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On the relativistic theory of the electron and the interpretation of quantum mechanics

By

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Translated by D. H. Delphenich

I. – Introduction.

It is my intention to present various ideas to this conference that are concerned with quantum mechanics and the interpretation that one gives to them at the present time. I shall mainly speak about the relativistic quantum theory of the motion of an electron. To the extent that we can account for it today, it seems almost certain that the quantum mechanics of the electron in its ideal form, *which we do not possess, moreover*, must one day define the basis for all physics. To that completely general advantage, I shall add, here in Paris, a particular advantage: All of you know that the bases for the modern theory of the electron were proposed in Paris by your celebrated compatriot Louis de BROGLIE.

The research that I would like to present hardly defines a neat and completely-successful theory ([1]). The common link that connects the topics to each other, which is a bit tenuous moreover and is the common source from which they are derived, is the discontent that one feels when one considers the present state of the theory, and above all that of the *present physical interpretation* of quantum mechanics. I would like to try to direct your attention to the great difficulties that it presents, and the most severe of them (and perhaps the most overlooked) is concerned with the reconciliation of the concepts of the theory of special relativity, on the one hand, and quantum mechanics, on the other.

We begin by fixing the fundamental notions in order to be assured that we are in complete agreement. Two types of mathematical entities figure in quantum mechanics: Functions $\psi(x, t)$ and operators (which are linear and Hermitian). $\mathcal{A} \cdot \psi$ is a complex function of the coordinates of a physical system and time t, which is treated as a parameter. ψ describes the *state* of that system

([1]) The original papers that define the basis for this presentation were published in the Sitzungsberichte der preussischen Akademie der Wissenschaften (1930), pp. 418; *ibid.*, (1931), pp. 63, 144, 238. In the following pages, some of the aspects of the problems described might seem a little less than precise; one will likewise find some new results here (see Notes I-III).

at a well-defined moment. An operator is a law that permits one to start from an arbitrary function ψ of the coordinates and form another function of the same arguments ([1]):

$$\psi \to \mathcal{A} \cdot \psi .$$

The physical significance of the operators is the following: Every physical quantity that one supposes to be *measurable* or observable in the system in question corresponds to a well-defined operator. For example, the coordinate x corresponds to the operator "multiplication by x," etc. Now, if one is given the state ψ, if one repeats the measurement of the "quantity $\mathcal{A}$" a large number of times then will not always find the same value, but only the *mean* of those measurements, which will be given by:

$$\overline{A} = \int \psi^* \mathcal{A} \psi \, dx ,$$

in which ψ^* is the complex conjugate of ψ. $\int dx$ indicates that the integration extends over all configurations of the system. *That is the interpretation that is generally adopted today*. It already contains the suggestion that ψ provides not only the mean value, but also all of the statistics of $\mathcal{A}$, i.e., of each observable. Assume that an analytic function $f(\mathcal{A})$ – viz., and *operator* – is defined by a power series:

$$f(\mathcal{A}) = b_0 + b_1 \mathcal{A} + b_2 \mathcal{A}^2 + \ldots,$$

in which $\mathcal{A}^n$ signifies the operation $\mathcal{A}$ being repeated n times. In particular, consider a function like the one that is indicated in the figure: It is zero everywhere, except between a_1 and a_2, where it takes the value 1. It is very easy to approximate such a function by a sequence of analytic functions. The mean value $\overline{f(\mathcal{A})}$ of the function $f(\mathcal{A})$ that is defined *by that sequence* is obviously the *probability* that the $\mathcal{A}$ is found between a_1 and a_2. One also shows very easily that the only possible values (i.e., the only ones with a non-zero probability) are the proper values of the operator $\mathcal{A}$. I shall not elaborate upon that point.

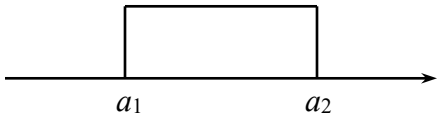

Figure 1.

([1]) $\mathcal{A}$ operates (in general) solely on the coordinates and not on time, i.e., it defines a law that relate the functions of the *coordinates* to each other, and not functions of coordinates *and time*. Meanwhile, the operator $\mathcal{A}$ might depend upon time, which would enter into its expression as a parameter (see the following).

The operator $\mathcal{H}$, which corresponds to the *energy* of the system defines the dynamical character of the system at a fixed time, i.e., the spontaneous variation of ψ with time, by the equation:

(I) $$\frac{h}{2\pi i}\frac{\partial \psi}{\partial t} = -\mathcal{H}\,\psi .$$

It is easy to solve that equation in a completely general manner. The solution is:

$$\psi(x, t) = e^{-\mathcal{H}t/\kappa}\,\psi(x, 0)\ , \qquad \kappa = \frac{h}{2\pi i} .$$

$e^{\mathcal{H}t/\kappa}$ is defined by a well-known power series. The verification of that solution is very easy. Naturally, it will also be very implicit.

There are two methods for arriving at more explicit statements in the problem that is posed by a "wave function" such as (I). One of them, which is well known, is to search for the proper values of the proper functions by means of the equation:

$$\mathcal{H}\,\psi = E\,\psi ,$$

or in other words, to develop the solution into a **Fourier** series in *time*. That is the method of wave mechanics.

The other method is that of the *operational calculus*. One seeks to avoid the study of the temporal variation of ψ, and one then demands to know: At each instant, is there no operator $\mathcal{A}(t)$, which must *depend* upon the parameter t, such that the operator will give the same mean value and the same statistics with the function $\psi(x, 0)$ that $\mathcal{A}$ gives with $\psi(x, t)$? The answer is in the *affirmative*, and the temporal variation of that $\mathcal{A}(t)$ for an arbitrary operator is given by the same simple formula:

(2) $$\kappa\frac{d\mathcal{A}(t)}{dt} = \mathcal{H}\,\mathcal{A}(t) - \mathcal{A}\,\mathcal{H}(t) ,$$

whose general solution is:

$$\mathcal{A}(t) = e^{\mathcal{H}t/\kappa}\mathcal{A}(0)\,e^{-\mathcal{H}t/\kappa} .$$

The advantage of this method is that everything that one can deduce will be valid for an arbitrary initial state $\psi(x, 0)$. One is not obliged to fix it in advance. In general, one *suppresses* the argument t in the statement of the relation (2).

Let us point out a special case of great importance, namely, the case in which $\mathcal{A}$ *commutes* with $\mathcal{H}$. The statistics of such a physical quantity will then be independent of time. We can then say that $\mathcal{A}$ is a "first integral" of the system in question.

II. – The Dirac electron.

We shall now apply those well-known facts to the **Dirac** electron in the case where external fields are absent. The Hamiltonian operator is then the following one in this case:

$$\mathcal{H} = c\,(\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \alpha_4 mc)\,.$$

The p_k are the operators $\kappa\dfrac{\partial}{\partial x_k}$, and x_1, x_2, x_3 represent x, y, z, resp. The α_k (k = 1, 2, 3, 4) operate on a variable that is different from x_1, x_2, x_3, which one can call ζ, and which has only four possible values. Therefore, the α_k are 4×4 matrices of elements. However, all that one needs to know about those matrices is their commutation relations:

1) Naturally, those matrices commute with each operator that does not operate on ζ, such as x_k, p_k, etc., and:

(2) $$\alpha_k\,\alpha_l + \alpha_l\,\alpha_k = 2\,\delta_{kl}\,.$$

Upon applying the operational calculation [equation (2)], one will easily show that the α_k (k = 1, 2, 3), or rather the $c\alpha_k$, are the operators that correspond to the components of the velocity of the electron. Indeed, from the well-known relation:

$$p_k\,x_k - x_k\,p_k = \kappa\cdot 1 \qquad (k = 1, 2, 3),$$

one finds that:

$$\kappa\frac{dx_k}{dt} = \mathcal{H}\,x_k - x_k\,\mathcal{H} = \kappa c\,\alpha_k\,,$$

$$\frac{dx_k}{dt} = c\,\alpha_k\,.$$

One of the more interesting aspects of the **Dirac** equation consists precisely of the fact that the notions of *momentum* and *velocity are separated*. The velocity operator commutes with the coordinates, so they are simultaneously observable, whereas x_k and p_k behave as they do in ordinary quantum mechanics.

What is even more interesting is that even in the absence of a field, the components of the velocity *are not first integrals*. They do not commute with $\mathcal{H}$. One will not find a simple value for their *commutators* with $\mathcal{H}$, but for their *anticommutators* one will find that:

$$\mathcal{H}\,\alpha_k + \alpha_k\,\mathcal{H} = 2\,c\,p_k \qquad (k = 1, 2, 3).$$

Therefore:

$$\kappa \frac{d\alpha_k}{dt} = 2\,\mathcal{H}\,\alpha_k - 2\,c\,p_k = 2\,\mathcal{H}\,(\alpha_k - c\,\mathcal{H}^{-1}\,p_k) = 2\,\mathcal{H}\,\eta_k\,,$$

in which one sets:

$$\eta_k = \alpha_k - c\,\mathcal{H}^{-1}\,p_k\,.$$

Since $\mathcal{H}$ and p_k commute with $\mathcal{H}$, one can infer that:

$$\kappa \frac{d\eta_k}{dt} = 2\,\mathcal{H}\,\eta_k\,,$$

which one can integrate to:

$$\eta_k = \alpha_k - c\,\mathcal{H}^{-1}\,p_k = e^{2\mathcal{H}t/\kappa}\,\eta_k^0\ .$$

Since α_k corresponds to $\frac{1}{c}\frac{dx_k}{dt}$, one will get x_k from a second integration:

$$x_k = a_k + c^2\mathcal{H}^{-1}p_k\,t + \tfrac{1}{2}c\kappa\mathcal{H}^{-1}\,e^{2\mathcal{H}t/\kappa}\,\eta_k^0\ .$$

Set:

$$a_k + c^2\mathcal{H}^{-1}p_k\,t = \hat{x}_k\ ,$$
$$\tfrac{1}{2}c\kappa\mathcal{H}^{-1}(\alpha_k - c\,\mathcal{H}^{-1}p_k) = \xi_k\,.$$

One will have:

$$x_k = \hat{x}_k + \xi_k\,.$$

The operator defined by the coordinates will then decompose into two parts. The first one is a linear function of time, as one would have expected of the operator x_k completely, and even with the correct factor of t : p_k corresponds to $\frac{m v_k}{\sqrt{1-\beta^2}}$, $\mathcal{H}^{-1}$ corresponds to $\frac{\sqrt{1-\beta^2}}{mc^2}$, and as a result $c^2\mathcal{H}^{-1}p_k$ will correspond to $\frac{1}{c^2}\,v_k \times c^2$, so to v_k.

