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Hamilton-Jacobi theory of forces whose measure depends upon the motion of the body

By

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As is known, it was first in the year 1834 that Hamilton published a new method for treating some mechanical problems by reducing the determination of the motion to the integration of a first-order partial differential equation and in that way arrived at an especially simple form for the differential equations for the elements of a motion that was acted upon by so-called perturbing forces. **Jacobi** summarized the basic ideas of that theory in a simpler form and generalized the applicability of the method, and in that way established a complete reshaping of the approach to those problems in that broader context, to which **Richelot**, **Liouville**, **Bertrand**, **Donkin**, and **Lipschitz** have added new discoveries.

In the present pages, I will present that method in such a form that I will consider the starting point for the problem to be the introduction of other variables that one can base the known differential equations of motion upon, which are such that the equations between them take on a simple form that is analogous to the one that they possessed originally. The condition equations for such a substitution can be represented in an especially simpler form when one generally appeals to different types of differentiation for the complete differentiation with respect to time, which represents an actual motion, and the variation, which represents a virtual motion. It was in my academic lectures in the Summer semester of 1862 that I first communicated the theory of these canonical substitutions and their application to the integration of the equations of motion for the effects of forces whose measure depends upon not only the mutual positions of the bodies, but also upon their changes in position, as well as the properties of the general equations for the variations of elements that are presented in Article IX, and then the equations that prove to special cases of the ones that were found by **Lagrange**, **Poisson**, **Hamilton**, and **Jacobi**.

In addition to those investigations, the following pages include a derivation of **Hamilton**'s equations from **Gauss**'s principle of least constraint. Another treatise will address the proof of the existence of a normal form for any canonical substitution in terms of only partially-given substitutions and the differential determinants of the canonical variables.

I. – Principle of least constraint.

Among the various fundamental laws of mechanics, **Gauss**'s principle of least constraint possesses several advantages. It takes exactly the same form for motion that it does for rest and for those conditions and restrictions on motion that might or might not possibly oppose any motion. It also suffices completely to determine the motion in all spaces in which the square of the element of length is represented by a homogeneous expression of degree two in the coordinate differentials that correspond to the element of length.

Gauss expressed his principle in the following form (v. 5 of his *Werke*):

> "The motion of a system of material points that are coupled in whatever way and whose motion is, at the same time, constrained by whatever sort of external restrictions will occur at each moment with the greatest possible agreement with the free motion, or the least possible constraint, when one considers the measure of that constraint that the entire system experiences at each point in time to be the sum of the products of the squares of the deviation of that point from its free motion with its mass."

The application of that fundamental law to the determination of the motion of bodies of stated kind then requires knowledge of the motion of an isolated free mass particle. The laws that are true for that come from the nature of the bodies and the effect of the forces that are present, so they are essentially physical. The assumptions that are most generally valid and most closely connected with the usual concept of force are the following two:

A free, isolated moving mass particle on which forces act that moves along a shortest line in space with unvarying velocity will describe equally-large path segments in equally-large time intervals.

A free, isolated moving particle with mass m that momentarily has no motion, but is under the influence of a force R will begin to move in the direction of the force R with an acceleration that is equal to R / m, so it will cover a path of $\frac{1}{2}(R / m)\, dt^2$ *in that direction during the next time element dt.*

Those two laws, in themselves, still do not determine the motion of a free mass particle under the assumption of an initial motion and the simultaneous effect of one or more forces, but those cases can be resolved with the assistance of the principle of least constraint in its most general interpretation. In the determination of the motion of a system, when one adds that principle, one will also be justified in replacing any given group of free motions with any other fictitious motions and the given conditions and restrictions for the motion of the system with other conceivable conditions such that the conditions will collectively continue to exist, and the motions of the total system that result from those fictitious free motions will be the same as the motions of the total system that result from that group of free motions. One of the most fruitful types of application of that process consists of imagining that the individual mass-particles m are once more decomposed into small mass-particles m_0, m_1, … such that $m = m_0 + m_1 + \ldots$, and one can add the new condition to the existing ones that m_0, m_1, … must remain inseparably coupled to each other. Any free motion that is immanent in the mass-particle m can then be replaced with an arbitrary well-defined free motion that can be ascribed to the particle – for example, m_0 .

Let the position of a point in space through which the motion goes be given by the values of the mutually-independent variables x_1, x_2, …, x_h, … Let the shortest lines be drawn from the point x_1, x_2, …, x_h, … to the point $x_1 + dx_1$, $x_2 + dx_2$, …, $x_h + dx_h$, … and to the point $x_1 + \delta x_1$, $x_2 + \delta x_2$, …, $x_h + \delta x_h$, …, and then construct the shortest line from the latter point to the first line, or by extension, the point at which it meets the latter. The shortest line that is drawn from the point x to that point of intersection is called the *projection* of the line that is drawn from x to $x + \delta x$ onto the line that is drawn from x to $x + dx$ and will considered to be positive when the projection and that line lie on the same side and negative when they lie on opposite sides. The product of the length of the projection times the length of that line will be denoted by $\mathfrak{D}$ and set equal to:

$$\sum_{h,k} X_{hk}\, dx_h\, \delta x_k \, ,$$

in which the X_{hk} depend upon the nature of the space and the chosen coordinates x_1, x_2, … and will generally satisfy the condition that $X_{hk} = X_{kh}$ and they are functions of x_1, x_2, … alone, but not dx_1, dx_2, …, δx_1, δx_2, …, and in which the summation $\sum$ is further extended over as many values 1, 2, 3, … of the indices h and k as the space has dimensions. If the point $x + dx$ coincides with $x + \delta x$ then that expression will go to:

$$\sum_{h,k} X_{hk}\, dx_h\, dx_k \, ,$$

which will be denoted by $\mathfrak{T}$, and shall mean the square of the length of the line that is drawn from the point x to $x + dx$, so it will always take on a positive value for arbitrary dx.

The length of a line whose points are given by the values of the x_1, x_2, …, x_h, … as functions of one independent variable is equal to:

$$\int \sqrt{\sum_{hk} X_{hk}\, dx_h\, dx_k} \, ,$$

so that integral must become a minimum for a shortest line that goes through two fixed points that correspond to the constant limiting values of the integral. If the variation δ denotes an arbitrary change in the functions x_1, x_2, …, x_h, … then a relation must exist between those variables for a shortest line such that $\delta\sqrt{\mathfrak{T}}$ will reduce to a complete differential d. Now when one takes the δ differentiation in the expression $\mathfrak{D}$ above to have the same sense as this variation:

$$\delta\sqrt{\mathfrak{T}} - d\,\frac{\mathfrak{D}}{\sqrt{\mathfrak{T}}} = \frac{\frac{1}{2}\delta\mathfrak{T} - d\mathfrak{D}}{\sqrt{\mathfrak{T}}} + \frac{\mathfrak{D}}{\mathfrak{T}}\,d\sqrt{\mathfrak{T}}$$

$$= \frac{1}{2\sqrt{\mathfrak{T}}}\sum_{h,k}\delta X_{hk}\cdot dx_h\cdot dx_k - \frac{1}{\sqrt{\mathfrak{T}}}\sum_{h,k} d(X_{hk}\,dx_h)\cdot\delta x_k + \frac{d\sqrt{\mathfrak{T}}}{\mathfrak{T}}\sum_{h,k} X_{hk}\,dx_h\,\delta x_k\,,$$

so that expression, which depends upon the δx, and no longer on its differentials $d\delta x$, and differs from $\delta\sqrt{\mathfrak{T}}$ by only a total differential, namely, $d\,\frac{\mathfrak{D}}{\sqrt{\mathfrak{T}}}$, must vanish for a shortest line.

If that is the path of freely-moving mass-particle, and one considers time to be the only independent variable in the d differentiation, while its differential dt is constant, then $\sqrt{\mathfrak{T}}$ will be equal to the velocity times dt, and $d\sqrt{\mathfrak{T}}$ will be equal to the acceleration times dt^2, so when the mass-particle moves freely without the influence of forces, from the fundamental law, one must have $d\sqrt{\mathfrak{T}} = 0$, and as a result of the equation above, one must then also have:

$$\tfrac{1}{2}\delta\mathfrak{T} - d\,\mathfrak{D} = 0$$

for any arbitrary system of values for the δx_1, δx_2, …, δx_h, …, and among others, for $\delta x_1 = d\,x_1$, $\delta x_k = d\,x_k$, $\delta x_h = d\,x_h$, as well, such that foregoing equation = 0 will again arise as a special case.

From the fundamental law, one can consider the motion of a mass-particle m that starts from rest and is provoked by a force R during the first time-element dt to initially coincide with the shortest line that is drawn from the point x to $x + dx$ when it has the same direction as the force R. It would emerge easily from the meaning of the notations that we have chosen here that this condition can be represented analytically by saying that $\mathfrak{D}/\sqrt{\mathfrak{T}}$ shall denote the length of the projection of a so-called virtual motion from the point x_1, x_2, x_3, … to an arbitrary infinitely-close point $x_1 + \delta x_1$, $x_2 + \delta x_2$, $x_3 + \delta x_3$, … in the direction of the force, so the motion that one cares to call the virtual motion δr that is performed from the mass-particle m in the direction of the force R. If one again takes time t to be the independent variable of the differentiation d and dt to be constant, moreover, then $\sqrt{\mathfrak{T}}$ will mean the product of the velocity times dt, so from the fundamental law, it will be zero at the onset of the motion $t = t_0$, which can only happen when the derivatives $\frac{dx_1}{dt}$, $\frac{dx_2}{dt}$, … vanish at that time-point. Under the same assumptions, $d\sqrt{\mathfrak{T}}$ will be the product of the acceleration times dt^2, so from the fundamental law, it will be equal to the value of $(R / m)\,dt^2$. If one multiplies the general equation for a shortest line above by $\sqrt{\mathfrak{T}}$ then one will get:

$$\tfrac{1}{2}\delta\mathfrak{T} - d\,\mathfrak{D} + \frac{\mathfrak{D}}{\sqrt{\mathfrak{T}}}\,d\sqrt{\mathfrak{T}} = -\sum_{h,k} X_{hk}\,ddx_h\cdot\delta x_k + \delta r\,\frac{R}{m}\,dt^2 = 0,$$

$$d\sqrt{\mathfrak{T}} = \frac{R}{m}dt^2,\ \mathfrak{T} = 0, \qquad \frac{dx_1}{dt} = 0,\ \frac{dx_2}{dt} = 0,\ \ldots,\ \frac{\mathfrak{D}}{\sqrt{\mathfrak{T}}} = \delta r, \qquad \text{for } t = t_0$$

and arbitrary δx_1, δx_2, … as the equations that determine the motion that starts from the state of rest at time $t = t_0$ and is provoked by the force R acting on the freely-moving mass-particle.

We now turn to the investigation of an arbitrary system of mass-particles and denote the coordinates of the isolated mass-particle m by x_1, x_2, x_3, … and consider the differentiation d to be with respect to dt, and indeed the change in the quantities that would actually arise as a consequence of motion. Any mass-particle m might possess an intrinsic motion by means of which, it would move from the point x_h to the point:

$$x_h + d_0\, x_h + \tfrac{1}{2} d_0\, d_0\, x_h + \frac{1}{1\cdot 2\cdot 3} d_0^3\, x_h + \ldots$$

if it were free at that time-point t and did not feel the effect of any force, and for which one has:

$$\tfrac{1}{2}\delta\mathfrak{T}_0 - d_0\,\mathfrak{D}_0 = 0$$

for a δx_h that is now arbitrary, namely, when $\mathfrak{T}_0$ and $\mathfrak{D}_0$ denote the same expressions that arise when the differentiation d is taken to mean d_0 , and indeed is again taken to be arbitrarily different for the different mass-particles m. A special group of forces R_i , R_n , … acts upon each mass-particle m that would move to the point:

$$x_h + d_i\, x_h + d_i\, d_i\, x_h + \frac{1}{1\cdot 2\cdot 3} d_i^3 x_h + \ldots$$

or

$$x_h + d_n\, x_h + d_n\, d_n\, x_h + \frac{1}{1\cdot 2\cdot 3} d_n^3 x_h + \ldots,$$

etc., in which:

$$\tfrac{1}{2}\delta\mathfrak{T}_i - d_i\,\mathfrak{D}_i + \frac{\mathfrak{D}_i}{\sqrt{\mathfrak{T}_i}} d_i\sqrt{\mathfrak{T}_i} = -\sum_{h,k} X_{hk}\, d_i d_i x_h \cdot \delta x_k + \delta r_i \frac{R_i}{m} dt^2 = 0,$$

$$\tfrac{1}{2}\delta\mathfrak{T}_n - d_n\,\mathfrak{D}_n + \frac{\mathfrak{D}_n}{\sqrt{\mathfrak{T}_n}} d_n\sqrt{\mathfrak{T}_n} = -\sum_{h,k} X_{hk}\, d_n\, d_n\, x_h \cdot \delta x_k + \delta r_n \frac{R_n}{m} dt^2 = 0,$$

when those forces act individually on m, and the latter is instantaneously in the rest state, but freely moving, and corresponding statements will be true for the remaining forces.

Any mass-particle m will be decomposed into smaller mass-particles m_0, m_1, … arbitrarily for each individual m, such that one must add to the original conditions the new one that the m_0, m_1, … that make up the components of a mass m must be rigidly coupled to each other. We would like to assume that the quantities $(m_i\ m_0)_h$ are determined in such a way that it would make no difference on the total motion whether the mass-point m possessed an intrinsic motion that would be given to the aforementioned point if it were free and exposed to no forces, or whether the components m_i, m_n, … possessed no intrinsic motion, but the component m_0 possessed one such that if it moved freely then it would arrive at the point:

$$x_h + (m, m_0)_h \left\{ d_0\, x_h + \tfrac{1}{2} d_0\, d_0\, x_h + \frac{1}{1\cdot 2\cdot 3} d_0^{\,3} x_h + \cdots \right\},$$

and furthermore, $(m, m_i)_h$, $(m, m_n)_h$, … might be determined to be such that the effects of the forces R_i, R_n on the masses could be replaced with forces that act upon the individual components m_i, m_n, … alone, such that it would arrive at the point:

$$x_h + (m, m_i)_h \left\{ d_i\, x_h + \tfrac{1}{2} d_i\, d_i\, x_h + \frac{1}{1\cdot 2\cdot 3} d_i^{\,3} x_h + \cdots \right\}$$

and the point

$$x_h + (m, m_n)_h \left\{ d_n\, x_h + \tfrac{1}{2} d_n\, d_n\, x_h + \frac{1}{1\cdot 2\cdot 3} d_n^{\,3} x_h + \cdots \right\}$$

under free motion.

