“Zur Kinematik und Dynamik der nichtlinearen Kontinuumstheorie,” Ann. Phys. (Leipzig) (7) **21** (1968), 57-69.

On the kinematics and dynamics of the nonlinear continuum theory of dislocations

By H.-A. BAHR and H.-G. SCHÖPF

Translated by D. H. Delphenich

Abstract. – The fundamental kinematical equation of the notion of dislocations will be derived and explained within the context of continuum theory. Information about the forces that act upon dislocations will be obtained with the help of a variational principle.

Introduction

An essential problem in the current state of the theory of dislocations is the extension of the static continuum theory [**1**] by way of kinematics and dynamics. Along with general arguments that extend the differential-geometric methods that are known in statics to four-dimensional processes [**2**], one is, above all, interested in physical consequences that are implied by the introduction of a velocity field for the dislocations. Here, one should cite the papers of KOSEVICH [**3**], which started by considering isolated dislocations in the linear theory and adapting the results to the continuum description. In addition to the investigations of MURA [**4**], we would especially like to stress the work of FOX [**5**]. In it, among other things, a nonlinear kinematics was developed that referred to the continuum from the outset. We consider the result, which we call the “fundamental kinematical equation,” to be quite important. Therefore, we shall first derive it briefly. In it, we shall place great value on an intuitive understanding of the equation by first treating the velocity field of the dislocations by analogy with the velocity field of a material medium, and then deriving the fundamental equation as a generalization of KRÖNER’s formulation of statics by way of the demand that the BURGERS vector should be conserved. In addition, the train of reasoning and physical understanding will require the use of a time derivative that was introduced for the first time by OLDROYD [**6**]. Finally, we can confirm the validity of the kinematical equation by additional arguments.

In the second part of the present paper, we will enter into the dynamics of moving dislocations in continuum theory by presenting and discussing a suitable variational principle. We first present it for conventional continuum mechanics and then generalize it to the case that is of interest here. In that way, the fundamental kinematical equation will be considered to be an auxiliary condition. The goal of this study is to obtain information about the structure of the forces that act upon dislocations. We will then obtain the PEACH-KOEHLER force for the nonlinear continuum theory, but extended by a term that can be interpreted as a generalization of the one that is found in KOSEVICH's linear theory [**3**] for isolated dislocations.

It should be mentioned that we shall adopt KRÖNER's notation [**1**] throughout.

1. – Kinematics.

1.1. – Basic kinematic notions. – We denote the LAGRANGE coordinates of a matter-point by X^L such that the course of motion will be described by:

$$x^k = x^k(X^L, t) , \tag{1}$$

if x^k are the coordinates in the laboratory system. With no loss of generality, they can be assumed to be rectangular-Cartesian ones. In the other cases that follow, it is reasonable to assume that partial derivatives are covariant ones.

According to the usual axioms of continuum mechanics, the deformation gradients will exist:

$$A_L^k \equiv \frac{\partial x^k}{\partial X^L}, \quad A_k^L \equiv \frac{\partial X^L}{\partial x^k}, \tag{2}$$

as well as higher derivatives, to the extent that is necessary. The velocity field of matter will then be given by:

$$v^k = \left.\frac{\partial x^k}{\partial t}\right|_{X^L = \text{const.}} . \tag{3}$$

Corresponding formulas will be true when a second type of matter comes under consideration and has the LAGRANGE coordinates $\tilde{X}^{\mathfrak{K}}$. It is preferable to formulate those equations four-dimensionally by additionally considering:

$$x^4 = X^4 = \tilde{X}^4 = t, \tag{4}$$

and writing:

$$v^\mu = \frac{\partial x^k}{\partial X^4}, \qquad \tilde{v}^\mu = \frac{\partial x^\mu}{\partial \tilde{X}^4}, \tag{5}$$

in which the three-dimensional velocities for $\mu = k$ have now been extended by:

$$v^4 = \tilde{v}^4 = 1.$$

We then get the relative velocity:

$$\left.\frac{\partial X^K}{\partial t}\right|_{\tilde{X}^{\mathfrak{L}}} = \frac{\partial X^K}{\partial \tilde{X}^4} = \frac{\partial X^K}{\partial x^\mu}\frac{\partial x^\mu}{\partial \tilde{X}^4}$$

$$= \frac{\partial X^K}{\partial x^\mu}\left(\frac{\partial x^\mu}{\partial \tilde{X}^4} - \frac{\partial x^\mu}{\partial X^4}\right)$$

$$= \frac{\partial X^K}{\partial x^k}\left(\tilde{v}^k - v^k\right). \tag{6}$$

It will prove to be extremely useful to employ not only the substantial time-derivative of a tensor:

