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On the differential geometry of the group of contact transformations: II. Normal form and main theorem for doubly-homogeneous contact transformations

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I. Introduction. – In the first article ([1]), following LIE, it was show that any contact transformation in the $2n - 1$ variables $\xi^1, \ldots, \xi^n, \zeta^2, \ldots, \zeta^n$ can be written as not only a homogeneous contact transformation in $2n$ variables $\xi^1, \ldots, \xi^n, \eta_1, \ldots, \eta_n$, but also as a doubly-homogeneous contact transformation in $2(n + 1)$ variables $x^0, \ldots, x^n, p_0, \ldots, p_n$ that satisfy the homogeneity conditions (I.25):

$$\left.\begin{aligned} x^\kappa \partial_\kappa x^{\kappa'} &= x^{\kappa'}, \\ p_\lambda \partial^\kappa x^{\kappa'} &= 0, \\ x^\kappa \partial_\kappa p_{\lambda'} &= 0, \\ p_\lambda \partial^\lambda p_{\lambda'} &= p_{\lambda'}, \end{aligned}\right\} \tag{1}$$

the conditions (I.28):

$$p_{\rho'} x^{\rho'} = p_\rho x^\rho, \tag{2}$$

and for $p_\rho x^\rho = 0$, the condition (I.14, 16):

$$\left.\begin{aligned} p_{\rho'} dx^{\rho'} &= p_\rho dx^\rho, \\ x^{\rho'} dp_{\rho'} &= x^\rho dp_\rho, \end{aligned}\right\} \tag{3}$$

and the conditions (I.24)

$$\left.\begin{aligned} p_{\rho'} \partial_\lambda x^{\rho'} &= p_\lambda, \\ p_{\rho'} \partial^\kappa x^{\rho'} &= 0, \\ x^{\rho'} \partial_\lambda p_{\rho'} &= 0, \\ x^{\rho'} \partial^\kappa p_{\rho'} &= x^\kappa \end{aligned}\right\} \tag{4}$$

([1]) "Zur Differentialgeometrie der Gruppe der Berührungstransformationen. I. Doppelthomogene Behandlung von Berührungstransformationen," Proc. Roy. Acad. Amsterdam **40** (1937), 100-107.

(which do *not* follow from (3), only for $p_\rho\, x^\rho = 0$). We will now show that we can always replace the doubly-homogeneous contact transformation thus-obtained with a transformation that satisfies condition (3), and therefore, also (4), in addition to (1) and (2), even for the general case of $p_\rho\, x^\rho \neq 0$, *without* changing the transformation of the elements (which are characterized by $p_\rho\, x^\rho = 0$). We call the form that is obtained in that way a *normal form* for the doubly-homogeneous contact transformation. We then prove that in such a way that it implies the main theorem that relates to the most general form of a finite, doubly-homogeneous contact transformation, as well.

2. The normal form. – Following LIE ([1]), one obtains the most general form of a homogeneous contact transformation in $\xi^1, \ldots, \xi^n, \eta_1, \ldots, \eta_n$ in the following way: One chooses any functions $\overset{1}{\Omega}, \ldots, \overset{q}{\Omega}$ of the ξ^h and $\xi^{h'}$ (in which q is a number ≥ 1 and $\leq n$) that are arranged so that the $(n + q)$-rowed determinant:

$$\begin{vmatrix} \partial_i \overset{\mathfrak{a}}{\Omega} & 0 \\ \underset{\mathfrak{a}}{\lambda}\, \partial_i \partial_{i'} \overset{\mathfrak{a}}{\Omega} & \partial_{i'} \overset{\mathfrak{a}}{\Omega} \end{vmatrix}; \quad \mathfrak{a} = 1, \ldots, q; \quad \partial_i = \frac{\partial}{\partial \xi^i}, \qquad \partial_{i'} = \frac{\partial}{\partial \xi^{i'}} \tag{5}$$

does not vanish identically in the $\underset{\mathfrak{a}}{\lambda}$ as a result of $\overset{\mathfrak{a}}{\Omega} = 0$, and then eliminates the $\underset{\mathfrak{a}}{\lambda}$ from the $2n + q$ equations:

$$\overset{\mathfrak{a}}{\Omega}(\xi^h, \xi^{h'}) = 0, \tag{6}$$

$$a) \quad \eta_i = -\, \underset{\mathfrak{a}}{\lambda}\, \partial_i \overset{\mathfrak{a}}{\Omega}, \qquad b) \qquad \eta_{i'} = +\, \underset{\mathfrak{a}}{\lambda}\, \partial_{i'} \overset{\mathfrak{a}}{\Omega}. \tag{7}$$

