"Das Prinzip der virtuellen Verrückungen und die Variationsprinzipien der Elastizitätstheorie," Der Stahlbau **11** (1938), 185-192.

The principle of virtual displacements and the variational principles of the theory of elasticity

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Translated by D. H. Delphenich

The principle of virtual displacements is as old as mechanics itself. In his *Elementaren Mechanik* ([1]), **Hamel** made note of the fact that **Aristotle**'s "Golden Rule" included the basic idea of the principle of virtual displacements. **Johann Bernoulli** ([2]) was the first to express the principle in the year 1717, and **Lagrange** ([3]) made an extended form of it the foundation for all mechanics in 1788.

One might think that it would be pointless to write about a principle that old in this day and age. However, experience has shown that despite the century-long application of the principle and its ever-increasing significance, and despite the classical shaping of the concept by **Lagrange**, ambiguities have emerged that cause confusion, and especially in the application of the principle to the problems of the theory of elasticity.

In the Eighteen-Eighties (1883 to 1886), a sharp rift appeared between **Mohr** and **Müller-Breslau**. At the start of the century, the discussion was revived in connection with a paper by **Weingarten** in the year 1901 ([4]), and took on a form that was, in some cases, quite vehement and unfriendly. At the time, **Weingarten**, **Weyrauch**, **Hertwig**, **Müller-Breslau**, **Koetter**, **Mohr**, **Mehrtens**, **Föppl**, and others tossed it around between themselves in various journals for about a year. In a paper in May 1914, **Domke** ([5]) presented the crux of that discussion clearly and consistently on the basis of variational principles. By contrast, some other questions in that paper remained open. **Föppl** ([6]) had also considered all of the points of contention in detail in *Drang und Zwang*. **Trefftz** ([7]) presented the majority of the problems that were connected with those questions in the *Handbuch der Physik* thoroughly and rigorously.

([1]) **Hamel**, *Elementare Mechanik*, pp. 471. Leipzig and Berlin 1912.
([2]) **Varignon**, *Nouvelle mécanique*, 1725. – Cf., **Hamel**, *loc. cit.*
([3]) **Lagrange**, *Mécanique analytique*, 1788.
([4]) **Weingarten**, "Rezension der Vorlesungen über technische Mechanik von A. Föppl," Archiv der Mathematik und Physik (1) **3** (1901), pp. 342.
([5]) **Domke**, "Über Variationsprinzipien in der Elastizitätslehre nebst Anwendungen auf die technische Statik," Zeit. Math. u. Phys. (1915), pp. 174, *et seq.*
([6]) **Föppl**, *Drang und Zwang*, Munich and Berlin, 1920, v. I, pp. 58, *et seq.*
([7]) **Trefftz**, *Handbuch der Physik*, Bd. VI., Chap. 2.

In recent years, a new discussion of the principle of virtual displacements has developed. In May 1936, **Pöschl** published a paper ([8]) with the goal of giving a clear interpretation of the minimal principles of the theory of elasticity. In it, **Pöschl** came to the conclusion that the principle led to fundamentally-different conclusions "according to whether one was dealing with ordinary problems of elastic equilibrium or problems of buckling." **Domke** proved the untenability of that conclusion in his response to that ([9]). **Marguerre** proved that from a completely-different viewpoint in an article that is complete in its own way ([10]). That paper was especially valuable for its application of the principle to finite deformations.

How strongly that the old principle has been once more addressed recently is shown by the fact that the agenda of this year's meeting of the Society for Applied Mathematics and Mechanics in Göttingen included a talk by **C. Weber**, Dresden, on those questions. **Kammüller** has also recently taken up a particular aspect of that topic ([11]). That prompted me to do some basic research ([12]) that led to a brief discussion between **Kammüller** and myself ([13]). That discussion convinced me of the necessity of an attempt to present a summary that is as brief as possible of the applications of the principle to the problems in elasticity that are most fundamental. Of especial interest to me was the represented the various forms in which the principle is applied and their reciprocal relationships and differences, and to show the limits within which each of those forms can be applied. Unfortunately, I must forgo including the effects of heat, since they would greatly expand the scope of the paper. The assumptions that I am stating from are briefly the following ones:

We consider an elastic body under the action of external forces that are applied to its surface. In order to simplify the equations, we shall ignore body forces, say the weight of the body. Assuming that all of the body is found in a state of rest, i.e., that is it supported in the required way. If we then calculate the reactions to the external forces at the supports and clamping points then they will be found to be in equilibrium with each other. Thus, when we speak of the occurrence of equilibrium or its perturbation in what follows, we will always mean only the equilibrium of internal forces (i.e., stresses) with the external forces.

If the material that the body is made of is given, i.e., if the stresses σ, τ are known as functions of the elongations ε and shears γ, and if the magnitudes, directions, and points of application of the external forces are further given then each point of the body will generally (i.e., when we exclude the case of so-called branching points of the elastic equilibrium) assume a *uniquely-determined equilibrium configuration*. The problem is then to determine it.

We assume that the loads are applied gradually, so they increase continuously from zero to their final value. We further assume that this process proceeds so slowly that the acceleration terms can be neglected in the expression for energy. For simplicity, we shall likewise avoid considering all effects of heat and temperature and assume *complete elasticity* for the material that the body is

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([8]) **Pöschl**, "Über die Minimalprinzipe der Elastizitätstheorie," Bauing. v. **17**, issue 17/18 (1936), pp. 160, *et seq.*

([9]) **Domke**, "Zum Aufsatz 'Über die Minimalprinzipe der Elastizitätstheorie' von Th. Pöschl," Bauing. **17** (1936), pp. 160, *et seq.*

([10]) **Marguerre**, "Über die Behandlung von Stabilitatätsproblemen mit Hilfe der energetischen Methode," Zeit. ang. Math. Mech., **18** (1938), pp. 57, *et seq.*

([11]) **Kammüller**, "Das Prinzip der virtuellen Verschiebungen. Eine grundsätzliche Betrachtung," B. u. E. **37** (1938), pp. 363, *et seq.*

([12]) **Schleusner**, "Zum Prinzip der virtuellen Verschiebungen," B. u. E. **27** (1938), pp. 252, *et seq.*

([13]) Letter from **Kammüller**, response by **Schleusner**, rebuttal by **Kammüller**. B. u. E. **37** (1938), pp. 271, *et seq.*

composed of, so we shall ignore internal and external frictional losses, although some of our equations will also remain valid when we consider such situations.

We shall further appeal to **St. Venant**'s principle, namely, that the external forces are treated as isolated forces, so the integration over the surface tractions is replaced by the summation over the isolated forces ([14]).

1. – The derivation of the main principle.

We consider a point-mass that is acted on by forces. We denote the forces (which can also include moments) as vectors by Fraktur symbols, and their magnitudes by the corresponding Antiqua symbols. Since the point-mass is coupled with other ones into a body, the applied forces can be either a force $\mathfrak{P}$ that acts on the body from the outside or internal forces $\mathfrak{Q}_m$ inside the body. The condition for the equilibrium of the forces at the point in question is:

$$\mathfrak{P} + \sum_m \mathfrak{Q}_m = 0 . \qquad (1)$$

Geometrically, that says that the starting point and endpoint of the polygonal path defined by the sequence of successive vectors $\mathfrak{P}$, $\mathfrak{Q}_1$, $\mathfrak{Q}_2$, … must coincide (cf., Fig. 1). Eq. (1) will remain correct when it is scalar-multiplied by a completely-arbitrary displacement vector $\delta\mathfrak{s}$ that we only assume to be *infinitely small*, as will be justified later. However, since it *not functionally coupled*, but *arbitrary*, we shall also not denote it by the differentiation symbol *d*, but with the variational symbol δ. We then get:

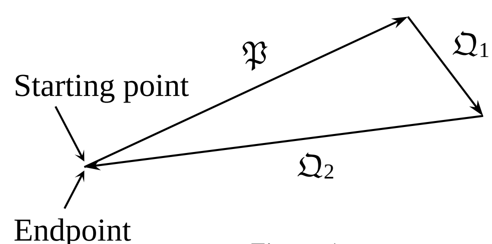

Figure 1.