Meanwhile, there are two remarks that should be made: The first one is that the *amplitude* of the vibration is always very small. Set:

$$x_k = \hat{x}_k + \xi_k\,.$$

One easily finds that:

$$\xi_k^2 = \frac{h^2\,c^2}{16\pi^2}\mathcal{H}^{-1}(1 - \mathcal{H}^{-2}c^2\,p_k^2) = \text{constant.}$$

However:

$$\mathcal{H}^2 = c^2\,(p_1^2 + p_2^2 + p_3^2 + m^2c^2)\,.$$

Therefore, the proper values of $\mathcal{H}^2$ are all greater than m^2c^4, so $|\mathcal{H}| > mc^2$. Therefore, those of $\mathcal{H}^{-2}$ will be less than $1/m^2c^4$, and one will have:

$$\text{proper values of } \xi_k^2 \leq \frac{h^2}{16\pi^2 m^2 c^2},$$

$$|\text{proper values of } \xi_k| \leq \frac{h}{4\pi mc} \sim 10^{-11} \text{ cm}.$$

The second remark refers to the fact that the jittering of the center of gravity is annulled in one special case, namely, the case in which the function ψ is what we shall call "a purely-positive function" later on, i.e., it contains only proper functions of $\mathcal{H}$ that belong to positive proper values. That will soon become clearer. For the moment, we note only the mathematical fact that ξ_k and η_k anticommute with $\mathcal{H}$:

$$\mathcal{H}\,\xi_k + \xi_k\,\mathcal{H} = 0,$$

$$\mathcal{H}\,\eta_k + \eta_k\,\mathcal{H} = 0,$$

which will become important in what follows.

III. – Even and odd operators.

The operators that commute with $\mathcal{H}$ (integrals in the absence of a field) and the operators that anticommute with $\mathcal{H}$ (like ξ_k, η_k) define special cases of an important classification that we shall now introduce: the classification into even and odd operators.

We just proved that the absolute values of the proper values of $\mathcal{H}$ are greater than:

$$|\mathcal{H}| \geq mc^2.$$

Therefore, their proper values are either $\geq mc^2$ or $\leq -mc^2$. It is easy to prove that, conversely, any value that is outside of the interval $(-mc^2, +mc^2)$ is a proper value of $\mathcal{H}$, and that it will correspond to a proper function (a plane wave, for example). We know that this is the greatest difficulty in **Dirac**'s theory. Indeed, negative proper values have no physical significance; one would even like to eliminate them. At least, it must be impossible for a "positive" proper function to transform in the course of time while giving rise to "negative" functions, or at the very least, that variation must be produced infinitely slowly, such that it would become sufficiently improbable for the *enormous* change in energy of $2mc^2$ to occur, which has never been observed.

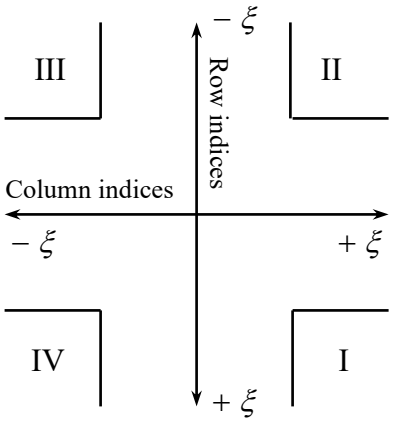

Figure 2.

Along that line of reasoning, one foresees that would be important to single out the operators that operate on a "positive" proper function to produce only a set of "positive" proper functions. The simplest method for making that distinction is to represent all of the operators in the system of matrices for which $\mathcal{H}$ is diagonal. Consider one such representation (see Fig. 2). The central cross is empty, and the field of matrix elements decomposes into four regions. We call a function $\psi(x)$ positive when its development into a series of proper functions of $\mathcal{H}$ (which is always the energy in the absence of a field!) contain only positive proper functions. A negative function is defined in the same way. An operator whose matrix elements belong to only the regions I and III will not change the positive or negative character of a positive or negative function. We then call the operator "even." The matrix elements of the "odd" operators belong to the regions II and IV. When they are applied to a positive function, those operators will make it negative, and *vice versa*. Now, it is easy to see that the operators that commute with $\mathcal{H}$ are even, while the ones that anticommute with $\mathcal{H}$ are odd. For example, for the equation:

$$\mathcal{H}\ \xi_k + \xi_k\ \mathcal{H} = 0\ ,$$

if we write the matrix element $(\rho\sigma)$ in the picture where $\mathcal{H}$ is diagonal then we will get:

$$(\mathcal{H}_\rho + \mathcal{H}_\sigma)\ (\xi_k)_{\rho\sigma} = 0\ .$$

Therefore, if $H_\rho \geq mc^2$, $H_\sigma \geq mc^2$, or if $H_\rho \leq mc^2$, $H_\sigma \leq mc^2$ then it would follow that $(\xi_k)_{\rho\sigma}$ must be zero, so the operator ξ_k must then be odd. The same argument also applies to η_k.

One likewise proves that in order for an operator to be even (or odd), it suffices that its second, third, or in general, its n^{th} commutator (anticommutator, resp.) with $\mathcal{H}$ should be zero. It will then follow that the decomposition of the coordinates to which we were led above, namely:

$$x_k = \hat{x}_k + \xi_k ,$$

is precisely the decomposition into even and odd operators. Indeed, we have proved that ξ_k is odd. As for $\hat{x}_k$, it is even because it is a linear function of time in the absence of the field. Therefore, its second commutator with $\mathcal{H}$ must vanish, so $\hat{x}_k$ must be even. Similarly, the decomposition:

$$\alpha_k = c\,\mathcal{H}^{-1}\,p_k + (\alpha_k - c\,\mathcal{H}^{-1}\,p_k) = c\,\mathcal{H}^{-1}\,p_k + \eta_k$$

represents a decomposition into even and odd operators.

We shall add some propositions that are almost obvious:

1. Any operator is decomposable into even and odd parts in only one way.

2. Any function is decomposable into even and odd parts in only one way.

3. Any positive function is orthogonal to any negative function.

4. The mean of an even operator will be zero when it is taken in a state that is represented by a "pure" function (i.e., it is either positive or negative).

5. The proper functions of an even operator can always be chosen to be "pure." (However, it might happen that positive and negative proper functions belong to one and the same proper value.)

6. An odd operator can never have pure proper functions, except perhaps for the proper value zero.

7. A product of powers of a certain number of "pure" operators is even or odd according to whether the number of *odd* operators that enter into it (or the sum of their exponents) is even or odd, respectively.

Proposition 4 proves what was said at the end of the last section about the behavior of the center of gravity in the case where the function ψ is purely positive (or negative).

IV. – Case of an external field.

We shall use the results that we obtained in order to the answer the following question, which has fundamental significance: If the initial state of an electron is given by a *positive* function ψ then will that function remain purely-positive in the course of time?

In the absence of a field, the answer is in the affirmative. The variation of ψ is governed by the equation of propagation:

$$\kappa \frac{\partial \psi}{\partial t} = - \mathcal{H}\ \psi$$

or

$$d\psi = - \frac{\mathcal{H}}{\kappa} \psi\, dt\ .$$

The operator $\mathcal{H}$ is even, by definition. Therefore, the increase in ψ will be positive at each moment, and ψ will remain positive.

Now, imagine the case of an arbitrary external field that is given by the potentials A_1, A_2, A_3, and V. **Dirac**'s theory accounts for the action of an external field on the electron by the following modifications of the operator $\mathcal{H}$:

1. One must add the potential energy $e\ V$ to it.

2. One must replace p_k with $p_k + \frac{e}{c} A_k$.

That manner of accounting for the existence of an external field is hardly satisfactory. Any conception of the field being "external" is lost. It is only a provisional measure, an "Ersatz," for a theory that we still do not possess. In truth, the "external field" is produced by other electrons and protons, and one must then treat the problem as a "many-body problem" and do that in full generality, i.e., in the case where the bodies are far enough from each other that **Coulomb**'s simple law will no longer suffice to describe their mutual actions. We still do not know how to attack that problem. At least, the methods that have been proposed up to now, such as hyper-quantization or repeated quantization and the quantization of the fields are so complicated and present such great difficulties that I cannot speak of them today.

We shall then employ **Dirac**'s method, which, as you know, has led to very good results, for example, as far as the fine structure of hydrogen is concerned, etc. Let $\mathcal{K}$ be the energy operator in an arbitrary field, i.e., the operator $\mathcal{H}$ when it has been subjected to the two modifications that were indicated above. (The symbol $\mathcal{K}$ was chosen in order to recall the name of **Kepler**.):

$$\mathcal{K} = c\ [\alpha_1\ (p_1 + \frac{e}{c} A_1) + \ldots + \alpha_3\ m\ c] + e\ V\ ,$$

in which A_1, A_2, A_3, V are given functions of x_1, x_2, x_3, t. The wave equation for ψ is then:

(4) $$d\psi = - \frac{\mathcal{K}}{\kappa} \psi\, dt\ .$$

Will the positive character of ψ be preserved when ψ is positive to begin with? That depends upon the parity of the operator $\mathcal{K}$. We will soon see that $\mathcal{K}$, although even *in the large*, still contains a small odd part.

