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On the differential geometry of the group of contact transformations: I. Doubly-homogeneous treatment of contact transformations

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I. Introduction. – As is known, LIE showed ([1]) that one can write a general contact transformation in the $2n - 1$ variables $\xi^1, \ldots, \xi^n, \zeta^2, \ldots, \zeta^n$ as a "homogeneous" contact transformation in the $2n$ variables $\xi^1, \ldots, \xi^n, \eta_1, \ldots, \eta_n$, in which $\zeta_2 = -\eta_2 / \eta_1$, etc. Here, the word "homogeneous" is intended to mean that the transformed ξ and η are homogeneous functions of order zero (one, resp.) in the old η. Now, the formulas of the homogeneous contact transformation that arises in that way exhibit a remarkable inclination towards a duality between ξ and η, which cannot, however, unfold completely for the simple reason that, in fact, the transformations ξ and η are in no way homogeneous functions of the old ξ. That raises the question of whether the coordinates cannot be chosen in a different way, and indeed, in such a manner that duality manifests itself completely in every relation. That question can be answered in the affirmative. If we choose homogeneous point coordinates $x^0, x^1, \ldots, x^n$; $\xi^1 = x^1 / x^0$, etc., instead of the usual point coordinates ξ^i of VAN DANTZIG's well-known geometry of H_n ([2]), and instead of the facet coordinates $\eta_1, \ldots, \eta_n$, we choose coordinates $p_0, p_1, \ldots, p_n$, which satisfy the equations:

$$\left.\begin{array}{l} p_1 : \ldots : p_n = \eta_1 : \ldots : \eta_n \\ x^0 p_0 + \cdots + x^n p_n = 0, \end{array}\right\} \tag{1}$$

and as a result of which, one will have the equations:

$$p_0 : p_1 : \ldots : p_n = -(\xi^1 \eta_1 + \ldots + \xi^n \eta_n) : \eta_1 : \ldots : \eta_n , \tag{2}$$

then the transformed x will be homogeneous functions of degrees one and zero, resp., and the transformed p be homogeneous functions of degrees zero and one, resp., in the old x

([1]) S. LIE, *Theorie der Transformationsgruppen*, Bd. II, pp. 139, *et seq.*

([2]) D. v. DANTZIG, "Theorie des projektiven Zusammenhangs n-dimensionaler Räume," Math. Ann. **106** (1932), 400-454; J. A. SCHOUTEN and J. HAANTJES, "Zur allgemeinen projektiven Differentialgeometrie," Comp. Math. **3** (1936), 1-51, and the literature that is cited therein.

(p, resp.), *and that will yield a completely dualistic treatment.* In this first article, only the structure of this new dualistic method shall be sketched out briefly ([1]).

2. The geometry of H_n . – We consider an X_{n+1} with the Ur-variables x^κ (κ; …, τ = 0, 1, …, n) and restrict the coordinate transformations to the group $\mathfrak{H}_{n+1}$ of the *homogeneous* transformations of degree 1 that are continuous and differentiable sufficiently often in the domain considered. The form of the equations of the "rays" in X_{n+1}, which are defined by:

$$x^\kappa = \lambda\, c^\kappa; \qquad c^\kappa = \text{constants}, \tag{3}$$

remains invariant under that group. We call the n-dimensional manifold of rays H_n, so each ray is a *point* of H_n, and each point of X_{n+1} , with the exception of $x^\kappa = 0$, is an *analytic point* of H_n . The point of H_n that corresponds to the analytic point x^κ will be denoted by $\lfloor x^\kappa \rfloor$ ([2]).

Along with the group $\mathfrak{H}_{n+1}$ of *coordinate* transformations, we consider the group $\mathfrak{F}$ of transformations of *analytic points* of the form ([3]):

$$'x^\kappa = \rho\, x^\kappa; \qquad \rho = \text{homogeneous of degree zero in } x^\kappa, \tag{4}$$

which leave each individual *point* invariant. The transformations of $\mathfrak{H}_{n+1}$ and $\mathfrak{F}$ commute with each other. Hence:

$$\left.\begin{aligned} x^{\kappa'} &= f^{\kappa'}(x^\kappa), \\ 'x^\kappa &= \rho\, x^\kappa, \end{aligned}\right\} \tag{5}$$

and it will then follow that:

$$'x^{\kappa} = f^{x'}('x^\kappa) = f^{x'}(\rho\, 'x^\kappa) = \rho\, f^{x'}('x^\kappa) = \rho\, 'x^{\kappa'}. \tag{6}$$

That is naturally related to the fact that we have restricted ourselves expressly to homogeneous functions of degree *one* in $\mathfrak{H}_{n+1}$.

