"Zur Differentialgeometrie der Gruppe der Berührungstransformationen. IV. Kovariante Ableitungen in der K_{2n-1}," Proc. Kon. Ned. Akad. Wet. Amst. **41** (1938), 577-584.

On the differential geometry of the group of contact transformations: IV. Covariant derivatives in K_{2n-1} [1].

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Translated by D. H. Delphenich

I. Introduction. – Let x^κ, $\kappa = 0, 1, \ldots, n$ be homogeneous coordinates in an n-dimensional space, and let p_λ , $\lambda = 0, 1, \ldots, n$ be homogeneous local facet coordinates. $\lfloor x^\kappa \rfloor$ [2] represents a *point*, while the combination $\lfloor x^\kappa \rfloor$, $\lfloor p_\lambda \rfloor$ represents an *element*, as long as $x^\rho p_\rho = 0$. The combination x^κ, p_λ is called the *analytic element* that is associated with the element. The totality of all elements defines a $(2n - 1)$-dimensional manifold with $2n + 2$ homogeneous coordinates, between which, a relation exists.

Two neighboring elements $\lfloor x^\kappa \rfloor$, $\lfloor p_\lambda \rfloor$ and $\lfloor x^\kappa + dx^\kappa \rfloor$, $\lfloor p_\lambda + dp_\lambda \rfloor$ *lie united* when one has $p_\rho\, dx^\rho = 0$, and therefore $x^\rho\, dp_\rho = 0$, as well. A set of elements, any two neighboring ones of which lie united, is called a *union of elements.* Any transformation of elements that takes every union of elements to a union of elements and possesses a unique inverse is called a *contact transformation.* Analytically, such a transformation is characterized by the invariance of the equation $p_\rho x^\rho = 0$ and the system of equations $p_\rho x^\rho = 0$, $p_\rho\, dx^\rho = 0$. They are *object transformations* (i.e., coordinates are unchanged, but the objects are transformed). We call the *coordinate transformations* (i.e., objects are unchanged, but the coordinates are transformed) that are characterized by the same invariance "contact transformations." In (K I) [3], it was proved that any coordinate transformation that is a contact transformation can be written in the following way:

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[1] It was the paper "Invariant theory of homogeneous contact transformations," by L. P. EISENHART and M. S. KNEBELMANN that gave rise to this investigation, and it was in that paper that covariant derivatives in the manifold of elements were considered for the first time. Our treatment differs from it by the fact that we start with doubly-homogeneous contact transformations and proceed to the fundamental theorem that allows one to establish any linear connection with the help of certain contact affinors.

[2] $\lfloor x^\kappa \rfloor$ means "except for an arbitrary numerical factor."

[3] J. A. SCHOUTEN, "Zur Differentialgeometrie der Gruppe der Berührungstransformationen, I. Doppelthomogene Behandlung von Berührungstransformationen," Proc. Roy. Acad. Amsterdam **40** (1937), 100-107.

$$\mathfrak{K}_{2n+2}\left\{\begin{array}{l} x^{\kappa'} = \varphi^{\kappa'}(x^\rho, p_\rho), \\ \qquad\qquad\qquad ; \\ p_{\lambda'} = \psi_{\lambda'}(x^\rho, p_\rho), \end{array}\right. \qquad \begin{vmatrix} \dfrac{\partial\varphi^{\kappa'}}{\partial x^\kappa} & \dfrac{\partial\varphi^{\kappa'}}{\partial p_\lambda} \\ \dfrac{\partial\psi_{\lambda'}}{\partial x^\kappa} & \dfrac{\partial\psi_{\lambda'}}{\partial p_\lambda} \end{vmatrix} \neq 0. \tag{1}$$

in which the $\varphi^{\kappa'}$ and $\psi_{\lambda'}$ are homogeneous functions of degree one (zero, resp.) in x^κ and degree zero (one, resp.) in p_λ that satisfy the following conditions:

$$\left.\begin{array}{l} V_{\rho'[\mu}T^{\rho'}_{\lambda]} = 0, \quad T^{[\nu}_{\rho'}U^{\kappa]\rho'} = 0, \quad V_{\rho[\mu'}T^{\rho}_{\lambda']} = 0, \quad T^{[\nu'}_{\rho}U^{\kappa]\rho} = 0, \\ T^{\nu}_{\rho'}T^{\rho'}_{\lambda} - V_{\lambda\rho'}U^{\kappa\rho'} = A^{\kappa}_{\lambda}, \qquad T^{\kappa'}_{\rho}T^{\rho}_{\lambda'} - V_{\lambda'\rho}U^{\kappa'\rho} = A^{\kappa'}_{\lambda'}, \end{array}\right\} \tag{2}$$

$$\left.\begin{array}{ll} p_{\lambda'} = T^{\lambda}_{\lambda'}p_\lambda, & p_\lambda = T^{\lambda'}_{\lambda}p_{\lambda'}, \\ x^{\kappa'} = T^{\kappa'}_{\kappa}x^\kappa, & x^\kappa = T^{\kappa}_{\kappa'}x^{\kappa'}, \\ U^{\kappa'\lambda}p_\lambda = 0, & U^{\kappa\lambda'}p_{\lambda'} = 0 \\ V_{\lambda'\kappa}x^\kappa = 0, & V_{\lambda\kappa'}x^{\kappa'} = 0, \end{array}\right\} \tag{3}$$