$$\frac{d}{dt}T_{l\cdots}^{k\cdots} \equiv \frac{\partial}{\partial t}T_{l\cdots}^{k\cdots} + T_{l\cdots,p}^{k\cdots}\, v^p , \tag{7}$$

but also a time derivative (convected time derivative) that was first introduced by OLDROYD [**6**]. It has a very intuitive meaning, namely, the temporal change in the tensor components that would be established by an observer that moves with the matter:

$$\begin{aligned} \frac{\eth}{\eth t}T_{l\cdots}^{k\cdots} &= A_{K\cdots}^{k\cdots} A_{l\cdots}^{L\cdots} \frac{\eth}{\eth t}T_{L\cdots}^{K\cdots} \\ &= \frac{\partial}{\partial t}T_{l\cdots}^{k\cdots} + T_{l\cdots,p}^{k\cdots}\, v^p + T_{p\cdots}^{k\cdots}\, v^p_{\ ,l} + \cdots - T_{l\cdots}^{p\cdots}\, v^k_{\ ,p} - \ldots \end{aligned} \tag{9}$$

The tensor character of OLDROYD's derivative is clear on the basis of (9). One generally arrives at a deeper understanding of it in a four-dimensional consideration (see [**7**] on that).

As an important example of an application, we mention that the so-called "deformation velocity tensor" is the OLDROYD derivative of the deformation tensor:

$$v_{(i,\,k)} = \frac{\eth}{\eth t}\,\varepsilon_{ik}\,, \tag{10}$$

$$\varepsilon_{ik} = \tfrac{1}{2}\,(\delta_{ik} - b_{ik}), \tag{11}$$

with

$$b_{ik} \equiv A_i^{\ A} A_k^{\ B} \delta_{AB} \ \Rightarrow \frac{\eth}{\eth t}\ b_{ik} = 0.$$

The substantial (OLDROYD, resp.) time derivative that relates to the velocity field will be denoted by $\tilde{d}/dt$ ($\tilde{\eth}/\eth t$, resp.) in what follows.

1.2. Consequence of the constancy of the BURGERS vector. – It is known that a basic idea of the continuum theory of dislocations is to extend the lattice vectors and their reciprocal to fields of triads $A_{(k)}^{k}$ and $A_l^{(l)}$, in order to adapt the crystalline structure to the continuum description in a rudimentary way. (Indices in parentheses serve to enumerate the various vectors.) The main problem in kinematics consists of making some statement about the time evolution of those vectors. That will not be determined by just the velocity v^k of matter. Among other things, it will depend significantly upon the motion of the dislocations, moreover.

In order to be able to work with the notations that were provided in the previous section, the following intuitive argument shall be put forth:

We consider a system of moving dislocation lines that do not intersect each other. They might be specified by, say, being given the coordinates, $\tilde{X}^1$, $\tilde{X}^2$ of the points at which they go through some surface at the time $t = t_0$. Furthermore, a curve parameter $\tilde{X}^3$ might be defined on it. With that:

$$x^k = x^k(\tilde{X}^1, \tilde{X}^2, \tilde{X}^3, t) \tag{1}$$

will be a parametric representation of such a dislocation line at time t when $\tilde{X}^1$, $\tilde{X}^2$ are constant. The transition to the continuum description will come about when one allows $\tilde{X}^1$, $\tilde{X}^2$ to take on a continuum of values. The $\tilde{X}^k$ might be chosen in such a way that (1) is soluble in a certain neighborhood of them. They can then be interpreted as LAGRANGE coordinates of the dislocations, and that will, in fact, exhibit the connection to the considerations above. In particular, $\tilde{v}^k$ can now be regarded as the velocity of the dislocations. In general, it will also contain a component in the directions tangent to the dislocation lines. Naturally, that component has no physical meaning, so it can be defined arbitrarily. However, in what follows, we shall refrain from making such a definition in order to not perturb the symmetry in our formulas. In addition, let it be pointed out that in the context of continuum theory, we can imagine a superposition of systems of dislocations of the kind being treated that have varying velocities [**5**].

The time evolution of the distribution of dislocations shall be determined by its motion alone; i.e., their creation and annihilation will not be considered explicitly in this paper. In that way, we can begin with the further consideration of the important convention that the BURGERS vector $b^{(k)}$ is constant in time:

$$\frac{\tilde{d}}{dt} b^{(k)} = 0 \tag{2}$$

for observers that move with the distribution of dislocations. In that condition, it is essential that $b^{(k)}$ should be defined originally in an ideal crystalline comparison state in which the lattice vectors suffer no temporal variation. In order to evaluate (2), we note the following known formulas:

$$b^{(k)} = -\oint dx^m A_m^{(k)} , \tag{3}$$

$$b^{(k)} = -\int \alpha_{lm}{}^{(k)} df^{lm} \tag{4}$$

$$= \int \alpha^{n(k)} df_n = \int \alpha^{N(k)} df_N ,$$

with which the dislocation density:

$$\alpha_{lm}{}^{(k)} = A_{[m,l]}^{(k)} = -\tfrac{1}{2} \varepsilon_{lmn} \, \alpha^{n(k)}, \tag{5}$$

$$\alpha^{n\,(k)} = -\,\varepsilon_{lmn}\,\alpha_{lm}{}^{(k)}$$

will be a measure of the number of dislocation lines that pierce the surface element $df^{\,lm}$ and have the glide direction $(-\,k)$.

