"Ueber die konforminvariante Gestalt der relativistischen Bewegungsgleichungen," Proc. Kon. Ned. Akad. Wet. Amsterdam **39** (1936), 1059-1065.

On the conformally-invariant form of the relativistic equations of motion

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1. Introduction. – In a previous paper ([1]), we showed that the MAXWELL equations and the impulse-energy equations can be written in a conformally-invariant manner. In this article, it will be shown that the relativistic equations of motion of a charged particle can also be brought into a conformally-invariant form, assuming that one transforms the mass in such a way that the product of mass and length remains invariant. h / c (dimension $[M\,L]$ then plays a role that is similar to that of c in the usual theory of relativity. Another invariant enters in place of the rest mass $\overset{\circ}{m}$, namely, the conformal mass $\overset{c}{m} = \overset{\circ}{m}(-\mathfrak{g})^{1/8}$, which has the dimension $[M\,L]$. Here, we shall give only the simple mathematical facts and avoid physical speculations.

We briefly recall the results that were obtained before. In a space-time with a conformal metric, one does not have a fundamental tensor g_{ih}, but a tensor density $\mathfrak{G}_{ih} = g_{ih}\,(-\mathfrak{g})^{1/4}$ ($\mathfrak{g}$ = det g_{ih}) of weight – 1 / 2. One does not have a line element $d\tau$, but a conformal (dimensionless) line element $d\mathfrak{s}$ that is defined by:

$$(d\mathfrak{s})^2 = \mathfrak{G}_{ih}\, d\xi^i\, d\xi^h. \tag{1}$$

The charge in a four-dimensional volume $d\omega$, which is itself conformally invariant, establishes a charge density of weight + 3 / 4 (dimension: $[M^{1/2}\, L^{3/2}\, T^{-1}]$) by means of the equation:

$$de\; d\mathfrak{s} = \rho\, d\omega, \tag{2}$$

and that will imply the current vector density of weight + 1:

$$\mathfrak{s}^h = \rho\, \frac{d\xi^h}{d\mathfrak{s}}. \tag{3}$$

One has the equations:

([1]) "Ueber die konforminvariante Gestalt der MAXWELLschen Gleichungen und der elektromagnetischen Impulsenergiegleichungen," Physica **1** (1935), 869-872.

$$\left.\begin{aligned}
F_{ji} &= 2\partial_{[j}\varphi_{i]}; \quad \partial_j = \frac{\partial}{\partial\xi^j}; \quad \text{(electromagnetic field)},\\
\partial_{[j}F_{ih]} &= 0,\\
\mathfrak{F}^{hi} &= \mathfrak{G}^{hl}\mathfrak{G}^{ij}F_{lj},\\
\mathfrak{s}^h &= -\partial_j\mathfrak{F}^{jh},\\
\partial_j\mathfrak{s}^h &= 0,\\
\mathfrak{S}^{h}_{\cdot i} &= -\mathfrak{F}^{hj}F_{ij} + \tfrac{1}{4}F_{lj}\mathfrak{F}^{lj}A_i^h \quad \text{(impulse - energy tensor density)}\\
-\nabla_h\mathfrak{S}^{h}_{\cdot i} &= \mathfrak{s}^jF_{ji} = \mathfrak{f}_i \quad \text{(force vector density)}.
\end{aligned}\right\} \tag{4}$$

Up to the last equation, no covariant differentiations appeared. The first six equations are then independent of any choice of a symmetric displacement, in the sense that one can replace ∂_j with ∇_j for any such choice. In the aforementioned paper, we showed that the last equation is true for any displacement for which one has $\nabla_j\,\mathfrak{G}^{hi} = 0$.

As is known, $\mathfrak{G}_{ih}$ by itself does not define a displacement, and therefore no geodetic lines either beyond the *null geodetic lines*, which are independent of any choice of displacement (proof in the next section). Hence, if a well-defined world-line is to result, something must be added, and it is known that the requirement of the linearity of the displacement will lead inevitably to a WEYL displacement.