$$\left.\begin{array}{l} T^{\kappa}_{\lambda'} = \partial_{\lambda'}x^\kappa = \partial^\kappa p_{\lambda'}, \quad T^{\kappa'}_{\lambda} = \partial_\lambda x^{\kappa'} = \partial^{\kappa'}p_\lambda; \quad \partial_\lambda = \dfrac{\partial}{\partial x^\lambda}, \quad \partial_\lambda = \dfrac{\partial}{\partial p_\lambda}, \\ U^{\nu\kappa'} = -U^{\kappa'\nu} = \partial^{\nu'}x^{\kappa'} = -\partial^{\kappa'}x^{\nu'}, \quad V_{\mu\lambda'} = -V_{\mu\lambda'} = \partial_\mu p_{\lambda'} = -\partial_{\lambda'}p_\mu. \end{array}\right\} \tag{4}$$

These transformations define a group $\mathfrak{K}_{2n+2}$. The expression $p_\rho\, x^\rho$ remains invariant under that group, and with the condition that $p_\rho\, x^\rho = 0$, the expression $x^\rho\, dp_\rho$, as well as $p_\rho\, dx^\rho$, are also invariant.

Along with them, we consider the transformations of the analytic elements:

$$\mathfrak{F}: {}'x^\kappa = \rho\, x^\kappa, \quad {}'p_\lambda = \rho^{-1}\, p_\lambda, \tag{5}$$

in which ρ is a homogeneous function of degree zero in x^κ and p_λ. Those transformations leave every individual element invariant and $p_\rho\, x^\rho$, in addition, while $x^\rho\, dp_\rho$ and $p_\rho\, x^\rho$ will also be invariant under the condition that $p_\rho\, x^\rho = 0$.

Those transformations define a group $\mathfrak{F}$. The manifold of all elements, when equipped with the groups $\mathfrak{K}_{2n+2}$ and $\mathfrak{F}$, is called K_{2n-1}.

2. The quantities in K_{2n-1}. – In what follows, we shall also write $x^{(\kappa)}$, instead of p_λ, and let the indices $\mathfrak{a}, \ldots, \mathfrak{g}$ run through the $2n+2$ values $1, \ldots, n + 1, (1), \ldots, (n+1)$. We then write:

$$\frac{\partial x^{\mathfrak{a}'}}{\partial x^{\mathfrak{b}}} = A^{\mathfrak{a}'}_{\mathfrak{b}}, \tag{6}$$

such that:

$$\left.\begin{array}{ll} A_{\lambda}^{\kappa'} = A_{(\kappa')}^{(\lambda)} = T_{\lambda}^{\kappa'}, & A_{(\lambda)}^{\kappa'} = -A_{(\kappa')}^{\lambda} = U^{\lambda\kappa'} = -U^{\kappa'\lambda}, \\ A_{\lambda}^{(\kappa')} = -A_{\kappa'}^{(\lambda)} = V_{\lambda\kappa'} = -V_{\kappa'\lambda}, & A_{(\lambda)}^{(\kappa')} = A_{\kappa'}^{\lambda} = T_{\kappa'}^{\lambda}, \end{array}\right\} \tag{7}$$

and now define the following quantities:

1. *Scalars:* They have degree $\mathfrak{r}$, are homogeneous of degree $\frac{1}{2}\mathfrak{r}$ in x^{κ} and p_{λ}, and are invariant under $\mathfrak{K}_{2n+2}$ and $\mathfrak{F}$.

2. *Contravariant contact vectors:* They have degree $\mathfrak{r}$ with $2n + 2$ components v^{κ}, $v_{\lambda} = v_{(\lambda)}$ that are homogeneous of degrees $\frac{1}{2}(\mathfrak{r} + 1)$, $\frac{1}{2}(\mathfrak{r} - 1)$ [$\frac{1}{2}(\mathfrak{r} - 1)$, $\frac{1}{2}(\mathfrak{r} + 1)$, resp.] in x^{κ}, p_{λ}, and the transformation equations:

$$\left.\begin{array}{l} \mathfrak{K}_{2n+2} : v^{\mathfrak{a}'} = A_{\mathfrak{b}}^{\mathfrak{a}'} v^{\mathfrak{b}} \;\text{ or }\; \begin{cases} v^{\kappa'} = T_{\kappa}^{\kappa'} v^{\kappa} + U^{\lambda\kappa'} v_{\lambda}, \\ v_{\lambda'} = V_{\kappa\lambda'} v^{\kappa} + T_{\lambda'}^{\lambda} v_{\lambda}, \end{cases} \\ \mathfrak{F} : \begin{cases} 'v^{\kappa'} = \rho v^{\kappa}, \\ 'v_{\lambda} = \rho^{-1} v_{\lambda}. \end{cases} \end{array}\right\} \tag{8}$$