$$\mathfrak{P} \cdot \delta\mathfrak{s} + \sum_m \mathfrak{Q}_m \cdot \delta\mathfrak{s} = 0 . \qquad (2)$$

Eq. (2) admits two interpretations according to how one combines the three factors in the scalar product $\mathfrak{Q}_m \cdot \delta\mathfrak{s} = Q_m \cdot \delta\mathfrak{s} \cdot \cos(\mathfrak{Q}_m, \delta\mathfrak{s})$. First of all, one can write that as $[Q_m \cdot \cos(\mathfrak{Q}_m, \delta\mathfrak{s})] \cdot \delta\mathfrak{s}$. Thus interpreted, eq. (2) expresses only the trivial fact that when a closed polygonal path is projected onto an arbitrary direction, the projections of the initial and final points will also coincide (Fig. 2). Secondly, one can write $\mathfrak{Q}_m \cdot \delta\mathfrak{s} = Q_m \cdot [\delta\mathfrak{s} \cdot \cos(\mathfrak{Q}_m, \delta\mathfrak{s})]$. If we then introduce a special notation for the projection of the displacement vector $\delta\mathfrak{s}$ onto the direction of the force $\mathfrak{Q}$: $\delta\mathfrak{s} \cdot \cos(\mathfrak{Q}, \delta\mathfrak{s}) = \delta q$ then eq. (2) will assume the following form:

$$P \cdot \delta p + \sum_m Q_m \cdot \delta q_m = 0 . \qquad (3)$$

([14]) Cf., the article by **Tedone-Timpe** in the *Enzyklopädie der Mathematischen Wissenschaften*, IV, 25, no. 15.

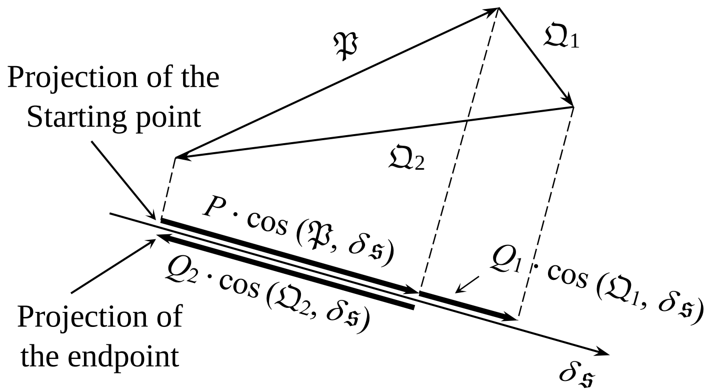

Figure 2.

If the point-mass in question, along with the forces $\mathfrak{P}$, $\mathfrak{Q}_1$, $\mathfrak{Q}_2$, … that are applied to it, are displaced by the vector $\delta\mathfrak{s}$ then the point of application of the force $\mathfrak{Q}_m$ will be displaced through the line segment δq_m *in the direction of the force* (Fig. 3). That is: Under that displacement, the force $\mathfrak{Q}_m$ will do work $Q_m \cdot \delta q_m$, just as the force $\mathfrak{P}$ will do work $P \cdot \delta p$. (In the example that is illustrated in Fig. 3, the forces $\mathfrak{P}$ and $\mathfrak{Q}_1$ do positive work, since they experience a displacement in the same sense as the force direction. The force $\mathfrak{Q}_2$ would do negative work since it experiences a displacement in the opposite sense to the direction of the force.) One can also interpret eq. (2) in the form (3) then as follows: Under a displacement of the point in question through the arbitrary vector $\delta\mathfrak{s}$, the sum of the works done in that way by all of the forces that act on the point must vanish. Since the point does not, in fact, experience that displacement, because the displacement is only an imaginary one and arbitrary, to boot, it will be called a *virtual displacement*. Likewise, the works done $P \cdot \delta p$ and $Q_m \cdot \delta q_m$ are not the works that are actually being done, but only imaginary works, and they will therefore be referred to by the classical expression that **Lagrange** coined ([3]) of *virtual works*.

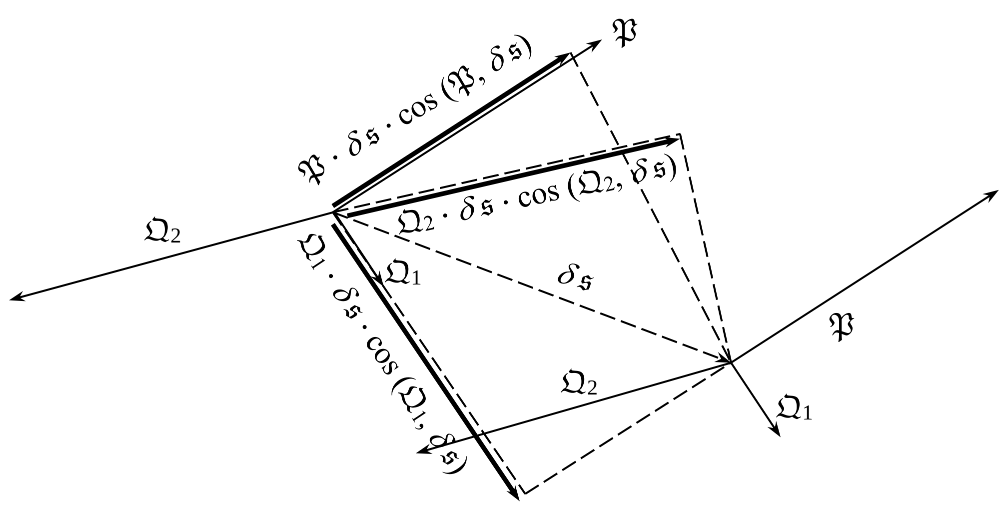

Figure 3.

Eq. (3) was derived as a necessary consequence of the equilibrium condition (1). If one demands that (3) must be true for not just any, but for *each arbitrary* displacement *d* s then eq. (1) will also follow conversely as a necessary consequence of eq. (3). We can the summarize the result as follows:

A necessary and sufficient condition for a point to be found in equilibrium under the action of applied forces is that the virtual work that is done by the applied forces must vanish for each arbitrary virtual displacement.

Lagrange ([3]) referred to that theorem as the *principle of virtual displacements*. However, one observes that this principle is nothing but an arbitrary interpretation (and as we saw, not the only one possible!) of an arbitrary mathematical operation on the equilibrium condition (1) that has no actual physical meaning.

From now on, we shall consider a volume element $dV = dv\ dy\ dz$ of a body that is found to be elastic equilibrium under the action of external forces. Normal stresses σ and shear stresses τ will act upon the volume element as a consequence of the deformation. If we choose the volume element to be small enough then the stresses $\sigma_x, \ldots, \tau_z$ will determine the internal forces $\mathfrak{Q}_m$ in eq. (3). As a virtual displacement, we choose an associated system of elongations $\delta\varepsilon_x$, $\delta\varepsilon_y$, $\delta\varepsilon_z$ and shears $\delta\gamma_x$, $\delta\gamma_y$, $\delta\gamma_z$. Those virtual elongations and shears are then *in addition to* the actual elongations ε_x, ε_y, ε_z and γ_x, γ_y, γ_z that are produced by the deformation of the body. We assume the compatibility of the $\delta\sigma_x, \ldots, \delta\tau_z$, i.e., they can distort the volume element, but not break it apart.

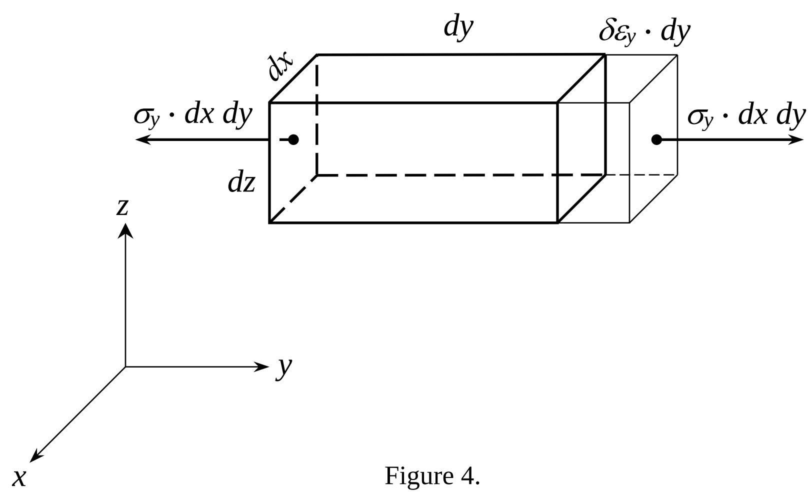

Figure 4.