All geometric objects are defined, in particular, on the two groups; e.g.:

Contravariant (covariant, resp.) projective vectors of degree $\mathfrak{r}$:

$$\mathfrak{H}_{n+1}\colon \left\{ \begin{aligned} v^{\kappa'} &= A_\kappa^{\kappa'}\, v^\kappa \\ w_{\lambda'} &= A_{\lambda'}^{\lambda}\, w_\lambda \end{aligned} \right. ; \quad A_\kappa^{\kappa'} = \frac{\partial x^{\kappa'}}{\partial x^\kappa}; \qquad \mathfrak{F}\colon \left\{ \begin{aligned} 'v^\kappa &= \rho^{\mathfrak{r}}\, v^\kappa \\ 'w_{\lambda'} &= \rho^{\mathfrak{r}}\, w_\lambda \end{aligned} \right\}. \tag{7}$$

([1]) The paper "Invariant theory of homogeneous contact transformations" by L. P. EISENHART and M. S. KNEBELMAN [Ann. of Math. **37** (1936), 747-765] afforded me especial inspiration, since they cast a whole new light upon the theory of homogeneous contact transformations (in LIE's sense).

([2]) $\lfloor A \rfloor$ means: "The ideal of A"; i.e., A, up to an arbitrary numerical factor. [D. v. DANTZIG, "On the general projective geometry III," Proc. Royal Acad. **37** (1934), 150-155.]

([3]) The type of index does not change under coordinate transformations, while the kernel symbol does not change under object transformations.

$\lfloor v^\kappa \rfloor$ represents a point in local space, and $\lfloor w_\lambda \rfloor$ represents a hyperplane. If $v^\kappa\, w_\kappa = 0$ then the point lies in the hyperplane. As a result of (5) and (7), x^κ is itself a contravariant projective vector of degree 1; $\lfloor x^\kappa \rfloor$ represents a *contact point* in any local space. By contrast, dx^κ is not a projective vector, since the transformation is appropriate to $\mathfrak{H}_{n+1}$, but inappropriate to $\mathfrak{F}$.

We now consider the set of all covariant vectors p_λ at each point $\lfloor x^\kappa \rfloor$ for which $x^\rho\, p_\rho = 0$. The $\lfloor p_\lambda \rfloor$ are all hyperplanes through the point $\lfloor x^\kappa \rfloor$ in local space. We call each individual $\lfloor p_\lambda \rfloor$ a *facet*, and the combination of a contact point $\lfloor x^\kappa \rfloor$ with an associated facet $\lfloor p_\lambda \rfloor$ is an *element.* The combination of x^κ and p_λ is called an *analytic element* that belongs to the element. Hence, ∞^2 analytic elements belong to an element.

Two neighboring elements $\lfloor x^\kappa \rfloor$, $\lfloor p_\lambda \rfloor$ and $\lfloor x^\kappa + dx^\kappa \rfloor$, $\lfloor p_\lambda + dp_\lambda \rfloor$ are said to *lie united* ([1]) when one has:

$$p_\rho\, dx^\rho = 0 \tag{8}$$

or, what amounts to the same thing:

$$x^\rho\, dp_\rho = 0. \tag{9}$$

A set of elements is called a *union of elements* when every two neighboring elements in it lie united.