The determinant condition guarantees that equations (6, 7) can be solved for $\xi^{h'}$, $\eta_{i'}$, and $\underset{\mathfrak{a}}{\lambda}$, as well as for ξ^h, η_i, $\underset{\mathfrak{a}}{\lambda}$. After eliminating the $\underset{\mathfrak{a}}{\lambda}$, (6. 7*a*) will give the $\xi^{h'}$ as functions of the ξ^h and η_i that are homogeneous of degree zero in the η_i. If one substitutes those values in (7*a*) then that will yield the $\underset{\mathfrak{a}}{\lambda}$ as functions of the ξ^h and η_i that are homogeneous of degree one in the η_i. Finally, substituting the $\xi^{h'}$ and the $\underset{\mathfrak{a}}{\lambda}$ in (7*b*) will yield the $\eta_{i'}$ as functions of the ξ^h and η_i that are homogeneous of degree one in η_i. The same thing will be true when one switches the $\xi^{h'}$, $\eta_{i'}$ with the ξ^h, η_i, resp. (LIE, *loc. cit.*, pp. 152). Eliminating the $\underset{\mathfrak{a}}{\lambda}$ will then yield a system of $2n$ equations that can be solved for $\xi^{h'}$, $\eta_{i'}$, as well as for ξ^h, η_i. Those equations represent a contact transformation that takes a point in general position to an $(n - q)$-dimensional manifold, and besides (6), there will be no further equations in the ξ^h, $\xi^{h'}$ (LIE, *loc. cit.*, pp. 158). The rank of the matrix $\partial \xi^{h'} / \partial \eta_i$ will then be equal to $n - q$.

([1]) *Theorie der Transformationsgruppen*, II, pp. 150.

We shall carry out the transition to the homogeneous coordinates x^κ, p_λ ; κ; λ, $\mu = 0, 1, \ldots, n$, so the element ($p_\rho x^\rho = 0$):

$$\left.\begin{aligned} &x^0 : x^1 : \ldots : x^n = 1 : \xi^1 : \ldots : \xi^n, \\ &p_0 : p_1 : \ldots : p_n = -(\xi^1\eta_1 + \cdots + \xi^n\eta_n) : \eta_1 : \ldots : \eta_n \end{aligned}\right\} \tag{8}$$

will go to:

$$\overset{\mathfrak{a}}{\Phi}(x^\kappa, x^{\kappa'}) = \overset{\mathfrak{a}}{\Phi}\left(\frac{x^\alpha}{x^0}, \frac{x^{\alpha'}}{x^{0'}}\right) = 0 \tag{9}$$

under (6), in which the $\overset{\mathfrak{a}}{\Phi}$ are homogeneous of degree zero in x^κ and $x^{\kappa'}$, and (7) will go to:

$$\left.\begin{aligned} &-\frac{p_\alpha}{p_0 x^0}(\xi^j\eta_j) = -\underset{\mathfrak{a}}{\lambda}\partial_\alpha \overset{\mathfrak{a}}{\Phi}, \quad -\frac{p_{\alpha'}}{p_{0'} x^{0'}}(\xi^{j'}\eta_{j'}) = +\underset{\mathfrak{a}}{\lambda}\partial_{\alpha'} \overset{\mathfrak{a}}{\Phi}, \qquad \alpha, \beta, \gamma = 1, \ldots, n, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad h, i, j = 1, \ldots, n, \\ &-\frac{1}{x^0}(\xi^j\eta_j) = -\underset{\mathfrak{a}}{\lambda}\partial_0 \overset{\mathfrak{a}}{\Phi}, \qquad -\frac{1}{x^{0'}}(\xi^{j'}\eta_{j'}) = +\underset{\mathfrak{a}}{\lambda}\partial_{0'} \overset{\mathfrak{a}}{\Phi}, \qquad \partial_\lambda = \frac{\partial}{\partial x^\lambda}, \quad \partial_{\lambda'} = \frac{\partial}{\partial x^{\lambda'}} \end{aligned}\right\} \tag{10}$$

or

$$p_\lambda :: -\underset{\mathfrak{a}}{\lambda}\partial_\lambda \overset{\mathfrak{a}}{\Phi}, \qquad p_{\lambda'} :: +\underset{\mathfrak{a}}{\lambda}\partial_{\lambda'} \overset{\mathfrak{a}}{\Phi}, \tag{11}$$