3. *Covariant contact vectors:* They have degree $\mathfrak{r}$ with $2n + 2$ components w_{λ}, $-w^{\kappa} = w_{(\kappa)}$ of degrees $\frac{1}{2}(\mathfrak{r} - 1)$, $\frac{1}{2}(\mathfrak{r} + 1)$, and $\frac{1}{2}(\mathfrak{r} + 1)$, $\frac{1}{2}(\mathfrak{r} - 1)$ in x^{κ}, p_{λ}, and the transformation equations:

$$\left.\begin{array}{l} \mathfrak{K}_{2n+2} : w_{\mathfrak{b}'} = A_{\mathfrak{b}'}^{\mathfrak{a}} w_{\mathfrak{a}} \;\text{ or }\; \begin{cases} w_{\lambda'} = T_{\lambda'}^{\lambda} w_{\lambda} + V_{\kappa\lambda'} w^{\kappa}, \\ w^{\kappa'} = U^{\lambda\kappa'} w_{\lambda} + T_{\kappa}^{\kappa'} w^{\kappa}, \end{cases} \\ \mathfrak{F} : \begin{cases} 'w_{\lambda} = \rho^{-1} w_{\lambda}, \\ 'w^{\kappa} = \rho w^{\kappa}. \end{cases} \end{array}\right\} \tag{9}$$

Corollary ([1]):

a. If v^{κ}, v_{λ} is a contravariant vector with degree $\mathfrak{r}$ then v_{λ}, $-v^{\kappa}$ will be a covariant vector with the degree $\mathfrak{r}$.

b. As a result of the facts that $V_{\kappa\lambda'} x^{\kappa} = 0$, $U^{\lambda\kappa'} p_{\lambda} = 0$, x^{κ}, 0; 0, p_{λ}; x^{κ}, p_{λ}, and $-x^{\kappa}$, p_{λ} are contravariant vectors, and 0, x^{κ}; p_{λ}, 0; p_{λ}, x^{κ}, and $-p_{\lambda}$, x^{κ} are covariant vectors with degree 1.

([1]) These properties will be partially lost when we replace (5) with the more general transformations $'x^{\kappa} = \rho x^{\kappa}$, $'p_{\kappa} = \sigma p_{\kappa}$, as we did in ($K$ I).

4. *Contact affinors and densities:* As usual, they are the sums of products of contact vectors. The degree is the product of the degrees of the factors. A component of an affinor $P^{\mathfrak{a}_1\cdots\mathfrak{a}_p}_{\mathfrak{b}_1\cdots\mathfrak{b}_q}$ with degree $\mathfrak{r}$, which has p_1 unbracketed and $p_2 = p - p_1$ bracketed upper indices and q_1 unbracketed and $q_2 = q - q_1$ bracketed lower indices, is homogeneous of degree $\frac{1}{2}\mathfrak{r} + \frac{1}{2}(p_1 - p_2 - q_1 + q_2)$ in p_λ, so $\mathfrak{F}$ will then take on a factor of $\rho^{p_1-p_2-q_1+q_2}$. The degree is then the sum of the degrees of each component in x^κ and p_λ. It will then follow that:

$$x^{\mathfrak{c}}\,\partial_{\mathfrak{c}} P^{\mathfrak{a}_1\cdots\mathfrak{a}_p}_{\mathfrak{b}_1\cdots\mathfrak{b}_q} = \mathfrak{r}\, P^{\mathfrak{a}_1\cdots\mathfrak{a}_p}_{\mathfrak{b}_1\cdots\mathfrak{b}_q} ,$$

and that the degree of a displacement is the sum of the degrees of the factors. The degree is then invariant under folding (“contraction”). The degree of the unit affinor $A^{\mathfrak{a}}_{\mathfrak{b}}$ is zero. Since the degree has nothing to do with the transformations of $\mathfrak{K}_{2n+2}$, , its definition can be generalized for geometric objects with upper and lower indices that are not affinors; e.g., the degree of $\partial_{\mathfrak{c}} P^{\mathfrak{a}_1\cdots\mathfrak{a}_p}_{\mathfrak{b}_1\cdots\mathfrak{b}_q}$ is equal to $\mathfrak{r} - 1$.

5. *The fundamental contravariant bivector* $f^{\mathfrak{a}\mathfrak{b}}$: It has the components:

$$f^{\kappa\lambda} = 0, \qquad f^{\kappa(\lambda)} = -f^{(\lambda)\kappa} = \delta^{\kappa}_{\lambda}, \quad f^{(\kappa)(\lambda)} = 0, \tag{10}$$

in particular, in any reference system, and as a result, in any other reference system, as well. Its degree is zero, and its components are invariant under $\mathfrak{F}$.

6. *The fundamental covariant bivector* $f_{\mathfrak{b}\mathfrak{a}}$: It has the components:

$$f_{\lambda\kappa} = 0\ , \qquad f_{\lambda(\kappa)} = -f_{(\kappa)\lambda} = \delta^{\kappa}_{\lambda}, \quad f_{(\lambda)(\kappa)} = 0, \tag{11}$$

in particular, in any reference system. Its degree is zero, and these components are also invariant under $\mathfrak{F}$.