The force $\sigma_y \cdot dx\ dz$ then acts on the left and right faces of the volume element in Fig. 4. When we think of the left face as fixed, the path-length of the displacement will be zero on the left and equal to $\delta\varepsilon_y \cdot dy$ on the right, so the virtual work done will be:

$$(\sigma_y \cdot dx\ dz) \cdot (\delta\varepsilon_y \cdot dy) = \sigma_y\ \delta\varepsilon_y \cdot dV.$$

By contrast, the sum of the virtual works will vanish under a total displacement of the volume element in the *y* direction since the contributions that come from the left and right faces will cancel each other.

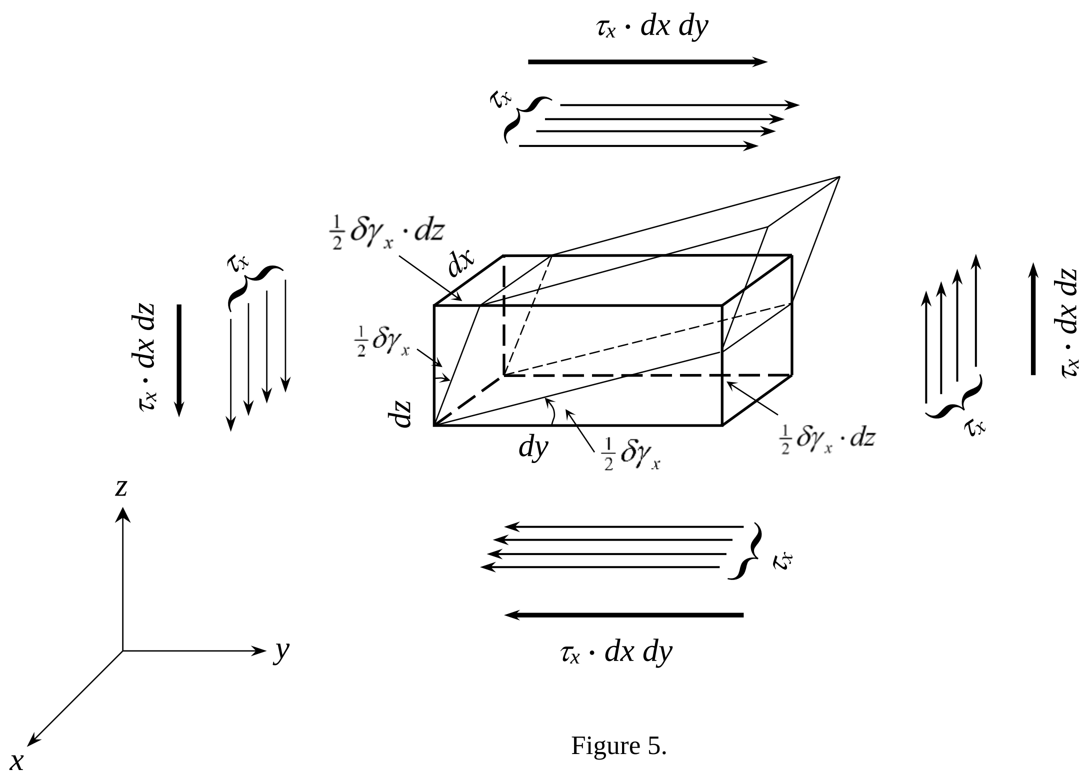

Figure 5.

$\tau_x \cdot dx\, dz$ acts on the left and right faces as a shearing force in Fig. 5. If we imagine that the lower left edge is fixed then the displacement path-length of the shearing force will be zero on the left face and $\frac{1}{2}\delta\gamma_x \cdot dy$ when the total shear is $\delta\gamma_x$. The virtual work is then:

$$(\tau_x \cdot dx\, dz) \cdot \left(\tfrac{1}{2}\delta\gamma_x \cdot dy\right) = \tfrac{1}{2}\tau_x\, \delta\gamma_x \cdot dV\,.$$

The shearing force $\tau_x \cdot dx\, dy$ acts on the upper and lower boundary faces. Their displacement path-length is zero below and $\frac{1}{2}\delta\gamma_x \cdot dz$ above, so the virtual work done will be:

$$(\tau_x \cdot dx\, dy) \cdot \left(\tfrac{1}{2}\delta\gamma_x \cdot dz\right) = \tfrac{1}{2}\tau_x\, \delta\gamma_x \cdot dV\,.$$

Since the various components of the virtual work will cancel each other under a total rotation of the volume element, the total virtual work that is done by the shearing stress τ_x will be:

$$\tau_x\, \delta\gamma_x \cdot dV\,.$$

Corresponding statements are true for the other four stress components. That work is done by the deformation of a body under the action of external forces. Its share of the work is then regarded as positive. By contrast, that deformation work will be done against the resistance of the internal forces. Its share of the work must then be given a negative sign. Eq. (3) will then go to:

$$(4) \qquad P\ \delta p - dV \cdot (\sigma_x\ \delta\varepsilon_x + \sigma_y\ \delta\varepsilon_y + \sigma_z\ \delta\varepsilon_z + \tau_x\ \delta\gamma_x + \tau_y\ \delta\gamma_y + \tau_z\ \delta\gamma_z) = 0$$

in our case. In that way, from our assumption, we can have $P \neq 0$ only for volume elements on the surface of the body.

An eq. (4) is true for every volume element in the body. If we would like that all of those equations should be able to exist simultaneously then we must merely take care that virtual displacements of the individual volume elements, which are mutually-independent, by their nature, should not have any breakdown in the connectivity of the body as a consequence. In order for that to be true, it is first necessary that the $\delta\varepsilon$, $\delta\gamma$ must be infinitely small. Secondly, they must be continuous functions of the position coordinates that satisfy the compatibility equations, and thirdly, that they agree must with the support conditions on the body. Other than that, they are arbitrary. The corresponding statement for the *dp* that they are determined by the virtual deformations of those volume elements on which the external forces *P* act will then follow automatically.

Under the assumptions that were made, we can integrate eq. (4) over the entire body. If we multiply by – 1 then we will finally get:

$$(5) \qquad \int dV\,(\sigma_x\,\delta\varepsilon_x + \sigma_y\,\delta\varepsilon_y + \sigma_z\,\delta\varepsilon_z + \tau_x\,\delta\gamma_x + \tau_y\,\delta\gamma_y + \tau_z\,\delta\gamma_z) - \sum_n P_n\,\delta p_n = 0\ .$$

That is the principle of virtual displacements for elastic bodies in its most general form, but it is also valid when the material in the body is not completely elastic, as well as for arbitrary laws of elasticity, and for finite deformations of the body. As we said, in that way, $\delta\varepsilon_x$, …, $\delta\gamma_z$ *are displacements and shears that are infinitely small, continuously-dependent upon the position coordinates, and satisfy the compatibility and support conditions, but can otherwise be chosen arbitrarily.*

In the special case of complete elasticity for the material in the body, the work done by the internal forces under the deformation that actually occurs will be:

$$(6) \qquad A_i = \int dV \int_0^{\varepsilon,\gamma} (\sigma_x\,d\varepsilon_x + \sigma_y\,d\varepsilon_y + \sigma_z\,d\varepsilon_z + \tau_x\,d\gamma_x + \tau_y\,d\gamma_y + \tau_z\,d\gamma_z)\ .$$

Therefore, the first term in eq. (5) will be the variation of the deformation work with respect to the displacement quantities $\delta_v A_i$ in this case. Likewise, the second term is the variation of the sum $\sum_n P_n\ p_n$ with respect to the displacement quantities p_n that appear in it, so it can be written as $\delta_v \left(\sum_n P_n\ p_n\right)$. We shall then introduce a special notation for the expression $\sum_n P_n\ p_n$. That quantity is not, say, the work that is done by external forces P_n by the deformation that actually takes place, because the forces P_n increase from zero to their final value P_n during the deformation.