and conversely, equations (7) can be derived from those proportionalities. The $\underset{\mathfrak{a}}{\lambda}$ can be determined as functions of the ξ^h, η_i that are homogeneous of degree one in η_i from (6, 7). However, only the ratios of the $\underset{\mathfrak{a}}{\lambda}$ can be calculated from (9, 11), and indeed, as homogeneous functions of degree zero in x^κ and p_λ, since equation (8) does not allow one to express the η_i in terms of ξ^h, but only to express the ratios of the η_i in terms of the ratios of the p_λ . In addition, $\xi^{h'}$ and $\eta_{i'}$ can be calculated from (6, 7) as functions of ξ^h and η_i that are homogeneous of degree zero (one, resp.) in η_i . The $\dfrac{x^{\kappa'}}{x^{0'}}$ and $-\dfrac{\eta_{i'}}{\xi^{j'}\eta_{j'}}$ (or, what amounts to the same thing, the $\dfrac{p_{\lambda'}}{p_{0'}}$) then follow from (9.11) as functions of the $\dfrac{x^\kappa}{x^0}$ and the $\dfrac{p_\lambda}{p_0}$. Likewise, the $\dfrac{x^\kappa}{x^0}$ and the $\dfrac{p_\lambda}{p_0}$ can be calculated as functions of $\dfrac{x^{\kappa'}}{x^{0'}}$ and $\dfrac{p_{\lambda'}}{p_{0'}}$. We then get all of the equations together as:

$$\overset{\mathfrak{a}}{\Phi}(x^\kappa, x^{\kappa'}) = 0 \qquad (\mathfrak{a} = 1, \ldots, q), \tag{12}$$

$$a)\quad p_\lambda = -\underset{1}{\lambda}\,\partial_\lambda \frac{\underset{\mathfrak{a}}{\lambda}}{\underset{1}{\lambda}}\overset{\mathfrak{a}}{\Phi}, \qquad\qquad b)\quad p_{\lambda'} = +\underset{1}{\lambda}\,\partial_{\lambda'} \frac{\underset{\mathfrak{a}}{\lambda}}{\underset{1}{\lambda}}\overset{\mathfrak{a}}{\Phi}, \tag{13}$$

in which $\underset{1}{\lambda}$ now plays the role of an undetermined parameter, and:

$$a)\quad x^{\kappa'} = \alpha\,\varphi^{\kappa'}(x^\kappa, p_\lambda), \qquad\qquad b)\quad p_{\lambda'} = \beta\,\psi_{\lambda'}(x^\kappa, p_\lambda), \tag{14}$$

$$a)\quad x^{\kappa} = \gamma\,\varphi^{\kappa}(x^\kappa, p_\lambda), \qquad\qquad b)\quad p_{\lambda} = \varepsilon\,\psi_{\lambda}(x^\kappa, p_\lambda). \tag{15}$$

After eliminating the ratios of the λ, $\frac{1}{\underset{1}{\lambda}}\underset{\mathfrak{a}}{\lambda}\overset{\mathfrak{a}}{\Phi}$ will be a known function. The $\varphi^{\kappa'}$ and $\psi_{\lambda'}$ can be chosen to be homogeneous of degree zero in x^κ and p_λ, and similarly, the φ^κ and ψ_λ can be chosen to be functions of degree zero in $x^{\kappa'}$ and $p_{\lambda'}$. The functions $\varphi^{\kappa'}$, $\psi_{\lambda'}$, φ^κ, ψ_λ are now known, except that the coefficients α, β, γ, ε are still undetermined. If a, b (c, d, resp.) are the degrees of α (β, resp.) in x^κ and p_λ, resp., and c', d' are the degrees of γ (ε, resp.) in $x^{\kappa'}$ and $p_{\lambda'}$, resp., then it will follow by substituting (14) in (15) that:

$$a'\mathfrak{e} = d, \qquad b'\mathfrak{e} = -b, \qquad c'\mathfrak{e} = -c, \qquad d'\mathfrak{e} = a \qquad (\mathfrak{e} = ad - bc), \tag{16}$$

and it will follow from this that γ and ε are established by the choice of a and b, when one assumes that $ad - bc \neq 0$. In addition, β is determined in terms of α as a result of the requirement that:

$$p_{\rho'}\,dx^{\rho'} = p_\rho\,dx^\rho \qquad \text{for} \qquad p_\rho\,x^\rho = 0. \tag{17}$$

If one substitutes (14*b*) and (14*a*) in the left-hand (right-hand, resp.) side of (13*b*) then λ can be calculated from that equation as a homogeneous function of x^κ and p_λ of degrees $a + c$, $b + d$, resp. The choice of α then establishes *all* coefficients. We would like to make that choice in such a way that $a = 1$, $b = 0$, and as a result, $c = 0$, $d = 1$, $a' = 1$, $b' = 0$, $c' = 0$, $d' = 1$. In order to do that, we need only to choose α to be an arbitrary homogeneous function of x^κ and p_λ of degree + 1, 0, resp. $\underset{1}{\lambda}$ then takes on the degrees + 1, + 1 in x^κ and p_λ, resp., and the same degrees when one writes it out in terms of $x^{\kappa'}$ and $p_{\lambda'}$.