2. Parallel displacement of vector densities for a WEYL displacement in X_n . – If Γ^h_{ji} are the parameters of a linear displacement then the covariant differential quotient of a density $\mathfrak{y}^\kappa$ of weight $\mathfrak{k}$ is given by the equation:

$$\nabla_j\,\mathfrak{y}^h = \partial_j\,\mathfrak{y}^h + \Gamma^h_{ji}\,\mathfrak{y}^i - \mathfrak{k}\,\Gamma^i_{ji}\,\mathfrak{y}^h\,. \tag{5}$$

Under a WEYL displacement, one has:

$$\Gamma^h_{ji} = \left\{ {h \atop j\,i} \right\} + \tfrac{1}{2}(Q_jA_i^h + Q_iA_j^h - \mathfrak{G}^{hk}\mathfrak{G}_{ij}Q_k)\,, \tag{6}$$

in which Q_i is a vector that transforms under the conformal transformation (“re-gauging”):

$$g'_{ih} = \sigma g_{ih}\,, \tag{7}$$

as follows:

$$Q'_j = Q_j - \partial_j \log \sigma. \tag{8}$$

Since $\left\{ {h \atop j\,i} \right\} = \tfrac{1}{2}\partial_j \log(-\mathfrak{g})$, the differential equation of a vector density under WEYL displacement will read:

$$\nabla_j \mathfrak{y}^h = \partial_j \mathfrak{y}^h + \begin{Bmatrix} h \\ j\,i \end{Bmatrix} \mathfrak{y}^i - \tfrac{1}{2} \mathfrak{k}\, \mathfrak{y}^h \partial_j \log(-\mathfrak{g}) + \tfrac{1}{2}[(1-4\mathfrak{k}) Q_j A_i^h + Q_i A_j^h - \mathfrak{G}^{hk} \mathfrak{G}_{ij} Q_k] \mathfrak{y}^i . \tag{9}$$

One easily derives from this equation that $\nabla_j \mathfrak{G}_{ih} = 0$ (independently of the choice of Q_i).

Since $d\xi^h / d\mathfrak{s}$ has weight 1/4, the equation of a geodetic line that is not a null line will be:

$$\left.\begin{aligned} &\frac{\delta}{d\mathfrak{s}} \frac{d\xi^h}{d\mathfrak{s}} = \frac{d\xi^j}{d\mathfrak{s}} \nabla_j \frac{d\xi^h}{d\mathfrak{s}} = \frac{d\xi^j}{d\mathfrak{s}} \partial_j \frac{d\xi^h}{d\mathfrak{s}} + \begin{Bmatrix} h \\ j\,i \end{Bmatrix} \frac{d\xi^j}{d\mathfrak{s}} \frac{d\xi^i}{d\mathfrak{s}} \\ &-\frac{1}{8} \frac{d\xi^h}{d\mathfrak{s}} \frac{d\xi^j}{d\mathfrak{s}} \partial_j \log(-\mathfrak{g}) + \frac{1}{2} (Q_i A_j^h - \mathfrak{G}^{hk} \mathfrak{G}_{ij} Q_k) \frac{d\xi^i}{d\mathfrak{s}} \frac{d\xi^j}{d\mathfrak{s}} = 0. \end{aligned}\right\} \tag{10}$$

However, if the geodetic is a null line then $d\mathfrak{s} = 0$. Nonetheless, one can assign an arbitrary scalar parameter z to the line. The equation then reads:

$$\left.\begin{aligned} &\frac{\delta}{dz} \frac{d\xi^h}{dz} \\ &= \frac{d\xi^j}{dz} \partial_j \frac{d\xi^h}{dz} + \begin{Bmatrix} h \\ j\,i \end{Bmatrix} \frac{d\xi^j}{dz} \frac{d\xi^i}{dz} + \tfrac{1}{2} (Q_i A_j^h + Q_j A_i^h - \mathfrak{G}^{hk} \mathfrak{G}_{ij} Q_k) \frac{d\xi^i}{dz} \frac{d\xi^j}{dz} :: \frac{d\xi^h}{dz} \end{aligned}\right\} \tag{11}$$

(:: = "proportional to"), and it will then follow that the geodetic null lines are independent of the choice of Q_i .