Before proceeding, I must emphasize the fact that in the definitions of "positive" and "negative," "even" and "odd," it is always the operator $\mathcal{H}$ of motion *in the absence* of external fields that is involved, and not $\mathcal{K}$. Indeed, what is interesting to us is the probability that a negative *internal* energy will appear in the course of time, which is an energy that corresponds classically to the well-known expression:

$$\frac{mc^2}{\sqrt{1-\beta^2}},$$

and not the probability of a negative *total* energy. Indeed, in order to avoid any contradiction with experiments, it is necessary to exclude the states of negative *internal* energy: That is the goal that we have in mind.

We must then study the operator $\mathcal{K}$ and find its decomposition into even and odd parts. That is made easier by the following proposition, which we shall soon prove. If we are given a decomposition of the operators $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$, …:

$$\begin{aligned}
\mathcal{A} &= p + i\,,\\
\mathcal{A}' &= p' + i'\,,\\
\mathcal{A}'' &= p'' + i''\,,\\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

then the decomposition of an operator $\mathcal{B}$ that is a *function* of $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$, … will be given by:

$$\begin{aligned}
\mathcal{B} &= f(\mathcal{A}, \mathcal{A}', \mathcal{A}'', \ldots)\\
&= \tfrac{1}{2}[f(p+i, p'+i', \ldots) + f(p-i, p'-i', \ldots)]\\
&+ \tfrac{1}{2}[f(p+i, p'+i', \ldots) - f(p-i, p'-i', \ldots)]\,.
\end{aligned}$$

That proposition is easily proved. One begins by forming the product $\mathcal{A}\mathcal{A}'$ for the function f and deducing that the theorem is valid for an arbitrary product of powers, and then for a sum of powers, and finally for an arbitrary analytic function.

Our operator $\mathcal{K}$ is given as a function of the α_k, the p_k, and the x_k. We obtained the decomposition of the α_k and the x_k in the preceding section. Now, the p_k are even (since they commute with $\mathcal{H}$), so the decomposition of $\mathcal{K}$ can be performed by means of the preceding theorem.

Let us examine the case of the hydrogen atom more closely. We have:

$$A_k = 0, \qquad e\,V = -\frac{e^2}{r}, \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}\ ,$$

$$\mathcal{K} = \mathcal{H} - \frac{e^2}{r} = \mathcal{P} + \mathcal{J},$$

$$\mathcal{P} = \mathcal{H} - \frac{e^2}{2}\left(r^{-1} + \frac{1}{\sqrt{(x_1 - 2\xi_1)^2 + \cdots}} \right),$$

$$\mathcal{J} = \quad - \frac{e^2}{2}\left(r^{-1} + \frac{1}{\sqrt{(x_1 - 2\xi_1)^2 + \cdots}} \right),$$

$$x_k = \hat{x}_k + \xi_k\,,$$
$$\hat{x}_k - \xi_k = x_k - 2\ \xi_k\,.$$

Granted, the operators x_k are somewhat complicated since they contain $\mathcal{H}^{-1}$, which is not a differential operator. Nevertheless, everything is well-defined. Extracting the square root presents no difficulties since the quantity under the radical is a positive operator, i.e., it has positive proper values, and it is almost obvious that one can take the square root in such a manner that all of its proper values will be positive. (The ambiguity that presents itself is, in fact, the same one that appeared in the definition of the operator r itself.)

We have posed the following question: Does equation (4) preserve the positive character of a positive function? Obviously not: $\mathcal{K}$ is not even. Its odd part $\mathcal{J}$, which we have isolated, must bring about the appearance of negative parts in the wave function in the course of time. Although $\mathcal{J}$ is very small compared to $\mathcal{P}$ (as we shall soon see), the probability of the appearance of an electron of negative mass will become *much too large* for it to be acceptable.

One can eliminate it by suppressing $\mathcal{J}$ or, in other words, by replacing the operator $\mathcal{K}$ with $\mathcal{P}$, its even part. Is that permissible? Is the operator $\mathcal{J}$ sufficiently *small* compared to $\mathcal{P}$ or $\mathcal{K}$ that suppressing $\mathcal{J}$ would not change the proper values perceptibly, i.e., the value of the terms in the fine structure of hydrogen?

The smallness of $\mathcal{J}$ results from the smallness of ξ_k, which we have estimated to have order of magnitude:

$$\{\xi\} = \frac{h}{4\pi\, m\, c}\ .$$

(We shall use the symbol { } to indicate the "order of magnitude.") The ratio of the orders of magnitude of $\mathcal{J}$ and $e^2/2r$ (which equal to precisely the value of the energy terms) will around:

$$2\left\{\frac{\xi_k}{r}\right\}.$$

For $\{r\}$, one can take the well-know "radius of the hydrogen atom":

$$\{r\} = \frac{h^2}{4\pi^2 m e^2}.$$

It then results that:

$$\frac{\{\mathcal{J}\}}{\{\text{term}\}} = \frac{2\pi e^2}{hc} = \alpha,$$

in which α denotes the fine structure constant, as usual.

We know that the ratio of the *fine structure* to the term is much smaller, viz., around α^2. Consequently, it would seem that $\mathcal{J}$ is much too large for its suppression to be permissible. It seems that the suppression of the odd part of $\mathcal{J}$ will change the proper values much more than separating the fine structure and would consequently completely destroy any agreement with experiments.

However, that is not true in reality. In order to see that, it would be slightly more convenient to not speak of the *suppression* of $\mathcal{J}$, but to start from the operator $\mathcal{P}$ and demand to know how the proper values of $\mathcal{P}$ will change when one *adds* the operator $\mathcal{J}$. We know that the perturbation of a proper value can be calculated *up to first order* by means of integrals ("perturbation matrices"):

$$\int \psi_i^* \mathcal{J} \psi_k \, dx,$$

in which the ψ_k are proper functions of the unperturbed problem that belong to the same proper value whose perturbation was just calculated. However, since the operator $\mathcal{J}$ is *odd, all of those matrix elements will disappear* if the proper functions ψ_k that belong to that proper value are all positive or all negative. Therefore, in that case, the perturbation of the proper values of an *even* operator by a small odd operator will be at most *of second order*.

Before studying what order of magnitude must come into play in our case, we must demand to know whether it is exactly true that the proper functions of $\mathcal{P}$ that belong to the same proper value are all positive or all negative. Let:

$$\chi = \chi_+ + \chi_-$$

be the decomposition of such one such proper function with the proper value P'. One will then have:

$$\mathcal{P}(\chi_+ + \chi_-) = P' \cdot (\chi_+ + \chi_-),$$

and upon decomposing into positive and negative parts:

$$\mathcal{P}\chi_+ = P'\chi_+, \qquad \mathcal{P}\chi_- = P'\chi_-.$$

Therefore, χ_+, χ_- are themselves proper functions of $\mathcal{P}$. One can always take the proper functions of $\mathcal{P}$ to be *pure*. However, it can happen that the same proper value will belong to a certain number of positive functions and also to some negative functions.

That possibility generally exists for an even operator, but not for $\mathcal{P}$. That is because it is not true for the operator $\mathcal{H}$, by definition, and $\mathcal{P}$ is sufficiently close to $\mathcal{H}$ that one can conclude that the same thing will be true for $\mathcal{P}$. If one regards $\mathcal{P}$ as the result of a perturbation of $\mathcal{H}$ then that perturbation will have an order of magnitude that is *much less than* $2mc^2$, which is the value that separates the two categories of proper values of $\mathcal{H}$. Thus, they cannot be mixed together by such a perturbation.

Let us return to the study of the order of magnitude of the perturbation that is produced by $\mathcal{J}$. We found that:

$$\frac{\{\mathcal{J}\}}{\{\text{term}\}} = \alpha,$$

and the perturbation has *second* order. Can we then say that it has order α^2 with respect to the term? Unfortunately not, since we must compare it to the complete operator $\mathcal{K}$, which has order mc^2. We will easily find that the *term* is only a fraction α^2 of mc^2. Therefore, compared to $\mathcal{K}$, the first order is α^2, the second order is α^6, and when compared to the term, that second order is only α^4. That is much less than the separation into fine structure, and even much less than the natural size of rays that are caused by the absorption of radiation, which has order α^3.

One can likewise prove that in the **Stark** and **Zeeman** effects, the perturbation is entirely insignificant. I shall not go into the details of that: It does not suggest any new viewpoint.

V. – The problem of relativity.

We have succeeded in deducing a wave equation that preserves the positive character of the function ψ by a well-known process, namely, by suppressing the odd part of the **Dirac** operator**,** which nonetheless produces results that agree with experiments for the fine structure of hydrogen, as well as the **Stark** and **Zeeman** effects.

Nevertheless, one should not overestimate the meaning of that success. In the first place, one must recall that the introduction of the notion of an "external field" into the **Dirac** equation is itself basically an artifice that permits us to avoid having to solve the *n*-body problem, which is a problem that we do not know how to further treat in a manner that accounts for relativity and the *retarded* potentials. In the second place, one should observe that the *existence* of an even operator with the correct proper values means nothing: It is even obvious that there exists an infinitude of such operator, because if one is given an arbitrary sequence of real numbers:

$$\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3,\ \ldots$$

and a complete system of arbitrary orthogonal functions:

$$\omega_1,\ \omega_2,\ \omega_3,\ \ldots$$

then one can always (at least formally) write out a function of two groups of variables:

$$K(x,\ \xi) = \sum_k \varepsilon_k\ \omega_k^*(x)\,\omega_k(\xi)\ ,$$

and *when it is regarded as an integral kernel*, it will be a linear Hermitian operator that has ε_k for its proper values and the ω_k for its proper functions. Now, one can choose the ω in such a fashion that they are all "pure," and in that case the operator will be even. What is so satisfying about our procedure consists solely of the fact that we have succeeded in defining such an operator that is relatively *simple*.