If we now substitute (13*a*) in (14*a*) then that will yield:

$$\left.\begin{aligned} x^{\kappa'} &= \alpha(x^\kappa, -\partial_\lambda \underset{\mathfrak{a}}{\lambda}\overset{\mathfrak{a}}{\Phi})\cdot\varphi^{\kappa'}(x^\kappa, -\partial_\lambda \underset{\mathfrak{a}}{\lambda}\overset{\mathfrak{a}}{\Phi}) \\ &= \alpha(x^\kappa, -\underset{\mathfrak{a}}{\lambda}\partial_\lambda \overset{\mathfrak{a}}{\Phi})\cdot\varphi^{\kappa'}(x^\kappa, -\underset{\mathfrak{a}}{\lambda}\partial_\lambda \overset{\mathfrak{a}}{\Phi}), \end{aligned}\right\} \tag{18}$$

and this will be a system of $n + 1$ equations in x^κ and $x^{\kappa'}$ that contains q parameters $\underset{\mathfrak{a}}{\lambda}$ ($\mathfrak{a} = 1, \ldots, q$), which is, in fact, homogeneous of degree zero. The ratios of those parameters can be determined as functions of x^κ and $x^{\kappa'}$ from $q - 1$ of those equations. If one substitutes those values in the remaining $n - q + 2$ equations then it will result from the fact that the $\underset{\mathfrak{a}}{\lambda}$ occur homogeneously of degree zero that the right-hand side will be homogeneous of degrees + 1, 0 in x^κ and $x^{\kappa'}$, and will therefore yield $n - q + 2$ equations of the form:

$$\overset{\alpha}{X}(x^\kappa, x^{\kappa'}) = 1 \qquad (\alpha = q - 1, \ldots, n) \tag{19}$$

whose left-hand sides will have degrees + 1, – 1 in x^κ, $x^{\kappa'}$. Since the quotient of two X represents a homogeneous function of degrees 0, 0 in x^κ and $x^{\kappa'}$, resp., and there can be only q homogeneous equations of degrees zero in x^κ and $x^{\kappa'}$ [namely, (12)], one can certainly derive $n - q + 1$ of equations (19) from the $(n - q + 2)^{\text{th}}$ one and (12). That $(n - q + 2)^{\text{th}}$ equation is, however, certainly independent of (12), since it has degrees + 1, – 1 in x^κ, $x^{\kappa'}$, resp. We write that equation as:

$$X(x^\kappa, x^{\kappa'}) = 1. \tag{20}$$

The q homogeneous equations (12) associate every point in general position in H_n with an $(n - q)$-dimensional manifold. Equation (20) changes nothing about that situation, since it only establishes the factor in the $x^{\kappa'}$. (20) has no meaning then for the geometric transformation of elements, which is also quite obvious, since it first arises when one establishes the choice of α, which is likewise inessential for the geometric transformation of the elements. The rank of the matrix of $\partial x^{\kappa'} / \partial p_\lambda$ will be $n - q$, and thus equal to the rank of the matrix of $\partial \xi^{h'} / \partial \eta_i$.

Equation (20) makes it possible for us to resolve our problem now, and to replace the contact transformation (14, 15) with another one that acts upon the elements precisely as (14, 15) do, but also satisfies equations (1), (2), and (3) for $p_\rho x^\rho \neq 0$. Namely, if we introduce the equations:

$$p_\lambda = -\underset{\mathfrak{a}}{\lambda}\,\partial_\lambda \overset{\mathfrak{a}}{\Phi} + p_{\rho'} x^{\rho'} \partial_\lambda X, \qquad p_{\lambda'} = +\underset{\mathfrak{a}}{\lambda}\,\partial_{\lambda'} \overset{\mathfrak{a}}{\Phi} - p_\rho x^\rho \partial_{\lambda'} X, \tag{21}$$

instead of equations (13), and equations (21) are equivalent to (13) for elements, then we will first have:

$$\left.\begin{aligned} x^{\rho'} p_{\rho'} &= 0 + p_\rho x^\rho X = p_\rho x^\rho, \\ x^\rho p_\rho &= 0 + p_{\rho'} x^{\rho'} X = p_{\rho'} x^{\rho'}, \end{aligned}\right\} \tag{22}$$

and secondly, if we consider (22):

$$-p_\rho dx^\rho + p_{\rho'} dx^{\rho'} = -p_\rho x^\rho dX = 0 \tag{23}$$

(and indeed, this is true even for $p_\rho x^\rho \neq 0$) then it will follow from (22) and (23) (likewise in all cases) that:

$$x^{\rho'} dp_{\rho'} = x^\rho dp_\rho. \tag{24}$$

We call (21) a *normal form* for a doubly-homogeneous contact transformation. The normal form is not determined uniquely since the choice of α is arbitrary. Since the form of X depends upon the choice of a, one can also proceed conversely, and choose X arbitrarily as a function of x^κ and $x^{\kappa'}$ of degree + 1, – 1, resp.

