"Zur Differentialgeometrie der Gruppe der Berührungstransformationen. III. Infinitesimale doppelthomogene Berührungstransformationen und ihre Beziehungen zur Mechanik und Elektrodynamik," Proc. Kon. Ned. Akad. Wet. Amst. **40** (1937), 470-480.

On the differential geometry of the group of contact transformations: III. Infinitesimal doubly-homogeneous contact transformations and their relationships to mechanics and electrodynamics.

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Translated by D. H. Delphenich

I. Introduction. – In this article, we shall discuss the infinitesimal doubly-homogeneous contact transformations and their applications. In the meantime, following the publication of the first two articles in this series (1), F. ENGEL kindly directed my attention to the fact that F. J. DOHMEN had already treated doubly-homogeneous contact transformations in his Greifswald dissertation in the year 1905 (2). Curiously, neither the results that he published in dissertation form nor the remarks of Lie that were inserted in Bd. III of *Transformationsgruppen* under "Kritik einiger neuerer Untersuchungen" ("critique of some recent investigations"), and which defined the starting point for that examination were followed up on in the literature. Therefore, we shall briefly report upon what was known before the publication of this series. In LIE (3), one initially finds a clear definition of the doubly-homogeneous *object* transformation. Naturally, he did not arrive at doubly-homogeneous *coordinate* transformations, which was consistent with the spirit of the times. However, he did obtain precisely the necessary and sufficient conditions that correspond to conditions [I, (25), (29), (30)] *mutatis mutandis*. LIE's discussion did not actually go any further, and it represented only a critique of a false Ansatz of LINDEMANN. In DOHMEN, the four coefficients a, b, c, d [I, pp. 103] were left free, while they had the values 1, 0, 0, 1 in LIE from the outset (which are the same values that I inferred from the demand of the commutability with the group $\mathfrak{F}$, which is a demand that can arise only upon considering the coordinate transformations), and DOHMEN arrived at the transition from ordinary to doubly-homogeneous contact transformations and its inverse. Instead of [I, (14), (15)], he started from equations that corresponded to the coordinate equations:

$$p_{\kappa'}\, dx^{\kappa'} = L\, p_\kappa\, dx^\kappa + M\, x^\lambda\, dp_\lambda\,,$$
$$x^{\kappa'}\, dp_{\kappa'} = N\, x^\lambda\, dp_\lambda + P\, p_\kappa\, dx^\kappa.$$

(1) "Zur Differentialgeometrie der Gruppe der Berührungstransformationen," Proc. Roy. Acad. Amsterdam **40** (1937), 100-107, 236-245.
(2) "Darstellung der Berührungstransformationen in Konnexkoordinaten."
(3) *Transformationsgruppen*, Bd. III, pp. 530.

Corresponding to his Ansatz, he dropped the condition that $p_\kappa x^\kappa = 0$ from the outset, and accordingly found the transformation in a form that did not make any additional use of that condition. He still did not find the main theorem [II, pp. 242], which allows one to represent any arbitrary doubly-homogeneous contact transformation with the help of $q + 1$ functions of x^κ, $x^{\kappa'}$ that are homogeneous of degree + 1, – 1, resp. By contrast, the derivation of an infinitesimal doubly-homogeneous contact transformation from a characteristic function was known to him, as well as the relationship between that function and the characteristic function of the corresponding ordinary contact transformation.

2. Infinitesimal doubly-homogeneous contact transformations. – For LIE ([1]), the most general infinitesimal contact transformation in ξ^h, ζ_a ($h, i, j = 1, \ldots, n$; $a, b, c = 2, \ldots, n$) had the form:

$$\boxed{\frac{d\xi^1}{d\tau} = \zeta_a \frac{\partial W}{\partial \zeta_a} - W, \quad \frac{d\xi^a}{d\tau} = \frac{\partial W}{\partial \zeta_a}, \quad \frac{d\zeta_b}{d\tau} = -\frac{\partial W}{\partial \xi^b} - \zeta_b \frac{\partial W}{\partial \xi^1}.} \tag{1}$$

