SELECTED PAPERS

ON

GAUSS'S PRINCIPLE OF LEAST CONSTRAINT

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TABLE OF CONTENTS

The Gibb's-Appell equations

Hertz's mechanics

INTRODUCTION

The most fundamental problem in theoretical mechanics (if not theoretical physics itself) is to observe some basic law in nature that is universal to either the equilibrium state of static systems or the time evolution of all dynamical systems and interpret it in a mathematical context that will lead to equations of equilibrium or motion for the states of any particular system. Such a basic law is what one calls a *first principle of mechanics.* The principal branches of mechanics then become kinematics, statics, and dynamics. Kinematics is essentially the geometry of motion, so it only seeks to describe the "state of motion," while dynamics seeks to explain the root causes for motion. Statics lies somewhere between kinematics and dynamics as the study of the equilibrium state of a mechanical system, although d'Alembert's principle defines a sort of "duality" between statics and dynamics in that it allows one to represent dynamics as a state of "equilibrium" in space-time, rather than simply space.

Ultimately though, the laws of mechanics should grow out of some "first principle" that borders upon a natural philosophy but must still suggest a mathematical formulation. The one that shall be addressed in this collection is Gauss's principle of least constraint, which basically says that the natural path of constrained motion is the path of least "constraint," where Gauss defines the function that measures the degree of constraint to be the sum of the squares of the differences between the acceleration of the free (i.e., unconstrained) motion of a point mass and its constrained acceleration, times those masses.

Before Gauss proposed his principle of least constraint, some of the attempts at defining such a first principle were Newton's laws of motion, the principle of virtual work, and the principle of least action. Following Gauss, Heinrich Hertz, in the final months of his short life, proposed a specialization of Gauss's principle to a principle of least curvature or least acceleration, which he intended to replace essentially both of Newton's first two laws of motion. Later, Paul Appell introduced the concept of "energy of acceleration" as a way of resetting Gauss's principle as a principle of least acceleration.

The purpose of this introduction is to give a more modern discussion of the issues that are discussed in the largely-classical works that follow. Although it assumes a certain familiarity with the basic notions of differentiable manifolds, the calculus of exterior differential forms, and the geometry of moving frames, every attempt has been made to stop short of the more advanced topics in those subjects ([1]). In particular, because all of the articles treat the geometry of manifolds (such as configuration manifolds) in terms of local expressions, the otherwise "basis-free" form of equations will also be presented in the local form that is closer to the spirit of the translated papers. Typically, the topology of manifolds will not be discussed in any detail since that is also the case in the readings. Similarly, although some mention of the symplectic approach to Hamiltonian mechanics will be made, that approach to theoretical mechanics will not be adhered to

([1]) Numerous textbooks on those subjects exist by now, such as [**1**]. Typically, the ones that adhere to the global, basis-free formulation of everything (which is in the spirit of the relativity principle) tend to avoid any discussion of the local forms of the same equations in terms of coordinates and components. However, since much of the physics literature is phrased exclusively in the language of local expressions, this author feels that a good compromise solution is to try to do both and not just choose up sides.

dogmatically, and for essentially two reasons: First of all, the selection of papers that follows involve the use of the calculus of variations consistently, so it is enough to generalize those techniques somewhat. Secondly, the scope of Hamiltonian mechanics is essentially the same as that of an action functional, namely, one must be dealing with conservative forces and perfect holonomic constraints. As a result, conservation of energy usually follows as a corollary to Hamilton's equations (when the Hamiltonian is time-independent, which is typical of the symplectic formulation), even though the scope of theoretical mechanics in general is not restricted to conservative systems by any means. However, the scope of Gauss's principle of least constraint is broader than that of the least action principle and can still be applied when one has non-conservative forces or imperfect or non-holonomic constraints, as well.

This introduction begins by defining the general setting for mechanical models. It then develops some of the key notions that will be used in the readings, such as variations (virtual displacements), virtual work, d'Alembert's principle, action functionals, the Euler-Lagrange equations, and a general discussion of constrained motion. The principles of Gauss, Appell, and Hertz will each be discussed, and the introduction will conclude with a survey of the translations that are included in this collection.

Some of the classical texts that were conferred in the process of producing this introduction, as well as referenced in the translations that follow are the ones by Lagrange [**2**], Jacobi [**3**], Helmholtz [**4**], Hertz [**5**], Boltzmann [**6**], Kirchhoff [**7**], Appell [**8**], Heun [**9**], Whittaker [**10**], Hamel [**11**], and Lanczos [**12**].

1. The general structure of mechanical models. – In this section, we shall describe the author's view of the "common ground" of all mechanical models (see also [**13**]), which consists of the following basic constructions:

1. A space of kinematical states.

2. A space of dynamical states.

3. A constitutive law that associates a dynamical state with a kinematical state, if only an infinitesimal one.

4. A set of equations that defines the equilibrium state in the case of statics or the time evolution of the kinematical state in the case of dynamics.

5. A first principle for generating the equations of equilibrium or motion.

a. Kinematical state spaces. – Typically, in theoretical mechanics, the state of a mechanical system is defined by position and velocity, in some general sense of the terms. For a single point moving in space, the position is simply that point at each instant of time, so the time sequence of positions becomes a curve in space. When that curve is assumed to be differentiable, the velocity at each point becomes the tangent vector to the curve that represents the time derivative of the position at that point. Hence, the state of the "system" (which is merely that one moving point) at each point in time is the triple that consists of the time point, the point in the ambient space that defines its position, and the associated velocity at that point in time. Any changes in the state of the system beyond simple inertia are generally attributed to the interaction of the point with its environment, such as gravity and electromagnetic fields, possible friction and viscous drag, or what we shall be dealing with mostly in this discussion, namely, *constraints* on the motion. Some examples of constraint are when the position of the point is constrained to remain along a curve or surface, or the velocity is constrained to some (typically algebraic) subspace of the tangent space at each point. The (kinematical) state of motion of the point can then become a point in the tangent bundle $T(M)$ to the ambient space of motion, which is a differentiable manifold M. If one is looking ahead to continuum mechanics, it can also be more useful to think of the kinematical state of the moving point at point in time as being a *1-jet of a differentiable curve in M* ([1]). That is, the equivalence class of differentiable curves through that point at that time that have the same tangent vector. That is also how tangent vectors are typically defined, but the difference between 1-jets and tangent vectors will become more noticeable when the dimensionless point is replaced with a spatially-extended region of space.

At a slightly higher level of complexity, one has systems that are composed of a finite set of distinct points that move in the ambient space M according an interaction model that accounts for not only the interaction of each point with its "external" environment, but also the mutual interactions of the various points with each other. The state of such a system is typically modelled by a Cartesian product of the positions and velocities of each individual point in the set. In such a case, the Cartesian product of the positions is more commonly referred to as the *configuration* of the system and the manifold that they define in the process, namely, $M \times \ldots \times M$, is its *configuration manifold*. If the dimension of the ambient space of motion is n and the number of points is N then the dimension of the configuration manifold will be nN. The motion of the configuration is then represented by N curves in the ambient space, which might very well intersect, such as in the case of collisions. The velocity of the moving configuration at any time will then consist of N tangent vectors at not-necessarily-distinct points in the ambient space. Hence, the kinematical state space in the case of free motion (i.e., in the absence of the interaction model or constraints) can be regarded as either the tangent bundle to the configuration manifold or the manifold of 1-jets of differentiable curves in the configuration manifold. As for the interaction model, we shall return to that in our discussion of equations of motion.

If one prefers to regard point-like matter as merely a simplifying approximation to extended matter then one must eventually address the extension of the preceding definitions to objects that move in an ambient space and have more dimensions to their spatial extent than none at all. Here, one must distinguish between a "prototype" for the object in question and the way that is "embedded" in the ambient space.

Previously, one could think of the prototype for a single point as simply a set with only one element in the case of statics or a timeline in the case of dynamics. The embedding of the point in the ambient space is simply the association of each time point with a spatial point in the case of

([1]) We shall be using only the most elementary concepts from the theory of jet manifolds, but those who wish to learn more about the subject can confer the book by Saunders [**14**].

a single point or a set of N spatial points in the case of N points, which can also be viewed as a point in the Cartesian product space, i.e., a point in the configuration manifold. When the material object is spatially extended, one generally needs not only a higher-dimensional space than a timeline in order to represent the prototype of the object, but one also generally needs a different space for each class of object. A particularly useful class of object is defined by topological equivalence, so one can regard all invertible, differentiable deformations of the object that have differentiable inverses (i.e., *diffeomorphisms* of the object) as "equivalent." For instance, the prototype space might be a sphere or ball, a solid or hollow cylinder, torus, cone, or any of the other shapes that one might encounter for real-world material objects. For the sake of simplicity in one's basic definitions, one might restrict a prototype space for the points of an object to be an open or closed subset $\mathcal{O}$ of the p-dimensional vector space $\mathbb{R}^p$ that shall be thought of as the *parameter space* of the object. It is most often defined by the level hypersurface of some algebraic function (or system of such equations) on $\mathbb{R}^p$ or the points that are obtained by turning that algebraic equation (or system) into an inequality. The parameter space might or might not include the time (or perhaps *proper time*) parameter that defines the curves depending upon whether one is dealing with dynamics or statics, respectively.

If the prototype object is a subset $\mathcal{O}$ in the parameter space $\mathbb{R}^p$ then an *embedding* of it in the ambient space M will be a differentiable map $x : \mathcal{O} \to M$, $(u^1, \ldots, u^p) \mapsto x\,(u^a)$ that is injective (i.e., one-to-one), so it does not intersect itself, and the map of $\mathcal{O}$ to its image $x\,(\mathcal{O})$ is a homeomorphism (i.e., topological equivalence). A consequence of this definition is that the differential map $dx|_u : T_u\mathcal{O} \to T_{x(u)}M$, $\mathbf{X} \mapsto dx|_u(\mathbf{X})$ will be a linear injection for each u . Granted, in the case of non-relativistic motion, one often encounters differentiable maps that are *immersions*, so the image of x will intersect itself in space, even though the differential maps $dx|_u$ are still injections. Examples of such things are figure eights, or circular or orbital motion, where the same set of points is traversed repeatedly. However, if one adds the time dimension to one's configuration manifold then one can eliminate the self-intersections by saying that the self-intersection involves spatial points that are associated with different time-points. Once again, this is another good reason to use manifolds of 1-jets of embeddings of $\mathcal{O}$ in M, since $\mathcal{O}$ will already include the time parameter in the case of dynamics.

If one has a local coordinate chart (U, x^i) on an open subset U in M then each point $x \in U$ can be represented by an n-tuple of real numbers $(x^1, \ldots, x^n)$ that define its *coordinates* with respect to that choice of chart. A curve $x : \mathbb{R} \to M$, $t \mapsto x(t)$ that passes through U can then be represented by the n coordinates $x^i\,(t)$ of each point, which is often represented by physicists and engineers as a system of n equations of the form:

$$x^i = x^i\,(t)\,, \qquad i = 1, \ldots, n\,. \tag{1.1}$$

In the case of a system of N distinct points $\{x_1, \ldots, x_N\}$ in M, one must generally define N local coordinate systems $(U_a,\ x_a^i,\ a = 1, \ldots, N)$ about each point, so a curve in $M \times \ldots \times M$ will become an ordered N-tuple $(x_1(t), \ldots, x_N(t))$ of points that are parameterized by t, which will then give rise to a system of nN equations of the form:

$$x_a^i = x_a^i(t), \qquad i = 1, \ldots, n, \qquad a = 1, \ldots, N. \tag{1.2}$$

When one goes to extended matter, in effect, the index a becomes another parameter in the parameter space of $\mathcal{O}$. Indeed, that is the soul of the mathematical notion of a *generalized Cartesian product*. Namely, if one thinks of the parameter space for N points in M as being simply the discrete index set for the values of a, namely, $\Pi = \{1, \ldots, N\}$ then a "point" in $M \times \ldots \times M$ can also be regarded as a function $x : \Pi \to M,\ a \mapsto x(a) = x_a$; in effect, the difference between $x(a)$ and x_a is merely a change of notation. The generalization of this from a finite set Π to an infinite one. such as $\mathcal{O}$, is then quite direct. That is, when the elements of Π are real numbers or p-tuples of real numbers, rather than write the images of the function x for each u in Π with real indices, such as x_u , one writes them as functions $x(u)$ of the continuous parameter u .

The local formulations of velocity in the three cases that have been described up to now – namely, a single point, N points, and an extended object – then become the systems of equations:

$$v^i(t) = \frac{dx^i}{dt}, \qquad i = 1, \ldots, n, \tag{1.3}$$

$$v_a^i(t) = \frac{dx_a^i}{dt}, \qquad i = 1, \ldots, n,\ a = 1, \ldots, N, \tag{1.4}$$

$$v^i(t, u^1, \ldots, u^{p-1}) = \frac{\partial x^i}{\partial t}(t, u^a), \qquad i = 1, \ldots, n,\ a = 1, \ldots, N, \tag{1.5}$$

respectively.

In the last case, one sees that the velocity becomes a time-varying vector field on the image of x in M, such as the flow velocity vector field of a fluid whose motion is not steady. Indeed, one can consider the other partial derivatives with respect to the parameters u^a as defining tangent vector fields to the curves that are obtained by holding all other parameters except for one constant. Such curves essentially define the "shape" of the image of the embedding of $\mathcal{O}$. Note that when time is included in the parameters, the number of shape parameters will be $p - 1$, which will define the spatial dimensions of the object.

A local coordinate chart (U, x^i) on M also defines a local coordinate chart on the portion $T(U)$ of its tangent bundle $T(M)$ that sits over U, namely, (x^i, v^i) . It also defines a *natural* frame field $\{\partial_i ,\ i = 1, \ldots, n\}$, where the basic tangent vectors ∂_i represent the directional derivatives in the directions of each coordinate line, so they are simply the partial derivative operators:

$$\partial_i = \frac{\partial}{\partial x^i}, \qquad i = 1, \ldots, n. \tag{1.6}$$

One can then locally express any tangent vector **v** at a point $x \in M$ as the linear combination:

$$\mathbf{v} = v^i \frac{\partial}{\partial x^i}, \tag{1.7}$$

in which the Einstein summation convention is assumed. In the case of a differentiable curve $x(t)$ in M, the vector field $\mathbf{v}(t)$ along the curve will have local components that are also functions of time $v^i(t)$. Therefore, the tangent vector **v** is associated with the directional derivative operator in the direction that **v** points to.

If we denote the manifold of all 1-jets of differentiable curves in M by $J^1(\mathbb{R}; M)$ then a local coordinate chart (U, x^i) on M will define a local chart on $J^1(\mathbb{R}; U)$ by way of (t, x^i, v^i). That makes it clear that the main difference between $J^1(\mathbb{R}; M)$ and $T(M)$ is only the addition of the time dimension. If the points of $\mathcal{O}$ in $\mathbb{R}^p$ are parameterized by u^a, $a = 1, \ldots, p$, which might include t, and U includes points in the image of the embedding of $\mathcal{O}$ by x then a local chart for $J^1(\mathcal{O}; M)$ will take the form (u^a, x^i, v^i_a).

One can define higher-order jets, which will then correspond to higher-order kinematical states. In the following discussion, the only extension that will be used, though, is the next one, namely, $J^2(\mathbb{R}; M)$. A point in that manifold – namely, a 2-jet j^2x of a twice-differentiable curve in M – will then be an equivalence class of curves through a point in M that have not only the same point and the same velocity in common, but also the same acceleration at that point. A local coordinate chart for $J^2(\mathbb{R}; M)$ will then look like (t, x^i, v^i, a^i). For a curve $x(t)$, and coordinates a^i will relate to the local components of the acceleration of $x(t)$:

$$a^i(t) = \frac{dv^i}{dt} = \frac{d^2x^i}{dt^2}. \tag{1.8}$$

We need to address a subtle aspect of the jet formulation of kinematical states that take the form of their integrability. Basically, it comes down to the difference between the *numbers* v^i or a^i and the *functions of time* $v^i(t)$ or $a^i(t)$, resp. When the functions $v^i(t)$ or $a^i(t)$ have the property that (1.3) is true in the one case, while (1.8) is true in the other, one says that they are *integrable*. An example of how the components of velocity might not be integrable in a common physical situation is the transformation of velocity to a rotation reference frame whose angular velocity with respect to some "inertial" frame might be represented by the antisymmetric 3×3 matrix of time-varying real functions $\omega^i_j(t)$, for which the component of velocity $v^i\,(t)$ with respect to the rotating frame will take the form:

$$v^i\,(t) = \frac{dx^i}{dt} + \omega^i_j\, x^j\,. \tag{1.9}$$

When kinematical states are described by points in a jet manifold $J^k\,(\mathcal{O};\, M)$, one can define a motion as a section of the *source projection* of $J^k\,(\mathcal{O};\, M)$ onto $\mathcal{O}$, which takes the k-jet $j^k_u x$ in $J^k(\mathcal{O};\, M)$ to the parameter point u in $\mathcal{O}$. A *section* $s : \mathcal{O} \to J^k\,(\mathcal{O};\, M)$ of that projection will then take each u in O to a k-jet $s\,(u) = j^k_u x$ in such a way that $j^k_u x$ will go back to u under the source projection. In the elementary case of a 1-jet of a curve, where $\mathcal{O} = \mathbb{R}$, so $u = t$, a section of the source projection will be a curve $s\,(t)$ whose local coordinate form will be:

$$s\,(t) = (t,\, x^i\,(t),\, v^i\,(t))\,. \tag{1.10}$$

The question of integrability is therefore something that first shows up when one considers sections of the source projection. Namely, one first defines the *1-jet prolongation* of the differentiable curve $x\,(t)$, which locally looks like:

$$j^1x\,(t) = (t,\, x^i\,(t),\, \dot{x}^i(t)\,)\,, \tag{1.11}$$

in which the dot over a variable denotes the time derivative.

One then says that a section $s\,(t)$ of the source projection $J^1\,(\mathcal{O};\, M) \to \mathcal{O}$ is *integrable* iff:

$$s\,(t) = j^1x\,(t)\,, \tag{1.12}$$

which locally becomes the condition (1.3).

When one goes to 2-jets, one sees that one has to differentiate twice to get the 2-jet prolongation of $x\,(t)$:

$$j^2x\,(t) = (t,\, x^i\,(t),\, \dot{x}^i(t), \ddot{x}(t))\,)\,, \tag{1.13}$$

and the condition for integrability is the predictable extension of (1.12):

$$s\,(t) = j^2x\,(t)\,. \tag{1.14}$$

However, it is entirely possible that one might have a section $s\,(t)$ in the case of 2-jets for which, one might have $v^i = \dot{x}^i$, but not $a^i = \dot{v}^i$, which means that there is more complexity to the integrability of sections when one has more than one derivative to consider.

b. Dynamical states. – Although there is a kind of mathematical duality at work in the relationship between kinematics and dynamics, nonetheless, the difference between them is physically essential, because the association of a dynamical state with a kinematical one has an unavoidably empirical contribution to it that defines the point at which one stops dealing with purely mathematical concepts and specifies a particular mechanical model. That association comes about by way of a mechanical constitutive law, which will be discussed shortly, but first we need to specify what goes into a more general dynamical state.

Whereas kinematics seeks only to *describe* motion mathematically by means of geometric and analytical concepts, dynamics seeks to *explain* that motion. For instance, in Newton's law of inertia, if one interprets the phrase "the state of an object in motion" as meaning its kinematical state then saying that it will remain in that state "unless acted upon by an external force" attempts to explain why the state of motion changed by introducing the dynamical concept of force. Actually, since the phrase "the state of an object in motion" is open to other interpretations, such as the difference between translational and rotational motion or interpreting the state in terms of linear or angular momentum, resp., the concept of "force" becomes just as ambiguous, since the same thing would be true for torque. Thus, one usually deals with generalized momenta and generalized forces.

Hence, one must proceed more heuristically than rigorously at a time like that. Some typical dynamical concepts are mass or moment of inertia, which describes the degree to which an object resists acceleration when acted upon by a force or torque (which is actually as close to a definition of inertia as Newtonian mechanics ever gets), the aforementioned linear or angular momentum, and translational or rotational kinetic energy. Similarly, potential energy – if it exists – is a dynamical concept.

Although from now on, the definitions will be made for the translational motion of point-like matter, they can be extended to rotational motion and extended matter. The reason for the restriction is simply based upon the limitations of the discussions of mechanics in the articles that follow.

In the case of the (non-relativistic) motion of a point in a space M along a differentiable curve $x(t)$, mass becomes simply a function of time $m(t)$ that allows one to associate each $m(t)$ with the corresponding position $x(t)$ of the point. In many cases, the mass is also assumed to be constant in time, but the most important counterexample is given by jet propulsion, for which the thrust is proportional to the mass flow rate $\dot{m}$.

The linear momentum that is associated with the motion of the mass along the curve is then proportional to the *covelocity 1-form:*

$$v = v_i \, dx^i, \qquad v_i = g_{ij} \, v^j, \tag{1.15}$$

in which a metric:

$$g = g_{ij}(x) \, dx^i \, dx^j \tag{1.16}$$

has been introduced into the tangent bundle $T(M)$ in order to associate tangent vectors, such as velocity, with covectors, such as covelocity. Of course, in most cases of non-relativistic mechanics, the metric is simply the Euclidian one, for which $g_{ij}(x) = \delta_{ij}$. Since that will not change the numerical values of contravariant components when they become covariant components, one is often encouraged to ignore the distinction between them, but that view of geometry has its flaws, even in the context of non-relativistic mechanics. (That will become obvious in the case of constrained motion.)

The linear momentum 1-form p is then defined by simply:

$$p = m\,v \qquad (p_i = m\,v_i = m\,g_{ij}\,v^j)\ . \tag{1.17}$$

The kinetic energy of the moving mass is then:

$$T = \tfrac{1}{2} mg\,(\mathbf{v},\mathbf{v}) = \tfrac{1}{2} m\,g_{ij}\,v^i v^j\ , \tag{1.18}$$

and for the Euclidian metric that reduces to the usual $\frac{1}{2}mv^2$. One sees that the duality between the kinematical concept of velocity and the dynamical concept of linear momentum can be expressed by noting that:

$$2T = p\,(\mathbf{v}) = p_i\,v^i\ . \tag{1.19}$$

That is, the natural bilinear pairing of a linear functional with a vector that evaluates the functional on the vector will give something that is essentially a form of energy. Actually, the number $2T$ is also what many of the following articles are calling *vis viva* ("force of life"), but that term has long since gone out of favor.

If one represents force as a 1-form:

$$F = F_i\,dx^i \tag{1.20}$$

then we will see an immediate difference between force and linear momentum that comes from the fact that whereas momentum is non-zero only where mass and velocity are non-zero, forces can be global, such as gravitation or electromagnetism, or localized to the mass, such as the force that one applies when pushing or pulling an object.

When one evaluates the force 1-form on the velocity vector, what one will get is the *kinematical power* that is being delivered to or dissipated from the moving mass:

$$P\,(t) = F\,(\mathbf{v}) = F_i\,v^i\ . \tag{1.21}$$

It is non-zero only where the velocity is non-zero, even when F is a global force field ([1]).

When $\mathbf{v}$ is integrable (so $v^i = dx^i / dt$), one sees that:

$$P\,dt = F. \tag{1.22}$$

Hence, when one integrates the 1-form F along a curve segment $\gamma(t)$ in M that starts at time t_0 and ends at time t_1, the resulting number:

([1]) Of course, there are some non-kinematical forms of power, such as the heat that is dissipated by a resistor while current is flowing through it. However, that heat relates to a "macrostate" of the collective *motion* of many electrons in the resistor that are individually interacting with the electric fields of the atomic ions in the crystal lattices.

$$W[\gamma] = \int_\gamma F = \int_{t_0}^{t_1} F_i(x(t))\, v^i(t)\, dt \tag{1.23}$$

will represent the *work done by F on m along γ.*

The force F is said to be *conservative* iff it is an exact (also called totally or completely integrable) 1-form, so there will exist some (non-unique) differentiable *potential function U* that makes:

$$F = -\,dU\,. \tag{1.24}$$

The negative sign is introduced, by convention, in order to make the potential energy drop in the direction of force, such as what one expects of a falling object.

One then sees that from Stokes's theorem:

$$W[\gamma] = -\int_\gamma dU = -\,[U(x(t_1)) - U(x(t_0))] = U(x(t_0)) - U(x(t_1))\,,$$

which says that the work done by F on m will be independent of the path taken between the starting point $x(t_0)$ and the end point $x(t_1)$.

When a potential function exists for F, one can define the *total energy* of the motion of m by adding the kinetic and potential energy at each point along the curve $x(t)$ of its motion:

$$E(t) = T(t) + U(t)\,. \tag{1.25}$$

That explains the choice of the word "conservative" for such a force, because if one starts with Newton's second law of motion ($F = m\,a$), evaluates it on $\mathbf{v}$ to get the power equation:

$$F(\mathbf{v}) = m\,a\,(\mathbf{v}),$$

and multiplies by dt then one will get:

$$F(\mathbf{v})\,dt = F = m\,a\,(\mathbf{v})\,dt = \tfrac{1}{2}m\,dv^2\,.$$

Under the assumption that m is constant in time and $F = -\,dU$, one will see that:

$$-\,dU = dT, \qquad \text{i.e.,} \qquad dE = 0.$$

Hence, the motion of a mass under a conservative force will satisfy the conservation of total energy.

More generally, when one integrates both sides of Newton's second law along a curve segment γ, one will get:

$$W[\gamma] = \int_\gamma ma = \int_{t_0}^{t_1} ma(\mathbf{v}(t))\,dt = \int_{v_0^2}^{v_1^2} \tfrac{1}{2}m\,dv^2 = \Delta T\,, \tag{1.26}$$

which is the *work-kinetic energy theorem; viz., that the total work done on the mass m along the curve segment* γ *is equal to the increase in its kinetic energy.* Hence, that also includes work done against friction or viscosity.

c. Constitutive laws. – Typically, the components of dynamical variables, such as F and p depend upon time, position, and velocity:

$$F_i = F_i(t, x^j, v^j), \qquad p_i = p_i(t, x^j, v^j) . \tag{1.27}$$

For instance, an electrostatic force might be due to a charge that changes in time, such as the charge on a capacitor plate or the globe on a Van de Graaf generator, as well as the position in space that another charge is located. Forces of viscous drag typically depend upon velocity, in addition, as well as the Lorentz force on an electric charge that moves in a magnetic field. From the form of the association (1.17), one sees that momentum can depend upon all three sets of variables, since m might vary in time, while g might vary with position, or even time.

From the form of the relationships in (1.27), one sees that it is most natural to define force and momentum to be 1-forms on the manifold $J^1(\mathbb{R}; M)$:

$$F = F_i(t, x^j, v^j)\, dx^i, \qquad p = p_i(t, x^j, v^j)\, dx^i . \tag{1.28}$$

However, if one wished to give both 1-forms the same units (of energy) then it would actually be more natural to make:

$$p = p_i(t, x^j, v^j)\, dv^i . \tag{1.29}$$

When the functions F_i and p_i are time-independent, one can also regard the force and momentum 1-forms as being defined on the cotangent bundle $T^*(M)$. Typically, if (U, x^i) is a coordinate chart on an open subset U in M then the local coordinates of $T^*(U)$ are usually written in the form (x^i, p_i). One must then note that the 1-form $p_i\, dx^i$ is defined globally on $T^*(U)$ [and in fact, $T^*(M)$ itself], and not merely along the curve of motion of some point-like mass. This approach to analytical mechanics (really, its Hamiltonian formulation) is preferred by the "symplectic" school of mechanics, but one finds that although it is an elegant way to formulate the mechanics of point-masses that move under the action of conservative forces, when one goes to extended matter (except for incompressible fluids) or non-conservative forces, the formulation becomes increasingly abstruse and mathematically axiomatic.

One can combine all of the dynamical 1-forms above that have the units of energy into a *fundamental 1-form* on the jet manifold $J^1(\mathbb{R}; M)$ (1):

(1) The author has pursued this approach to mechanics in a number of articles [**15**]. It generalizes the calculus of variations by starting with the first variation functional as the fundamental concept, instead of the action functional, which amounts to integrating the virtual work along a curve.

$$\phi = P\,dt + F_i\,dx^i + p_i\,dv^i. \tag{1.30}$$

d. Equations of motion. – In their most general form for the purposes of point mechanics, equations of motion are typically systems of ordinary differential equations whose solutions will be the coordinates $x^i(t)$ of the position of the moving object as a function of time. In order for them to be well-determined, the number of equations must be equal to the number of independent variables. The order of the equations (i.e., the highest-order time derivatives that appear in them) generally depends upon the order of the kinematical state that one is determining. For instance, when the kinematical state is first-order (i.e., position and velocity), the system of differential equations will typically be second-order.