There are a certain number of specialized questions that must be examined. For example, upon introducing the A_k that correspond to a light wave and studying the wave that is propagated, it would be interesting to see whether one would reproduce **Rayleigh**'s classical formula for the limiting case of high frequencies. **I. Waller** has proved that one can reproduce the effect with the **Dirac** equation, but only thanks to the possibility of transitions between the states of positive and negative energy. In the present case, *if one reproduces* the same formula (which I hope and believe but have not proved yet) then the mathematical mechanism must be different from the preceding one because those transitions are strictly forbidden by the "purification" that we have undertaken.

However, there are some other disquieting questions. One can consider one of the most important properties of the **Dirac** equation, or even perhaps *the most important* one, is its invariance under the relativistic **Lorentz** transformation. If one introduces the well-known **Lorentz** transformation of the:

$$x_1, x_2, x_3, t \rightarrow x_1',\ x_2',\ x_3',\ t'$$

and at the same time, one performs a certain linear transformation on the four components of ψ:

$$\psi_1,\ \psi_2,\ \psi_3,\ \psi_4 \rightarrow \psi_1',\ \psi_2',\ \psi_3',\ \psi_4',$$

then the equation that the ψ'_k must satisfy is exactly the same as the one that the ψ_k must satisfy.

Will that invariance be preserved after one suppresses $\mathcal{J}$?

The answer is that it is not preserved exactly. Naturally, in the case that is interesting to us, the lack of invariance cannot be very big since the change in the operator is not big. However, it is not possible to restore the exact form to the equation.

In the first place, one is inclined to believe that this demoralizing result will make all of the previous results useless. However, I do not believe that this is true. Indeed, I am convinced that in quantum mechanics, the question of relativistic invariance is much more complicated than has been imagined up to now, and in the next section, I intend to recast the problem of the relationship between quantum mechanics and special relativity from a somewhat-more-general viewpoint.

For the moment, one can make the following remarks in order to excuse the non-invariance of our equation to some extent.

In the first place, we can demand to know what form our equation would have to take in order for it to be invariant. What sort of equation must a function ψ satisfy if it is "positive" for all values of the parameter t and remains positive under an arbitrary **Lorentz** transformation?

That question admits a precise answer, and even a *unique* answer, that is very striking. I would only like to indicate the result here ([1]): If one imposes the conditions on ψ that is must be "positive" at any moment and invariant under an arbitrary **Lorentz** transformation then that will inevitably imply that ψ must be a solution to:

$$\kappa \frac{\partial \psi}{\partial t} = - \mathcal{H} \psi ,$$

i.e., the problem in the *absence* of a field. Thus, the search for a wave equation that is devoid of negative proper values and is also invariant would be pointless. The preceding equation is *the only one* that enjoys those properties, and it obviously applies only to the case in which a field is absent.

Must one renounce the postulate of relativity? Obviously not. However, the **Lorentz** transformation seems to be something that is much more complicated in quantum mechanics than it was in ordinary mechanics. One senses that the operators:

$$x_k, \qquad x_k - 2 \xi_k ,$$

or rather:

$$\hat{x}_k + \xi_k , \qquad \hat{x}_k - \xi_k ,$$

play an entirely symmetric role in the new theory. I must thank **von Neumann** for the remark that one can exhibit a certain canonical transformation that changes *all* of the operators $p + i$ into $p - i$ at once, i.e., that it simply changes the signs of all odd operators. Consequently, all of the relations that exist between the operators $p + i$ (for example, commutation relations) will likewise exist between the $p - i$. That will make it almost inevitable that in the new theory, one must conclude that it is not the x_k that must correspond to the coordinates of the electron, but rather the even

([1]) See Note I.

operators $\hat{x}_k$. Consequently, the **Lorentz** transformation must be performed on the x_k, but not on the $\hat{x}_k$.

However, the **Lorentz** transformation will then become something that is indeed more complicated and difficult, *not to formulate, but only to imagine*. When one performs transformations on the x_k, t, it is permissible to close one's eyes to the fact that the x_k are not ordinary numbers because they commute with each other in such a way that one can choose a system of matrices in which they are simultaneously diagonal. The transformation is then performed on the *proper values* of x_k, i.e., on the values of the coordinates in the elementary sense. That is what one always does tacitly. It is not possible to apply that way of looking at things to the $\hat{x}_k$ since they do not commute with each other, in such a way that there exists no representation in which all three of them are diagonal. It is difficult to glimpse how the transformation of the $\hat{x}_k$ and t must be formulated, and above all, in a theory that treats time as an ordinary variable, and not as an operator.

The fact that the true operator-coordinates of the electron do not commute with each other seems to me to be very interesting in its own right because it would have to say that it is not generally possible to measure two coordinates or all three coordinates at once exactly. The correlative uncertainty in the two observables is mainly determined, as one knows, by the order of magnitude of their commutator. It is not difficult to calculate the commutators of the $\hat{x}_k$, and that is quite interesting since they are intimately linked with the "spin" operators ([1]).

The problem that is posed seems to me to be that of formulating the **Lorentz** transformation as an operator equation between the non-commuting operators $\hat{x}_k$.

VI. – Relativity and quantum mechanics.

The difficulties that we encountered upon attempting to account for the relativistic viewpoint in quantum mechanics seem all the more interesting to me due to the fact that they are completely *unexpected*. We know that the new mechanics, in the form of *wave* mechanics, which is what it is almost universally called today, owes its origin to the famous research of **L. DE BROGLIE** and his ingenious concept of the electronic wave that must accompany the motion of the electron. The research of **L. DE BROGLIE** was supported by the theory of special relativity. It was thus impregnated by relativity, so to speak. When one takes that to be the starting point for deriving the wave equation and the problems of proper values, one might feel a bit ashamed that one is obliged to first *suppress* the relativistic viewpoint, and one would hope that this is only a temporary situation of short duration and that it would not be too difficult to reintroduce relativity back into the equations. However, instead of reducing the difficulty, it would seem that the difficulty has increased from one year to the next up to today, when it has taken on frightening proportions.

Meanwhile, the form in which we have encountered it in the preceding discussion is complicated and very specialized. One can object that the **Dirac** equation that we started from probably does not represent the only possible form for the relativistic equation of the electron and

([1]) See Note II.

that our method for suppressing the odd part of the Hamiltonian operator is perhaps not the only method for modifying the **Dirac** equation in such a fashion that it would eliminate the negative energies. One can suppose that it is necessary to completely abandon that line of reasoning, which is, despite its success, quite simply condemned by the relativistic difficulties that it gave rise to.

That is why I would like to ask your permission to present some more-general ideas regarding the relationship between the concepts in the theory of relativity and quantum mechanics. I believe that from an entirely-general perspective, while using only the fundamental theorems of those two theories, one can recognize that the difficulty in reconciling them is not based upon the special form of the equations that were adopted, but on the essentially different nature of the fundamental concepts in the two theories.

Since the time of **Einstein**'s great discovery, one has become accustomed to subject any physical theory to the postulate of special relativity, i.e., to demand that it must be invariant under **Lorentz** transformations. One never encounters any serious difficulties in that. On the contrary, one is invariably led to the problem of generalizing the theory in question to bodies in motion. However, upon attempting to subject quantum mechanics to the same postulate, one does clash with some difficulties. Why is that? It is not so surprising that this would be true. In order to know what one must call a **Lorentz** transformation in some special case, one must introduce two systems of **Lorentz** coordinates. Their introduction is based upon the idea that it is, in principle, possible to measure the coordinates of a material point and time with unlimited accuracy, and also as often as one desires, and then to infer conclusions in regard to its velocity. However, that is not permitted in quantum mechanics. Hence, quantum mechanics is not obligated to be subject to the relativity postulate with an arbitrarily high accuracy. It has, in turn, the right to demand to examine the degree of error to which one can define a coordinate system. It is required to satisfy the relativity postulate only to that degree of error.

Let us examine that idea more closely. Among the operations that are necessary if one is to establish a **Lorentz** system, one finds the operation that consists of synchronizing clocks at different points of the system by means of light signals; it might even be the most important of all of them. Now, it is easy to see that this operation can have only limited accuracy unless the clocks are *infinitely heavy*. The reason for that is as follows:

Imagine a system of mass m (which I shall call a "clock") and two "events" that occur in that system: for example, two positions of the hands. One wishes to measure the time that elapses between those two events ("setting the clock"). In order to announce the first one, the clock must emit an optical signal that must be short enough that one can demand to know the instant when the event took place more exactly. One first demands a precision of τ seconds.

The spatial length of a signal must not exceed $c\ \tau$ centimeters. However, one then knows from the fundamental principles of optics that light cannot be monochromatic. The frequency interval that comes into play will be at least:

$$\Delta\nu = \frac{1}{4\pi\tau}.$$

Now, when a system emits light of frequency ν, one must assume that it recoils, and that the corresponding quantity of motion will be:

$$\frac{h\nu}{c}.$$

(**Einstein**, in his famous proof of **Planck**'s formula in 1917, showed that this result is one of the most immediate consequences of quantum theory. It is made almost inevitable by the experimental facts that are known about blackbody radiation. If the system did not experience that recoil then the thermal agitation of a molecule in equilibrium with the blackbody radiation would not be the same as it would be in a gas, so a gas would not be in equilibrium with the blackbody radiation at the same temperature.)

The *non-monochromatic* character of the light that is emitted as a signal will then imply an uncertainty in the recoil of:

(1) $$\Delta p = \frac{h}{4\pi c \tau},$$

from which one can infer β. However, that uncertainty in the *velocity* will lead to a *second* lack of precision in the measurement of the *time* that passes between two events because the clock is slowed down by the motion, and it is obvious that what is interesting to us will be the time that *would have* elapsed if the clock had remained at rest; in other words, the time in the "rest system" of the clock, which we know with an error of β.