The displacement is called *pseudo-WEYLIAN* when the vector Q_i can be gauged to zero; i.e., when it is a gradient vector. If that gauging has been performed and one has introduced the particular fundamental tensor for which the displacement is a RIEMANNIAN one then (9) will go to:

$$\nabla_j \mathfrak{y}^h = \partial_j \mathfrak{y}^h + \begin{Bmatrix} h \\ j\,i \end{Bmatrix} \mathfrak{y}^i - \tfrac{1}{2} \mathfrak{k}\, \mathfrak{y}^h \partial_j \log(-\mathfrak{g}). \tag{12}$$

One can define the curvature affinor $R_{kji}^{\cdot\cdot\cdot h}$ from the Γ_{ji}^h of a general WEYL displacement in a known way, and from it, the quantities $R_{ji} = R_{hji}^{\cdot\cdot\cdot h}$, and finally the density $\mathfrak{R} = R_{ji} \mathfrak{G}^{ji}$ of weight 1 / 2. That density defines a fundamental tensor $\varepsilon^2 \overset{\circ}{g}_{ih} = \mathfrak{G}_{ih} \mathfrak{R}$, and then an absolute (i.e., cosmologically-determined) mass. The constant ε (dimension $[L^{-1}]$) is chosen so that the usual mass will result – so ε will be very small, since $\mathfrak{R}$ is, in any event, very small in matter-free domains ([1]).

([1]) H. WEYL, *Raum, Zeit, Materie*, 4th ed., pp. 269.

If we introduce the notation $\varepsilon^2 S$ for the scalar $\mathfrak{R}\,(-\,\mathfrak{g})^{-1/4}$ that the factor σ^{-1} takes on under gauging then we will have:

$$\partial_j \log S' = \partial_j \log S - \partial_j \log \sigma \tag{13}$$

under gauging, and it will emerge from this that:

$$Q_j = P_j + \partial_j \log S, \tag{14}$$

in which P_j is a gauge-invariant vector. Under gauging with the absolute mass, one will have $S = 1$ and $Q_j = P_j$.

3. Derivation of the conformally-invariant form of the equations of motion. – The classical relativistic equations of motion for a point of rest mass $\overset{\circ}{m}$ and charge e in empty space read:

$$\overset{\circ}{m}\left(\frac{d^2\xi^h}{d\tau^2} + \begin{Bmatrix} h \\ j\,i \end{Bmatrix} \frac{d\xi^j}{d\tau}\frac{d\xi^i}{d\tau}\right) = \frac{e}{c}\frac{d\xi^i}{d\tau} F_{ij}\, g^{hj}\,. \tag{15}$$

If we introduce the conformally-invariant line element $d\mathfrak{s} = (-\,\mathfrak{g})^{1/2}\, c\, d\tau$ in place of $d\tau$ then the equation will go to:

$$\frac{d^2\xi^h}{d\tau^2} + \begin{Bmatrix} h \\ j\,i \end{Bmatrix} \frac{d\xi^j}{d\tau}\frac{d\xi^i}{d\tau} - \frac{1}{8}\frac{d\xi^h}{d\tau}\frac{d\xi^j}{d\tau}\partial_j \log(-\mathfrak{g}) = \frac{e}{\overset{\circ}{m} c^2}(-\mathfrak{g})^{-1/8}\frac{d\xi^i}{d\mathfrak{s}}\; F_{ij}\,\mathfrak{G}^{hj}. \tag{16}$$

Since $\dfrac{e}{\overset{\circ}{m} c^2}(-\mathfrak{g})^{-1/8} F_{ij}$ is dimensionless, and $\mathfrak{G}^{hj}$ and $d\xi^i / d\mathfrak{s}$ is conformally-invariant, the right-hand side will be conformally-invariant. A comparison with (10) will teach us that the left-hand side is equal to precisely $\dfrac{\delta}{d\mathfrak{s}}\dfrac{d\xi^h}{d\mathfrak{s}}$ for a pseudo-WEYL displacement whose Q_i is, at the same time, reduced to exactly zero. The form of the equation that is invariant under an arbitrary conformal transformation then reads:

$$\left.\begin{aligned} \frac{\delta}{d\mathfrak{s}}\frac{d\xi^h}{d\mathfrak{s}} = \frac{d^2\xi^h}{d\mathfrak{s}^2} + \begin{Bmatrix} h \\ j\,i \end{Bmatrix} \frac{d\xi^j}{d\mathfrak{s}}\frac{d\xi^i}{d\mathfrak{s}} - \frac{1}{8}\frac{d\xi^h}{d\mathfrak{s}}\frac{d\xi^i}{d\mathfrak{s}}\partial_i \log(-\mathfrak{g}) \\ +\tfrac{1}{2}(A_j^h\, P_i - \mathfrak{G}^{hk}\mathfrak{G}_{ij}P_k)\frac{d\xi^j}{d\mathfrak{s}}\frac{d\xi^i}{d\mathfrak{s}} + \tfrac{1}{2}(A_j^h\,\partial_i \log S - \mathfrak{G}^{hk}\mathfrak{G}_{ij}\partial_k \log S)\frac{d\xi^j}{d\mathfrak{s}}\frac{d\xi^i}{d\mathfrak{s}} \\ = \frac{e}{\overset{\circ}{m} c^2}(-\mathfrak{g})^{-1/8}\frac{d\xi^i}{d\mathfrak{s}} F_{ij}\mathfrak{G}^{hj}, \end{aligned}\right\} \tag{17}$$

in which P_i is regarded as a gradient vector, for the moment. If one again introduces the parameter τ then the equation will read:

$$\frac{d^2\xi^h}{d\tau^2}+\left\{\begin{matrix} h \\ j\,i \end{matrix}\right\}\frac{d\xi^j}{d\tau}\frac{d\xi^i}{d\tau}+\tfrac{1}{2}(A_j^h\,Q_i-\mathfrak{G}^{hk}\mathfrak{G}_{ij}Q_k)\frac{d\xi^i}{d\tau}\frac{d\xi^j}{d\tau}=\frac{e}{\overset{\circ}{m}c^2}\frac{d\xi^i}{d\tau}F_{ij}g^{jh}\,. \tag{18}$$

Since MAXWELL's equations are conformally-invariant, it follows that e, as well as F_{ij}, must be conformally invariant. Since the conformal invariance of c is also out of the question, $\frac{e}{\overset{\circ}{m}c^2}(-\mathfrak{g})^{-1/8}F_{ij}$ can only be dimensionless when $\overset{\circ}{m}$ takes on a factor of $\sigma^{-1/2}$ under the transformation (7). The conformally-invariant mass $\overset{c}{m}=\overset{\circ}{m}(-\mathfrak{g})^{1/8}$, with the dimension [$M$ L], enters into the denominator in the right-hand side of (17). The conformally-invariant mass density $\overset{c}{m}$ that is defined by the equation $c\,d\,\overset{\circ}{m}\,d\tau=\,d\,\overset{\circ}{m}\,d\mathfrak{s}$ $=\overset{c}{\mu}\,d\omega$ belongs to that mass, and it likewise has a dimension of [M L]. $\overset{c}{\mu}c$ is the conformally-invariant action density. Transforming the lengths by $\sigma^{1/2}$ must then imply a transformation of the mass by $\sigma^{-1/2}$. h / c will remain invariant in that, and that constant will then play a role when one goes over to the conformal theory of relativity that is similar to the role of c when one goes over to the usual one. Since the dimensions of e and F_{ij} are both [$M^{1/8}$ $L^{3/2}$ T^{-1}] = [$M^{1/2}$ $L^{1/2}\cdot LT^{-1}$], the demand of the conformal invariance of those quantities will also already lead to the invariance of [M L], moreover.