He transformed the differential form $\underset{1}{d}\,\xi^1 - \zeta_a \underset{1}{d}\,\xi^a$ as follows ([2]):

$$\frac{d}{d\tau}\left(\underset{1}{d}\,\xi^1 - \zeta_a \underset{1}{d}\,\xi^a\right) = -\frac{\partial W}{\partial \xi^1}\left(\underset{1}{d}\,\xi^1 - \zeta_a \underset{1}{d}\,\xi^a\right). \tag{2}$$

If we go over to the variables η_i by means of the equation:

$$1 : -\zeta_2 : \ldots : -\zeta_n = \eta_1 : \eta_2 : \ldots : \eta_n\,, \tag{3}$$

and we add the extra condition that one should have:

$$\frac{d}{d\tau}\left(\eta_h \underset{1}{d}\,\xi^1\right) = 0 \tag{4}$$

then we will get:

$$\frac{d\xi^h}{d\tau} = \frac{\partial \mathfrak{B}}{\partial \eta_h}, \quad \frac{d\eta_i}{d\tau} = -\frac{\partial \mathfrak{B}}{\partial \xi^i} \tag{5}$$

uniquely, in which:

$$\mathfrak{B} = -\eta_1\, W\left(\xi^h, \frac{-\eta_a}{\eta_1}\right). \tag{6}$$

([1]) *Transformationsgruppen II*, pp. 252.

([2]) We write $\underset{1}{d}$, in order to distinguish it from the symbol d.

The most general infinitesimal homogeneous contact transformation then has the form (5), in which $\mathfrak{B}$ is an arbitrary function of ξ^h, η_i that is homogeneous of degree one in η_i. (LIE, *loc. cit.*, pp. 263).

We now go over to the variables x^α, p_α by means of:

$$\left.\begin{aligned} 1 : \xi^1 : \ldots : \xi^n &= x^0 : x^1 : \ldots : x^n, \\ -\eta_i \xi^i : \eta_1 : \ldots : \eta_n &= p_0 : p_1 : \ldots : p_n, \end{aligned}\right\} \tag{7}$$

and initially set:

$$\left.\begin{aligned} x^0 &= 1, & x^h &= \xi^h, \\ p_0 &= -\eta_j\, \xi^j, & p_i &= \eta_j. \end{aligned}\right\} \tag{8}$$

If we then set:

$$\mathfrak{T}(x^\kappa, p_\lambda) = -\, x^0\, p_1\, W\left(\frac{x^h}{x^0}, \frac{-p_a}{p_1}\right) \tag{9}$$

then $\mathfrak{T}$ will be homogeneous of degree one in x^κ, as well as in p_λ, and will not contain p_0, such that:

$$\left.\begin{aligned} \frac{dx^h}{d\tau} &= \frac{\partial\mathfrak{T}}{\partial p_h}, \qquad \frac{dx^0}{d\tau} = \frac{\partial\mathfrak{T}}{\partial p_0}, \\ \frac{dp_i}{d\tau} &= -\frac{\partial\mathfrak{T}}{\partial x^i}, \quad \frac{dp_0}{d\tau} = -p_i\,\frac{\partial\mathfrak{T}}{\partial p_i} + x^h\,\frac{\partial\mathfrak{T}}{\partial x^i}, \\ &\qquad\qquad = -\mathfrak{T} + p_0\,\frac{\partial\mathfrak{T}}{\partial p_0} + \mathfrak{T} + x^0\,\frac{\partial\mathfrak{T}}{\partial x^0}, \\ &\qquad\qquad = -\frac{\partial\mathfrak{T}}{\partial x^0}. \end{aligned}\right\} \tag{10}$$

Since:

$$\left.\begin{aligned} \frac{d}{d\tau}(p_\kappa x^\kappa) &= p_\kappa\,\frac{\partial\mathfrak{T}}{\partial p_\kappa} - x^\kappa\,\frac{\partial\mathfrak{T}}{\partial x^\kappa} = 0, \\ \frac{d}{d\tau}(p_\kappa \underset{1}{d}\, x^\kappa) &= \frac{dp_\kappa}{d\tau}\underset{1}{d}\, x^\kappa + p_\kappa \underset{1}{d}\,\frac{dx^\kappa}{d\tau} = -\frac{\partial\mathfrak{T}}{\partial x^\kappa}\underset{1}{d}\, x^\kappa - p_\kappa\,\frac{\partial\mathfrak{T}}{\partial p_\kappa}, \\ &= -\underset{1}{d}\,\mathfrak{T} + \underset{1}{d}\,\mathfrak{T} = 0, \end{aligned}\right\} \tag{11}$$

the differential form $p_\kappa \underset{1}{d}\, x^\kappa$ will be invariant and will be, in fact, *independent of the equation* $p_\kappa x^\kappa = 0$.

