is to remain on a given surface then the varied coordinates must also fulfill the equation of the surface; i.e., the varied position must belong to the surface. Corresponding statements will be true for the "varied position" of several mass-points. Any motion that consists of a series of positions that are compatible with the condition equations will be called a *possible* motion of the system. Any of those possible motions will be distinguished as *unvaried*, and the values of the variables at each time point that belong to that special motion will be called *unvaried* values of the variables for that time point. Those values are only to be regarded as *arbitrary zero-points* of the variations ([3]). The remaining possible motions are called *varied motions*, and correspondingly, the term *varied value* of a variable at a time point will now mean each possible value that is different from the *unvaried* value.

It is known that the principles of the same kind (in the terminology that was used in the introduction) differ from each other by the individual variational conditions that are

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([1]) *Mech.*, II, pp. 2, *et seq.*

([2]) *Mech.*, II, pp. 228, *et seq.*

([3]) The unvaried motion is often referred to as "actual," which is, however, premature and unclear. Indeed, it is only by means of the principle in question that one can first show which of all possible motions are actual, since one can choose any possible motion to be unvaried at the outset.

associated with each of them, that is, by well-defined instructions for isolating certain manifolds of narrower scope from the infinite manifold of all motions of the system that are compatible with the conditions from which the variations are to arise, and which must still be infinite in order to justify the necessary arbitrariness. For example, only those variations of the coordinates should be employed in Hamilton's principle that *do not include a variation of the time*; i.e., each state of a varied motion will be assigned to a state of the unvaried motion that are each passed through simultaneously, while the general case is the one in which a state of unvaried motion is assigned an entirely arbitrary state varied motion. Another case of variational conditions is this one: For Gauss's principle of least constraint, the so-called "Gaussian variation" will be employed, whose variational conditions read: For the state of motion in question, the variations of all coordinates and velocities will vanish, and only the variations of the acceleration should be non-zero. Now, the latter variational conditions are the ones that must be set down in order to succeed in exhibiting the desired identity for the case in question, because the right-hand side of the identity to be exhibited includes only variations of the accelerations [(2)].

III. – The variational conditions in special cases.

In what follows, Gauss's method of variation shall be considered more closely by using **Boltzmann**'s representation ([1]).

A *motion that is varied in the Gaussian manner* shall be characterized by the fact that, just as the coordinates are certain functions of time:

$$x_i = f_i(t), \qquad i = 1, 2, 3, \ldots, 3n \tag{3}$$

for the unvaried motion, here, they will be somewhat different functions $\varphi_i(t)$ of time that likewise fulfill the condition equations of the system, and one further requires of them that at a well-defined time (which can be chosen arbitrarily), the values of the functions themselves and their first derivatives of the coordinates with respect to time (so, the velocities) are all equal to the corresponding values for the unvaried motion. Only their second derivatives with respect to time (i.e., the accelerations) at that time point shall have somewhat different values from the ones in the unvaried motion, and indeed, the increases that the accelerations experience under the transition from the unvaried motion to the varied one shall be denoted $\delta\ddot{x}_i$, which is consistent with the previous conventions. All δx_i and $\delta\dot{x}_i$ shall be equal to zero for any arbitrarily-chosen, but well-defined, time point, while the $\delta\ddot{x}_i$ should be non-zero. Naturally, that is true for only the precise moment in time considered, but for, say, a finite time interval, since otherwise all also $\delta\ddot{x}_i$ would have be zero during that interval. That type of variations is therefore *point-wise*; i.e., it initially makes no sense to speak of a global variation of the motion. In that regard, however, there exists no fundamental distinction from the variational method that appears for Hamilton's (d'Alembert's, respectively) principle, because in the latter, an