The motion that is actually performed then takes each component m_0, m_i, m_n of the mass m from the point x_h to:

$$x_h + d\, x_h + \tfrac{1}{2} d\, dx_h + \frac{1}{1\cdot 2\cdot 3} d^{\,3} x_h + \ldots,$$

so we can then represent any other position of the system that is compatible with the internal coupling of the masses and with the conditions and external restrictions in such a way that the mass m, and therefore each of its components m_0, m_i, m_n, …, will assume the position:

$$x_h + \delta x_h + dx_h + \tfrac{1}{2} d\, dx_h + \frac{1}{1\cdot 2\cdot 3} d^{\,3} x_h + \ldots,$$

such that because δ and d mean infinitely-small changes, the position:

$$x_h + \delta x_h$$

will also be compatible with the conditions for the mass m. Those differentials of the coordinates that correspond to any possible deviation from the free motion of the particle will then be:

$$(m, m_0)_h\ (d_0\, x_h + \tfrac{1}{2} d_0\, d_0\, x_h + \cdots) - (\delta x_h + dx_h + \tfrac{1}{2} d\, dx_h + \cdots)$$

for m_0, and:

$$(m, m_i)_h\ (d_i\, x_h + \tfrac{1}{2} d_i\, d_i\, x_h + \cdots) - (\delta x_h + dx_h + \tfrac{1}{2} d\, dx_h + \cdots)$$

for m_i , and so forth, so from the assumption that was made for space and those coordinates, the square of that deviation will be:

$$\sum_{h,k} X_{hk} \,\{(m, m_0)_h \; (d_0\, x_h + \tfrac{1}{2} d_0\, d_0\, x_h + \cdots) - (\delta x_h + dx_h + \tfrac{1}{2} d\; dx_h + \cdots)\}$$
$$\times \{(m, m_0)_k \; (d_0\, x_k + \tfrac{1}{2} d_0\, d_0\, x_k + \cdots) - (\delta x_k + dx_k + \tfrac{1}{2} d\; dx_k + \cdots)\} \}$$

for m_0 , and:

$$\sum_{h,k} X_{hk} \,\{(m, m_i)_h \; (d_i\, x_h + \tfrac{1}{2} d_i\, d_i\, x_h + \cdots) - (\delta x_h + dx_h + \tfrac{1}{2} d\; dx_h + \cdots)\}$$
$$\times \{(m, m_i)_k \; (d_i\, x_k + \tfrac{1}{2} d_i\, d_i\, x_k + \cdots) - (\delta x_k + dx_k + \tfrac{1}{2} d\; dx_k + \cdots)\} \}$$

for m_i , so the measure of the constraint for that motion is equal to:

$$\sum_m [m_0 \sum_{h,k} X_{hk} \,\{(m, m_0)_h \; (d_0\, x_h + \tfrac{1}{2} d_0\, d_0\, x_h + \cdots) - (\delta x_h + dx_h + \tfrac{1}{2} d\; dx_h + \cdots)\}$$
$$\times \{(m, m_0)_k \; (d_0\, x_k + \tfrac{1}{2} d_0\, d_0\, x_k + \cdots) - (\delta x_k + dx_k + \tfrac{1}{2} d\; dx_k + \cdots)\} \}$$
$$+ \; m_i \sum_{h,k} X_{hk} \,\{(m, m_i)_h \; (d_i\, x_h + \tfrac{1}{2} d_i\, d_i\, x_h + \cdots) - (\delta x_h + dx_h + \tfrac{1}{2} d\; dx_h + \cdots)\}$$
$$\times \{(m, m_i)_k \; (d_i\, x_k + \tfrac{1}{2} d_i\, d_i\, x_k + \cdots) - (\delta x_k + dx_k + \tfrac{1}{2} d\; dx_k + \cdots)\} \}$$
$$+ \text{ etc.}]$$

From **Gauss**'s principle, the dx_h and ddx_h , … are to be determined in such a way that among all possible values of the δx_h , this expression will assume its smallest value for $\delta x_1 = 0$, $\delta x_2 = 0$, …, $\delta x_h = 0$, so when one develops that expression in powers of δx, the sum of the linear terms that this yields, namely:

$$-2 \sum_m \sum_{h,k} X_{hk} \,\{m_0 \,(m, m_0)_h \; (d_0\, x_h + \tfrac{1}{2} d_0\, d_0\, x_h + \cdots)$$
$$+ \, m_i \,(m, m_i)_k \; (d_i\, x_k + \tfrac{1}{2} d_i\, d_i\, x_k + \cdots) - m\,(dx_k + \tfrac{1}{2} d\; dx_k + \cdots)\} \delta x_h \},$$

in which $m_0 + m_i + m_n + \ldots$ is replaced with m, will never be negative.

One obtains the still-unknown $(m, m_0)_h$, $(m, m_i)_h$, … by considering the fact that if the mass-point m, which consists of m_0 , m_i , m_n , …, were free and acted upon by no forces then it would move to:

$$x_h + dx_h + \tfrac{1}{2} d\; dx_h + \frac{1}{1\cdot 2\cdot 3} d^3 x_h + \ldots = x_h + d_0\, x_h + \tfrac{1}{2} d_0\; d_0\, x_h + \frac{1}{1\cdot 2\cdot 3} d_0^3 x_h + \ldots,$$

and if m_0, along with its rigidly-coupled components, moves freely from there then it must also arrive at:

$$x_h + (m, m_0)\, (d_0\, x_h + \tfrac{1}{2} d_0\; d_0\, x_h + \frac{1}{1\cdot 2\cdot 3} d_0^3 x_h + \ldots),$$

while none of the remaining components possess free motions, so one must have:

$$d_i x_h + \tfrac{1}{2} d_i\, d_i x_h + \frac{1}{1\cdot 2\cdot 3} d_i^{\,3} x_h + \ldots = 0 \qquad \text{for all } h\,.$$

From the principle of least constraint:

$$-2\sum_{h,k} X_{hk}\ \{m_0\,(m, m_0)_h\ (d_0\, x_h + \tfrac{1}{2} d_0\, d_0\, x_h + \cdots) - m\,(dx_k + \tfrac{1}{2} d\, dx_k + \cdots)\}\delta x_h$$

cannot become negative for any system of values $\pm\,\delta x_1$, …, $\pm\,\delta x_1$, …, so it must be equal to zero. The factor of $X_{hk}\ \delta x_k$ in it is only a special value of δx_k , so it must be equal to zero, since the sum is proportional to the square of a length-element in space for that special case, so when one refers to the equation above, that will imply the relation:

$$\begin{aligned} m_0\,(m, m_0)_h\ (d_0\, x_h + \tfrac{1}{2} d_0\, d_0\, x_h + \cdots) &= m\,(dx_k + \tfrac{1}{2} d\, dx_k + \cdots) \\ &= m\,(d_0\, x_h + \tfrac{1}{2} d_0\, d_0\, x_h + \cdots)\,, \end{aligned}$$

so one will have:

$$(m, m_0)_h = \frac{m}{m_0}\,.$$

The same argument will imply that:

$$(m, m_i)_h = \frac{m}{m_i}, \qquad (m, m_n)_h = \frac{m}{m_n}, \ \ldots$$

When one substitutes those values, the sum of the terms that are linear in δx in the measure of the constraint will assume the form:

$$\begin{aligned} -2\sum_m m \sum_{h,k} X_{hk}\ \{&(d_0\, x_h + \tfrac{1}{2} d_0\, d_0\, x_h + \cdots) + (d_i\, x_h + \tfrac{1}{2} d_i\, d_i\, x_h + \cdots) + \ldots \\ &- (dx_k + \tfrac{1}{2} d\, dx_k + \cdots)\}\delta x_h\,. \end{aligned}$$

One still has $\dfrac{d_i\, x_h}{dt} = 0$, $\dfrac{d_n\, x_h}{dt} = 0$ for all indices h in that expression. If there are no such internal couplings of the masses and external restrictions on the motion that would give rise to a discontinuity in the magnitude or direction of motion then one will have:

$$d_0\, x_h = dx_h$$

for all indices h and all mass-particles m.

The part of the measure of the constraint that is linear in δx will then reduce to:

$$-2\sum_m m \sum_{h,k} X_{hk}\,(\tfrac{1}{2} d_0\, d_0 x_h + \tfrac{1}{2} d_i\, d_i x_h + \tfrac{1}{2} d_n\, d_n x_h + \cdots - \tfrac{1}{2} d\, dx_h)\,\delta x_k\,,$$

or when one considers that, from the above, one has:

$$\tfrac{1}{2}\delta\mathfrak{T} - d_0\mathfrak{D}_0 = \tfrac{1}{2}\sum_{h,k}\delta X_{hk} d_0 x_h \cdot d_0 x_h - \sum_{h,k} d_0 X_{hk} \cdot d_0 x_h \cdot \delta x_k - \sum_{h,k} X_{hk} d_0\, d_0 x_h \cdot \delta x_k = 0,$$

$$\tfrac{1}{2}\delta\mathfrak{T}_i - d_i\mathfrak{D}_i + \frac{\mathfrak{D}_i}{\sqrt{\mathfrak{T}_i}} d_i\sqrt{\mathfrak{T}_i} = -\sum_{h,k} X_{hk} d_i\, d_i x_h \cdot \delta x_k + \delta r_i \frac{R_i}{m} dt^2 = 0,$$

$$\tfrac{1}{2}\delta\mathfrak{T}_n - d_n\mathfrak{D}_n + \frac{\mathfrak{D}_n}{\sqrt{\mathfrak{T}_n}} d_n\sqrt{\mathfrak{T}_n} = -\sum_{h,k} X_{hk} d_n\, d_n x_h \cdot \delta x_k + \delta r_n \frac{R_n}{m} dt^2 = 0,$$

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$$\begin{aligned}\tfrac{1}{2}\delta\mathfrak{T} - d\mathfrak{D} &= \tfrac{1}{2}\sum_{h,k}\delta X_{hk} dx_h \cdot dx_h - \sum_{h,k} dX_{hk} \cdot dx_h \cdot \delta x_k - \sum_{h,k} X_{hk} d\, dx_h \cdot \delta x_k \\ &= \tfrac{1}{2}\sum_{h,k}\delta X_{hk} d_0 x_h \cdot d_0 x_h - \sum_{h,k} d_0 X_{hk} \cdot d_0 x_h \cdot \delta x_k - \sum_{h,k} X_{hk} d\, dx_h \cdot \delta x_k ,\end{aligned}$$

it will reduce to:

$$\sum_m m\,(\tfrac{1}{2}\delta\mathfrak{T} - d\mathfrak{D}) - \sum_i R_i\,\delta r_i\, dt^2$$

or

$$-\delta\sum_m \tfrac{1}{2} m \sum_{h,k} X_{hk} dx_h \cdot dx_h + d\sum_m n \sum_{h,k} X_{hk} dx_h \cdot \delta x_k - \sum_i R_i\,\delta r_i\, dt^2 ,$$

which is an expression that must never be negative then for a continuous motion of the mass-particle m, namely, if x_h are the coordinates of m at time t, while:

$$x_h + dx_h + \tfrac{1}{2} d\, dx_h + \frac{1}{1\cdot 2\cdot 3} d^3 x_h + \ldots$$

are the coordinates of m at time $t + dt$, and:

$$x_h + \delta x_h$$

mean the coordinates of m that determine a position of that mass-particle that is possible under its given internal couplings and the external conditions and restrictions on the motion of the system. The R_i mean all of the forces that act upon the mass-particles, and the δr_i mean the virtual motion that the point of application of R_i would describe in the direction of that force for the virtual motions $\delta x_1, \delta x_2, \ldots, \delta x_h, \ldots$

For a space with the property that under an n-fold extension, the length element in it can be represented by the ν^{th} root of an irreducible expression that is homogeneous of degree ν in the differential of the n coordinates, namely:

$$\sqrt[\nu]{\mathfrak{T}} = \sqrt[\nu]{\sum_h X_{h_1 h_2 \cdots h_\nu} dx_{h_1} \cdot dx_{h_2} \cdots dx_{h_\nu}} ,$$

the way that one determines the motion will have its closest analogy with the one that was just considered when one denotes the actual forces by R_i and lets $\mathfrak{D}$ denote the ν-fold sum:

$$\sum_h X_{h_1 h_2 \cdots h_\nu} \delta x_{h_1} \cdot dx_{h_2} \cdots dx_{h_\nu} ,$$

which extends over all indices $h_1, \ldots, h_\nu$ that are taken from the sequence 1, 2, 3, …, n, and one does not let the expression:

$$-\sum_m m\left(\tfrac{1}{\nu}\delta\mathfrak{T} - d\mathfrak{D}\right) - \sum_i R_i\, \delta r_i\, dt^\nu \tag{1}$$

become negative for any virtual motion δx_1, δx_2, …, δx_n of the mass-particle m that is compatible with the given restrictions. Here as well, the free motion of a mass-particle m on which no forces act would take place with equal velocity along a shortest line, because one has:

$$\delta\sqrt[\nu]{\mathfrak{T}} - d(\mathfrak{D}\cdot\mathfrak{T}^{1/\nu-1}) = \left(\tfrac{1}{\nu}\delta\mathfrak{T} - d\mathfrak{D}\right)\mathfrak{T}^{1/\nu-1} + (\nu-1)\mathfrak{D}\cdot\mathfrak{T}^{-1}\cdot d\sqrt[\nu]{\mathfrak{T}} .$$

If the conditions on the motion that are given are such that the opposite of any possible motion is also possible then the expression (1) above, which contains opposite signs in the two cases and can never be negative, must be equal to zero.

II. – Force function.

The first two terms in the expression [1] above are related to each other in such a way that the first term, taken with the opposite sign, $\frac{1}{\nu}\delta\mathfrak{T}$, contains the complete δ variation that appears in the second term $d\,\mathfrak{D}$ after one performs the suggested differentiation, and conversely, the second term contains the complete d differentiation that appears in the first term (when taken with the opposite sign) after one performs the δ variation. Each of the two terms is already determined by the other one with that rule for forming the terms.

If one denotes all coordinates x_h of all mass-particles m by ξ_1, ξ_2, …, ξ_l, …, and one sets:

$$\frac{dx_h}{dt} = x'_h, \qquad \frac{d\xi_l}{dt} = \xi'_l ,$$

in general, and sets the quantity that **Leibnitz** called the *vis viva* for the case $\nu = 2$ equal to:

$$\sum_m m \sum_h X_{h_1 h_2 \cdots h_\nu} x'_{h_1} x'_{h_2} \cdots x'_{h_\nu} = \nu\, T$$

then the basic equation will be:

$$-\delta T + \frac{d}{dt}\sum_l \frac{\vartheta T}{\vartheta \xi_l'}\delta\xi_l - \sum_i R_i\,\delta p_i = 0,$$

when the partial differentiation ϑ of a function considers the quantities ξ and ξ_l' to be mutually independent, and the ξ_l in the sum have been set equal to all coordinates of all mass-particles *m*, in succession.