Therefore, the time t that we would measure is too large and must be corrected by multiplying t times a factor that is found between $\sqrt{1-\beta^2}$ and 1. Hence, the second error will be:

(3) $$t\,(1-\sqrt{1-\beta^2}).$$

Upon inferring β from (2) and substituting it in (3), one will find:

$$t\left(1-\frac{\tau}{\sqrt{\tau^2+\tau_0^2}}\right), \qquad \text{with} \qquad \tau_0 = \frac{h}{4\pi mc^2}.$$

The *total* error is then:

$$\Delta t = \tau + t\left(1-\frac{\tau}{\sqrt{\tau^2+\tau_0^2}}\right).$$

It will be a minimum (since t is greater than or equal to τ_0) for:

$$\tau = \tau_0\sqrt{\left(\frac{t}{\tau_0}\right)^{2/3}-1}\ .$$

It has the value:

$$\Delta t = t\left\{1-\left[1-\left(\frac{\tau_0}{t}\right)^{2/3}\right]^{3/2}\right\}.$$

One easily proves that one always has $\Delta t \geq \tau_0$. When t becomes equal to τ_0, one will have $\tau = 0$, $\Delta t = t$, so the error will be 100 %. For $t < \tau_0$, the most favorable value will be $\tau = 0$, so the error remains 100 %.

I believe that one infer the conclusion that a clock of mass m is not capable of being set with a precision of τ_0, and even perhaps that the events that happen in a system of mass m cannot be localized in time with a precision that is greater than τ_0 (or rather an error that is less than τ_0) by an external observer. Nonetheless, that is a somewhat-abbreviated manner of speaking. From the viewpoint of the external observer, the localization can be performed to any desired precision. However, as for what happens in the interior of the system (or in other words: as for the proper time of the system), it would have no significance beyond a precision of τ_0. In an entirely analogous way, one can show that the measurement of the spatial distance between two points that belong to a material system of total mass m cannot be performed with an error that is less than around:

$$\frac{h}{4\pi mc}.$$

That is because from **Heisenberg**'s "law of uncertainty," if the position of the two points is observed with a precision λ then that would imply an error in the quantity of motion of the system that is at least:

$$\Delta p = \frac{h}{4\pi\lambda} = \frac{mc\beta}{\sqrt{1-\beta^2}},$$

in which β is the uncertainty in the speed of the system. That would imply an uncertainty in the **Lorentz** contraction, so an additional uncertain of:

$$\lambda\left(1-\frac{1}{\sqrt{1-\beta^2}}\right)$$

in the observed length l if one would like to deduce a true length or rest length from the distance one would like to measure. One will indeed see that upon reducing the former of those two errors, one will increase the latter, and one can glimpse that the inevitable error has the indicated order of magnitude. Note that if m is equal to the mass of the electron then it would be precisely the order of magnitude of the operators ξ_k that would represent the change in the operator-coordinates in our "purified" equation compared to the **Dirac** equation. Therefore, the lack of invariance in our equation in the conventional sense is rightfully produced in a domain where the determination of

the coordinates themselves, and consequently the concept of the **Lorentz** transformation, would become problematic.

I said: "If *m* is equal to the electronic mass." Naturally, if one assumes the use of clocks and yardsticks that are as heavy as one desires then the time and the location of the electron would be observable with the desired arbitrary precision. However, I do not believe that this is the case, i.e., I do not believe that this is admissible ([1]).

It then appears that from the quantum-mechanical viewpoint the theory of relativity belongs on the same level as classical mechanics, in the sense that it represents only an approximation that relates to the macroscopic scale. One must not assume quite simply the formulas of relativity (for example, the **Lorentz** formulas) and suppose that they are valid with no changes at the intra-atomic scale. They must be subjected to some modifications that are probably analogous to the ones that ordinary mechanics submits to in order to transform it into quantum mechanics. One must "quantize" the **Lorentz** transformation.

Quantization of the **Lorentz** formula: What does that say? What *can* it mean? I believe that it can have no other significance than the following one: One must reasonably regard the well-known formulas as having been established between *the operators*, rather than between *the numbers* (viz., *c*-numbers) that belong to the coordinates. Indeed, in classical theory, those formulas serve to calculate the values of the coordinates and time for a point (or rather, for a point-like event) in *one* reference system if one *knows* their values in another system. However, in quantum mechanics, one does not generally know the *value* of an observable quantity, but only the probability for it to have this or that value. What one demands is to be able to calculate the probabilities in one reference system when they are known in another system. Since probability is calculated in quantum mechanics by means of the *operator* that belongs to the observable in question, it is quite natural to think that the **Lorentz** transformation might take the form of equations between the operators that represent x, y, z, t in the two reference systems.

From a much more general viewpoint, we are then led to the same problem as the one that we encountered upon starting from the **Dirac** equations. What gave us the difficulty in it was the non-commutation of the coordinates, which would become very likely from the modification of the **Dirac** theory that I proposed. However, even if one does not believe it, nonetheless, if one ignores it then another difficulty would persist in the "quantization of the **Lorentz** transformation." One indeed knows that *time* and the spatial coordinates enter into that transformation in an entirely symmetric fashion. They are quantities of the same type that constitute four components of a world-vector. One can even say that the **Lorentz** equations express nothing more than that fundamental fact. However, in quantum mechanics (at least with its current interpretation), that is not true: Time is completely different from the coordinates. In the current interpretation, all of the statements in quantum mechanics reduce to the following form: The probability for the observable *q* to have this or that value *at the instant t* is this or that. One never deals with the probability that *time* should have this or that value. Time is not treated as an observable, so there is no operator that would give one its "statistics." It is a parameter whose value is supposed to be known exactly: In reality, that is the now-obsolete time of **Newton**, and quantum mechanics is not worried about

([1]) See Note III.

the existence of the now-obsolete pendulum that one would need in order to know the value of that parameter *t*.

At the moment, I do not know what to tell you in regard to how one can relinquish the latter non-statistical parameter that quantum mechanics has allowed to survive. One still seems to need it since its statement of the probabilities are certainly not *invariant*, and consequently they must be expressed as functions of *something*. Perhaps one might suspect that they are expressed as functions of each other, but that is only a very vague manner of speaking.

However, it seems to me to be completely beyond all doubt that one *must* relinquish that much-too-classical notion of time, and not only because of relativity. That notion of time is a serious shortcoming with some ramifications in quantum mechanics (at least in its current interpretation), and even when one ignores the postulates of relativity. That is essentially because one's knowledge of the variable *t* is acquired in the same way as any other variable, i.e., by observing a certain physical system, namely, a clock. *t* is then an observable and must be treated as an observable. Time must generally have a "statistic" and not a "value." The exceptional role of time is therefore unjustified.

One can object that in quantum mechanics, it is always possible that *one* of the observables (or even half of the observables) has values that are determined completely (the other half would then be completely indeterminate). One can say that the ideal clock that we have need for is quite simply a system in the variable "position of the hand" always has a precisely-determined value. The exceptional role of time would then be, if not justified, then at least permissible.

But that is not true. I believe that one can prove that the state of the "ideal clock" is not possible for a real system. I could base that upon the only the well-known statement that time and energy are conjugate canonical variables. However, I would not like to do that since in that statement (and only in that statement), one speaks of time *as if it were treated as an observable*. Now, that would be hypocritical: It is not. That is why that statement has always seemed a bit obscure to me. In addition, what I would like to show is that time *must* be treated as an observable, and not as a parameter. That would then be a *petitio principii* since one is trying to prove the theorem that one has tacitly supposed.

In order to examine the properties of an ideal clock *while it is working*, recall that with the interpretation of quantum mechanics, the probability of an arbitrary value of a variable at the instant *t* must be calculated by starting from the function ψ of *x* and *t* :

$$\psi(x, t) .$$

In a word, all that one can expect from an observation of the system must be something that can be deduced from its wave function. Let us apply that to our ideal clock. That is perfectly admissible, except that it is necessary to suppose that there are two of them: one to read the time and another that is described by the function $\psi(x, t)$, and to which one applies our argument. Now recall that when one develops the function ψ into a **Fourier** series (or **Fourier** integral):

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\omega t} c(x, \omega)\, d\omega \ ,$$

the integral:

$$\int dx \int_{\omega_1}^{\omega_2} |c(x,\omega)|^2 \, d\omega$$

represents the probability that one can find a value for the energy of the system that is found between:

$$\frac{h\omega_0}{2\pi} \quad \text{and} \quad \frac{h\omega_1}{2\pi}.$$

Now, an ideal clock is a system that must satisfy the following postulate: There are some events that one must expect with certainty at a given instant, and which one is likewise certain that they will not happen at any other instant. For example, a certain position of the hand would need to have a probability of 1 at a certain instant and a probability of zero at any other instant if the clock were perfectly accurate. However, it would be better to begin by supposing that the precision is limited to τ seconds. It would then follow that the function ψ must have a certain quality: namely, the one that confers the value of zero to the probability of the event in question, which is a quality that ψ must *always* possess, except in the interval τ. However, one easily sees that this would be impossible if, for example, only one value of ω entered into the development of ψ since no probability magnitude will change when one multiplies ψ by a factor of modulus 1, such as $e^{i\omega t}$, and almost the same thing would be true if only an interval:

$$\Delta\omega \ll \frac{1}{\tau}$$

entered into the development of ψ. At the very least, one must have that the interval is:

$$\Delta\omega \sim \frac{1}{\tau}.$$

Consequently, if one would like to pass to the limit of a perfectly-accurate clock then $\Delta\omega$, and as a result, the uncertainty interval for the energy, would become infinite.

That would not suffice, perhaps, to further make the state of the ideal clock physically impossible because it might be the case that despite the infinitude in $\Delta\omega$, the probability that an energy would exceed a given value ω_1 would tend to zero as ω_1 becomes infinite, and that is all that would be reasonable for us to demand. However, up to now, we have postulated the property of the "ideal clock" for only one instant. Upon applying that to all instants, we would infer the conclusion that the value of:

$$\int |c(x,\omega)|^2 \, dx$$

must be independent of ω, in such a way that all energies will become equally probable ([1]).

