"Ueber die Herstellung von Wirbelbewegungen in idealen Flüssigkeit durch conservative Kräfte," Ann. Phys. (Leipzig) **56** (1895), 144-147.

On the creation of vorticial motions in ideal fluids by conservative forces

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1. – A fundamental theorem in the hydrodynamics of vortex theory is this one:

No particle in an ideal fluid mass in which arbitrary density distribution might prevail and upon which arbitrary external conservative forces act can be in rotation when it was not already in rotation to begin with ([1]).

If vorticial motions can nonetheless be created with remarkable ease in arbitrary fluids, *especially by means of rapidly-decaying pressure differences*, then that is related to the fact that this anomalous behavior of the fluids can be attributed to their lack of *ideal* behavior, and so great is one's confidence in the absolute admissibility of the theorem that was cited above that **Lord Kelvin** had based his celebrated ideas regarding the constitution of ponderable matter upon it.

Meanwhile, it has remained unnoticed that it is precisely in the experimental arrangements on which vortex rings have been created most regularly and reliably that *the degree* of ideal behavior in the fluid is largely irrelevant to the outcome of the experiment, and the great significance that is assigned to the theorem in its own right would seem to make it desirable to place a sharp limit on its domain of evidence.

The result of the following little study of that situation is the remark that *the initially-cited theorem indeed possesses unrestricted validity for any ideal fluid that is everywhere and continually found to be in complete* ([2]) *thermodynamic equilibrium regardless of whether the fluid is compressible or incompressible. However, in general, it cannot be applied to other fluids when they are completely ideal and incompressible then.*

2. – We understand the quantities ρ, p, X, Y, Z, and u, v, w to mean the density, pressure, external forces, and flow velocity, respectively, that prevail in the fluid, which is assumed to be completely frictionless, at time t and at the point x, y, z. The equations of motion will then be true:

([1]) **Von Helmholtz** first expressed that theorem for *incompressible* fluids. For ideal *compressible* fluids, it is generally accepted everywhere in the literature of hydrodynamics, sometimes expressly and sometimes tacitly.

([2]) In order for that to be true, it is necessary that the fluid should coexist with a second thermodynamical phase.

$$(1)\qquad \begin{cases} X - \dfrac{1}{\rho}\dfrac{\partial p}{\partial x} = \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z}, \\ Y - \dfrac{1}{\rho}\dfrac{\partial p}{\partial y} = \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z}, \\ Z - \dfrac{1}{\rho}\dfrac{\partial p}{\partial z} = \dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z}, \end{cases}$$

$$(2)\qquad \frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right).$$

Since one differentiates the second of those equations with respect to z and the third one with respect to $(-y)$, adds the two equations thus-obtained, and suitably reorders them then one will get:

$$(3)\qquad \begin{cases} \dfrac{\partial Y}{\partial z} - \dfrac{\partial Z}{\partial y} - \dfrac{\partial}{\partial z}\left(\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}\right) \\ \qquad = \dfrac{d}{dt}\cdot\left(\dfrac{\partial v}{\partial z} - \dfrac{\partial w}{\partial y}\right) - \dfrac{\partial u}{\partial x}\left(\dfrac{\partial v}{\partial z} - \dfrac{\partial w}{\partial y}\right) - \dfrac{\partial v}{\partial x}\left(\dfrac{\partial w}{\partial x} - \dfrac{\partial u}{\partial z}\right) - \dfrac{\partial w}{\partial z}\left(\dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial x}\right) \\ \qquad\quad + \left(\dfrac{\partial v}{\partial z} - \dfrac{\partial w}{\partial y}\right)\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial x}\right). \end{cases}$$

The d / dt in that means a *total* differentiation with respect to time. The assumption that the external forces are *conservative* makes the term $\partial Y / \partial z - \partial Y / \partial z$ vanish, and the introduction of the rotational velocity:

$$(4)\qquad \begin{cases} \xi = \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z} - \dfrac{\partial w}{\partial y}\right), \\ \eta = \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x} - \dfrac{\partial u}{\partial z}\right), \\ \zeta = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial x}\right) \end{cases}$$

will simplify equation (3) to the following one:

(5)
$$\begin{cases} \dfrac{\partial}{\partial y}\left(\dfrac{1}{2\rho}\dfrac{\partial p}{\partial z}\right)-\dfrac{\partial}{\partial z}\left(\dfrac{1}{2\rho}\dfrac{\partial p}{\partial y}\right) \\ \qquad =\dfrac{d\xi}{dt}-\xi\dfrac{\partial u}{\partial x}-\eta\dfrac{\partial v}{\partial x}-\zeta\dfrac{\partial w}{\partial x}+\xi\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}\right), \\ \text{and} \\ \dfrac{\partial}{\partial z}\left(\dfrac{1}{2\rho}\dfrac{\partial p}{\partial x}\right)-\dfrac{\partial}{\partial x}\left(\dfrac{1}{2\rho}\dfrac{\partial p}{\partial z}\right) \\ \qquad =\dfrac{d\eta}{dt}-\xi\dfrac{\partial u}{\partial y}-\eta\dfrac{\partial v}{\partial y}-\zeta\dfrac{\partial w}{\partial y}+\eta\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}\right), \\ \dfrac{\partial}{\partial x}\left(\dfrac{1}{2\rho}\dfrac{\partial p}{\partial y}\right)-\dfrac{\partial}{\partial y}\left(\dfrac{1}{2\rho}\dfrac{\partial p}{\partial x}\right) \\ \qquad =\dfrac{d\zeta}{dt}-\xi\dfrac{\partial u}{\partial z}-\eta\dfrac{\partial v}{\partial z}-\zeta\dfrac{\partial w}{\partial z}+\zeta\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}\right), \\ \text{respectively.} \end{cases}$$

3. – If ξ, η, ζ are simultaneously zero for a fluid particle then the following differential equations will be true for that point in time:

(6)
$$\begin{cases} \dfrac{d\xi}{dt}=\dfrac{\partial}{\partial y}\left(\dfrac{1}{2\rho}\dfrac{\partial p}{\partial z}\right)-\dfrac{\partial}{\partial z}\left(\dfrac{1}{2\rho}\dfrac{\partial p}{\partial y}\right), \\ \dfrac{d\eta}{dt}=\dfrac{\partial}{\partial z}\left(\dfrac{1}{2\rho}\dfrac{\partial p}{\partial x}\right)-\dfrac{\partial}{\partial x}\left(\dfrac{1}{2\rho}\dfrac{\partial p}{\partial z}\right), \\ \dfrac{d\zeta}{dt}=\dfrac{\partial}{\partial x}\left(\dfrac{1}{2\rho}\dfrac{\partial p}{\partial y}\right)-\dfrac{\partial}{\partial y}\left(\dfrac{1}{2\rho}\dfrac{\partial p}{\partial x}\right). \end{cases}$$

However, when one makes no special assumptions, the expressions on the right of the equal sign will generally differ from zero by a finite amount, and indeed not only for compressible fluids, but also for completely incompressible ones, to the extent that this concept even has an intrinsically physical sense ([1]).

However, we can make those expressions vanish by the requirement that the expression $dp/2\rho$ should be *a complete differential*. That would be the case:

([1]) Cf., on this, also my following note regarding a caveat that is required by the theoretical introduction of incompressible fluids. Whoever loves paradoxes can say that vorticial motions are impossible in an *incompressible* fluid, but they can be easily generated in an *infinitesimally-compressible* fluid.

I. For any fluid in which all possible fluctuations in pressure are compensated by either *completely-isothermal* or *completely-adiabatic* ones. One should not remotely expect that one of those extremes should generally exist, even only approximately, in *moving* fluids (let alone ones in which there are discontinuous perturbations that are necessarily coupled with the creation of vortices).

II. However, the second case, where a possible realization of the requirement above does not generally contradict any theoretical consideration, is the one in which the fluid coexists with a second thermodynamic phase and is found to be in complete thermodynamic equilibrium with it everywhere and continually.

4. – Allow me the further remark that *the remaining fundamental laws* of the hydrodynamical laws of vortices, which refer to the phenomena of motion in completed vortex rings, will not be touched by the essentially-physical considerations above. Rather, they will be true in full rigor as soon as the effect of the pressure impulse has ceased. In fact, those laws will express purely geometric (kinematic, resp.) truths that can basically dispense with any physical assumptions.

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