We now go on to the arbitrary homogeneous coordinates:

$$'x^\kappa = \kappa x^\kappa, \qquad 'p_\lambda = \lambda\, x_\lambda, \tag{12}$$

in which the coefficients κ and λ are arbitrary homogeneous functions of degree zero in x^κ and p_λ. One will then have:

$$\frac{d\,'x^\kappa}{d\tau} = \frac{d\log\kappa}{d\tau}\,'x^\kappa + \kappa\frac{d\mathfrak{T}}{dp_\kappa}, \qquad \frac{d\,'p_\lambda}{d\tau} = \frac{d\log\lambda}{d\tau}\,'p_\kappa - \lambda\frac{d\mathfrak{T}}{dx^\kappa}. \tag{13}$$

However, we can change that infinitesimal transformation without altering its geometric meaning when we add a term $\alpha\,'x^\kappa$ ($\beta\,'p_\lambda$, resp.) on the right. (So the transformation of $\frac{'x^\kappa}{'x^0}$ and $\frac{'p_\lambda}{'p_0}$ will not change as a result.) We would like to choose that additional term in such a way that $'p_\kappa\,'x^\kappa$, as well as $'p_\kappa \underset{1}{d}\,'x^\kappa$, are, in turn, invariant, independently of the equation $'p_\kappa\,'x^\kappa = 0$. In view of the homogeneity of $\mathfrak{T}$, we can write:

$$\left.\begin{aligned} \frac{d\,'x^\kappa}{d\tau} &= \frac{d\log\kappa}{d\tau}\,'x^\kappa + \frac{\partial\mathfrak{T}('x^\kappa,'p_\lambda)}{\partial\,'p_\kappa} + \alpha\,'x^\kappa, \\ \frac{d\,'p_\lambda}{d\tau} &= \frac{d\log\lambda}{d\tau}\,'p_\lambda - \frac{\partial\mathfrak{T}('x^\kappa,'p_\lambda)}{\partial\,'x^\lambda} + \beta\,'p_\lambda. \end{aligned}\right\} \tag{14}$$

Since:

$$\frac{d}{d\tau}('p_\kappa\,'x^\kappa) = \frac{d\log\kappa}{d\tau}\,'x^\kappa\,'p_\kappa + \alpha\,'x^\kappa\,'p_\kappa + \frac{d\log\lambda}{d\tau}\,'x^\kappa\,'p_\kappa + \beta\,'x^\kappa\,'p_\kappa \tag{15}$$

and

$$\left.\begin{aligned} \frac{d}{d\tau}('p_\kappa \underset{1}{d}\,'x^\kappa) &= \frac{d\log\lambda}{d\tau} - 'p_\kappa \underset{1}{d}\,'x^\kappa - \underset{1}{d}\,'x^\kappa \frac{\partial\mathfrak{T}('x^\kappa,'p_\lambda)}{\partial\,'x^\kappa} + \beta\,'p_\kappa \underset{1}{d}\,'x^\kappa \\ &\quad + 'p_\kappa \underset{1}{d}\left(\frac{d\log\kappa}{d\tau}\,'x^\kappa + \frac{\partial\mathfrak{T}('x^\kappa,'p_\lambda)}{\partial\,'p_\kappa} + \alpha\,'x^\kappa\right) \\ = \left(\frac{d\log\kappa}{d\tau} + \beta\right)'p_\kappa \underset{1}{d}\,'x^\kappa &+ \left(\frac{d\log\kappa}{d\tau} + \alpha\right)'x^\kappa \underset{1}{d}\,'p_\kappa + \underset{1}{d}\left\{\left(\frac{d\log\kappa}{d\tau} + \alpha\right)'x^\kappa\,'p_\kappa\right\}, \end{aligned}\right\} \tag{16}$$

those two expressions will be independent of the equation $p_\kappa\, x^\kappa = 0$, and will be zero for each differential $\underset{1}{d}$ when:

$$\alpha = -\frac{d\log\kappa}{d\tau}, \qquad \beta = -\frac{d\log\lambda}{d\tau}, \tag{17}$$

such that the desired infinitesimal transformation will read:

$$\left.\begin{aligned} \frac{d\,'x^\kappa}{d\tau} &= \frac{\partial}{\partial' p_\kappa}\mathfrak{T}('x^\kappa, 'p_\lambda), \\ \frac{d' p_\lambda}{d\tau} &= -\frac{\partial}{\partial' x^\lambda}\mathfrak{T}('x^\kappa, 'p_\lambda). \end{aligned}\right\} \tag{18}$$