With the use of (3) and:

$$\frac{\tilde{d}}{dt}\,dx^m = dx^l\,\tilde{v}^m{}_{,l}\,, \tag{6}$$

after a simple conversion, one will see that the following expression must be a gradient:

$$\frac{\partial}{\partial t}A_m^{(k)} + 2\tilde{v}^p A_{[m,p]}^{(k)} = T^{(k)}{}_{,m}\,. \tag{7}$$

An application of the operator $-\,\varepsilon^{lmn}\,\partial_l$, along with (5), will imply that:

$$-\,\varepsilon^{lmn}\,\frac{\partial}{\partial t}\alpha_{lm}{}^{(k)} = \frac{\partial}{\partial t}\alpha^{n(k)} = -2\,\partial_p(\alpha^{[n(k)}\,\tilde{v}^{p]})\,. \tag{8}$$

That equation can be regarded as the continuity equation for the dislocation density, which has its roots in the demand that the BURGERS vector must be constant. One will also get (8) when one calculates (2) by means of (4) and observes that:

$$\frac{\tilde{d}}{dt}\,df_n = -\,\tilde{v}^m{}_{,n}\,df_m + \tilde{v}^m{}_{,m}\,df_n\,, \tag{9}$$

in so doing. One can then likewise conclude (7) from (8) by that process. If one now substitutes (7) in:

$$\begin{aligned}\frac{\eth}{\eth t}A_m^{(k)} &= \frac{\partial}{\partial t}A_m^{(k)} + A_{m,p}^{(k)}\,v^p + A_p^{(k)}\,v^p{}_{,m} \\ &= \frac{\partial}{\partial t}A_m^{(k)} + 2v^p A_{[m,p]}^{(k)} + (A_p^{(k)}\,v^p)_{,m}\end{aligned} \tag{10}$$

then it will result that:

$$\frac{\eth}{\eth t}A_m^{(k)} = 2\,(v^p - \tilde{v}^p)\,A_{[m,p]}^{(k)} + \Phi_{,m}^{(k)} \tag{11}$$

$$[\Phi^{(k)} = A_p^{(k)}\,v^p + T^{(k)}]\,.$$

From (7), the OLDROYD derivative of $A_m^{(k)}$ with respect to $\tilde{v}^p$ must also be a gradient [cf., (10)]:

$$\begin{aligned}\frac{\eth}{\eth t}A_m^{(k)} &= \frac{\partial}{\partial t}A_m^{(k)} + 2\,\tilde{v}^p A_{[m,p]}^{(k)} + (A_p^{(k)}\tilde{v}^p)_{,m} \\ &= (T^{(k)} + A_p^{(k)}\tilde{v}^p)_{,m} \equiv \Psi_{,m}\,.\end{aligned} \tag{12}$$

In analogy with the CLEBSCH transformation that is known in hydrodynamics, that relation allows one to show that in the representation of the covariant vector by the MONGE potential:

$$A_m^{(k)} \equiv C^{(k)}{}_{,m} + F^{(k)} G^{(k)}{}_{,m} , \tag{13}$$

one will have the relations:

$$\begin{gathered} F^{(k)} = F^{(k)}(\tilde{X}^{\mathfrak{K}}) , \qquad G^{(k)} = G^{(k)}(\tilde{X}^{\mathfrak{K}}) , \\ \Leftrightarrow \frac{\tilde{d}}{dt} F^{(k)} = 0, \qquad \frac{\tilde{d}}{dt} G^{(k)} = 0. \end{gathered} \tag{14}$$

Namely, one integrates the equation:

$$\frac{\partial}{\partial t} A_{\mathfrak{M}}^{(k)} = \Psi_{,\mathfrak{M}} \qquad (A_{\mathfrak{M}}^{(k)} = A_m^{(k)} A_{\mathfrak{M}}^m),$$

which is identical to (12), and gets:

$$A_{\mathfrak{M}}^{(k)} = A_{\mathfrak{M}}^{(k)}(t_0) + \frac{\partial}{\partial \tilde{X}^{\mathfrak{M}}} \int_{t_0}^{t} \Psi \, dt .$$