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([1]) *Mech. I*, pp. 209, *et seq.*

arbitrary, but well-defined, point in the unvaried motion is initially associated with a point of the varied motion for which the variations of the coordinates have certain non-zero values. Now, in both cases, one therefore treats the transition from the variation of the motion at one time point to the *global* variation of the motion ([1]). In the case of Hamilton's (d'Alembert's, respectively) principle, that will be effected as follows ([2]): One advances from one moment of time to the next and applies the aforementioned point-wise variation everywhere. One will then obtain values of the variations δx_i that are arranged with no connection to time, such that the integral $\int \delta x_i \cdot dt$ would make no sense, since the quantity δx_i under the integral sign is an absolutely-discontinuous function of time. However, one can always select infinitely many arrangements of variations from the infinite manifold of those variations that are subject to the restriction that all δx_i should be continuous, but otherwise arbitrary, functions of time, so ones that might be represented in the form:

$$\delta x_i = \varepsilon f_i(t), \qquad i = 1, 2, 3, \ldots, 3n, \tag{4}$$

in which the $f_i(t)$ represent arbitrary, but continuous and finite functions of time, and ε is a positive quantity that is common to all coordinates and all times, and which can be made arbitrarily small in order to correspond to the requirement that was imposed above that the variations should be "infinitely small." Now, one can call the temporal sequence of those varied positions a "varied motion" and then the integral that was defined above will make sense as an integral of a continuous function.

However, if one would like to take advantage of the same notions for the variations $\delta \ddot{x}_i$ in the Gaussian method of variation then one would find that this process would not be possible, because when the variations of the accelerations are continuous functions of time, the variations of the velocities, and therefore those of the coordinates, as well, would *not be continually zero* or "infinitely-small" of higher order than the variations of the accelerations, as one must have for the corresponding advance from one moment in time to the next in the case of the Gaussian method of variation. Namely, if one $\delta \ddot{x}_i$ is a continuous function then it must be always positive or always negative in a time interval that is chosen to be correspondingly-small (as long as it is not, say always zero, which must be excluded here, however). As a result of the relation:

$$\delta \ddot{x}_i = \frac{d\,\delta \dot{x}_i}{dt}, \tag{5}$$

the $\delta \dot{x}_i$ that belongs to that time interval must increase or decrease, so it must be distinct from the value zero. However, as was mentioned expressly here already, the tacit assumption is made in that ([3]) that *time is not varied*, which was just expressed by equation (5).

([1]) The variational method for the principle of least action also behaves in precisely that way, since the variation of the time will first become essential when one considers (exhibits, respectively) the globally-varied motion.

([2]) **Boltzmann**, *Mech.*, II, pp. 3, *et seq.*

([3]) **Boltzmann**, *Mech.*, II, pp. 4, remark.

One must then look for another method in order to make it possible to construct integrals of the type $\int \delta \ddot{x}_i \, dt$ in order to go from the point-wise variations of the motion to global ones, respectively.

To that end, the aforementioned advance from one moment in time to the next in the motion shall be subjected to a more detailed consideration for the case in question. A mass-point will be considered that traverses a given path with a given velocity at each point under the unvaried motion. The motions that are varied in the Gaussian manner in this case can be exhibited as follows: One imagines a second path that has a point in common with the original one and whose tangent at that point coincides with the tangent to the original path at that point. Hence, if the mass-point at the point common to both paths advances simultaneously under the two motions (namely, the unvaried motion and the one that arises by traversing the varied path) with velocities that have the same absolute value and the same direction then one will, in fact, be dealing with motion that is varied in the Gaussian manner, which is generally the case, so the acceleration of the mass-point at the common point to the paths will not be the same for the varied and unvaried motions. That can be ensured, e.g., in such a way that the accelerations in the common direction of the path are equal and the varied path possesses a different curvature from the original path at the common point. Now, Gauss' principle of least constraint says that for every individual point that is considered in that way, of the infinite manifold of possible motions that are varied in that way, the motion that actually takes place will be the one for which the relation:

$$\sum_{i=1}^{3n} (m \ddot{x}_i - X_i) \, \delta \ddot{x}_i = 0 \tag{6}$$

exists.