The basic equation for motion will then take on an especially simple form when the last term $\sum_i R_i\,\delta r_i\,dt^\nu$ can also be represented in the form of the difference between a total variation and a total differential. As **Lagrange** first pointed out, for most of the forces in nature, $\sum_i R_i\,\delta r_i$ is the total variation of a function that depends upon only the coordinates of the mass-particle *m*, and not on its state of motion, so either the variation of that function will not contain any differential or it will be set equal to zero.

Gauss first considered forces whose measure depends upon not only the position of the mass-particle *m*, but also on its state of motion. For our further investigations, we would like to assume that this dependency is such that:

$$\sum_i R_i\,\delta r_i$$

is the difference between a total variation and a total derivative with respect to time. If the total variation is:

$$= \delta V$$

then the total derivative must be:

$$\frac{d}{dt}\left\{\sum_l \frac{\vartheta V}{\vartheta \xi_l'}\delta\xi_l + \sum_l \frac{\vartheta V}{\vartheta \xi_l''}\delta\xi_l' + \cdots\right\},$$

in which $\xi_l'' = \dfrac{dd\xi_l}{dt}$, etc. The quantity *V* might be called the "potential" for the given forces under the motion of a system, as a generalization of the name that **Gauss** introduced, or the "force function," as a generalization of **Hamilton**'s terminology. We would like to restrict our examination to the case in which *V* contains no derivatives that are higher than the first ξ_l', such that we will then have:

$$\sum_i R_i\,\delta r_i = \delta V - \frac{d}{dt}\sum_l \frac{\theta V}{\theta \xi_l'}\delta\xi_l\,,$$

and the fundamental equation (1) of motion will assume the form:

$$-\delta(T+V)+\frac{d}{dt}\sum_l \frac{\vartheta(T+V)}{\vartheta\xi_l'}\delta\xi_l = 0. \tag{2}$$

The expression:

$$\sum_l \frac{\vartheta(T+V)}{\vartheta\xi_l'}\delta\xi_l$$

in this equation possesses the property that its value will remain unchanged, which can also be based on coordinates ξ that are fixed or moving in space and dependent or independent of each other.

Namely, if $q_1, q_2, \ldots, q_\lambda, \ldots$ denote mutually-independent variables then one must be able to represent $\ldots, \xi_l, \ldots$ as functions of t and the q, so one must have:

$$\frac{d\xi_l}{dt} = \frac{\partial\xi_l}{\partial t}+\sum_h \frac{\partial\xi_l}{\partial q_h}\frac{dq_h}{dt} \qquad \text{or} \qquad \xi_l' = \frac{\partial\xi_l}{\partial t}+\sum_h \frac{\partial\xi_l}{\partial q_h}q_h',$$

in which ∂ denotes the partial differentiations with respect to t and the q, and $\frac{\partial\xi_l}{\partial t}$, $\frac{\partial\xi_l}{\partial q_h}$ are independent of all $\ldots, q_h', \ldots$, such that one will have:

$$\frac{\partial\xi_l'}{\partial q_h'} = \frac{\partial\xi_l}{\partial q_h},$$

in general, and in that way one will have:

$$\sum_l \frac{\vartheta(T+V)}{\vartheta\xi_l'}\delta\xi_l = \sum_l \frac{\vartheta(T+V)}{\vartheta\xi_l'}\sum_h \frac{\partial\xi_l}{\partial q_h}\delta q_h = \sum_l \frac{\vartheta(T+V)}{\vartheta\xi_l'}\sum_h \frac{\partial\xi_l'}{\partial q_h'}\delta q_h = \sum_l \frac{\vartheta(T+V)}{\vartheta q_l'}\delta q_l,$$

as will be proved.

If we now set:

$$\frac{\vartheta(T+V)}{\vartheta q_l'} = p_l, \tag{3}$$

followinh the path that was first taken by **Lagrange**, then equation (1) will become:

$$0 = -\delta(T+V)+\frac{d}{dt}\sum_l \frac{\vartheta(T+V)}{\vartheta q_l'}\delta q_l$$

$$= -\sum \frac{\vartheta(T+V)}{\vartheta q_l}\delta q_l - \sum \frac{\vartheta(T+V)}{\vartheta q_l'}\delta q_l' + \sum \frac{d}{dt}\left[\frac{\vartheta(T+V)}{\vartheta q_l'}\right]\cdot\delta q_l + \sum \frac{\vartheta(T+V)}{\vartheta q_l'}\frac{d\delta q_l}{dt}$$

$$= \sum \left\{ \frac{\vartheta(T+V)}{\vartheta q_l} + \frac{d}{dt}\left[\frac{\vartheta(T+V)}{\vartheta q'_l}\right]\right\} \delta q_l = \sum \left\{ \frac{dp_l}{dt} - \left[\frac{\vartheta(T+V)}{\vartheta q_l}\right]\right\} \delta q_l \,, \tag{4}$$

in which the summations extend over all values 1, 2, 3, …, n of the only index l that appears in the expression, and in which we shall denote the number of variable quantities q by n from now on.

III. – General differentials.

The study of many remarkable properties of the function $T + V$ will be simplified considerably when one introduces the concept of a general differential D, in the sense that it represents any sort of changes in a function and the quantities that enter into it that are required by the form of that function such that when the integral equations for the function and its argument that the given differential equations satisfy are added, the integration constants must also be subjected to that general differentiation.

The variation δ that was used up to now, which means an arbitrary virtual motion, is a more general differentiation then the so-called complete differentiation with respect to time t, but of the general differentiations, it encompasses only the ones for which the coordinates experience an infinitely-small change that is compatible with the given conditions.

After one introduces the quantities q, which determine the position of the system of moving masses at the time t, and might be called the coordinates in the general sense for that reason, the function $T + V$, which will be initially given as a function of t, …, q_l , …, q'_l , so when we once more denote partial differentiation with respect to those quantities by q, the general differential will become:

$$D\,(T + V) = \frac{\vartheta(T+V)}{\vartheta t} Dt + \sum \frac{\vartheta(T+V)}{\vartheta q_l} Dq_l + \sum \frac{\vartheta(T+V)}{\vartheta q'_l} Dq'_l \,,$$

or when one recalls the differential equations (3) for the p and the equation of motion (4) that was just found:

$$D\,(T + V) = \frac{\vartheta(T+V)}{\vartheta t} Dt + \sum p'_l \, Dq_l + \sum p_l \, Dq'_l \,,$$

in which the Dq'_l and Dt mean completely-independent differentials, while Dq_1, Dq_2, …, Dq_n must satisfy the restrictions that are given for the motion.

The two differentiations D and d that enter here are mutually independent in their sequence, so then can be switched, and in that way the last equation will imply that:

$$D\,(T + V) = \frac{\vartheta(T+V)}{\vartheta t} Dt + \frac{d}{dt} \sum p_l \, Dq_l \,.$$

If one takes the general differentiation D in this to have the special sense of complete differentiation d with respect to t, and one divides the equation that arises in that way by the factor dt, which is constant for the complete differentiation d with respect to time t then one will have:

$$\frac{d(T+V)}{dt} = \frac{\vartheta(T+V)}{\vartheta t} + \frac{d}{dt}\sum p_l\, Dq'_l\,,$$

and when one substitutes the value of the partial derivative of $T + V$ with respect to t that this yields, the general equation will go to:

$$D\,(T+V) = \frac{d}{dt}\{(T+V-\sum p_l\, q'_l)Dt + \sum p_l\, Dq_l\} \tag{5}$$

or

$$D\,(T+V) = \frac{d}{dt}(T+V-\sum p_l\, q'_l)Dt + \sum p'_l\, Dq_l + \sum p_l\, Dq'_l\,, \tag{6}$$

and under special assumptions on the general differentiations Dt, …, Dq_l , …, Dq'_l , …, the defining equations above (3) for the p will yield the equations of motion [4] and the value of $\vartheta(T+V) / \vartheta t$ that was found before.

If one subtracts from the two sides of that equation, the corresponding sides of the identity equation:

$$D\sum p_l\, q'_l = \sum q'_l\, Dp_l + \sum p_l\, Dq'_l\,,$$

then that will give:

$$D(T+V-\sum p_l\, q'_l) = \frac{d}{dt}(T+V-\sum p_l\, q'_l)\cdot Dt + \sum p_l\, Dq'_l - \sum q'_l\, Dp_l\,,$$

or, when one sets:

$$-T-V+\sum p_l\, q'_l = -(T+V) + \sum q'_l\,\frac{\vartheta(T+V)}{\vartheta q'_l} = H \tag{7}$$

and

$$\frac{dH}{dt} = H',$$

to abbreviate, that will give:

$$DH = H'Dt - \sum p'_l\, Dq_l + \sum q'_l\, Dp_l\,. \tag{8}$$

For the case in which the variables q are mutually independent in regard to the internal couplings and the given external restrictions, when one thinks of the quantities q' in the expression above for H, which **Jacobi** called the **Hamiltonian** function, as being determined by t, …, q_l, …, p_l, … with the help of the defining equation (3) for the p, and one denotes the partial differentiation with respect to the latter variables by ∂, those equations will contain the following equations that **Hamilton** presented:

$$\frac{\partial H}{\partial p_l} = q_l' = \frac{dq_l}{dt},$$

$$-\frac{\partial H}{\partial q_l} = p_l' = \frac{dp_l}{dt} = \frac{\vartheta(T+V)}{\vartheta q_l}, \qquad (8^*)$$

$$\frac{\partial H}{\partial t} = H' = \frac{dH}{dt} = -\frac{\vartheta(T+V)}{\vartheta t}$$

as special cases under special assumptions that pertain to how they are defined here.

The general differential of $T + V$ is represented by a complete derivative with respect to t in (6) above, but if one now restricts the meaning of that general differentiation to that of a variation then that will imply the generalized **Hamilton** theorem:

$$0 = \delta\int (T+V)\,dt = \delta\int\left(\sum p_l \frac{dq_l}{dt} - H\right)dt,$$

namely, when the values of the quantities at the limits of this **Hamiltonian** integral are assumed to be unvarying. Upon performing the variation, one will get:

$$0 = \delta\int (T+V)\,dt = \int \frac{d}{dt}\left(\sum \frac{\vartheta(T+V)}{\vartheta q_l'}\delta q_l\right)dt + \int\sum\left\{\frac{\vartheta(T+V)}{\vartheta q_l'} - \frac{d}{dt}\left[\frac{\vartheta(T+V)}{\vartheta q_l'}\right]\right\}\delta q_l\,dt,$$

such that the equations of motion that were exhibited before will once more follow from the condition of the vanishing of the variation.

In his "Untersuchung eines Problem der Variationsrechnung, in welchem das Problem der Mechanik enhalten ist" Borchardt's Journal, Bd. 74, **Lipschitz** took that generalization of **Hamilton**'s theorem to be the basis for determining the motion when the motion takes place under the influence of forces that depend upon the position, and not the evolution, of the system and possess a force function V, and when space is further thought of as being constructed in such a way that its element of length is represented by the ν^{th} root of a homogeneous expression of degree ν in the coordinate differentials.

If follows from the equation $\dfrac{dH}{dt} = \dfrac{\theta(T+V)}{\theta t}$ that when $T + V$ does not contain the quantity t explicitly along with the quantities q and q':

$$\sum p_l\, q_l' - (T + V) = H = \text{const.}$$

will be an integral of equations (8^*) for the motion of the system, and that will define a generalization of the principle of the conservation of *vis viva* that **Johann Bernoulli** first found.

If the q are fixed coordinates in space then T will not contain time t explicitly, so in that case, one needs only for the potential V to not contain time t explicitly in order for the integral above to be valid.

If the potential V is independent of the motion (so it does not contain q') then …, q_l, … will be fixed coordinates in space, and when one applies **Euler**'s theorem to T as a homogeneous function of degree ν in the quantities q', one will get:

$$\sum_l p_l\, q_l' = \sum_l q_l' \frac{\vartheta(T+V)}{\vartheta q_l'} = \sum_l q_l' \frac{\vartheta T}{\vartheta q_l'} = \nu\, T.$$

If the potential V does not contain time t explicitly, either, so H = constant, then one will have:

$$\int \sum (p_l\, q_l' - H)\, dt = \int \nu\, T\, dt - H \int dt$$
$$= \int \sum m_i\, v_i^{\nu}\, dt - H \int dt = \int \sum m_i\, v_i^{\nu-1} ds_i - H \int dt\,,$$

when ds_i or $v_i\, dt$ means the path that is traversed by the mass-particle m_i during the time dt. Since the variation of the first term in this equation vanishes, from the aforementioned generalized **Hamilton** theorem, the variation of $\int \sum m_i\, v_i^{\nu-1} ds_i$ must also become zero with the aid of the integral equation H = const., as **Maupertuis**'s principle of least effort would require for $\nu = 2$.

Under the assumptions that were made here, and for $\nu = 2$, one will also get the principle of the conservation of *vis viva*:

$$\text{const.} = H = \sum p_l\, q_l' - T - V = (n-1)\, T - V = T - V\,.$$

One can add two more systems of differential equations to the two systems that were exhibited above. If one subtracts equation (6), after introducing the function H in equation (7), namely:

$$D\,(T + V) = -\sum p_l'\, Dq_l + \sum p_l\, Dq_l' + \sum q_l'\, Dp_l + \sum q_l\, Dp_l'\,,$$

from the identity equation:

$$D \frac{d}{dt} \sum p_l\, q_l = \sum p_l'\, Dq_l + \sum p_l\, Dq_l' + \sum q_l'\, Dp_l + \sum q_l\, Dp_l'$$

then that will give:

$$D\left(\frac{d}{dt} \sum p_l\, q_l - T - V\right) = H'\, Dt + \sum q_l'\, Dp_l + \sum q_l\, Dp_l'\,,$$

so $\frac{d}{dt}\sum p_l\, q_l - T - V$ will be represented as a function of the variables t, p_1, …, p_n, p_1', …, p_n', and its partial derivatives with respect to those variables will be equal to H', q_1', …, q_n', q_1, …, q_n, respectively.