As is known, in WEYL's theory, Q_i will be identified with the undetermined electromagnetic potential vector φ_i from the outset. Except for the fact that the intrinsically completely free transformation of φ_i must be connected with gauging in a physically not-well-founded way, a WEYL displacement will arise in that way that likewise can have not meaning for the world-lines of free particles. If one sets $P_i+\partial_i\log S$ equal to the potential vector in (17) then that will yield world-lines that cannot coincide with the correct world-lines for any definition of the potential vector. Rather, we would like to demand that equation (17), when written briefly as:

$$\frac{\delta}{d\mathfrak{s}}\frac{d\xi^h}{d\mathfrak{s}}=\frac{e}{\overset{\circ}{m}c^2}\frac{d\xi^i}{d\mathfrak{s}}F_{ij}\,\mathfrak{G}^{hj}, \tag{19}$$

must yield the correct experimentally-verified world-lines with sufficient precision when the cosmologically-determined natural mass is used as a basis. It will then emerge from this that whether or not the vector P_i, which is invariant under gauging, is a gradient vector, the other quantities that occur in the equation will likewise be very small in matter-free regions. Hence, the term with S must vanish under gauging in the natural mass, and the equation must go to (15), up to a small deviation that is consistent with the measured results. It would not be impossible that P_i contains, *inter alia*, a terms of the form $\alpha\,\partial_i\log\frac{\overset{c}{\mu}{}^{1/2}}{\mathfrak{R}}$, in which α represents any constant. That will yield a possible experimentally-accessible deviation of the world-lines that is independent of the conformally-invariant action density.

4. The conformally-invariant form of the DIRAC equation. – In latter years, various authors have tried to exhibit a conformally-invariant DIRAC equation ([1]).

DIRAC came to the result that there is no simple way for arriving at such an equation ([2]). We will thus understand the phrase "conformally-invariant" to mean only "independent of $\mathfrak{G}_{ih}$, but not of any field Q_i." We let the latter restriction drop, since there are actually times that nature seems to give world-lines, and thus any type of displacement, and we then consider the DIRAC equation in the form:

$$\left(\frac{h}{i}\alpha^j\nabla_j + \overset{\circ}{m}c\alpha^0\right)\psi = 0. \tag{20}$$

In a conformal geometry, one must employ $'\alpha^j = (-\mathfrak{g})^{1/8}\,\alpha^j$, instead of α^j, since one has:

$$'\alpha^{(h}\,'\alpha^{i)} = \mathfrak{G}^{hi}, \tag{21}$$

and only $\mathfrak{G}^{hi}$ is available. However, the equation will then read:

$$\left(\frac{h}{i}(-\mathfrak{g})^{1/8}\alpha^j\nabla_j + \overset{\circ}{m}(-\mathfrak{g})^{1/8}c\,'\alpha^0\right)\psi = 0, \tag{22}$$

with:

$$'\alpha^0 = {}'\alpha^{[1}\,'\alpha^2\,'\alpha^3\,'\alpha^{4]} = \alpha^0, \tag{23}$$

and this equation is no longer conformally-invariant under constant mass in the second term, since $'\alpha^0$ has weight zero. From our Ansatz, however, $\overset{c}{m} = \overset{\circ}{m}(-\mathfrak{g})^{1/8}$ is precisely conformally invariant, and we will then arrive at the conformally-invariant DIRAC equation:

$$\left(\frac{h}{i}\,'\alpha^j\nabla_j + \overset{c}{m}c\,'\alpha^0\right)\psi = 0. \tag{24}$$

The equation is identical to the usual DIRAC equation, up to the possible influence of vectors P_i in ∇_j, such that one can say that if one ignores the known replacement of ∂_j with ∇_j then the usual DIRAC equation will already be conformally-invariant, as long as one understands "conformal invariance" in our sense of the term and introduces the correct transformation of mass.

([1]) P. A. M. DIRAC, "Wave equations in conformal space," Annals of Math. **37** (1936), 429-442.
O. VEBLEN, "A conformal wave equation," Proc. Nat. Acad. Sci. **21**(1935), 484-487.

([2]) *Loc. cit.*, pp. 442.

When using the conformal equation, one must naturally expect that $\bar{\psi}\psi$ would represent the conformally-invariant electrical probability density ρ of weight 3/4; i.e., ψ must be normalized as a density of weight 3/8. Since the three-dimensional spatial element $d\omega_3$ is a density of weight – 3/4, $\rho\, d\omega_3$ properly has weight zero. However, $\bar{\psi}\ \alpha^h \psi$ also has, in fact, weight + 1, as one would demand of the current vector $\mathfrak{s}^h$.