Typically, differential equations of order higher than two have anomalies associated with causality, though. An example of that is the Lorentz-Dirac equations for the motion of an electric charge in an electromagnetic field, when corrected for the radiation reaction ("radiation damping") that accompanies the acceleration of the charge. The radiation damping term raises the order of the equations of motion to three, and some of the solutions that the equations can admit include charges anticipating the force that acts upon them or accelerating in the absence of forces.

The most common type of problem that one poses for in a problem of motion is the *initial-value problem*. In that problem, one is given the initial kinematical state (such as initial position and velocity) and then looks for the kinematical state at any later time point. However, it is also common to pose a *two-point boundary-value problem*. In that case, one is given the initial and final position and looks for a curve that will connect them, such as the trajectory of a ballistic missile. In such a case, one finds that the existence of a solution is not as easy to guarantee as it is in the initial-value problem, where the continuous differentiability of the functions involved is enough to ensure a unique solution, at least for a short enough time interval. Depending upon the maximum range of a missile, there might be no solution, one solution, or two solutions to the two-point boundary-value problem.

We shall now go on to the deepest problem of mechanics, which defines most of the discussions in the selected articles. That is the problem of discerning first principles in nature that will imply the equations of motion as their consequences.

2. First principles of motion. – A first principle of physics is essential an axiom that one forms from the otherwise-undefinable logical primitives that one simply agrees to start with, such as the concepts of space, time, and the most elementary properties of matter, such as mass, charge, and spin. Similarly, concepts such as force also have that sort of primitive character that makes it impossible to define them in terms of anything more fundamental.

The best sort of first principle for the motion of material objects in space is one that will imply a set of equations of motion as a consequence in some way. Ideally, it should apply to all of the dynamical scenarios that one might encounter in nature, but in some cases although the models are limited in scope, such as dynamical models that imply the conservation of energy, that scope is nonetheless sufficiently broad as to make the first principle worthy of consideration.

We shall now examine some of the more established first principles and how they allow one to define equations of motion.

a. Newton's laws. – The first truly fundamental set of first principles for motion were due to Newton, who proposed the following three, which are presumably known to anyone who would be reading this, but we shall restate them in a slightly more precise form, in order to discuss them:

1. The law of inertia: A system in a state of motion will remain in that state of motion unless acted upon by an external force.

2. The acceleration of a material body is proportional to the resultant of the external forces that act upon it, and the proportionality factor is one over its inertial mass.

3. To every action there is an equal and opposite reaction.

One thing that must be pointed out about these laws immediately is that the terms that are introduced are quite ambiguous in several cases. In particular, the concepts of "state of motion," "action," and "reaction" are open to interpretation. If one interprets the state of a system's motion as its total momentum then not only does one have to specify whether that means linear or angular momentum, as well as specifying whether one truly means force or torque, but that would also make the first law a corollary to the second law, and thus somewhat redundant.

It is the third law that is the most open to interpretation. One can see that the concepts of "action" and "reaction" are clearly not as broad in scope as those of "cause" and "effect," as in the example of a geometric progression of dominos. If the "action" is tipping over the first one then the "reaction" might be that an increasing number of dominos tip over in succession, which is neither equal nor opposite to the action in that sense. Indeed, it is intriguing that nowadays the preferred explanation for jet propulsion is that it is due to Newton's third law when one interprets the action as the linear momentum of the exhaust gases that escape from the rocket nozzle or jet engine, while the reaction is the linear momentum of the vehicle from which the gases are escaping. Nonetheless, in 1920, when the American rocket pioneer Robert Goddard had been interviewed by the New York Times and had been discussing the possibility of exo-atmospheric rocket flight, his comments provoked an indignant reply from an editor who wrote an op-ed piece for that distinguished newspaper entitled "A Severe Strain on Credulity." In it, he said of Goddard (who had a Ph. D. in physics and had been a professor at Princeton) "Of course, he only seems to lack the knowledge that is ladled out daily in high schools," as in Newton's laws of motion. To the editor, the "action" was still the escaping exhaust gases, but the "reaction" took the form of the collision of the exhaust gases with the molecules of the atmosphere, which is what he believed was the source of the thrust. Consequently, without an atmosphere to react to the gases, he thought that there could be no thrust. Amusingly, many years later, when Neil Armstrong first stepped down on the lunar surface, the New York Times felt compelled to publish a belated "retraction" to the earlier commentary, which they reproduced with the added comment "Apparently, Newton was right." More precisely, the editor was not contradicting Newton's third law, but merely misinterpreting it.

At any rate, it is the second law that is practically a system of differential equations to begin with, since one can represent the acceleration as the second derivative with respect to time of the position. In its simplest form, for a single mass m that moves along a curve $x(t)$ in space, the second law becomes the vector equation:

$$\mathbf{F} = m\frac{d^2x}{dt^2}, \tag{2.1}$$

in which **F** is the resultant of the forces that act upon m. That can also be expressed in component form when one chooses a local frame field about the points of the curve, such as the natural frame field of the coordinate system that allows one to express the points $x(t)$ as coordinates $x^i(t)$:

$$F^i = m\frac{d^2x^i}{dt^2}. \tag{2.2}$$

In this form, it becomes clear that if the dimension of the space M is n (which is typically 1, 2, or 3) then this will represent a system of n second-order ordinary differential equations for the n coordinate functions $x^i(t)$. A unique solution to the initial-value problem will then require $2n$ independent pieces of initial data, namely, the n initial coordinates $x_0^i = x^i(0)$ and the n initial velocities $\dot{x}_0^i = \dot{x}^i(0)$.

If one interprets the resultant force **F** as the resultant torque $\boldsymbol{\tau}$ in the case of rotational motion, so the mass becomes the moment of inertia I and the acceleration is angular acceleration, which is $\boldsymbol{\alpha} = d^2\theta / dt^2$ in the simple case of planar rotations, then the second law will take the form:

$$\boldsymbol{\tau} = I\frac{d^2\theta}{dt^2}. \tag{2.3}$$

Of course, there is a crucial difference between the case of planar rotation and spatial rotation that is based in the fact that when one is dealing with spatial translations or planar rotations, the group of motions that acts upon space is Abelian, so the order in which the motions are performed is irrelevant, but that is no longer the case for spatial rotations, only some of which commute.

The third law becomes more meaningful for systems of more than one interacting body, when it typically pertains to the internal forces of interaction between masses and not to the external forces that act upon them. For instance, a force of tension or compression in an elastic link that connects two masses will be equal and opposite at each mass, which has the subtle corollary that the force of tension (or compression) in that link cannot be represented by a *unique* vector, except up to sign. If one wants to describe the tension or compression that acts upon an intermediate point of the link (or really, its cross-section) then one must imagine cutting a section through the link at that point and assigning equal and opposite force vectors on its opposing faces.

Note that one cannot think of the acceleration of a mass as a *reaction* to the applied forces and still be consistent with Newton's third law, since although the length of the acceleration vector is proportional to the length of the force vector, nonetheless, they both point in the same direction. That subtle fact will become more crucial when we discuss d'Alembert's principle.

b. Conservation laws, balance principles. – If one assumes that the mass of a point-like object is constant in time then one can rewrite the right-hand side of Newton's second law (2.1) as:

$$m\frac{d^2x}{dt^2} = m\frac{d\mathbf{v}}{dt} = \frac{d(m\,\mathbf{v})}{dt} = \frac{d\mathbf{p}}{dt}\,.$$

Hence, the second law can be expressed in the form:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}\,. \tag{2.4}$$

That can be regarded as the principle of the *balance of (linear) momentum*. When $\mathbf{F} = 0$, it becomes the simple statement that $\mathbf{p}$ is constant in time, which is then the principle of *conservation of momentum*. Analogous statements apply to torque and angular momentum when the moment of inertia is constant.

Note that when one starts from (2.4) and drops the assumption that $m(t)$ is constant, one will get:

$$\mathbf{F} = \frac{dm}{dt}\mathbf{v} + m\frac{d\mathbf{v}}{dt}\,. \tag{2.5}$$

The extra term essentially accounts for the "thrust" that is produced by the expulsion of mass, so although one can think of (2.4) as being more general in scope than (2.1), in effect, if one includes the thrust in the resultant of the external forces then that will not necessarily be true.

If one forms the power equation that (2.4) implies, while assuming that the mass is constant and that the spatial metric g does not change along the curve $x(t)$ then one will get:

$$P = g(\mathbf{F}, \mathbf{v}) = g\left(\frac{d\mathbf{p}}{dt}, \mathbf{v}\right) = m\,g\left(\frac{d\mathbf{v}}{dt}, \mathbf{v}\right) = \frac{1}{2}m\frac{d}{dt}g(\mathbf{v}, \mathbf{v}) = \frac{dT}{dt}\,,$$

which can be expressed in terms of 1-forms as:

$$F(\mathbf{v})\,dt = dT\,, \tag{2.6}$$

although the two sides are defined only along the curve of motion. Integrating both sides along the second γ of that curve that goes from t_0 to t_1 will give:

$$W[\gamma] = \int_{t_0}^{t_0} F(\mathbf{v})\,dt = \int_\gamma F = \Delta T, \tag{2.7}$$

which is the work-kinetic energy theorem again.

When the force 1-form F is conservative ($F = - dU$), one will get:

$$- dU(\mathbf{v}) = -\frac{dU}{dt} = \frac{dT}{dt},$$

or:

$$\frac{dE}{dt} = 0 \qquad (E \equiv T + U), \tag{2.8}$$

which amounts to the principle of the *conservation of energy*. Of course, since not all forces are conservative and mass is not always constant (and indeed, momentum does not always take the "convective" form $m\,\mathbf{v}$), one sees that the work-kinetic energy theorem is more general in scope than conservation of energy, which is not as general in scope as the balance of momentum.

c. Virtual work. – If one represents the force 1-form in its component form for a natural coframe field as:

$$F = F_i\, dx^i \tag{2.9}$$

then one can also think of the 1-form F as an infinitesimal amount of work that is done by the various force components F_i along the infinitesimal displacements dx^i. Similarly, if $\delta\mathbf{x}(t)$ is any vector field along a curve $x(t)$ then the evaluation of the 1-form F on that vector field will take the form:

$$\delta W \equiv F(\delta\mathbf{x}) = F_i\, \delta x^i. \tag{2.10}$$

Since, presumably, the only "actual" displacements of the points of the curve point in the direction of the velocity vector at each point along the way, one might think of the vector field $\delta\mathbf{x}$ as a *virtual displacement* of the curve when it is transverse (i.e., non-collinear) to the velocity, while the function of time $\delta W(t)$ is the *virtual work* that is done by the force F along that virtual displacement. Note that only the part of F that is transverse to the curve $x(t)$ will contribute to the virtual work it does along a virtual displacement.

The *principle of virtual work* is a first principle that first pertains to only the equilibrium state of a system of interacting masses. It comes down to the idea that a linear functional (covector, 1-form) on a vector space vanishes iff its evaluation on each vector in that space vanishes. Hence, the virtual work that is done by F on any virtual displacement will vanish iff $F = 0$.

Of course, the usual discussion of virtual work also includes the idea that the system of masses is subject to some set of constraints, which then has the effect of reducing the possible virtual displacements to only the ones that are consistent with the constraints. That, in turn, has the effect of defining some distinguished subsets of the tangent spaces along the curve, such as linear subspaces in the case of linear constraints. However, since the main concern of this collection of articles is the theory of constrained motion, we shall return to that aspect of the principle of virtual work in due time.

Note that in many of the following articles, the principle of virtual work is referred to as the principle of "virtual velocities," since in effect, the virtual displacements are like the restrictions of the velocity vector fields of finite deformations of the original curve (viz., differentiable homotopies) to the original curve. Hence, they are the infinitesimal generators of one-parameter families of finite deformations of the original curve (at least, for holonomic constraints).

d. D'Alembert's principle. – As was just pointed out, the principle of virtual work is a first principle for the equations of statics since it characterizes the equilibrium state. Namely, equilibrium is characterized by the vanishing of the resultant of the external forces on a system and the vanishing of the resultant moment of all forces that act upon the individual masses with respect to some conveniently-chosen reference point O :

$$\mathbf{M} = \sum_{a=1}^{N} \mathbf{M}_a = \sum_{a=1}^{N} \mathbf{r}_a \times \mathbf{F}_a \,, \tag{2.11}$$

where the summation extends over all masses upon which the forces act and $\mathbf{r}_a$ is the displacement vector that takes one from O to the position of the mass m_a .

The equilibrium condition is then:

$$\sum_{a=1}^{N} \mathbf{F}_a = 0, \qquad \sum_{a=1}^{N} \mathbf{M}_a = 0. \tag{2.12}$$

Note that this implies that a state of equilibrium is not the same thing as a state of *rest*, since the vanishing of the accelerations still allows for uniform motions of constant velocity, while rest would imply the vanishing of velocity as well. However, one can say that if a system in equilibrium is in a state of rest at one point in time then it will remain at rest for all following times.

The credit for the principle of virtual work was typically given to Johann Bernoulli (1667-1748). What Jean-Baptiste de la Rond d'Alembert (1717-1783) added to the principle of virtual work was the idea that if one were to regard each $-\, m_a\, a_a$ as an *inertial force* and include them in the resultant of the forces then the principle of virtual work would also serve as a first principle of dynamics, as well as statics. In particular, the total virtual work that is done by all forces – applied and inertial – along the virtual displacements $\delta\mathbf{x}_a$ would take the form:

$$\delta W = \sum_{a=1}^{N} (F_a - m_a\, a_a)(\delta \mathbf{x}_a)\,. \tag{2.13}$$

Hence, the vanishing of this sum for all admissible $\delta\mathbf{x}_a$ would imply Newton's second law.

Let us rewrite (2.13) in the form:

$$\sum_{a=1}^{N} \frac{d}{dt}(m_a v_a)(\delta \mathbf{x}_a) = \sum_{a=1}^{N} F_a(\delta \mathbf{x}_a),$$

or

$$\sum_{a=1}^{N} \frac{d}{dt}[(m_a v_a)(\delta \mathbf{x}_a)] - \sum_{a=1}^{N} m_a v_a \left(\frac{d}{dt}\delta \mathbf{x}_a \right) = \sum_{a=1}^{N} F_a(\delta \mathbf{x}_a). \tag{2.14}$$

When we introduce the kinetic energy of each mass m_a :

$$T_a = \tfrac{1}{2} m_a \, v_a(\mathbf{v}_a) = \tfrac{1}{2} m_a \, \delta_{ij} \, v_a^i \, v_a^j, \tag{2.15}$$

we will see that:

$$\delta T_a = m_a \, v_a \, (\delta \mathbf{v}_a) = m_a \, v_a \left(\frac{d}{dt}\delta \mathbf{x}_a \right) + m_a \, v_a \, (\delta \mathbf{v}_a - \frac{d}{dt}\delta \mathbf{x}_a). \tag{2.16}$$

Hence, (2.14) can be put into the form:

$$\sum_{a=1}^{N} \frac{d}{dt}[(m_a v_a)(\delta \mathbf{x}_a)] - \delta T + \sum_{a=1}^{N} m_a v_a \left(\delta \mathbf{v}_a - \frac{d}{dt}\delta \mathbf{x}_a \right) = \sum_{a=1}^{N} F_a(\delta \mathbf{x}_a), \tag{2.17}$$

which is the general form of what Karl Heun [**9**] called the *central equation*, which was also used by Hamel [**11**]. One can regard it as an equivalent statement to the principle of virtual work, combined with d'Alembert's principle.

When the virtual displacement of each velocity $\mathbf{v}_a$ is *integrable*, so the virtual displacements $\delta\mathbf{v}_a$ are defined to make:

$$\delta \mathbf{v}_a = \frac{d}{dt}\delta \mathbf{x}_a, \tag{2.18}$$

the last term on the left-hand side of (2.17) will vanish, and the central equation will take the form:

$$\sum_{a=1}^{N} \frac{d}{dt}[(m_a v_a)(\delta \mathbf{x}_a)] - \delta T = \sum_{a=1}^{N} F_a(\delta \mathbf{x}_a). \tag{2.19}$$

In the case of a single point-mass m moving under the influence of external forces, whose result is the 1-form F, this will take the simpler form:

$$\frac{d}{dt}[(m\, v)(\delta \mathbf{x})] - \delta T = F\,(\delta \mathbf{x}), \tag{2.20}$$

or

$$\frac{d}{dt}(m\, v_i\, \delta x^i) - \delta T = F_i\, \delta x^i, \tag{2.21}$$

in component form. Typically, in the following discussions of constrained motion, we shall consider only that most elementary system, since one can basically sum over all individual contributions when there is more than one mass to consider. The main difference would be in the mutual interactions of the masses, but we shall be more concerned with the contribution to the resultant force that comes from the force of constraint.

e. Least action. – One must notice that Newton's laws of motion include an axiom (namely, the second one) that is not so much a principle of natural philosophy as a set of equations of motion in its own right. Hence, if one prefers to start with some general pattern in the motion of natural systems that does not sound as much like a quantitative statement as a qualitative one then one must look elsewhere.

Perhaps it was the emergence of the "optimist" school of philosophy (which was so mordantly satirized by Voltaire in *Candide*) that led to the emerge of optimization principles as the first principle of natural motion ([1]). The optimists felt that we live in the best of all possible worlds, so in order to make that into a more mathematical statement, one must first clarify the terms "best" and "world," if not also "possible." That is where the burgeoning mathematical discipline called the calculus of variations, which was originally due to mainly Leonhard Euler (1707-1783) and Joseph-Louis Lagrange (1736-1813), was ideally suited to the task. In particular, the latter produced the monumental two-volume work *Mécanique analytique*, which was published over the span of time from 1788 to 1789, and which set down not only his approach to the calculus of variations, but its applications to mechanics.

In effect, one considers all "possible" curves in space that connect a starting point to a final point ([2]) and associates a number with them that represents the "action" that is associated with that curve and looks for the curve (or curves) with the least action. Note that this usage of the word "action" is entirely distinct from its use in Newton's third law. Moreover, the action generally takes the form of the integral of some suitable 1-form along the possible curve. Thus, one has at least narrowed down the field of consideration to possible curves in space between two points, but there is still considerable ambiguity in the choice of suitable 1-form, i.e., the definition of the action itself.

One should first be aware that it is tempting in this modern era to first define an "infinite-dimensional manifold" whose points are the curves between two points and then regard the action functional, which associates each point in that manifold with a number, as basically a differentiable function on that manifold. The problem of finding its minimum then looks like a natural extension of the usual problem in the calculus of several variables, namely, one looks for the critical points of the function, where its differential (otherwise known as the *first variation*) vanishes, and considers the second derivatives of the function (viz., the *second variation* of the action functional) at a critical point in order to determine whether it is a minimum, maximum, inflection point, or whatever. However, although that view of the calculus of variations has a definite heuristic value

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([1]) Anyone who cannot imagine how such an abstract concept as the principle of least action could be the basis for a drama that involved kings and famous writers should read Adolph Mayer's history of the least-action principle [**16**], as well as that of Helmholtz [**17**].

([2]) Strictly speaking, that situation refers to only the "fixed endpoint" class of variational problems. One can also consider "variable endpoint" problems, which introduce "transversality" conditions, but we shall let that pass for now.

in terms of visualizing what is going on, nonetheless, except for mathematical studies of a topological or analytical nature (also known as "global analysis"), the infinite-dimensional manifold picture is not as useful in practical applications as other approaches to the topic. By contrast, the method of jet manifolds is quite natural and has the advantage that the jet manifolds are typically finite-dimensional.

The earliest form that the principle of least action took was due to Pierre Louis Moreau de Maupertuis (1698-1759). In effect, the 1-form that he used to define the integrand of the action functional was the momentum 1-form:

$$p = p_i(q^j)\, dq^i , \tag{2.22}$$

in which we have reverted to the traditional Lagrangian notation, in which the coordinates of the points in the parameter manifold $\mathcal{O}$ are denoted by q^i, instead of u^a, and referred to as *generalized coordinates*. Therefore, t might be included in the generalized coordinates. Of course, nowadays since differentiable manifolds are not always represented by embedded submanifolds in $\mathbb{R}^n$, the distinction between Cartesian coordinates and generalized (i.e., curvilinear) ones often becomes moot. However, the difference between the coordinates q^i on the parameter space and the coordinates x^i in $\mathbb{R}^n$ will become more essential when we discuss holonomic constraints.

The action functional for the curve $q(t)$ between two points $q_0 = q(t_0)$ and $q_1 = q(t_1)$ then takes the form:

$$S[q(t)] = \int_q p = \int_{t_0}^{t_1} p(\mathbf{v})\, dt = 2\int_{t_0}^{t_1} T(t)\, dt . \tag{2.23}$$

Hence, the definition of the action along the curve becomes twice the time integral of the kinetic energy of the motion of the mass along that curve. That means that the units of action in this case are those of energy times time.

In order to find the extremum of that action functional, we now revert to the slightly more modern Euler-Lagrange formulation of the problem, in which we start with a differentiable function $\mathcal{L}(t, q^i, \dot{q}^i)$ that we call the *Lagrangian* of the mechanical system. The action functional for a path will then take the form:

$$S[x(t)] = \int_{t_0}^{t_1} \mathcal{L}(t, q^i(t), \dot{q}^i(t))\, dt . \tag{2.24}$$

One can already see how the concept of jets enters into this picture quite naturally, since the Lagrangian is now clearly a function on the manifold $J^1(\mathbb{R}; M)$ of 1-jets of differentiable curves in M that we discussed above; that is:

$$\mathcal{L} = \mathcal{L}(t, q^i(t), \dot{q}^i(t)) = \mathcal{L}(j^1 q(t)) , \tag{2.25}$$

so we are also using the 1-jet prolongation $j^1q\,(t)$ of the curve $q\,(t)$ in M to a curve in $J^1(\mathbb{R}; M)$:

$$j^1q\,(t) = (t, q^i(t), \dot{q}^i(t)). \tag{2.26}$$

In order to find an extremum of this functional, one must first define its first variation δS (i.e., the differential of the function on an infinite-dimensional manifold of curves), which becomes a linear functional on variations δq (i.e., virtual displacements) of the curve that is defined by:

$$\delta S\,[\delta q] = \int_{t_0}^{t_1} d\mathcal{L}\,(\delta j^1 x)\,dt\ . \tag{2.27}$$

One first computes:

$$d\mathcal{L} = \frac{\partial \mathcal{L}}{\partial t}\,dt + \frac{\partial \mathcal{L}}{\partial q^i}\,dq^i + \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\,d\dot{q}^i . \tag{2.28}$$

The vector field $\delta j^1q\,(t)$ along the curve $j^1q\,(t)$ is defined by:

$$\delta j^1q\,(t) = \delta q^i(t)\frac{\partial}{\partial q^i} + \delta\dot{q}^i(t)\frac{\partial}{\partial \dot{q}^i} . \tag{2.29}$$

Note the absence of a component that would take the form of $\delta t\,(t)$. That would amount to including a reparameterization of the curve, along with the purely spatial deformation.

One now finds that:

$$\delta\mathcal{L} = d\mathcal{L}\,(\delta j^1 q) = \frac{\partial \mathcal{L}}{\partial q^i}\,\delta q^i + \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\,\delta\dot{q}^i . \tag{2.30}$$

In (2.29), if one assumes that the variation $\delta j^1 q$ is *integrable*, so:

$$\delta\dot{q}^i = \frac{d}{dt}\,\delta q^i , \tag{2.31}$$

then (2.30) will take the form:

$$\delta\mathcal{L} = \frac{\delta \mathcal{L}}{\delta q^i}\,\delta q^i - \frac{d}{dt}\left(p_i\,\delta q^i\right), \tag{2.32}$$

in which one has introduced the *variational derivative* of $\mathcal{L}$ and the *generalized momentum*, which are defined by:

$$\frac{\delta \mathcal{L}}{\delta q^i} = \frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} , \qquad p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}^i} , \tag{2.33}$$

respectively. This step, which is traceable to the product rule of differentiation, is typically referred to as "integration by parts," since that is what it turns into upon integration.

The first variation then takes the form:

$$\delta S\,[\delta q] = \int_{t_0}^{t_1} \frac{\delta \mathcal{L}}{\delta q^i}\,\delta q^i\, dt + \left[\,p_i\,\delta q^i\,\right]_{t_0}^{t_1} . \tag{2.34}$$

For the fixed endpoint problem, one does not vary the endpoints, so $\delta q^i\,(t_0) = \delta q^i\,(t_1) = 0$, and the term in square brackets will vanish. If the first variation is to vanish for all δq^i then that will imply that:

$$\frac{\delta \mathcal{L}}{\delta q^i} = 0, \tag{2.35}$$

which are the usual *Euler-Lagrange equations*.

If one also introduces the *generalized force* that is associated with the Lagrangian $\mathcal{L}$ by way of:

$$F_i = \frac{\partial \mathcal{L}}{\partial q^i} \tag{2.36}$$

then one can put these equations into the "generalized" Newtonian form:

$$F_i = \frac{dp_i}{dt} . \tag{2.37}$$

Hence, to some extent, the principle of least action is equivalent to Newton's second law.

To return to Maupertuis's principle, one should note that one can generally define the Lagrangian by means of kinetic energy:

$$\mathcal{L}\,(q^i, \dot{q}^i) = T\,(q^i, \dot{q}^i) = \tfrac{1}{2}\, m\, g_{ij}(q)\, \dot{q}^i\, \dot{q}^j . \tag{2.38}$$

The path of least action in this case becomes a *geodesic* (i.e., a path of least distance) through the two endpoints for the spatial metric. That is because:

$$F_i = \tfrac{1}{2}\, m\, g_{jk,i}\, \dot{q}^j\, \dot{q}^k , \qquad\qquad p_i = m\, g_{ij}\, \dot{q}^j , \tag{2.39}$$

so the Euler-Lagrange equations will take the form:

$$0 = \tfrac{1}{2}\, m\, g_{jk,i}\, \dot{q}^j\, \dot{q}^k - \frac{d}{dt}(m\, g_{ij}\, \dot{q}^j) = m[\tfrac{1}{2}\, g_{jk,i}\, \dot{q}^j\, \dot{q}^k - g_{ij,k}\, \dot{q}^j \dot{q}^k - g_{ij}\, \ddot{q}^j],$$

and after symmetrizing the second term, one will get the covariant equations:

$$0 = \ddot{q}_i + [i, jk]\dot{q}^j \dot{q}^k , \tag{2.40}$$

into which, the *Christoffel symbol of the first kind* has been introduced:

$$[i, jk] = \tfrac{1}{2}(g_{ij,k} + g_{ik,j} - g_{jk,i}) . \tag{2.41}$$

Note that equations (2.40) no longer include the mass, so they are purely kinematical equations now. However, that is an artifact of the restriction to a constant point-mass that would not apply to extended mass densities in general.

When one raises the index *i*, equations (2.40) will take the contravariant form:

$$0 = \ddot{q}^i + \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\} \dot{q}^j \dot{q}^k , \tag{2.42}$$

into which the *Christoffel symbol of the second kind* has been introduced:

$$\left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\} = \tfrac{1}{2} g^{il}(g_{lj,k} + g_{lk,j} - g_{jk,l}) . \tag{2.43}$$

Equations (2.42) are the traditional equations for geodesics of the spatial metric *g*, and play a fundamental role in Einstein's theory of gravitation.

Of course, Maupertuis's definition of the action functional involves only kinetic energy, but not potential energy. That amounts to saying that it pertains to only the free (i.e., unforced) motion of a mass, although one might say that it is constrained by the geometry that the metric *g* describes. More generally, if a conservative force $-dU$ acts upon the mass then one can define a Lagrangian of the form:

$$\mathcal{L} = T - U . \tag{2.44}$$

Since the potential function *U* is assumed to be a function of only the spatial position of the mass-point, it will not contribute to the generalized momentum of $\mathcal{L}$, but only the generalized force, which will now put the Euler-Lagrange equations into the form:

$$\frac{\delta T}{\delta q^i} = \frac{\partial U}{\partial q^i} . \tag{2.45}$$

This is an example of *Lagrange's equations of the second kind*. We shall discuss the Lagrange equations of the first and second kind in the next section on constrained motion.