If one subtracts the same equation (6) from the identity equation:

$$D\sum p_l'\, q_l = \sum p_l'\, Dq_l + \sum q_l\, Dp_l'$$

then that will give:

$$D(\sum p_l'\, q_l - T - V) = H'\cdot Dt + \sum q_l\, Dp_l' - \sum p_l\, Dq_l',$$

so $\sum p_l'\, q_l - T - V$ will then be represented as a function of the variables t, p_1', …, p_n', q_1', …, q_n', and its partial derivatives with respect to those variables will be equal to:

$$H,\ q_1,\ \ldots,\ q_n,\ -p_1,\ \ldots,\ -p_n\,.$$

IV. – Substitution function. Integration. Perturbation theory.

The especially simple form of the differential equations that are presented by a mechanical problem comes from the fact that a suitable system of variables …, p_l, … was introduced for a system of independent coordinates …, q_l, …, and indeed the original …, q_l, … can be chosen entirely arbitrarily, so there will always be associated …, p_l … However, systems of associated variables can also be found in an even more general way that have the property that they give that simple form to the differential equations, and for that reason, **Jacobi** gave them the name of *canonical variables*. In fact, the equation:

$$D\,(T + V) = \frac{d}{dt}\{(T + V - \sum p_l \frac{dq_l}{dt})Dt + \sum p_l\, Dq_l\},$$

which includes all of the remaining ones, shows that if the φ and ψ are to define a new system of independent canonical variables, instead of the p and q, then it would only be necessary for the function $T + V$ to either be the same function, but expressed in terms of t, φ, ψ after replacing the p and q with the φ and ψ in that equation, or set equal to a new function. We can give that new function the form $T + V - S$, in which S remains to be determined more precisely, and we will then get:

$$D\,(T + V - S') = \frac{d}{dt}\{(T + V - S' - \sum \varphi_l \frac{d\psi_l}{dt})Dt + \sum \varphi_l\, D\psi_l\},$$

and after subtracting that equation from the foregoing, we will get:

$$DS' = \frac{d}{dt}\{(S' + \sum \varphi_l \frac{d\psi_l}{dt} - \sum p_l \frac{dq_l}{dt})Dt - \sum \varphi_l\, D\psi_l + \sum p_l\, Dq_l\}\,.$$

Should that equation for the substitution of canonical variables φ_l, ψ_l for the p_l, q_l be true in general, that is, independently of the special equations for a certain mechanical problem, then since the one side is a complete derivative with respect to time t, the other one DS', and therefore S', must also be so. There must then be a function S that fulfills the equations:

$$\frac{dS}{dt} = S',$$

$$DS = \left(\frac{dS}{dt} + \sum \varphi_l \frac{D\psi_l}{dt} - \sum p_l \frac{Dq_l}{dt}\right)Dt - \sum \varphi_l\, D\psi_l + \sum p_l\, Dq_l\,, \tag{9}$$

$$DS = -E\,Dt + \sum p_l\, Dq_l - \sum \varphi_l\, D\psi_l\,,$$

in which one sets:

$$E = \sum p_l \frac{dq_l}{dt} - \sum \varphi_l \frac{d\psi_l}{dt} - \frac{dS}{dt}\,. \tag{10}$$

Conversely, if equation (9) is satisfied for arbitrary functions S and E then the variables ψ and φ that were introduced will be a canonical system, since equation (10) will follow from (9) as a special case of D differentiation, and the fundamental equation that was exhibited above for …, ψ_l, …, φ_l, … will arise from both of the fundamental equations (6) for …, q_l, …, p_l, …, which can also be presented in the form:

$$D\,(T + V - S') = \frac{d}{dt}(T + V - S' - \sum \varphi_l\, \psi'_l) \cdot Dt + \sum \varphi'_l\, D\psi_l + \sum \varphi_l\, D\psi'_l$$

or

$$D(T + V - S' - \sum \varphi_l\, \psi'_l) = \frac{d}{dt}(T + V - S' - \sum \varphi_l\, \psi'_l) \cdot Dt + \sum \varphi'_l\, D\psi_l - \sum \psi'_l\, D\varphi_l$$

$$= -D\,(H - E) = (H' - E')\,Dt - \sum \psi'_l\, D\varphi_l + \sum \varphi'_l\, D\psi_l\,. \tag{11}$$

It follows from the first of those two equations that when $T + V - S'$ is regarded as a function of t, ψ'_l, ψ_l, its partial derivatives with respect to those quantities will be equal to $-H' + E'$, φ_l, φ'_l. If one considers $T + V - S' - \sum \varphi_l\, \psi'_l$ or $-H + E$ to be a function of t, φ_l, ψ_l, and denotes the partial derivatives with respect to those variables by ϑ then one will obtain from the second (11) of those two equations that:

$$\frac{\vartheta(H - E)}{\vartheta\varphi_l} = \psi'_l = \frac{d\psi_l}{dt},$$

$$-\frac{\vartheta(H-E)}{\vartheta\psi_l} = \varphi'_l = \frac{d\varphi_l}{dt}, \tag{12}$$

$$\frac{\vartheta(H-E)}{\vartheta t} = H' - E' = \frac{d(H-E)}{dt}.$$

The fundamental equation of motion, the equation of substitution, and the equation of motion that is transformed in that way agree in form in such a way that the general relations that exist between the quantities $q_1, \ldots, q_n, p_1, \ldots, p_n, \psi_1, \ldots, \psi_n, \varphi_1, \ldots, \varphi_n$ alone, and which will be developed more thoroughly in the following articles, will also exist between the quantities $q_1, \ldots, q_n, p'_1, \ldots, p'_n, q'_1, \ldots, q'_n, -p_1, \ldots, -p_n$, and likewise between $q_1, \ldots, q_n, p'_1, \ldots, p'_n, p_1, \ldots, p_n, q'_1, \ldots, q'_n$, and furthermore between $\psi_1, \ldots, \psi_n, \varphi'_1, \ldots, \varphi'_n, \varphi_1, \ldots, \varphi_n, \psi'_1, \ldots, \psi'_n$, and so forth.

The general equation of substitution includes the special case in which the quantities ψ, like the q, have the meaning of coordinates in such a form that S, as well as S', will be zero, and that furthermore the quantities q will be given as functions of t and the ψ, and indeed in such a way that they will be independent of how the ψ are represented as functions of t and q, and that finally, the quantities E and φ are determined by the equation of substitution.

Another very general, and especially important, type of substitution is the one for which the relations between the two systems of variables can be represented in such a way that the p will become functions of the quantities t, q, ψ. All remaining quantities can also be determined by the latter then when one substitutes the expressions that are obtained for p_l. Due to the importance of that kind of representation of the various variables, we would like to introduce a special symbol for the partial derivatives with respect to t, q, ψ, namely, δ, since that differentiation will include the variation that was considered above as a special case. The general equation of substitution will then give:

$$\frac{\delta S}{\delta q_l} = p_l, \qquad \frac{\delta S}{\delta \psi_l} = -\varphi_l, \qquad \frac{\delta S}{\delta t} = -E = \frac{dS}{dt} - \sum \varphi_l \frac{d\psi_l}{dt} + \sum p_l \frac{dq_l}{dt},$$

and it is clear from this how *when the* $p_1, \ldots, p_n$ *are given as functions of* $t, q_1, \ldots, q_n, \psi_1, \ldots, \psi_n$ *in such a way that they can be the partial derivatives of one and the same function with respect to* $q_1, \ldots, q_n$*, the remaining variables can then be determined as a canonical system of variables* $\psi_1, \ldots, \psi_n, \varphi_1, \ldots, \varphi_n$.

If $H - E$ is independent of one or more, or even all, of the quantities ψ and φ under such a substitution then it will follow from equations (12) for the partial derivatives of $H - E$ that the quantities that correspond to $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n$ and are provided with the same index will be integration constants in each case. If $H - E$ were zero or also only independent of $\psi_1, \ldots, \psi_n, \varphi_1, \ldots, \varphi_n$, then the latter would all be integration constants and define a complete system of integrals of the differential equations:

$$\frac{\partial H}{\partial p_l} = \frac{dq_l}{dt},$$

$$-\frac{\partial H}{\partial q_l} = \frac{dp_l}{dt}.$$

The problem of integrating these equations completely can also be expressed in such a form that the quantities – H, p_1, …, p_n are represented as functions of t, q_1, …, q_n, and a number of quantities ψ_1, …, ψ_n that is equal to the number of q such that they can be the partial derivatives of a single function, and indeed the partial derivatives with respect to t, q_1, …, q_n, respectively. The multi-termed quadrature:

$$\int (\sum p_l \, Dq_l - H \, Dt),$$

whose lower limits are absolute constants or depend upon the quantities ψ, which are regarded as constant for only the integration, will then yield a substitution function S whose partial derivatives with respect to ψ, together with the ψ, will define a complete system of integrals of the given differential equations.

A special form of that solution consists of representing the quantities p as functions of the q and an equal number of quantities ψ such that they can be the partial derivatives of a common function, as before, and at the same time, H can reduce to a function of t and ψ alone, so the multi-term quadrature:

$$\int (\sum p_l \, Dq_l - H \, Dt)$$

will then give the same sort of substitution function that it did before.

The problem can also be expressed in the form that **Hamilton** and **Jacobi** employed: In the given equation:

$$H = \text{funct.}\,(t, q_1, \dots, q_l, \dots, q_n, p_1, \dots, p_l, \dots, p_n),$$

one substitutes:

$$-H = \frac{\delta W}{\delta t}, \qquad p_l = \frac{\delta W}{\delta q_l}$$

and converts it into a partial differential equation:

$$0 = \frac{\delta W}{\delta t} + \text{funct.}\left(t, q_1, \dots, q_n, \frac{\delta W}{\delta q_1}, \dots, \frac{\delta W}{\delta q_n}\right),$$

whose general integral W is a function of the quantities t, q_1, …, q_n that depends upon one additive constant and m other integration constants ψ_1, …, ψ_n. That function W will then be a substitution function like S, and the remaining integrals of the equations of motion will come about when one sets $\delta W / \delta\psi_l$ = const.

In the study that is being carried out here, the force function V can depend upon the quantities $\frac{dq_1}{dt}, \ldots, \frac{dq_n}{dt}$ in an arbitrary way, and therefore so can $T + V$, as well as $H = -\frac{\delta W}{\delta t}$ can depend upon $p_l = \frac{\delta W}{\delta q_l}$ in an arbitrary way, so the following general developments will be directly applicable to any first-order partial differential equation when one further observes that, following **Jacobi**, one can reduce a differential equation that includes not only the independent variables and the partial derivatives of the desired function W^*, but also the function W^* itself, to a differential equation that includes the function W without differentiations by the substitution:

$$W = \tau W^*, \qquad \text{so} \qquad W^* = \frac{\delta W}{\delta \tau}, \qquad \frac{\delta W^*}{\delta t} = \frac{1}{\tau}\frac{\delta W}{\delta t}, \qquad \frac{\delta W^*}{\delta q_l} = \frac{1}{\tau}\frac{\delta W}{\delta q_l}.$$

If the principle of conservation of *vis viva* is valid then H will be a constant, and the partial differential equation:

$$0 = -H + \text{funct.}\left(q_1, \ldots, q_n, \frac{\delta W}{\delta q_1}, \ldots, \frac{\delta W}{\delta q_n}\right)$$

will give a general integral that takes the form of a function W that depends upon an additive constant and $n - 1$ other constants $\psi_1, \ldots, \psi_{n-1}$, and:

$$W - H \cdot t$$

will be a substitution function S, in which H appears in place of ψ or a function of $\psi_1, \ldots, \psi_{n-1}, \psi_n$.

The first form of the problem that was given here, which coincides with the complete integration of $2n$ equations:

$$\frac{\partial H}{\partial p_l} = \frac{dq_l}{dt},$$

$$-\frac{\partial H}{\partial q_l} = \frac{dp_l}{dt},$$

then includes the entirely-special case, which is nonetheless capable of many applications, in which each of the quantities p_k is, in a sense, to be determined as a function of the q_k (if that is even possible) that are equipped with the same index, and a system of n quantities $\psi_1, \ldots, \psi_n$, in such a way that the function H will become independent of the q when one substitutes those expressions for the p. The integrals in:

$$\int p_1\, Dq_1 + \cdots + \int p_n\, Dq_n - \int H\, Dt = S$$

will then become simple quadratures for unvarying $\psi_1, \ldots, \psi_n$, and when one takes the integrals of the functions of the ψ with fixed limits, S will become a substitution function, and $\psi_1, \ldots, \psi_n, \frac{\delta S}{\delta \psi_1}, \ldots, \frac{\delta S}{\delta \psi_n}$ will define a complete system of integration constants for the given differential equations.

In that form, one can determine the motion of a free mass-particle, which can be inferred directly from **Newton**'s laws for one or two fixed mass-particles, or also one that is constrained to remain on an ellipsoidal surface without the action of forces when one introduces ellipsoidal coordinates as independent variables, as **Jacobi** did.

The **Hamilton-Jacobi** form of perturbation theory is obtained from the canonical substitution in the following way: If H denotes the **Hamiltonian** function (7) for the completely-mechanical problem (8^*), so when one includes the so-called perturbing forces, while E is the **Hamiltonian** function for the motion that would arise if the perturbing forces were not present, and furthermore $\psi_1, \ldots, \psi_n, \varphi_1, \ldots, \varphi_n$ are the canonical integrals for the latter problem, so for the $2n$ equations:

$$\frac{\partial E}{\partial p_l} = \frac{\vartheta q_l}{\vartheta t},$$

$$-\frac{\partial E}{\partial q_l} = \frac{\vartheta p_l}{\vartheta t},$$

and finally, if:

$$S = \int \left(\sum p_l \frac{\vartheta q_l}{\vartheta t} - E \right) dt$$

is the associated **Hamiltonian** integral, so:

$$DS = -E\, Dt + \sum p_l\, Dq_l - \sum \varphi_l\, D\psi_l ,$$

then, as in equations (12), the elements ψ and φ that are altered by the perturbing forces will be determined by means of the $2n$ differential equations:

$$\frac{\vartheta (H - E)}{\vartheta \varphi_l} = \frac{d\psi_l}{dt},$$

$$-\frac{\vartheta (H - E)}{\vartheta \psi_l} = \frac{d\varphi_l}{dt},$$

in which $H - E$ is thought of as representing a function of $t, \psi_1, \ldots, \psi_n, \varphi_1, \ldots, \varphi_n$.

V. – Forces whose measure depends upon motion.

In the year 1835, **Gauss** (as would emerge from his handwritten notes that were published in volume five of his complete works, which I edited) was the first to think of determining forces that would depend upon not only the mutual position of the interacting bodies, but also upon the motion itself. His investigations, which were directed along those lines on many occasions, had the goal of explaining forces such as the ones that appear in the phenomena associated with galvanic currents. Under the assumption that the interactions between the galvanic currents and its carrier would be such that every force that acts upon the current would be transmitted to the carrier, and that furthermore the two forces that act in opposite directions upon two different types of electrical particle at the same place would provoke galvanic currents whose intensity is just as large throughout the entire linear current conductor and is proportional to the sum of the two forces, in my prize essay "Zur mathematischen Theorie electrischer Ströme" in the year 1857, I was the first to prove rigorously how the electrodynamical and electromotive laws that were discovered by **Ampère**, **Faraday**, **Lenz**, and **Franz Neumann** could be explained by the sort of forces that **Gauss** examined. Unfortunately, **Gauss**'s handwritten notes were still not available to me at that time, since otherwise I would have been spared some investigations, although a proof of the lemma of the coincidence of the potential for the interaction between galvanic currents with the potential for the interaction between magnetic surfaces, which I gave in that essay, was still not found in **Gauss**, but only the proof of the coincidence between the force components that were parallel to the coordinate axes (**Gauss**'s *Werke*, Bd. V, pp. 624).