([1]) See Note IV.

VII. – An analogy between wave mechanics and some problems of probabilities in classical physics.

The subject that I would now like to enter into is not intimately linked with the questions that were addressed in the preceding chapters. First of all, one gets the impressions that things are not coupled so completely. One deals with a classical problem: the problem of probabilities in the theory of Brownian motion. However, at the end of the day, an analogy with wave mechanics will emerge that seemed quite striking to me when I discovered it, so it is difficult for me to believe that it is purely accidental.

By way of introduction, I would like to cite a remark that I found in the "Gifford Lectures" of **A. S. Eddington** (Cambridge, 1928. pp. 216, *et seq.*). **Eddington**, while speaking of the interpretation of wave mechanics, made the following remark in a note at the bottom of the page:

> "The whole interpretation is very obscure, but it seems to depend on whether you are considering the probability *after you know what has happened* or the probability for the purposes of prediction. The $\psi\psi^*$ is obtained by introducing two symmetrical systems of ψ waves travelling in opposite directions in time. One of these must presumably correspond to probable inference from what is known (or is stated) to have been the condition at a later time."

(D. H. D.: A translation of that into French follows that would be unnecessary here.)

One knows beyond a doubt that the Brownian motion of a particle that is subject to no other forces besides molecular collisions that originate in the motion is governed by the diffusion equation:

$$\frac{\partial \omega}{\partial t} = D \cdot \Delta \omega \,. \tag{1}$$

D is the diffusion constant, and ω, or rather:

$$\omega\,(x, y, z, t)\; dx\; dy\; dz\;,$$

is the probability that the particle will be found between x and $x + dx$, y and $y + dy$, z and $z + dz$ at the instant t. One assumes that one knows the coordinates of the particle at a given moment t_0, or more generally, that one knows the probability $\omega\,(x, y, z, t)$ at the moment t_0. The probability at an arbitrary instant t that is later than t_0 is the solution to (1) that takes the given value for t_0.

One also knows that **Smoluchowski** and **Fokker**, and then **Planck**, have generalized that problem to some cases that are much more complicated, such as when external forces act on the particle, or they do not all act on the particle, but only on a completely-general system that is subject to influences that are, on the one hand, regular, such as gravitation for example, and have an irregular, random character, on the other, such as molecular collisions or, for example, the

charges that are communicated to an electrometer by an ionization current that is due to the presence of a radioactive body.

In all of those problems, it is the equation that one calls the "**Fokker** equation" that governs the probability of finding the state of a system that is found between well-defined limits at the instant *t*. That equation is a generalization of the diffusion equation, and it always has the form:

$$\frac{\partial \omega}{\partial t} = \mathcal{H}\omega ,$$

in which $\mathcal{H}$ is a differential operator that is determined by the nature of the system and the regular and irregular influences to which it is subjected. ω is a function of the state of the system and *t*.

The superficial analogy that exists between that theory of classical probability and wave mechanics when it is interpreted statistically has probably not escaped any physicist who knows about both of them. The form of the wave equation for an arbitrary system:

$$-\frac{h}{2\pi i}\frac{\partial \psi}{\partial t} = \mathcal{H}\ \psi$$

is almost the same as that of the **Fokker** equation.

Even the form of the operator $\mathcal{H}$ is the same in the simplest case, in which it reduces to the **Laplace** operator. Nevertheless, the analogy can be regarded as only a superficial one for two reasons, and that is why it has not been discussed very much up to now. In the first place, the imaginary coefficient has the consequence that the wave equation, despite its *apparent* parabolic character, is *essentially* hyperbolic, and consequently describes phenomena that are essentially reversible, whereas the phenomena of diffusion, etc., are essentially irreversible. In the second place, ψ is not the probability, but only what one calls the probability amplitude: The probabilities themselves are bilinear expressions in ψ and ψ^*. In the simplest case, that probability is equal to:

$$\psi\psi^* dx .$$

Therefore, despite the analogy in *the equations*, the mathematical machinery for calculating probabilities is very different in the two theories, and that has the effect that the most characteristic traits of quantum mechanics are lacking from the classical theory completely.

In this presentation, I would like to draw your attention to the fact that there exist classical problems that present a much better analogy with wave mechanics and have not been treated at all up to now. One deals with problems that are concerned with exactly the same systems with the same influences, both regular and irregular, but with just one difference that pertains to how one poses the question. In order to simplify the discussion, let us take the simplest case: one-dimensional Brownian motion, in which the **Fokker** equation reduces to:

(2) $$\frac{\partial \omega}{\partial t} = D \frac{\partial^2 \omega}{\partial x^2}, \qquad \omega \text{ is a function of } x \text{ and } t.$$

Suppose that someone (e.g., observer A) has observed the state of the system, i.e., the coordinate x, at the instant t_0 *and at the instant* t_1. Someone else (e.g., observer B) observes x at the instant t *that is between* t_0 *and* t_1 :

$$t_0 < t < t_1 \ .$$

The observer A must deduce the probability of a result that is determined by the observation of B *from his own* observations. That is an entirely reasonable problem, and it can be further generalized in the following way: One can assume that the observer A has not observed the coordinate x at the instants t_0, t_1 *exactly*, but he knows only the *probability* for those two instants:

$$\omega\,(x, t_0) = \omega_0\,(x)\ , \qquad \omega\,(x, t_1) = \omega_1\,(x)\ .$$

With one *or the other* of those two givens, the problem will become one of a well-known type. With just the first given, ω will become the solution to (2) that reduces to $\omega_0\,(x)$ for t_0. With just the second given, ω will become the solution of:

(3) $$-\frac{\partial \omega}{\partial t} = D \frac{\partial^2 \omega}{\partial x^2}$$

that reduces to ω_1 for t_1 . However, if one possesses both of the two givens then ω will be one or the other of them because the solution to (2) is determined uniquely by its initial value and the solution to (3), by its final value. Neither of them will be so indulgent as to reduce to an arbitrarily-prescribed value at a *second* instant. One will then be dealing with a type of problem that is completely new. I would like to exhibit the solution before I prove that it exists. ω is the *product* of a solution of (2) and a solution of the adjoint equation (3). The two solutions must be chosen in such a fashion that their product satisfies both the initial condition and the final condition. I have not succeeded in proving either that such solutions always exist or that they are unique. However, I am completely convinced of those facts. That boundary-value problem even seems to be new to mathematics.

The proof is somewhat lengthy, so I hope that it will not be too fatiguing for you. I shall call the function $\omega\ (x,\ t)$ whose final and initial probabilities we seek, or rather, the terminal probabilities ω_0 and ω_1, the *intermediate probability.* We begin by treating the particular case that was pointed out before, in which the terminal probabilities are both known *with certainty*. If the particle is observed to be at the point x_0 at the instant t_0 and at the point x_1 at the instant t_1 then what is the probability that it will be found between x and $x + dx$ at the instant t ? Let a great number N of particles (which displace independently) start from x_0 at the instant t_0. We are interested in only the ones that are found near x_1 at the instant t_1, namely, between x_1 and $x_1 + dx_1$.Their number is given by a well-known formula as the fundamental solution to the diffusion equation:

(4) $$n_1 = N \frac{1}{\sqrt{4\pi D(t_1 - t_0)}} e^{-\frac{(x_1 - x_0)^2}{4D(t_1 - t_0)}} dx_1 = N \cdot g\,(x_1 - x_0,\, t_1 - t_0)\, dx_1 ,$$

to abbreviate. On the other hand, the fraction of the N particles that satisfies the double condition that:

1. It is found in the interval $(x, x + dx)$ at the instant t, and

2. It is found in the interval $(x_1, x_1 + dx_1)$ at the instant t_1 ,

is given by:

$$n = N \cdot g\,(x - x_0,\, t - t_0)\, dx \cdot g\,(x_1 - x_0,\, t_1 - t_0)\, dx_1 ,$$

since the two probabilities are obviously independent. Now, the probability that we seek is obviously n / n_1 . We will then find in this special case that:

(5) $$\omega\,(x,\, t) = \frac{g\,(x - x_0, t - t_0)\, g\,(x_1 - x, t_1 - t)}{g\,(x_1 - x_0, t_1 - t_0)} .$$

It is easy to see that ω is the product of a solution of (2) and a solution of (3), which agrees with the general result that we have suggested.

In order to treat the general case with terminal probabilities ω_0, ω_1, one must again deal with a very large number N of particles, but they do not all have to start from the same location. One must start with the number:

(A) $$N\, \omega_0\,(x_0)\, dx_0$$

of them in the "cell" $(x_0, x_0 + dx_0)$ and repeat that experiment *a large number of times* because in most cases, one would not get the desired result, namely, that one has:

(B) $$N\, \omega_1\,(x_1)\, dx_1$$

particles in $(x_1, x_1 + dx_1)$ at the instant t_1. Nonetheless, the probability that *the latter* distribution is realized at the instant t_1 is not zero, and what one must do (but only mentally, unfortunately!) is to repeat the experiment a sufficient number of times until the same small proportion of the experiments that lead to the desired result define a very large number. That small fraction will then represent precisely the statistical information that agrees with the question that was posed.

Now, all that we must know is how many particles (A) are found among the particles (B). If we know that number for an arbitrary value of x_0 and x_1 then it would suffice to multiply it by the expression (5) and dx in order to find the proportion of those particles (which are produced at x_0 and arrive at x_1) that are found between x and $x + dx$ at the instant t. Upon adding them up over all

values of x_0 and x_1, i.e., upon integrating from x_0 to x_1, we will find the number that we seek to determine.