3. The main theorem. – If we set:

$$\left.\begin{array}{ll} X = \overset{0}{X}, & X(\overset{\mathfrak{a}}{\Phi}+1) = \overset{\mathfrak{a}}{X}, \\ \underset{\mathfrak{a}}{\lambda} = X\underset{\mathfrak{a}}{\mu}, & p_\rho x^\rho + \underset{\mathfrak{a}}{\mu}\dfrac{\overset{\mathfrak{a}}{X}}{X} = -\underset{0}{\mu} \end{array}\right\} \tag{25}$$

then the $\overset{\mathfrak{p}}{X}$ ($\mathfrak{p}$ = 0, 1, …, q) will be homogeneous of degrees + 1, – 1 in x^κ, $x^{\kappa'}$, resp., and equations (12), (20), and (21) can be written as:

$$\overset{\mathfrak{p}}{X}(x^\kappa, x^{\kappa'}) = 1 \qquad (\mathfrak{p} = 0, 1, \ldots, q), \tag{26}$$

$$p_\lambda = -\underset{\mathfrak{p}}{\mu}\,\partial_\lambda \overset{\mathfrak{p}}{X}, \qquad p_{\lambda'} = +\underset{\mathfrak{p}}{\mu}\,\partial_{\lambda'} \overset{\mathfrak{p}}{X}. \tag{27}$$

The contact transformation will be obtained by eliminating the $\underset{\mathfrak{p}}{\mu}$ and solving (26), (27) for $x^{\kappa'}$, $p_{\lambda'}$, as well as for x^κ, p_λ.

One now asks whether one can, conversely, always get a doubly-homogeneous contact transformation with degrees + 1, 0; 0, + 1 in x^κ, $x^{\kappa'}$, resp., from q + 1 arbitrary homogeneous functions of degrees + 1, – 1 in those variables. First of all, the equations must naturally be soluble for $x^{\kappa'}$, $p_{\lambda'}$, $\underset{\mathfrak{p}}{\mu}$, and likewise for x^κ, p_λ, $\underset{\mathfrak{p}}{\mu}$; i.e., the determinant

$$\begin{vmatrix} \partial_\lambda \overset{\mathfrak{p}}{X} & 0 \\ \underset{\mathfrak{p}}{\mu}\,\partial_{\lambda'}\partial_\lambda \overset{\mathfrak{p}}{X} & \partial_{\lambda'} \overset{\mathfrak{p}}{X} \end{vmatrix} \tag{28}$$

must not vanish identically in $\underset{\mathfrak{p}}{\mu}$ as a result of (26). If that requirement is met then, according to LIE, that will imply a homogeneous contact transformation in any case, and the degrees of $x^{\kappa'}$ and $p_{\lambda'}$ in p_λ will be 0 (+ 1, resp.). Since we started with functions that were homogeneous in x^κ, $x^{\kappa'}$, the transformation will also be doubly-homogeneous, and all that must be proved is that the degrees of $x^{\kappa'}$ and $p_{\lambda'}$ in p_λ are + 1 and 0, resp. If the equations that are obtained by solving for $x^{\kappa'}$, $p_{\lambda'}$ and their inverses are:

$$a)\quad x^{\kappa'} = \varphi^{\kappa'}(x^{\kappa}, p_{\lambda}), \qquad b)\quad p_{\lambda'} = \psi_{\lambda'}(x^{\kappa}, p_{\lambda}), \tag{29}$$

$$a)\quad x^{\kappa} = \varphi^{\kappa}(x^{\kappa'}, p_{\lambda'}), \qquad b)\quad p_{\lambda} = \psi_{\lambda}(x^{\kappa'}, p_{\lambda'}) \tag{30}$$

then it will follow from the fact that one can also write the left-hand sides of equations (26) homogeneously with degrees – 1, + 1 in x^{κ}, $x^{\kappa'}$, resp., that the degrees of (29) and (30) must then be equal (cf., pp. 4):

$$a = a', \qquad b = b' = 0, \qquad c = c', \qquad b = b' = 1. \tag{31}$$

It will then follow from (16) that either:

$$a = a' = +\,1, \quad c = c' = 0 \tag{32}$$

or

$$a = a' = -\,1, \quad c = c' = \text{freely chosen.} \tag{33}$$

If we set $a = -\,1$ from now on then we will have:

$$\frac{x^{\kappa'}}{x^{0'}}\, x^0\, x^{0'} = \varphi^{\kappa'}\left(\frac{x^{\kappa}}{x^0}, p_{\lambda}\right), \tag{34}$$

and eliminating the p_{λ} can then give only equations that contain $\dfrac{x^{\kappa'}}{x^{0'}}$, $\dfrac{x^{\kappa}}{x^0}$, and $x^0\, x^{0'}$, and thus equations of the form:

$$\frac{1}{x^0 x^{0'}}\, F\left(\frac{x^{\kappa'}}{x^{0'}}, \frac{x^{\kappa}}{x^0}\right) = 1. \tag{35}$$

That system of equations must be equivalent to (26). However, those equations are homogeneous of degree zero in the $2n + 2$ variables x^{κ}, $x^{\kappa'}$, and as a result, a system of equations of the form (35) can never be equivalent to (26). Only the values (32) then remain.