(14) follows directly from:

$$A_{\mathfrak{M}}^{(k)}(t_0) \equiv \partial_{\mathfrak{M}} C'^{(k)}(\tilde{X}^{\mathfrak{K}}) + F^{(k)}(\tilde{X}^{\mathfrak{K}}) \partial_{\mathfrak{M}} G^{(k)}(\tilde{X}^{\mathfrak{K}})$$

and

$$C^{(k)} = C'^{(k)} + \int_{t_0}^{t} \Psi \, dt .$$

1.2 Fundamental kinematical equation. – Our goal of determining the time evolution of the lattice vectors will be attained when we define the gradients $\Phi^{(k)}{}_{,m}$, which have been undetermined up to now, in a suitable way. To that end, we first write (1.2.11) as:

$$\frac{\partial A_M^{(k)}}{\partial X^4} \equiv \left. \frac{\partial A_M^{(k)}}{\partial t} \right|_{X^L} = -2 \frac{\partial X^R}{\partial X^4} \alpha_{RM}{}^{(k)} + \Phi^{(k)}{}_{,M} , \tag{1}$$

in which we have recalled (1.1.8), (1.1.9), (1.1.6). The left-hand side obviously describes the temporal change in the (reciprocal) lattice vectors from the standpoint of the observer that moves with the matter. Now, according to Fox [**5**], it is very physically reasonable to require that this gliding velocity should be non-zero only when a relative velocity $(\partial X^R / \partial \tilde{X}^4)$ exists between the dislocations and the matter. The simplest Ansatz is to set:

$$\Phi^{(k)}{}_{,M} = 0. \tag{2}$$

The fundamental kinematical equation will then read:

$$\frac{\partial}{\partial t} A_m^{(k)} = 2(v^p - \tilde{v}^p)\, A_{[m,p]}^{(k)}, \tag{3}$$

$$\frac{\partial}{\partial t} A_m^{(k)} + 2\,\tilde{v}^p A_{[m,p]}^{(k)} + (A_p^{(k)}\tilde{v}^p)_{,m} = 0, \tag{4}$$

in an equivalent formulation, or when expressed in terms of the MONGE potentials (12.13), (12.14):

$$H^{(k)} \equiv \frac{d}{dt} C^{(k)} + F^{(k)} \frac{d}{dt} G^{(k)} = 0. \tag{5}$$

There is a certain heuristic merit to briefly pursuing relationships for an isolated dislocation. In that case, one can write [**8**]:

$$\begin{aligned} A_K^{(k)} &= \delta_K^{(k)} - b^{(k)}\, n_K\, \delta(\zeta) \\ &= \delta_K^{(k)} - \int_F df'_K\, b^{(k)} \frac{\delta(X^Q - X'^Q)}{\Delta} \end{aligned} \tag{6}$$

as a generalization of the linear theory. In that, F means an arbitrary surface with normal n_K that is based upon the dislocation loop, ζ is a coordinate in the direction of n_K, $\Delta = \sqrt{\det(A_R^i\, A_S^k\, \delta_{ik})}$, and df'_K is the surface element $\Delta\, \varepsilon_{ABC}\, dX'^B\, dX'^C$. Left-multiplying by $(\varepsilon^{KLM} / \Delta)\, \partial_L$ and employing the properties of the δ-function and STOKES's theorem will yield:

$$\begin{aligned} \frac{\varepsilon^{KLM}}{\Delta} \partial_L A_K^{(k)} &= \alpha^{M(k)} \\ &= \oint dX'^M b^{(k)} \frac{\delta(X^Q - X'^Q)}{\Delta}. \end{aligned} \tag{7}$$

If one now calculates the temporal change in (6) by merely adding the lateral surface to F that arises from the motion of the dislocation loop then that will yield:

$$\frac{\partial A_K^{(k)}}{\partial X^4} = \oint dX'^L \varepsilon_{MKL}\, \Delta \frac{\partial X'^M}{\partial \tilde{X}^4} b^{(k)} \frac{\delta(X^Q - X'^Q)}{\Delta}$$

$$= \varepsilon_{MKL}\, \Delta \frac{\partial X^M}{\partial \tilde{X}^4}\, \alpha^{L(k)} = -\,2 \frac{\partial X^M}{\partial \tilde{X}^4}\, \alpha_{MK}^{(k)}.$$

That is our formula (1), but without the gradients $\Phi^{(k)}{}_{,K}$. We can arrive at it, in addition, when we let the surface vary in time, as well, but that would be physically absurd.