Rather, $\sum_n P_n\, p_n$ is the work that the forces P_n would have done under the deformation if they had acted with their final value P_n during the entire displacement p_n from the outset. I have therefore chosen the terminology *final-value work done by external forces* for that quantity ([15]):

$$A_{ea} = \sum_n P_n\, p_n\,. \tag{7}$$

As opposed to that, the *work that is actually done by external forces* under the deformation is:

$$A_a = \sum_n \int_0^{P_n} P_n\, dp_n\,. \tag{8}$$

Finally, the *virtual work done by external forces* under the arbitrary virtual displacements $\delta\varepsilon_x$, …, $\delta\tau_z$, δp_n [cf., eq. (5)] is:

$$A_{va} = \sum_n P_n\, \delta p_n = \delta_v A_{ea}\,. \tag{9}$$

We shall correspondingly refer to the quantity:

$$A_{vi} = \int dV\,(\sigma_x\,\delta\varepsilon_x + \sigma_y\,\delta\varepsilon_y + \sigma_z\,\delta\varepsilon_z + \tau_x\,\delta\gamma_x + \tau_y\,\delta\gamma_y + \tau_z\,\delta\gamma_z) = \delta_v A_i \tag{10}$$

as the *final-value work done by internal forces* ([15]). The *work actually done by internal forces* under the deformation is given by the quantity A_i in eq. (6). For complete elasticity and in the absence of heat effects of any sort, from the energy principle, we must always have:

$$A_i = A_a\,. \tag{11}$$

Finally, from eq. (5), the *virtual work done by internal forces* under the arbitrary virtual displacements $\delta\varepsilon_x$, …, $\delta\tau_z$ is:

$$A_{vi} = \int dV\,(\sigma_x\,\delta\varepsilon_x + \sigma_y\,\delta\varepsilon_y + \sigma_z\,\delta\varepsilon_z + \tau_x\,\delta\gamma_x + \tau_y\,\delta\gamma_y + \tau_z\,\delta\gamma_z) = \delta_v A_i\,. \tag{12}$$

One then notes that, in particular, only the quantities (5) and (8) represent actual physical works done. By contrast, the quantities (7), (9), (10), (12) are only quantities for calculation that one can interpret as imaginary works due to the fact that they have the dimensions of kg · cm.

([15]) **Engesser** [Zeit. d. Architekten- u. Ingenieur-Vereins zu Hannover, **35** (1889), pp. 733, *et seq.*] and **Domke** [*loc. cit.*, footnote ([5])] referred to the quantities (7) and (10) as "virtual works" with no restricting qualifiers. That terminology cannot be justified, because since the time of Lagrange, the concept of virtual work has been established in the sense of eqs. (9) and (12). It is only for a special system of virtual displacements (that we shall have more to say about) that (9) and (12) will *formally* look like (7) and (10).

On the basis of the latter discussion, we can give the principle of virtual displacement [eq. (5)] the following form *in the case of complete elasticity:*

(13) $$\delta_v (A_i - A_{ea}) = 0 \,.$$

The index v means that each of the quantities in the parentheses is varied with respect to the displacement quantities that appear in it, so A_i is varied with respect to the ε and γ, while A_{ea} is varied with respect to the p_n . Moreover, eq. (13) is nothing but a mathematically-derived prescription for calculation. *It will also be true in that form when the deformations ε_x, …, τ_z have finite magnitudes in the equilibrium configuration, and it will be true for an arbitrary law of elasticity.*

2. – The interpretation of the main principle as a condition for the minimum of potential energy.

In eq. (13), the expression in parentheses is the potential energy of the total system, because if we denote the deformation work per unit volume by A_i^*:

(1) $$A_i^* = \int_0^{\varepsilon,\gamma} (\sigma_x \, d\varepsilon_x + \sigma_y \, d\varepsilon_y + \sigma_z \, d\varepsilon_z + \tau_x \, d\gamma_x + \tau_y \, d\gamma_y + \tau_z \, d\gamma_z)$$

then the expression under the integral sign will be a complete differential for completely-elastic bodies, and it will follow that:

(2) $$+\frac{\partial A_i^*}{\partial \varepsilon_x} = \sigma_x \,, \quad +\frac{\partial A_i^*}{\partial \varepsilon_y} = \sigma_y \,, \quad +\frac{\partial A_i^*}{\partial \varepsilon_z} = \sigma_z \,,$$
$$+\frac{\partial A_i^*}{\partial \gamma_x} = \tau_x \,, \quad +\frac{\partial A_i^*}{\partial \gamma_y} = \tau_y \,, \quad +\frac{\partial A_i^*}{\partial \gamma_z} = \tau_z \,.$$

The plus sign says that an increase in A_i^* means that the capacity of the internal forces to do work has increased. A_i is then the total potential energy of the internal forces:

(3) $$A_i = \Pi_i \,.$$

Likewise, (1.7) implies that:

$$-\frac{\partial(-A_{ea})}{\partial p_n} = P_n \,.$$

The minus sign says that an increase in A_{ea} (corresponding to work that has already been done by the external forces P_n) means that the capacity of the external forces to do further work has decreased. Hence, $-A_{ea}$ is the potential energy of external forces:

$$-A_{ea} = \Pi_a \,. \tag{5}$$

It follows from (3) and (5) that the potential energy of the total system is:

$$\Pi = \Pi_i + \Pi_a = A_i - A_{ea}\,, \tag{6}$$

and eq. 1.(13) can now be written:

$$\delta_v \,\Pi = 0\,. \tag{7}$$

The vanishing of the first variation of an expression means that this expression will be an extremum. We can then express the principle of virtual displacements for completely-elastic bodies in the following form: *The state of deformation that occurs in the equilibrium configuration is the state in which the potential energy of the total elastic system is a minimum.*

That formulation is also nothing but an *interpretation* of a purely-mathematical operation. That formulation has the advantages of brevity and memorability. It has the further advantage that it subordinates the appearance of elastic equilibrium to a general physical axiom that is always being confirmed by experiments, namely, the axiom of the minimum of potential energy. However, that *interpretation* (and not with the computational Ansatz!) implies a complication in our case that has already led to much confusion.

If one speaks of a minimum of the potential energy then that means: In the desired equilibrium state, the potential energy of the total system is smaller than it is in any other (neighboring) state of displacement. If we consider a pendulum that swings about its equilibrium state, or any other oscillating system, then it will, in fact, assume a comparison state in which the potential energy is greater than it is in the equilibrium configuration during its oscillation. The comparison state is then a *physically-possible state* in its own right. In our case, we have expressly excluded oscillatory processes in order to obtain no acceleration terms that would depend upon time in the energy theorem. Indeed, the loading shall result so gradually that the acceleration components can be neglected. Every transitional state up to a state of complete loading will be itself an equilibrium state, namely, the one that corresponds to the degree of loading that has been achieved at that time. Since only a single state of displacement is possible for a given material (i.e., a given law of elasticity) and a given loading P_n, namely, just the desired equilibrium state ε_x, …, γ_z, p_n, out *comparison states are not physically possible under the conditions on the system.* In order to be able to speak of comparison states at all, we must remove any of the geometric and physical conditions that were imposed on the problem. In our case, we have already made that decision: *Among all of the displacement states that are compatible with the geometric conditions on the system, we seek the ones that correspond to equilibrium between internal and external forces with* *the law of elasticity that is valid.* We thus remove the condition of equilibrium for the comparison state ([16]) and characterize it by saying that it is not, in fact, a physically-possible state.