One consequence of the least-action principle is that when the Lagrangian $\mathcal{L}$ does not depend upon time, the total energy of the system will be conserved, i.e., constant in time. In order to see that, one starts by looking at the total derivative of $\mathcal{L}$ along a curve $x\,(t)$:

$$\frac{d\mathcal{L}}{dt} = \frac{\partial\mathcal{L}}{\partial t} + \frac{dq^i}{dt}\frac{\partial\mathcal{L}}{\partial q^i} + \frac{d\dot{q}^i}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}^i}\,. \tag{2.46}$$

From the Euler-Lagrange equations, one can replace $\frac{\partial\mathcal{L}}{\partial q^i}$ with $\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}^i}$, which will make:

$$\frac{d\mathcal{L}}{dt} = \frac{\partial\mathcal{L}}{\partial t} + \frac{dq^i}{dt}\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}^i} + \frac{d}{dt}\frac{dq^i}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}^i} = \frac{\partial\mathcal{L}}{\partial t} + \frac{d}{dt}\left(\frac{dx^i}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}^i}\right). \tag{2.47}$$

The second term in the final expression can be rewritten in the form:

$$\frac{d}{dt}\left(\frac{dq^i}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}^i} - \mathcal{L}\right) = \frac{d}{dt}\left(p_i\,\dot{q}^i - \mathcal{L}\right). \tag{2.48}$$

However, one immediately recognizes that the final expression in parentheses defines the Legendre transformation that takes the Lagrangian to its corresponding Hamiltonian:

$$H = p_i\,\dot{q}^i - \mathcal{L}\,, \tag{2.49}$$

which also describes the total energy of the system.

Hence, (2.47) and (2.48) imply that $d\mathcal{L}\,/\,dt$ will vanish:

$$\frac{dH}{dt} = -\frac{\partial\mathcal{L}}{\partial t} = \frac{\partial H}{\partial t}\,, \tag{2.50}$$

which says that the total energy of the system is constant along any integral curve of the Euler-Lagrange equations when the Lagrangian is time-independent.

f. Hamilton's canonical equations. – Since neither p_i nor $\dot{q}^i$ are functions of q^i (for point-like motion) and $\mathcal{L}$ does not depend upon p_i, one infers from (2.49) that:

$$\frac{\partial H}{\partial q^i} = -\frac{\partial\mathcal{L}}{\partial q^i}\,, \qquad \frac{\partial H}{\partial p_i} = \dot{q}^i\,. \tag{2.51}$$

When one combines this with (2.50) and the definition (2.33) of the generalized (or conjugate momentum) p_i to $\mathcal{L}$ and the Euler-Lagrange equations, one gets the set of equations:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}\,, \qquad \frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}\,, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}\,. \tag{2.52}$$

These are Hamilton's canonical equations of motion, which are called that because they are expressed in terms of the "canonical" variables q^i and p_i .

Once again, conservation of energy follows from those equations when H is not a function of t. Furthermore:

$$\frac{dH}{dt} = \frac{dq^i}{dt}\frac{\partial H}{\partial q^i} + \frac{dp_i}{dt}\frac{\partial H}{\partial p_i} = \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial H}{\partial p_i} = 0. \tag{2.53}$$

One identifies the *Poisson bracket* of two differentiable functions in the penultimate expression, namely:

$$\{f, g\} \equiv \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}\,. \tag{2.54}$$

The vanishing of $\{H, H\}$ then follows from the antisymmetry of the bracket.

The translations that follow make little use of Poisson brackets, so we shall not pursue that aspect of Hamiltonian mechanics any further.

3. Constrained motion in general. – In many cases that occur in nature, as well as in engineering practice, the motion of the various components in a mechanical system is restricted in some way. For instance, one can consider the difference between a body falling freely and one that rolls down the side of a hill. Indeed, one might even rephrase the notion of a mass-point moving along a curve as saying that it is constrained to that curve, such as a bead that slides along a stiff wire.

There are many different types of constraints that can be imposed upon motion, and a rough classification of the most common distinctions in mechanics might take the form:

1. Linear vs. nonlinear.

2. Holonomic vs. non-holonomic.

3. Time-varying vs. time-independent.

4. Perfect vs. imperfect.

5. Two-sided vs. one-sided.

6. Regular vs. singular.

We shall clarify the meanings of those terms in due time, once we have discussed the nature of constraints as bundles of submanifolds in tangent spaces.

In order to specify a constraint on motion in a mechanical system, one typically specifies a restriction on the acceptable velocities or virtual displacements for motion in M that is represented by differentiable curves. Thus, one specifies a submanifold $\mathcal{D}_x$ of each tangent space T_xM to the configuration manifold M, which will call the *tangent constraint space* or simply *constraint space*. The dimension of the constraint space at a given point then represents the *degrees of freedom* in the motion. The disjoint union of all of the $\mathcal{D}_x$ over all of the $x \in M$ will be called the *constraint bundle* and will be denoted by $\mathcal{D}(M)$.

The most commonly introduced constraint manifolds are linear subspaces, since that is what one would get by differentiating a holonomic constraint on position (i.e., the embedding of a submanifold). Hence, if the dimension of the constraint space $\mathcal{D}_x$ is p then it can be defined by a set of $n - p$ 1-forms C^α, $\alpha = 1, \ldots, n - p$ that annihilate any vector $\mathbf{X}$ that belongs to $\mathcal{D}_x$:

$$C^\alpha(\mathbf{X}) = 0 \qquad \text{for all } \alpha. \tag{3.1}$$

If there is a local coordinate system (U, x^i) about x then the C^α can be expressed in terms of the natural coframe field dx^i that is defined by the coordinate while $\mathbf{X}$ can be expressed in terms of the natural frame field ∂_i :

$$C^\alpha = C_i^\alpha(x)\, dx^i, \qquad \mathbf{X} = X^i(x)\, \partial_i, \tag{3.2}$$

so the condition (3.1) will take the local form:

$$C_i^\alpha X^i = 0 \qquad \text{for all } \alpha. \tag{3.3}$$

Although nonlinear constraints on position are commonplace, since that basically amounts to saying that the tangent constraint space is not a linear subspace of a linear space, until the theory of relativity became common knowledge to theoretical physics, nonlinear constraints on *velocity* were regarded as having a more speculative character. (Notice the discussion of nonlinear constraints by Appell in the articles that follow.) One example of a nonlinear constraint on velocity is given by saying that it must have constant speed, such as the unit speed of an arc-length parameterization. The constraint space in each tangent space is then a sphere. In relativistic mechanics, one typically imposes a quadratic constraint on velocity that takes the form of saying that massive matter moves along curves that are parameterized by proper time, so the four-velocity vector $\mathbf{u}$ must satisfy the constraint:

$$c^2 = g\,(\mathbf{u}, \mathbf{u}) = g_{\mu\nu}\, u^\mu\, u^\nu, \tag{3.4}$$

while massless matter moves along light-like ones, for which the c^2 gets replaced by 0. Of course, in the case of non-holonomic constraints, one sees that as long as the constraints are defined by systems of linear partial differential equations, the constraints on velocity will automatically be linear.

In general, one might define $n - p$ differentiable functions ϕ^α of $(x, \mathbf{v})$ on the tangent bundle $T(M)$ to configuration space. Their differentials would then take the form:

$$d\phi^\alpha = \frac{\partial \phi^\alpha}{\partial x^i}\, dx^i + \frac{\partial \phi^\alpha}{\partial v^i}\, dv^i\,. \tag{3.5}$$

In the linear case, this would reduce to:

$$C_i^\alpha = \frac{\partial \phi^\alpha}{\partial x^i}\,, \qquad \frac{\partial \phi^\alpha}{\partial v^i} = 0. \tag{3.6}$$

However, as we shall see, that would only give one a linear holonomic constraint.

We shall deal with the cases of holonomic and non-holonomic constraints in separate sections below, and only in the context of linear constraints. Basically, the issue is one of the integrability of the constraint, in a sense that will be clarified. One finds that the maximal dimension of an integral submanifold for a constraint will be less than or equal to the number of degrees of freedom in the motion, and equality will be obtained only for holonomic constraints. Although many of the basic steps in developing the mathematics of holonomic and non-holonomic constraints are essentially the same, it is the ones that differ that deserve the most attention, since they usually represent contributions from the non-integrability of the constraint.

When the constraints depend upon time, such as when something is constrained to move on something that is moving in its own right (e.g., an ant moving on a rotating phonograph record or the Foucault pendulum, which oscillates in a plane that is fixed on a rotating surface), one calls the constraints *rheonomic*. When they are independent of time, one calls them *scleronomic*. Of course, if one is going to include time as a dimension in the configuration manifold then the distinction will become less meaningful, especially in the relativistic context, where the splitting of space-time into space and time is not as canonical, due to the non-existence of a universal time scale. However, it should be pointed out that when the configuration manifold does not include a time coordinate, but the constraint is time-dependent, one is not often dealing with an embedding of a submanifold in M, but a submersion; i.e., the differential map is a projection, not an injection. For instance, orbits are self-intersecting curves, but in order to make them non-self-intersecting, one can simply add the time dimension to Π so that they would become helices.

One often sees linear rheonomic constraints defined by a set of $n - p$ 1-forms:

$$C^\alpha = C_i^\alpha dx^i + a^\alpha dt \, . \tag{3.7}$$

One could either think of that as the constraint on velocity that:

$$C^\alpha (\mathbf{v}) = a^\alpha, \tag{3.8}$$

which is affine (i.e., inhomogeneous linear) or introduce another generalized coordinate $q^0 = t$ and define $C_0^\alpha = a^\alpha$, $v^0 = 1$, which would then make the constraint linear:

$$C_\mu^\alpha \, v^\mu = 0, \qquad \mu = 0, \ldots, p \, . \tag{3.9}$$

Because of that, we shall not typically distinguish between scleronomic and rheonomic constraints, but we shall assume that time is included in the generalized coordinates, if necessary.

The question of whether a constraint is imperfect or perfect comes down to the question of whether the *force of constraint* does work or not, respectively. That is, when motion is kinematically confined to a submanifold of an ambient manifold (or at least, its velocity vectors are confined to a sub-bundle of its tangent bundle), one can derive corresponding constraints on the acceleration vectors by time-differentiation, which can be regarded as introducing forces of constraint that are proportional to the masses that are in motion. In the case of a perfect constraint, the acceleration vector will always be normal to the linear subspace that constrains the velocity vectors, while an imperfect constraint will also include a "tangential" component that lies in that constraint subspace.

As we will see, that is also the difference between a curve that is geodesic and one that is not. A common example of an imperfect constraint is that of a box sliding across a rough floor. The force of constraint consists of not only the normal force that is equal and opposite to the weight, but also the force of friction, which is horizontal and points in the opposite direction to either the horizontal component of the applied force or the velocity of motion. The work done by the force of constraint is then the work done by friction. Another example of an imperfect constraint that is not based in friction or viscosity is that of what are now called "servo-constraints," which were introduced by Henri Beghin [**18**] in his dissertation under Gabriel Koenig, and were also discussed by Appell in his own treatise on rational mechanics [**8**].

The issue of two-sided versus one-sided constraints comes down to whether the constraint is defined by a set of equations or a set of equations and inequalities, respectively. So far, we have been implicitly discussing the two-sided constraints since we have been posing systems of equations. However, their scope can be enlarged to include inequalities, which would make some of them one-sided. For instance, when a marble rolls on the surface of a bowl that is concave upwards under the influence of gravity, it will always remain in contact with the constraint surface, but when the bowl is inverted, if the initial velocity of the marble is sufficiently high then it can leave the constraint surface and move freely in space. Hence, in the latter case, the constraint on the position of the marble will take the form of an inequality that says that its height above the surface of the bowl must be greater than or equal to 0. Some other common examples of one-sided constraints are a ball bouncing on a floor and a pair of objects in space that are connected by a flexible, inextensible string such that their separation distance must lie between 0 and the length of the string. A complementary arrangement might be that of rigid spheres of finite radii that move freely in space, but whose centers can approach each other only as close as the sum of the radii.

Finally, a constraint will be called *regular* when the dimension of the constraint space in each tangent space is the same at each point of the configuration manifold. When there are "singular" points at which the dimension of the constraint space is different, the constraint itself will be called *singular*. In the case of linear constraints, that will usually come about when the $n - p$ 1-forms C^α that collectively annihilate the constraint subspace cease to be linearly independent at some points, so the dimension of that subspace becomes higher than p. That can also affect the degree of determinacy of the equations of motion in the process. A simple example of a singular constraint is given by motion on a cone, say, with the equation:

$$C(x^1, x^2, x^3) = (x^1)^2 + (x^2)^2 - (x^3)^2 = 0, \tag{3.10}$$

which clearly includes the origin as one of its points, and that will, in fact, be the vertex of the cone. The differential of C is:

$$dC = 2\,(x_1\, dx^1 + x_2\, dx^2 - x_3\, dx^3)\,, \tag{3.11}$$

which will vanish at the vertex of the cone. At any other point on the cone, the 1-form dC will annihilate a plane of tangent vectors to the surface of the cone, but at the vertex, $dC(\mathbf{X})$ will vanish for any vector $\mathbf{X}$. Hence, the dimension of the annihilating subspace of dC will increase to three at the singular point. Of course, one might wish to restrict oneself to the only the lines through the vertex that lie in the cone itself, which is not a linear space, but a "conical" tangent space at the singular point. (That is a common situation with algebraic sets of the kind that are defined by C.)

4. Complete integrability of a linear constraint. – When the constraint is linear, it defines a vector bundle $\mathcal{D}(M)$ that is a sub-bundle of $T(M)$ that we have called the constraint bundle, which then associates a p-dimensional linear subspace in each tangent space to M. Such a thing can also be regarded as a *differential system* on M. Furthermore, since the linear subspaces are annihilated by the set of 1-forms C^α, one can also think of $\mathcal{D}(M)$ as being defined by the *exterior differential system* $C^\alpha = 0$.

a. Frobenius's theorem ([1]). – An *integral submanifold* for the differential system $\mathcal{D}(M)$ is a differentiable map $x : \Pi \to M$, $q \mapsto x(q)$, where Π is a subset of $\mathbb{R}^k$ ($k \leq p$) such that the image $dx|_q$

([1]) In addition to Frenkel [**1**], one might also confer some other elementary texts on differential forms, such as Warner [**19**] and (Henri) Cartan [**20**].

$(T_q(\Pi))$ of each tangent space $T_q(\Pi)$ to Π is a subspace of the linear subspace $\mathcal{D}_{x(q)}M$ that is annihilated by the 1-forms C^α. Note that in general the dimension k of Π does not have to be equal to p, since that would be true only for a *completely-integrable differential system.* The maximum dimension of Π for which integral submanifolds exist is called the *degree of integrability* of the differential system $\mathcal{D}$, and at the very least, there will always be integral curves, for which that degree will be one.

One can then see that when the differential system is defined by a linear constraint, the dimensions of its integral submanifolds will be less than or equal to the degrees of freedom in the motion, and equality will obtain only for the case of complete integrability. That is the case of holonomic constraints, by definition.

The condition for complete integrability is given by *Frobenius's theorem*, which was due to Ferdinand Georg Frobenius (1849-1917), whose study of the Pfaff problem, which we shall discuss briefly below, introduced many ideas that were fundamental to the development of analysis and geometry that followed. It can be expressed in terms of either the vector fields on M that take their values in the subspaces of $\mathcal{D}$ or the set of 1-forms that annihilate them.

In the former case, if $\mathbf{X}$, $\mathbf{Y}$ are vector fields on M that take their values in $\mathcal{D}$ then they are said to be *in involution* if their Lie bracket:

$$[\mathbf{X}, \mathbf{Y}] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \tag{4.1}$$

is another vector field that takes it values in $\mathcal{D}$. Frobenius's theorem then says that the differential system $\mathcal{D}$ is completely integrable iff all vector fields that take their values in $\mathcal{D}$ are in involution.

Dually, the condition on the 1-forms C^α that annihilate the subspaces of $\mathcal{D}$ is that the 2-forms that take the form of their exterior derivatives $d_\wedge C^\alpha$ should also annihilate those subspaces. In order for that to be true, it is necessary and sufficient that there should exist a set of $n - p$ 1-forms $\eta^\alpha_\beta = \eta^\alpha_{\beta i}(x)\, dx^i$ that make:

$$d_\wedge C^\alpha = \eta^\alpha_\beta \wedge C^\beta , \qquad \text{for all } \alpha = 1, \ldots, n - p . \tag{4.2}$$

The relationship between the two formulations of the theorem can be obtained directly from the so-called "intrinsic" formula for the exterior derivative $d_\wedge C^\alpha$. For any two vector fields $\mathbf{X}$ and $\mathbf{Y}$, it says that the value of $d_\wedge C^\alpha$, when it is evaluated on those two vector fields, will be:

$$d_\wedge C^\alpha (\mathbf{X}, \mathbf{Y}) = \mathbf{X}\,[C^\alpha (\mathbf{Y})] - \mathbf{Y}\,[C^\alpha (\mathbf{X})] - C^\alpha ([\mathbf{X}, \mathbf{Y}]) , \tag{4.3}$$

in which the first two brackets only serve to indicate that the vector field $\mathbf{X}$ or $\mathbf{Y}$ acts upon the functions in brackets as a directional derivative, while the last one refers to the Lie bracket of the vector fields. When **X** and **Y** are consistent with the constraints, the functions $C^\alpha(\mathbf{X})$, $C^\alpha(\mathbf{Y})$ will vanish, and one will be left with:

$$d_\wedge C^\alpha(\mathbf{X}, \mathbf{Y}) = -C^\alpha([\mathbf{X}, \mathbf{Y}]) . \tag{4.4}$$

Hence, $d_\wedge C^\alpha(\mathbf{X}, \mathbf{Y})$ will vanish iff $[\mathbf{X}, \mathbf{Y}]$ does. However, that does not imply that $d_\wedge C^\alpha$ must vanish on *all* vector fields.

b. The Pfaff problem (1). – In the case where $n - p = 1$, so there is just the one 1-form C, that 1-form is often referred to as a *Pfaffian*, and the equation $C = 0$ is referred to as the *Pfaff equation.* The Pfaff problem is the problem of finding integral submanifolds to the Pfaff equations, and in particular, the maximum dimension of such things. Frobenius's theorem can then be expressed by saying that a necessary and sufficient condition for the completely integrability of the Pfaff equation is the vanishing of the *Frobenius 3-form* $C \wedge d_\wedge C$. Note that this condition will be satisfied trivially when the dimension n of the ambient manifold M is less than three, such as a surface, because if $k > n$ then one will always have the vanishing of k-forms as a general property of exterior forms (2).

5. Frame fields adapted to linear constraints. – A concept that will prove useful from now on is that of a p-frame field $\{\mathbf{e}_a(q), a = 1, \ldots, p\}$ on the ambient manifold M that is *adapted* to the constraints that are imposed on its tangent vectors. Hence, such a concept will be applicable only in the case of linear constraints, since only they have constraint submanifolds that are linear subspaces that can be spanned by frames. Since the contributions to the various geometric objects that will be defined that come from the geometry of the ambient manifold tend to be additive contributions, in order to focus on the geometry of the constraint bundle, we shall assume that the ambient manifold is simply n-dimensional Euclidian space E^n, namely, $\mathbb{R}^n$ with a metric δ that makes the canonical frame on $\mathbb{R}^n$ orthogonal. Recall that the canonical frame on $\mathbb{R}^n$ is composed of the vectors $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 0)$, … $(0, \ldots, 0, 1)$, which will also be vectors that generate the coordinate axes for a Cartesian coordinate system $\{x^i, i = 1, \ldots, n\}$. The natural frame field that is associated with those coordinates is then denoted by $\{\partial_i , i = 1, \ldots, n\}$, where ∂_i represents the partial (or directional) derivative $\partial / \partial x^i$ with respect to the coordinate in question. The reciprocal natural coframe field will then be $\{dx^i, i = 1, \ldots, n\}$, and the Euclidian metric will take the form:

$$\delta = \delta_{ij}\, dx^i\, dx^j \tag{5.1}$$

in that natural coframe field.

(1) The author has produced a monograph [**21**] on the applications of the theory of the Pfaff equation to physics that include more details on the subject.

(2) Basically, the exterior product of k 1-forms will be non-zero iff they are all linearly independent. However, one can never find more than n linearly-independent 1-forms on an n-dimensional vector space.

a. Definition of a frame field adapted to a linear constraint. – If those linear constraints take the form of $n-p$ 1-forms C^α that collectively annihilate the constraint space $\mathcal{D}_x E^n$ at each $x \in E^n$ then the frame $\mathbf{e}_a$ will be adapted to those constraints iff each member is annihilated by all of the C^α:

$$C^\alpha(\mathbf{e}_a) = 0 \qquad \text{for all } \alpha = 1, \ldots, n-p . \tag{5.2}$$

One can generally express the frame field $\mathbf{e}_a$ in component form with respect to the natural frame field on E^n:

$$\mathbf{e}_a(x) = x_a^i(x)\,\partial_i . \tag{5.3}$$

If $C^\alpha = C_i^\alpha(x)\,dx^i$ then one can write (5.2) in local component form as:

$$C_i^\alpha\, x_a^i = 0. \tag{5.4}$$

From the linearity of the 1-forms C^α, one sees that any linear combination of the basis vectors $\mathbf{e}_a$ (i.e., any tangent vector to the constraint submanifold) will also be consistent with the constraint. That is, if $\mathbf{X} = X^a\,\mathbf{e}_a$ then:

$$C^\alpha(\mathbf{X}) = X^a\, C^\alpha(\mathbf{e}_a) = 0.$$

Another concept that will prove useful in what follows is that of an extension of the adapted p-frame field $\mathbf{e}_a$ to an n-frame field. Certainly, at least one n-frame field exists on on E^n, namely, the natural frame field ∂_i . Since we already have the component matrix $x_a^i(x)$ for the adapted frame field $\mathbf{e}_a$, what we need to do is to extend that matrix, which has rank p, to one that has rank n, and we shall call that invertible $n\times n$ matrix x_j^i. The remaining components of x_j^i define $n-p$ linearly-independent vector fields $\{\mathbf{e}_{p+\alpha},\ \alpha = 1, \ldots, n-p\}$:

$$\mathbf{e}_{p+\alpha}(x) = x_{p+\alpha}^i(x)\,\partial_i \tag{5.5}$$

that are initially arbitrary, except for being linearly independent of each other and the original vector fields $\mathbf{e}_a$, but if we want them to also be adapted to the constraints then we might also specify that when we go to the reciprocal coframe field e^i [which is the one that makes $e^i(\mathbf{e}_j) = \delta_j^i$], the dual covector fields $e^{p+\alpha}$ will agree with the 1-forms C^α that have already been defined ([1]):

$$C^\alpha = e^{p+\alpha}, \qquad \text{so} \qquad C_i^\alpha = \tilde{x}_i^{p+\alpha}, \tag{5.6}$$

since the reciprocal coframe field can be expressed in components as:

$$e^i(x) = \tilde{x}_j^i\, dx^j . \tag{5.7}$$

([1]) From now on, the tilde will indicate the inverse of a matrix when used in that context.

The linear subspace $N_x E^n$ of $T_x E^n$ that is spanned by the $n - p$ vectors $\mathbf{e}_{p+\alpha}$ at each point x (q) is complementary to the subspace $\mathcal{D}_x E^n$ that is spanned by the first p of them, namely, $\mathbf{e}_a$. Hence, one has a direct sum splitting:

$$T_x E^n = \mathcal{D}_x E^n \oplus N_x E^n$$

at each point of the constraint submanifold. One can then think of the vectors in $N_x E^n$ as being normal to the tangent constraint spaces $\mathcal{D}_x E^n$. A different choice of extension to the adapted p-frame field would define a different normal subspace $N_x E^n$ and a different splitting, so the projection of any tangent vector in $T_x E^n$ into its components in $\mathcal{D}_x E^n$ and $N_x E^n$ would also change. However, since we will typically be starting with vectors in $\mathcal{D}_x E^n$, as long as we do not differentiate the normal frame vector fields, the only real issues concerning the normal parts of tangent vectors will be whether they vanish or are equal, more than their exact components, which will change by conventional linear-algebraic laws under a different choice of extension.

b. Lie brackets of adapted frame members. – One often needs to know the Lie brackets $[\mathbf{e}_a , \mathbf{e}_b]$ of the adapted frame members. In general, one will have:

$$[\mathbf{e}_a , \mathbf{e}_b] = c^c_{ab}\, \mathbf{e}_c + c^\alpha_{ab}\, \mathbf{e}_\alpha , \tag{5.8}$$

in which the symbols c^i_{ab}, $i = 1, \ldots, n$ are functions of x that are called the *structure functions* of the adapted frame field $\mathbf{e}_a$. Hence, the condition for involution of the frame members is the vanishing of c^α_{ab} , while c^c_{ab} can be arbitrary, except that it must be antisymmetric in its lower indices and obey an identity that follows from the Jacobi identity, although we shall not elaborate upon that here.

If we represent $\mathbf{e}_a$ (x) in the component form (5.3) then:

$$[\mathbf{e}_a, \mathbf{e}_b] = \left(x^j_a\, \partial_j x^i_b - x^j_b\, \partial_j x^i_a\right)\partial_i = \left(x^j_a\, \partial_j x^k_b - x^j_b\, \partial_j x^k_a\right)\tilde{x}^i_k\, \mathbf{e}_i ,$$

so that will make the structure functions of the adapted frame field equal to:

$$c^i_{ab} = \left(x^j_a\, \partial_j x^k_b - x^j_b\, \partial_j x^k_a\right)\tilde{x}^i_k . \tag{5.9}$$

One can also relate those Lie brackets to the exterior derivatives of the constraint 1-forms $C^\alpha = e^\alpha$, which we then extend to $C^i = e^i$, namely:

$$d_\wedge C^\alpha = \tfrac{1}{2}\gamma^\alpha_{jk}\, C^j \wedge C^k = \tfrac{1}{2}\gamma^\alpha_{ab}\, C^a \wedge C^b + \tfrac{1}{2}\gamma^\alpha_{a\beta}\, C^a \wedge C^\beta + \tfrac{1}{2}\gamma^\alpha_{\beta\gamma}\, C^\beta \wedge C^\gamma, \tag{5.10}$$

and when one uses the "intrinsic" formula (4.4) for the exterior derivatives of C^α, one will see that:

$$\gamma^\alpha_{ab} = d_\wedge C^\alpha (\mathbf{e}_a , \mathbf{e}_b) = - C^\alpha [\mathbf{e}_a , \mathbf{e}_b] = - c^\alpha_{ab} , \tag{5.11}$$

in which we have used (5.8). Similarly:

$$\gamma^{\alpha}_{a\beta} = - c^{\alpha}_{a\beta}, \quad \gamma^{\alpha}_{\beta\gamma} = - c^{\alpha}_{\beta\gamma} . \tag{5.12}$$

When c^{α}_{ab} vanishes, the C^{β} will factor out of the exterior products on the right-hand side of (5.10), and that will make:

$$d_{\wedge}C^{\alpha} = \eta^{\alpha}_{\beta} \wedge C^{\beta} , \tag{5.13}$$

in which we have set:

$$\eta^{\alpha}_{\beta} = - c^{\alpha}_{a\beta} C^{a} - c^{\alpha}_{\gamma\beta} C^{\gamma}. \tag{5.14}$$

(The factor of 1/2 appears when one antisymmetrizes the exterior product.)

c. Equations of the adapted frame field. – It is useful to know the system of partial differential equations that the p-frame field $\mathbf{e}_a$ (x) must satisfy. We start with the one that the extended n-frame field $\mathbf{e}_i$ $(x) = x^j_i(x)\, \partial_j$ must satisfy.

Upon differentiation, we will get:

$$d\mathbf{e}_i = dx^j_i \otimes \partial_j , \tag{5.15}$$

and when we introduce the matrix of 1-forms:

$$\omega^j_i = dx^j_k\, \tilde{x}^k_i, \tag{5.16}$$

that will give the differential equations of the frame field $\mathbf{e}_i$:

$$d\mathbf{e}_i = \omega^j_i \otimes \mathbf{e}_j . \tag{5.17}$$

In particular, the equations of the p-frame field $\mathbf{e}_a$ will take the form:

$$d\mathbf{e}_a = \omega^b_a \otimes \mathbf{e}_b + \omega^{\alpha}_a \otimes \mathbf{e}_{p+\alpha} . \tag{5.18}$$

We can also express ω^j_i in the form:

$$\omega^j_i = \omega^j_{ik}\, C^{k}, \tag{5.19}$$

since from (5.16), we have:

$$\omega^j_i = \partial_k x^j_l\, \tilde{x}^l_i\, dx^k = \partial_m x^j_l\, \tilde{x}^l_i\, x^m_k C^{k} ,$$

so:

$$\omega^j_{ik} = x^m_k \partial_m x^j_l\, \tilde{x}^l_i = (\mathbf{e}_k x^i_l)\, \tilde{x}^l_i . \tag{5.20}$$

If we compare this to (5.9) then we will see that:

$$c^i_{ab} = \omega^i_{ab} - \omega^i_{ba}. \tag{5.21}$$

One can regard the matrix of 1-forms as ω^j_i as a connection 1-form, and it is characterized by the being the connection that will make the frame field $\mathbf{e}_i$ consist of parallel frames. It is often referred to as the *teleparallelism* connection or *absolute parallelism*; a manifold (which must then be parallelizable) that is given such a connection is also referred to as a *Weitzenböck space* ([1]). A vector field $\mathbf{X} = X^i\,\mathbf{e}_i$ is then parallel iff its components X^i with respect to $\mathbf{e}_i$ are constant.