2. – A path to dynamics

2.1. Balance of energy. – It is known from the theory of dislocations that the compatible total deformation ε_{ik} can be decomposed into two incompatible pieces:

$$\varepsilon_{ik} = \varepsilon_{ik}^{el} + \varepsilon_{ik}^{pl} \tag{1}$$

namely, the elastic part:

$$\varepsilon_{ik}^{el} = \tfrac{1}{2}(\delta_{ik} - g_{ik}) \tag{2}$$

and the plastic one:

$$\varepsilon_{ik}^{pl} = \tfrac{1}{2}(g_{ik} - b_{ik}) . \tag{3}$$

in which:

$$g_{ik} = A_i^{(r)} A_k^{(s)} \delta_{(r)(s)} . \tag{4}$$

In order to arrive at the balance of energy, we postulate the existence of a potential energy $U(\varepsilon_{AB}^{el})$ with the known property:

$$\sigma^{AB} = \rho \frac{\partial U}{\partial \varepsilon_{AB}^{el}} , \tag{5}$$

in which:

$$\varepsilon_{AB}^{el} = A_A^i A_B^k \varepsilon_{ik}^{el} , \qquad \sigma^{AB} = A_i^A A_k^B \sigma^{ik} . \tag{6}$$

If we left-multiply the dynamical equation:

$$\rho \frac{dv^i}{dt} = \sigma^{ik}{}_{,k} \tag{7}$$

by v_i then we will get:

$$\begin{aligned} \rho \frac{d}{dt} v^2/2 - (\sigma^{ik} v_i)_{,k} &= \sigma^{ik} v_{i,k} \\ &= - \sigma^{ik} \left(\frac{\partial}{\partial t} \varepsilon_{ik}^{el} + \frac{\partial}{\partial t} \varepsilon_{ik}^{pl} \right) \\ &= - \rho \frac{dU}{dt} - \sigma^{ik} \frac{\partial}{\partial t} \varepsilon_{ik}^{pl} . \end{aligned} \tag{8}$$

In this, (1.1.10), (1), (5) were used in succession. The balance of energy will then read:

$$\rho \frac{d}{dt} v^2/2 + \rho \frac{d}{dt} U = (\sigma^{ik} v_i)_{,k} - \sigma^{ik} \frac{\partial}{\partial t} \varepsilon_{ik}^{pl} . \tag{9}$$

The sum of the kinetic and elastic energy of a material volume element will then change as a result of the work done on the outer surface and the plastic energy dissipation.

Obviously, the local time derivative that is employed in the linear theory in this dissipative term will be simply replaced with the OLDROYD derivative in the nonlinear theory.

It follows from (3), (4), (1.1.12), (1.3.3) that:

$$\begin{aligned} -\sigma^{ik}\frac{\partial}{\partial t}\varepsilon_{ik}^{pl} &= -\sigma^{ik}\,\delta_{(r)(s)}\,A_i^{(r)}\frac{\partial}{\partial t}A_k^{(s)} \\ &= -2\sigma^{ik}\,\delta_{(r)(s)}\,A_i^{(r)}A_{[k,p]}^{(s)}(v^p-\tilde{v}^p)-\sigma^{ik}\delta_{(r)(s)}A_i^{(r)}\,\Phi_{,k}^{(s)}. \end{aligned} \tag{10}$$

The appearance of such dissipation must be physically excluded from our model, since the dislocations do not move relative to the matter. We consider that to be one more reason to set $\Phi^{(s)}{}_{,k} = 0$.

The term:

$$k_p^{P.K.} = -2\sigma^{ik}\delta_{(r)(s)}A_i^{(r)}A_{[k,p]}^{(s)} \tag{11}$$

that appears in (10) is obviously the nonlinear generalization of the PEACH-KOEHLER force to the continuum theory; i.e., the force that is exerted on the dislocations. The dissipative term:

$$k_p^{P.K.}(v^p-\tilde{v}^p) \tag{12}$$

can now be interpreted as follows: $k_p^{P.K.}\,\tilde{v}^p$ is the power that the matter transfers to the dislocations, and as a result $-k_p^{P.K.}\,v^p$ will be the power that the dislocations transfer to matter. Dissipation will then take places when the matter transfers more power to the dislocations than the opposite process.

2.2 Heuristic considerations. – According to (2.1.12), energetic considerations will produce the force that acts upon the dislocations only up to a term that is perpendicular to the relative velocity of matter and dislocations. Naturally, little can be said about such a term in the context of a purely-phenomenological theory. However, the use of variational principles has proved to be a heuristic tool in electrodynamics, for example, for gaining information about the forces that appear [**9**]. In principle, one reduces the known equations to a variational problem in order to arrive at new terms by suitable variation of the initially-unvaried quantities. That process makes sense insofar as the LAGRANGE function includes the coupling between the interacting systems.

KOSEVICH [**3**] applied already such a method to the isolated dislocation in the linear case and adapted to the results to the continuum case. In what follows, we would like to give a variational principle that refers directly to the nonlinear continuum theory of dislocations. KOSEVICH's procedure [**3**] will then prove to be unsuitable.