The function $\mathfrak{T}$ does not contain p_0 . However, if we now define an arbitrary function that can go to $\mathfrak{T}$ by means of $'p_\lambda\,'x^\lambda = 0$, so:

$$'\mathfrak{T}\,('x^\kappa, 'p_\lambda) = \mathfrak{T}\,('x^\kappa, 'p_\lambda) + ('x^\kappa, 'p_\lambda)\,F\,('x^\kappa, 'p_\lambda), \tag{19}$$

then

$$\left.\begin{aligned} \frac{d\,'x^\kappa}{d\tau} &= \frac{\partial'\mathfrak{T}}{\partial' p_\kappa} = \frac{\partial\mathfrak{T}}{\partial' p_\kappa} + 'x^\kappa F + ('x^\kappa, 'p_\lambda)\frac{\partial F}{\partial' p_\kappa}, \\ \frac{d' p_\lambda}{d\tau} &= -\frac{\partial'\mathfrak{T}}{\partial' x^\lambda} = -\frac{\partial\mathfrak{T}}{\partial' x^\lambda} - 'p_\lambda F - ('x^\kappa, 'p_\lambda)\frac{\partial \mathfrak{T}}{\partial' x^\lambda} \end{aligned}\right\} \tag{20}$$

will represent an infinitesimal transformation that has the same meaning as (18), as a result of $'p_\lambda\,'x^\lambda = 0$. Since a function of that sort that does not contain p_0 can be exhibited in that way from an arbitrary homogeneous function of degree one in $'x^\kappa$ and $'p_\lambda$ with the help of $'p_\lambda\,'x^\lambda = 0$, we have then obtained the theorem:

Main theorem:

The most general infinitesimal doubly-homogeneous contact transformation has the form:

$$\frac{dx^\kappa}{d\tau} = \frac{\partial\mathfrak{T}}{\partial p_\kappa}, \qquad \frac{dp_\lambda}{d\tau} = -\frac{\partial\mathfrak{T}}{\partial x^\lambda}, \tag{21}$$

in which $\mathfrak{T}$ represents an arbitrary homogeneous function of degree one in x^κ and p_λ.

In addition, we have found the path that leads from the function $W\,(\xi^h, \zeta_a)$ to the function $\mathfrak{T}$: $\mathfrak{T}$ must be taken to be just that function that is homogeneous of degree one in x^κ and p_λ and that can go to:

$$-x^0\,p_1\,W\left(\frac{x^h}{x^0}, \frac{-p_a}{p_1}\right) \tag{22}$$

by means of $p_\kappa\,x^\kappa = 0$ (1).

(1) F. J. DOHMEN, *loc. cit.*, pp. 40.

3. Symmetrization of a mechanical problem in t**, in particular** (1)**.** – Suppose that a mechanical (or electrodynamical) problem has been posed in the coordinates ξ^p ($p, q, r = 1, \ldots, n - 1$) with a LAGRANGE function L that depends upon ξ^p, $\dot{\xi}$, and that t and does not vanish identically. The equations of motion can be written in LAGRANGIAN form:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{\xi}^q} L - \frac{\partial L}{\partial \xi^q} = 0, \tag{23}$$

as well as in HAMILTONIAN form:

$$\left.\begin{array}{c} \partial^p H = \dot{\xi}^p, \quad \partial_p H = -\dot{\zeta}_p; \quad \partial^p = \dfrac{\partial}{\partial \zeta_p}, \quad \partial_p = \dfrac{\partial}{\partial \xi^p}, \\ H = \zeta_p \dot{\xi}^p - L; \quad \zeta_q = \dfrac{\partial L}{\partial \dot{\xi}^p}. \end{array}\right\} \tag{24}$$