By specialization to the constraint subspaces, one also sees that the constraint subspaces themselves will be parallel under that connection. From (5.21), the structure functions of the adapted frame field can be obtained from the connection 1-form that makes the frame field parallel.

d. The kinematics of motion in an adapted frame. – Now, suppose that $x\,(t)$ is a twice-differentiable curve in E^n that is constrained by a linear constraint that is defined by $n - p$ 1-forms C^α, as above. Furthermore, let $\mathbf{e}_a$ be a frame field on E^n that is adapted to that constraint. Hence, the velocity vector field along $x\,(t)$:

$$\mathbf{v}\,(t) = \left.\frac{dx}{dt}\right|_t = v^i(t)\,\partial_i \tag{5.22}$$

can then be expressed in terms of the adapted frame as:

$$\mathbf{v}\,(t) = \upsilon^a\,(t)\,\mathbf{e}_a\,(t)\,, \tag{5.23}$$

in which:

$$v^i\,(t) = x^i_a(x(t))\,\upsilon^a\,, \quad \mathbf{e}_a\,(t) = \mathbf{e}_a\,(x\,(t))\,. \tag{5.24}$$

Therefore, in order to get the velocity of a constrained curve, one starts by defining a curve $\upsilon^a\,(t)$ in $\mathbb{R}^p$ and "injecting" it into the tangent spaces along $x\,(t)$ by way of the matrices $x^i_a(x(t))$.

One can then get the acceleration of the curve $x\,(t)$ by differentiating both sides of (5.23):

$$\mathbf{a} = \dot{\upsilon}^a\,\mathbf{e}_a + \upsilon^a\,\dot{\mathbf{e}}_a. \tag{5.25}$$

In order to get the time derivative of $\mathbf{e}_a$ along the curve, we use essentially the chain rule for differentiation:

$$\dot{\mathbf{e}}_a = d\mathbf{e}_a(\mathbf{v}) \qquad \left(\frac{d\mathbf{e}_a}{dt} = \frac{dx^i}{dt}\frac{\partial \mathbf{e}_a}{\partial x^i}\right). \tag{5.26}$$

The differential equation of the moving frame (5.18) will then give:

$$\dot{\mathbf{e}}_a(t) = \omega^b_a(\mathbf{v})\,\mathbf{e}_b + \omega^\alpha_a(\mathbf{v})\,\mathbf{e}_{p+\alpha}\,. \tag{5.27}$$

([1]) For more details, the author has also compiled an anthology of translations of early papers in the geometry and physics of teleparallelism [**22**] that includes a more modern introduction to the topic.

Substituting that into (5.25) gives **a** in the form:

$$\mathbf{a} = \mathfrak{a}^a \, \mathbf{e}_a + \mathfrak{a}^{p+\alpha} \, \mathbf{e}_{p+\alpha} \, , \tag{5.28}$$

with:

$$\mathfrak{a}^a = \dot{\upsilon}^a + \omega^a_b(\mathbf{v})\upsilon^b, \qquad \mathfrak{a}^{p+\alpha} = \omega^\alpha_a(\mathbf{v}) \, \upsilon^b. \tag{5.29}$$

Hence, even when the velocity of the curve is constrained to a linear subspace of each tangent space, the acceleration will generally have a tangential part and a normal part.

One can put the tangential components into the form:

$$\mathfrak{a}^a = \dot{\upsilon}^a + \omega^a_{(bc)}\upsilon^b \, \upsilon^c, \tag{5.30}$$

in which the indices bc were symmetrized since the product $\upsilon^b \, \upsilon^c$ is symmetric in those indices. Hence:

$$\omega^a_{(bc)} = \tfrac{1}{2}(\omega^a_{bc} + \omega^a_{cb}) \, . \tag{5.31}$$

One will then see that the demanding that the acceleration must always be normal to the curve will imply the vanishing of its tangential components, which amounts to the system of ordinary differential equations that make the curve a geodesic for the connection in question.

6. Holonomic constraints. – Holonomic constraints typically bear upon the positions of the various components of a mechanical system directly. Some of the common ones involve requiring that the bodies in motion should all lie along a common curve or surface. Another common holonomic constraint is rigidity, in which one constrains the distances between some or all pairs of masses to remain constant in time. That is an example of a constraint that can be applied to discrete or continuous distributions of masses.

a. Defining holonomic constraints. – Basically, a holonomic constraint is one whose constraint sub-bundle $\mathcal{D}$ (M) is completely integrable. We shall concentrate on the special cases of linear constraints and $M = E^n$.

There are two ways of defining an integral submanifold of a completely-integrable linear constraint that can be thought of as the difference between a *locus* of points and an *envelope* of a set of tangent spaces.

Defining a submanifold as a locus amounts to defining a differentiable embedding $x : \Pi \to E^n$, $q \mapsto x(q)$, from a p-dimensional prototype (i.e., parameter) manifold Π into the ambient space, where we are parameterizing Π by what Lagrange called the *generalized coordinates* q^a. Since generalized coordinates exist for only integral submanifolds of the constraints, in order for the dimension of Π to equal the number of degrees of freedom for the constrained motion, one would have to be dealing with a holonomic constraint.

When one has a local coordinate system (U, q^a) about the point $q \in \Pi$ and one uses the natural coordinate system in E^n, one can locally represent the constraint submanifold that is defined by x in the form of a system of n equations in p independent variables:

$$x^i = x^i\,(q^a)\ . \tag{6.1}$$

As a consequence of the assumption that x is injective, one also has that its differential map $dx|_q : T_u\Pi \to T_{x(q)}\,E^n$, $\mathbf{X} \mapsto dx|_q\,(\mathbf{X})$, is also a linear injection of each tangent space to Π into a p-dimensional linear subspace $\Xi_{x(q)}M$ of the corresponding tangent space to M. That implies that just as the positions of moving points are constrained to the image Ξ of the submanifold x, the tangent vectors, such as the velocity of any constrained curve, are constrained to the linear subspaces that take the form of each $\Xi_{x(q)}\,E^n$. Hence, the differential form of the local system of equations (6.1) is the system of n linear equations in the p components X^a of a vector field $\mathbf{X} = X^a(q)\,\partial_a$ on Π :

$$\frac{\partial x^i}{\partial q^a}\,X^a = 0\ . \tag{6.2}$$

In order to define a frame field $\mathbf{e}_a\,(x)$ on the image of the integral submanifold $x\,(q)$ in $\mathbb{R}^n$ that is adapted to the constraints, when we are dealing with holonomic constraints, we can use the component matrix:

$$x^i_a(x(q)) = x^i_{,a}(q) = \frac{\partial x^i}{\partial q^a}\,, \tag{6.3}$$

which will make $\mathbf{e}_a = \mathbf{e}_a\,(q)$ the "push-forward" $x_*\partial_a$ of the natural frame field on $\mathbb{R}^p$, which is $\partial_a = \partial_a / \partial q^a$, by the embedding x ; hence:

$$\mathbf{e}_a\,(q) = x_*\partial_a \equiv x^i_{,a}(q)\,\partial_i = \frac{\partial x^i}{\partial q^a}\frac{\partial}{\partial x^i}\,. \tag{6.4}$$

One immediately has that the vector fields $\mathbf{e}_a\,(q)$, which are push-forwards of natural frame vector fields, still commute:

$$[\mathbf{e}_a\,, \mathbf{e}_b] = (x^j_{,a}\,\partial_j x^i_{,b} - x^j_{,b}\,\partial_j x^i_{,a})\,\partial_i = (x^i_{,b,a} - x^j_{,b,a})\,\partial_i = 0, \tag{6.5}$$

which follows from the general rule for mixed second partial derivatives since we are assuming that the component functions $x^i_{,a}(q)$ are continuously differentiable. The fact that the adapted frame members commute in this case is also due to the general property of push-forwards that they "commute" with Lie brackets; i.e., the Lie bracket of two push-forwards is the push-forward of their Lie bracket:

$$[x_*\mathbf{X}, x_*\mathbf{Y}] = x_*[\mathbf{X}, \mathbf{Y}]\ . \tag{6.6}$$

Therefore, since the natural frame field on $\mathbb{R}^p$ has members that commute, their push-forwards by x will also commute.

Defining a submanifold as an envelope amounts to an indirect, or implicit, definition, as opposed to the direct route that a locus represents. Typically, one has a space of $\mathbb{R}^{n-p} = (\phi^1, \ldots, \phi^{n-p})$ for functions on E^n to take their values in and a differentiable function $\phi : E^n \to \mathbb{R}^{n-p}$, $x \mapsto \phi^\alpha(x)$, so a constraint submanifold Ξ in E^n will consist of a level set of that function. Therefore, it will be the set of all $x \in E^n$ that give the same value ϕ_0^α to the function ϕ. Hence, the submanifold will be representable locally as the system of $n - p$ equations in n unknowns:

$$\phi^\alpha(x^i) = \phi_0^\alpha . \tag{6.7}$$

Often $n - p = 1$, in which case the constraint submanifold will be a constraint hypersurface. It will be said to have either dimension p or codimension 1.

By differentiation, that will imply the constraint on tangent vectors $\mathbf{X}$ to Ξ that:

$$d\phi|_x(\mathbf{X}) = 0 . \tag{6.8}$$

In other words, the constraint on the tangent vectors is that they must lie in the *annihilating subspaces* of the differential map $d\phi$. Those are then p-dimensional linear subspaces of each tangent space $T_x E^n$. Locally, the constraint (6.8) takes the form of the $n - p$ linear equations in the n components X^i of the tangent vector $\mathbf{X} = X^i \partial_i$:

$$\frac{\partial \phi^\alpha}{\partial x^i} X^i = 0. \tag{6.9}$$

One sees that there is a partial duality to the concepts of locus and envelope, at least as far as their dimensions are concerned. The fact that the duality is not complete comes from the fact that when the constraint submanifold is defined as a locus, one defines only one submanifold, but when one defines an envelope, one has a family of submanifolds that "foliate" the manifold E^n like the pages of book.

If one defines the set of $n - p$ 1-forms:

$$C^\alpha = d\phi^\alpha = \frac{\partial \phi^\alpha}{\partial x^i}(x)\, dx^i \tag{6.10}$$

then one can represent the differential system $\mathcal{D} = T(\Xi)$ as the algebraic solution to the exterior differential system:

$$C^\alpha = 0, \qquad \alpha = 1, \ldots, n - p . \tag{6.11}$$

That is, each tangent subspace of $\mathcal{D}_x = T_x(\Xi)$ is the linear subspace that is annihilated by all $C^\alpha = C_i^\alpha(x)\,dx^i$; i.e. the set of all $\mathbf{X} \in T_x(\Xi)$ such that:

$$C^\alpha(\mathbf{X}) = C_i^\alpha(x)\,X^i(x) = 0, \qquad \text{for all } \alpha = 1, \dots, n-p\,. \tag{6.12}$$

If $C^\alpha = C_i^\alpha(x)\,dx^i = \phi_{,i}^\alpha(x)\,dx^i$ then the condition (5.2) will become:

$$0 = \phi_{,i}^\alpha\, x_{,a}^i = \frac{\partial \phi^\alpha}{\partial x^i}\frac{\partial x^i}{\partial q^q} = \frac{\partial \phi^\alpha}{\partial q^a}\,, \tag{6.13}$$

which says that the functions ϕ^α are constant on the image of $x(q)$. Of course, that is the envelope definition of a holonomic constraint.

When the constraint is holonomic, so $C^\alpha = d\phi^\alpha$, $d_\wedge C^\alpha$ will vanish identically, from a basic property of the exterior derivative operator $d_\wedge$, and the differential system that is defined by the annihilating hyperplanes of C^α will be completely integrable. In particular, they will be the intersections of the hypersurfaces that are defined by the functions ϕ^α.

c. Constrained curves in E^n. – When a mechanical system has a holonomic constraint that takes the form of an embedding $x : \Pi \to E^n$, $q \mapsto x(q)$ or a set of differentiable functions $\phi^\alpha : E^n \to \mathbb{R}^{n-p}$, $x \mapsto \phi^\alpha(x)$, one can define a constrained curve $x(t)$ in E^n either directly by first defining a curve $q(t)$ in Π and mapping it into E^n by way of x or indirectly by defining $x(t)$ in E^n and subjecting it to the constraint defined by ϕ^α ; those alternatives will then take the forms:

$$x(t) = x(q(t)) \qquad \text{and} \qquad \phi^\alpha(x(t)) = \text{const.}, \tag{6.14}$$

respectively. For consistency, one must have $\phi^\alpha \cdot x = \text{const.}$

By differentiation, one finds that when one defines the velocity vector field $\mathbf{v}(t)$ along $x(t)$ to be the push-forward of the velocity vector field $\dot{\mathbf{q}}(t)$ on $q(t)$ by the map x, namely:

$$\mathbf{v}(t) = (x_*\,\dot{\mathbf{q}})(t) \qquad \left(v^i = \frac{\partial x^i}{\partial q^a}\dot{q}^a\right), \tag{6.15}$$

the vector field $\mathbf{v}(t)$ will automatically satisfy the constraints. That is because when one differentiates (6.7), one will get:

$$\frac{\partial \phi^\alpha}{\partial x^i}\frac{\partial x^i}{\partial q^a} = 0\,. \tag{6.16}$$

Hence:

$$C^\alpha(\mathbf{v}(t)) = d\phi^\alpha(\mathbf{v}(t)) = d\phi^\alpha(x_*\dot{\mathbf{q}}(t)) = \frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial x^i}{\partial q^a}\dot{q}^a(t) = 0\,. \tag{6.17}$$

Otherwise, when the embedding x has not been defined, one has only the implicit constraint on $\mathbf{v}(t)$ that:

$$d\phi^\alpha(\mathbf{v}(t)) = \frac{\partial\phi^\alpha}{\partial x^i}\dot{x}^i(t) = 0\,. \tag{6.18}$$

If one defines the adapted frame field $\mathbf{e}_a(q) = x_*\partial_a$, as before, then one can also express $\mathbf{v}(t)$ in the form:

$$\mathbf{v} = \dot{q}^a\mathbf{e}_a\,. \tag{6.19}$$

This form is more convenient for discussing the acceleration of a constrained curve, since we initially get:

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \ddot{q}^a\mathbf{e}_a + \dot{q}^a\dot{\mathbf{e}}_a\,,$$

and when we include the differential equation of the moving frame (5.27), that will take the form:

$$\mathbf{a}(t) = [\ddot{q}^a + \dot{q}^b\omega_b^a(\dot{\mathbf{q}})]\,\mathbf{e}_a + \dot{q}^b\omega_b^{p+\alpha}(\dot{\mathbf{q}})\,\mathbf{e}_{p+\alpha}\,. \tag{6.20}$$

Once again, this vector field along the curve $x(q(t))$ has a tangential part:

$$\mathfrak{a}^b = \ddot{q}^b + \dot{q}^c\omega_c^b(\dot{\mathbf{q}}) = \ddot{q}^b + \omega_{(cd)}^b\,\dot{q}^c\dot{q}^d \tag{6.21}$$

and a normal part:

$$\mathfrak{a}^{p+\alpha} = \dot{q}^b\omega_b^{p+\alpha}(\dot{\mathbf{q}}) = \omega_{(cd)}^{p+\alpha}\,\dot{q}^c\,\dot{q}^d\,. \tag{6.22}$$

Of particular interest are the curves for which the acceleration is always normal. Hence, its tangential part must always vanish, which gives the system of second order ordinary differential equations for the generalized coordinates $q^a(t)$:

$$0 = \ddot{q}^a + \omega_{(bc)}^a\,\dot{q}^b\dot{q}^c\,. \tag{6.23}$$

These clearly resemble the geodesic equations (2.42) that we obtained before from a variational principle (viz., least distance), except that here the components ω_{bc}^a do not take the same form as the Christoffel symbols $\left\{ \begin{matrix} a \\ b\ c \end{matrix} \right\}$. That is because we are defining a different connection from the Levi-Civita connection, which required that we had to first define a metric on the space of generalized coordinates, namely, the "teleparallelism" connection, which makes the frame field $\mathbf{e}_i(q)$ parallel.

d. Virtual displacements and virtual paths. – A virtual displacement of a constrained curve $x(t)$ is a vector field $\delta\mathbf{x}\,(t)$ along that curve that is transverse to the velocity vector field $\mathbf{v}\,(t)$. Since a virtual displacement is assumed to have the character of a velocity (i.e., an infinitesimal generator of a one-parameter family of finite displacements), one typically assumes that any virtual displacement is subject to the same constraint as a velocity.

Once again, that can be accomplished in the previous two ways when the constraints are holonomic, namely, when the constraint submanifold is defined to be a locus or an envelope. In the former case, if the constraint is defined by an embedding $x : \Pi \to E^n$ then one can start with a vector field $\delta\mathbf{q}\,(t)$ along the curve $q\,(t)$ in Π and push it forward to a vector field $\delta\mathbf{x}\,(t)$ on $x\,(t)$ using the differential of the embedding:

$$\delta\mathbf{x}\,(t) = (\,x_*\,\delta\mathbf{q})(t) \qquad \left(\delta x^i = \frac{\partial x^i}{\partial q^a}\,\delta q^a\right) . \tag{6.24}$$

The vector field $\delta\mathbf{x}\,(t)$ will automatically satisfy the constraint that $d\phi^\alpha\,(\delta\mathbf{x}\,(t))$ vanishes for all t for the same reason that the push-forward of velocity did before. Otherwise, when there is no embedding, the vector field $\delta\mathbf{x}\,(t)$ will have to satisfy:

$$0 = d\phi^\alpha\,(\delta\mathbf{x}\,(t)) = \frac{\partial\phi^\alpha}{\partial x^i}\,\delta x^i(t) \qquad \text{for all } t. \tag{6.25}$$

That is also how one would define the constraint on the $\delta\mathbf{x}\,(t)$ when the constraint submanifold is defined to be an envelope, i.e., ϕ^α = const.

Suppose that one represents the vector field $\delta\mathbf{x}\,(t)$ along $x\,(t)$ as the infinitesimal generator of a differentiable one-parameter family of paths $h\,(t, s)$ in E^n, so $h\,(t, 0) = x\,(t)$ and:

$$\delta\mathbf{x}\,(t) = \left.\frac{\partial h}{\partial s}\right|_{s=0} . \tag{6.26}$$

For each value of the second parameter s, one has another path that one might denote by $x_s(t) = h(t,s)$. If that path is also consistent with the constraints that are imposed then one can call it a *virtual path*; that is, the virtual displacement of that path:

$$\delta\mathbf{x}_s\,(t) = \left.\frac{\partial h}{\partial s}\right|_{s} \tag{6.27}$$

must also satisfy the constraint (6.25):

$$0 = d\phi^\alpha(\delta\mathbf{x}_s(t)) = \frac{\partial\phi^\alpha}{\partial x^i}\,\delta x_s^i(t) \tag{6.28}$$

for all t and s .

One can also ensure that the constraints are satisfied by defining a differentiable one-parameter family of curves $\bar{h}(t, s)$ in Π whose infinitesimal generator is:

$$\delta\mathbf{q}_s(t) = \left.\frac{\partial\bar{h}}{\partial s}\right|_s \tag{6.29}$$

and pushing $\delta\mathbf{q}_s(t)$ forward to $\delta\mathbf{x}_s(t)$ using dx, as in (6.24).

When one takes the Lie bracket of the vector field $\delta\mathbf{x}(t, s) = \delta\mathbf{x}_s(t)$ on the surface patch in E^n that is defined by the image of $\bar{h}(t, s)$ under x with the velocity vector field:

$$\mathbf{v}(t, s) = \left.\frac{\partial\bar{h}}{\partial t}\right|_t , \tag{6.30}$$

one will get:

$$[\delta\mathbf{x}, \mathbf{v}] = [x_*\delta\mathbf{q}, x_*\dot{\mathbf{q}}] = x_*[\delta\mathbf{q}, \dot{\mathbf{q}}] \ .$$

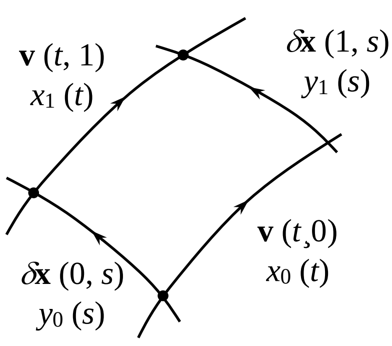
Now:

$$[\delta\mathbf{q}, \dot{\mathbf{q}}] = \left(\frac{\partial h^a}{\partial s}\frac{\partial h^b}{\partial t}\frac{\partial}{\partial q^b} - \frac{\partial h^a}{\partial t}\frac{\partial h^b}{\partial s}\frac{\partial}{\partial q^b}\right)\partial_a = \frac{\partial h^a}{\partial s}\frac{\partial h^b}{\partial t}\frac{\partial^2}{\partial q^b\partial q^a} - \frac{\partial h^a}{\partial t}\frac{\partial h^b}{\partial s}\frac{\partial^2}{\partial q^b\partial q^a}$$

$$= \left(\frac{\partial h^a}{\partial s}\frac{\partial h^b}{\partial t} - \frac{\partial h^a}{\partial s}\frac{\partial h^b}{\partial t}\right)\frac{\partial^2}{\partial q^a\partial q^b} = 0 \ .$$

Therefore:

$$[\delta\mathbf{x}, \mathbf{v}] = x_*[\delta\mathbf{q}, \dot{\mathbf{q}}] = 0 \ . \tag{6.31}$$



One can interpret that geometrically by forming the curvilinear quadrilateral that is defined by the four curves in E^n: $y_0(s) = \bar{h}(0, s)$, $x_1(s) = \bar{h}(t, 1)$, $x_0(s) = \bar{h}(t, 0)$, $y_1(s) = \bar{h}(1, s)$. By definition, those curves close into the boundary of the surface $\bar{h}(t, s)$. When one pushes them forward to curves in E^n by way of the embedding x, they will also define a curvilinear quadrilateral, and the condition (6.31) says that it will also be a closed polygon. Indeed, this is typical of holonomic constraints. We illustrate this situation in the accompanying figure.

e. Virtual work. – As usual, the applied force 1-form $F = F_i\,dx^i$ will be defined on E^n, either globally or only on the curve $x\,(t)$. If the embedding x exists then one can, however, "pull back" the 1-form F on E^n to a 1-form Q on Π:

$$Q \equiv x^*F = Q_a\,dq^a, \qquad Q_a \equiv F_i\,\frac{\partial x^i}{\partial q^a}. \tag{6.32}$$

That is because, from the chain rule for differentiation, one also has that the 1-forms dq^a are the pull-backs of the 1-forms dx^i :

$$x^*dx^i = \frac{\partial x^i}{\partial q^a}\,dq^a. \tag{6.33}$$

As a result, when one evaluates the pull-back of F on the virtual displacement $\delta\mathbf{q}$, one will get the same number as when one evaluates F on the virtual displacement $\delta\mathbf{x}$:

$$F\,(\delta\mathbf{x}) = F_i\;\delta x^i = F_i\,\frac{\partial x^i}{\partial q^a}\,\delta q^a = Q_a\;\delta q^a = (x^*F)(\delta\mathbf{q})\,. \tag{6.34}$$

In other words, the virtual work that is done by F along the constrained virtual displacement $\delta\mathbf{x}$ can be expressed in terms of either of the coordinates q^a or x^i . Note that although F does not have to be consistent with the constraints, nonetheless, the only part of F that will contribute to the virtual work done along $\delta\mathbf{x}$ is the part that is consistent. Hence, the normal part of the external force will do no virtual work, which implies that the constraint must be perfect.

The statement of the principle of virtual work for constrained virtual displacements is now:

When a system of forces is in equilibrium, the virtual work done by the forces that act upon the system under any virtual displacement that is consistent with the constraints will always vanish.

Since $\delta\mathbf{q}$ is free of constraints, one can infer from (6.34) that when a system of forces F is in equilibrium, one must have:

$$Q_a = 0 \qquad \text{for all } a\,. \tag{6.35}$$

f. Lagrange multipliers. – When the constraints on the motion are defined as an envelope, rather than a locus (so by a set of $n-p$ 1-forms C^α that must vanish when they are evaluated on any vector in the constraint subspaces), the first technique that one learns about finding critical points of functions that are subject to constraints is the technique of the Lagrange multipliers (cf., Lanczos [**12**]). The way that one introduces them into the principle of virtual work in the present case amounts to saying that the 1-form $\lambda_\alpha\,C^\alpha$ has the character of a force of constraint. Therefore, when it is evaluated on any tangent vector $\delta\mathbf{x}$, the resulting number:

$$\delta W_c = \lambda_\alpha\,C^\alpha\,(\delta\mathbf{x}) \tag{6.36}$$

will represent the virtual work done by the force of constraint along the virtual displacement $\delta\mathbf{x}$.

If one decomposes the arbitrary virtual displacement $\delta\mathbf{x}$ into a sum $\delta\mathbf{x}_c + \delta\mathbf{x}_n$ of a component $\delta\mathbf{x}_c$ that is consistent with the constraints and a component $\delta\mathbf{x}_n$ that is normal to them then one will have:

$$\delta W_c = \lambda_\alpha C^\alpha (\delta\mathbf{x}_n) = \lambda_\alpha \delta x^\alpha , \tag{6.37}$$

since $C^\alpha (\delta\mathbf{x})$ will vanish when that virtual displacement is consistent with the constraints. Thus, only the normal part of the virtual displacement will do any virtual work, so in a sense the force that is described by λ_α is a sort of "restoring" force that maintains the constraint. One can also say that *the force of constraint will do no work for any virtual displacement that is consistent with the constraint.* Hence, the constraint must be, by definition, perfect.

An important subtlety that associated with the introduction of Lagrange multipliers is that it allows one to treat the otherwise-unconstrained virtual displacements $\delta\mathbf{x}$ as if they were free. Hence, if one adds the forces of constraint to the external forces then the principle of virtual work can be restated by saying that:

$$(F_i + \lambda_\alpha C^\alpha) \delta x^i = 0 \tag{6.38}$$

for *any* virtual displacement δx^i.

g. D'Alembert's principle. – D'Alembert's extension of the principle of virtual work involves only the inclusion of the virtual work that is done by inertial forces under that same displacement. That is, one must include $- ma (\delta\mathbf{x}) = - m a_i \delta x^i$, where $a = a_i dx^i$ is the co-acceleration 1-form that is metric dual to the acceleration vector field (i.e., $a_i = \delta_{ij} a^j$), but with the conditions that $\mathbf{v}(t)$ and $\delta\mathbf{x}(t)$ must be consistent with the constraint. Hence, one must have that:

$$(F - ma) (\delta\mathbf{x}) = 0 \tag{6.39}$$

for every $\delta\mathbf{x}$ that is consistent with the constraint.

When the constraints are defined by an envelope, one can introduce Lagrange multipliers λ_α and replace the external force F with $F + \lambda_\alpha d\phi^\alpha$, and then put (6.39) into the form:

$$(F + \lambda_\alpha d\phi^\alpha - m a) (\delta\mathbf{x}) = 0 . \tag{6.40}$$

Since $d\mathbf{x}$ is free now, if that equation is true for all $d\mathbf{x}$ then one will get the equations of motion in the form:

$$m a = F + \lambda_\alpha d\phi^\alpha. \qquad (m a_i = F_i + \lambda_\alpha \partial_i \phi^\alpha). \tag{6.41}$$

However, since the addition of the $n - p$ undetermined multipliers λ_α means that these are now n equations in $2n - p$ unknowns, one must add $n - p$ more equations into order to also solve for all of the unknown functions. Those extra equations can take the form of the constraint on the velocity vector field:

$$\frac{\partial \phi^\alpha}{\partial x^i} v^i = 0, \qquad a = 1, \ldots, n - p . \tag{6.42}$$

When one introduces generalized coordinates, one will have an embedding x, as above. That will then allow us to pull back F to Q as in (6.32). We can now express a and $\delta\mathbf{x}$ in terms of generalized coordinates as:

$$a = \mathfrak{a}_a \, dq^a, \qquad \delta\mathbf{x} = \delta q^a \, \partial_a . \tag{6.43}$$

Equation (6.39) will then take the form:

$$0 = (Q_a - m \, \mathfrak{a}_a) \, \delta q^a = g_{ab} \, (Q^a - m \, \mathfrak{a}^b) \, \delta q^b \qquad \text{for all } \delta q^b. \tag{6.44}$$

The metric g_{ab} on Π is the pull-back of the Euclidian metric on E^n by the embedding x:

$$g_{ab} = \delta_{ij} \frac{\partial x^i}{\partial q^a} \frac{\partial x^j}{\partial q^b} . \tag{6.45}$$

Since (6.44) must be true for all δq^b, we can infer that:

$$Q^a = m \, \mathfrak{a}^a, \tag{6.46}$$

which is essentially Newton's second law, except that we have added a contribution to the tangential part of the acceleration that comes from the existence of a constraint. From (6.21), that tangential part is:

$$\mathfrak{a}^a = \ddot{q}^a + \omega^a_{(bc)} \, \dot{q}^b \dot{q}^c , \tag{6.47}$$

in which the connection that is used is the teleparallelism connection that is defined by the adapted frame field and its extension to an n-frame field. Thus, we could also write (6.46) in the form:

$$m \, \ddot{q}^a = Q^a - m \, \omega^a_{(bc)} \, \dot{q}^b \dot{q}^c . \tag{6.48}$$

Hence, the second term on the right-hand side can be interpreted as the force of constraint. Since the components of ω^a_{bc} can be interpreted as curvatures in the various directions, in effect, it represents a generalization of the centrifugal force that is associated with constraining motion to a circle to a higher-dimensional constraint space.

h. Lagrange's equations of the first and second kind. – When one confers more than one reference on the subject of the two forms of Lagrange's equations of motion, one immediately finds that there is a certain degree of disagreement as to what truly constitutes the proper form of those systems of equations. The approach that we shall take is based upon the difference between representing the constraint as an envelope, which will also be applicable to the case of non-holonomic coordinates, and representing the constraint as a locus, which is only applicable to holonomic constraints. The forms that we derive will allow us to show how the Euler-Lagrange equations can be generalized when one does not start with an action functional.