The advance from one moment in time to the next in the motion shall now be carried out. As was suggested, one should not regard the path as being globally varied, but rather, one should regard it as composed of very many pieces that always deviate from the paths that arise from the point-wise variation that was just described, and which one represents at each point of the unvaried path. Geometrically, that is expressed by saying: One thinks of each point of the original path as being endowed with a family of paths that are *varied in the Gaussian manner* (those paths shall also be referred to in that way now). In that way, one will get a doubly-infinite manifold of curves that possesses an *envelope*, however (say, when they are all represented in a plane), that is just the original path. Now, the motion of the mass-point at any point of that path is to be varied in such a way that the state of motion is taken, not from the unvaried motions, but from the simply-infinite manifold of motions that are varied in the Gaussian manner whose motions at that point are appropriate. However, the mass-point should continually remain on the original path and possess the prescribed velocity at each point. One can see from this that the choice of varied path, from which, the state of motion shall be taken will no longer be entirely arbitrary, since the acceleration of the varied motion (i.e., now that would be the motion that is assembled, so to speak, from the individual motions that are varied in the Gaussian manner) must not be always greater than or always less than the unvaried motion in any small enough time interval, since otherwise the mass-point would reach a velocity that is different from the prescribed one. If one now lets the pieces that are associated with the individual point-wise-varied paths decrease without limit then the variations of the acceleration when one traverses the path that one thinks of as composite, which now represents the globally-varied path, cannot be always positive or always negative in any sufficiently-small time interval. If they are not to be constantly zero then they can only be represented by a function of time that *changes its sign arbitrarily often in any sufficient small time interval*. However, such a function is *not integrable*, and that is, in turn, the same result that Boltzmann considered in his remark.

As was mentioned before, this argument is true only for the case in which time is not varied. If one were to introduce a *variation of time* [i.e., one would no longer compare the acceleration of the unvaried motion at any well-defined moment in time t with the acceleration of the (globally) varied motion that takes place at the same moment in time, but with an acceleration that belongs to the moment in time $t + \delta t$ of the varied motion] then one could no longer justify the statement that was just made about the variation of the acceleration as a function of time, and that suggests the idea of seeking to exhibit the varied motion with the help of the variation of time.

IV. – Exhibiting the variational conditions that are suitable to the case in question.

It shall now be shown that one can always arrive at a varied motion for which the variations of the accelerations are *integrable* and which likewise correspond to the other requirements that were just imposed by introducing a variation of time.

The argument will proceed along the following train of thought:

1. It will be shown that there are always infinitely many different system of $\delta \ddot{x}_i$ that are defined with no variation of time, yield a Gaussian variation, and therefore, as was shown, are not integrable.

2. It will be shown that for each such arrangement of $\delta \ddot{x}_i$, there is a variation of time (which will be denoted by δt) by whose introduction, the distribution of variations of the accelerations in time will become integrable, which shall be suggested by the notation $\bar{\delta} \ddot{x}_i$.

3. It will be shown that the time variation δt can be chosen in such a way that the variations of the velocities and coordinates that it produces will become infinitely small of higher order than the variations of the accelerations when the latter decrease without limit.

4. It will be shown that, in addition, the variation δt can be chosen in such a way that the integral of the form $\int F \, \bar{\delta} \ddot{x}_i \, dt$ will not always be zero, regardless of the function F, which will be important for our later conclusions.

It will then follow that:

1. It was already recognized to be necessary that the variations of the accelerations should be functions of time that change their signs arbitrarily often in any sufficiently small time interval. (From now on, in this argument, a single function will be considered by which the variations of a single acceleration can be represented. The results will then be valid for all accelerations directly.) Such a function can be given in the following way: One divides the interval t_a to t_b in which that function is to be regarded into an even number n of equal pieces τ_n and then determines that: The values of the functions $\delta \ddot{x}_i$ at a point of the subdivision that is separated from the starting point t_a of the interval by an even number of sub-intervals τ_n, and thus, at those points that are determined by:

$$t_+ = t_a + \mu\, \tau_n\,, \qquad \mu = 0, 2, 4, \ldots, n,$$

have finite, positive values that are taken from an arbitrarily-given function $f(t)$ that is to be continuous. [$f(t)$ must then have values that are always positive.] In that, one can ignore a constant that can be made arbitrarily small and which multiplies the all values of the function. Furthermore (with the same addendum), the values of the function shall be negative at the remaining "odd" points of the subdivision, namely:

$$t_- = t_a + \nu\, \tau_n\,, \qquad \mu = 1, 3, 5, \ldots, n-1,$$

which will be (up to absolute value) the arithmetic mean of the values of the function at the two neighboring "even" points of the subdivision:

$$\left|(\delta \ddot{x}_i)_\nu\right| = \frac{(\delta \ddot{x}_i)_{\nu-1} + (\delta \ddot{x}_i)_{\nu+1}}{2}.$$