Mathematically, that is not the most serious complication. We must simply replace the $\delta\varepsilon$, $\delta\gamma$, δp in eq. (1.5) with any quantities that lie between the aforementioned limits. However, for the *interpretation* in the sense of eq. (2.7) [(1.13), resp.], the complication arises that we must define the concept of potential energy, so the capacity to do work, for a state of our system that is not even physically possible at all. That definition cannot be a physical one, accordingly. It is a mathematical one, and indeed an arbitrary one, with the single restriction that it must lead to the mathematical operations in eq. (1.5, 13) [(2.7), resp.] when one establishes the minimum. In the present case, the solution is obvious: We define the potential energy formally using eq. (6) when we make the comparison state:

$$(8)\qquad \begin{cases} A_i' = \int dV \int\limits_0^{\varepsilon+\delta\varepsilon,\gamma+\delta\gamma} (\sigma_x\, d\varepsilon_x + \sigma_y\, d\varepsilon_y + \sigma_z\, d\varepsilon_z + \tau_x\, d\gamma_x + \tau_x\, d\gamma_x + \tau_x\, d\gamma_x) \\ A_{ea}' = \sum_n P_n\,(p_n + \delta p_n). \end{cases}$$

Since we have assumed the *reversibility* of the deformation process, the path of integration can always be continued beyond the equilibrium state $\varepsilon_x, \ldots, \gamma_z$. Since the law of elasticity, which determines the σ, τ as functions of the ε, γ, is also assumed to be valid for the comparison state, that definition will then be unambiguous, and when it is substituted in eq. (1.13), it will again lead back to eq. (1.5) in the neighborhood of the equilibrium state. The difference consists of the fact that according to our definition, the $\varepsilon_x, \ldots, \gamma_z$ in the comparison states no longer correspond to their values $\varepsilon_x, \ldots, \gamma_z$ in the equilibrium state, but to the quantities that correspond to the displacements $\varepsilon_x + \delta\varepsilon_x, \ldots, \gamma_z + \delta\gamma_z$ under the law of elasticity. However, the difference is a second-order infinitesimal, while eq. (1.5) represents a relation between first-order quantities.

If one chooses the arbitrary displacements in such a way that the points of application of the external forces are not displaced, so $\delta p_n = 0$ for all n, then one will get the equation $\delta_v A_i = 0$ from (1.5), instead of (2.7), so a law of minimum deformation work, instead of a law of minimum potential energy.

3. – The second variational principle.

Under the restricting assumption that the displacements ε, γ, p that actually occur in the equilibrium state are infinitely small, one can introduce them as a system of virtual displacements in eq. (1.5), since they certainly satisfy the remaining necessary assumptions for the $\delta\varepsilon$, $\delta\gamma$, δp

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([16]) **Kammüller** erred when he said [*loc. cit.*, footnote ([13]), pp. 272, last paragraph] that with the Ansatz $\sigma_x\ \delta\varepsilon_x$, "$\delta\varepsilon_x$ is not compatible with the connectivity of the body." Precisely the opposite is true! We *require* that the $\delta\varepsilon_x, \ldots$ *must* be compatible with the connectivity of the body. However, the stresses that correspond to the varied state of deformation $\varepsilon_x + \delta\varepsilon_x, \ldots, \gamma_z + \delta\gamma_z$, $p_n + \delta p_n$ under the law of elasticity that is valid can no longer be in equilibrium with the external forces.

(viz., compatibility and the fulfillment of the support conditions). When one recalls (1.10) and (2.7), (1.5) will then become:

$$(1)\qquad \begin{cases} \int dV\,(\sigma_x\,\delta\varepsilon_x+\sigma_y\,\delta\varepsilon_y+\sigma_z\,\delta\varepsilon_z+\tau_x\,\delta\gamma_x+\tau_y\,\delta\gamma_y+\tau_z\,\delta\gamma_z)-\sum_n P_n\,\delta p_n=0, \\ A_{ei}-A_{ea}=0. \end{cases}$$

That equation ([17]) is true only in the case of equilibrium, since eq. (1.5), from which it was obtained, is the mathematical formulation of the equilibrium condition. On the other hand, (1.5) is also true for an arbitrary, neighboring system of external forces $P_n + \delta P_n$ and the equilibrium system of stresses $\sigma_x + \delta\sigma_x$, …, $\tau_z + \delta\tau_z$ that corresponds to them:

$$\int dV\,[(\sigma_x+\delta\sigma_x)\,\delta\varepsilon_x+(\sigma_y+\delta\sigma_y)\,\delta\varepsilon_y+(\sigma_z\,\delta\sigma_z)\,\delta\varepsilon_z$$
$$+(\tau_x+\delta\tau_x)\,\delta\gamma_x+(\tau_y+\delta\tau_y)\,\delta\gamma_y+(\tau_z+\delta\tau_x)\,\delta\gamma_z]-\sum_n(P_n+\delta P_n)\,\delta p_n=0.$$

We can replace the $\delta\varepsilon$, $\delta\gamma$, δp in that with an arbitrary system of compatible displacements, so in particular the system ε_x, …, γ_z that belongs to the stresses and forces σ_x, …, τ_z, p_n, as long as we assume that those quantities are infinitely small:

$$\int dV\,[(\sigma_x+\delta\sigma_x)\,\varepsilon_x+(\sigma_y+\delta\sigma_y)\,\varepsilon_y+(\sigma_z\,\delta\sigma_z)\,\varepsilon_z$$
$$+(\tau_x+\delta\tau_x)\,\gamma_x+(\tau_y+\delta\tau_y)\,\gamma_y+(\tau_z+\delta\tau_x)\,\gamma_z]-\sum_n(P_n+\delta P_n)\,p_n=0.$$

If one subtracts eq. (1) from that equation then it will follow that:

$$(2)\qquad \int dV\,(\varepsilon_x\,\delta\sigma_x+\varepsilon_y\,\delta\sigma_y+\varepsilon_z\,\delta\sigma_z+\gamma_x\,\delta\tau_x+\gamma_y\,\delta\tau_y+\gamma_z\,\delta\tau_x)-\sum_n p_n\,\delta P_n=0\,.$$

With that, we have arrived at the second variational principle of the theory of elasticity, in which it is not the displacements that are varied, but the forces (stresses). One can then contrast the *principle of virtual forces* with the principle of virtual displacements. In that equation, the $\delta\sigma$, $\delta\tau$, δP are *stresses and forces that are infinitely small, continuously dependent upon the position coordinates, and found to be in equilibrium with each other, but can be otherwise chosen arbitrarily.*

The second variational principle is valid in the form of eq. (2) for an arbitrary law of elasticity, and indeed even when the material that the body is composed of is not completely elastic. By

([17]) That is the aforementioned special system of virtual displacements under which the virtual work will be formally equal to the final work, and which led **Engesser** and **Domke** [*loc. cit.*, footnote ([15])] to employ the concept of virtual work in a way that was not generally consistent with **Lagrange**'s classical definition.

contrast, *it is not valid for finite deformations*, in general, since we must assume that the $\varepsilon_x, \ldots, \gamma_z$, p_n are infinitesimal in its derivation.

Here, we already see a crucial difference between the second principle and the first one, which shows that despite the formal similarity between eqs. (3.2) and (1.5), they are essentially different.

In the special case of complete elasticity, the quantity in parentheses in eq. (2) will be a complete differential. In that case, we introduce a new quantity B_i by the following definition:

$$(3) \qquad B_i = \int dV\,(\varepsilon_x\,\delta\sigma_x + \varepsilon_y\,\delta\sigma_y + \varepsilon_z\,\delta\sigma_z + \gamma_x\,\delta\tau_x + \gamma_y\,\delta\tau_y + \gamma_z\,\delta\tau_x)\ .$$

B_i has the dimension of work and will be referred to as the *extension work done by internal forces*, following **Engesser** ([18]). We correspondingly definition the *extension work done by external forces* by:

$$(4) \qquad B_a = \sum_n \int_0^{P_n} p_n\,dP_n\ .$$

One obtains from (1.6), (1.10), and (3.3., as well as (1.8), (1.7), and (3.4), by partial integration:

$$(5) \qquad A_i = A_{ei} - B_i\ ,$$
$$(6) \qquad A_a = A_{ea} - B_a\ ,$$

in full generality, and even for finite deformations. From (1.11) and (3.1), that will imply that $B_i = B_a$. However, that equation is true only when (3.1) is true, i.e., it is true for only infinitely-small deformations, in general ([18]). If one introduces (3) into (2) then one will get the second variational principle in the form:

$$(7) \qquad \delta_k\,(B_i - A_{ea}) = 0$$

in the case of complete elasticity and for an arbitrary law of elasticity.