The very incisive investigations that were most recently carried out by **Helmholtz** into the nature of electrodynamical forces have shown that when one does not determine the interaction between the electrical bodies and their carriers completely (which is what has been done up to now), the assumption that there are forces that depend upon the motion must lead to phenomena that contradict our conception of the nature of the forces that provoke the motions.

At this point, I would like to determine the forces that depend upon the motion only in regard to the fact that their analytical treatment agrees with the treatment of the forces that depend upon the mutual positions of the bodies that act upon each other as much as possible. The principle of action and reaction is directly applicable then. The principles of the conservation of the motion of the center of mass and the conservation of the areal velocity will then persist when the force between two mass-particles is proportional to the mass, its direction lies along the connecting line between the two masses or its extension, and the magnitude of the force depends upon only the distance between the two masses, moreover, so when the distance between two mass-particles with intensities ε', ε'' is r, the sum of the virtual moments of the two reciprocal forces that are exerted upon the mass-particles will be represented by:

$$\varepsilon' \varepsilon'' F\left(r, \frac{dr}{dt}, \frac{ddr}{dt^2}, \frac{d^2 r}{dt}, \dots\right) \delta r .$$

In the derivation of the equation of motion above, I showed that the simplicity of its form was essentially based upon the fact that the virtual moments of the forces, can be represented as the sum of a total variation of a function and the total derivative with respect to time of a sum of functions, multiplied by the variation of the coordinates. If that simple form for the equation of motion for the forces that are considered here is to remain valid then:

$$\varepsilon' \varepsilon'' F\left(r, \frac{dr}{dt}, \frac{ddr}{dt^2}, \frac{d^2 r}{dt}, \ldots\right) \delta r$$

$$= \delta V\left(r, \frac{dr}{dt}, \frac{ddr}{dt^2}, \frac{d^2 r}{dt}, \ldots\right) + \frac{d}{dt}\left\{V_1\left(r, \frac{dr}{dt}, \frac{ddr}{dt^2}, \frac{d^2 r}{dt}, \ldots\right)\right\} \delta r$$

$$= \frac{\partial V}{\partial r}\delta r + \frac{\partial V}{\partial \frac{dr}{dt}} \cdot \delta \frac{dr}{dt} + \frac{\partial V}{\partial \frac{ddr}{dt^2}} \cdot \delta \frac{ddr}{dt^2} + \ldots + \frac{\partial V_1}{\partial r}\frac{dr}{dt} \cdot \delta r + \frac{\partial V_1}{\partial \frac{dr}{dt^2}}\frac{ddr}{dt^2} \cdot \delta r + \ldots + V_1 \frac{d\delta r}{dt} + \ldots$$

can be an identity, and therefore:

$$\varepsilon' \varepsilon'' F \cdot \delta r = \frac{\partial V}{\partial r}\delta r + \frac{\partial V_1}{\partial t}\frac{dr}{dt}\delta r + \frac{\partial V_1}{\partial \frac{dr}{dt}}\frac{ddr}{dt^2}\delta r + \ldots,$$

$$0 = \frac{\partial V}{\partial \frac{dr}{dt}}\delta \frac{dr}{dt} + V_1 \delta \frac{dr}{dt},$$

$$0 = \frac{\partial V}{\partial \frac{ddr}{dt}}\delta \frac{ddr}{dt^2},$$

$$0 = \frac{\partial V}{\partial \frac{d^2 r}{dt^2}}\delta \frac{d^2 r}{dt^2},$$

so one has:

$$V = \text{function}\,(r, dr/dt),$$

$$V_1 = -\frac{\partial V}{\partial \frac{dr}{dt}},$$

$$\varepsilon' \varepsilon'' F\left(r, \frac{dr}{dt}, \frac{ddr}{dt^2}\right) = \frac{\partial V}{\partial r} - \frac{\partial}{\partial r}\left(\frac{\partial V}{\partial \frac{dr}{dt}}\right) \cdot \frac{dr}{dt} - \frac{\partial}{\partial \frac{dr}{dt}}\left(\frac{\partial V}{\partial \frac{dr}{dt}}\right) \cdot \frac{ddr}{dt^2},$$

$$\varepsilon'\varepsilon'' F\left(r, \frac{dr}{dt}, \frac{ddr}{dt^2}\right)\delta r = \frac{\partial V}{\partial r} - \frac{d}{dt}\left\{\frac{\partial V}{\partial \frac{dr}{dt}}\delta r\right\}.$$

If, for example:

$$V = V_0 + \sum_n V_n \cdot \left(\frac{dr}{dt}\right)^n,$$

in which V_0 and V_n are independent of dr / dt, then one will have:

$$\varepsilon'\varepsilon'' F\left(r, \frac{dr}{dt}, \frac{ddr}{dt^2}\right)\delta r = \delta\{V_0 + \sum_n V_n \cdot \left(\frac{dr}{dt}\right)^n\} - \frac{d}{dt}\left\{\sum n V_n \left(\frac{dr}{dt}\right)^{n-1} \cdot \delta r\right\},$$

$$\varepsilon'\varepsilon'' F\left(r, \frac{dr}{dt}, \frac{ddr}{dt^2}\right) = \frac{\partial V_0}{\partial r} - \sum_n (n-1)\frac{\partial V_n}{\partial r}\left(\frac{dr}{dt}\right)^n - \sum_n n(n-1) V_n \left(\frac{dr}{dt}\right)^n \frac{ddr}{dt^2}\},$$

and for $n = 2$ and constant values of $r\, V_0$ and $r\, V_n$, that will imply the law that **W. Weber** published in the year 1852.

VI. – Two free mass particles.

In order to keep in mind the complete determination of the motion under the action of forces that depend upon the motion of the bodies, I would like to work through two simply-soluble problems using the special method that was given in Art. IV, and first consider two mass-particles that move in an ν-fold extended flat space.

If:

$m, x_1, \ldots, x_\nu$	are the inertial mass and rectangular rectilinear coordinates of a mass-point
$M, X_1, \ldots, X_\nu$	are the corresponding things for the other point

then the distance r between the two points will satisfy:

$$rr = \sum_{\lambda=1}^{\nu} (x_\lambda - X_\lambda)^2,$$

and by assumption, the force function V depends upon only m, M, r, and dr / dt. The total *vis viva* will be:

$$2T = m\sum_{\lambda=1}^{\nu} x'_\lambda x'_\lambda + M\sum_{\lambda=1}^{\nu} X'_\lambda X'_\lambda .$$

If we set:

$$m + M = L^{-2}, \quad \frac{m+M}{mM} = NN,$$

to abbreviate, and introduce the quantities $q_1, \ldots, q_{2n}$ by the equations:

$$m\,x_1 = m\,L\,q_{\nu+1} + \frac{1}{N}\,q_1 \cos q_2\,,$$

$$m\,x_\lambda = m\,L\,q_{\nu+\lambda} + \frac{1}{N}\,q_1 \sin q_2 \cos q_3 \ldots \sin q_\lambda \cos q_{\lambda+1} \qquad \text{for } 1 < \lambda < \nu$$

$$m\,x_\nu = m\,L\,q_{2\nu} + \frac{1}{N}\,q_1 \sin q_2 \cos q_3 \ldots \sin q_{\nu-1} \sin q_\nu\,,$$

$$M\,X_1 = M\,L\,q_{\nu+1} - \frac{1}{N}\,q_1 \cos q_2\,,$$

$$M\,X_\lambda = M\,L\,q_{\nu+\lambda} - \frac{1}{N}\,q_1 \sin q_2 \cos q_3 \ldots \sin q_\lambda \cos q_{\lambda+1} \qquad \text{for } 1 < \lambda < \nu,$$

$$M\,X_\nu = M\,L\,q_{2\nu} - \frac{1}{N}\,q_1 \sin q_2 \cos q_3 \ldots \sin q_{\nu-1} \sin q_\nu\,,$$

then we will have $r = N\,q_1$, and the total *vis viva* will be:

$$2T = q_1'\,q_1' + \sum_{\lambda=2}^{\nu} \left(q_1 \sin q_2 \cdots \sin q_{\lambda-1}\,q_\lambda'\right)^2 + \sum_{\nu=\nu+1}^{2\nu} q_\mu'\,q_\mu'\,,$$

so

$$p_1 = \frac{\vartheta(T+V)}{\vartheta q_1'} = q_1' + \frac{\vartheta V}{\vartheta q_1'}\,,$$

$$p_\lambda = \frac{\vartheta(T+V)}{\vartheta q_\lambda'} = (q_1 \sin q_2, \ldots, \sin q_{\lambda-1})^2\,q_\lambda' \qquad \text{for } 1 < \lambda \leq \nu,$$

$$p_\mu = \frac{\vartheta(T+V)}{\vartheta q_\mu'} = q_\mu' \qquad \text{for } \nu + 1 \leq \mu \leq 2\nu,$$

and therefore:

$$H = \sum_{l=1}^{2\nu} p_l\,q_l' - T - V = T - V + q_1'\,\frac{\vartheta V}{\vartheta q_1'}\,,$$

$$= -V + q_1' \frac{\vartheta V}{\vartheta q_1'} + \tfrac{1}{2} q_1' q_1' + \tfrac{1}{2} \sum_{\lambda=2}^{\nu} (q_1 \sin q_2 \ldots \sin q_{\lambda-1})^{-2} p_\lambda\, p_\lambda + \tfrac{1}{2} \sum_{\mu=\nu+1}^{2\nu} p_\mu\, p_\mu\,.$$

If we set:

$$\tfrac{1}{2} p_\mu p_\mu = \psi_\mu \qquad \text{for } \nu \leq \mu \leq 2\nu$$

$$\tfrac{1}{2} p_\lambda p_\lambda + \psi_{\lambda+1} \csc\, q_\lambda^2 = \psi_\lambda \qquad \text{for } 1 \leq \lambda \leq \nu$$

$$-V + q_1' \frac{\vartheta V}{\vartheta q_1'} + \tfrac{1}{2} q_1' q_1' + \frac{\psi_2}{q_1\, q_1} = \psi_1\,,$$

in analogy with the **Jacobi** process, then:

$$H = \psi_1 + \sum_{\mu=\nu+1}^{2\nu} \psi_\mu\,,$$

and when we represent the quantity p_1 in the equation:

$$S = -\psi_1\, t - \sum_{\mu=\nu+1}^{2\nu} \psi_\mu t + \int p_1\, dq_1 + \sum_{\lambda=2}^{\nu-1} \int (2\psi_\lambda - 2\psi_{\lambda+1} \csc q_\lambda^2)^{1/2}\, dq_\lambda + \sum_{\mu=\nu}^{2\nu} q^\mu \sqrt{2\psi_\mu}$$

as a function of q_1 and the ψ with the help of the introductory equation for ψ_1, all of the integrals in that equation will become quadratures whose upper limits are once more q_1, q_λ for constant ψ.

The **Hamiltonian** function H can then be represented in terms of the mutually-independent quantities ψ alone, so the differential expression $\sum p_l\, Dq_l$ will then become a complete differential for unvarying ψ by that substitution, and the functions that are determined by the equations above and are set to:

$$\psi_l = \text{const.}, \qquad \frac{\delta S}{\delta \psi_l'} = -\varphi_l = \text{const.}$$

for all indices $l = 1, 2, 3, \ldots, 2n$ will be the $4n$ integral equations by which one determines the motion of the free, mutually-interacting, mass-particles m and M in ν-fold extended space according to the law of the force function V.

For the special case in which the force function has the simple form:

$$V = V_0 + V_1 \frac{dr}{dt} + V_2 \frac{dr^2}{dt^2}\,,$$

and V_0, V_1, V_2 are functions of only r, one will have:

$$p_1 = NV_1 + (2 + N\,N\,V^2)^{1/2}\left(V_0 - \frac{\psi_2}{q_1\,q_1} + \psi_1\right)^{1/2}.$$

VII. – Two mass particles in multiply-extended Gaussian and Riemannian spaces.

If one finds that a mass-particle is fixed at the coordinate origin and the radius vector r is drawn from that point to the moving point, the shortest lines are drawn from its midpoint to the ν mutually-rectangular coordinate axes that are composed of shortest lines, and one measures out the segments ξ_1, ξ_2, …, ξ_ν along those axes from the coordinate origin in well-defined directions, measured positively, then from my investigations into the multiply-extended Gaussian and Riemannian spaces in the Nachrichten von der Königlichen Gesellschaft der Wissenschaftern zu Göttingen 1873 January, no. 2, Lehrsatz IV, one will have:

$$\sin \tfrac{1}{2} i\, r^2 = \frac{\sum \tan i\, \xi_\mu^2}{1 + \sum \tan i\, \xi_\mu^2},$$

and the square of the element of length will be equal to:

$$\frac{4}{ii}\,\frac{\sum (d \tan i\, \xi_\mu)^2}{(1 + \sum \tan i\, \xi_\mu^2)^2},$$

namely, when the summations are extended over $\mu = 1, 2, 3, \ldots, \nu$, and i means the reciprocal value of the absolute unit of length for a Riemannian or homogeneous finite space, while it means the reciprocal value of the absolute unit of length, multiplied by $\sqrt{-1}$ for Gaussian or infinite space.