With that, we have proved the following theorem:

Main theorem:

The most general doubly-homogeneous contact transformation of x^{κ}, p_{λ} into $x^{\kappa'}$, $p_{\lambda'}$ that has degrees 1, 0 (0, 1, resp.) *in x^{κ} and p_{λ} will be obtained in normal form when one starts with* $q + 1$ $(1 \leq q \leq n)$ *equations of the form* (26), *in which the* $\overset{\mathfrak{p}}{X}$ *are homogeneous functions of the x^{κ}, $x^{\kappa'}$ of degrees* + 1, – 1, *and chooses them such that the determinant* (28) *does not vanish identically in the* $\underset{\mathfrak{p}}{\mu}$ *as a result of* (28), *and then eliminates the* $\underset{\mathfrak{p}}{\mu}$ *from equations* (26) *and* (27) *and solves those equations for $x^{\kappa'}$, $p_{\lambda'}$ (x^{κ}, p_{λ} , resp.).*

Since a complete duality exists between the x^κ and the p_λ, one can switch x^κ and p_λ in the formulation of the main theorem. One can then just as well begin with functions of p_λ, $p_{\lambda'}$ with degrees + 1, – 1, resp.

4. Example. – The path that goes from a homogeneous contact transformation to its associated doubly-homogeneous one is quite simple. One initially writes down the functions $\overset{\mathfrak{a}}{\Phi}$ in x^κ and $x^{\kappa'}$, chooses an arbitrary function X of degrees + 1, – 1 in x^κ and $x^{\kappa'}$, resp., and then constructs $\overset{\mathfrak{p}}{X}$.

As an example, we treat the transformation:

$$\xi^{1'} = \xi^1 - \xi^a \zeta_a , \qquad \xi^{2'} = \zeta_2 , \qquad \xi^{3'} = \zeta_3 , \qquad \zeta_{2'} = -\xi^2, \qquad \zeta_{3'} = -\xi^3. \qquad (a = 2, 3) \tag{36}$$

Converting to the ξ^h, η_i yields the transformation:

$$\begin{aligned} \xi^{a'} &= -\frac{\eta_a}{\eta_1}, \qquad \eta_{a'} = \xi^a \eta_1 , \\ \xi^{1'} &= \frac{\xi^j \eta_j}{\eta_1}, \qquad \eta_{1'} = \eta_1 , \end{aligned} \qquad (h, i, j = 1, 2, 3) \tag{37}$$

which is established uniquely by the requirement that $\eta_{j'}\, d\xi^{j'} = \eta_j\, d\xi^j$. There is only one function Ω, namely:

$$\Omega \equiv \xi^{1'} - \xi^1 + \xi^2 \xi^{2'} + \xi^3 \xi^{3'} = 0, \tag{38}$$

that goes to:

$$\Phi \equiv \frac{x^{1'}}{x^{0'}} - \frac{x^1}{x^0} + \frac{x^2 x^{2'}}{x^0 x^{0'}} + \frac{x^3 x^{3'}}{x^0 x^{0'}} = 0 \tag{39}$$

under the transition to x^κ, p_λ. We then get the equations:

$$\left.\begin{aligned} &a) \quad \begin{aligned} p_0 &= -\lambda \frac{x^{1'}}{x^0 x^{0'}} + \lambda \frac{\Phi}{x^0} = -\lambda \frac{x^{1'}}{x^0 x^{0'}}, \\ p_1 &= \lambda \frac{1}{x^0}, \\ p_2 &= -\lambda \frac{x^{2'}}{x^0 x^{0'}}, \\ p_3 &= -\lambda \frac{x^{3'}}{x^0 x^{0'}}, \end{aligned} \qquad b) \quad \begin{aligned} p_{0'} &= -\lambda \frac{x^1}{x^0 x^{0'}} + \lambda \frac{\Phi}{x^0} = -\lambda \frac{x^1}{x^0 x^{0'}}, \\ p_{1'} &= \lambda \frac{1}{x^{0'}}, \\ p_{2'} &= \lambda \frac{x^2}{x^0 x^{0'}}, \\ p_{3'} &= \lambda \frac{x^3}{x^0 x^{0'}} \end{aligned} \end{aligned}\right\} \tag{40}$$