3. The geometry of K_{2n-1}. – The totality of all elements defines a manifold of dimension $2n - 1$ when the two times $(n + 1)$ homogeneous coordinates x^κ, p_λ are bound by the relation:

$$x^\rho\, p_\rho = 0. \tag{10}$$

A transformation of the elements that:

A. Takes every element to another element,

B. Does not disturb the united position of two neighboring elements,

C. Possesses a single-valued inverse

is called a *contact transformation.*

Therefore, a contact transformation will always have the form:

$$\left.\begin{aligned} {}'x^\kappa &= \varphi^\kappa(x^\rho, p_\sigma), \\ {}'p_\lambda &= \psi_\lambda(x^\rho, p_\sigma), \end{aligned}\right\} \tag{11}$$

with a non-vanishing functional determinant (cf., Condition *C*), in which the φ^κ and ψ_λ must be *homogeneous* functions of any degree in x^ρ and p_σ, since one is indeed dealing

([1]) LIE, *loc. cit.*, pp. 65.

with a transformation of elements (not of analytic elements), and changing the choice of numerical factor in x^ρ, p_σ should not lead to any change in the elements $\lfloor 'x^\kappa \rfloor$, $\lfloor 'p_\lambda \rfloor$. The choice of degree x^ρ and p_σ is geometrically irrelevant, since one can multiply φ^κ, as well as ψ_λ, by an arbitrary factor that is homogeneous of any degree in x^ρ and p_σ without affecting $\lfloor 'x^\kappa \rfloor$, $\lfloor 'p_\lambda \rfloor$. Furthermore, Condition *A* says that $'x^\rho\, 'p_\rho = 0$ must follow from $x^\rho\, p_\rho = 0$, and Condition *B* says that $'p_\rho\, d'x^\rho = 0$ must likewise follow from $p_\rho\, dx^\rho = 0$ and $x^\rho\, p_\rho = 0$.

Instead of those object transformations, here we shall consider the corresponding coordinate transformations (in which the elements themselves do not change then, but only their coordinates):

$$\left.\begin{aligned} x^{\kappa'} &= \varphi^{\kappa'}(x^\rho, p_\sigma), \\ p_{\lambda'} &= \psi_{\lambda'}(x^\rho, p_\sigma), \end{aligned}\right\} \tag{12}$$

with a non-vanishing functional determinant, in which the $\varphi^{\kappa'}$ and $\psi_{\lambda'}$ are now homogeneous functions of x^ρ and p_ρ, $'x^\rho\, 'p_\rho = 0$ follows from $x^\rho\, p_\rho = 0$, and $'p_\rho\, d'x^\rho = 0$ follows from $p_\rho\, dx^\rho = 0$.

We also call these transformations "contact transformations." In addition, (as in H_n) we consider the group of transformations of analytic elements:

$$\mathfrak{F}: \quad 'x^\kappa = \rho\, x^\kappa; \quad 'p_\lambda = \sigma\, p_\lambda . \tag{13}$$

We would now like to try to restrict the choice of the functions $\varphi^{\kappa'}$ and $\psi_{\lambda'}$ without losing the contact transformations in that way. Since the choice of the degree of $\varphi^{\kappa'}$ and $\psi_{\lambda'}$ and in x^ρ, p_ρ is geometrically irrelevant, we can initially restrict that choice with two requirements:

a. Contact transformations shall commute with the transformations of $\mathfrak{F}$.

b. The proportionality factor in the transition from $p_\rho\, dx^\rho$ to $p_{\rho'}\, dx^{\rho'}$ for ($p_\rho\, x^\rho = 0$) shall be equal to unity:

$$p_{\rho'}\, dx^{\rho'} = p_\rho\, dx^\rho. \tag{14}$$

Since one has:

$$p_{\rho'}\, x^{\rho'} = 0, \qquad\qquad p_\rho\, x^\rho = 0 \tag{15}$$

for elements, in any case, when one differentiates (15), while considering (14), it will follow that:

$$x^{\rho'}\, dp_{\rho'} = x^\rho\, dp_\rho \tag{16}$$

for elements. If the degrees of $\varphi^{\kappa'}$ are equal to *a* and *b*, and those of $\psi_{\lambda'}$ are equal to *c* and *d* – in particular, to those of x^κ (p_λ, resp.) – then requirement (a) will say that:

$$\left.\begin{aligned} 'x^{\kappa'} &= \varphi^{\kappa'}(\rho x^\kappa, \sigma p_\lambda) = \rho^a \sigma^b x^{\kappa'} = \rho x^{\kappa'}, \\ 'p_{\lambda'} &= \psi_{\lambda'}(\rho x^\kappa, \sigma p_\lambda) = \rho^c \sigma^d\, p_{\lambda'} = \sigma\, p_{\lambda'}, \end{aligned}\right\} \tag{17}$$