We now write $t = \xi^n$, $H = -\zeta_n$ and introduce a new infinitesimal quantity $d\tau$ in order to avoid the use of differentials (2). Then let:

$$\mathfrak{L} = -H\frac{dt}{d\tau} + \zeta_p \dot{\xi}^p \frac{dt}{d\tau} = \zeta_a \dot{\xi}^a \frac{dt}{d\tau}; \quad a, b, c = 1, \ldots, n, \tag{25}$$

such that one will have:

$$\mathfrak{L}\, d\tau = L\, dt, \tag{26}$$

and let:

$$\mathfrak{H}\,(\zeta_a, \xi^a) = \frac{dt}{d\tau}(-L + \zeta_p \dot{\xi}^p + \zeta_n) = -\mathfrak{L} + \zeta_a \frac{d\xi^a}{d\tau}. \tag{27}$$

$\mathfrak{L}$ is the new LAGRANGIAN function that is homogeneous of order one in the velocities, and one works with the HAMILTONIAN equation $\mathfrak{H} = 0$, instead of with a HAMILTONIAN function.

As one easily verified, the equations of motion now read, in LAGRANGIAN form:

$$\frac{d}{d\tau}\frac{\partial \mathfrak{L}}{\partial \dot{\xi}^b} - \frac{\partial \mathfrak{L}}{\partial \xi^b} = 0, \tag{28}$$

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(1) In the article "Homogeneous variables in classical dynamics" by P. A. M. DIRAC [Proc. Camb. Phil. Soc. **29** (1933), 389-400], this symmetrization was applied to the first homogenization, which makes the LAGRANGIAN function homogeneous of degree one in the velocities. However, since the contact transformation here is still not made homogeneous, we shall prefer the expression "symmetrization."

(2) One can choose $d\tau$ in a manner that is adapted to the problem, but one can also work purely with the differentials without introducing $d\tau$.

in which the dot now represents differentiation with respect to τ. $\mathfrak{L}$ is a function of ξ^a and $d\xi^a / d\tau$ that is homogeneous of degree one in the "velocities" $\dot{\xi}^b$. In HAMILTONIAN form, the equations read:

$$\partial^a \mathfrak{H} = \frac{d\xi^a}{d\tau}, \quad \partial_a \mathfrak{H} = -\frac{d\zeta_b}{d\tau}, \qquad \partial_b = \frac{\partial}{\partial \xi^b}, \qquad \partial^a = \frac{\partial}{\partial \zeta_a}. \tag{29}$$

The function $\mathfrak{H}$ is not the only function that satisfies the HAMILTONIAN equations. If $\mathfrak{T}(\mathfrak{H})$ is a function of $\mathfrak{H}$ that vanishes for $\mathfrak{H} = 0$ then:

$$\partial^a \mathfrak{F} = \frac{\partial \mathfrak{F}}{\partial \mathfrak{H}} \frac{d\xi^a}{d\tau}, \qquad \partial_b \mathfrak{F} = -\frac{\partial \mathfrak{F}}{\partial \mathfrak{H}} \frac{d\zeta_b}{d\tau}, \tag{30}$$

and the equation $\mathfrak{F} = 0$ will then be employed as the HAMILTON equation when one replaces $d\tau$ with $d\tau' = d\tau \Big/ \frac{\partial \mathfrak{F}}{\partial \mathfrak{H}}$.

4. The first homogenization ([1]). – Equations (29) represent an infinitesimal contact transformation in the variables ξ^{n+1}, ξ^a, ζ_a, assuming that one extends it by an equation in ξ^{n+1} that has the form:

$$\frac{d\xi^{n+1}}{d\tau} = \zeta_a \, \partial^a \mathfrak{H} - \mathfrak{H} \, . \tag{31}$$