One now lets n get bigger and bigger, which will make τ_n become smaller and smaller. The arrangement of values that arises in that way when $\lim n = \infty$ or $\lim \tau_n = 0$ will then represent a function with the required properties. However, due to the arbitrary choice of the positive values of the function, there will obviously be infinitely many such functions. Such a function can be represented graphically as two quasi-curves that are reflected in the axis $\delta \ddot{x}_i = 0$; i.e., the t-axis.

2. The variation δt, which is to be regarded as a function of time, shall be chosen in such a way that it has the value zero for all $t_+ = t_a + \mu\, \tau_n$, for which $\delta \ddot{x}_i$ is positive, by assumption, and has the value τ_n for all $t_- = t_a + \nu\, \tau_n$, for which $\delta \ddot{x}_i$ is negative. In that way, one will find that *all* points of the subdivision will be assigned a *positive* value of the function now. In the $\lim \tau_n = 0$, the new assignment of values (which is not expressed by $\bar{\delta} \ddot{x}_i$) will go to the positive quasi-branch of the function $\delta \ddot{x}_i$, which is, however, an *actual curve* now. As is immediately clear, the function $\bar{\delta} \ddot{x}_i$ is continuous, and therefore *integrable*. It should be remarked here that this time variation δt has no connection whatsoever to the accelerations $\delta \ddot{x}_i$, and indeed should not be put on the same level as them in that regard.

3. Naturally, the velocity $\dot{x}_i$ of the unvaried motion will be associated with another velocity of the varied motion, which will be denoted by $\dot{x}_i + \bar{\delta}\dot{x}_i$, point-by-point, by the introduction of a variation δt of time, as long as δt is not just equal to zero. Whereas $\delta \dot{x}_i$ would be equal to zero at any time (which is an assumption of the Gaussian variation), that is not the case for $\bar{\delta}\dot{x}_i$. One has the relation:

$$\bar{\delta}\dot{x}_i = \ddot{x}_i \cdot \delta t . \tag{7}$$

One now remarks that: The variations $\delta \ddot{x}_i$ (and naturally, the variations $\bar{\delta}\ddot{x}_i$, as well) shall be *infinitely small*; i.e., they shall be represented by:

$$\delta \ddot{x}_i = \varepsilon \, \varphi_i \, (t),$$

in which ε is a constant that can be made infinitely small. Likewise, one has:

$$\bar{\delta}\ddot{x}_i = \varepsilon \, \bar{\varphi}_i (t) .$$

Now, the arguments that were presented in 1. and 2. are also true for finite values of $\delta \ddot{x}_i$, since the constant ε is virtually absent from it. In contrast to those finite values of the function, the quantity τ_n, which converges to zero, is infinitely small. There is therefore nothing that prevents one from assuming that in the case where the $\delta \ddot{x}_i$ themselves become infinitely small upon multiplying by the constant ε, the $\tau_n = \delta t$ will become *infinitely small of higher order* than the $\delta \ddot{x}_i$ (say, finite functions that are multiplied by ε^2), since no special assumptions at all were make about the type of passage to the limit $\lim \tau_n = 0$. As a result of the relation (7) and the further one:

$$\bar{\delta} x_i = \dot{x}_i \cdot \delta t, \tag{8}$$

the variations $\bar{\delta}\dot{x}_i$ and $\bar{\delta} x_i$ will now also be *infinitely small of higher order* than the variations $\delta \ddot{x}_i$ (the variations $\bar{\delta}\ddot{x}_i$, respectively), since $\ddot{x}_i$ and $\dot{x}_i$ are finite.