If one now sets:

$$\tan i\,\xi_1 = \tan \tfrac{1}{2} i\, q_1 \cos q_2,$$

$$\tan i\,\xi_2 = \tan \tfrac{1}{2} i\, q_1 \sin q_2 \cos q_3,$$

$$\tan i\,\xi_\mu = \tan \tfrac{1}{2} i\, q_1 \sin q_2 \sin q_3 \ldots \sin q_\mu \cos q_{\mu+1} \quad \text{for } \mu < \nu,$$

$$\tan i\,\xi_{\nu-1} = \tan \tfrac{1}{2} i\, q_1 \sin q_2 \sin q_3 \ldots \sin q_{\nu-1} \cos q_\nu,$$

$$\tan i\,\xi_\nu = \tan \tfrac{1}{2} i\, q_1 \sin q_2 \sin q_3 \ldots \sin q_{\nu-1} \sin q_\nu$$

then one will have:

$$\sum_{\mu=1}^{\nu} \tan i\,\xi_\mu^2 = \tan \tfrac{1}{2} i\, q_1^2 , \qquad q_1 = r,$$

and when one assumes that the mass of the moving particles is unity, the *vis viva* will be equal to:

$$2T = q_1' q_1' + \frac{1}{i\,i} \sin i\, q_1^2\, q_2' q_2' + \frac{1}{i\,i} \sin i\, q_1^2 \cdot \sin q_2^2 \cdot q_2' q_2'$$
$$+ \frac{1}{i\,i} \sin i\, q_1^2 \cdot \sin q_2^2 \cdot \sin q_3^2 \cdot \sin q_{\mu-1}^2 \cdot q_\mu' q_\mu'$$
$$+ \frac{1}{i\,i} \sin i\, q_1^2 \cdot \sin q_2^2 \cdot \sin q_3^2 \cdot \sin q_{\nu-1}^2 \cdot q_\nu' q_\nu' ,$$

so:

$$p_1 = \frac{\vartheta(T+V)}{\vartheta q_1'} = \frac{\vartheta V(q_1, q_1')}{\vartheta q_1'} + q_1' ,$$

$$p_2 = \frac{\vartheta(T+V)}{\vartheta q_2'} = \frac{1}{ii} \sin i\, q_1^2 \cdot q_2' ,$$

$$p_\mu = \frac{\vartheta(T+V)}{\vartheta q_\mu'} = \frac{1}{ii} \sin i\, q_1^2 \cdot \sin q_2^2 \cdot \sin q_3^2 \cdots \sin q_{\mu-1}^2 \cdot q_\mu' , \qquad \text{for } 1 < \mu \le \nu,$$

and therefore:

$$H = \sum_{l=1}^{n} p_l\, q_l' - T - V = T - V + q_1' \frac{\vartheta V}{\vartheta q_1'}$$

$$= -V + q_1' \frac{\vartheta V}{\vartheta q_1'} + \tfrac{1}{2} q_1' q_1' + \sum_{\mu=2}^{\nu} \tfrac{1}{2} ii \csc i\, q_1^2 \cdot \csc i\, q_2^2 \cdot \csc i\, q_3^2 \cdots \csc i\, q_{\mu-1}^2 \cdot p_\mu\, p_\mu .$$

The substitution:

$$\tfrac{1}{2} p_\nu\, p_\nu = \psi_\nu ,$$

$$\tfrac{1}{2} p_{\nu-1}\, p_{\nu-1} + \psi_\nu \csc\, q_{\nu-1}^2 = \psi_{\nu-1} ,$$

$$\tfrac{1}{2} p_\lambda\, p_\lambda + \psi_{\lambda+1} \csc\, q_\lambda^2 = \psi_\lambda \qquad \text{for } 1 < \lambda < \nu,$$

$$-V + q_1' \frac{\vartheta V}{\vartheta q_1'} + \tfrac{1}{2} q_1' q_1' + ii\, \psi_2 \csc i\, q_1^2 = \psi_1$$

yields:

$$H = \psi_1 ,$$

and for constant ψ:

$$DS = -H\,Dt + \sum_{l=1}^{n} p_l\,Dq_l\,.$$

The substitution function is:

$$S = -\,\psi_1\,t + \int p_1\,Dq_1 + \sum_{\mu=2}^{\nu-1}\int (2\psi_\mu - 2\psi_{\mu+1}\csc q_\mu^{\,2})^{1/2}\,dq_\mu + q_\nu\sqrt{2\psi_\nu}\,,$$

since p_1 will be a function of q_1 and the quantities ψ alone, when one consults the equation for ψ_1. The upper limits of the integrals are q_μ.

The motion of a free mass-particle in a homogeneous ν-fold extended space when a force-function $V(r, dr/dt)$ acts according to a fixed law is then determined completely by the equations:

$$\psi = \text{const.},\qquad \frac{\delta S}{\delta\psi_l} = -\,\varphi_l = \text{const.},$$

in which l means the indices 1, 2, 3, …, ν in succession.

VIII. – General differential equations for the substitutions.

In the theory of general perturbations, the perturbation formulas that **Lagrange** and **Poisson** found assume an important position. They relate to the variations of those quantities – viz., the so-called "elements" – that would be integration constants for the unperturbed motion. As **Jacobi** pointed out, they take on especially simple values for the canonical integration constants that **Hamilton** employed.

Those relations, along with the new equations that **Hamilton** and **Jacobi** added to them, are obtained very simply from the substitution equation (9) that was given above:

$$DS = \sum p_l\,Dq_l - \sum \varphi_l\,D\psi_l - E\,Dt\,.$$

If one differentiates this using a general differentiation Δ, which is nonetheless independent of the D differentiation, then that will give:

$$\Delta DS = \sum p_l\,\Delta Dq_l - \sum \varphi_l\,\Delta D\psi_l - E\,\Delta Dt + \sum \Delta p_l\,Dq_l - \sum \Delta\varphi_l\,D\psi_l - \Delta E\,Dt\,.$$

However, if one imagines that the general differential Δ was used in the first equation then:

$$\Delta S = \sum p_l\,\Delta q_l - \sum \varphi_l\,\Delta\psi_l - E\,\Delta t\,,$$

and if one then differentiates by D then that will imply:

$$D\Delta S = \sum p_l\,\Delta Dq_l - \sum \varphi_l\,D\Delta\psi_l - E\,\Delta Dt + \sum Dp_l\,\Delta q_l - \sum D\varphi_l\,\Delta\psi_l - DE\,\Delta t\,.$$

The two differentiations D and Δ are independent of each other, so the sequence in which they are performed will have no influence on the value, and when one subtracts the two second-order differential equations from each other, one will get the equation:

$$\sum (Dq_l\,\Delta p_l - \Delta q_l\,Dp_l) = \sum (D\psi_l \Delta\varphi_l - \Delta\psi_l\,D\varphi_l) + Dt \cdot \Delta E - \Delta t \cdot DE\,, \tag{13}$$

or, when one calls the expression $Dq_l\,\Delta p_l - \Delta q_l\,Dp_l$ a *differential determinant* of the function-pair q_l and p_l, one can express that in words:

If the $q_1, \ldots, q_n$ and $p_1, \ldots, p_n$ define a system of canonical variables then in order for the quantities $\psi_1, \ldots, \psi_n$ and $\varphi_1, \ldots, \varphi_n$ that are introduced by the substitution equations to also define a system of canonical variables, in general, it is necessary and sufficient that the sums of the general two-parameter differential determinants of all associated pairs q_l and p_l should differ from the sums of the ψ_n and φ_l that are formed in the same way by only the two-parameter differential determinant of the variables t and any function E.

That theorem will also be true when one restricts the concept of general differentiation in such a way that the time t remains unchanged. The two sums of the differential determinants will be equal to each other, and there will always be a function E that fulfills the conditions for that complete lemma.

We will prove that the differential equation (13) is also sufficient for the quantities φ and ψ to stay a system of canonical variables by distinguishing six cases:

1. The p and q are given as functions of the q, ψ, and t. Then let:

$$\chi_l = p_l\,, \qquad \chi_{n+l} = -\varphi_l\,, \qquad \chi_{2n+1} = \chi_m = -E,$$
$$x_l = q_l\,, \qquad x_{n+l} = \psi_l\,, \qquad x_{2n+1} = x_m = t\,.$$

2. If the q and φ are given as functions of the p, ψ, and t then let:

$$\chi_l = -q_l\,, \qquad \chi_{n+l} = -\varphi_l\,, \qquad \chi_{2n+1} = \chi_m = -E,$$
$$x_l = p_l\,, \qquad x_{n+l} = \psi_l\,, \qquad x_{2n+1} = x_m = t\,.$$

3. If the p and ψ are given as functions of the q, φ, and t then let:

$$\chi_l = p_l\,, \qquad \chi_{n+l} = \psi_l\,, \qquad \chi_{2n+1} = \chi_m = -E,$$
$$x_l = q_l\,, \qquad x_{n+l} = \varphi_l\,, \qquad x_{2n+1} = x_m = t\,.$$

4. If the q and ψ are given as functions of the p, φ, and t then let:

$$\chi_l = -q_l\,, \qquad \chi_{n+l} = \psi_l\,, \qquad \chi_{2n+1} = \chi_m = -E,$$
$$x_l = p_l\,, \qquad x_{n+l} = \varphi_l\,, \qquad x_{2n+1} = x_m = t\,.$$

5. If the q and p are given as functions of the ψ, φ, and t then let:

$$\chi_l = \sum_h p_h \frac{\vartheta q_h}{\vartheta \psi_l} - q_l\,, \qquad \chi_{n+l} = \sum_h p_h \frac{\vartheta q_h}{\vartheta \varphi_l}\,, \qquad \chi_{2n+1} = \chi_m = \sum_h p_h \frac{\vartheta q_h}{\vartheta t} - E,$$
$$x_l = p_l\,, \qquad x_{n+l} = \varphi_l\,, \qquad x_{2n+1} = x_m = \quad t\,.$$

6. If the ψ and φ are given as functions of the q, p, and t then let:

$$\chi_l = q_l - \sum_h \varphi_h \frac{\vartheta \psi_h}{\vartheta q_l}\,, \qquad \chi_{n+l} = -\sum_h \varphi_h \frac{\vartheta \psi_h}{\vartheta p_l}\,, \qquad \chi_{2n+1} = \chi_m = -\sum_h \varphi_h \frac{\vartheta \psi_h}{\vartheta t} - E,$$
$$x_l = p_l\,, \qquad x_{n+l} = \varphi_l\,, \qquad x_{2n+1} = x_m = \quad t\,.$$

In all cases, the condition equation (13) goes to one of the form:

$$\sum_{k=l}^{m} (Dx_k\, \Delta\chi_k - \Delta x_k\, D\chi_k) = 0,$$

so if one now takes the $\xi_1, \ldots, \xi_m$ to be any functions of the x that do not make the expression $\xi_1\,\chi_1 + \xi_2\,\chi_2 + \ldots + \xi_m\,\chi_m$ vanish, and one imagines that the equations:

$$\frac{dy}{\sum_{k=1}^{m} \xi_k\,\chi_k} = \frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \ldots = \frac{dx_m}{\xi_m}\,,$$

are integrated completely then m integration constants $y_1, y_2, \ldots, y_m$ will appear in that way, one of which – say, y_m – is coupled with y by addition, and the variables x can be considered to be functions of the quantities $y, y_1, y_2, \ldots, y_m$. That will then imply:

$$\chi_1 \frac{\partial x_1}{\partial y} + \chi_2 \frac{\partial x_2}{\partial y} + \cdots + \chi_m \frac{\partial x_m}{\partial y} = \chi_1 \frac{\partial x_1}{\partial y_m} + \chi_2 \frac{\partial x_2}{\partial y_m} + \cdots + \chi_m \frac{\partial x_m}{\partial y_m} = 1,$$

so for a general differentiation D:

$$\chi_1\, Dx_1 + \chi_2\, Dx_2 + \ldots + \chi\, Dx_m = D\,(y + y_m) + Y_1\, Dy_2 + \ldots + Y_{m-1}\, Dy_{m-1}\,,$$

in which $Y_1, \ldots, Y_{m-1}$ are functions of $y, y_1, y_2, \ldots, y_m$ that must fulfill the equation:

$$\sum_{k=l}^{m-1} (Dy_k\, \Delta Y_k - \Delta y_k\, DY_k) = 0$$

between the χ and x. In the special case in which all quantities y are constant for the D differentiation, except for y_l, where $1 \leq l \leq m - 1$, and all quantities y are constant for the Δ differentiations, with the exception of y, in one case, and then y_m, the equation will become:

$$Dy_l \cdot \frac{\partial Y_l}{\partial y}\Delta y = 0 \qquad\qquad Dy_l \cdot \frac{\partial Y_l}{\partial y_m}\Delta y_m = 0,$$

so for every index l between 1 and $m - 1$, Y_l will be independent of y and y_m. Therefore:

$$Y_1\, Dy_1 + Y_2\, Dy_2 + \ldots + Y_{m-1}\, Dy_{m-1}$$

will be a differential expression with only $m - 1$ independent variables, and the coefficients $Y_1, \ldots, Y_{m-1}$, along with their independent variables $y_1, \ldots, y_{m-1}$, will satisfy the corresponding condition as the coefficients $\chi_1, \ldots, \chi_m$ in the linear expression with the m independent variables x. The differential expression with $m - 1$ terms can then be decomposed once more by the same process into a differential and a linear differential expression with $m - 2$ independent variables, and with a corresponding condition. By carrying out that process, one will then arrive at a representation of the linear expression as the differential of a single function:

$$\chi_1\, Dx_1 + \chi_2\, Dx_2 + \ldots + \chi_m\, Dy_m = Dw.$$

If we denote the functions that arise each time in the six cases that were distinguished above by $w_1, w_2, \ldots, w_6$, respectively, in the application of that theorem to our investigations then we will have:

$$\sum p_l\, Dq_l - \sum \varphi_l\, D\psi_l - E\, DT = Dw_1\,,$$

$$-\sum q_l\, Dp_l - \sum \varphi_l\, D\psi_l - E\, DT = Dw_2\,,$$

$$\sum p_l\, Dq_l + \sum \psi_l\, D\varphi_l - E\, DT = Dw_3\,,$$

$$-\sum q_l\, Dp_l + \sum \psi_l\, D\varphi_l - E\, DT = Dw_4\,,$$

$$\sum_l \left(\sum_h p_h \frac{\partial q_h}{\partial \psi_l} - \varphi_l \right) D\psi_l + \sum_l \sum_h p_l \frac{\partial q_h}{\partial \varphi_l} D\varphi_l + \left(\sum_h p_h \frac{\partial q_h}{\partial t} - E \right) Dt = Dw_5\,,$$

$$\sum_l \left(p_l - \sum_h \varphi_l \frac{\partial \psi_h}{\partial q_l} \right) Dq_l - \sum_l \sum_h \varphi_h \frac{\partial \psi_h}{\partial p_l} D\, p_l - \left(\sum_h \varphi_h \frac{\partial \psi_h}{\partial t} + E \right) Dt = Dw_6\,,$$

or with the assistance of the identity equations:

$$D\sum p_l\, q_l = \sum p_l\, Dq_l + \sum q_l\, Dp_l\,, \qquad D\sum \varphi_l\, \psi_l = \sum \varphi_l\, D\psi_l + \sum \psi_l\, D\varphi_l\,,$$

when one adds them together, the partial differentials:

$$\sum p_l\, Dq_l - \sum \varphi_l\, D\psi_l - E\, dt = Dw_1 = D\left(w_2 + \sum p_l\, q_l\right) = D\left(w_3 - \sum \varphi_l\, \psi_l\right)$$
$$= D\left(w_4 + \sum p_l\, q_l - \sum \varphi_l\, \psi_l\right) = Dw_5 = Dw_5$$

will exist in all cases, so one will have a substitution function S which couples the two systems of variables (namely, the q, p and the ψ), φ in such a way that when the one is a canonical system, the other one will also be so.