They have the special property that $\mathfrak{H}$ does not depend upon ξ^{n+1}. As a result, $\underset{1}{d}\, \xi^{n+1} - \zeta_a \underset{1}{d}\, \xi^n$ is invariant:

$$\left.\begin{aligned} d(\underset{1}{d}\, \xi^{n+1} - \zeta_a \underset{1}{d}\, \xi^n) &= \underset{1}{d}\, d\xi^{n+1} - d\zeta_a \underset{1}{d}\, \xi^a - \zeta_a \underset{1}{d}\, d\xi^{n+1} \\ &= \{\underset{1}{d}\, (\xi^{n+1} - \zeta_a \xi^n) + (\underset{1}{d}\, \xi^a)\, \partial_a \mathfrak{H} - \zeta_a \underset{1}{d}\, \partial^a \mathfrak{H}\} d\tau \\ &= \{(\underset{1}{d}\, \zeta_a)\, \partial^a \mathfrak{H} - \underset{1}{d}\, \mathfrak{H} + (\underset{1}{d}\, \xi^a)\, \partial_a \mathfrak{H} + \zeta_a \underset{1}{d}\, \partial^a \mathfrak{H} - \zeta_a \underset{1}{d}\, \partial^a \mathfrak{H}\} d\tau = 0. \end{aligned}\right\} \tag{32}$$

It follows from (31) that:

$$\frac{d\xi^{n+1}}{d\tau} = \zeta_a \frac{d\xi^a}{d\tau} - \mathfrak{H} = \mathfrak{L}, \tag{33}$$

([1]) The first homogenization appeared as the second homogenization in DIRAC, *loc. cit.*, namely, homogenization of the momenta.

and the differential $d\xi^{n+1}$ of the auxiliary variable ξ^{n+1} will then have the meaning of $\mathfrak{L}$ $d\tau = L\ dt$. We now go on to the homogeneous momenta η_i (h, i = 1, …, n + 1) and define:

$$\zeta_a = -\frac{\eta_a}{\eta_{n+1}}. \tag{34}$$

It will then emerge from (33) that:

$$\mathfrak{H} = -\frac{\eta_a}{\eta_{n+1}}\dot{\xi}^a - \mathfrak{L} = -\frac{\eta_h}{\eta_{n+1}}\dot{\xi}^h. \tag{35}$$

If we then introduce the function:

$$H = -\eta_{n+1}\,\mathfrak{H}\left(\xi^h, \frac{-\eta_a}{\eta_{n+1}}\right) = \eta_h\,\dot{\xi}^h, \tag{36}$$

which is homogeneous of degree one in η_i , then [cf., (5)]:

$$\boxed{\frac{\partial H}{\partial \eta_h} = \dot{\xi}^h, \quad \frac{\partial H}{\partial \xi^i} = -\dot{\eta}_i,} \tag{37}$$

and those are the equations of motion in HAMILTONIAN form after the first homogenization. They represent a homogeneous infinitesimal contact transformation. They are joined with the equation:

$$\boxed{H = 0.} \tag{38}$$

One can also succeed in writing down the equation in LAGRANGIAN form here ([1]).

5. The second homogenization. – We now go on to the coordinates x^κ and the momenta p_λ , which are defined by:

$$\left.\begin{aligned} &x^0 : x^1 : \ldots : x^{n+1} = 1 : \xi^1 : \ldots : \xi^{n+1}, \\ &p_0 : p_1 : \ldots : p_{n+1} = -\eta_i\,\xi^i : \eta_1 : \ldots : \eta_{n+1}, \end{aligned}\right\} \tag{39}$$

and introduce the function:

$$\mathbf{H} = -x^0\,p_{n+1}\,\mathfrak{H}\left(\frac{x^h}{x^0}, \frac{-p_a}{p_{n+1}}\right), \tag{40}$$

([1]) DIRAC, *loc. cit.*, pp. 394.

which is homogeneous of degree on in x^κ and p_λ , in place of $\mathfrak{H}$. That will make [cf., (10)]:

$$\frac{\partial \mathbf{H}}{\partial p_\kappa} = \dot{x}^\kappa, \qquad \frac{\partial \mathbf{H}}{\partial x^\lambda} = -\dot{p}_\lambda, \tag{41}$$

and these are the equations of motion in the HAMILTONIAN form after the second homogenization. They represent a doubly-homogeneous contact transformation. They are joined with the equation:

$$\mathbf{H} = 0, \tag{42}$$

and one can also naturally find equations in LAGRANGIAN form here.

6. The electrodynamical equations of motion in the usual general theory of relativity. – As is known, the symmetrization of the electrodynamical equations of motion lead to a function $\mathfrak{H}$ that has the form:

$$\mathfrak{H} = \tfrac{1}{2} mc^2 - \frac{1}{2m} g^{hi} \left(\eta_h - \frac{e}{c}\varphi_h\right)\left(\eta_i - \frac{e}{c}\varphi_i\right), \tag{43}$$

when one chooses $d\tau$ according to:

$$d\tau^2 = g_{hi}\, d\xi^h\, d\xi^i \tag{44}$$

(signature $- - - +$).