(in which use has been made of the equation $\Phi = 0$ in the first rows), and those equations imply that:

$$\left.\begin{array}{llll}
x^{0'} = \alpha, & p_{0'} = -\beta\dfrac{x^1}{x^0}, & x^0 = \gamma, & p_0 = -\delta\dfrac{x^{1'}}{x^{0'}}, \\
x^{1'} = -\alpha\dfrac{p_0}{p_1}, & p_{1'} = \beta, & x^1 = -\gamma\dfrac{p_{0'}}{p_{1'}}, & p_1 = \delta, \\
a) & b) & c) & d) \\
x^{2'} = -\alpha\dfrac{p_2}{p_1}, & p_{2'} = \beta\dfrac{x^2}{x^0}, & x^2 = \gamma\dfrac{p_{2'}}{p_{1'}}, & p_2 = -\delta\dfrac{x^{2'}}{x^{0'}}, \\
x^{3'} = -\alpha\dfrac{p_3}{p_1}, & p_{3'} = \beta\dfrac{x^3}{x^0}, & x^3 = \gamma\dfrac{p_{3'}}{p_{1'}}, & p_3 = \beta\dfrac{x^{3'}}{x^{0'}}.
\end{array}\right\} \tag{41}$$

γ and δ can be determined as functions of $x^{\kappa'}$, $p_{\lambda'}$, as long as α and β are given as functions of x^κ and p_λ, assuming that $ad - bc \neq 0$ (cf., pp. 4). We determine β from α when we demand that:

$$p_{\rho'}\,dx^{\rho'} = p_\rho\,dx^\rho = -x^\rho\,dp_\rho \quad (\rho = 0, 1, 2, 3), \tag{42}$$

or (when one considers $p_\rho\,x^\rho = 0$):

$$\left.\begin{array}{r}
\beta\,d\alpha\left(-\dfrac{x^1}{x^0}-\dfrac{p_0}{p_1}-\dfrac{p_2\,x^2}{p_1\,x^0}-\dfrac{p_3\,x^3}{p_1\,x^0}\right)+\beta\,\alpha\left(-\dfrac{1}{p_1}dp_0-\dfrac{p_0}{(p_1)^2}dp_1\right) \\
-\dfrac{x^2}{p_1\,x^0}dp_2+\dfrac{p_2\,x^2}{(p_1)^2\,x^0}dp_1-\dfrac{x^3}{p_1\,x^0}dp_3+\dfrac{p_3\,x^3}{(p_1)^2\,x^0}dp_1 \\
=-\dfrac{\beta\alpha}{p_1\,x^0}x^\rho dp_\rho = -x^\rho dp_\rho,
\end{array}\right\} \tag{43}$$

from which, it will follow that:

$$\alpha\beta = +\,p_1\,x^0. \tag{44}$$

We likewise find that:

$$\gamma\,\delta = +\,p_{1'}\,x^{0'}, \tag{45}$$

and upon substituting (41.*a*, *b*) in (40*b*):

$$\lambda = \alpha\beta, \tag{46}$$

and likewise upon substituting (41.*c*, *d*) in (40*a*):

$$\lambda = \gamma\,\delta. \tag{47}$$

If we now choose α to be any function of x^κ and p_λ of degrees 1, 0 – e.g., $\alpha = x^0$ – then it will follow that:

$$\alpha = x^0, \qquad \beta = p_1, \qquad \gamma = x^{0'}, \qquad \delta = p_1, \qquad \lambda = p_1\, x^0 = p_{1'}\, x^{0'}, \tag{48}$$

and

$$\left.\begin{array}{llllllll}
 & x^{0'} = x^0, & & p_{0'} = -\dfrac{p_1\,x^1}{x^0}, & & x^0 = x^{0'}, & & p_0 = -\dfrac{p_{1'}\,x^{1'}}{x^{0'}}, \\
 & x^{1'} = -\dfrac{p_0\,x^0}{p_1}, & & p_{1'} = p_1, & & x^1 = -\dfrac{p_{0'}\,x^{0'}}{p_{1'}}, & & p_1 = p_{1'}, \\
a) & & b) & & c) & & d) & \\
 & x^{2'} = -\dfrac{p_2\,x^0}{p_1}, & & p_{2'} = \dfrac{p_1\,x^2}{x^0}, & & x^2 = \dfrac{p_{2'}\,x^{0'}}{p_{1'}}, & & p_2 = -\dfrac{p_{1'}\,x^{2'}}{x^{0'}}, \\
 & x^{3'} = -\dfrac{p_3\,x^0}{p_1}, & & p_{3'} = \dfrac{p_1\,x^3}{x^0}, & & x^3 = \dfrac{p_{3'}\,x^{0'}}{p_{1'}}, & & p_3 = -\dfrac{p_{1'}\,x^{3'}}{x^{0'}}.
\end{array}\right\} \tag{49}$$