If one restricts the definitions of the general differentials D and Δ in such a way that one leaves time t unchanged then for a canonical substitution:

$$\sum (Dp_l\, \Delta p_l - \Delta q_l\, Dp_l) = \sum (D\psi_l\, \Delta\varphi_l - \Delta\, \psi_l\, D\varphi_l)\,, \qquad Dt = 0, \qquad \Delta t = 0.$$

That form satisfies the condition equation that would make the substitution a canonical one. In fact, if one assumes that $Dt = 0$, $\Delta t = 0$ in the foregoing proof then that will yield the result that there exist functions w_1, w_2, …, w_6 that satisfy the equations that were found before under the assumption that $Dt = 0$, and will then be completely independent of E. If one then sets:

$$E = -\frac{\delta w_1}{\delta t}, \qquad E = -\frac{\partial_2 w_2}{\partial_2 t}, \qquad E = -\frac{\partial_3 w_3}{\partial_3 t}, \qquad E = -\frac{\partial_4 w_4}{\partial_4 t},$$
$$E = -\frac{\vartheta w_5}{\vartheta t} + \sum p_h \frac{\vartheta q_h}{\vartheta t}, \qquad E = -\frac{\partial w_6}{\partial t} - \sum \varphi_h \frac{\partial \psi_h}{\partial t},$$

in which the partial differentiations δ, ∂_2, ∂_3, ∂_4, ϑ, ∂ refer to those systems of variables that are considered to be mutually independent and by which the remaining quantities in each of the six cases are represented as functions, then S will be determined in the same way as before.

IX. – Jacobi's perturbation formulas.

The general differential equation (13):

$$\sum (Dq_l\, \Delta p_l - \Delta q_l\, Dp_l) = \sum (D\psi_l\, \Delta\varphi_l - \Delta\, \psi_l\, D\varphi_l) + Dt\, \Delta E - \Delta t\, DE,$$

when one performs the differentiations in the special senses:

$$\text{all } Dq = 0, \qquad Dp_l = 0 \quad \text{for} \quad l \neq h, \quad Dt = 0,$$
$$\Delta\psi_l = 0 \quad \text{for } l \neq h, \qquad \text{all} \quad \Delta\psi_l = 0, \; \Delta t = 0,$$

will become:

$$-\frac{\vartheta q_h}{\vartheta \psi_k} \Delta\psi_k \cdot Dp_h = -\Delta\psi_k \frac{\vartheta \varphi_k}{\vartheta p_h} Dp_h\,,$$

so one will have:

$$\frac{\vartheta q_h}{\vartheta \psi_k} = \frac{\vartheta \varphi_k}{\vartheta p_h}.$$

If one imposes various special assumptions on the differentiations in such a way that one assumes that all of the quantities p, q, t, except one, are unvarying under the D differentiations, and likewise all of the ψ, φ, t, except one, are unvarying under the Δ differentiations, then one will get the new system of equations:

$$\begin{aligned}
&\frac{\vartheta q_h}{\vartheta \psi_k} = \frac{\partial \varphi_k}{\partial p_h}, &\quad& \frac{\vartheta q_h}{\vartheta \varphi_k} = -\frac{\partial \psi_k}{\partial p_h}, &\quad& \frac{\vartheta q_h}{\vartheta t} = \frac{\partial E}{\partial p_h}, \\
&\frac{\vartheta p_h}{\vartheta \psi_k} = -\frac{\partial \varphi_k}{\partial q_h}, && \frac{\vartheta p_h}{\vartheta \varphi_k} = \frac{\partial \psi_k}{\partial q_h}, && \frac{\vartheta p_h}{\vartheta t} = -\frac{\partial E}{\partial q_h}, \\
&\frac{\vartheta E}{\vartheta \psi_k} = \frac{\partial \varphi_k}{\partial t}, && \frac{\vartheta E}{\vartheta \varphi_k} = -\frac{\partial \psi_k}{\partial t}, && \frac{\vartheta E}{\vartheta t} = \frac{\partial E}{\partial t}
\end{aligned} \tag{14}$$

that **Jacobi** exhibited, which are valid for all indices h and k. In order to give a common form to those various systems, we would like to introduce the notations:

$$q_{-\nu} = p_\nu, \qquad q_{+0} = E, \qquad q_{-0} = t,$$

$$\psi_{-\nu} = \varphi_\nu, \qquad \psi_{+0} = E, \qquad \psi_{-0} = t,$$

$$[h] = +1 \quad \text{for } h \geq \pm 0, \qquad [h] = -1 \quad \text{for } h < 0,$$

so the common form will become:

$$[h]\frac{\vartheta q_h}{\vartheta \psi_{-k}} = [-k]\frac{\vartheta \psi_k}{\vartheta q_{-h}}, \qquad h = +0, \pm 1, \pm 2, \ldots, \pm n, \quad k = +0, \pm 1, \pm 2, \ldots, \pm n. \tag{14*}$$

Conversely, one also has the theorem that when the **Jacobi** equations are fulfilled, that substitution of the quantities q, p with the ψ, φ will be canonical, because when one performs the summation over the stated values of h and k, one will have:

$$\sum_h \sum_k \left\{[h]\frac{\vartheta q_h}{\vartheta \psi_{-k}} - [-k]\frac{\vartheta \psi_k}{\vartheta q_{-h}}\right\} \Delta q_{-h}\, D\psi_{-k} = \sum_h [h]\, Dq_h\, \Delta q_{-h} - \sum_k [-k]\, \Delta \psi_k\, D\psi_{-k}$$

identically, so the two sides of this equation will be zero, with which the differential equation (13), which is true for the canonical substitution in general, will arise when one reintroduces the original notations.

If the function E is not given then one needs to assume only that $Dt = 0 = \Delta t$ in the development that was just carried out. The differential equation that then arises will not contain the function E, and it can be determined in the way that was done in article VIII.

X. – Poisson's perturbation formulas.

If q, p can be represented as function of ψ, φ, t, and conversely, ψ, φ can also be represented as functions of q, p, t, and Φ denotes a function of the $4n + 1$ quantities q, p, ψ, φ, t, and Ψ is a function of Φ, then one will have:

$$\frac{\vartheta\Psi}{\vartheta\Phi} = \frac{\partial\Psi}{\partial\Phi} + \sum \frac{\partial\Psi}{\partial q_l}\frac{\vartheta q_l}{\vartheta\Phi} + \sum \frac{\partial\Psi}{\partial p_l}\frac{\vartheta p_l}{\vartheta\Phi},$$

$$\frac{\partial\Psi}{\partial\Phi} = \frac{\vartheta\Psi}{\vartheta\Phi} + \sum \frac{\vartheta\Psi}{\vartheta\psi_l}\frac{\partial\psi_l}{\partial\Phi} + \sum \frac{\vartheta\Psi}{\vartheta\varphi_l}\frac{\partial\varphi_l}{\partial\Phi},$$

identically, when the summations are extended over the indices $l = 1, 2, 3, \ldots, n$. If one takes the Ψ and Φ in these equations to be any two of the quantities φ, ψ, t, in succession, and replaces the $\frac{\vartheta q_l}{\vartheta\Phi}$ and $\frac{\vartheta p_l}{\vartheta\Phi}$ with the analogous derivatives then one will get the following conditions for a canonical substitution:

$$\begin{aligned}
&\sum_l \left(\frac{\partial\psi_h}{\partial q_l}\frac{\partial\psi_k}{\partial p_l} - \frac{\partial\psi_h}{\partial p_l}\frac{\partial\psi_k}{\partial q_l}\right) = 0,\\
&\sum_l \left(\frac{\partial\psi_h}{\partial q_l}\frac{\partial\varphi_k}{\partial p_l} - \frac{\partial\psi_h}{\partial p_l}\frac{\partial\varphi_k}{\partial q_l}\right) = \begin{cases} 0 & h \neq k, \\ 1 & h = k, \end{cases},\\
&\sum_l \left(\frac{\partial\varphi_h}{\partial q_l}\frac{\partial\varphi_k}{\partial p_l} - \frac{\partial\varphi_h}{\partial p_l}\frac{\partial\varphi_k}{\partial q_l}\right) = 0,\\
&\sum_l \left(\frac{\partial E}{\partial q_l}\frac{\partial\psi_k}{\partial p_l} - \frac{\partial E}{\partial p_l}\frac{\partial\psi_k}{\partial q_l}\right) = \frac{\partial\psi_h}{\partial t},\\
&\sum_l \left(\frac{\partial E}{\partial q_l}\frac{\partial\varphi_k}{\partial p_l} - \frac{\partial E}{\partial p_l}\frac{\partial\varphi_k}{\partial q_l}\right) = \frac{\partial\varphi_h}{\partial t}
\end{aligned} \tag{15}$$

for $l = 1, 2, 3, \ldots, n$.

If we employ the same notations as in the previous article and use $\left(\frac{\partial\psi_h}{\partial\psi_{-k}}\right)$ and $\left(\frac{\partial\psi_h}{\partial q_\lambda}\right)$ to mean that:

$$\left(\frac{\partial\psi_h}{\partial\psi_{-k}}\right) = 1 \qquad \text{for } h = -k,$$

$$\left(\frac{\partial\psi_h}{\partial\psi_{-k}}\right) = \frac{\partial\psi_h}{\partial t} \qquad \text{for } h = +0,$$

but in all other cases:

$$\left(\frac{\partial\psi_h}{\partial\psi_{-k}}\right) = 0,$$

and that:

$$\left(\frac{\partial\psi_h}{\partial q_\lambda}\right) = 1 \qquad \text{for } h = \lambda = +0,$$

but in all other cases:

$$\left(\frac{\partial\psi_h}{\partial q_\lambda}\right) = \frac{\partial\psi_h}{\partial q_\lambda},$$

and we set $[\lambda] = +1$ for a positive λ, while $[\lambda] = -1$ for a negative value of λ, and $[+0] = [-0] = +1$, then we can give the five systems of equations above the common form:

$$[k]\left(\frac{\partial\psi_h}{\partial\psi_{-k}}\right) - \sum_{\lambda=+0}^{\mp n}[-\lambda]\left(\frac{\partial\psi_h}{\partial q_\lambda}\right)\left(\frac{\partial\psi_k}{\partial q_{-\lambda}}\right) = 0, \tag{15*}$$

and on the other hand, it will follow that this equation remains valid for all systems of values $\pm 0, \pm 1, \pm 2, \ldots, \pm n$ of the h and k, with the exception of $h = -k = -0$.

Poisson was the first to exhibit differential expressions of the type that appear above in the summations in (15) that relate to l in his "Mémoire sur la variation des constantes arbitraires dans les questions de Mécanique," 16 October 1809, Journal de l'École polytechnique, Cah. 15.

If one excludes the system of values $h = +0$ and $h = -k = -0$ then the term for $l = +0$ will always vanish in the summation, and equation (15*) will assume the simpler form:

$$[k]\left(\frac{\partial\psi_h}{\partial\psi_{-k}}\right) + \sum_{\lambda=\pm 1}^{\mp n}[\lambda]\frac{\partial\psi_h}{\partial q_\lambda}\frac{\partial\psi_k}{\partial q_{-\lambda}} = 0.$$

If equations (15) or (15*) are fulfilled then conversely, the substitution will be a canonical one, because when one lets:

$$\left(\frac{\vartheta q_\nu}{\vartheta \psi_h}\right) = 1 \quad \text{for } \nu = h = +\,0,$$

but:

$$\left(\frac{\vartheta q_\nu}{\vartheta \psi_h}\right) = \frac{\vartheta q_\nu}{\vartheta \psi_h}$$

for all other systems of values for ν and h in the expression:

$$\sum_{h,k,\nu} [k][\nu]\left(\frac{\vartheta q_\nu}{\vartheta \psi_{+k}}\right) Dq_{-\nu}\, D\psi_{-k} \cdot \left\{[k]\left(\frac{\partial \psi_{+h}}{\partial \psi_{-k}}\right) - \sum_{\lambda=+0}^{\pm n} [-\lambda]\left(\frac{\partial \psi_h}{\partial q_\lambda}\right)\left(\frac{\partial \psi_k}{\partial q_{-\lambda}}\right)\right\},$$

then if one deals with the individual cases in which the bracketed terms have a different sense from the derivatives separately and then performs the summations over h for the values $\pm\,0, \pm\,1, \pm\,2, \ldots, \pm\,n$, and the summations over λ, ν, k for the values $\pm\,0, \pm\,1, \pm\,2, \ldots, \pm\,n$, except for the combination $h = -\,k = -\,0$, then that will imply:

$$-\sum_\nu [\nu]\,\Delta q_\nu\, Dq_{-\nu} + \sum_k [-k]\, D\psi_k\, \Delta\psi_{-k}\,.$$

That expression must then become zero and in that way, once more imply the differential equation (13) that is true for a canonical substitution. If the function E is not known then one needs only to set $Dt = \Delta t = 0$ in that development and exclude the indices $\pm\,0$, and the equations that include E will not enter into the calculations then, and that function will first be determined from the substitution S that was calculated before in article VIII.

XI. – Lagrange's perturbation formulas.

If one takes the differentiations D and Δ in the general differential equation (13) for the canonical substitution to have the special meaning that any two of the quantities ψ_1, $\ldots$, ψ_n, φ_1, $\ldots$, φ_n, and t vary independently, but the remaining ones can be considered to be unvarying, then one will get:

$$\sum_l \left(\frac{\vartheta q_l}{\vartheta \psi_h}\frac{\vartheta p_l}{\vartheta \psi_k} - \frac{\vartheta p_l}{\vartheta \psi_h}\frac{\vartheta q_l}{\vartheta \psi_k}\right) = 0,$$

$$\sum_l \left(\frac{\vartheta q_l}{\vartheta \psi_h}\frac{\vartheta p_l}{\vartheta \varphi_k} - \frac{\vartheta p_l}{\vartheta \psi_h}\frac{\vartheta q_l}{\vartheta \varphi_k}\right) = \left\{\begin{array}{ll} 0 & h \neq k \\ 1 & h = k \end{array}\right., \tag{16}$$

$$\sum_l \left(\frac{\vartheta q_l}{\vartheta \varphi_h}\frac{\vartheta p_l}{\vartheta \varphi_k} - \frac{\vartheta p_l}{\vartheta \varphi_h}\frac{\vartheta q_l}{\vartheta \varphi_k}\right) = 0,$$

$$\sum_l \left(\frac{\vartheta q_l}{\vartheta t} \frac{\vartheta p_l}{\vartheta \psi_h} - \frac{\vartheta p_l}{\vartheta t} \frac{\vartheta q_l}{\vartheta \psi_h} \right) = \frac{\vartheta E}{\vartheta \psi_h},$$

$$\sum_l \left(\frac{\vartheta q_l}{\vartheta t} \frac{\vartheta p_l}{\vartheta \varphi_h} - \frac{\vartheta p_l}{\vartheta t} \frac{\vartheta q_l}{\vartheta \varphi_h} \right) = \frac{\vartheta E}{\vartheta \psi_h}.$$

Conversely, those five systems of equations characterize that substitution as a canonical one, because when one multiplies those equations by:

$$\begin{aligned}
& D\psi_h \, \Delta\psi_k , \\
& D\psi_h \, \Delta\varphi_k - \Delta\psi_h \, D\varphi_k , \\
- & D\psi_h \, \Delta\varphi_k , \\
& Dt \, \Delta\psi_h - \Delta t \, D\psi_h , \\
& Dt \, \Delta\varphi_h - \Delta t \, D\varphi_h ,
\end{aligned}$$

respectively, and then sums over all indices, adds the equations obtained together and assembles the sums of partial differentials, one will again get the general differential equation (13) that is true for the canonical substitution.