One has the equation:

$$f^{\mathfrak{a}\mathfrak{c}} f_{\mathfrak{b}\mathfrak{c}} = A^{\mathfrak{a}}_{\mathfrak{b}} . \tag{12}$$

We employ $f^{\mathfrak{a}\mathfrak{b}}$ and $f_{\mathfrak{b}\mathfrak{a}}$ for the raising and lowering of indices, and agree that it will always be the *first* index of $f^{\mathfrak{a}\mathfrak{b}}$ that acts and the *second* index of $f_{\mathfrak{b}\mathfrak{a}}$.

Hence:

$$\left.\begin{array}{lll} v_{\mathfrak{b}} = f_{\mathfrak{b}\mathfrak{a}} v^{\mathfrak{a}} & \text{or} & \left\{\begin{array}{l} v_\lambda = f_{\lambda(\kappa)} v^{(\kappa)} = v^{(\lambda)}, \\ v_{(\lambda)} = f_{(\lambda)\kappa} v^{\kappa} = -v^{\kappa}, \end{array}\right. \\ v^{\mathfrak{a}} = f^{\mathfrak{b}\mathfrak{a}} v_{\mathfrak{b}} & \text{or} & \left\{\begin{array}{l} v^{\kappa} = v_{(\lambda)} f^{(\lambda)\kappa} = -v_{(\lambda)}, \\ v^{(\lambda)} = v_{\kappa} f^{\kappa(\lambda)} = v_\lambda, \end{array}\right. \end{array}\right\} \tag{13}$$

which agrees with the conditions that we posed in (2) and (3). It follows from (12) that:

$$f_{\cdot\mathfrak{b}}^{\mathfrak{a}} = -f_{\mathfrak{b}}^{\cdot\mathfrak{a}} = A_{\mathfrak{b}}^{\mathfrak{a}} \qquad \text{or} \qquad \left.\begin{aligned} f_{\cdot\lambda}^{\kappa} &= -f_{\lambda}^{\cdot\kappa} = \delta_{\lambda}^{\kappa}, \\ f_{\cdot(\lambda)}^{(\kappa)} &= -f_{(\lambda)}^{\cdot(\kappa)} = \delta_{(\lambda)}^{(\kappa)}. \end{aligned}\right\} \tag{14}$$

In general, one has the rule that a change of sign will appear under raising or lowering of indices if and only if the upper index is not bracketed and the lower one is bracketed ([1]). Just as with the introduction of a fundamental tensor, here, as well, the distinction between covariant and contravariant quantities vanishes with the introduction of a fundamental bivector, and only the distinction between covariant, contravariant, and mixed components will remain. Unlike the situation in RIEMANNIAN geometry, the various types of components will differ by at most a sign, since their absolute values will not change under raising and lowering of indices.

3. Connection in K_{2n-1} **.** – A linear connection in K_{2n-1} is established by the equation:

$$\nabla_{\mathfrak{c}}\, v^{\mathfrak{a}} = \partial_{\mathfrak{c}}\, v^{\mathfrak{a}} + \Pi_{\mathfrak{c}\mathfrak{b}}^{\mathfrak{a}}\, v^{\mathfrak{b}}, \tag{15}$$

in which the $\Pi_{\mathfrak{c}\mathfrak{b}}^{\mathfrak{a}}$ define a geometric object of degree – 1 with the transformation rule:

$$\mathfrak{K}_{2n+2}:\ \Pi_{\mathfrak{c}'\mathfrak{b}'}^{\mathfrak{a}'} = A_{\mathfrak{c}'\mathfrak{b}'\mathfrak{a}}^{\mathfrak{c}\mathfrak{b}\mathfrak{a}'}\,\Pi_{\mathfrak{c}\mathfrak{b}}^{\mathfrak{a}} + A_{\mathfrak{b}}^{\mathfrak{a}'}\partial_{\mathfrak{c}'}A_{\mathfrak{b}'}^{\mathfrak{b}}, \tag{16a}$$

$$\left.\begin{aligned} '\Pi_{\mu\lambda}^{\kappa} &= \rho^{-1}\,\Pi_{\mu\lambda}^{\kappa}, && (-1,0), & '\Pi_{(\mu)(\lambda)}^{(\kappa)} &= \rho\,\Pi_{(\mu)(\lambda)}^{(\kappa)}, && (0,-1), \\ '\Pi_{\mu\lambda}^{(\kappa)} &= \rho^{-3}\,\Pi_{\mu\lambda}^{(\kappa)}, && (-2,1), & '\Pi_{(\mu)(\lambda)}^{\kappa} &= \rho^{3}\,\Pi_{(\mu)(\lambda)}^{\kappa}, && (1,-2), \\ '\Pi_{(\mu)\lambda}^{(\kappa)} &= \rho^{-1}\,\Pi_{(\mu)\lambda}^{(\kappa)}, && (-1,0), & '\Pi_{\mu(\lambda)}^{\kappa} &= \rho\,\Pi_{\mu(\lambda)}^{\kappa}, && (0,-1), \\ '\Pi_{\mu(\lambda)}^{(\kappa)} &= \rho^{-1}\,\Pi_{\mu(\lambda)}^{(\kappa)}, && (-1,0), & '\Pi_{(\mu)\lambda}^{\kappa} &= \rho\,\Pi_{(\mu)\lambda}^{\kappa}, && (0,-1). \end{aligned}\right\} \tag{16b}$$