so

$$a = 1, \quad b = 0, c = 0, d = 1. \tag{18}$$

Obviously, these values are consistent with the group property of contact transformations. We now give $\varphi^{\kappa'}$, as well as $\psi_{\lambda'}$, an arbitrarily-chosen factor that is homogeneous of degree zero in x^κ and p_λ. These factors can then always be chosen in many ways that will satisfy the requirement (b). If we write out the differentials on the left-hand sides of (14) and (16) then it will follow that:

$$\left.\begin{aligned} p_{\rho'}\,\partial_\kappa\varphi^{\rho'}dx^\kappa + p_{\rho'}\,\partial^\lambda\varphi^{\rho'}dp_\lambda &= p_\rho dx^\rho, \\ x^{\rho'}\,\partial_\kappa\psi_{\rho'}\,dx^\kappa + x^{\rho'}\,\partial^\lambda\psi_{\rho'}\,dp_\lambda &= x^\rho dp_\rho, \end{aligned}\right\} \quad \left.\begin{aligned} \partial_\kappa &= \frac{\partial}{\partial x^\kappa}, \\ \partial^\lambda &= \frac{\partial}{\partial p_\lambda}. \end{aligned}\right\} \tag{19}$$

However, we must not conclude from these equations that $p_{\rho'}\ \partial_\kappa\,\varphi^{\rho'} = p_\kappa$, etc., since the dx^κ and dp_λ are not independent, as a result of the fact that $x^\rho\ dp_\rho + p_\rho\ dx^\rho = 0$. If one considers that:

$$p_{\rho'}\ x^{\rho'} = \alpha\, p_\rho\, x^\rho \tag{15a}$$

then one can infer only that:

$$\left.\begin{aligned} p_{\rho'}\,\partial_\lambda\varphi^{\rho'} &= (1+\beta)\,p_\gamma, \\ p_{\rho'}\,\partial^\kappa\varphi^{\rho'} &= \beta x^\kappa, \\ x^{\rho'}\partial_\lambda\psi_{\rho'} &= (\alpha-1-\beta)\,p_\lambda, \\ x^{\rho'}\partial^\kappa\psi_{\rho'} &= (\alpha-\beta)\,x^\kappa, \end{aligned}\right\} \tag{20}$$

in which β is an arbitrary factor that is homogeneous of degree zero in x^κ and p_λ.

It would now be convenient to restrict the choice of $\varphi^{\rho'}$ and $\psi_{\lambda'}$ from now on in such a way that the right-hand sides of all these equations will become as simple as possible. To that end, we remark that any transformation of the form (12) can be replaced with:

$$\left.\begin{aligned} x^{\kappa'} &= \varphi^{\kappa'}(x^\rho, p_\sigma) + x^\tau p_\tau \eta^{\kappa'}(x^\rho, p_\sigma), \\ p_{\lambda'} &= \psi_{\lambda'}(x^\rho, p_\sigma) + x^\tau p_\tau \zeta_{\lambda'}(x^\rho, p_\sigma), \end{aligned}\right\} \tag{21}$$

in which $\eta^{\kappa'}$ and $\zeta_{\lambda'}$ are functions of degree 0, – 1 (– 1, 0, resp.) in x^κ, p_λ that remain finite for $x^\rho\, p_\rho = 0$ and are otherwise chosen arbitrarily, *without the transformation of the coordinates of the elements* suffering a change in that way, and without the conditions (a, b) breaking down. One will then have:

$$\left.\begin{aligned} p_{\rho'}\,\partial_\lambda x^{\rho'} &= (1+\beta+\psi_{\kappa'}\,\eta^{\kappa'})\,p_\lambda, \\ p_{\rho'}\,\partial^\kappa x^{\rho'} &= (\beta+\psi_{\kappa'}\,\eta^{\kappa'})x^\kappa, \\ x^{\rho'}\partial_\lambda\,p_{\rho'} &= ((\alpha-\beta-1)+\varphi^{\lambda'}\zeta_{\lambda'})\,p_\lambda, \\ x^{\rho'}\partial^\kappa\,p_{\rho'} &= ((\alpha-\beta)+\varphi^{\lambda'}\zeta_{\lambda'})\,x^\kappa \end{aligned}\right\} \tag{22}$$

for these new transformations (as long as they act upon elements, so one must again set $x^\rho p_\rho = 0$ in the result).