The first homogenization leads to [1]:

$$H = -\tfrac{1}{2} mc^2 \eta_5 + \frac{1}{2m} \eta_5\, g^{hi} \left(-\frac{\eta_h}{\eta_5} - \frac{e}{c}\varphi_h\right)\left(-\frac{\eta_i}{\eta_5} - \frac{e}{c}\varphi_i\right), \tag{45}$$

and the second one leads to:

$$\mathbf{H} = -\tfrac{1}{2} mc^2 \eta_5 + \frac{1}{2m} p_5\, x^0\, g^{hi} \left(-\frac{p_h}{p_5} - \frac{e}{c}\varphi_h\right)\left(-\frac{p_i}{p_5} - \frac{e}{c}\varphi_i\right). \tag{46}$$

Hence, the homogenizations do not lead to forms that are symmetric in ξ^h, η_i (x^κ, p_λ, resp.). However, we would now like to show that this is related to only the particular method that was employed here in order to compel homogenization, and that a completely symmetric doubly-homogeneous representation can be found in a different way.

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[1] Cf., DIRAC, *loc. cit.*, pp. 400.

7. The electrodynamical equations of motion in projective field theory. – In projective field theory ([1]), we have homogeneous coordinates x^κ (κ; …, τ = 0, 1, …, 4) from the outset.

Correspondingly, the local space is initially a projective space. The metric will be introduced by means of a quantity $G_{\lambda\kappa}$, for which one has:

$$G_{\lambda\kappa} x^\lambda x^\kappa = -\chi^2, \tag{47}$$

in which $G_{\lambda\kappa}$ is homogeneous of degree – 2 in x^κ, and χ^2 is a positive constant that has the dimension $[L^2]$. The relations between the usual fundamental tensor g_{ih} and $G_{\lambda\kappa}$ read:

$$g_{ih} \frac{\partial \xi^i}{\partial x^\lambda} \frac{\partial \xi^h}{\partial x^\kappa} - q_\lambda\, q_\kappa = G_{\lambda\kappa}, \tag{48}$$

in which:

$$q_\lambda = G_{\lambda\kappa} q^\kappa, \qquad q^\kappa = \chi^{-1} x^\kappa. \tag{49}$$

There is a (projective) displacement that leaves $G_{\lambda\kappa}$ invariant, and for whose parameters $\Pi^\kappa_{\mu\lambda}$, one will have:

$$\left.\begin{aligned} \Pi^\kappa_{\mu\lambda} &= \begin{Bmatrix} \kappa \\ \mu\,\lambda \end{Bmatrix} + (q-1)\, q_{\mu\lambda} q^\kappa + (1-p)\, q_\mu q^{\cdot\kappa}_\lambda + (1-q)\, q_\lambda q^{\cdot\kappa}_\mu, \\ x^\mu \Pi^\kappa_{\mu\lambda} &= -\chi\, p\, q^\kappa_{\cdot\lambda} - \mathcal{A}^\kappa_\lambda, \quad q_{\mu\lambda} = \partial_{[\mu} q_{\lambda]}, \\ x^\lambda \Pi^\kappa_{\mu\lambda} &= -\chi\, q\, q^\kappa_{\cdot\lambda} - \mathcal{A}^\kappa_\lambda, \end{aligned}\right\} \tag{50}$$

in which the CHRISTOFFEL symbol { } refers to $G_{\lambda\kappa}$, p and q are constants, and $q_{\lambda\kappa}$ is connected with the bivector F_{ji} of the electromagnetic field as follows:

$$\left.\begin{aligned} q_{\mu\lambda} &= \frac{1}{2} \frac{k}{c} \frac{\partial \xi^j}{\partial x^\mu} \frac{\partial \xi^i}{\partial x^\lambda} F_{ji}, \\ k = \frac{q}{\sqrt{q^2 - 2pq + p}} \sqrt{\frac{\kappa}{2}}, &\qquad \kappa = \text{gravitational constant} = 1.87 \times 10^{-27}\, [M^{-1} L]. \end{aligned}\right\} \tag{51}$$

The vector of four-velocity is:

$$i^\kappa = \frac{1}{c} \left(\frac{dx^\kappa}{d\tau} + q^\kappa q_\lambda \frac{dx^\lambda}{d\tau} \right) = i^\mu\, \partial_\mu\, x^\kappa, \tag{52}$$

([1]) J. A. SCHOUTEN, "La théorie projective de la relativité," Annales de l'Institut HENRI POINCARÉ **5** (1935), 51-88.

and one will then have $i^\kappa q_\kappa = 0$ and $i^\kappa i_\kappa = +1$.