Hence, the main problem is to find the number of (A) that start from (B). In order to do that, it would not suffice to fix one's attention on *one* group (A) and *one* group (B): One must imagine the entire ensemble. Divide the level of x into equal cells that we take to have unit length, to simplify the notation. Let a_k be the number of particles that are produced in the k^{th} cell, and let b_l be the number of particles that arrive at the l^{th} cell. Let c_{kl} be the number of particles that are produced in the k^{th} cell and *arrive* at the l^{th} one. We let g_{kl} denote the *a priori* probability for a particle that is produced in the k^{th} cell to arrive at the l^{th} one [g_{kl} is therefore nothing but an abbreviation for the expression (4)]. The c_{kl} satisfy the relations:

(6) $$\sum_l c_{kl} = a_k\,, \qquad \sum_k c_{kl} = b_l\,,$$

and no other relation: An arbitrary system of c_{kl} that agrees with the preceding relations is *possible* and is sometimes realized. Therefore, the number c_{kl} that are exactly the ones that we have need for are not all fixed uniquely: Indeed, we must study the *statistics* of the c_{kl} . However, I have not done that. It seems very obvious to me that in the limiting case where the number N is very large, the only systems of numbers c_{kl} that are realized are the ones that are very close to the most-probable system, i.e., the system c_{kl} for which the most-improbable event that we assume to be confirmable (namely, the numbers a_k and b_l) acquits at least the greatest probability (i.e., greater than for any other system c_{kl}).

If we know the cell of origin for each individual particle, as well as the destination cell, then the probability for the observed event, i.e., the transformation of the distribution a_k into b_l, to be realized in *that* particular manner will be:

$$\prod_k \prod_l g_{kl}{}^{c_{kl}}\,.$$

However, that is not the total probability for that event to be realized by means of a well-defined system of c_{kl} . That is because one can permute the a_k particles that are produced in the k^{th} cell:

$$\frac{a_k\,!}{\prod_l c_{kl}\,!}$$

times, and similarly for all k, so:

$$\prod_k \frac{a_k\,!}{\prod_l c_{kl}\,!} = \frac{\prod_k a_k\,!}{\prod_k \prod_l c_{kl}\,!}$$

times. Therefore, the total probability is:

$$\prod_k a_k ! \prod_k \prod_l \frac{g_{kl}^{\,c_{kl}}}{c_{kl}!} .$$

One must determine the maximum of that expression when one varies the c_{kl} while taking the relations (6) into account. Since $\prod_k a_k !$ does not vary, it suffices to find the maximum of the double product, or its logarithm, which is more convenient. Therefore:

$$\delta \sum_k \sum_l \{c_{kl} \log g_{kl} - c_{kl} (\log c_{kl} - 1)\} + \delta \sum_k \lambda_k \sum_l c_{kl} + \delta \sum_l \mu_l \sum_k c_{kl} = 0 ,$$

or

$$\sum_k \sum_l \delta c_{kl} \{\log g_{kl} - \log c_{kl} + \lambda_k + \mu_l\} = 0 .$$

Therefore:

$$c_{kl} = e^{\lambda_k + \mu_l} g_{kl} .$$

Set:

$$\psi_k = e^{\lambda_k} , \qquad \varphi_l = e^{\mu_l} ,$$

in such a fashion that:

(7) $$c_{kl} = \psi_k \varphi_l g_{kl} .$$

The ψ_k, φ_l are determined by:

(8) $$\varphi_l \sum_k \psi_k g_{kl} = b_l , \qquad \psi_k \sum_l \varphi_l g_{kl} = a_k .$$

The solution is contained in (7) and (8). We must only translate it into the language of continua. We write:

$c(x_0, x_1)\, dx_0\, dx_1$	for	c_{kl}
$N\, \omega_0(x_0)\, dx_0$	"	a_k
$N\, \omega_1(x_1)\, dx_1$	"	b_l
$\sqrt{N}\, \psi(x_0)\, dx_0$	"	ψ_k
$\sqrt{N}\, \psi(x_1)\, dx_1$	"	φ_l

Indeed, it is obvious that ψ_k, φ_l must correspond to functions of x_0 and x_1. The square root $\sqrt{N}$ is introduced only to simplify the notations. We will then find that:

(9)
$$\psi(x_0) \int \varphi(x_1)\, g(x_1 - x_0, t_1 - t_0)\, dx_1 = \omega_0(x_0) ,$$

$$\varphi(x_1) \int \psi(x_0)\, g(x_1 - x_0, t_1 - t_0)\, dx_0 = \omega_1(x_1) ,$$

and

$$c\,(x_0, x_1)\,dx_0\,dx_1 = N\,g\,(x_1 - x_0,\, t_1 - t_0)\;\psi\,(x_0)\;\varphi\,(x_1)\,dx_0\,dx_1\;.$$

That is the number of particles that are produced at x_0 and arrive at x_1. As we have suggested, we must multiply it by the expression (5) and by dx (and divide by N), and then integrate from x_0 and x_1 in order to find the "intermediate probability." It will then have the following value:

$$\omega\,(x,\,t) = \int g\,(x_1 - x_0, t_1 - t_0)\;\psi\,(x_0)\,dx_0 \cdot \int g\,(x_1 - x_0, t_1 - t_0)\;\varphi(x_1)\,dx_1\,.$$

One easily sees that the first factor is a solution to (2), while the second one is a solution to (3).

φ and ψ are determined by the integral equations (9), which seem very complicated to me, although they express nothing but the boundary conditions that are imposed upon the product of the two solutions.

One can add some remarks to what was just proved that are quite interesting beyond any analogy with wave mechanics. In the first place, one can easily prove that $\omega\,(x,\,t)$ satisfies the condition that its integral will give a constant value, namely 1 (if ω_0 and ω_1 have been normalized in the same way). One can then show that the "center of gravity," i.e., $\int x\,\omega(x,t)\,dx$, moves with constant velocity upon starting from its initial position until it reaches its final position. In the simple case where the probabilities at the limits are certainties, the maximum probability is itself also a linear function of time: At each instant, one is dealing with a **Gaussian** distribution that grows larger until half of the time has elapsed and then contracts to the final point.

Some other very interesting remarks pertain to the fact that the role of the two functions ω_0, ω_1 is absolutely symmetric. One can say that neither of the two directions of time is privileged. Upon exchanging ω_0 and ω_1, one will find exactly the same evolution of the probability, but in the opposite sense from t_1 to t_0.

If the terminal probabilities ω_1, ω_2 are given in such a way that ω_1 is exactly the distribution at the instant t_1 that would result automatically (i.e., from the diffusion equation) from the distribution ω_0 that is given at t_0 then the solution to our two integral equations will be almost obvious: One can take:

$$\varphi \equiv 1, \quad \psi \equiv \omega_0,$$

and the evolution of the probability will take place as it would in the usual diffusion problem.

However, the "inverse" case, so to speak, is just as simple. If ω_0 and ω_1 are given in such a manner that ω_1 results from ω_0 over the time interval $t_1 - t_0$ according to the adjoint equations, or what amounts to the same thing, if ω_0 results from ω_1 according to the ordinary diffusion equation then the solution will be likewise almost obvious: One must take:

$$\varphi \equiv \omega_1, \quad \psi \equiv 1,$$

and everything will happen as it would according to the "inverse" equation.

One can deduce a very curious conclusion from that regarding the manner by which the considerable thermodynamic fluctuations that sometime occur (although quite rarely) are produced in a system that is in equilibrium. Imagine that one is observing a system of particles that are in a state of diffusion and are in thermodynamic equilibrium. Assume that one has found them to be distributed almost uniformly at a given instant t_0, and that one has found a spontaneous and *considerable* deviation from that uniformity at $t_1 > t_0$. One wants to know the manner by which that deviation is produced. What manner would be the most probable?

The answer is the following: The most probable is the one that is produced by a complete reversal of the laws of diffusion in such a way that the diffusion current at each instant will point in the direction of the *gradient* (and not the "descent") of the concentration and will have an intensity that corresponds to exactly the usual value of the diffusion constant D (up to sign).

I even believe that the degree of certainty with which one can exhibit that probability is exactly the same as in the direct case. In addition, I believe that all of that is applicable to not only the case of diffusion, but to all other irreversible laws of physics.

It still remains for me to draw your attention to the great analogy that presents itself between the probabilities that are envisioned in quantum mechanics and what we have called *intermediate probabilities* in problems of the preceding type.

That analogy is much more pronounced in problems of that type than it was in the older problems that **Smoluchowski**, **Fokker**, and **Planck** had treated. In the new problems, not only does the **Fokker** equation resemble the fundamental equation of quantum mechanics, but just as with the latter, the intermediate probability is given by the *product* of two solutions to *two* equations that differ from each other only by their signs and time variables. It is precisely that symmetry with respect to time that accounts for the fact that the evolution of the intermediate probability is *reversible* in the same manner as the one in quantum probability. That would say that upon changing the sign of time in the expression for an intermediate probability that evolves according to the general equations of the problem, one will arrive at an expression that likewise defines an evolution that is compatible with those equations, and which consequently can be realized by conveniently choosing the boundary conditions. The same thing is not at all true in the usual problems of Brownian motion, which reduce mathematically to problems of diffusion or the conduction of heat and define an essentially irreversible evolution.

Nonetheless, a considerable difference between them still persists that must be pointed out: The evolution of the intermediate probability *is not wave-like*. That amounts to saying that the constant D in equations (2) and (3) is essentially real, whereas the square root of negative unity $\sqrt{-1}$ enters into the wave equation of quantum mechanics in a very mysterious fashion. Mathematically, that implies:

1. The wave-like character of the evolution of the function ψ, and that

2. Its character is essentially complex.

The complex function ψ corresponds to *two* real functions in such a way that it would suffice to define the boundary conditions by giving the value of ψ at *just one* well-defined instant. That is

the way of looking at things that is generally adopted in quantum mechanics. Is it the only admissible one? In our problem, that would amount to regarding the values of ψ and φ as being given at a well-defined instant (instead of the values of their product at two different instants), which would be inadmissible and would make absolutely no sense.