The degree will be reduced by 1 under covariant differentiation. Since $dx^{\mathfrak{a}}$ is not a contact vector, there is, in general, no covariant differential. As in an H_{2n-1}, one can derive three affinors from $\Pi_{\mathfrak{c}\mathfrak{b}}^{\mathfrak{a}}$ ([2]):

$$\left.\begin{aligned} P_{\cdot\mathfrak{b}}^{\mathfrak{a}} &\overset{h}{=} x^{\mathfrak{c}}\,\Pi_{\mathfrak{c}\mathfrak{b}}^{\mathfrak{a}} + A_{\mathfrak{b}}^{\mathfrak{a}}, \\ Q_{\cdot\mathfrak{c}}^{\mathfrak{a}} &\overset{h}{=} \Pi_{\mathfrak{c}\mathfrak{b}}^{\mathfrak{a}}\,x^{\mathfrak{b}} + A_{\mathfrak{c}}^{\mathfrak{a}}, \end{aligned}\right\} \qquad \text{degree } 0, \tag{17}$$

$$S_{\mathfrak{c}\mathfrak{b}}^{\cdot\cdot\mathfrak{a}} \overset{h}{=} \Pi_{[\mathfrak{c}\mathfrak{b}]}^{\cdot\cdot\mathfrak{a}}, \qquad \text{degree} - 1. \tag{18}$$

([1]) Observe that one therefore always has $v^{\mathfrak{a}}\, w_{\mathfrak{a}} = -\, v_{\mathfrak{a}}\, w^{\mathfrak{a}}$.

([2]) $\overset{h}{=}$ means "equality is valid only for holonomic reference systems."

If we write:

$$\nabla_{\mathfrak{c}} f_{\mathfrak{ba}} = F_{\mathfrak{cba}} , \qquad F_{\mathfrak{c}\,(\mathfrak{ba})} = 0 \tag{19}$$

then it will follow that:

$$2\Pi^{\mathfrak{a}}_{\mathfrak{c}[\mathfrak{b}} f_{\mathfrak{a}]\mathfrak{b}} = F_{\mathfrak{cba}} , \tag{20}$$

or

$$2\,\Pi_{\mathfrak{c}[\mathfrak{ba}]} = F_{\mathfrak{cba}} , \qquad \Pi_{\mathfrak{cba}} = \Pi^{\mathfrak{a}}_{\mathfrak{cb}} f_{\mathfrak{ab}} . \tag{21}$$

It follows that:

$$S_{[\mathfrak{cba}]} = \Pi_{[\mathfrak{cba}]} = F_{[\mathfrak{cba}]} . \tag{22}$$

If the connection is symmetric then it will follow from (21) that $\Pi_{\mathfrak{cba}}$ is symmetric in all indices for the case in which $F_{\mathfrak{cba}}$ vanishes:

$$\Pi_{\mathfrak{cba}} = \Pi_{(\mathfrak{cba})} . \tag{23}$$

One can then derive from (21) that:

$$-\Pi^{\mathfrak{a}}_{\mathfrak{cb}} x_{\mathfrak{a}} = Q_{\mathfrak{bc}} - f_{\mathfrak{bc}} + F_{\mathfrak{cba}}\, x^{\mathfrak{a}}, \tag{24}$$

such that a third quantity of valence two with degree 0 can be derived from $\Pi^{\mathfrak{a}}_{\mathfrak{cb}}$ here, and which can, however, be expressed in terms of $Q_{\mathfrak{ba}}$, $f_{\mathfrak{ba}}$, and $F_{\mathfrak{cba}}$. Applying $f^{\,\mathfrak{ba}}$ to (20) yields:

$$-\Pi^{\mathfrak{a}}_{\mathfrak{cb}} = F_{\mathfrak{cba}} f^{\,\mathfrak{ba}} = F_{\mathfrak{c}\cdot\mathfrak{a}}^{\;\cdot\,\mathfrak{a}} . \tag{25}$$

4. Establishing a connection with the help of a double plane (*Doppelblatte*). – An affinor B^{κ}_{λ} of rank ρ in an E_n , $n = 2p$ that satisfies the equation:

$$B^{\kappa}_{\rho} B^{\rho}_{\lambda} = B^{\kappa}_{\lambda} \tag{26}$$

determines a unique *double plane* in this E_n – i.e., a system of two E_p that have no common direction – and conversely, B^{κ}_{λ} is determined uniquely by the double plane. In the same way, a contact affinor field $B^{\mathfrak{a}}_{\mathfrak{b}}$ with degree zero and rank $n + 1$ that satisfies the equation:

$$B^{\mathfrak{a}}_{\mathfrak{c}} B^{\mathfrak{c}}_{\mathfrak{b}} = B^{\mathfrak{a}}_{\mathfrak{b}} \tag{27}$$

will define a structure in every local space that we will likewise call a *double plane*. For the affinor $C^{\mathfrak{a}}_{\mathfrak{b}} = A^{\mathfrak{a}}_{\mathfrak{b}} - B^{\mathfrak{a}}_{\mathfrak{b}}$, one obviously has:

$$C^{\mathfrak{a}}_{\mathfrak{c}} C^{\mathfrak{c}}_{\mathfrak{b}} = C^{\mathfrak{a}}_{\mathfrak{b}} , \qquad B^{\mathfrak{a}}_{\mathfrak{c}} C^{\mathfrak{c}}_{\mathfrak{b}} = 0. \tag{28}$$