If one now chooses $\eta^{\kappa'}$ and $\zeta_{\lambda'}$ such that:

$$\psi_{\kappa'}\eta^{\kappa'} = -\beta, \quad \varphi^{\lambda'}\zeta_{\lambda'} = 1 - \alpha + \beta \tag{23}$$

then one will arrive at:

$$\left.\begin{aligned} p_{\rho'}\partial_\lambda x^{\rho'} &= p_\lambda, \\ p_{\rho'}\partial^\kappa x^{\rho'} &= 0, \\ x^{\rho'}\partial_\lambda p_{\rho'} &= 0, \\ x^{\rho'}\partial^\kappa p_{\rho'} &= x^\kappa. \end{aligned}\right\} \tag{24}$$

We introduce these equations (24) as the *third restricting condition* (c). In addition, one also has the homogeneity conditions:

$$\left.\begin{aligned} x^\kappa\partial_\kappa x^{\kappa'} &= x^{\kappa'}, \\ p_\lambda\partial^\kappa x^{\kappa'} &= 0, \\ x^\kappa\partial_\kappa p_{\lambda'} &= 0, \\ p_\lambda\partial^\lambda p_{\lambda'} &= p_{\lambda'}. \end{aligned}\right\} \tag{25}$$

Moreover, as a result of (15a), (21), and (23), one will have:

$$\left.\begin{aligned} x^{\rho'}p_{\rho'} &= \varphi^{\rho'}\psi_{\rho'} + \eta^{\rho'}\psi_{\rho'}(x^\tau p_\tau) + \zeta_{\rho'}\varphi^{\rho'}(x^\tau p_\tau) + \zeta_{\rho'}\eta^{\rho'}(x^\tau p_\tau)^2, \\ &= x^\rho p_\rho + \zeta_{\rho'}\eta^{\rho'}(x^\tau p_\tau)^2. \end{aligned}\right\} \tag{26}$$

If one then chooses $\eta^{\kappa'}$ and $\zeta_{\lambda'}$ such that:

$$\zeta_{\kappa'}\eta^{\kappa'} = 0, \tag{27}$$

in addition, then one will have:

$$x^{\rho'}p_{\rho'} = x^\rho p_\rho\,. \tag{28}$$

We introduce this equation as the *fourth restricting condition* (d). For given $\varphi^{\kappa'}$ and $\psi_{\lambda'}$, there will obviously always be infinitely many values of $\eta^{\kappa'}$ and $\zeta_{\lambda'}$ that satisfy the equations (23) and (27).

One easily shows that two successive transformations that satisfy (c, d) will once more yield a transformation that satisfies (c, d). We have not therefore lost any contact transformations by introducing the conditions (a, b, c, d), and the group property has not been forfeited, but the analytic properties of the transformation functions have been simplified considerably.

It follows by differentiating (24) that:

$$(\partial_{[\mu}\, p_{|\rho'|})\,\partial_{\lambda]}\, x^{\rho'} = 0; \qquad (\partial^{[\nu}\, p_{\rho'})\,\partial^{\kappa]}\, x^{\rho'} = 0, \tag{29}$$

$$(\partial^\kappa p_{\rho'})\,\partial_\lambda x^{\rho'} - (\partial_\lambda p_{\rho'})\,\partial^\kappa x^{\rho'} = A_\lambda^\kappa; \qquad A_\lambda^\kappa = \partial_\lambda x^\kappa = \partial^\kappa p_\lambda, \tag{30}$$

and it will follow from going over to *dx* and *dp* in these equations and suitably combining the terms that:

$$dx^\kappa = (\partial^\kappa p_{\rho'})\, dx^{\rho'} - (\partial^\kappa x^{\rho'})\, dp_{\rho'}\,; \qquad dp_\lambda = -(\partial_\lambda p_{\rho'})\, dx^{\rho'} + (\partial_\lambda x^{\rho'})\, dp_{\rho'}\,. \tag{31}$$