The first step is to represent the kinetic energy in the form:

$$T(\mathbf{v}) = \tfrac{1}{2} m \delta_{ij} v^i v^j \,, \tag{6.49}$$

so the inertial force can be represented in the form:

$$m \frac{dv_i}{dt} = \frac{d}{dt}\left(\frac{\partial T}{\partial v^i}\right) = -\frac{\delta T}{\delta x^i} \,. \tag{6.50}$$

That will allow one to rewrite the equations of motion (6.41) in the form:

$$-\frac{\delta T}{\delta x^i} = F_i + \lambda_\alpha \frac{\partial \phi^\alpha}{\partial x^i} \,. \tag{6.51}$$

That is what we shall regard as the *Lagrange equations of the first kind.*

In the event that the external force F is conservative ($F = - dU$), one can add $- U$ to T and get a Lagrangian $\mathcal{L}$, in which case the Lagrange equations of the first kind will take the quasi-Euler-Lagrange form:

$$\frac{\delta \mathcal{L}}{\delta x^i} = \lambda_\alpha \frac{\partial \phi^\alpha}{\partial x^i} \,. \tag{6.52}$$

Now, one has:

$$d(\lambda_\alpha \phi^\alpha) = d\lambda_\alpha \phi^\alpha + \lambda_\alpha d\phi^\alpha \,, \tag{6.53}$$

and if one expresses the constraint in the form $\phi^\alpha(x) = 0$ (which one can always do, since an arbitrary constant on the right-hand side would not affect the differential, which is what is definitive) then the first term on the right-hand side will vanish, and one can replace $\lambda_\alpha d\phi^\alpha$ with $d(\lambda_\alpha \phi^\alpha)$. Thus, the expression $\lambda_\alpha \phi^\alpha$ has the character of a potential function for the force of constraint. (Of course, it will only exist in the case of holonomic constraints.)

If the force F is conservative then one can express the right-hand side of (6.51) in the form $-d(U - \lambda_\alpha \phi^\alpha)$, and one can define a constrained Lagrangian for the equations of motion:

$$\mathcal{L}_c = \tfrac{1}{2} m v^2 - U + \lambda_\alpha \phi^\alpha, \tag{6.54}$$

which will allow one to express the constrained equations of motion in Euler-Lagrange form:

$$\frac{\delta \mathcal{L}_c}{\delta x^i} = 0. \tag{6.55}$$

As we shall see in a later section, a slight modification of equations (6.51) and (6.52) can also be used in the case of non-holonomic constraints. However, equations (6.55) will break down when there are no ϕ^α. Thus, they apply to only conservative forces and perfect holonomic constraints.

By contrast, in order to get the Lagrange equations of the second kind, one must introduce generalized coordinates, so they can only apply to holonomic constraints. The first thing to be done is to express the inertial force in terms of a variation derivative of kinetic energy. One has:

$$\frac{\delta T}{\delta q^a} = \frac{\partial T}{\partial q^a} - \frac{d}{dt}\frac{\partial T}{\partial q^a} = m[\tfrac{1}{2}\partial_a \, g_{bc}\, \dot{q}^b \, \dot{q}^c - \frac{dg_{ab}}{dt}\dot{q}^b - g_{ab}\,\ddot{q}^b],$$

and since:

$$\frac{dg_{ab}}{dt}\dot{q}^b = \partial_c \, g_{ab}\, \dot{q}^b \dot{q}^c = \tfrac{1}{2}(\partial_c \, g_{ab}\, \dot{q}^b \dot{q}^c + \partial_c \, g_{ab}\, \dot{q}^b \dot{q}^c),$$

one can write:

$$-\frac{\delta T}{\delta q^a} = m(g_{ab}\,\ddot{q}^b + [a, bc]\dot{q}^b\,\dot{q}^b) = m\, g_{ab}\left(\ddot{q}^b + \begin{Bmatrix} b \\ c\, d \end{Bmatrix}\dot{q}^c\,\dot{q}^d\right) = m\,\mathfrak{a}_a, \tag{6.56}$$

into which we have once more introduced the two types of Christoffel symbols that are associated with the metric g_{ab} on Π. Thus, we have shown that the inertial force, when expressed in generalized coordinates, equals the variational derivative of kinetic energy, when it is also expressed in those coordinates.

Thus, the equations of motion (6.46) in generalized coordinates can be put into the form:

$$-\frac{\delta T}{\delta q^a} = Q_a, \tag{6.57}$$

which are the *Lagrange equations of the second kind.* We point out that the existence of the generalized coordinates q^a implies that the constraints must be holonomic, so that defines one limitation of the use of these equations. Furthermore, the constraints must also be perfect since the normal part of the external forces must do no virtual work.

Traditionally, one says that the effect of the constraint is to change the ordinary derivative with respect to time into a covariant derivative that uses the Levi-Civita connection that is defined by the metric g. However, if one writes (6.57) in the form:

$$m\,\ddot{q}^a = Q^a - m\begin{Bmatrix} a \\ b\, c \end{Bmatrix}\dot{q}^b\,\dot{q}^c \tag{6.58}$$

then one will see that the contribution to the acceleration that comes from the Levi-Civita connection can also be interpreted as a force of constraint that effectively substitutes for the $\lambda_\alpha C^\alpha$, which would vanish on Π . Hence, the use of the Levi-Civita connection is also appropriate only for perfect holonomic constraints.

When the applied force is conservative (thus, global) and the components Q_a take the form:

$$Q_a = -\frac{\partial U}{\partial q^a} = -\frac{\delta U}{\delta q^a}, \tag{6.59}$$

if one defines the Lagrangian $\mathcal{L} = T - U$ then one will get the Euler-Lagrange equations:

$$\frac{\delta \mathcal{L}}{\delta q^a} = 0 . \tag{6.60}$$

Hence, one sees that these equations are also limited to conservative forces and perfect holonomic constraints, as (6.55) were.

i. Heun's central equation. – When one includes the constraint in the principle of virtual work by introducing generalized coordinates, combined with d'Alembert's principle, one can also obtain the Lagrange equations of the second kind Heun's central equation (2.17). One finds that the basic expressions in that equation convert into:

$$v_i\,\delta x^i = \dot{q}_a\,\delta q^a, \qquad F_i\,\delta x^i = Q_a\,\delta q^a, \qquad T = \tfrac{1}{2} m\,\delta_{ij} v^i v^j = \tfrac{1}{2} m\,g_{ab}\,\dot{q}^a\,\dot{q}^b . \tag{6.61}$$

One must first deal with the term that expresses the integrability of the virtual displacement of the velocity vector field:

$$\delta v^i - \frac{d}{dt}\delta x^i = \delta\left(\frac{\partial x^i}{\partial q^a}\dot{q}^a\right) - \frac{d}{dt}\left(\frac{\partial x^i}{\partial q^a}\delta q^a\right) = \delta\left(\frac{\partial x^i}{\partial q^a}\dot{q}^a\right) - \frac{\partial^2 x^i}{\partial q^a\,\partial q^b}\dot{q}^a\,\delta q^b - \frac{\partial x^i}{\partial q^a}\frac{d}{dt}\delta q^a .$$

Now, the operator δ can be represented on Π by the directional derivative in the direction of $\delta\mathbf{q}$, so:

$$\delta = \delta q^a \frac{\partial}{\partial q^a} . \tag{6.62}$$

It then has the properties as an operator that if f and g are arbitrary differentiable functions of q then:

$$\delta f = \frac{\partial f}{\partial q^a}\delta q^a, \qquad \delta(f\,g) = \delta f\,g + f\,\delta g , \tag{6.63}$$

which are basically just the chain rule and the product rule for differentiation. Hence:

$$\delta\left(\frac{\partial x^i}{\partial q^a}\dot{q}^a\right) = \frac{\partial^2 x^i}{\partial q^a\,\partial q^b}\dot{q}^a\delta q^b + \frac{\partial x^i}{\partial q^a}\delta\dot{q}^a,$$

and we finally get what Hamel [**11**] called the *transition equation* (*Überträgungsgleichung*):

$$\delta v^i - \frac{d}{dt}\delta x^i = \frac{\partial x^i}{\partial q^a}(\delta\dot{q}^a - \frac{d}{dt}\delta q^a). \tag{6.64}$$

That amounts to saying that the vector field on E^n whose components are on the left-hand side of that equation is the push-forward of the vector field on Π whose components are in parentheses on the right-hand side. As a result, if the expression on the right vanishes then so will the one on the left:

$$\text{If} \quad \delta\dot{q}^a = \frac{d}{dt}\delta q^a \qquad \text{then} \qquad \delta v^i = \frac{d}{dt}\delta x^i. \tag{6.65}$$

As long as both sides of the final equation are consistent with the constraints, the converse statement is also true, since although the matrix $\partial x^i / \partial q^a$ is not generally invertible, it still represents a linear injection (i.e., it is one-to-one). Hence, the only vector in any $T_q\Pi$ that will map to zero under it is zero itself.

The general form for the central equation in terms of the q coordinates that applies to non-integrable virtual displacements $\delta\dot{\mathbf{q}}(t)$ will then be:

$$\frac{d}{dt}(m\,\dot{q}_a\,\delta q^a) - \delta T + m\,\dot{q}_a\left(\delta\dot{q}^a - \frac{d}{dt}\delta q^a\right) = Q_a\,\delta q^a. \tag{6.66}$$

Since one can simply *define* the vector field $\delta\dot{\mathbf{q}}(t)$ to be the time derivative of $\delta\mathbf{q}\,(t)$, one can consider the central equation in its simpler form that applies to integrable virtual displacements:

$$\frac{d}{dt}(m\,\dot{q}_a\,\delta q^a) - \delta T = Q_a\,\delta q^a. \tag{6.67}$$

Now:

$$\delta T = \frac{\partial T}{\partial q^a}\delta q^a + \frac{\partial T}{\partial \dot{q}^a}\delta\dot{q}^a = \frac{\delta T}{\delta q^a}\delta q^a + \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^a}\delta q^a\right) = \frac{\delta T}{\delta q^a}\delta q^a + \frac{d}{dt}\left(m\,\dot{q}_a\,\delta q^a\right). \tag{6.68}$$

Hence, the central equation (6.67) will reduce to the form of Lagrange's equation in its second form (6.57) when (6.67) is true for all δq^a.

7. Non-holonomic constraints. – In the case of non-holonomic constraints, the differential system that is defined by the 1-forms $C^\alpha = 0$ is not completely integrable. Hence, integral submanifolds of maximum dimension do not exist. However, one finds that since most of the constructions that one defines for the holonomic case are infinitesimal in character – i.e., defined in terms of tangent and cotangent objects – there are many aspects of holonomic constraints that carry over to the non-holonomic case. At every step, however, one must consider the contribution that comes from non-integrability, which is usually comes from the non-involution of vector fields or the non-vanishing of $d_\wedge C^\alpha$.

a. Examples of non-holonomic constraints. – For Hamel [**11**], the simplest example of a system with non-holonomic constraints was a blade slicing in a plane that is not allowed to scrape transversely. If the coordinates of the point of contact of the blade (imagine that its cutting edge is convex) are x and y and the angle that the blade makes with the x-axis is θ then a point in the configuration manifold can be described by the coordinates (x, y, θ). Hence, the dimension of the configuration manifold is three, which is minimal for the existence of non-holonomic constraints.

The constraint that is imposed upon the motion of the blade is that its velocity must be collinear with its longitudinal axis. That can be expressed by the condition that:

$$\frac{dy}{dx} = \tan\theta \qquad \text{or} \qquad C = \sin\theta\, dx - \cos\theta\, dy = 0\,. \tag{7.1}$$

Since that makes:

$$d_\wedge C = \cos\theta\, d\theta \wedge dx + \sin\theta\, d\theta \wedge dy\,, \tag{7.2}$$

the Frobenius 3-form will be:

$$C \wedge d_\wedge C = - dx \wedge dy \wedge d\theta\,, \tag{7.3}$$

which does not vanish, because all three 1-forms involved are linearly independent.

Although the constraint subspaces are tangent planes, their non-integrability implies that although integral curves can exist, integral surfaces cannot. Such a non-integrable distribution of tangent planes on a three-dimensional manifold is sometimes referred to as a "pseudo-surface" ([1]).

The "canonical" example of a non-holonomic constraint is defined by a disc (or sphere) rolling without slipping on a plane or surface; for simplicity, we assume that the disc remains vertical throughout its motion and moves in a plane. The constraint takes the form of requiring the velocity $\mathbf{v}$ of translation at the point of contact to be proportional to the angular velocity of rotation $\boldsymbol{\omega}$, which is always perpendicular to $\mathbf{v}$ in its plane (which is the plane of translation), by way of the radius vector $\mathbf{r} = - R\,\mathbf{k}$ of the disc when it points downward, where R is the radius of the disc.

Hence, the position of the disc can be described by a point in a four-dimensional configuration manifold that is described by the two planar coordinates (x, y) of the point of contact and the two angular coordinates (ψ, θ), which describe the angle ψ through which the radius vector of the disc

([1]) One can also generalize to "pseudo-hypersurfaces" (see [**23**]) and even "pseudo-submanifolds."

has rolled about the rotational axis since some fixed initial time point and the steering angle θ that describes how much that rotational axis has rotated in the *xy*-plane from its initial position at that time point.

In order to formulate the constraints on the tangent vectors to the manifold $\mathbb{R}^4$ of points (x, y, ψ, θ), it is convenient to introduce a moving frame $\{\mathbf{e}_\alpha, \alpha = 1, 2\}$ in the plane that is defined by the disc, namely, $\mathbf{e}_1$ is a unit vector that points in the direction of translational motion, while $\mathbf{e}_2$ is the unit vector along the axis of rotation, which is oriented to make $\{\mathbf{e}_1, \mathbf{e}_2\}$ a right-handed orthonormal frame with its origin at the point of contact (x, y) of the disc with the plane. Hence, if $\{\mathbf{i}, \mathbf{j}\}$ is the right-handed orthonormal frame at (x, y) that is parallel to the x and y axes, respectively, then one will have:

$$\mathbf{e}_1 = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \qquad \mathbf{e}_2 = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\,. \tag{7.4}$$

The velocity vector $\mathbf{v} = \mathbf{v}_t + \mathbf{v}_r$ to a path in $\mathbb{R}^4$ consists of two parts: $\mathbf{v}_t$, which describes the translational velocities along the axes in the plane, and $\mathbf{v}_r$, which describes the rotational velocities in the space of angles; hence:

$$\mathbf{v}_\mathrm{t} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{k} = v^1\,\mathbf{e}_1 + v^2\,\mathbf{e}_2\,, \qquad \mathbf{v}_\mathrm{r} = \dot{\psi}\,\partial_\psi + \dot{\theta}\,\partial_\theta\,. \tag{7.5}$$

The constraint on the velocity of the curve $x(t)$ can then be put into the form:

$$v^1 = R\dot{\psi}\,, \qquad v^2 = 0\,. \tag{7.6}$$

If we denote the reciprocal coframe to $\mathbf{e}_\alpha$ by e^α [so $e^\alpha(\mathbf{e}_\beta) = \delta^\alpha_\beta$] then we can put those two constraints into the forms $e^1(\mathbf{v}) = R\,d\psi(\mathbf{v})$ and $e^2(\mathbf{v}) = 0$. That means that the constraints on velocity are defined by the vanishing of the two 1-forms:

$$C^1 = e^1 - R\,d\psi, \qquad C^2 = e^2\,. \tag{7.7}$$

The constraint sub-bundle $\mathcal{D}$ of $T(\mathbb{R}^4)$ is therefore composed of two-dimensional planar subspaces. In order to discover whether that constraint is holonomic or not, we need to examine the exterior derivatives of the constraint 1-forms, which are $d_\wedge C^1 = d_\wedge e^1$ and $d_\wedge C^2 = d_\wedge e^2$. Hence, we must first get the expressions for the coframe members in terms of dx and dy, which are:

$$e^1 = \cos\psi\,dx - \sin\psi\,dy\,, \qquad e^2 = \sin\psi\,dx + \cos\psi\,dy\,. \tag{7.8}$$

These take the form $e^\alpha = R^\alpha_\beta\,dx^\beta$, in which the planar rotation matrix is:

$$R^\alpha_\beta = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} . \tag{7.9}$$

The exterior derivatives of the coframe members can then be expressed in the form:

$$d_\wedge e^\alpha = \omega^\alpha_\beta \wedge e^\beta , \tag{7.10}$$

in which the infinitesimal planar rotation matrix is defined by:

$$\omega^\alpha_\beta = dR^\alpha_\gamma \, \tilde{R}^\gamma_\beta = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} d\psi \ . \tag{7.11}$$

The exterior derivatives of the constraint 1-forms are then:

$$d_\wedge C^\alpha = \omega^\alpha_\beta \wedge C^\beta + \Theta^\alpha, \qquad \text{where } \ \Theta^1 \equiv 0, \ \ \Theta^2 \equiv R \, d\psi \wedge d\theta . \tag{7.12}$$

Although the non-vanishing of Θ^α appears to prevent the complete integrability of the constraint sub-bundle into integral surfaces, one should keep in mind that the condition for complete integrability only said that there was *some* matrix of 1-forms η^α_β that would make $d_\wedge C^\alpha$ take the form $\eta^\alpha_\beta \wedge C^\beta$, not that the condition would have to be true for *all* matrices of 1-forms. Thus, it is always possible that a better choice of η^α_β than ω^α_β would put $d_\wedge C^\alpha$ into the desired form. However, in this case, that is not true, although we shall not elaborate here.

Note that the example of a disc rolling without slipping on a line or curve would not suffice to exhibit a non-holonomic constraint, because the dimension of the configuration manifold would then reduce to just two, namely, the space of the coordinates (x, ψ). Hence, any 3-form (such as the Frobenius 3-form) would have to vanish automatically, and the constraint would be holonomic in that case.

b. Adapted frame fields. – Although, by definition, an integral submanifold $x : \Pi \to E^n$ of dimension p will not exist in the non-holonomic case, nonetheless, one can still define a substitute for the differential maps $dx|_q : T_q\Pi \to T_{x(q)} E^n$ that will allow one to duplicate many of the constructions that one made in the holonomic case, namely, a frame field on E^n that is adapted to the constraint.

One first notices that a linear injection $\iota_x : \mathbb{R}^p \to T_x E^n$ defines a p-frame $\{\mathbf{e}_a (x), a = 1, \ldots, p\}$ in the p-dimensional linear subspace $\mathcal{D}_x$ of $T_x E^n$ that is the image of ι_x . One simply defines the frame vectors $\mathbf{e}_a(x)$ to be the images of the corresponding canonical basis vectors [1] $\boldsymbol{\delta}_a$ in $\mathbb{R}^p$ under ι_x:

$$\mathbf{e}_a(x) = \iota_x(\boldsymbol{\delta}_a) . \tag{7.13}$$

Since the p-frame $\mathbf{e}_a(x)$ is defined for every x in E^n, it will be a p-frame field on E^n.

One can express the p-frame field $\mathbf{e}_a(x)$ in terms of the natural frame field in the form:

$$\mathbf{e}_a(x) = x_a^i(x)\,\partial_i . \tag{7.14}$$

Thus, the component functions $x_a^i(x)$ can substitute for the functions $x_{,a}^i(q)$ that only exist in the holonomic case. In particular, if $\xi^a(x)$ are p scalar functions on E^n then one can define a vector field $\mathbf{X}(x)$ on E^n by:

$$\mathbf{X}(x) = \xi^a(x)\,\mathbf{e}_a(x) = \xi^a(x)\,x_a^i(x)\,\partial_i . \tag{7.15}$$

That is, the components with respect to the two frame fields on E^n are related by:

$$X^i = x_a^i\,\xi^a . \tag{7.16}$$

The frame field $\mathbf{e}_a(x)$ is adapted to the constraints when all of the frame members are annihilated by the 1-forms C^α:

$$C^\alpha(\mathbf{e}_a) = 0, \qquad \text{for all } \alpha, a . \tag{7.17}$$

Locally that will then take the matrix form:

$$C_i^\alpha\,x_a^i = 0, \tag{7.18}$$

which replaces the corresponding holonomic condition that $\phi_{,i}^\alpha\,x_{,a}^i = \partial_a(\phi \cdot x)^\alpha$, which is no longer well-defined.

As a consequence of the definition of an adapted frame field, any linear combination of adapted frame vectors, such as in (7.15), will also be consistent with the constraint that is defined by the C^α. As a result, one can regard the scalars $X^\alpha = C^\alpha(\mathbf{X})$ as the normal components of the vector $\mathbf{X}$, which must then vanish when $\mathbf{X}$ is consistent with the constraint.

Extending the adapted p-frame field $\mathbf{e}_a(x)$ to an n-frame field $\mathbf{e}_i(x)$ works the same way as in the holonomic case, except that the invertible $n\times n$ matrix $x_j^i(x)$ that extends $x_a^i(x)$ is a function of the points x in E^n where $\mathbf{e}_a(x)$ is defined, not the points q in Π. The definition of the reciprocal coframe field to $\mathbf{e}_i$ is now:

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[1] Recall that the canonical basis on $\mathbb{R}^p$ is defined by the vectors $\boldsymbol{\delta}_1 = (1, 0, \ldots, 0), \ldots, \boldsymbol{\delta}_p = (0, \ldots, 1)$.

$$e^i = \tilde{x}^i_j(x)\,dx^j\,, \tag{7.19}$$

which is assumed to be adapted to the constraint 1-forms, so:

$$e^{p+\alpha} = C^\alpha, \qquad \tilde{x}^{p+\alpha}_j = C^\alpha_j\,. \tag{7.20}$$

The differential equation of the total moving frame $\mathbf{e}_i$ will be:

$$d\mathbf{e}_i = \omega^j_i \otimes \mathbf{e}_j\,, \tag{7.21}$$

in which one now defines:

$$\omega^j_i = dx^j_k\,\tilde{x}^k_i \qquad (\omega^j_{ik} = x^l_{i,k}\,\tilde{x}^j_l). \tag{7.22}$$

In particular, the adapted frame field $\mathbf{e}_a$ will then have the equations:

$$d\mathbf{e}_a = \omega^b_a \otimes \mathbf{e}_b + \omega^{p+\alpha}_a \otimes \mathbf{e}_{p+\alpha}\,. \tag{7.23}$$

Note that since:

$$\omega^{p+\alpha}_{bc} = (\mathbf{e}_c x^k_b)\,\tilde{x}^{p+\alpha}_k = (\mathbf{e}_c x^k_b)\,C^\alpha_k\,, \tag{7.24}$$

the components of the matrices $\omega^{p+\alpha}_{bc}$ can be expressed in terms of geometric objects that were defined by the constraints to begin with, so they do not depend upon a choice of extension of the adapted frame field $\mathbf{e}_a$.

One can already see that the components of $\omega^{p+\alpha}_{bc}$ will not be symmetric in the lower two indices, and in fact:

$$\omega^{p+\alpha}_{bc} - \omega^{p+\alpha}_{cb} = (\mathbf{e}_c x^k_b - \mathbf{e}_b x^k_c)\,C^\alpha_k = [\mathbf{e}_c, \mathbf{e}_b]^k\,C^\alpha_k = c^k_{cb}\,C^\alpha_k\,, \tag{7.25}$$

which will not vanish, in general, when the constraints are non-holonomic.

c. Constrained curves. – As we said before, integral curves can exist for the case of non-holonomic constraints. Thus, one can at least have integral curves that are consistent with those constraints, but not necessarily higher-dimensional integral submanifolds.

If the constraint subspaces $\mathcal{D}_x E^n$ are the tangent subspaces to the configuration manifold E^n that are annihilated by the set of linearly-independent 1-forms $\{C^\alpha, \alpha = 1, \ldots, n-p\}$, and $\mathbf{v}(t) = v^i(t)$ is the velocity vector field to a differentiable curve $x(t)$ in E^n then the curve is constrained to the $\mathcal{D}_{x(t)} E^n$ when its velocity satisfies:

$$0 = C^\alpha(\mathbf{v}(t)) = C^\alpha_i(x(t))\,v^i(t) \qquad \text{for all } t, \alpha\,. \tag{7.26}$$

By differentiation, that will imply the corresponding constraint on the acceleration of the curve:

$$0 = \frac{dC^\alpha}{dt}(\mathbf{v}) + C^\alpha\left(\frac{d\mathbf{v}}{dt}\right).$$

If we define the acceleration vector field to be $\mathbf{a} = d\mathbf{v} / dt$ then we can express that constraint in the form:

$$C^\alpha(\mathbf{a}) = -\frac{dC^\alpha}{dt}(\mathbf{v}). \tag{7.27}$$

However, we must still make sense of what we mean by dC^α / dt. Since they are the derivatives of the 1-forms C^α along the curve $x(t)$, we shall interpret that as the total derivative with respect to $\mathbf{v}(t)$. We will have:

$$\frac{dC_i^\alpha}{dt} = \frac{dx^j}{dt}\frac{\partial C_i^\alpha}{\partial x^j} = C_{i,j}^\alpha v^j. \tag{7.28}$$

When we apply those 1-forms to $\mathbf{v}$ that will make the constraint on acceleration take the form:

$$C^\alpha(\mathbf{a}) = -C_{(i,j)}^\alpha v^i v^j. \tag{7.29}$$

That is because when the constraint was holonomic (so $C^\alpha = d\phi^\alpha$), the second derivatives $\phi_{,i,j}^\alpha$ were all symmetric in the i and j, but in the non-holonomic case the $C_{i,j}^\alpha$ are not necessarily symmetric, so one must symmetrize them. That also means that the antisymmetric part of the differential dC^α (i.e., $d_\wedge C^\alpha$) does not contribute to the constraint on acceleration, since that antisymmetric part will only appear in the Lie derivatives of the C^α with respect to $\mathbf{v}$:

$$\mathrm{L}_{\mathbf{v}}C^\alpha = i_{\mathbf{v}}d_\wedge C^\alpha + d\, i_{\mathbf{v}}C^\alpha = i_{\mathbf{v}}d_\wedge C^\alpha = v^i(\partial_i C_j^\alpha - \partial_j C_i^\alpha)\, dx^j.$$

Since v^i is not a function of position, but only time, it can be absorbed into the second partial derivative on the right, and since v^i is assumed to be consistent with the constraint, the second term will vanish, which will leave:

$$\mathrm{L}_{\mathbf{v}}C^\alpha = \frac{dC_i^\alpha}{dt}dx^i \tag{7.30}$$

When one evaluates the 1-form $\mathrm{L}_{\mathbf{v}}C^\alpha$ on $\mathbf{v}$, that will give:

$$(i_{\mathbf{v}}d_\wedge C^\alpha)(\mathbf{v}) = d_\wedge C^\alpha(\mathbf{v}, \mathbf{v}) = 0,$$

from antisymmetry.