The index k in that means that each of the quantities in parentheses is varied with respect to the forces (stresses, resp.) that appear in it, so the extension work with respect to the σ, τ and the final work with respect to the external forces P_n ; eq. (7) corresponds to eq. (1.13) formally. Nonetheless, the meaning of the two equations is completely different, which would follow from their derivation. (1.13) was the condition for the quantity ($A_i - A_{ea}$) to be a *minimum*. By contrast, (3.7) is the condition for the quantity ($B_i - A_{ea}$) to be a *maximum*. One convinces oneself of that most simply when one derives (3.2) directly by the complete variation of A_{ei} and A_{ea} using (1.5). From (3.1), the complete variations of A_{ei} and A_{ea} will then cancel each other out when one imposes the condition that only equilibrium systems of forces should be considered. That is because ($A_{ei} - A_{ea}$) is constantly equal to zero then, and as a result, the variation of that expression will also

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([18]) **Engesser**, [*loc. cit.*, footnote ([13]), pp. pp. 743] still did not distinguish between the extension work done by the internal and external forces, since he restricted his considerations to infinitely-small deformations, and both quantities will be equal to each other in that case. Cf., also **Domke** [*loc. cit.*, footnote ([5]), pp. 177].

vanish. One then gets the result that when ($-B_i + A_{ea}$) is varied with respect to the forces (stresses), it must be a minimum, so ($B_i - A_{ea}$) must be a maximum.

Furthermore, the expression ($B_i - A_{ea}$) is by no means equal to the potential energy in the total system. *Therefore, the second variational principle, like the first one, cannot be subordinate to a general law of nature that is confirmed by experiment*. It is nothing but a mathematical conversion of the first principle, which was valid only under severely-restricting conditions, to boot, namely, the assumption that the deformations of the body were infinitely small. Above all, it completely absurd to speak of a "minimum of potential energy" with the second principle (which is applied much more often than the first one, in practice), since on the one hand, the expression to be varied is not the potential energy, and on the other, it is not its minimum that is sought, but its maximum.

One then sees that despite the formal analogy between eqs. (3.2) and (1.5) [(3.7) and (1.13), resp.], there is no analogy at all between the domain of validity and the interpretation of those equations, and one will also see the reason why ([19]). In particular, it is expressed by the fact that the second principle admits no transition to eq. (2.7) or an equation that is analogous to it that can be interpreted physically. However, that is further expressed by the fact that one cannot derive the second principle directly from physical foundations in a way that would correspond to section 1. It is a rule of computation that cannot be interpreted physically.

Nothing in that is altered by the fact that in the most-frequently-occurring case in practice, one will have $B_i - A_i = \frac{1}{2} \cdot A_{ea}$ by the validity of **Hooke**'s law. One will indeed make no computational error then when one replaces the extension work in (3.7) with the deformation work, and in so doing, make the analogy to eq. (1.13) complete. However, one would violate the spirit of that equation if one referred to the expression thus-obtained as "potential energy." Above all, it would be simply false if one stated that one then seeks the minimum of that "potential energy" with eq. (7). That is because (7) determines the *maximum* of the expression in parentheses in all circumstances, regardless of whether Hooke's law is true or not. The fact that this mix-up also leads to no error in calculation is merely due to the fact that the necessary condition for the occurrence of a maximum is mathematically the same as it is for the occurrence of a minimum, namely, the vanishing of the first variation. The two cases are distinguished by only the sufficient conditions that pertain to them, namely, the sign of the *second* variation, and the practical calculator probably never cares to test it, and with good reason, since that question has practically no definitive meaning, in general.

However, at this point [as before, *loc. cit.* ([20])], I would like to stress that it seems pedagogically dubious and inconsistent with the foundations of a science in this era to express theorems whose assumptions and limitations one knows precisely in a form that indeed produces correct results in its domain of validity that is most important today, but will lead to false results in a different part of its domain of validity and will not reproduce the actual content of the theorem in question in each case.

It still remains for us to examine the type of comparison state under which eq. (7) will single out the unique state that actually occurs. Even in that case, the comparison states are not physically possible under the conditions on the system, on the grounds that were cited in Section 2.

([19]) **Kammüller**, [*loc. cit.*, footnote ([13]), pp. 271, rebuttal, no. 4].
([20]) **Schleusner** [*loc. cit.*, footnote ([12]), pp. 254 and *loc. cit.* footnote ([13]), pp. 271].

Corresponding to eq. (2.8), we define the extension work and the final work for the comparison state by:

(8) $$\begin{cases} B'_i = \int dV \int_0^{\sigma+\delta\sigma,\tau+\delta\tau} (\varepsilon_x\, d\sigma_x + \varepsilon_y\, d\sigma_y + \varepsilon_z\, d\sigma_z + \gamma_x\, d\tau_x + \gamma_x\, d\tau_x + \gamma_x\, d\tau_x) \\ B'_{ea} = \sum_n (p_n + \delta p_n)\, p_n \ . \end{cases}$$

In that, we assume that the law of elasticity that determines ε, γ as functions of σ, τ is also valid for the comparison states. Moreover, in the derivation of the second variational principle, we assumed that the varied stresses and external forces were also in equilibrium with each other. We must necessarily abandon the condition of compatibility, i.e., the consistency with the geometric conditions on the system, for the comparison states. That is, we must imagine that the connectivity of the body is lost, or the support conditions are abandoned at one or more places along one or more surfaces. The result is then:

In the application of the second variational principle, among all systems of forces and stresses in equilibrium, we seek that one of them that corresponds to a compatible state of displacement (i.e., consistent with the connectivity of the body and the support conditions) under the law of elasticity that is valid.

Indeed, the second variational principle has a narrower domain of validity than the first one. However, its domain of validity generally overlaps with that of the first. That is because with the condition of equilibrium of the forces that is required by the second principle, it is generally easier to mathematically formulate it in a complicated system than with the condition of the compatibility of displacements that is required by the first principle. An especially-important realm of applications of the second principle is the calculation of statically-indeterminate quantities. One can calculate them with either a direct application of the second principle in the form of eq. (3.2) or (3.7) or with the help of **Castigliano**'s theorem that can be derived most simply from the second principle.

The deformation state depends on external forces, among which we also include the support forces, as remarked before. We can then represent the extension work done by the internal forces B_i as a function of the external forces: $B_i = B_i\,(P_n)$. By varying that function with respect to the forces, we will get:

(9) $$\delta_k\, B_i = \sum_n \frac{\partial B_i}{\partial P_n} \cdot \delta P_n \ .$$

On the other hand, it follows from (3.2) with (3.3):

(10) $$\delta_k\, B_i = \sum_n p_n \cdot \delta P_n \ .$$

If we substitute (10) in (9) then it will follow that:

(11) $$\frac{\partial B_i}{\partial P_n} = p_n .$$

We now choose the variations of the forces in such a way that the given external loads are not varied. One will then have $\delta P_n = 0$ for those quantities in (9) and (10). By contrast, we imagine that the connectivity of the body is broken at the points or surfaces of application of statically-indeterminate forces and that those statically-indeterminate forces (which we will denote by X_n in the usual way) act as external forces. Eq. (11) will then go to:

(12) $$\frac{\partial B_i}{\partial X_n} = x_n ,$$

when x_n means the displacement of the force X_n in the direction of the force. In that way, from the assumptions of the second principle, the functional dependency of the extension work B_i on the statically-indeterminate forces X_n must be understood in the following way: If the connectivity of the body is lost in the way that was described above then one chooses the X_n to be an arbitrary system of forces that is in equilibrium with the given external forces P_n . One then determines the system of stresses in equilibrium σ_x, …, τ_z that is associated with the extended system of external forces P_n, X_n . B_i will then be determined uniquely as a function of the X_n by eq. (3.2) from the law of elasticity, and with that, the sense of eq. (12) is also established uniquely. The condition of compatibility is that all of the cuts made on the body close, so all of the displacement path-lengths x_n vanish. (12) will then be:

(13) $$\frac{\partial B_i}{\partial X_n} = 0 \qquad (n = 1, 2, \ldots).$$

One can once more interpret the system in equilibrium (130 as the condition for the occurrence of an extremum, and indeed one can show that one is dealing with a minimum. One can then express the content of eq. (13), namely, **Casitgliano**'s theorem, as:

The statically-indeterminate quantities do, in fact, assume the value that would make the extension work done by internal forces be a minimum.