Since a suitable variational process for pure elastomechanics is largely unknown, we shall first give such a thing in order to then generalize it for our own purposes. In it, we will treat the NEWTONian approximation to the general-relativistic method that SCHÖPF published [**9**]. It can be characterized by saying that the LAGRANGE coordinates can be employed as field variables. Intuitively, it adapts the techniques that one encounters in point mechanics to continuum mechanics. At the same time, we shall give the necessary relationships in the context of NEWTONian mechanics.

2.3 Variational techniques in continuum mechanics. – In the particle description, the position of the K^{th} point-mass x^i (K, t) will be displaced through a segment δx^i (K, t) according to:

$$x^i\,(K, t) \to x^i\,(K, t) + \delta x^i\,(K, t)\,. \tag{1}$$

The corresponding variation in the field description is described analogously:

$$x^i\,(X^K, t) \to x^i\,(X^K, t) + \delta x^i\,(x^k, t)\,. \tag{2}$$

Here, the point-mass x^i (X^K, t) that belongs to the initial position X^K will be displaced through a segment δx^i (x^k, t). In that, it is desirable for the transition from (1) to (2) that the field description δx^i can be regarded as a function of x^k by means of (11.1).

One gets the variation of the velocity field v^i (x^k, t) with the help of (2) when one reverts to its definition (1.1.3). After the variation, with (2), one will have:

$$v^i\,(x^k + \delta x^k, t) + \delta v^i\,(x^k + \delta x^k, t) = \frac{\partial}{\partial t}(x^k + \delta x^k(x^l), t)\Big|_{X^K}\,, \tag{3}$$

for the velocity field $v^i + \delta v^i$ (due to the shift in dependency to the argument $x^i + \delta x^i$). If one neglects higher powers of δx^k in this then it will follow that:

$$\delta v^i = \frac{\partial}{\partial t}\,\delta x^i + v^l\,\partial_l\,\delta x^i - \delta x^l\,\partial_l\,v^i. \tag{4}$$

One likewise gets the variation of A_K^i by solving:

$$A_K^i\,(x^k + \delta x^k, t) + \delta A_K^i = \frac{\partial}{\partial X^K}(x^i + \delta x^i)$$

in the form:

$$\delta A_K^i = A_K^l\,\partial_l\,\delta x^i - \delta x^l \partial_l A_K^i\,. \tag{5}$$

The variation of the mass density $\rho\,(x^k, t)$ is implied by the conservation of mass that one must require:

$$\delta(\rho\, d\tau) = 0$$

and gives:

$$\delta\rho = -\,\partial_i\,(\delta x^k\,\rho) \tag{6}$$

and corresponds to the continuity equation for actual displacements $\delta x^k = v^i\,\delta t$.

2.4 Application to conventional continuum mechanics. – In the conventional theory of elasticity, we start from the LAGRANGIAN density:

$$L = \tfrac{1}{2}\rho\, v_i\, v^i - \rho\, U\,(\varepsilon_{AB}) \tag{1}$$

and employ (2.1.5), in which we must observe that in this case we will have $\varepsilon_{AB}^{pl} = 0$. Therefore, we must employ:

$$\varepsilon_{AB} = \tfrac{1}{2}\left(A_A^i A_B^k\, \delta_{ik} - \delta_{AB}\right). \tag{2}$$

The variation can be performed immediately with the help of (2.3.4), (2.3.5), and (2.3.6). After performing the partial integrations, one will get:

$$\delta L = \delta x^l \left\{\partial_n \sigma_l^n - \frac{\partial}{\partial t}(\rho\, v_l) - \partial_i(\rho\, v_l\, v^i)\right\}.$$

It will then follows with the continuity equation for mass that:

$$\rho \frac{d}{dt} v^i = \partial_i\, \sigma^{il}. \tag{3}$$

2.5 Variational principle for the continuum theory of moving dislocations. – In the presence of dislocations, one must substitute the elastic deformation in (2.4.1) and write (2.1.2) as:

$$\varepsilon_{AB}^{el} = \tfrac{1}{2} A_A^i A_B^i (\delta_{ik} - A_i^{(i)} A_k^{(k)} \delta_{(i)(k)}). \tag{1}$$

The quantities $A_i^{(i)}$ are regarded as new field quantities. Due to the fundamental kinematical equation (1.3.4), they cannot be varied freely. We consider the latter to be an auxiliary condition with the help of LAGRANGE multipliers $\lambda_{(k)}^k$. With that, our generalized LAGRANGE density will now read:

$$L = \tfrac{1}{2}\rho\, v_i\, v^i - \rho U(\varepsilon_{AB}^{el}) - \lambda_{(k)}^k \left\{\frac{\partial}{\partial t} A_k^{(k)} + 2 A_{[k,l]}^{(k)}\, \tilde{v}^l + (A_l^{(k)}\, v^l)_{,k}\right\}. \tag{2}$$