However, since, one has, on the other hand:

$$dx^\kappa = (\partial_{\rho'} x^\kappa)\, dx^{\rho'} + (\partial^{\rho'} x^\kappa)\, dp_{\rho'}\,; \qquad dp_\lambda = (\partial_{\rho'} p_\lambda)\, dx^{\rho'} + (\partial^{\rho'} p_\lambda)\, dp_{\rho'}\,, \tag{32}$$

it will follow that:

$$\partial^\kappa p_{\lambda'} = \partial_{\lambda'} x^\kappa\,; \qquad \partial^\nu x^{\kappa'} = -\,\partial^{\kappa'} x^\nu\,; \qquad \partial_\mu p_{\lambda'} = -\,\partial_{\lambda'} p_\mu\,; \qquad \partial_\lambda x^{\kappa'} = \partial^{\kappa'} p_\lambda\,, \tag{33}$$

which is a system of equations that is equivalent to (29, 30), since conversely (29) and (30) can be derived from (33). We shall prove that (33), and therefore also (29, 30), represent the necessary and sufficient conditions for a transformation of the form (12) to be a contact transformation, in which the φ' are homogeneous of degrees 1 and 0, and the ψ' are homogeneous of degrees 0 and 1 in x^κ (p_λ, resp.). If one goes from (33) to $x^{\kappa'}$ and $p_{\lambda'}$ then equations (24) will follow as a result of homogeneity and:

$$\left.\begin{aligned} \partial_\lambda (p_{\rho'}\, x^{\rho'}) &= p_\lambda, \\ \partial^\kappa (p_{\rho'}\, x^{\rho'}) &= x^\kappa \end{aligned}\right\} \tag{34}$$

or

$$d\,(p_{\rho'}\, x^{\rho'}) = p_\rho\, dx^\rho + x^\rho\, dp_\rho = d\,(p_\rho\, x^\rho) \tag{35}$$

will follow from those equations. Now, since $p_\rho\, x^\rho$, as well as $p_{\rho'}\, x^{\rho'}$, are homogeneous of degree 1 in x^κ and p_λ , (28) will then be fulfilled; i.e., elements will go to elements. Furthermore, as a result of (33), one will have:

$$\left.\begin{aligned} p_{\rho'}\, dx^{\rho'} &= p_{\rho'}\,(\partial_\rho x^{\rho'})\, dx^\rho + p_{\rho'}\,(\partial^\rho x^{\rho'})\, dp_\rho, \\ &= p_{\rho'}\,(\partial^{\rho'} p_\lambda)\, dx^\lambda - p_{\rho'}\,(\partial^{\rho'} x^\rho)\, dp_\rho, \\ &= p_\rho\, dx^\rho, \end{aligned}\right\} \tag{36}$$

such that (14) is also fulfilled; i.e., the united position of neighboring elements will not be perturbed.

In light of (33), we write:

$$\left.\begin{aligned} &\partial_{\lambda'} x^\kappa = \partial^\kappa p_{\lambda'} = T^\kappa_{\lambda'}, && \partial_\lambda x^{\kappa'} = \partial^{\kappa'} p_\lambda = T^{\kappa'}_\lambda, \\ &\partial^\nu x^{\kappa'} = -\partial^{\kappa'} x^\kappa = U^{\nu\kappa'} = -U^{\kappa'\nu}, && \partial_\mu p_{\lambda'} = -\partial_{\lambda'} p_\mu = V_{\mu\lambda'} = -V_{\lambda'\mu}, \end{aligned}\right\} \tag{37}$$

to abbreviate, in which equations (29, 30) then go over to the handier form:

$$\left.\begin{array}{c} V_{\rho'[\mu}T^{\rho'}_{\lambda]} = 0; \quad T^{[\nu}_{\rho'}U^{\kappa]\rho'} = 0, \\ T^{\kappa}_{\rho'}T^{\rho'}_{\lambda} - V_{\lambda\rho'}U^{\kappa\rho'} = A^{\kappa}_{\lambda}, \end{array}\right\} \tag{38}$$

and the equations (24), with their inverses, will go to:

$$\left.\begin{array}{ll} p_{\lambda'} = T^{\lambda}_{\lambda'}p_{\lambda}, & p_{\lambda} = T^{\lambda'}_{\lambda}p_{\lambda'}, \\ x^{\kappa'} = T^{\kappa'}_{\kappa}x^{\kappa}, & x^{\kappa} = T^{\kappa}_{\kappa'}x^{\kappa'}, \\ U^{\kappa'\lambda}p_{\lambda} = 0, & U^{\kappa\lambda'}p_{\lambda'} = 0, \\ V_{\lambda'\kappa}x^{\kappa} = 0, & V_{\lambda\kappa'}x^{\kappa'} = 0, \end{array}\right\} \tag{39}$$

which are equations that now express the homogeneity conditions (25), at the same time as their inverses.