The first three systems also satisfy the equations (16) that would make the substitution canonical, as one will find when one assumes that $Dt = 0 = \Delta t$ in the foregoing investigation and determines the functions S and E as in article VIII.

If one applies the general differential equation (13) to the case in which $\psi_1, \ldots, \psi_n$, $\varphi_1, \ldots, \varphi_n$, are integration constants and represents them in terms of functions of any other $2n$ integration constants $c_1, c_2, \ldots, c_{2n}$, and the takes the differentiations D and Δ to mean that only c_μ varies for D and only c_ν varies for Δ, while the remaining c and t remain unchanged, then when one multiplies both sides of the general differential equation (13) by the product of $Dc_\mu \, \Delta c_\nu$ with a function of the integration constants, one will get **Lagrange**'s theorem:

$$\sum_l \left(\frac{dq_l}{dc_\mu} \frac{dp_l}{dc_\nu} - \frac{dp_l}{dc_\mu} \frac{dq_l}{dc_\nu} \right) = \text{const.}$$

XII. – Hamilton's perturbation formulas.

If the quantities p and φ can be represented as functions of the q, ψ, and t then one can take:

$$\sum (Dq_l \, \Delta p_l - \Delta q_l \, Dp_l) = \sum (D\psi_l \, \Delta\varphi_l - \Delta\psi_l \, D\varphi_l) + Dt \, \Delta E - \Delta t \, DE,$$

$$\begin{array}{lllll}
Dq_l = 0 & \text{for } l \neq h, & \text{all} & D\psi = 0, & Dt = 0, \\
\Delta q_l = 0 & \text{for } l \neq k, & \text{all} & \Delta\psi = 0, & \Delta t = 0
\end{array}$$

in the general equations, which will make:

$$Dq_h \cdot \frac{\delta p_h}{\delta q_k} \Delta q_k - \Delta q_h \cdot \frac{\delta p_k}{\delta q_h} Dq_k = 0,$$

so that will imply:

$$\frac{\delta p_h}{\delta q_k} = \frac{\delta p_k}{\delta q_h},$$

if the partial derivatives with respect to the variables q, ψ, and t are again denoted by δ.

If one sets:

$$\begin{array}{lllll} Dq_l = 0 & \text{for } l \neq h, & \text{all} & D\psi = 0, & Dt = 0, \\ \Delta\psi_l = 0 & \text{for } l \neq k, & \text{all} & \Delta q = 0, & \Delta t = 0 \end{array}$$

then equation (13) will go to:

$$Dq_h \cdot \frac{\delta p_h}{\delta \psi_k} \Delta \psi_k = -\Delta \psi_k \cdot \frac{\delta \varphi_k}{\delta q_h} Dq_k,$$

so:

$$\frac{\delta p_h}{\delta \psi_k} = -\frac{\delta \varphi_k}{\delta q_h}.$$

If one sets:

$$\begin{array}{llllll} & Dq_l = 0 & \text{for } l \neq h, & \text{all} & D\psi = 0, & Dt = 0, \\ \text{all} & \Delta q = 0 & \text{all} & \Delta\psi = 0 & & \end{array}$$

then the general equation will imply that:

$$Dq_h \cdot \frac{\delta p_h}{\delta t} \Delta t = -\Delta t \cdot \frac{\delta E}{\delta q_h} Dq_h,$$

so

$$\frac{\delta p_h}{\delta t} = -\frac{\delta E}{\delta q_h}.$$

If one carries out the examination of all permissible special assumptions of that kind for the D and Δ then one will get the five systems of equation that Hamilton presented under special assumptions:

$$\begin{gathered} \frac{\delta p_h}{\delta q_k} = \frac{\delta p_k}{\delta q_h}, \quad \frac{\delta p_h}{\delta \psi_k} = -\frac{\delta \varphi_k}{\delta q_h}, \quad \frac{\delta \varphi_h}{\delta \psi_k} = \frac{\delta \varphi_k}{\delta \psi_h}, \\ \frac{\delta p_h}{\delta t} = -\frac{\delta E}{\delta q_h}, \quad \frac{\delta \varphi_h}{\delta t} = \frac{\delta E}{\delta \psi_h}, \end{gathered} \tag{17}$$

which are true for all indices of h and k.

However, if, conversely, those equations are satisfied for an arbitrary function E then, as before, it will follow that the assumed representation of the q and p as functions of the ψ, φ, and t will then define a canonical substitution, so those equations will be the known condition equations for the existence of a function S whose partial derivatives with respect to $q_1, \dots, q_n, \psi_1, \dots, \psi_n$, and t are equal to $p_1, \dots, p_n, -\varphi_1, \dots, -\varphi_n$, and $-E$.

If we set:

$$Q_\nu = q, \qquad Q_{-\nu} = \psi_\nu, \qquad Q_0 = t,$$
$$P_\nu = p_\nu, \qquad P_{-\nu} = \varphi_\nu, \qquad P_0 = E,$$

for a positive ν then the five systems of **Hamilton** equation can be written in the common form:

$$[-h]\frac{\delta P_h}{\delta Q_k} = [-k]\frac{\delta P_k}{\delta Q_h} \qquad \text{for } h \text{ and } k \text{ equal to } 0, \pm 1, \pm 1, \dots, \pm n. \tag{17*}$$

If we multiply the two sides of that equation by DQ_k and ΔQ_h and sum over all values of h and k then we will get:

$$\sum [-h]\, DP_h\, \Delta Q_h = \sum [-k]\, DQ_k\, \Delta P_k,$$

which is once more the general differential equation for a canonical substitution.

The five systems of equations above are complete, in the sense that arbitrarily many of the functions $p_1, \dots, p_n, \varphi_1, \dots, \varphi_n, E$ that are expressed in terms of $q_1, \dots, q_n, \psi_1, \dots, \psi_n$, and t can be given such that only the equations between those given functions that are valid for that system are fulfilled, and the remaining functions can then be determined in such a way that they collectively define a canonical substitution.

In fact, in the last equation, one needs only to assume that those Dq_h and Δq_h, $D\psi_\lambda$ and $\Delta\psi_\lambda$ are equal to zero for which the respective p_h and φ_λ that are provided with the same index are not given. Likewise, Dt and Δt are set equal to zero when E is not given, so the p_h and φ_λ, and perhaps E, as well, that are not given will not enter into that equation, and for just the given ones:

$$p_1, p_2, \dots, p_m, \varphi_1, \dots, \varphi_\mu, \qquad \text{and possible } E,$$

one will get the equation:

$$0 = \sum_{l=1}^{m} (Dq_l\, \Delta p_l - \Delta q_i Dp_l) - \sum_{\lambda=1}^{\mu} (D\psi_\lambda\, \Delta\varphi_\lambda - \Delta\psi_\lambda D\varphi_\lambda) + Dt \cdot \Delta E - \Delta t \cdot DE,$$

and from article VIII, no. 1, that is the condition for the expression:

$$\sum_{l=1}^{m} p_l\, Dq_l - \sum_{\lambda=1}^{\mu} \varphi_\lambda D\psi_\lambda - E\, Dt$$

for constant $q_{m+1}, \dots, q_n, \psi_{\mu+1}, \dots, y_n$, to be the complete differential DS^* of a function S^* whose partial derivatives are:

$$\frac{\delta S^*}{\delta q_{m+1}} = p_{m+1}\,,\ \ldots,\ \frac{\delta S^*}{\delta q_n} = p_n\,,\ \ \frac{\delta S^*}{\delta \psi_{\mu+1}} = -\,\varphi_{\mu+1}\,,\ \ldots,\quad \frac{\delta S^*}{\delta \psi_n} = -\,\varphi_n\,,$$

and to set:

$$\frac{\delta S^*}{\delta t} = -\,E$$

when E is not given.

XIII. – New differential equations for the canonical substitution.

Three different systems of independent variables come under consideration in the **Jacobi** and **Hamilton** differential equations: first of all, the quantities q, p, t, then ψ, φ, t, and finally the q, ψ, t ; we have denoted the three different corresponding differentiations by ∂, ϑ, and δ. Now, even more groupings of the independent variables are required for many investigations.

If we set:

$$\begin{array}{llll} p_\nu = q_{-\nu}\,, & E = \psi_{+0} & \text{or} & E = q_{+0}\,, \\ \varphi_\nu = \psi_{-\nu}\,, & t = \psi_{-0} & \text{or} & t = q_{-0}\,, \end{array}$$

for ease of understanding, then we would like to imagine choosing $2n$ of the quantities $q_{\pm 1}$, …, $q_{\pm n}$, $\psi_{\pm 1}$, …, $\psi_{\pm n}$, and one of the q_{-0} , ψ_{-0} as a system of $2n + 1$ independent variables, and denote them with:

$$q_{h_1}\,,\ \ldots,\ q_{h_\nu}\,,\ \psi_{k_1}\,,\ \ldots,\ \psi_{k_\mu}\,,$$

while their partial derivatives are denoted by $\mathfrak{d}$, such that one will then have:

$$\frac{\partial P}{\partial q_l} = \frac{\mathfrak{d} P}{\mathfrak{d} q_l} + \sum_k \frac{\mathfrak{d} P}{\mathfrak{d} \psi_k}\frac{\partial \psi_k}{\partial q_l}\,,$$

$$\frac{\vartheta P}{\vartheta \psi_l} = \frac{\mathfrak{d} P}{\mathfrak{d} \psi_l} + \sum_h \frac{\mathfrak{d} P}{\mathfrak{d} q_h}\frac{\partial q_h}{\partial \psi_l}\,,$$

$$\frac{\mathfrak{d} P}{\mathfrak{d} q_l} = \sum_l \frac{\vartheta P}{\vartheta \psi_l}\frac{\partial \psi_l}{\partial q_h}\,,$$

$$\frac{\mathfrak{d} P}{\mathfrak{d} \psi_l} = \sum_l \frac{\partial P}{\partial q_l}\frac{\mathfrak{d} q_l}{\mathfrak{d} \psi_k}$$

identically for every function P, in which the summations over h and k are extended over all q_h and y_k that appear as independent variables, and the summation over l is extended over all values $-\,0, \pm\,1, \pm\,2, \ldots, \pm\,n$.

From the first of the two formulas, the equation:

$$\sum_h \sum_k \frac{\partial\Phi}{\partial q_h}\frac{\partial P}{\partial \psi_k}\left([-k]\frac{\vartheta q_h}{\vartheta\psi_{-k}} - [h]\frac{\vartheta\psi_k}{\vartheta\psi_{-h}}\right) = 0,$$

which follows immediately from the **Jacobi** equations (14), art. IX, will go to:

$$\sum_h [h]\frac{\partial\Phi}{\partial q_h}\left(\frac{\partial P}{\partial q_{-h}} - \frac{\partial P}{\partial q_{-h}}\right) + \sum_k [-k]\frac{\partial P}{\partial \psi_k}\left(\frac{\vartheta\Phi}{\vartheta\psi_{-k}} - \frac{\partial\Phi}{\partial \psi_{-k}}\right) = 0. \tag{18}$$

That equation includes **Jacobi**'s equation as a special case when one takes the ∂ differentiation to mean that the independent variables are, among others, e.g., q_l and ψ_λ, but not q_{-l} and $\psi_{-\lambda}$, and that one then sets $\Phi = q_l$, $P = \psi_\lambda$. Equation (18) goes to the second **Hamilton** equation (17) when one refers the ∂ differentiation to the independent variables $q_1, \ldots, q_n, \psi_1, \ldots, \psi_n, t$, and sets $P = p_l$, $\Phi = \varphi_\lambda$. With the help of the equation that is obtained in that way, equation (18) above also implies the first **Hamilton** equation when one sets $P = p_l$, $\Phi = p_\lambda$, as well as the third when one sets $P = \varphi_l$, $\Phi = \varphi_\lambda$, and one can also derive the fourth equation directly when one refers the ∂ differentiation to the quantities $E, \psi_1, \ldots, \psi_n, q_1, \ldots, q_n$ as the independent variables, and one sets:

$$\Phi = t, \quad P = p_l = q_{-\lambda}, \quad \psi_0 = E, \qquad \psi_{-0} = t$$

in equation (18) above; one would then have:

$$0 = -\frac{\partial t}{\partial q_l} + \frac{\partial p_l}{\partial E} = \frac{\partial t}{\partial E}\frac{\delta E}{\delta q_l} + \frac{\delta p_l}{\delta t}\frac{\partial t}{\partial E}.$$

One gets the fifth **Hamilton** equation in an analogous way.

We would not like to examine the general form for the equation in the case where E proves to be the independent variables in (18) here.

What is remarkable about the general relation (18) above is that it will also arise from the expression above for those ψ that appear to be independent under the ∂ differentiation and those ψ that enter into Φ when one performs the summation over l in:

$$\sum_k \sum_l \left(\frac{\vartheta\Phi}{\vartheta\psi_l} - \frac{\partial\Phi}{\partial \psi_l}\right)\frac{\partial P}{\partial \psi_k}\left\{[-k]\left(\frac{\partial\psi_l}{\partial\psi_{-k}}\right) + \sum_h [-h]\left(\frac{\partial\psi_l}{\partial q_h}\right)\left(\frac{\partial\psi_k}{\partial q_{-h}}\right)\right\}$$

over the ψ_l that come under consideration. In regard to that, from the general equation (18) above, one can choose the quantities:

$$q_1, q_2, \ldots, q_n, \psi_1, \psi_2, \ldots, \psi_i, p_{i+1}, p_{i+2}, \ldots, p_n$$

to be independent for the ∂ differentiation, and set $P = p_l$, Φ = funct. (ψ_1 , …, ψ_n) = f, and derive the equation:

$$0 = -\frac{\partial f}{\partial q_\lambda} + \sum_{h=i+1}^{n} \left(\frac{\partial f}{\partial q_h} \frac{\partial p_\lambda}{\partial p_h} - \frac{\partial f}{\partial p_h} \frac{\partial p_\lambda}{\partial q_h} \right),$$

which is valid for every $\lambda \leq i$, that **Jacobi** presented in his treatise "Nova methodus, aequationes differentiales partiales primi ordinis inter numerum variabilium quemcunque propositas integrandi," Borchardt's Journal, Bd. 60, as a special case of (18).