The double plane is called *involutory*, in particular, in $f_{\mathfrak{ba}}$ when:

$$B_{\mathfrak{ba}}^{\mathfrak{bc}} f_{\mathfrak{bc}} = 0, \qquad C_{\mathfrak{ba}}^{\mathfrak{bc}} f_{\mathfrak{bc}} = 0, \tag{29}$$

or, when written otherwise:

$$f_{\mathfrak{ba}} = B_{\mathfrak{b}}^{\mathfrak{b}} C_{\mathfrak{a}}^{\mathfrak{c}} f_{\mathfrak{bc}} + C_{\mathfrak{b}}^{\mathfrak{b}} B_{\mathfrak{a}}^{\mathfrak{c}} f_{\mathfrak{bc}}, \tag{30}$$

it will emerge from this that:

$$\left. \begin{aligned} B_{\mathfrak{b}}^{\mathfrak{e}} f_{\mathfrak{ea}} &= B_{\mathfrak{b}}^{\mathfrak{e}} C_{\mathfrak{a}}^{\mathfrak{b}} f_{\mathfrak{eb}}, \\ C_{\mathfrak{b}}^{\mathfrak{e}} f_{\mathfrak{ea}} &= C_{\mathfrak{b}}^{\mathfrak{e}} B_{\mathfrak{a}}^{\mathfrak{b}} f_{\mathfrak{eb}}. \end{aligned} \right\} \tag{31}$$

Similar equations are true for $f^{\mathfrak{ab}}$, since:

$$B_{\mathfrak{c}}^{\mathfrak{d}} f^{\mathfrak{ca}} f_{\mathfrak{bd}} = B_{\mathfrak{c}}^{\mathfrak{d}} f^{\mathfrak{ca}} C_{\mathfrak{b}}^{\mathfrak{e}} B_{\mathfrak{d}}^{\mathfrak{f}} f_{\mathfrak{ef}} = f^{\mathfrak{ca}} C_{\mathfrak{b}}^{\mathfrak{e}} f_{\mathfrak{ec}} = -C_{\mathfrak{b}}^{\mathfrak{a}}, \tag{32}$$

such that (29) is equivalent to:

$$C_{\mathfrak{cd}}^{\mathfrak{ab}} f^{\mathfrak{cb}} = 0, \qquad B_{\mathfrak{cd}}^{\mathfrak{ab}} f^{\mathfrak{cd}} = 0. \tag{33}$$

We say that a quantity with one index *lies in the B-domain* (*C-domain*, resp.) when it does not change when one applies B (C, resp.) to that index. Applying C (B, resp.) to the same index will then give zero.

We shall now prove the theorem:

Any linear connection in K_{2n-1} is determined uniquely by being given the following quantities:

1. *The covariant differential quotient of* $f_{\mathfrak{ba}}$:

$$\nabla_{\mathfrak{c}} f_{\mathfrak{ba}} = F_{\mathfrak{cba}} . \tag{34}$$

2. *The covariant differential quotient of the quantity* $B_{\mathfrak{b}}^{\mathfrak{a}}$ *that belongs to any arbitrary double plane that is involutory relative to* $f_{\mathfrak{ba}}$:

$$\nabla_{\mathfrak{c}} B_{\mathfrak{b}}^{\mathfrak{a}} = -\nabla_{\mathfrak{c}} C_{\mathfrak{b}}^{\mathfrak{a}} = E_{\mathfrak{cb}}^{\cdot\cdot\mathfrak{a}} . \tag{35}$$

3. *The two following components of* $S_{\mathfrak{cba}}$:

$$\left. \begin{aligned} S_{\mathfrak{fed}} B_{\mathfrak{c}}^{\mathfrak{f}} C_{\mathfrak{ba}}^{\mathfrak{ed}} &= T_{\mathfrak{cba}}, \\ S_{\mathfrak{fed}} C_{\mathfrak{c}}^{\mathfrak{f}} B_{\mathfrak{ba}}^{\mathfrak{ed}} &= U_{\mathfrak{cba}}. \end{aligned} \right\} \tag{36}$$