4. One can see immediately that an integral of the form $\int F \, \bar{\delta}\ddot{x}_i \, dt$ will certainly not be zero, in general, since the function $\bar{\delta}\ddot{x}_i$ is positive over the entire interval.

It should be added that, as is immediately clear, one can also impose the condition on the functions $\bar{\delta}\ddot{x}_i$ that their values should be zero at two well-defined fixed time-points, which might be called t_1 and t_2.

With that, it is shown that, as predicted, there are always infinitely many different systems of motions that will yield a global variation of the motion when they are varied (point-wise) in the Gaussian manner and that correspond to the following conditions on the variations: The variations of all coordinates and velocities are equal to zero at all times, and the variations of the accelerations are non-zero, integrable functions of time.

One can add yet another condition on the variations arbitrarily that is expressed by:

$$\bar{\delta}\dddot{x}_i = \frac{d\bar{\delta}\ddot{x}_i}{dx} . \tag{9}$$

The justification for it will become clear immediately, since that convention implies no consequences for the variations of the derivatives of the coordinates with respect to time of order less than three.

It is now possible to address the formal calculation, and indeed on the grounds of the following four variational conditions:

$$\delta x_i = 0 \text{ for all times,} \tag{I}$$

$$\bar{\delta}\dot{x}_i = 0 \text{ for all times,} \tag{II}$$

$$\bar{\delta}\ddot{x}_i \text{ will be non-zero and integrable,} \tag{III}$$

$$\bar{\delta}\dddot{x}_i = \frac{d\bar{\delta}\ddot{x}_i}{dx} . \tag{IV}$$

V. – Formal implementation.

We shall now address the problem of establishing the identity that was mentioned in section II, whose right-hand side is given already.

To that end, we shall first establish the assumptions of a mechanical nature. The *vis viva* of the system of *n* mass-points will be considered to be a quadratic form in the velocities $\dot{x}_i$:

$$L \equiv \tfrac{1}{2}\sum_{i=1}^{3n} m_i\, \dot{x}_i^2 . \tag{10}$$

The virtual work of the explicit forces X_i shall be denoted by δA :

$$\delta A \equiv \sum_{i=1}^{3n} X_i\, \delta x_i . \tag{11}$$

(From now on, for the sake of brevity, only the indices will be written in the summations, without specifying the range over which they vary, since it will always be 1 to 3*n*.)

One now forms the following expressions:

$$\frac{dL}{dt} = \sum_i m_i \, \dot{x}_i \, \ddot{x}_i \, ,$$

$$\frac{d^2L}{dt^2} = \sum_i m_i \, [\ddot{x}_i^2 + \dot{x}_i \, \dddot{x}_i] \, ,$$

$$\delta \frac{d^2L}{dt^2} = \sum_i m_i \, [2\ddot{x}_i \, \delta\ddot{x}_i + \dot{x}_i \, \delta\dddot{x}_i + \dddot{x}_i \, \delta\dot{x}_i] \, , \tag{12}$$

and furthermore:

$$\frac{d\delta A}{dt} = \sum_i \left(\frac{dX_i}{dt} \delta x_i + X_i \frac{d\delta x_i}{dt} \right),$$

$$\frac{d^2\delta A}{dt^2} = \sum_i \left(\frac{d^2X_i}{dt^2} \delta x_i + 2 \frac{dX_i}{dt} \frac{d\delta x_i}{dt} + X_i \frac{d^2\delta x_i}{dt^2} \right). \tag{13}$$

Those relations will be valid in full generality when the operation on a function of several variables that is suggested by the symbol δ is performed according to the rule:

$$\delta F\,(x_1, x_2, \ldots) = \frac{\partial F}{\partial x_1} \delta x_1 + \frac{\partial F}{\partial x_2} \delta x_2 + \ldots \tag{14}$$