The symmetric, doubly-covariant tensor fields:

$$H^\alpha = C_{(i,j)}^\alpha\, dx^i dx^j \tag{7.31}$$

can be regarded as a generalization of the second fundamental form for a p-dimensional submanifold in E^n, which would only exist in the holonomic case, by definition. Hence, one sees that one can still define a substitute for that second fundamental form in the non-holonomic case, since it is constructed from tangent and cotangent objects that will still exist, even though they might not be completely integrable. One can then refer to the distribution $\mathcal{D}(E^n)$ of p-dimensional tangent subspaces as a "pseudo-submanifold" of E^n. Equation (7.29) basically says that the normal component a^α of the acceleration vector field can be obtained from the second fundamental form by means of:

$$a^\alpha = - H^\alpha(\mathbf{v}, \mathbf{v}) . \tag{7.32}$$

Now, let us assume that we have a p-frame field $\mathbf{e}_a(x)$ on E^n that is adapted to the constraint subspaces, and let $x(t)$ be a differentiable curve in E^n. If the velocity vector field $\mathbf{v}(t)$ of the curve is constrained to the subspaces $\mathcal{D}_x(t)$ then there will be scalar functions $\upsilon^a(t)$ that make:

$$\mathbf{v}(t) = \upsilon^a(t)\,\mathbf{e}_a(x(t)) = \upsilon^a(t)\,x_a^i(x(t))\,\partial_i . \tag{7.33}$$

Since $\mathbf{v}(t) = v^i(t)\,\partial_i$, that will imply that the components v^i must satisfy:

$$v^i = \upsilon^a x_a^i \tag{7.34}$$

if they are to be consistent with the constraint.

The constraint on acceleration will now take the form:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \dot{\upsilon}^a\,\mathbf{e}_a + \upsilon^a\,\dot{\mathbf{e}}_a . \tag{7.35}$$

When we observe that $\dot{\mathbf{e}}_a = d\mathbf{e}_a(\mathbf{v})$ and substitute the equation (7.23) of the moving frame $\mathbf{e}_a$ that will give

$$\mathbf{a} = \dot{\upsilon}^a\,\mathbf{e}_a + \upsilon^b[\omega_b^a(\mathbf{v})\,\mathbf{e}_a + \omega_b^{p+\alpha}(\mathbf{v})\,\mathbf{e}_{p+\alpha}],$$

and after some reorganization, we will have:

$$\mathbf{a} = [\dot{\upsilon}^a + \upsilon^b\,\omega_b^a(\mathbf{v})]\,\mathbf{e}_a + \upsilon^b\omega_b^{p+\alpha}(\mathbf{v})\,\mathbf{e}_{p+\alpha} . \tag{7.36}$$

Once again, this acceleration has tangential and normal components, namely:

$$\mathfrak{a}^a = \dot{\upsilon}^a + \upsilon^b\,\omega_b^a(\mathbf{v}) = \dot{\upsilon}^a + \omega_{(bc)}^a\,\upsilon^b\upsilon^c , \qquad \mathfrak{a}^{p+\alpha} = \omega_{(bc)}^{p+\alpha}\,\upsilon^b\upsilon^c , \tag{7.37}$$

respectively. Note that, once again, since ω^i_{bc} does not have to be symmetric in b and c, while the product $\upsilon^b\upsilon^c$ is symmetric in those indices, one must symmetrize ω^i_{bc}:

$$\omega^a_{(bc)} = \tfrac{1}{2}(\omega^a_{bc} + \omega^a_{cb}) = \tfrac{1}{2}(\mathbf{e}_c x^k_b + \mathbf{e}_b x^k_c)\,\tilde{x}^\alpha_k\,, \tag{7.38}$$

$$\omega^{p+\alpha}_{(bc)} = \tfrac{1}{2}(\omega^{p+\alpha}_{bc} + \omega^{p+\alpha}_{cb}) = \tfrac{1}{2}(\mathbf{e}_c x^k_b + \mathbf{e}_b x^k_c)\,\tilde{x}^{p+\alpha}_k = \tfrac{1}{2}(\mathbf{e}_b x^k_c + \mathbf{e}_c x^k_b)\,C^\alpha_k\,. \tag{7.39}$$

If we return to equation (7.31), which gives the components of H^α with respect to the natural coframe field dx^i and convert to its components with respect to the extended adapted coframe field e^i then we will get:

$$H^\alpha_{i\,j} = \tfrac{1}{2}(C^\alpha_{k,m} + C^\alpha_{m,k})\,x^k_i\,x^m_j = \tfrac{1}{2}(\,x^m_j\,C^\alpha_{k,m}\,x^k_i + x^k_i C^\alpha_{m,k}\,x^m_j\,) = \tfrac{1}{2}[(\mathbf{e}_j C^\alpha_k)\,x^k_i + (\mathbf{e}_i C^\alpha_k)\,x^k_j\,]\,.$$

Of course, when one evaluates the 1-forms e^i on an adapted velocity vector field $\mathbf{v}$ (t), one will get the components υ^i of $\mathbf{v}$ with respect to the adapted frame field $\mathbf{e}_i$, so only υ^α will be non-vanishing, in general, and one can concentrate on the components:

$$H^\alpha_{bc} = \tfrac{1}{2}[(\mathbf{e}_b C^\alpha_k)\,x^k_c + (\mathbf{e}_c C^\alpha_k)\,x^k_b\,]\,. \tag{7.40}$$

However, since $C^\alpha_k = \tilde{x}^{p+\alpha}_k$, one will have $C^\alpha_k\,x^k_c = \tilde{x}^{p+\alpha}_k\,x^k_c = \delta^{p+\alpha}_c$, which will make:

$$(\mathbf{e}_b\,C^\alpha_k)\,x^k_c = -\,C^\alpha_k\,(\mathbf{e}_b\,x^k_c)\,. \tag{7.41}$$

Therefore:

$$H^\alpha_{bc} = -\,\tfrac{1}{2}(\mathbf{e}_b\,x^k_c + \mathbf{e}_c\,x^k_b)\,C^\alpha_k = -\,\omega^{p+\alpha}_{(bc)}\,. \tag{7.42}$$

Thus, what we were calling the second fundamental form of this pseudo-submanifold contains the same information as the normal part of the connection that we have defined.

An important special case of a constrained curve is then defined by curves whose acceleration vector field is always normal to the constraint spaces. Since that means that the tangential part of the acceleration vanishes, the equation for the components of the velocity vector field for such a curve will be:

$$0 = \dot{\upsilon}^a + \omega^a_b(\mathbf{v})\,\upsilon^b = \dot{\upsilon}^a + \omega^a_{(bc)}\,\upsilon^b\,\upsilon^c\,. \tag{7.43}$$

One sees that equation (7.43) is the geodesic equation for the connection on the frame bundle that is defined by the 1-forms ω^j_i. However, even if one imposes the constraint that the n-frame field $\mathbf{e}_i$ must be orthonormal, so the connection will become a metric connection, it is not generally the same as the Levi-Civita connection that is defined by the metric g_{ab}, which must always have zero torsion. The present connection ω^j_i is the one that is defined most naturally by the frame field $\mathbf{e}_i$, namely, the aforementioned teleparallelism connection. Its torsion 2-form S^a is obtained from the antisymmetric part of ω^a_{bc}:

$$S^a = \tfrac{1}{2}(\omega^a_{bc} - \omega^a_{cb})\, e^b \wedge e^c = \tfrac{1}{2} c^a_{bc}\, e^b \wedge e^c\,. \tag{7.44}$$

Now:

$$\omega^a_{bc} - \omega^a_{bc} = (\mathbf{e}_c\, x^k_b - \mathbf{e}_b\, x^k_c)\, \tilde{x}^a_k\,,$$

while:

$$[\mathbf{e}_b, \mathbf{e}_c] = (\mathbf{e}_b\, x^k_c - \mathbf{e}_c\, x^k_b)\, \tilde{x}^l_k\, \mathbf{e}_l = (\mathbf{e}_b\, x^k_c - \mathbf{e}_c\, x^k_b)\, \tilde{x}^a_k\, \mathbf{e}_a + (\mathbf{e}_b\, x^k_c - \mathbf{e}_c\, x^k_b)\, \tilde{x}^{p+\alpha}_k\, \mathbf{e}_{p+\alpha}\,,$$

which makes:

$$\omega^a_{bc} - \omega^a_{bc} = -\, C^\alpha([\mathbf{e}_b, \mathbf{e}_c]) = d_\wedge C^\alpha(\mathbf{e}_b, \mathbf{e}_c)\,; \tag{7.45}$$

that is:

$$S^a = d_\wedge C^\alpha. \tag{7.46}$$

Therefore, the torsion of the teleparallelism connection that is defined by an adapted frame field originates in the anholonomity of the constraints.

Note that a general property of the geodesic equation is that since the second term is quadratic in the velocities, and therefore symmetric in them, if the connection that one uses has any non-vanishing torsion then the torsion will not contribute to the geodesic equation. Hence, since the Levi-Civita connection is the unique metric connection that has zero torsion, if the adapted frame field is orthonormal then the teleparallelism connection will imply the same geodesics as the Levi-Civita connection.

Meanwhile, the normal components of the acceleration will always be:

$$a^{p+\alpha}_n = \omega^{p+\alpha}_{(bc)}\, \upsilon^b \upsilon^c = -\, H^\alpha(\mathbf{v}, \mathbf{v})\,, \tag{7.47}$$

which is also symmetric in bc, so one can say that the torsion of the connection does not contribute to acceleration in general, and not merely to the geodesic equation, in particular.

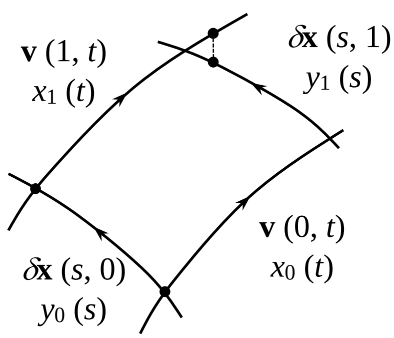
When the adapted frame field is defined for holonomic constraints, one replaces $x^i_a(x)$ with $x^i_{,a}(q)$ and C^α with $d\phi^\alpha$, and one sees that S^α will vanish in that case.

d. – Virtual displacements and virtual paths. – One can still define a virtual displacement of a curve $x(t)$ in E^n to be a vector field $\delta\mathbf{x}(t) = \delta x^i(t)\,\partial_i$ along the curve that is transverse to the velocity vector field $\mathbf{v}(t)$. The condition that the virtual displacement should be consistent with the constraint that is defined by the 1-forms C^α is then:

$$0 = C^\alpha(\delta\mathbf{x}(t)) = C^\alpha_i\, \delta x^i\,. \tag{7.48}$$

However, one should be aware that if one were to obtain the vector field $\delta\mathbf{x}\,(t)$ by differentiating a differentiable homotopy $h\,(s, t)$ with respect to s at $t = 0$ then unless the differential system $\mathcal{D}(E^n)$ that defines the constraint were integrable to the extent that it admits integral surfaces, the virtual paths that h defines for each fixed s would not generally be consistent with the constraint.

Basically, the issue is whether extensions of $\mathbf{v}\,(t)$ and $\delta\mathbf{x}\,(t)$ to vector fields $\mathbf{v}\,(x)$ and $\delta\mathbf{x}\,(x)$ on some neighborhood of $x\,(t)$ that were both consistent with the constraints could have a vanishing Lie bracket $[\mathbf{v}, \delta\mathbf{x}]$ on that neighborhood. If not then if one followed an integral curve $x_0\,(t)$ of $\mathbf{v}$ along some finite segment (say, from t_0 to t_1) and then followed the integral curve $y_1\,(s)$ of $\delta\mathbf{x}$ that went through $y_1\,(s_0) = x_0\,(t_1)$ to the point at $y_1\,(s_1)$ then one would not reach the same point as if one first followed the integral curve $y_0\,(s)$ of $\delta\mathbf{x}$ that went through $y_0\,(s_0) = x_0\,(t_0)$ until the point $y_0\,(s_1)$ and then followed the integral curve of $\mathbf{v}$ that went through $y_0\,(s_1) = x_1\,(t_0)$ until one reached the point $x_1\,(t_1)$; i.e., $x_1\,(t_1) \neq y_1(s_1)$. Hence, the curvilinear rectangle that is defined by those four curve segments would not close and could not bound a surface patch. We illustrate that situation in the accompanying figure. One should compare it with the corresponding figure that was depicted in the case of holonomic constraints.



The best way to evaluate $[\mathbf{v}, \delta\mathbf{x}]$ is in an adapted frame field $\mathbf{e}_a$, because $\mathbf{v} = \upsilon^a\,\mathbf{e}_a$ and $\delta\mathbf{x} = \delta q^a\,\mathbf{e}_a$ will automatically be consistent with the constraints. That is, the components of $\delta\mathbf{x}$ with respect to the natural frame field will be:

$$\delta x^i = x^i_a\,\delta q^a\,. \tag{7.49}$$

One would then have:

$$[\mathbf{v}, \delta\mathbf{x}] = (\upsilon^b\,\mathbf{e}_b\,\delta q^a - \upsilon^b\,\mathbf{e}_b\,\delta q^a)\,\mathbf{e}_a + \upsilon^a\,\delta q^b\,[\mathbf{e}_a\,, \mathbf{e}_b]\,. \tag{7.50}$$

Although the first term on the right-hand side will be consistent with the constraints, in the non-holonomic case, the Lie bracket $[\mathbf{e}_a\,, \mathbf{e}_b]$ will include a normal component, which will prevent the aforementioned curvilinear rectangle that is defined by the integral curves of $\mathbf{v}$ and $\delta\mathbf{x}$ from closing into the boundary of a surface patch.

In fact, when one uses the "intrinsic" formula for the exterior derivative of C^α when it is evaluated on $\mathbf{v}$ and $\delta\mathbf{x}$, one will get:

$$d_\wedge C^\alpha\,(\mathbf{v}, \delta\mathbf{x}) = \mathbf{v}\,(C^\alpha\,(\delta\mathbf{x})) - \delta\mathbf{x}\,(C^\alpha\,(\mathbf{v})) - C^\alpha\,([\mathbf{v}, \delta\mathbf{x}]) = -\,C^\alpha\,([\mathbf{v}, \delta\mathbf{x}]), \tag{7.51}$$

in which the first two terms on the right-hand side of the first equation vanish because both vector fields $\mathbf{v}$ and $\delta\mathbf{x}$ are consistent with the constraints. Hence, as long as $d_\wedge C^\alpha$ does not vanish, the Lie bracket $[\mathbf{v}, \delta\mathbf{x}]$ will have a non-vanishing normal component.

Of course, the fact that the curvilinear rectangle does not close will not be a problem as long as one considers matters infinitesimally along the curve $x\,(t)$.

e. Virtual work. – If $F(t) = F_i(x(t))\, dx^i$ is the applied force 1-form along the curve $x(t)$ then one can form the virtual work $F(\delta\mathbf{x})$ done by F along the virtual displacement $\delta\mathbf{x}$, and if the virtual displacement is consistent with the constraints then one can express that virtual work in the form:

$$\delta W = F(\delta\mathbf{x}) = F_i\, \delta x^i = F_i\, x_a^i\, \delta q^a = Q_a\, \delta q^a, \tag{7.52}$$

when one defines:

$$Q_a = F_i\, x_a^i\,. \tag{7.53}$$

Note that whereas the matrix x_a^i represents an *injection* of vectors, it represents a *projection* of covectors. That is, if F has normal components, they will not contribute to the virtual work that it does along $\delta\mathbf{x}$ when $\delta\mathbf{x}$ is consistent with the constraints, but only its components that are "tangential."

One can also use an adapted frame field $\mathbf{e}_a$ to represent the vector field $\delta\mathbf{x}$ in a manner that is consistent with the constraints, while representing the 1-form F, which does not have to be consistent with the constraints, in terms of the reciprocal coframe field e^i to the completion of $\mathbf{e}_a$:

$$F = F_i\, e^i = Q_a\, e^a + Q_{p+\alpha}\, e^{p+\alpha}, \quad \delta\mathbf{x} = \delta q^a\, \mathbf{e}_a\,. \tag{7.54}$$

Of course, since $e^{p+\alpha}(\mathbf{e}_a) = 0$ for every α and a, δW will again reduce to $Q_a\, \delta q^a$. That is, the normal component of F does no virtual work when the virtual displacement is consistent with the constraints. Hence, the constraints must be implicitly perfect constraints.

One can also employ Lagrange multipliers in the case of non-holonomic constraints. The force of constraint still takes the form:

$$F_c = \lambda_\alpha\, C^\alpha, \tag{7.55}$$

which then allows one to express the principle of virtual work in the form:

$$(F + \lambda_\alpha\, C^\alpha)(\delta\mathbf{x}) = 0 \qquad\qquad \text{for all } \delta\mathbf{x}, \tag{7.56}$$

because the introduction of Lagrange multipliers has made it possible to treat the virtual displacements $\delta\mathbf{x}$ as free now. Hence, equilibrium will imply the equations:

$$F = -\lambda_\alpha\, C^\alpha. \tag{7.57}$$

Thus, equilibrium implies that the resultant of the external forces that act upon each point of the configuration space must be equal and opposite to the force of constraint.

However, even when F is conservative, one cannot generally regard the sum:

$$F + \lambda_\alpha\, C^\alpha = -\, dU + \lambda_\alpha\, C^\alpha \tag{7.58}$$

as an exact 1-form, since its exterior derivative is $d\lambda_\alpha \wedge C^\alpha + \lambda_\alpha \, d_\wedge C^\alpha$, which is typically non-vanishing.

f. D'Alembert's principle. – In order to treat d'Alembert's principle in the case of non-holonomic constraints, one must first address the issue of the work done by acceleration. In general, one has:

$$\delta W_a = m\, a_i\, \delta x^i = m\, \delta_{ij}\, \frac{dv^i}{dt}\, \delta x^j \tag{7.59}$$

with respect to a natural frame field.

When we include the forces of constraint using Lagrange multipliers, the principle of virtual work, when combined with d'Alembert's principle, will take the form:

$$m\, \delta_{ij}\, \frac{dv^i}{dt}\, \delta x^j = (F_i + \lambda_\alpha C_i^\alpha)\; \delta x^i\,, \tag{7.60}$$

and if that is to be true for all δx^i then we must have:

$$m\, a_i = F_i + \lambda_\alpha C_i^\alpha\,, \tag{7.61}$$

which are then the equations of constrained motion. Of course, since we have introduced $n - p$ new unknown variables, in the form of λ_α, we will have to include the equations of constraint with the equations of motion in order to solve for all of the unknowns, namely:

$$C^\alpha(\mathbf{v}) = 0 \qquad\qquad (C_i^\alpha v^i = 0) \qquad (a = 1, \ldots, n - p). \tag{7.62}$$

Now, let us examine the form that the principle of virtual work, combined with d'Alembert's principle takes in a frame field $\mathbf{e}_a$ that is adapted to the constraints, which we extend to an n-frame field $\{\mathbf{e}_i,\, i = 1, \ldots, n\}$, and adapt its reciprocal coframe field e^i to C^α (so $e^\alpha = C^\alpha$). We represent F, a, and $\delta\mathbf{x}$ in terms of that frame and its reciprocal coframe:

$$F = Q_a\, e^a + Q_\alpha\, e^\alpha, \qquad a = \mathfrak{a}_a\, e^a + \mathfrak{a}_\alpha\, e^\alpha, \qquad \delta\mathbf{x} = \delta q^a\, \mathbf{e}_a\,, \tag{7.63}$$

in which we have defined $\delta\mathbf{x}$ to be consistent with the constraints, and $\mathfrak{a}_i = g_{ij}\, \mathfrak{a}^j$ with $\mathbf{a}$ expressed as in (7.36).

The virtual work that is done by all forces (viz., external and inertial) under the virtual displacement $\delta\mathbf{x}$ will then take the form:

$$\delta W = \delta_{ij}(F^i - m a^i)\, \delta x^j = g_{ab}\, (Q^a - m\, \mathfrak{a}^a)\; \delta q^a\,. \tag{7.64}$$

Hence, if the virtual work done by a virtual displacement $\delta\mathbf{x}$ that is consistent with the constraints vanishes for all δq^b then that will once more give:

$$Q^a = m\,\mathfrak{a}^a \tag{7.65}$$

for the equations of motion, as it did in the holonomic case. However, the tangential acceleration will now take the form:

$$\mathfrak{a}^a = \dot{\upsilon}^a + \omega^a_{(bc)}\,\upsilon^b\upsilon^c\,, \tag{7.66}$$

since the absence of integral submanifolds will imply that one cannot define generalized coordinates anymore, but only the components of the velocity vector with respect to the adapted frame field. Furthermore, the only thing that changes in the definition of the components ω^a_{bc} is that the components x^i_a of the members $\mathbf{e}_a$ of the adapted frame field with respect to the natural one can no longer take the form $x^i_{,a}$ of partial derivatives of an embedding. As a result, the partial derivatives $\partial_b\, x^i_a$ will not be symmetric, so the resulting teleparallelism connection will have non-vanishing torsion. However, the torsion part will not contribute to the geodesic equation.

If we put the equations of motion into the form:

$$m\,\dot{\upsilon}^a = Q^a - m\,\omega^a_{(bc)}\,\upsilon^b\upsilon^c \tag{7.67}$$

then we can regard the second term on the right-hand side as a force of constraint that replaces the one that gets defined by Lagrange multipliers in the natural frame field, namely:

$$\mathfrak{Q} = -[m\,\omega^a_{(bc)}\,\upsilon^b\upsilon^c]\,\mathbf{e}_a\,. \tag{7.68}$$

g. Lagrange's equations of the first and second kind. – In order to get Lagrange's equations in the first form when one has non-holonomic constraints, the only change that must be made is to replace the exact constraint 1-forms $d\phi^\alpha$ with more general 1-forms C^α. That would then imply the equations of motion:

$$-\frac{\delta T}{\delta x^i} = F_i + \lambda_\alpha C^\alpha_i\,, \tag{7.69}$$

which are Lagrange's equations of the first kind in a natural frame field. Thus, the first form of Lagrange's equations (6.51) can still be used in the case of non-holonomic constraints, as long as one replaces the exact 1-forms $d\phi^\alpha$ with the more general ones C^α.

When the force 1-form F is conservative, one can combine its potential function with the kinetic energy and get a Lagrangian $\mathcal{L}$ that will put the latter equations into the quasi-Euler-Lagrange form:

$$\frac{\delta \mathcal{L}}{\delta x^i} = \lambda_\alpha C^\alpha_i\,. \tag{7.70}$$

However, since the C^α are not exact now, one can go no further.

Strictly speaking, Lagrange's equations of the second kind cannot be defined in the usual way in the case of non-holonomic constraints, because the generalized coordinates will not typically exist. However, one must be cautioned that sometimes authors have been known to introduce "quasi-coordinates," which are obtained by integrating 1-forms along paths connecting pairs of points. The problem with that is that unless the 1-form that is being integrated is exact, the integral will not be path-independent, so the resulting number will be ambiguous. Nonetheless, as we have pointed out before, as long as one is dealing with only tangent and cotangent objects to the configuration manifold, one can introduce an adapted frame field as a substitute to the push-forward of the natural frame field on the parameter space and still define some of the same things.

In order to get the Lagrange equations of the second kind in an adapted frame field from the equations of motion in the form (7.65), the first challenge is to replace $m\,\mathfrak{a}^a$ with an expression that will substitute for the variational derivative $\delta T / \delta q^a$ when there are no q^a. Basically, we introduce it as merely a symbolic representation of a more rigorously defined expression.

In a natural frame, the variation of T will take the form:

$$\delta T = \frac{\partial T}{\partial x^i}\,\delta x^i + \frac{\partial T}{\partial v^i}\,\delta v^i\,, \tag{7.71}$$

and if we replace δx^i with $x^i_a\,\delta q^a$ and δv^i with $x^i_a\,\delta \upsilon^a$ then we can write:

$$\delta T = \mathbf{e}_a T\;\delta q^a + \frac{\partial T}{\partial v^i}\,x^i_a\,\delta \upsilon^a\,,$$

but:

$$\frac{\partial T}{\partial v^i}\,x^i_a \;=\; m v_i\,x^i_a = m\;\upsilon_a = \frac{\partial T}{\partial \upsilon^a}\,,$$

so:

$$\delta T = \mathbf{e}_a T\;\delta q^a + \frac{\partial T}{\partial \upsilon^a}\,\delta \upsilon^a\,. \tag{7.72}$$

If we define $\delta\upsilon^a$ to be integrable:

$$\delta\upsilon^a = \frac{d}{dt}\,\delta q^a \tag{7.73}$$

then that will make:

$$\delta T = \frac{\delta T}{\delta q^a}\,\delta q^a + \frac{d}{dt}\left(\frac{\partial T}{\partial \upsilon^a}\,\delta q^a\right), \tag{7.74}$$

in which we now define:

$$\frac{\delta T}{\delta q^a} \equiv \mathbf{e}_a T - \frac{d}{dt}\frac{\partial T}{\partial \upsilon^a}\,, \tag{7.75}$$

this time. Thus, the directional derivatives in the directions of the frame members have substituted for the partial derivatives with respect to the generalized coordinates.

In order to go further and deduce a non-holonomic replacement for Lagrange's equations in their second form, it helps to start with Heun's central equation.

h. Heun's central equation. – Let us now see how Heun's central equation is expressed in an adapted frame. First, it is important to examine what happens to the transition equation, which relates the non-integrability of the virtual displacement $\delta\dot{x}^i$ to that of $\delta\upsilon^a$. One has:

$$\frac{d}{dt}\delta x^i - \delta\dot{x}^i = \frac{d}{dt}(x^i_a\,\delta q^a) - \delta(x^i_a\,\upsilon^a) = \frac{dx^i_a}{dt}\delta q^a + x^i_a\frac{d}{dt}\delta q^a - \delta x^i_a\,\upsilon^a - x^i_a\,\delta\upsilon^a$$
$$= x^i_a\left(\frac{d}{dt}\delta q^a - \delta\upsilon^a\right) + \frac{dx^i_a}{dt}\delta q^a - \delta x^i_a\,\upsilon^a .$$

The last two terms can be expressed in the form:

$$\frac{dx^i_a}{dt}\delta q^a - \delta x^i_a\,\upsilon^a = \mathbf{e}_b x^i_a\,\upsilon^b\delta q^a - \mathbf{e}_b\,x^i_a\,\upsilon^a\delta q^b = \mathbf{e}_a x^i_b\,\upsilon^a\delta q^b - \mathbf{e}_b\,x^i_a\,\upsilon^a\delta q^b = [\mathbf{e}_a, \mathbf{e}_b]^i\,\upsilon^a\delta q^b ,$$

so the transition equation will become:

$$\frac{d}{dt}\delta x^i - \delta\dot{x}^i = x^i_a\left(\frac{d}{dt}\delta q^a - \delta\upsilon^a\right) + [\mathbf{e}_a, \mathbf{e}_b]^i\,\upsilon^a\delta q^b \tag{7.76}$$

in the non-holonomic case, where one can clearly see that this will reduce to the holonomic version (6.64), for which $x^i_a = x^i_{,a}$ and $[\mathbf{e}_a , \mathbf{e}_b]$ will vanish. Hence, the integrability of $\delta\upsilon^a$ does not have to imply that of $\delta\dot{x}^i$ or *vice versa*. In fact, since υ^a and δq^b are functions of time, not space, one can say that:

$$[\mathbf{e}_a, \mathbf{e}_b]^i\,\upsilon^a\delta q^b = [\boldsymbol{\upsilon}, \delta\mathbf{q}] , \tag{7.77}$$

although that is stretching the definition of the Lie bracket somewhat.