Proof: Naturally, $F_{\mathfrak{cba}}$ must satisfy the demand that $F_{\mathfrak{c}(\mathfrak{ba})} = 0$, $T_{\mathfrak{cba}}$ must lie in the C-domain with $\mathfrak{a}$ and $\mathfrak{b}$ and in the B-domain with $\mathfrak{c}$, and likewise, $U_{\mathfrak{cba}}$ must lie in the B-

domain with $\mathfrak{a}$ and $\mathfrak{b}$ and in the C-domain with $\mathfrak{c}$. Moreover, differentiation (29) will yield the following relations between $F_{\mathfrak{cba}}$ and $E_{\mathfrak{cb}}^{\cdot\cdot\mathfrak{a}}$:

$$\left.\begin{aligned} F_{\mathfrak{ced}}\, B^{\mathfrak{ed}}_{\mathfrak{ba}} &= E_{\mathfrak{cbd}}\, B^{\mathfrak{d}}_{\mathfrak{a}} - E_{\mathfrak{cad}}\, B^{\mathfrak{d}}_{\mathfrak{b}}\,, \\ F_{\mathfrak{ced}}\, C^{\mathfrak{ed}}_{\mathfrak{ba}} &= -E_{\mathfrak{cbd}}\, C^{\mathfrak{d}}_{\mathfrak{a}} + E_{\mathfrak{cad}}\, C^{\mathfrak{d}}_{\mathfrak{b}}\,, \end{aligned}\right\} \tag{37}$$

from which, it will follow that:

$$\left.\begin{aligned} E_{\mathfrak{ced}}\, C^{\mathfrak{e}}_{\mathfrak{b}} B^{\mathfrak{d}}_{\mathfrak{a}} = 0, \quad E_{\mathfrak{ceb}}\, B^{\mathfrak{e}}_{\mathfrak{b}} C^{\mathfrak{d}}_{\mathfrak{a}} = 0, \\ F_{\mathfrak{ced}}\,(B^{\mathfrak{ed}}_{\mathfrak{ba}} - C^{\mathfrak{ed}}_{\mathfrak{ba}}) = 2E_{\mathfrak{c}[\mathfrak{ba}]}. \end{aligned}\right\} \tag{38}$$

If those requirements are satisfied then $\frac{1}{2}N^3 - \frac{1}{2}N^2$ free components ($N = 2n + 2$) will remain for $F_{\mathfrak{cba}}$, $\frac{1}{4}N^3 + \frac{1}{2}N^2$ for $E_{\mathfrak{cba}}$ (when $F_{\mathfrak{cba}}$ is fixed already), $\frac{1}{8}N^3$ for $E_{\mathfrak{cba}}$, and $\frac{1}{8}N^3$ for $U_{\mathfrak{cba}}$, so there will be N^3 in all, and there are also N^3 unknowns, namely, the $\Pi^{\mathfrak{a}}_{\mathfrak{cb}}$. A tensor:

$$G_{\mathfrak{ba}} = 2\, f_{(\mathfrak{b}\,|\,\mathfrak{d})}\, B^{\mathfrak{d}}_{\mathfrak{a})} \tag{39}$$

can be defined from $f_{\mathfrak{ba}}$ and $B^{\mathfrak{a}}_{\mathfrak{b}}$, whose rank is equal to $2n + 2$, as a result of the involutory position of $B^{\mathfrak{a}}_{\mathfrak{b}}$. Differentiating that equation yields:

$$\left.\begin{aligned} \nabla_{\mathfrak{c}} G_{\mathfrak{ba}} &= F_{\mathfrak{cbd}} B^{\mathfrak{d}}_{\mathfrak{a}} + E_{\mathfrak{cba}} \\ &\quad + F_{\mathfrak{cad}} B^{\mathfrak{d}}_{\mathfrak{b}} + E_{\mathfrak{cab}}\,. \end{aligned}\right\} \tag{40}$$

When (35) is written out, it will read:

$$E_{\mathfrak{cb}}^{\cdot\cdot\mathfrak{a}} - \partial_{\mathfrak{c}} B^{\mathfrak{a}}_{\mathfrak{b}} = -\Pi^{\mathfrak{d}}_{\mathfrak{cb}} B^{\mathfrak{a}}_{\mathfrak{b}} + \Pi^{\mathfrak{a}}_{\mathfrak{cd}} B^{\mathfrak{d}}_{\mathfrak{b}}\,, \tag{41}$$

from which, it will emerge that:

$$\left.\begin{aligned} C^{\mathfrak{d}}_{\mathfrak{b}}(E_{\mathfrak{cd}}^{\cdot\cdot\mathfrak{a}} - \partial_{\mathfrak{c}} B^{\mathfrak{a}}_{\mathfrak{b}}) = \Pi^{\mathfrak{e}}_{\mathfrak{cd}}\, B^{\mathfrak{a}}_{\mathfrak{e}}\, C^{\mathfrak{d}}_{\mathfrak{b}} &= \Pi_{\mathfrak{cdf}} C^{\mathfrak{d}}_{\mathfrak{b}}\, f^{\,\mathfrak{fe}} B^{\mathfrak{a}}_{\mathfrak{e}} \\ &= \Pi_{\mathfrak{cdg}} C^{\mathfrak{dg}}_{\mathfrak{bf}}\, f^{\,\mathfrak{fa}}\,, \end{aligned}\right\} \tag{42}$$

and

$$C^{\mathfrak{a}}_{\mathfrak{e}}(E_{\mathfrak{cb}}^{\cdot\cdot\mathfrak{e}} - \partial_{\mathfrak{c}} B^{\mathfrak{e}}_{\mathfrak{b}}) = \Pi^{\mathfrak{e}}_{\mathfrak{cd}}\, C^{\mathfrak{a}}_{\mathfrak{e}}\, B^{\mathfrak{d}}_{\mathfrak{b}} = \Pi_{\mathfrak{cdg}} C^{\mathfrak{dg}}_{\mathfrak{bf}}\, f^{\,\mathfrak{fa}}\,, \tag{43}$$