We call the group of coordinate transformations that are characterized by (24, 25) $\mathfrak{K}_{2n+2}$. The manifold of elements that is equipped with the groups $\mathfrak{K}_{2n+2}$ and $\mathfrak{F}$ is called K_{2n-1} . A special case is defined by the subgroup of extended point transformations. One will then have $T^{\kappa'}_{\lambda} = A^{\kappa'}_{\lambda}$, $U^{\kappa'\kappa} = 0$, $V_{\lambda\lambda'} = p_{\rho}\,\partial_{\lambda}A^{\kappa'}_{\lambda}$, and (39) will go to:

$$x^{\kappa'} = A^{\kappa'}_{\kappa}x^{\kappa}, \quad p_{\lambda'} = A^{\lambda}_{\lambda'}p_{\lambda}. \tag{40}$$

4. Relationships with general contact transformations. – If:

$$\left.\begin{array}{l} \xi^1 = \varphi(\xi^2,\dots,\xi^n), \\ f(\xi^2,\dots,\xi^n) = 0, \\ F(x^0,\dots,x^n) = 0; \quad \xi^1 = \dfrac{x^1}{x^0}, \text{etc.} \end{array}\right\} \tag{41}$$

are the equations of one and the same hypersurface then one easily shows that:

$$-1 : \frac{\partial\varphi}{\partial\xi^2} : \dots : \frac{\partial\varphi}{\partial\xi^n} = \frac{\partial f}{\partial\xi^1} : \dots : \frac{\partial f}{\partial\xi^n} \tag{42}$$

and

$$\frac{\partial F}{\partial x^0} : \frac{\partial F}{\partial x^1} : \dots : \frac{\partial F}{\partial x^n} = -\left(\xi^1\frac{\partial f}{\partial\xi^1} + \dots + \xi^n\frac{\partial f}{\partial\xi^n}\right) : \frac{\partial f}{\partial\xi^1} : \dots : \frac{\partial f}{\partial\xi^n}. \tag{43}$$

It will then follow, first of all, that:

$$\left.\begin{array}{l} -1 : \zeta_2 : \dots : \zeta_n = \eta_1 : \dots : \eta_n, \\ p_0 : p_1 : \dots : p_n = -(\xi^1\eta_1 + \dots + \xi^n\eta_n) : \eta_1 : \dots : \eta_n, \end{array}\right\} \tag{44}$$

and secondly, that:

$$p_{\rho'}\, dx^{\rho'} = p_\rho\, dx^\rho \tag{45}$$

is equivalent to:

$$\eta_{j'}\, d\xi^{j'} = \eta_j\, d\xi^j, \qquad \text{in which } j = 1, \ldots, n, \tag{46}$$

and to:

$$d\xi^{1'} - \zeta_{c'}\, d\xi^{c'} = \tau(\xi, \zeta)\,(d\xi^1 - \zeta_c\, d\xi^c), \qquad c = 2, \ldots, n, \tag{47}$$

in which:

$$\tau(\xi, \zeta) = \frac{\eta_1}{\eta_{1'}} \tag{48}$$

is, in fact, a function of only ξ^κ and ζ_b, since $\eta_{1'}$ is homogeneous of degree one in η_i.

It emerges from this that a doubly-homogeneous contact transformation in x^κ, p_λ is a homogeneous contact transformation in the LIE sense in ξ^h, η_i, and a general contact transformation in ξ^h, ζ_b. It can also be easily proved from (44) that any general contact transformation in ξ^h, ζ_b (which can, as is known, always be written as a homogeneous contact transformation in the LIE sense in ξ^h, η_i) can also be always written as a doubly-homogeneous contact transformation in x^κ, p_λ. All investigations into contact transformations in the $(2n - 1)$-dimensional element manifold can then be carried out completely with the $2n + 2$ homogeneous coordinates x^κ, p_λ.