The variation of ρ, v^i, A_A^i results as it did in **2.3**. Furthermore, the quantities $\lambda_{(k)}^k$ and $A_k^{(k)}$ are varied independently. The additional velocity components $\tilde{v}^l$ that appear will be treated by analogy with (2.3.4) by varying the dislocation position according to:

$$\delta \tilde{v}^l = \frac{\partial}{\partial t} \delta \tilde{x}^i + \tilde{v}^l\, \partial_l\, \delta \tilde{x}^i - \delta \tilde{x}^l\, \partial_l\, \tilde{v}^i. \tag{3}$$

The use of (2.3.4), (2.3.5), (2.3.6), and (3) yields the following expressions for the coefficients of δx^k, $\delta\tilde{x}^k$, $\delta A_k^{(k)}$, $\delta\lambda_{(k)}^k$:

$$\delta x^k \Big\{ -\rho \frac{d}{dt} v_k + \sigma^l_{k,l} - \partial_l \,(\sigma^{il} A_i^{(r)} A_k^{(s)})\, \delta_{(r)(s)} + \sigma^{il} A_i^{(r)} A_{l.k}^{(s)}\, \delta_{(r)(s)} - \frac{\partial}{\partial t}(\lambda^l_{(k),l} A_k^{(k)}) - \partial_i (v^i \lambda^l_{(k),l} A_k^{(k)}) - \lambda^l_{(k),l} A_i^{(k)} \partial_k v^i \Big\}, \tag{4}$$

$$\delta\tilde{x}^k \left\{ 2\frac{\partial}{\partial t}\left(\lambda^l_{(k)} A^{(k)}_{[l,k]}\right) + 2(\lambda^l_{(k)} A^{(k)}_{[l,m]} \tilde{v}^m)_{,m} + 2\lambda^l_{(k)} A^{(k)}_{[l,m]}\, \tilde{v}^m_{,k} \right\}, \tag{5}$$

$$\delta A_k^{(k)} \left\{ \sigma^{ik} A_i^{(r)} \delta_{(r)(k)} + \frac{\partial}{\partial t} \lambda^k_{(k)} + 2\left(\lambda^{[k}_{(k)} \tilde{v}^{l]}\right)_{,l} + \lambda^l_{(k),l} v^k \right\}, \tag{6}$$

$$\delta\lambda^k_{(k)} \left\{ \frac{\partial}{\partial t} A_k^{(k)} + 2A^{(k)}_{[k,l]} \tilde{v}^l + \left(A_l^{(k)} v^l\right)_{,k} \right\}. \tag{7}$$

With:

$$\lambda^k_{(k),k} \equiv \rho\, \lambda_{(k)} \tag{8}$$

and the continuity equation for mass, one can put the term in (4) that is independent of $\lambda^k_{(k)}$ into the form:

$$-\left(\rho \frac{d}{dt} \lambda_{(k)} A_k^{(k)} + \rho \lambda_{(k)} \frac{\partial}{\partial t} A_k^{(k)} \right). \tag{9}$$

In advance of a later discussion, we require that $\int dt\, d\tau\, L$ must be stationary, such that all coefficients (4), (5), (6), (7) will vanish. In particular, (6) will imply a determining equation for $\lambda^k_{(k)}$. Taking the divergence will imply that:

$$\rho \frac{d}{dt} \lambda_{(k)} = -\left(\sigma^{il} A_i^{(r)}\right)_{,l} \delta_{(r)(k)} . \tag{10}$$

With that and (8), (4) will go to:

$$\delta x^k \left\{ -\rho \frac{d}{dt} v_k + \sigma^l_{k,l} + 2\sigma^{il} A_i^{(r)} A^{(s)}_{[l,m]}\, \delta_{(r)(s)} - \rho\, \lambda_{(k)} \frac{\partial}{\partial t} A_k^{(k)} \right\}. \tag{11}$$

In a similar way, after a somewhat-lengthy calculation, (5) can be brought into the form:

$$\delta\tilde{x}^k \left\{ \rho\, \lambda_{(k)} \frac{\partial}{\partial t} A_k^{(k)} - 2\sigma^{il} A_i^{(r)} A^{(s)}_{[l,k]}\, \delta_{(r)(s)} \right\}. \tag{12}$$

Without going into the details, we mention that we will arrive at the same result when we consider the fundamental kinematical equation in the form (1.3.5) and thus put the LAGRANGIAN density into the form:

$$L = \tfrac{1}{2}\rho\, v_i\, v^i - \rho\, U\left(\varepsilon^{el}_{AB}\right) - \rho\, \lambda_{(k)}\, H^{(k)}. \tag{13}$$

In that way, the $A_i^{(k)}$ that are included in ε^{el}_{AB} (1) can be expressed in terms of the MONGE potentials according to (1.2.13) and (1.2.14). In contrast to the previous procedure, one must then vary the potentials $C^{(k)}$ independently, while the potentials $F^{(k)}$, $G^{(k)}$ are regarded as functions of only the $\tilde{X}^{\mathfrak{K}}$ in the variation of those LAGRANGE coordinates of the dislocations, which will once more result in (12).