When one brings the variational conditions (I) to (IV) into play, one can see that one can calculate formally as if the time were not varied at all. Namely, one must first perform the operations of differentiation and variations according to the rule in equation (14), which implies the results in equations (12) and (13) in a completely general way. One then replaces the differential quotients $\frac{d\delta x_i}{dt}$ ($\frac{d^2\delta x_i}{dt^2}$, respectively) with $\delta\dot{x}_i$ ($\delta\ddot{x}_i$, respectively) in the right-hand side of equation (13), since they are considered to have no time variation, corresponding to equation (5). One now carries out the time variation that was described thoroughly above by setting the variations δx_i and $\delta\dot{x}_i$ equal to zero, which means that the $\delta\ddot{x}_i$ must be replaced with the $\bar{\delta}\ddot{x}_i$, and from 3. in section IV, that must produce no variations in the values of the velocities and coordinates. To abbreviate, the variations $\bar{\delta}\ddot{x}_i$ will once more be denoted by $\delta\ddot{x}_i$ now. That will legitimize the aforementioned formal method of calculation.

One also gets equations (12) and (13) by performing the aforementioned operations:

$$\delta \frac{d^2L}{dt^2} = \sum_i m_i \, [2\ddot{x}_i \, \delta\ddot{x}_i + \dot{x}_i \, \delta\dddot{x}_i] \, , \tag{15}$$

$$\frac{d^2\delta A}{dt^2} = \sum_i X_i \, \delta\ddot{x}_i \, . \tag{16}$$

If one subtract equation (16) from equation (15) and integrates the difference between fixed, but arbitrary, time limits t_1 and t_2, for which, as was suggested above, the convention is made that:

$$(\delta \ddot{x}_i)_{t_1} = (\delta \ddot{x}_i)_{t_2} = 0, \qquad i = 1, 2, 3, \ldots, 3n, \tag{V}$$

then one will get:

$$\int_{t_1}^{t_2} \left(\delta \frac{d^2 L}{dt^2} - \frac{d^2 \delta A}{dt^2} \right) dt = \int_{t_1}^{t_2} \sum_i \left(2 m_i \, \ddot{x}_i \, \delta \ddot{x}_i + m_i \, \dot{x}_i \frac{d \delta \ddot{x}_i}{dx} - X_i \, \delta \ddot{x}_i \right) dt \, . \tag{17}$$

If one integrates the second term with the help of partial integration then it will give:

$$\left| \sum_i m_i \, \dot{x}_i \, \delta \ddot{x}_i \right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_i m_i \, \ddot{x}_i \, \delta \ddot{x}_i \cdot dt \, .$$

The first sum is equal to zero, as a result of the condition (V). One will then get:

$$\int_{t_1}^{t_2} \left(\delta \frac{d^2 L}{dt^2} - \frac{d^2 \delta A}{dt^2} \right) dt = \int_{t_1}^{t_2} \sum_i (m_i \, \ddot{x}_i - X_i) \, \delta \ddot{x}_i \cdot dt \, . \tag{VI}$$

That is the desired identity. It shows the *complete* equivalence of the principle that corresponds to setting the left-hand side equal to zero:

$$\int_{t_1}^{t_2} \left(\delta \frac{d^2 L}{dt^2} - \frac{d^2 \delta A}{dt^2} \right) dt = 0 \tag{VII}$$

and *Gauss's principle of least constraint* under the assumptions that were made here. That requires that one must prove that the principle in equation (VII) follows from Gauss's principle, and conversely. The first part of that proof is obvious. If one starts, conversely, from the validity of the form (VII) then one must reason as follows: The integral is a definite integral between fixed, but arbitrary, limits, so it must assume the value zero for all pairs t_1 and t_2 that correspond to the condition (V), which can happen only when the integrand itself is zero; i.e., Gauss's principle of least constraint will be true.

That proves the equivalence of the form (VII) and Gauss's principle of least constraint, and the form (VII) now represents the *desired integral form of that principle.*

One can also arrive at an extension of that result to the case of generalized coordinates, as well as the inclusion of non-mechanical processes, on the basis of lengthy and cumbersome developments by introducing the kinetic potential in a manner that is analogous to what **A. Wassmuth** ([1]) did for the principle of least action.

([1]) "Die Bewegungsgleichungen des Elektrons und das Prinzip der kleinsten Aktion," Wien. Ber. CXX, Abt. IIa (1911).