Heun's central equation:

$$\frac{d}{dt}(m\,v_i\,\delta x^i) - \delta T = (F_i + \lambda_\alpha C^\alpha_i)\,\delta x^i \tag{7.78}$$

first becomes:

$$\frac{d}{dt}(m\,v_i)\,\delta x^i + m v_i\left(\frac{d}{dt}\delta x^i - \delta\dot{x}^i\right) = (F_i + \lambda_\alpha C^\alpha_i)\,\delta x^i, \tag{7.79}$$

and when the transition equation (7.76) is substituted, one will have:

$$\frac{d}{dt}(m\,v_i)\,\delta x^i + m v_i\,x^i_a\left(\frac{d}{dt}\delta q^a - \delta\upsilon^a\right) + m v_i\,[\mathbf{e}_a, \mathbf{e}_b]^i\,\upsilon^a\delta q^b = (F_i + \lambda_\alpha C^\alpha_i)\,\delta x^i,$$

With the substitutions $\delta x^i = x^i_a\,\delta q^a$, $\upsilon_a = v_i\,x^i_a$, $Q_a = x^i_a\,F_i$, the term with the Lagrange multipliers will drop out, and what will be left is:

$$\frac{d}{dt}(m\,v_i)\,x^i_a\,\delta q^a + m\,\upsilon_a\left(\frac{d}{dt}\delta q^a - \delta\upsilon^a\right) + m v_i\,[\mathbf{e}_a, \mathbf{e}_b]^i\,\upsilon^a\delta q^b = Q_a\,\delta q^a,$$

and with a little reorganizing, one will have:

$$\frac{d}{dt}(m\,\upsilon_a\,\delta q^a) - m\,\upsilon_a\,\delta\upsilon^a - m v_i\left(\frac{dx^i_a}{dt} + [\mathbf{e}_a, \mathbf{e}_b]^i\,\upsilon^b\right)\delta q^a = Q_a\,\delta q^a.$$

The term in parentheses can be simplified to:

$$\frac{dx^i_a}{dt} + [\mathbf{e}_a, \mathbf{e}_b]^i\,\upsilon^b = (\mathbf{e}_b\,x^i_a + [\mathbf{e}_a, \mathbf{e}_b]^i\,)\upsilon^b = (\mathbf{e}_b\,x^i_a + \mathbf{e}_a x^i_b - \mathbf{e}_b x^i_a)\upsilon^b = \mathbf{e}_a x^i_b\,\upsilon^b,$$

which makes:

$$\frac{d}{dt}(m\,\upsilon_a\,\delta q^a) - m v_i\,\mathbf{e}_a x^i_b\,\upsilon^b\delta q^a - m\,\upsilon_a\,\delta\upsilon^a = Q_a\,\delta q^a.$$

Now, the variation of kinetic energy:

$$T = \tfrac{1}{2} m\,g_{ab}\,\upsilon^a\upsilon^b \tag{7.80}$$

will take the form:

$$\delta T = \tfrac{1}{2} m\,\delta g_{ab}\,\upsilon^a\upsilon^b + m\,g_{ab}\,\upsilon^a\delta\upsilon^b = \tfrac{1}{2} m\,\mathbf{e}_a\,g_{bc}\,\upsilon^b\upsilon^c\delta q^a + m\,\upsilon_a\,\delta\upsilon^a, \tag{7.81}$$

but:

$$\tfrac{1}{2}\mathbf{e}_a\,g_{bc}\,\upsilon^b\upsilon^c = \tfrac{1}{2}\delta_{ij}(\mathbf{e}_a x^i_b\,x^j_c + x^i_b\,\mathbf{e}_a x^j_c)\,\upsilon^b\,\upsilon^c = \tfrac{1}{2}\delta_{ij}(\mathbf{e}_a x^i_b\,v^j + v^i\,\mathbf{e}_a x^j_b)\,\upsilon^b = v_i\,\mathbf{e}_a x^i_b\,\upsilon^b,$$

so we finally have:

$$\frac{d}{dt}(m\,\upsilon_a\,\delta q^a) - \delta T = Q_a\,\delta q^a, \tag{7.82}$$

which is the form that the central equation will take in an adapted frame.

We can go from this equation to the Lagrange equations of motion in their second form directly by noting that (7.74) will imply that:

$$\frac{d}{dt}(m\,\upsilon_a\,\delta q^a) - \delta T = -\frac{\delta T}{\delta q^a}\,\delta q^a \tag{7.83}$$

when one performs the actual differentiation $\partial T / \partial v^a$, namely:

$$-\frac{\delta T}{\delta q^a} = Q_a\,. \tag{7.84}$$

Of course, one is cautioned that the variational derivative does not have its usual definition, since there are no generalized coordinates in the present case, but the form (7.75) that it takes in an adapted anholonomic frame field.

8. Gauss's principle of least constraint. – We finally come to the main topic of the translations that follow, namely, Gauss's principle of least constraint and its subsequent alterations.

As is often the case, Gauss's brief, but seminal, 1829 paper "Über ein neues allgemeines Grundsetz der Mechanik" ("One a new general foundation for mechanics") was more descriptive than analytical. Mostly, he suggested that mechanics might be derivable from the principle that:

> "The motion of a system of material points that are always coupled to each other in some way, and whose motion is, at the same time, always subject to external constraints, agrees with the free motion at each moment to the greatest possible extent or with the least possible constraint when one considers a measure of the constraint that the system suffers at each point in time to be the sum of the products of the squares of the deviations of each point from the free motion of its mass."

He did, however, make a start towards giving that principle a more analytical formulation. Basically, he attempted to apply the methods of least-squares that he felt was so fundamental to much of the mathematics that he dealt with in order to obtain an analytical expression for the constraint function that must be minimized. Although most of the discussions that follow are generalized to systems of points that are subject to constraints, nonetheless, many of the essential features are already present in the simplest system, namely, a single point in space that is subject to constraints of the aforementioned types, in addition to external forces. In particular, Gauss's principle of least constraint does not specify a particular class of constraint, so in that sense it is at least as general in scope as the principle of virtual work, when combined with d'Alembert's principle, in the case of a moving point. In fact, it was later shown by Josiah Willard Gibbs that Gauss's principle was, in fact, *more* general than the latter principles, since it also applies when the constraints are one-sided (i.e., defined by inequalities), and not only when they are two-sided (i.e., defined by equalities).

The essence of Gauss's principle is that, in the absence of constraints, the point-mass m in a space that is permeated by external forces whose resultant F acts upon m would move in a manner that one could characterize as "free" motion. Such a state of motion is defined by a *second*-order jet:

$$s_f(t) = (t, x_f^i(t), v_f^i(t), a_f^i(t)) \ , \tag{8.1}$$

and in particular, the free acceleration will be:

$$a_f^i(t) = \frac{1}{m} F^i(t, x_f(t), v_f(t)) \, . \tag{8.2}$$

When one imposes the constraints on the motion of m, the resultant motion will be represented by another second-order jet:

$$s_c(t) = (t, x_c^i(t), v_c^i(t), a_c^i(t)) \ . \tag{8.3}$$

When one looks at a given point $x^i(t)$ along a curve and a given velocity $v^i(t)$, the two kinematical states will take the forms:

$$s_f(t) = (t, x^i(t), v^i(t), \frac{1}{m} F^i(t)) \ , \qquad s_c(t) = (t, x^i(t), v^i(t), a_c^i(t)) \ , \tag{8.4}$$

respectively.

Since the kinematical state of free motion $s_f(t)$ is fixed by the position and velocity, the only thing that is left indeterminate is the constrained acceleration $a_c^i(t)$. In order to obtain it, Gauss first defined the deviation between the free and constrained motion at that point and with that velocity to be ([1]):

$$\mathfrak{C}(\mathbf{a}_c) = \tfrac{1}{2}\sum_{i=1}^{n} m\left(a_c^i - \frac{1}{m}F^i\right)^2 = \tfrac{1}{2}\sum_{i,j=1}^{n} m\,\delta_{ij}\left(a_c^i - \frac{1}{m}F^i\right)\left(a_c^j - \frac{1}{m}F^j\right) \ . \tag{8.5}$$

The constrained acceleration $\mathbf{a}_c$ is then obtained by assuming that it will minimize this (presumably) differentiable function on the n-dimensional vector space that $\mathbf{a}_c$ belongs to, which then becomes a routine problem in multivariable calculus. In particular, one should note that the principle of least constraint, unlike the principle of least action, is not a variational principle. That is, one is only looking for an extremal *vector*, not an extremal *curve*.

Differentiating $\mathfrak{C}$ with respect to the components of $\mathbf{a}_c$ gives:

$$\frac{\partial \mathfrak{C}}{\partial a_c^i} = m\,\delta_{ij}\left(a_c^j - \frac{1}{m}F^j\right) = \delta_{ij}\left(m\,a_c^j - F^j\right), \tag{8.6}$$

([1]) We have included a factor of 1/2 for the sake of later convenience.

and if that vanishes then one will be left with Newton's second law for the constrained acceleration. In order to see that the extremum is indeed a minimum, one needs only to differentiate a second time to get:

$$\frac{\partial^2 \mathfrak{C}}{\partial a_c^i \, \partial a_c^j} = m \delta_{ij} , \tag{8.7}$$

which is clearly positive-definite.

Of course, one should notice that since F^j / m is also defined to be the free acceleration, the only thing that one gets from a naïve minimization of $\mathfrak{C}$ is that the constrained acceleration will be equal to the free acceleration, which seems to ignore the contribution from the constraints themselves. In the case of two-sided linear constraints, we find that the most direct route to their introduction is to express the constrained acceleration $\mathbf{a}_c$ and the external forces $\mathbf{F}$ in terms of a frame field $\mathbf{e}_a$ that is adapted to the constraint subspaces, along with an extension of it to an n-frame field $\mathbf{e}_i$ on $\mathbb{R}^n$. Hence, when we express everything in terms of the adapted frame field and its reciprocal coframe field:

$$\mathbf{v} = \upsilon^a \, \mathbf{e}_a , \qquad \mathbf{a} = \mathfrak{a}^a \, \mathbf{e}_a + \mathfrak{a}^\alpha \, \mathbf{e}_\alpha , \qquad \mathbf{F} = Q^a \, \mathbf{e}_a + Q^\alpha \, \mathbf{e}_\alpha , \qquad g = g_{ij} \, e^i \, e^j , \tag{8.8}$$

we can express the constraint function in terms of the adapted frame:

$$\mathfrak{C}\,(\mathbf{a}) = \tfrac{1}{2}\, m \, g_{ij} \, (\mathfrak{a}^i - \frac{1}{m} Q^i) \, (\mathfrak{a}^j - \frac{1}{m} Q^j) \, . \tag{8.9}$$

Since we are only differentiating with respect to $\mathbf{a}$, the fact that g_{ij} are functions of x will not affect anything, and we will have:

$$\frac{\partial \mathfrak{C}}{\partial \mathfrak{a}^i} = m \, g_{ij} \, (\mathfrak{a}^i - \frac{1}{m} Q^i) = m \, \mathfrak{a}_i - Q_i \, . \tag{8.10}$$

The critical points of the function $\mathfrak{C}$ will then be defined by the equations:

$$Q^a = m \, \mathfrak{a}^a = m \, \nabla_t \, \upsilon^a , \qquad Q^{p+\alpha} = m \, \mathfrak{a}^{p+\alpha} = m \, \omega_a^{p+\alpha}(\mathbf{v}) \, \upsilon^a \, . \tag{8.11}$$

These are not only the constrained equations of motion that we got before from the principle of virtual work, plus d'Alembert, but also an equation that expresses the equilibrium between the normal components of the external force and the reaction force of the constraint to the velocity of constrained motion. The latter takes the form of a generalized centrifugal force with a radius of rotation that equals the radius of curvature in the chosen direction, which is the reciprocal of the curvature in that direction.

We shall now turn to the question of how the principle of least constraint relates to the principle of virtual work, when combined with d'Alembert's principle.

First of all, in some of the articles that follow, the quantity $\Delta\mathbf{F} = m\,\mathbf{a}_c - \mathbf{F}$ is referred to as the "lost force," which basically amounts to the force of constraint that confines the mass m. That is, one can thing of the applied force $\mathbf{F}$ as being a difference:

$$\mathbf{F} = \mathbf{F}_c - \Delta\mathbf{F}\,, \tag{8.12}$$

in which the component $\mathbf{F}_c = m\,\mathbf{a}_c$ is the part that account for the actual acceleration of the constrained mass. The "lost virtual work" that is done by the lost force along a virtual displacement $\delta\mathbf{x}$ will then be:

$$\delta W = \Delta F\,(\delta\mathbf{x}) = d\mathfrak{C}\,(\delta\mathbf{x}), \tag{8.13}$$

in which:

$$d\mathfrak{C} = \frac{\partial\mathfrak{C}}{\partial a_c^i}\,da_c^i = \delta_{ij}\left(m\,a_c^i - F^i\right)da_c^j\,. \tag{8.14}$$

However, one sees that the differentials da_c^i in this expression do not act upon $\delta\mathbf{x}$ as much they act upon $\delta\mathbf{a}_c$. In order to resolve that issue, one assumes that:

$$\delta\mathbf{x}\,(s) = \delta\mathbf{x}_0 + \delta\mathbf{v}_0\,s + \tfrac{1}{2}\,\delta\mathbf{a}_c s^2 = \tfrac{1}{2}\,\delta\mathbf{a}_c s^2\,, \tag{8.15}$$

since the initial position and velocity are not being varied. Hence:

$$\delta W = \tfrac{1}{2}\,s^2\,\delta_{ij}\left(m\,a_c^i - F^i\right)\delta a_c^j\,. \tag{8.16}$$

One then sees that as long as s is not zero, the vanishing of δW for all virtual displacements $\delta\mathbf{x}$ that are consistent with the constraints is equivalent to the principle of virtual work, augmented by d'Alembert's principle.

However, as the American physicist Josiah Willard Gibbs (1839-1903) pointed out in an 1879 paper [**24**], the principle of least constraint goes beyond the scope of virtual work and d'Alembert, because it can also be used in cases of one-sided constraints, as well as impulsive forces, such as in collisions. The example of the former that he gave was essentially the one above, namely, the difference between a marble rolling in a bowl that is concave upwards and rolling on its outside when it is concave downward. Thus, Gauss's principle has the potential to go deeper into the fundamental nature of motion than all of the other first principles that came before it.

In order to see that Gauss's principle can be used with inequality constraints, while the principle of virtual work plus d'Alembert requires equality constraints, one first notes that the latter principle is expressed as the *vanishing* of the virtual work that is done by all virtual displacements that are consistent with the constraints, so the condition in the principle is itself an equality, which tends to imply that the constraints themselves must be equalities, as well. On the other hand, Gauss's principle can be rephrased as an inequality in its own right. Namely, it says that if $\mathbf{a}_c$ is the true, constrained acceleration and $\mathbf{a}$ is any other acceleration then:

$$\mathfrak{C}(\mathbf{a}_c) \leq \mathfrak{C}(\mathbf{a}) . \tag{8.17}$$

The argument that Gibbs proposed is easiest to understand in an adapted frame with one normal direction. If one were to impose the constraint on the virtual displacements that the tangential components $\delta\mathbf{x}_t$ are arbitrary, but the normal component is non-negative – so $\delta x^n \geq 0$ – and one were to weaken the principle of virtual work, plus d'Alembert, to an inequality: $\delta W \geq 0$ for all permissible $\delta\mathbf{x}$ then that would give only that:

$$\mathfrak{a}_t = \frac{1}{m} Q_t , \qquad \mathfrak{a}_n \geq \frac{1}{m} Q_n , \tag{8.18}$$

which is not sufficient to determine the acceleration uniquely.

However, if one replaces the principle of virtual work (plus d'Alembert) with the inequality:

$$d\mathfrak{C}(\delta\mathbf{a}) = (m\,\mathfrak{a}_t - Q_t)\,\delta\mathfrak{a}^t + (m\,\mathfrak{a}_n - Q_n)\,\delta\mathfrak{a}^n \geq 0 \tag{8.19}$$

and imposes the constraint on $\mathfrak{a}_n$ that it must be non-negative ($\mathfrak{a}_n \geq 0$) then one will have two cases for how that will affect $\delta\mathbf{a}$, namely, the tangential components $\delta\mathfrak{a}^t$ will still be arbitrary, but as for the normal component, one will have:

1. If $\mathfrak{a}_n = N = \omega_a^{p+\alpha}(\mathbf{v})\, \upsilon^a$ (i.e., the reaction of the constraint as an equality) then one will only be free to displace $\mathfrak{a}^n$ upwards. Hence, $\delta\mathfrak{a}^n \geq 0$, which will make:

$$\mathfrak{a}_n = N \geq \mathfrak{a}_n \geq \frac{1}{m} Q_n . \tag{8.20}$$

However, that still defines $\mathfrak{a}_n$ by an equality.

2. If $\mathfrak{a}_n > N$ then that will mean that one is also free to displace $\mathfrak{a}^n$ downwards, as well. Hence, $\delta\mathfrak{a}^n$ will also be free and that will make:

$$\mathfrak{a}_n = \frac{1}{m} Q_n , \tag{8.21}$$

which is also an equality.

In either case, one can set:

$$\mathfrak{a}_n = \max \{N, \frac{1}{m} Q_n \}, \tag{8.22}$$

and the acceleration will be determined uniquely.

9. The Gibbs-Appell equations. – Along with discussing the principle of least constraint in his cited article, Gibbs also introduced a concept that is closely related to Gauss's definition of the constraint function $\mathfrak{C}$, namely, the function of acceleration:

$$S(\mathbf{a}) = \tfrac{1}{2} m a^2 = \tfrac{1}{2} m \delta_{ij} a^i a^j . \tag{8.23}$$

One then sees that:

$$\frac{\partial S}{\partial a^i} = m\, a_i . \tag{8.24}$$

If one expands the definition (8.5) of $\mathfrak{C}(\mathbf{a})$ into:

$$\mathfrak{C}(\mathbf{a}) = \tfrac{1}{2} m \delta_{ij} a^i a^j - F_i a^i + \frac{1}{2m} F^2 = S(\mathbf{a}) - F_i a^i + \frac{1}{2m} F^2 \tag{8.25}$$

then one will see that, since forces are not generally presumed to depend upon accelerations ([1]), the critical points of the constraint function $\mathfrak{C}(\mathbf{a})$ will occur for the **a** that make:

$$F_i = \frac{\partial S}{\partial a^i}, \tag{8.26}$$

which can also be regarded as another form of Newton's second law.

One sees that since F is not presumably a function of **a**, the critical points of $\mathfrak{C}(\mathbf{a})$ will be the same as the critical points of:

$$\mathfrak{R}(\mathrm{a}) = S(\mathbf{a}) - F_i a^i , \tag{8.27}$$

i.e.:

$$\frac{\partial \mathfrak{C}}{\partial a^i} = \frac{\partial \mathfrak{R}}{\partial a^i} . \tag{8.28}$$

Since one also has:

$$\frac{\partial^2 \mathfrak{C}}{\partial a^i \partial a^j} = \frac{\partial^2 \mathfrak{R}}{\partial a^i \partial a^j} = m\, \delta_{ij} \tag{8.29}$$

then, the critical points of $\mathfrak{R}$ will have the same positive definite character as the critical points of $\mathfrak{C}$. Hence, a minimum of $\mathfrak{R}$ will be a minimum of $\mathfrak{C}$, and conversely. In that sense, the minimization problem for Gauss's constraint function is equivalent to the minimization problem for the function $\mathfrak{R}$, which implies the equations (8.26). Indeed, one can also extend that equivalence to the case of one-sided constraints in both cases.

([1]) Of course, that is not true of the radiation damping (or radiation reaction) force that is associated with an accelerated charge, but strictly speaking that would require a relativistic treatment, if not a wave-mechanical one, which would the lie beyond the scope of the present monograph.

Some years after Gibbs published his treatment of the laws of dynamics (around 1899), the French mathematician Paul Émile Appell (1855-1930) expanded considerably upon the function S (**a**) that Gibbs had introduced. At the suggestion of A. de Saint Germain (see the translation below), he called it the "energy of acceleration," since it bore a formal similarity to the kinetic energy that one associates with velocity. (Of course, the units are not actually those of energy.) Hence, one now refers to equations (8.26) as the *Gibbs-Appell equations.*

10. Hertz's mechanics. – Heinrich Hertz (1857-1894) only lived to the age of thirty-six, having met his end as a result of "a painful abscess," as Helmholtz described it (or "granulomatosis with polyangiitis," as the Wikipedia entry [**25**] describes it), but he still managed to make far-reaching inroads into the foundations of physics, especially the nature of electromagnetic waves, which had been predicted by James Clerk Maxwell in 1864, and Hertz's work on the subject culminated in the publication of his 1893 *magnum opus Untersuchungen über die Ausbreitung der Elektrischen Kraft,* which was translated by D. E. Jones into *Electric waves – Being researches on the propagation of electric action with finite velocity through space* [**26**]. Amusingly, Hertz was another example of a scientist who, like Einstein, clearly did not see the practical ramifications of his work. When asked about those practical applications of his electromagnetic waves, Hertz was quoted as saying "Nothing, I guess."

However, one of his more enigmatic contributions was his final treatise on the fundamental principles of mechanics *Prinzipien der Mechanik* [**5**], which was published posthumously in 1894. Even in translation, it is a hard read, and tends to reflect the inchoate character of a book that probably needed more editing, but it is possible to distill out a selected causal chain of related ideas that point to a deep and radical rethinking of the basic axioms of mechanics. Indeed, it is in that work that one finds the terms "holonomic" and "non-holonomic" being first coined.

One of the aspirations Hertz had in his mechanics was to obviate the introduction of the concept of force, which only met with so much widespread acceptance, but perhaps the most far-reaching of the alterations to the basic axioms of mechanics is essentially an extension of the scope of Newton's law of inertia. Hertz calls it his Fundamental Law (cf., § 309 of his book), which takes the form:

A free system persists in a state of rest or moves uniformly along a straightest path.

In order to make this more mathematical, one first interprets the term "free" to mean "acted upon by only forces of constraint," while a "state of rest" means $v = 0$, and "uniform motion" means "constant speed ($\dot{v} = 0$)." It is the addition of the term "straightest path" that needs further clarification and occupies a good part of the first part of the book.

Basically, a constrained path $x(s)$ is "straightest" if the absolute value of its curvature $\kappa(s)$ at each point is a minimum for all paths through that point that have the same tangent vector $\mathbf{t}(s)$ at that point. Hence, one is dealing with a constrained minimization problem from elementary differential calculus, not the calculus of variations.

Recall that the Frenet-Serret definition of the curvature of a curve $x(s)$ that is parameterized by arc length, so the tangent vector:

$$\mathbf{t}(s) = \left.\frac{dx}{ds}\right|_s \tag{9.1}$$

has unit length for all s, is:

$$\frac{d\mathbf{t}}{ds} = \kappa\,\mathbf{n}\,, \qquad \text{so} \qquad \kappa = \left\langle \frac{d\mathbf{t}}{ds}, \mathbf{n} \right\rangle, \tag{9.2}$$

in which $\mathbf{n}(s)$ is the unit normal vector field to the curve.

That also makes:

$$\kappa^2 = \left\| \frac{d\mathbf{t}}{ds} \right\|^2, \tag{9.3}$$

and to minimize κ^2 is to minimize $|\kappa|$.

If the constraint on the tangent vector field is defined by the annihilating subspace of some set of p 1-forms $C^\alpha = C_i^\alpha\, dx^i$, $\alpha = 1, \ldots, p$ (so the constraint is linear and homogeneous, but not necessarily holonomic) then it will take the form:

$$C^\alpha(\mathbf{t}(s)) = 0 \quad \text{for all } s \text{ and } \alpha\,. \tag{9.4}$$

By differentiating with respect to s, that constraint on the tangent vector will imply the constraint on the curvature:

$$0 = \frac{d}{ds}[C^\alpha(\mathbf{t})] = \frac{dC^\alpha}{ds}(\mathbf{t}) + C^\alpha\left(\frac{d\mathbf{t}}{ds}\right) = dC^\alpha(\mathbf{t}, \mathbf{t}) + \kappa\, C_n^\alpha\,, \tag{9.5}$$

in which $C_n^\alpha = C^\alpha(\mathbf{n})$ are the normal components of the 1-forms C^α, in the Frenet-Serret sense of the word "normal," which is not necessarily the sense that is defined by the constraints.

Hence, when one introduces the Lagrange multipliers λ_α, the function to be minimized will take the form:

$$f(\kappa) = \tfrac{1}{2}\kappa^2 + \lambda_\alpha\,(dC^\alpha(\mathbf{t}, \mathbf{t}) + \kappa\, C_n^\alpha)\,. \tag{9.6}$$

Differentiation with respect to κ gives a critical point whenever the curvature takes the form:

$$\kappa = -\lambda_\alpha\, C_n^\alpha\,. \tag{9.7}$$

Since a second differentiation of f will give 1, which is positive, the critical point is a minimum.

Naturally, one asks how the straightest path relates to the shortest path – i.e., the geodesic through each point that has the same tangent vector at that point. That becomes a problem in the calculus of variations, namely, one seeks to extremize the path functional that takes the form of the constrained arc length of the path:

$$S\,[x\,(s)] = \int_{x(s)} ds + \lambda_\alpha C^\alpha \,. \tag{9.8}$$

In order to highlight the contribution of the constraint itself, we shall assume that the path $x\,(s)$ lives in n-dimensional Euclidian space; the alteration that would involve a more general Riemannian manifold (viz., the introduction of the Levi-Civita connection) would affect only the definition of ds). Although one could express C^α in the form:

$$C^\alpha = C_i^\alpha \frac{dx^i}{ds} ds\,, \tag{9.9}$$

and arrive at a Lagrangian in that way, nonetheless, it is more informative to reparametrize the curve $x\,(s)$ as $x\,(\tau)$, with:

$$v = \frac{ds}{d\tau}\,. \tag{9.10}$$

The integrand will then take the form $\mathcal{L}\,dt$, with:

$$\mathcal{L}\;(x^i, \dot{x}^i) \;=\; (\delta_{ij}\,\dot{x}^i\,\dot{x}^j)^{1/2} + \lambda_\alpha(x) C_i^\alpha(x)\,\dot{x}^i\,. \tag{9.11}$$

A straightforward calculation gives the variation derivative of $\mathcal{L}$ with respect to x^i :

$$\frac{\delta\mathcal{L}}{\delta x^i} = \frac{1}{v}\ddot{x}_i - \frac{\dot{v}}{v^2}\dot{x}_i + \frac{d\lambda_\alpha}{d\tau}C_i^\alpha + \lambda_\alpha \frac{dC_i}{d\tau}\,. \tag{9.12}$$

If one reverts back to the arc-length parameterization ($\tau = s$, $v = 1$, $\dot{v} = 0$) and notes that:

$$\frac{dC}{ds} = \mathrm{L}_{\mathbf{t}}C = i_{\mathbf{t}}\, d_\wedge C, \tag{9.13}$$

since $C\,(\mathbf{t}) = 0$, then the equations of the constrained geodesic will take the form ([1]):

$$0 = \frac{dt}{ds} + \frac{d\lambda_a}{d\tau}C^a + \lambda_a\, i_{\mathbf{t}} d_\wedge C^a\,, \tag{9.14}$$

in which $t = t_i\, dx^i$ is the 1-form that is metric-dual to the tangent vector field. Evaluating the 1-form on the right-hand side of this equation on the normal vector field **n** will give a corresponding constraint on the curvature of the path:

([1]) For a more general Riemannian manifold, the only change in this equation will be in the definition of $dt\,/\,ds$, which will involve using covariant differentiation using the Levi-Civita connection, rather than ordinary differentiation.

$$0 = \kappa + \frac{d\lambda_\alpha}{d\tau} C_n^\alpha + \lambda_\alpha \, d_\wedge C^\alpha (\mathbf{t}, \mathbf{n}) \,. \tag{9.15}$$

Here, we can consider the issue of the holonomity of the constraints, and note that for holonomic constraints, the last term on the right-hand side of the equation above will vanish. Since the remaining equation:

$$\kappa = -\frac{d\lambda_a}{d\tau} C_n^a \tag{9.16}$$

differs from (9.7) only in that the undetermined Lagrange multipliers λ_α have been replaced with their equally-undetermined derivatives with respect to s, one gets a fundamental relationship that Hertz pointed out (cf., § 190):

Theorem:

When the constraints on a path are holonomic, it will be a straightest path iff it is a shortest path.

By contrast, when the constraints are non-holonomic, the two concepts do not generally coincide. Whether the straightest path is longer than or equal to the shortest one will then depend upon the nature of the anholonomity $d_\wedge C^\alpha$ of the constraints by way of $d_\wedge C^\alpha (\mathbf{t}, \mathbf{n})$.

Note that one can weaken the condition of holonomity to simply:

$$0 = \lambda_\alpha \, d_\wedge C^\alpha (\mathbf{t}, \mathbf{n}) \qquad \text{for all } \alpha \,. \tag{9.17}$$

In order to convert these purely-geometric results (i.e., parameterized by arc-length) into kinematical ones, all that one has to do is to parameterize the curve x (s) by time t, instead ([1]). That will make:

$$\mathbf{v} = v\,\mathbf{t}, \qquad\qquad \mathbf{a} = \frac{d\mathbf{v}}{dt} = \dot{v}\,\mathbf{t} + v^2 \kappa\,\mathbf{n} \,. \tag{9.18}$$

The constraint (9.4) on velocity:

$$C^\alpha (\mathbf{v}) = 0 \qquad \text{for all } \alpha \tag{9.19}$$

is not fundamentally different from before, since the two vector fields **t** and **v** are proportional by a non-zero scalar multiple. The new form of the constraint (9.5) on acceleration will be:

$$0 = \frac{d}{dt} [C^\alpha (\mathbf{v})] = \frac{dC^\alpha}{dt} (\mathbf{v}) + C^\alpha \left(\frac{d\mathbf{v}}{dt} \right) = dC^\alpha (\mathbf{v}, \mathbf{v}) + v^2 \, \kappa \, C_n^\alpha \ ,$$

([1]) Or proper time τ, in the case of relativistic motion.

since the normal component of **a** does not include the new tangential contribution. Hence, the new condition is:

$$0 = v^2\,[dC^\alpha(\mathbf{t}, \mathbf{t}) + \kappa\, C_n^\alpha]\,, \tag{9.20}$$

which is not fundamentally different since v is assumed to never vanish.