Here, equation (20) is:

$$\frac{x^0}{x^{0'}} = 1, \tag{50}$$

such that we will arrive at the generalized transformation:

$$\left.\begin{array}{ll}
p_0 = -\lambda\dfrac{x^{1'}}{x^0\,x^{0'}} + p_{\rho'}\,x^{\rho'}\,\dfrac{1}{x^{0'}}, & p_{0'} = -\lambda\dfrac{x^1}{x^0\,x^{0'}} + p_\rho\,x^\rho\,\dfrac{1}{x^{0'}}, \\
p_1 = \lambda\dfrac{1}{x^0}, & p_{1'} = \lambda\dfrac{1}{x^0}, \\
p_2 = -\lambda\dfrac{x^{2'}}{x^0 x^{0'}}, & p_{2'} = -\dfrac{p_{1'}\,x^{2'}}{x^0 x^{0'}}, \\
p_3 = -\lambda\dfrac{x^{3'}}{x^0 x^{0'}}, & p_{3'} = \lambda\dfrac{x^{3'}}{x^0 x^{0'}}.
\end{array}\right\} \tag{51}$$

Since $\lambda = p_1\,x^0$, that implies that:

$$\left.\begin{array}{llll}
x^{0'} = x^0, & p_{0'} = p_0 + \dfrac{p_2\,x^2 + p_3\,x^3}{x^0}, & x^0 = x^{0'}, & p_0 = p_{0'} + \dfrac{p_{2'}\,x^{2'} + p_{3'}\,x^{3'}}{x^{0'}}, \\
x^{1'} = x^1 + \dfrac{p_2\,x^2 + p_3\,x^3}{p_1}, & p_{1'} = p_1, & x^1 = x^{1'} + \dfrac{p_{2'}\,x^{2'} + p_{3'}\,x^{3'}}{p_{1'}}, & p_1 = p_{1'}, \\
x^{2'} = -\dfrac{p_2\,x^0}{p_1}, & p_{2'} = \dfrac{p_1\,x^2}{x^0}, & x^2 = \dfrac{p_{2'}\,x^{0'}}{p_{1'}}, & p_2 = -\dfrac{p_{1'}\,x^{2'}}{x^{0'}}, \\
x^{3'} = -\dfrac{p_3\,x^2}{p_1}, & p_{3'} = \dfrac{p_1\,x^3}{x^0}, & x^3 = \dfrac{p_{3'}\,x^{0'}}{p_{1'}}, & p_3 = -\dfrac{p_{1'}\,x^{3'}}{x^{0'}},
\end{array}\right\} \tag{52}$$

and equations (2), (3), (4), are, in fact, true, even for $p_\rho\, x^\rho \neq 0$, for that transformation, which will be identical to (49) when it is applied to elements. One also obtains the normal form that was just found when one solves the equations:

$$\left.\begin{aligned}
&\overset{0}{X} \equiv \frac{x^0}{x^{0'}} = 1, \quad \overset{1}{X} \equiv \frac{1}{(x^{0'})^2}(x^0 x^{1'} - x^1 x^{0'} + x^2 x^{2'} + x^3 x^{3'} + x^0 x^{0'}) = 1,\\
&\qquad p_0 = -\underset{0}{\mu}\frac{1}{x^{0'}} + \underset{1}{\mu}\frac{x^{0'} + x^{1'}}{(x^{0'})^2}, \quad p_{0'} = -\frac{\underset{0}{\mu}}{x^{0'}} + \underset{1}{\mu}\frac{x^0 + x^1}{x^{0'}},\\
&\qquad p_1 = -\underset{1}{\mu}\frac{1}{x^{0'}}, \qquad\qquad\qquad p_{1'} = \underset{1}{\mu}\frac{1}{x^{0'}},\\
&\qquad p_2 = -\underset{1}{\mu}\frac{x^{2'}}{(x^{0'})^2}, \qquad\qquad\quad p_{2'} = \underset{1}{\mu}\frac{x^{2'}}{(x^{0'})^2},\\
&\qquad p_3 = -\underset{1}{\mu}\frac{x^{3'}}{(x^{0'})^2}, \qquad\qquad\quad p_{3'} = \underset{1}{\mu}\,\lambda\frac{x^{3'}}{(x^{0'})^2},
\end{aligned}\right\} \tag{53}$$

after eliminating $\underset{0}{\mu}$ and $\underset{1}{\mu}$ from $x^{\kappa'}$, $p_{\lambda'}$.