The total (potential + kinetic) impulse will then be represented by a point in the local projective space, namely, the *impulse-energy point:*

$$p^\kappa = m\,c\,i^\kappa + \frac{e}{k} q^\kappa. \tag{53}$$

The demand that the auto-geodetic lines of the displacement, which are defined by:

$$p^\mu \nabla_\mu p^\kappa = 0, \tag{54}$$

should be the paths of charged particles will lead to the condition that:

$$p + q = 2 \tag{55}$$

and to the equation:

$$i^\kappa \partial_\mu p_\lambda = \frac{1}{mc} \Pi^\rho_{\sigma\lambda} p^\sigma p_\rho + \frac{\rho}{\chi mck} p_\lambda . \tag{56}$$

It follows from (53) that:

$$\left.\begin{aligned} G^{\lambda\mu} p_\lambda p_\mu &= + m^2 c^2 - \frac{e^2}{k^2}, \\ p_\lambda q^\lambda &= -\frac{e}{k}, \end{aligned}\right\} \tag{57}$$

from which, it will emerge that:

$$\mathbf{F}\,(p, q) = \frac{1}{2mc}\left\{ G^{\kappa\lambda} p_\kappa p_\lambda - \frac{2e}{\chi k} p_\lambda x^\lambda + \chi^{-2}\left(\frac{e^2}{k^2} + m^2 c^2\right) G_{\lambda\kappa} x^\lambda x^\kappa \right\} = 0. \tag{58}$$

Now, if one considers (53) and the fact that $p + q = 2$ then:

$$\left.\begin{aligned} \frac{\partial \mathbf{F}}{\partial p_\kappa} &= \frac{1}{mc}\left(p^\kappa - \frac{e}{k} q^\kappa \right) = i^\kappa = \frac{d' x^\kappa}{d\tau}, \quad \frac{d'}{d\tau} = i^\mu \partial_\mu, \\ \frac{\partial \mathbf{F}}{\partial x^\lambda} &= \frac{1}{2mc}\left(p_\rho p_\sigma \partial_\lambda G^{\rho\sigma} - \frac{2e}{\chi k} p_\lambda \right) \\ &= \frac{1}{mc}\left(-\Pi^\sigma_{\rho\lambda} p^\rho p_\sigma - \frac{e}{\chi k} p_\lambda \right) = -\frac{d' p_\lambda}{d\tau}, \end{aligned}\right\} \tag{59}$$

such that $\mathbf{F}$ will prove to be the HAMILTONIAN function. Now, $\mathbf{F}$ is not homogeneous in p_λ and x^κ. However, one now has $p_\lambda x^\lambda = -\chi e / k$. If one sets:

$$\mathbf{H} = \frac{1}{2mc}\left\{-\frac{\chi e}{k}\frac{G^{\kappa\lambda}}{p_\rho x^\rho} - \frac{e}{\chi k}p_\lambda x^\lambda + \frac{m^2c^2}{\chi^2}\frac{k}{e\chi}G_{\lambda\kappa}x^\lambda x^\kappa p_\rho x^\rho\right\} \tag{60}$$

then it will follow by differentiation (and after some reorganization of terms) that:

$$\frac{\partial \mathbf{H}}{\partial p_\kappa} = \frac{d' x^\kappa}{d\tau}, \qquad \frac{\partial \mathbf{H}}{\partial x^\lambda} = -\frac{d' p_\lambda}{d\tau}, \tag{61}$$

in which the desired double homogenization is achieved with a homogeneous $\mathbf{H}$.

One should note that $\mathbf{H}$ does not vanish, and instead of that condition, one will have the equation:

$$\mathbf{H} = m\,c, \tag{62}$$

which follows from (58) and (60).

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