2.6 Discussion of the results. – From (2.5.12), the variation of the position of the dislocation will then yield the expression:

$$k_k \equiv \rho\, \lambda_{(k)}\, \frac{\eth}{\eth x} A_k^{(k)} - 2\sigma^{il}\delta_{(r)(s)} A_i^{(r)} A_{[l,k]}^{(s)}\,. \tag{1}$$

The second term is, as expected, the nonlinear generalization of the PEACH-KOEHLER force for the continuum theory, which is now extended by an additional term:

$$\rho\, \lambda_{(k)}\, \frac{\eth}{\eth x} A_k^{(k)} = 2\rho\, \lambda_{(k)} A_{[k,l]}(v^l - \tilde{v}^l)\,. \tag{2}$$

As we likewise expect, it is orthogonal to $v^l - \tilde{v}^l$. Furthermore, it proves to be the generalization of the force term that extends the PEACH-KOEHLER force for moving isolated dislocations in KOSEVICH's theory. Namely, in the linear approximation, (2.5.10) will simplify to:

$$\rho \frac{\partial}{\partial t}\lambda_k = -\,\rho \frac{\partial}{\partial t} v_k\,. \tag{3}$$

In so doing, we have considered:

$$A_k^{(k)} = \delta_k^{(k)} - \beta_k^{(k)}, \qquad \beta^2 \approx 0.$$

in particular. If we integrate (3) to:

$$\lambda_k = -\, v_k$$

and consider that $\tilde{v}^k \gg v^k$ then the term that KOSEVICH gave will follow. Since we required the stationarity of the action integral, k_k will vanish here precisely as it did for KOSEVICH. We will then obtain the fundamental dynamical equations from (2.5.11) in the form (2.4.3).

From **2.1**, the vanishing of k_k has the consequence that no dissipation will appear. That is understandable insofar as a variational principle can yield only a conservative theory. In reality, one must naturally go over to a non-conservative theory in which the k_k does not necessarily need to vanish and in which no further forces will be considered either (e.g., the resistance of the crystal lattice to the motion of dislocations). However, we are convinced that our procedure will yield the correct structure for k_k , since the path that was taken here in order to consider the fundamental kinematical equation (1.3.4) by means of LAGRANGE multipliers is correct. It is possible that we might want to impose additional conditions on the $A_k^{(k)}$, as is the case in, e.g., hydrodynamics, in which we cannot simply vary the density ρ and the velocity $\mathfrak{v}$ freely while we consider the continuity equation to be the auxiliary condition [**10**]. Work along those lines is still in progress.

References

1. K. KONDO, RAAG Memoirs, Vol. I, II, Divisions B, C, D, Tokyo (1955), (1958).
   B. A. BILBY, R. BULLOUGH, and E. SMITH, Proc. Roy. Soc. London A**231** (1955), 263.
   B. A. BILBY and E. SMITH, Proc. Roy. Soc. London A**236** (1956), 482; **244** (1958), 538.
   E. KRÖNER, Arch. Rat. Mech. Anal. **4** (1959/60), 273.
2. H. GÜNTHER, Schriftenreihe d. Institute f. Math. d. DAW **A4** (1967).
   S. AMARI, RAAG Research Notes (3) **52** (1962).
3. A. M. KOSEVICH, UFN, **84**, 4 (1964), 579.
4. T. MURA, Int. J. Eng. Sci. **1** (1963), 371.
5. N. FOX, J. Inst. Math. Appl. **2** (1966), 285.
6. J. G. OLDROYD, Proc. Roy. Soc. London A**200** (1950), 583.
7. C. TRUESDELL, R. TOUPIN, "Classical Field Theories," *Handbuch der Physik*, Flügge (ed.) III/1 (1960), 448.
   H.-G. SCHÖPF, Abhand. d. DAW, Kl. Math. Phys., Techn. (1967), Heft 3.
8. L. D. LANDAU, E. M. LIFSCHITZ, Lehrb. d. Theor. Phys. VII, *Elastizitätstheorie*, Akademie Verlag, Berlin, 1967.
9. H.-G. SCHÖPF, Ann. Phys. (Leipzig) **12** (1964), 377; **13** (1964), 41.
10. *Handbuch der Physik*, Flügge (ed.) VIII/1 (1959) .

Dresden, Chair of mathematical physics at the Technical University.

(Submitted to the editor on 21 August 1967)