In that way, B_i is regarded as a function of the X_n in the sense that was described above. **Castigliano**'s theorem, like the variational principle from which it was derived, is true in that form only for infinitely-small deformations and only under the assumption of complete elasticity of the material that the body is composed of, but for an arbitrary law of elasticity.

4. – A third form for the variational principle. The work equation.

We have referred to the fact that the principle of virtual displacements, in its original form that was treated in sections 1 and 2 (viz., varying the displacements), is true for not only arbitrary laws of elasticity, but also for finite deformations $\varepsilon_x, \ldots, \gamma_z, p_n$. Now, in practical statics, very often (but not by any means always) the following assumptions are fulfilled, at least approximately:

1. The displacement quantities $\varepsilon_x, \ldots, \gamma_z, p_n$ are regarded as vanishingly-small in comparison to the dimensions of the body.

2. The elongations and shears ε, γ are proportional to the stresses σ, τ (**Hooke**'s law of elasticity).

If those two assumptions are fulfilled then it is not necessarily true, but true in the majority of cases that occur in practice, that:

3. The displacement path-lengths p_n of the external forces P_n are proportional to the magnitudes of those forces.

One can easily verify that the third assumption does not necessarily follow from the first two in some examples. For instance, it is not true for a straight rod with a center joint that is fixed, but articulated, at the ends and loaded transversely to the axis of the rod at the center joint. It is also not true for an axially-compressed straight rod after it exceeds the limits of buckling ([21]).

If all three assumptions are fulfilled simultaneously then (and only then, in general) one can *formally* substitute finite quantities for the $\delta\varepsilon, \delta\gamma, \delta p$ in (1.5), in which one imagines that a common infinitely-small proportionality factor has been dropped from the equations. If one replaces the notations $\delta\varepsilon, \delta\gamma, \delta p$ with the notations ε, γ, p in (1.5), in that spirit, then one will get the equation:

$$(1) \qquad \int dV\,(\sigma_x\,\varepsilon_x + \sigma_y\,\varepsilon_y + \sigma_z\,\varepsilon_z + \tau_x\,\gamma_x + \tau_y\,\gamma_y + \tau_z\,\gamma_z) - \sum_n P_n\,p_n = 0\,.$$

That is *formally* the same equation as (3.1). However, the meaning of the quantities is essentially different in the two equations. The σ, τ, P means the stresses and forces that are actually present in the equilibrium state in both equations. However, the ε, γ, p in (3.1) mean the ones that correspond to the σ, τ, P under the law of elasticity that is valid, so the displacement quantities that actually occur. By contrast, in (4.1), they mean an arbitrary, but compatible, system of

([21]) It was perhaps this fact that led **Pöschl** to the strange conclusion that one required a special form for the variational principle for the buckling problem [cf., footnote ([8])]. Of course, for the buckling problem, it is not the special, much-used forms of the variational theorems that we derived conceptually that are valid, but probably the generally-valid first form of the principle (1.5) [(1.13) or (2.7), resp.], from which all other forms are first derived by introducing restricting special assumptions. If one restricts oneself to infinitely-small deflections then the second variational principle (3.2) [(3.7), resp.] will still be true for the buckling problem.

displacements that does not need to have any relationship to the actual stresses and forces σ_x, …, τ_z, P_n .

If the first of the three aforementioned conditions is fulfilled then the second variational principle will also be applicable. If the other two conditions are fulfilled, in addition, then one can formally replace the $\delta\sigma$, $\delta\tau$, δP in (3.2) with finite quantities, when one imagines that a common, infinitely-small proportionality factor has been dropped from the equation. If one replaces the notations $\delta\sigma$, $\delta\tau$, δP in (3.2) with σ, τ, P, accordingly, then that equation will *formally* assume the same form as (4.1). However, the meaning of those quantities in this case is, in turn, completely different from before. That is because the ε, γ, p now mean the displacement quantities that actually occur in the equilibrium state, while the σ, τ, P mean an arbitrary system of stresses and forces in equilibrium that does not need to have any relationship to the displacements ε_x, …, γ_z, p_n that actually occur.

This third form of the principle of virtual displacements is the form that is applied most often in practical statics ([22]). **Müller-Breslau** introduced the term "work equation" for the elastic system being investigated for it ([23]). That third form is, as its derivation shows, *not an autonomous variational principle*, but according to whether one couples the actual stresses and forces with imaginary displacements or one couples the actual displacements with imaginary stresses and forces, it will be a *modified form of the first or second variational principle, respectively*, with a strongly-restricted domain of validity (e.g., infinitely-small deformations, **Hooke**'s law, proportionality of forces and displacements, and even for the external forces). One can then express the double meaning to the content of eq. (1) as:

We either seek, among all compatible systems of displacements, the unique one that corresponds to system of stresses and forces that actually occurs under Hooke's law or we seek, among all systems of forces stresses in equilibrium, the unique one that corresponds to the system of displacements that actually occurs in the equilibrium state under Hooke's law.

The usefulness of this simple combination of the two variational principles, in a slightly-modified form, into a single equation is beyond question. However, one must not forget that its domain of applicability is restricted considerably in comparison to the domain of validity of the two variational principles and that the *combination is a purely-formal one*. In that form, the principle is not even capable of being given a unified, intuitive interpretation. It says things that are intrinsically quite different according to whether one applies it in the spirit of the first principle (actual stresses and forces, imaginary displacements) or in the spirit of the second principle (actually displacements, imaginary stresses and forces).

The work equation can be employed in practice only when one is dealing with the determination of a finite number of unknown quantities (say, the displacements of isolated points or the magnitude of isolated statically-indeterminate support forces and clamping moments). The work equation will become unusable in practice as soon as one deals with the determination of

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([22]) **Kammüller** also started from this form of the principle. [*Loc. cit.*, footnote ([11]), pp. 363, eq. (1)]

([23]) **Müller-Breslau**, *Die neueren Methoden der Festigkeitslehre und der Statik der Baukonstruktionen*, Leipzig 1904, pp. 23, eq. (6).

unknown functions (say, of bending lines or deformations of two or three-dimensional structures), or as soon as one of the aforementioned assumptions is no longer fulfilled. One would then have to revert to the variational Ansätze (3.7) [(1.13), resp.].

5. – The concept of virtual work and its relationship to the variational principles of the theory of elasticity.

We proposed to the combine two variational principles, namely, the principle of virtual displacements and the principle of virtual forces, under the generic term of "principle of virtual works" ([24]). I have already remarked ([25]) that this proposal does not seem felicitous, and I would like to base that claim rigorously here.

Obviously, that proposal was inspired by the third form of the principle, namely, the work equation (4.1), in which the fundamental difference between the two variational principles seemed formally blurred. In order to avoid any error in our consideration of the principles in the form of the work equation, we must always revert to the explicit form of the first (second, resp.) variational principle then, according to which sense of the work equation we are applying at the time.

The first objection to the aforementioned proposal is based on the different domains of validity of the two principles and the work equation. The first principle is true in general. The second one is true only for infinitely-small deformations, i.e., only for a subset of the domain of validity of the first principle. The two principles are not on an equal par with each other, but the second one is subordinate to the first, so it is only a consequence of the first one that is connected with a special assumption. Finally, the work equation is true for only a subset of the domain of the second principle, so it is subordinate to both principles. It is therefore impossible to build a new generic theoretical concept that subsumes both principles and is based on the domain of validity of the work equation, since it would not encompass either of the two principles completely. However, the *practical* desire to have a simplest-possible union of the principles into a form that would be sufficient for the vast majority of cases that arise in daily requirements suggests that eq. (4.1), with the simple name of "work equation" that **Müller-Breslau** gave to it, should suffice.