such that:

$$\left.\begin{aligned} \Pi_{\mathfrak{ced}} C^{\mathfrak{ed}}_{\mathfrak{ba}} &= -f_{\mathfrak{ae}}\, C^{\mathfrak{d}}_{\mathfrak{b}}\,(E_{\mathfrak{cd}}^{\cdot\cdot\mathfrak{e}} - \partial_{\mathfrak{c}} B^{\mathfrak{e}}_{\mathfrak{b}}), \\ \Pi_{\mathfrak{ced}} B^{\mathfrak{ed}}_{\mathfrak{ba}} &= f_{\mathfrak{ab}}\, C^{\mathfrak{d}}_{\mathfrak{e}}\,(E_{\mathfrak{cb}}^{\cdot\cdot\mathfrak{e}} - \partial_{\mathfrak{c}} B^{\mathfrak{e}}_{\mathfrak{b}}). \end{aligned}\right\} \tag{44}$$

Only $\Pi_{\mathfrak{ced}}\, B^{\mathfrak{e}}_{\mathfrak{b}} C^{\mathfrak{d}}_{\mathfrak{a}}$ and $\Pi_{\mathfrak{ced}}\, C^{\mathfrak{e}}_{\mathfrak{b}} B^{\mathfrak{d}}_{\mathfrak{a}}$ are now left to be determined.

It follows from (36) that:

$$\Pi_{\mathfrak{fed}}\, B^{\mathfrak{f}}_{\mathfrak{b}} C^{\mathfrak{ed}}_{\mathfrak{ba}} - \Pi_{\mathfrak{efd}}\, B^{\mathfrak{f}}_{\mathfrak{c}} C^{\mathfrak{ed}}_{\mathfrak{ba}} = 2T_{\mathfrak{cba}} \tag{45}$$

or

$$\Pi_{\mathfrak{fed}}\, C_{\mathfrak{c}}^{\mathfrak{f}}\, B_{\mathfrak{b}}^{\mathfrak{e}}\, C_{\mathfrak{a}}^{\mathfrak{d}} = -\,2T_{\mathfrak{bca}} + \Pi_{\mathfrak{fed}}\, B_{\mathfrak{b}}^{\mathfrak{f}} C_{\mathfrak{ca}}^{\mathfrak{ed}}, \tag{46}$$

and with consideration given to (21):

$$\Pi_{\mathfrak{fed}}\, C_{\mathfrak{ca}}^{\mathfrak{fe}}\, B_{\mathfrak{b}}^{\mathfrak{d}} = \Pi_{\mathfrak{fed}}\, C_{\mathfrak{c}}^{\mathfrak{f}}\, B_{\mathfrak{b}}^{\mathfrak{e}}\, C_{\mathfrak{a}}^{\mathfrak{d}} + F_{\mathfrak{fed}} C_{\mathfrak{ca}}^{\mathfrak{fe}} B_{\mathfrak{b}}^{\mathfrak{d}}. \tag{47}$$

One can get $\Pi_{\mathfrak{fed}}\, B_{\mathfrak{c}}^{\mathfrak{f}}\, C_{\mathfrak{b}}^{\mathfrak{e}}\, B_{\mathfrak{a}}^{\mathfrak{d}}$ and $\Pi_{\mathfrak{fed}}\, B_{\mathfrak{cb}}^{\mathfrak{fe}}\, C_{\mathfrak{a}}^{\mathfrak{d}}$ from (36) and (21) in the same way, and one will the get the $\Pi_{\mathfrak{cb}}^{\mathfrak{a}}$, expressed in terms of known quantities, by addition.

5. Reinterpretation of the result for the linear displacements in an X_n with even n. – In an X_n (n = even) with a fundamental bivector $f_{\lambda\kappa}$ of rank n and a double plane field B_λ^κ in involutory position, in particular, one can apply similar considerations to $f_{\lambda\kappa}$. A fundamental tensor $g_{\lambda\kappa}$ of rank n can be derived from $f_{\lambda\kappa}$ and B_λ^κ. Every linear displacement is established by $\nabla_\mu f_{\lambda\kappa}$, $\nabla_\mu B_\lambda^\kappa$, $S_{\tau\sigma\rho} B_\mu^\tau C_{\lambda\kappa}^{\sigma\rho}$, and $S_{\tau\sigma\rho} C_\mu^\tau B_{\lambda\kappa}^{\sigma\rho}$. For the case in which all of those quantities vanish, one has the theorem:

There exists one and only one linear displacement in an X_n that leaves a fundamental bivector and a double plane field that lies involutorily with respect to that bivector, and in addition, admits infinitesimal parallelograms in any 2-direction that has one common direction with every two p-directions of the double plane. That displacement leaves a fundamental tensor invariant, and therefore a metric. It is not symmetric, in general, but satisfies the equation $S_{[\mu\lambda\kappa]} = 0$.