Must one interpret the remark of Eddington that was cited above as if it pointed to the necessity of modifying that way of looking at things in wave mechanics and taking the boundary conditions to be the values of just one real probability at two different instants?

MATHEMATICAL NOTES

NOTE I

The proof is very easy when we make use of our concepts of even, odd, etc. I would like to exhibit that proof since it has not been published.

One must know the manner by which a **Lorentz** transformation acts on ψ. First, one transforms the coordinates x_1, x_2, x_3, t in the well-known way. One must then transformation the variable ζ, i.e., apply a certain linear transformation to ψ_1, ψ_2, ψ_3, ψ_4 whose coefficients depend upon those of the **Lorentz** transformations.

For the purpose that we have in mind, it suffices to utilize an infinitesimal transformation. We can write the operator that is applied to ψ as:

$$\mathcal{L} = 1 + \varepsilon_{kl}\left(x_k \frac{\partial}{\partial x_l} - x_l \frac{\partial}{\partial x_k} \right), \qquad x_4 = i\,c\,t$$

(summed over k and l).

However, as I have suggested, one must once more apply another operator that operates on only ζ, namely, the following one:

$$\Lambda = 1 + \tfrac{1}{2}\varepsilon_{kl}\,\gamma_k\,\gamma_l\,, \qquad \gamma_1 = -\,i\,\alpha_4\,\alpha_1\,,$$
$$\gamma_2 = -\,i\,\alpha_4\,\alpha_2\,,$$
$$\gamma_3 = -\,i\,\alpha_4\,\alpha_3\,,$$
$$\gamma_4 = \alpha_4\,.$$

Take the special case:

$$k = 1, \qquad l = 4,$$

which will suffice for us, i.e., suppose that only ε_{14} (which we shall call ε) is non-zero:

$$\gamma_k\,\gamma_l = -\,i\,\alpha_4\,\alpha_1\,\alpha_4 = +\,i\,\alpha_1\,,$$

$$\Lambda\,\mathcal{L} = 1 + \tfrac{1}{2}\varepsilon\,\alpha_1 + \varepsilon\left(x_k \frac{\partial}{\partial(ict)} - ict\frac{\partial}{\partial x_1}\right).$$

That operator must *leave* ψ positive. Thus, its odd part must give zero when it is applied to ψ.

In all of that, the multiplication by t, just like the differentiation $\partial / \partial t$, must be regarded as an even operator since:

1. Multiplication by a parameter does not change sign.

2. ψ is supposed to be *identically* positive in t.

The last term is therefore even. The odd parts in the second and third terms are obviously (upon suppressing $i\varepsilon$):

$$\tfrac{1}{2}\eta_1 - \frac{1}{c}\xi_1\frac{\partial}{\partial t},$$

but

$$\xi_1 = -\tfrac{1}{2}\,c\kappa\eta_1\mathcal{H}^{-1}.$$

Therefore (upon suppressing the factor 1/2):

$$\left(\eta_1 + \kappa\eta_1\mathcal{H}^{-1}\frac{\partial}{\partial t}\right)\psi = 0\,,$$

$$\eta_1\mathcal{H}^{-1}\left(\mathcal{H} + \kappa\frac{\partial}{\partial t}\right)\psi = 0\,,$$

$$-\mathcal{H}^{-1}\eta_1\left(\mathcal{H} + \kappa\frac{\partial}{\partial t}\right)\psi = 0\,,$$

or upon applying the operator $\mathcal{H}$:

$$\eta_1\left(\mathcal{H} + \kappa\frac{\partial}{\partial t}\right)\psi = 0\,.$$

η_1 has no proper value equal to zero (at least if the rest energy is finite). Therefore, it must be true that:

$$\left(\mathcal{H} + \kappa\frac{\partial}{\partial t}\right)\psi = 0\,.$$

Q.E.D.

NOTE II

The spin operators that have been in question in this lecture are *products* of the α_k, in such a fashion that, for example:

$$s_1 = -\,i\,\alpha_2\,\alpha_3 = +\,i\,\alpha_3\,\alpha_2\,.$$

Determine the *even part* of the s_k, which we denote by $\hat{s}_k$. Since the decomposition of α_k is:

$$\alpha_k = c\mathcal{H}^{-1}p_k + \eta_k\,,$$

one will find $\hat{s}_k$ by adding the product of the even parts of α_2 and α_3 to the product of the odd parts:

$$\begin{aligned}\hat{s}_k &= -\,i(c^2\mathcal{H}^{-2}p_2\,p_3 + \eta_2\,\eta_3)\\ &= +\,i(c^2\mathcal{H}^{-2}p_2\,p_3 + \eta_3\,\eta_2)\,.\end{aligned}$$

Upon adding those two equations:

$$\hat{s}_1 = \frac{i}{2}(\eta_3\,\eta_2 - \eta_2\,\eta_3)\,.$$

The even part of the spin is the commutator of the η_k (just as the spin itself is the commutator of the α_k). One likewise infers from that equation that:

1. Spin is even *for the most part*.

2. Its even part is an integral of the motion in the absence of an external field.

Indeed, the $\hat{s}_k$ commute with $\mathcal{H}$ because the η_k *anticommute*.

Since:

$$\xi_k = \tfrac{1}{2}c\kappa\mathcal{H}^{-1}\eta_k\,,$$

the commutators of the η_k are intimately coupled with the commutators of the ξ_k. One easily finds that (pay attention to the sign!):

$$\xi_2\,\xi_3 - \xi_3\,\xi_2 = -\tfrac{1}{2}ic^2\kappa^2\mathcal{H}^{-2}\,\hat{s}_1\,.$$

The commutators of the ξ_k are the same as those of the x_k, but with the opposite sign. In order to see that, write down that the x_k commute:

$$x_k\,x_l - x_l\,x_k = 0\,.$$

Replace them by their decompositions:

$$\left(\hat{x}_k + \xi_k\right)\left(\hat{x}_l + \xi_l\right) - \left(\hat{x}_l + \xi_l\right)\left(\hat{x}_k + \xi_k\right) = 0\,,$$

and equate the *even* part of that equation to zero. Consequently:

$$\hat{x}_2\,\hat{x}_3 - \hat{x}_3\,\hat{x}_2 = +\,\tfrac{1}{2} i c^2 \kappa^2 \mathcal{H}^{-2}\,\hat{s}_1 = -\,\frac{i}{8\pi} c^2 \kappa^2 \mathcal{H}^{-2}\,\hat{s}_1\,.$$

Since $\mathcal{H}^{-}2$ has, in general, an order of magnitude of $m^{-2}c^{-4}$, the commutators will have an order of magnitude of:

$$\frac{1}{2}\left(\frac{h}{4\pi m c}\right)^2.$$

That is perhaps the order of magnitude that one might have expected since the commutation is not perturbed by the addition of operators whose order of magnitude is $h / 4\pi mc$.

NOTE III

What persists in any case is the **Heisenberg** uncertainty principle. If one desires to have a precision of λ in the coordinates then that would imply an uncertainty in the velocity that could be roughly estimated to be:

$$\Delta v = \frac{h}{4\pi m \lambda}\,,$$

in which m is the electron mass, and upon suppressing the factor of $\sqrt{1-\beta^2}$, since only the order of magnitude is of any interest. It is true that nothing prevents one from making λ as small as one desires. However, in order for that precision to have any physical significance, one must simultaneously determine the *time* with a precision of τ, for which:

$$\tau \cdot \Delta v \leq \lambda\,.$$

We shall no longer speak of the *mass* of the clock that serves to measure time from now on, but only its *size*. I would like to prove that it must be very small.

Let l be that size. Obviously, it is also the *uncertainty* in the *location* at which it displays the time. Upon applying the time that it displays to a body moving with a velocity of Δv, that would imply an uncertainty of:

$$\frac{l\,\Delta v}{c^2} \leq \tau\,,$$

which must be equal to at most τ, so:

$$\frac{l(\Delta v)^2}{c^2} \leq \lambda ,$$

$$\frac{l}{\lambda^4}\left(\frac{h}{4\pi mc}\right)^2 \leq \lambda ,$$

$$l \leq \frac{\lambda^3}{\left(\frac{h}{4\pi mc}\right)^2} .$$

If one would like to increase the precision to something greater than $h / 4\pi mc$ then one would need to have a clock that is *much smaller* than that, i.e., 10^{-11} cm. (And at the same time, very heavy.)

NOTE IV

Formally, the essence of the proof is the same as for the analogous proposition concerning observables that are canonically conjugate.

There must exist a Hermitian operator $\mathcal{A}$ (for "*aiguille*" = "hand") such that $\psi(x, t)$ is identically a proper function of $\mathcal{A}$ with the proper value t :

$$\mathcal{A}\ \psi(x, t) - t\ \psi(x, t) = 0 .$$

Upon replacing ψ with the Fourier integral, one will find:

$$\int e^{-i\omega t}(\mathcal{A}c - tc)\, d\omega = 0 .$$

An integration by parts of the second term will give:

$$\int e^{-i\omega t}\left(\mathcal{A}c - i\frac{\partial c}{\partial \omega}\right) d\omega = 0 ,$$

so:

$$\mathcal{A}c - i\frac{\partial c}{\partial \omega} = 0 ,$$

and also:

$$\mathcal{A}^* c^* + i\frac{\partial c^*}{\partial \omega} = 0 .$$

Upon multiplying the last two equations by c^* and c, resp., and adding them, one will have:

$$\frac{\partial}{\partial \omega} |c|^2 = i(c^* \mathcal{A}^* c - c \mathcal{A}^* c^*) \ .$$

The integral $\int dx$ in the right-hand side is zero as a result of the properties of Hermitian (or self-adjoint) operators. The integral $\int |c|^2 \, dx$ is therefore independent of ω, as we asserted.

Since *all* values of energy are equally probable, it would follow that a bounded interval of energy would belong to a probability of zero. Hence, it is infinitely improbable that we would find a *finite* value for the energy of our clock. It is therefore physically impossible!

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