If the first objection was directed against the basis for the proposed new generic concept then the second one is directed against its terminology. Since the time of **Lagrange**, one understands virtual work to mean the work that is done by the *forces or stresses that are actually present in the equilibrium states* of a system (so in our case, the quantities $\sigma_x, \ldots, \tau_z, P_n$) under a displacement of the equilibrium configuration that is thought to be small (so for the present problem [cf., (1.9) and (1.12)], the quantities $\int dV\,(\sigma_x\,\delta\varepsilon_x + \cdots + \tau_x\,\delta\gamma_x)$ and $\sum_m P_n\,\delta p_n$). In that representation, we have learned of a whole series of other quantities that likewise have the dimension of work and are likewise-imaginary quantities, so imaginary works:

([24]) **Kammüller** [*loc. cit.*, footnote ([13]), pp. 271, letter, section 5] "Instead of the principle of virtual displacements, one would be more correct in saying the principle of virtual works, which would then subsume the variation with respect to deformations, as well as with respect to forces, from the theoretical standpoint." Cf., also, *loc. cit.*, pp. 272, rebuttal, last sentence.

([25]) **Schleusner** [*loc. cit.*, footnote ([13]), pp. 271, reply, last paragraph].

$$\left[\int dV\,(\sigma_x\,\varepsilon_x + \cdots), \sum\nolimits_m P_n\, p_n, \int dV\,(\varepsilon_x\,\delta\sigma_x + \cdots), \sum\nolimits_m p_n\,\delta P_n, \int dV\,(\varepsilon_x\, d\sigma_x + \cdots)\right] .$$

If one would also like to refer to those quantities as "virtual works" ([26]) then one would succeed in only blurring a concept that has been uniquely and clearly delineated for 150 years, and causing confusion in so doing.

The equations of the first principle, viz., (1.5) and (1.13), and likewise the work equation (4.1), when one applies it in the spirit of the first principle (i.e., actual stresses and forces, imaginary displacements), are then equations between virtual works done on the system being investigated. However, they are the equations of the second principle, viz., (3.2) and (3.7), or those of the work equation (4.1), when one applies it in the spirit of the second principle (i.e., actual displacements, imaginary stresses and forces)!

The quantities $\int dV\,(\varepsilon_x\,\delta\sigma_x + \cdots)$ and $\sum\nolimits_m p_n\,\delta P_n$ that appear in the second principle [cf., (3.2)] indeed have the dimension of work and are imaginary quantities. However, they are not imaginary works done by the *stresses and forces that are actually present in the equilibrium state.*

Naturally, one can also interpret the quantities in the second principle as virtual works if one would even like to do that. Since the deformations $\varepsilon_x, \ldots, \gamma_z$ that actually occur in the second principle must be assumed to be infinitely small, one can regard them as a *special* system of displacements (as we did in the derivation of the work equation). With that assumption, the quantities $\int dV\,(\varepsilon_x\,\delta\sigma_x + \cdots)$, $\sum\nolimits_m p_n\,\delta P_n$ are also virtual works in the classical sense of the concept. However, they are not virtual works done by the system of forces $\sigma_x, \ldots, \tau_z, P_n$ that are actually present in equilibrium, but a completely-different system of forces $\delta\sigma_x, \ldots, \delta\tau_z, \delta P_n$. That system of forces is not only different from the one that is actually present in equilibrium, it is not even close to it. That is because even when one assumes that the deformations $\varepsilon_x, \ldots, \gamma_z, p_n$ are infinitely small, (as is known) the associated stresses and forces $\sigma_x, \ldots, \tau_z, P_n$ can definitely have finite magnitudes. (One needs only to imagine a one-axis stress state that corresponds to the relation $\sigma = \varepsilon \cdot E$ under **Hooke**'s law and the appreciable magnitude of E in the units of measurement that one uses.) By contrast, the stresses and forces $\delta\sigma_x, \ldots, \delta\tau_z, \delta P_n$ must be assumed to be infinitely small. If one would then like to already interpret the quantities in the second principle as virtual works then they will not, in any event, be the virtual works that are done by the forces that are actually present in the equilibrium state of the system in question, but a system of forces that is completely different from it, even in terms of orders of magnitude. Even when one completes the transition from the second principle to the work equation (4.1), nothing will change in that. Properly speaking, one must indeed imagine that all of the stresses and forces are multiplied by a common infinitely-small proportionality factor when one applies the work equation in the sense of the second principle. The wording of the arguments up to now will then be preserved. However, if one imagines that the proportionality factor has been dropped then the imaginary

([26]) **Kammüller**, *loc. cit.*, footnote ([13]), pp. 272, last sentence.

stresses and forces (which are now formally finite in size) will likewise be by no means close to the actual system of forces ([27]).

If one would then like to interpret the second principle as the "principle of virtual work" in the classical sense then one would have to apply it, not to the forces that are actually present in equilibrium, but to any arbitrary system of forces in equilibrium that is not even close to it, which would lose the character of a variation of the forces that are actually present under that interpretation.

Whereas one then obtains equations between the virtual works in the system being examined from the first principle, the second principle will yield equations between the virtual works in the virtual replacement system that are basically different from the actual system and merely selected in such a way that that the mathematical operations produce the desired results for the actual system.

It seems to me that there is nothing to be gained in theoretical clarity by constructing a new theoretical concept on such a foundation (which is the only one on which one can justify the concept of virtual work for the quantities in the second principle).

Moreover, we suspect that this interpretation of eq. (3.2) also contradicts its interpretation as an extremal condition. We have already mentioned the fact that with eq. (3.2), and therefore the work equation (4.1), as well, when we apply them in the spirit of the second principle (actual displacements, imaginary stresses and forces), we can no longer speak of a minimum of the potential energy, because first of all it is a maximum, and secondly it is not the potential energy that we seek. However, if we interpret eq. (3.2) as an equation between virtual works then we will no longer be dealing with either a minimum or a maximum, nor will we be dealing with either the potential energy or any other quantity. That is because the interpretation of eq. (3.2) as an extremal condition is expressed mathematically by (3.7). However, the B_i and A_{ea} in that equation are not the extension work (final work, resp.) of the arbitrary system of forces $\delta\sigma_x, \ldots, \delta\tau_z, \delta P_n$, but the system of forces that is actually present in the equilibrium state $\sigma_x, \ldots, \tau_z, P_n$, the systems of forces that is immediately-close to it $\sigma_x + \delta\sigma_x, \ldots, \tau_z + \delta\tau_z, P_n + \delta P_n$.

There are only three possibilities then: Firstly, one might refer eq. (3.2), and correspondingly eq. (4.1), in the one case, to an interpretation as the actual system being examined, and in the other, to a virtual replacement system that no longer has anything to do with the actual system, by an application in the spirit of the second principle. I cannot imagine that this would contribute to the clarity of the theoretical concept. Second, one gives weight to the intuitive and memorable interpretation as an extremal condition. One would probably need to have compelling reasons for introducing a new theoretical concept that also seems untenable (according to the first objection). Thirdly, one abandons the artificial interpretation of the quantities in the second principle as virtual works, as one desires, and in that way also abandons the combination of both principles into a general term for a principle of virtual works. The result is then:

There is only one principle that subsumes all phenomena: It is the principle of virtual displacements, which can be interpreted as the condition of the minimum of potential energy in

([27]) If one has a simple statically-indeterminate system then one might care to choose the system of virtual forces to be the system $P_1 = 0$, $P_2 = 0$, …, $X = -1$, and the system of stresses that is in equilibrium with external forces. It is obvious that this system is not at all close to the system of forces $P_1 = P_1$, $P_2 = P_2$, …, $X = X$ that is actually present in equilibrium.

the case of complete elasticity. A second principle can be derived from it under severely-restricting assumptions: It is the principle of virtual works, which can be interpreted as the condition for the maximum of the expression ($B_i - A_{ea}$) in the case of complete elasticity. Both principles can be formally combined into single equation by repeated gross restrictions on the domain of validity, namely, the work equation, which does not admit a consistent interpretation, but can be interpreted in different way according to whether it is applied in the spirit of the first or second principle. For practical applications, the work equation is still the most important equation of all of them. However, whoever would like to appeal to the theoretical clarity of variational principles could find no object more unsuitable as a foundation than the work equation. It must always revert to the basic form of the two principles, and eventually it always reverts to the first principle, namely, the principle of virtual displacements.