However, since acceleration is not merely proportional to the normal vector field, the norm-squared of the acceleration will take the form:

$$a^2 = \|\,\mathbf{a}\,\|^2 = \dot{v}^2 + v^4\kappa^2\,, \qquad \text{or} \qquad \kappa^2 = \frac{1}{v^4}(a^2 - \dot{v}^2)\,, \tag{9.21}$$

and the condition on the acceleration can be written in the time-parameterized form:

$$0 = dC^\alpha(\mathbf{v}, \mathbf{v}) + (a^2 - \dot{v}^2)^{1/2} C_n^\alpha\ . \tag{9.22}$$

If one wishes to find the path of least acceleration then the function $f(a)$ will now take the form:

$$f(a) = \tfrac{1}{2}a^2 + \lambda_\alpha[dC^\alpha\left(\mathbf{v}, \mathbf{v}\right) + (a^2 - \dot{v}^2)^{1/2} C_n^\alpha]\ . \tag{9.23}$$

The condition on a for a path of least acceleration will then take the form:

$$0 = a + \frac{a}{\sqrt{a^2 - \dot{v}^2}}\lambda_\alpha C_n^\alpha\,, \tag{9.24}$$

and for uniform motion ($\dot{v} = 0$) that becomes simply:

$$a = -\lambda_\alpha C_n^\alpha\,. \tag{9.25}$$

Since one also has that $a = v^2 k$ when $\dot{v}$ vanishes, one has the theorem (cf., § 344):

Theorem:

A system whose motion is subject to only constraints will remain at rest or move uniformly along a path of least acceleration.

That is then an equivalent form to Hertz's Fundamental Law of motion.

As before, the path of least acceleration will be a geodesic when the constraints are holonomic, since the new form of the geodesic equation (9.14) is now:

$$0 = \frac{1}{v}a - \frac{\dot{v}}{v^2}v + \frac{d\lambda_\alpha}{dt}C^\varepsilon + \lambda_\alpha\, i_{\mathbf{v}} d_\wedge C_i^\alpha\,, \tag{9.26}$$

and for uniform motion, this will become:

$$a = -v \frac{d\lambda_\alpha}{dt} C^\alpha - \lambda_\alpha v \, i_{\mathbf{v}} d_\wedge C_i^\alpha , \tag{9.27}$$

which is essentially (9.25) when the second term on the right-hand side vanishes.

One can now see that Hertz's principle of least curvature is a specialization of Gauss's principle of least constraint, as Hertz himself pointed out (cf., §§ 388-391). Indeed, Hertz contended that the combination of his law of least curvature, in conjunction with Newton's law of inertia was equivalent to Gauss's principle of least constraint. In order to see that, one simply notes that Newton's law generally refers to unconstrained motion in Euclidian space, for which the straightest path will be, in fact, straight; that is, uniform, rectilinear motion. Hence, its curvature will vanish, or equivalently, its acceleration. The difference between the constrained acceleration and the unconstrained acceleration will then be the constrained acceleration and Gauss's definition of the constraint function to be minimized will coincide with the function that gives the last acceleration when differentiated.

Hertz's ambition to eliminate the concept of force from mechanics seems to have been the aspect of his treatise that attracted what little attention that it did attract. Georg Hamel [**11**] felt that mechanics without force is not mechanics, but other commentators (such as Poincaré) surmised that Hertz was trying to replace the concept of force with that of constraint. Interestingly, when one formulates gauge field theories in the modern way, the "horizontal sub-bundle" of the tangent space to the total space of the principal fiber bundle that defines the gauge structure, which is defined by the annihilating subspace of the connection 1-form, can be regarded as a constraint on curves in that total space. Its anholonomity is then described by its curvature 2-form, and a "horizontal lift" of a curve in the base manifold (i.e., space-time) to the total space becomes a curve that is consistent with the constraint.

In particular, when the field theory is electromagnetism, the connection 1-form is the electromagnetic potential 1-form A, which is then the constraint 1-form, and its "anholonomity" becomes the electromagnetic field strength 2-form $F = d_\wedge A$; i.e., the field strength is due to the anholonomity of the constraint. A more recent development in the theory of gravitation is the experimental verification of "gravito-electromagnetism." Hence, Maxwell's equations can also serve as the field equations for weak gravitational fields. Thus, there might be something to the idea that the global, universal forces of nature are, in a sense, constraints, even if perhaps the localized, artificial forces (e.g., pushes and pulls by animals or machines) might not be so.

The last chapter of Hertz's book, which was concerned with "cyclic" systems, contains the material that got the most attention, even if it was less convincing than his more fundamental discussions of constrained kinematics and dynamics. Basically, he was attempting to replace forces acting at a distance with propagating effects, which is reasonable, and in that regard, he was probably following up on the ideas of his mentor Helmholtz. However, his way of accomplishing that seemed strongly reminiscent of the mechanical ether models for the propagation of electromagnetic waves, which were doomed. In effect, he was trying to replace potential energy with the kinetic energy of hidden masses that were coupled together. Of course, a popular way of regarding wave motion is essentially that of hidden oscillators, although typically one does not think of them as being massive, except in the sense that they have finite frequencies.

Nonetheless, several subsequent researches by distinguished figures of physics pursued various aspects of the theory of cyclic systems, such as Helmholtz's work on monocyclic and polycyclic systems [**27**], Ehrenfest's approach to quantum mechanics by way of adiabatic invariants [**28**], and de Broglie's theory of the "hidden thermodynamics" of isolated particles [**29**]. One should not be too quick to dismiss the latter study, because it contained an intriguing footnote in which the author pointed out that what he was calling the "hidden thermostat" that isolated particles exchanged energy with was what others in that era were calling the "quantum vacuum." Certainly, the idea that isolated particles can exchange energy with the quantum vacuum seems entirely plausible.

11. – Hierarchy of the first principles for constrained motion. – It would be instructions to organize the various first principles that we have discussed for constrained motion in terms of the types of forces and types of constraints that they can be applied to. In descending order of generality, we have:

1. Gauss's principle of least constraint: all forces and types of constraints.

2. Gibbs-Appell equations: equivalent to least constraint.

3. Virtual work, plus d'Alembert's principle: all forces, but only two-sided perfect constraints.

4. Lagrange's equations of the first kind: equivalent to virtual work, plus d'Alembert.

5. Lagrange's equations of the second kind: perfect holonomic constraints but can be formulated for non-holonomic constraints with a different definition for $\delta\mathcal{L} / \delta q^a$ using adapted frame fields.

6. Least action principle: conservative forces and perfect holonomic constraints.

We could also mention Hertz's mechanics, although as he envisioned it, there would be no forces, except essentially constraint forces. However, as far as constraints are concerned, the principle of least curvature is equivalent to Gauss's principle of least constraint.

12. Overview of the translations. – We shall now include a brief discussion of the various papers that follow for the sake of navigating them in a more logically-organized way. They are sorted into papers on least constraint, the Gibbs-Appell equations, and Hertz's mechanics, and within each category, they appear in chronological order. However, various chains of logical connection exist between them that one can infer by looking at how they refer to each other.

a. Least constraint. – Naturally, the translations must begin with the original 1829 article by Gauss in which he first proposed the principle of least constraint. Although it is admittedly brief, and represents more of a suggested path of research, nonetheless, it already included a number of key points that were developed more analytically by the people who followed up on them. In particular, he already suggested that his principle would generalize the principle of virtual work, plus d'Alembert, to the case in which the constraints were one-sided, not two-sided.

The next two papers by Reuschle and Scheffler address the lack of an analytical formulation for the principle of least constraint, as well as any worked examples that would show how one might apply it in some cases that would be reasonable from a physical standpoint.

The paper by Reuschle, which was published in 1845, began by observing that in the sixteen years since Gauss published his brief note on the principle of least constraint, apparently the only mention of it was in an 1839 book on dynamics by the Cambridge scholar Earnshaw [**30**], although Reuschle confessed that he found the proof that was given by the latter author incomprehensible. Reuschle then summarized the proofs that had been given by Gauss and Earnshaw, which had a more synthetic-geometric character, and then gave his own formulation of it in the language of differential calculus as it was practiced at the time. To remedy the lack of examples of the application of the principle, he applied it to the case of a pair of masses suspended at opposite ends of an inextensible rope that hangs from a pulley and the case of a level. He then showed how one could derive the principle of virtual work, combined with d'Alembert's principle, from Gauss's principle, although as was typical of the era, he referred to the virtual *moment* of a force and a virtual displacement, not the virtual *work* done by the force along that virtual displacement. He then discussed the introduction of Lagrange multipliers into the formulation of the principle and concluded by specializing the principle of least constraint to the scope of statics problems.

The Scheffler paper followed thirteen years later in 1858 and started off pursuing much the same agenda as the Reuschle paper, although with a somewhat different approach to explaining the principle of least constraint. One of the points that the author raised was that Gauss's principle had an advantage over the combination of virtual work and d'Alembert that it was, after all, a *single* principle that specialized to both statics and dynamics, as opposed to a *pair* of first principles. However, he also noted that Gauss's principle is not always the most practical one to use as a basis for mathematical modelling in applications. Reuschle also included rotations as virtual displacements, as well as translations, as well as pointing out that Gauss's principle can be used with one-sided constraints, while virtual work, plus d'Alembert, cannot. He added some more worked examples in the form of the pendulum, the lever, a point-mass that moves along a fixed line or surface, and the collision of inelastic bodies. He also compared and contrasted Gauss's principle of least constraint with Maupertuis's principle of least action (in its pre-Lagrangian form). Finally, Reuschle proposed yet another first principle that he felt generalized Gauss's principle, as well. Namely, he posited that the total work done by applied forces under the motion of a system must be "complete," as in suffering no losses or gains. He then showed how that would imply the principle of virtual work, plus d'Alembert, as well as generalizing Gauss's principle to include the possibility of massless points, which is intriguing, since the concept of a photon did not come about until decades later.

The paper by Schering is included largely for the sake of completeness, since relatively little attention was being given to the principle of least constraint at that point in time, so the fact that it begins with a discussion of the topic would make it seem appropriate. However, although the title would suggest that author is going to formulate that principle in terms of the Hamilton-Jacobi equation (which would have to involve a reduction in the generality of Gauss's principle), in effect the general flow of ideas through the article seems to leave the principle of least constraint behind without returning to it. Eventually it evolved into mostly a discussion of how Hamilton-Jacobi theory relates to various approaches to the theory of perturbations. The article is also hampered by Schering's idiosyncratic introduction of a large number of alternative notations for differentials and partial derivatives that make the resulting equations rather opaque to modern readers, as well as his use of the term "substitution function" to refer to a canonical transformation.

The Lipschitz papers show that he was explicitly addressing the interface between mechanics and geometry, since the concepts in differential geometry that he develops are often motivated by corresponding considerations in mechanics, and in particular, the principle of least constraint. However, it is also clear that his work was not widely known, or at least accepted by his contemporaries, and the reason for that is fairly clear from a remark that Wassmuth made in one of the later translations to the effect that Lipschitz was so self-referent in his papers that in order to read any of the later ones, one would typically have to have mastered a number of his papers that preceded the one in question. Two in particular [**31**] defined Lipschitz's approach to differential geometry, and the first of the two appeared in the same journal as Christoffel's paper [**32**] in which he introduced the Christoffel symbols. Although Lipschitz eventually made note of the overlap between the two approaches, one must still get used to his personal notations for things that are commonly notated by different symbols nowadays. Another subtlety in his approach to differential geometry is the fact that he was investigating alternatives to the Riemannian extension of the Gaussian curvature of a surface that are more along the lines of what Kronecker was doing toward that goal. The key to understanding it is to note that the Gaussian curvature of a surface, as the product of the principal curvatures, is also the determinant of its second fundamental form. Hence, another dimensional extension of Gaussian curvature would be to use the determinant of the second fundamental form in n dimensions. One sees that Lipschitz seemed to be the only researcher at the time who was suggesting that one could also formulate the constraint function in the principle of least constraint in terms of accelerations that involve the covariant time derivative of the velocity when it is defined by the metric on a more general ambient space than E^n.

Lipschitz also critiqued Schering's approach to the principle of least constraint and found that it was flawed. One suspects that the two researchers must have been somewhat close because in addition to each of them referring to the other in their papers, one sees that Schering also employs a generalization of the usual length as a square root of a sum of squares to a p^{th} root of a sum of p^{th} powers, which is also sometimes referred to as the "Lipschitz norm," and is the basis for the definition of L^p spaces in functional analysis.

As the title of the paper by Rachmaninoff suggests, what he was proposing was an alternative to Gauss's principle of least constraint that he called the "principle of least work done by lost forces." It generalized the principle of virtual work, plus d'Alembert's principle, by including both one-sided and two-sided constraints, and Rachmaninoff proceeded to show how his principle would relate to not only Gauss's and virtual work, plus d'Alembert, but also to the principle of least action.

The first paper by Anton Wassmuth is concerned with how one might apply the principle of least constraint to electrodynamics. However, one should be cautioned that although one might be expecting a generalization of the formulation of Maxwell's equations in terms of the least action principle to a formulation in terms of least constraint, that is not actually the case. Indeed, the actual application was to the theory of electrical circuits. The other two papers by Wassmuth are basically in logical sequence, although they are separated by some other papers chronologically, including the one by Michael Radaković, who succeeded Wassmuth as professor of physics at the University of Graz in Austria. They were concerned with adapting the problem that had been addressed by Lipschitz of expressing the least constraint principle to the case of generalized coordinates, so they are restricted to holonomic constraints. The third paper by Wassmuth, which is perhaps the more definitive of the two on that subject, addressed the decomposition of the constraint function, when expressed in general coordinates, into a term that involves only the (generalized) accelerations, which is then similar to the Gibbs-Appell energy of acceleration, and a "remainder term," which does not include any accelerations. No actual general expression for that remainder term was obtained for k generalized coordinates and n configuration space coordinates, although he did develop successively more involved expressions for the lower-dimensional cases.

The papers by (Christian) Adolph Mayer (1839 – 1908) were published as successive articles in the same issue (1899) of the journal in which they appeared. The first one addressed the general problem of obtaining actual equations of motion in the form of a system of ordinary differential equations for motion that is constrained by perfect, but one-sided constraints; that is, although there is no friction in the system, some of the constraints are expressed by inequalities, rather than equalities of the form:

$$f_r(t, x, \dot{x}) \leq 0.$$

To some extent, the first paper was based upon some earlier research on motion with one-sided constraints by Mikhail Vasilyevich Ostragradsky (1801 – 1862) in St. Petersburg in 1834 and 1838 [**33**]. That work had left open an ambiguity, which Edouard Study had explained to Mayer in a private communication was not an oversight, but an actual fallacy in the formulation. Hence, Mayer was expanding that unpublished work of Study into the first of the two papers here. His approach to dealing with the one-sided constraints on the motion was to expand the constraint function in a Taylor series and eventually derive a corresponding restriction on the variation of the accelerations that took the form:

$$\delta f_r'' \leq 0 .$$

He then went on to introduce essentially Lagrange multipliers that produced a system of differential equations with a basically Newtonian form when one includes forces of constraint and a system of algebraic equations that allow one to solve for the Lagrange multipliers. He proved the uniqueness of the solutions to the differential equations for the cases of one and two constraints but admitted that a general proof would probably be quite tedious. He then discussed the integration of the differential equations over a finite time interval.

At the conclusion of the first article, Mayer pointed out that when the velocities of the moving points are constrained by inequalities of the "less than or equal to" form, there is the possibility that when the velocity goes from the "less than" state to the "equal to" state, there will generally be a jump discontinuity to deal with, such as when one approximates a collision by an impulsive force. The second paper by Mayer then continued the analysis of the first paper with a discussion of how to deal with those jump discontinuities in the velocities that also referred back to work of Ostragradsky on the theory of shocks. He cited the case of two masses that are connected by a flexible, inextensible string of finite length when their separation distance equals that length; one can also think of a point-mass colliding with a wall. The approach that Mayer took to regularizing the jump discontinuity in velocity was to start with Gauss's principle of least constraint and follow an argument with velocity that ran parallel to the one that he had presented with acceleration in the previous article by deriving a constraint on $\delta f_r'$. He then derived an analogous system of ordinary differential equations involving the velocities and a system of algebraic equations for the multipliers in the latter. He then dealt with the difference between external collisions and internal ones and applied his equations to two physical examples, namely, the aforementioned pair of point-masses connected by a flexible, inextensible string of finite length and a rigid circular ring that moves in a plane and collides with a point-mass. In the latter case, he distinguished between internal and external collisions between the ring and the point.

Logically, the brief paper by Paul Appell on least constraint from 1900 probably belongs with his other papers on the Gibbs-Appell equations, but it is included in the current sequence since it specifically addressed Gauss's principle. One can also see that he published it as a supplement to one of his early works on his own approach to dynamics. Mostly, Appell cited two examples of mechanical systems in which the kinetic and potential energies have the same expressions, although they have different equations of motion. Basically, the difference came down to the fact that one of them involved non-holonomic constraints (viz., rolling without slipping), while the other involved holonomic constraints (viz., sliding without friction).

The 1899 paper by the German mathematician Ernst Zermelo (1871 – 1953) was essentially a response to the first of the two Mayer papers. What he addressed was the method of proof that Mayer had used in obtaining a minimum for the constraint function in the presence of inequality constraints, and in particular, the fact that Mayer's method of proof was indirect, if not heuristic, and was only established for a few low-dimensional cases. Zermelo, by contrast, derived a concise direct proof of the existence and uniqueness of an acceleration that would minimize the constraint function that includes the general cases. An interesting subtlety in that proof is that the actual uniqueness of the minimum was contingent upon the assumption that the region in configuration space that was defined by the inequalities was simply-connected and everywhere-convex.

The 1901 paper by Aurel Voss (1845-1931) is basically a survey of how energetic techniques relate to the various approaches to obtaining systems of equations, such as the principles of virtual work, plus d'Alembert's principle, and the principle of least action. It concludes with a discussion of Gauss's principle that refers to the work of Lipschitz and Wassmuth.

The 1907 paper by Richard Leitinger also comes out of the University of Graz, where one also found Wassmuth and Radaković. It was also influenced by not only their work on the subject, but also Lipschitz. However, in order to understand why the author claims that the same expression for Gauss's principle of least constraint, when one derives it from the second form of Lagrange's equations, will result for either holonomic or non-holonomic constraints, as well as rheonomic or scleronomic constraints, one must refer to the lectures by Boltzmann [**6**] that Leitinger repeatedly cited. One finds that although Boltzmann discussed the difference between formulating Lagrange's equations in their two forms for both holonomic and non-holonomic constraints, he did not seem to address the fact that generalized coordinates will only exist in the case of holonomic constraints. Hence, one might read the Leitinger paper with some skepticism in regard to whether some things might need to be reformulated in terms of adapted frames, rather than generalized coordinates.

The paper by Ernst Schenkl (1913) also came out of the mathematical-physical department at the University of Graz, and while Wassmuth was its chair. Although it was emphasized above that Gauss's principle of least constraint is not strictly a variational principle, in the sense that one looks for an instantaneous value of the acceleration that will minimize a function on a finite-dimensional manifold, not a curve that will minimize a functional on curves, the paper by Schenkl was in response to a suggestion of Wassmuth that one might extend Gauss's principle to a true variational principle by integrating the constraint function along a curve between two time points. Although the ultimate result of the study was perhaps only so definitive in that regard, one must nonetheless consider it as an interesting first attempt.

The paper by Paul Stäckel from 1919 is distinguished from the others in this collection by its treatment of singular linear constraints; that is, there are points at which the system of linear constraints is not linearly-independent. One finds that although the principle of least constraint can still be used, nonetheless, the uniqueness of the acceleration that defines a minimum of the constraint function will break down at a singular point.

b. Gibbs-Appell equations. – Although the 1879 paper by Josiah Willard Gibbs [**24**] was historically the first to suggest what Paul Appell later expanded upon, it has not been included since it was published in English and did not require translation.

The papers by Appell tend to have considerable overlap to them, and in fact, the survey article that he wrote in 1925 is probably the best one to read first. The ones that precede it historically are included in order to show the process of development that went into his idea. The first paper that is included here, which comes from the *Compte rendus* of 1899, was not actually the exact first time that he suggested his formulation of the equations of dynamics, as its first sentence suggests, but seems to be the starting point that he cited later. Indeed, the roots of his new approach to obtaining equations of motion in physical mechanics were already present to some extent in his previous work on rolling motion and non-holonomic constraints [**34**]. Although Appell's paper "Développements sur une forme nouvelle des équations de la dynamique," [J. Math. pures appl. (9) **6** (1900), 5-40] was an early survey of the progress that he had made at that point in time, it is also not included since it overlaps considerably with the later paper in 1925.

The brief (one page) note by Saint-German in the *Compte rendus* (1900) is also included for the sake of completeness, since it mostly just proposes the term "energy of acceleration" for the Appell function.

The 1910 paper by Gian Antonio Maggi essentially related Appell's approach to motion with non-holonomic constraints to some previous work by Vito Volterra on the same topic [**35**].

The next four papers by Appell and Edouard Guillaume that were published between 1912 and 1916 are concerned with various applications of the Appell equations to other things than systems of point-masses. The papers on its possible applications to electrodynamics are reminiscent of the paper by Wassmuth that was referred to above, although neither author specifically cites that paper. They both based their discussion upon an earlier analysis in a 1907 booklet by Emmanuel Carvallo [**36**], in which he used a mechanical model called the "Barlow wheel" to illustrate the fact that one could use the principle of virtual work, but not the Euler-Lagrange equations, when dealing with certain phenomena that relate to two and three-dimensional electrical conductors. They also referred to some work of Hendrik Antoon Lorentz on the theory of electrons [**37**] in which he attempted to recast Maxwell's equations as equations of constrained motion. There is also an extension of the Appell function from finite systems of point-masses to continuum mechanics that addressed the perfect fluid as a particular case and arrived at the Euler equations of motion as a consequence. Basically, the finite set of point masses was replaced with a mass density function and the summation over all masses became an integral over the support of the mass density. The 1916 paper by Appell related the forces that enter into Lagrange's equations of the first kind when one is dealing with non-holonomic constraints to what William Thomson (later Lord Kelvin) had called "gyroscopic forces" in his 1867 *Treatise on Natural Philosophy* [**38**] with Peter Guthrie Tait. The final 1925 paper by Appell essentially summarized the content of all of his papers on his equations of dynamics that had preceded it.

One might also confer the 1930 paper [**39**] on Appell's equations by Seeger, which is not included because it is in English.

c. Hertz's mechanics. – As pointed out before, Hertz's treatise on mechanics was not initially met with any widespread acceptance, and perhaps that is due to its inchoate character as an unedited posthumous book. However, one must note that the few people who made note of his theory were certainly distinguished figures.

The 1896 paper by Otto Hölder made repeated references to Hertz's work, although it is primarily concerned with how one must adapt the usual least-action principle when one is dealing with non-holonomic constraints. Keep in mind that although the topic of rolling motions had been discussed by both mathematicians and physicists for some time up to that point, nonetheless, it was Hertz who first introduced the terminology "holonomic" and "non-holonomic" specifically.

The 1897 paper by Henri Poincaré is a largely philosophical discourse. He summarized the fundamental points of Hertz' justification for his system of mechanics by comparing the three fundamental logical systems at the foundations of mechanics: viz., the classical system, the energetic system, and the Hertzian system. The classical system involves the concepts of space, time, force, and mass as its logical primitives. The energetic system introduces kinetic and potential energy as its primitive, undefined notions. Hertz's system is based upon the assumptions that the only forces in nature are forces of constraint and that if some bodies in nature appear to act in response to forces then that is because they coupled by bodies that are invisible to us. Of course, of the two assumptions, it is the second one that seems far more questionable, and Poincaré pointed out that it was reminiscent of Lord Kelvin's attempt to model the electromagnetic vacuum as a "gyrostatic ether."

The one-page note by Ludwig von Boltzmann (1899) addressed the lack of examples in Hertz's treatise. He first showed that a certain linkage that consists of two articulated links of equal length a, the first of which has one fixed end and the second of which has a mass μ at its free end, could serve as a model for a mass μ that is constrained to move inside of a hollow sphere of radius $2a$. He then posed the problem of finding a picture in the spirit of Hertz's mechanics that would exhibit the elastic collision of elastic balls.

The first paper by the German mathematician Alexander von Brill in Tübingen, which was published in the next year after Boltzmann had posed his problem, began with his response to that problem, and then concluded with some observations about Hertz's program of replacing forces-at-a-distance with a medium of "hidden masses." In the second paper, which was also published in 1900, Brill continued his commentary on Hertz's substitute for forces, but also made some observations about the more fundamental axioms of Hertz's laws of motion that related to extending Newton's law of inertia with the principle of least curvature.

A translation that was not included here, although it is still quite intriguing in its own right, is the author's translation of Erwin Schrödinger's unpublished handwritten notes [**40**] that bore the compelling title of "Hertzian mechanics and Einstein's theory of gravitation." Although it is certainly true that Hertz's concept of the path of least curvature is closely related to that of geodesic, which plays a fundamental role in Einstein's theory, by the time one finishes reading Schrödinger's notes, one recognizes why the notes were unpublished: Past the title, no further mention was made of either Hertz's mechanics or Einsteinian gravitation. However, one could see how Schrödinger was developing the foundations of wave mechanics by way of the optical-mechanical analogy that centers around the Hamilton-Jacobi equation.

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THE TRANSLATIONS

“Über ein neues allgemeines Grundsetz der Mechanik,” J. reine angew. Math. **4** (1829), 232-235.

On a new general foundation for mechanics

(By Herrn Hofrat and Prof. Dr. **Gauss** in Göttingen)

Translated by D. H. Delphenich

As is known, the principle of virtual velocities converts all of statics into a mathematical problem, and **d'Alembert**'s principle for dynamics reduces that study, in turn, to statics. Therefore, it is in the nature of things that no one has given any new basic principle for the study of motion and equilibrium that would already include both of the latter principles, and from which they could be derived. In the meantime, however, due to that situation, a new principle would not seem to be worthless. It would always remain interesting and instructive to abstract a new and preferable viewpoint for the laws of nature such that one would easily solve this or that problem from it, or that it would add a special reasonableness to that problem. The great geometer who built the structure of mechanics on the basis of the principle of virtual velocities so brilliantly, did not reject the raising of **Maupertuis**'s principle of least action to a state of greater determinacy and generality, which is a principle to which one can appeal with great advantage from time to time ([*]).

The peculiar character of the principle of virtual velocities consists of the fact that it is a general formula for solving all static problems and a paradigm for all other principles, without taking the credit for them so directly that it would already recommend itself as something plausible, as long as one only expresses it.

In that regard, the principle that I have proposed here seems to have one advantage. However, it also has a second one, namely, that it encompasses the law of motion and rest in completely the same way and in greatest generality. Therefore, it would very much be in order that from the gradual development of science and the teaching of individuals, simple things should come before complicated things and specialized things should come before the generalities. Nonetheless, once one has reached the higher standpoint, the spirit will demand the converse process, whereby all of statics appears to be only an entirely special case of mechanics. Even the aforementioned geometer seem to place some value upon that notion when he regarded one of the advantages of the principle of least action as being that it included both equilibrium and motion at the same time when one

([*]) Remark by the author: However, permit me to point out here that I was not satisfied by the way that another great geometer attempted to prove **Huyghens** law for the extraordinary refraction of light in crystals with double refraction by means of the principle of least action. In fact, the admissibility of that basic law depends essentially upon the conservation of *vis viva*, which would merely constrain the positions and speeds of moving points without having any influence on the direction of motion, which was, however, assumed in the aforementioned attempt. It seems to me that in systems of emanations, all endeavors to link the phenomena of double refraction with the general laws of dynamics must remain fruitless as long as one considers light particles to be merely points.

expresses them in such a way that the *vis viva* is smallest for both of them, which is a remark that seems to be more clever than true, however, since the minimum in both cases occupies an entirely different place in both of them.

The new principle is the following one:

The motion of a system of material points that are always coupled to each other in some way, and whose motion is, at the same time, always subject to external constraints, agrees with the free motion at each moment to the greatest possible extent or with the least possible constraint when one considers a measure of the constraint that the system suffers at each point in time to be the sum of the products of the squares of the deviations of each point from the free motion of its mass.

Let $m, m', m'', \ldots$ be the masses of the points. Let $a, a', a'', \ldots$, be their positions at time t, and let $b, b', b'', \ldots$, be the positions that they would assume after the infinitely-small time interval dt as a result of the forces that act upon them during that time and the speeds and directions that would be attained at tome t, in the event that they were all completely free. The actual positions $c, c', c'', \ldots$ will then be the ones that are compatible with all of the conditions on the system and for which $m\,(b\,c)^2 + m'(b'c')^2 + m''(b''c'')^2 + \ldots$ is a minimum.

Equilibrium is obviously only a special case of the general law, and the condition for:

$$m\,(b\,c)^2 + m'(b'c')^2 + m''(b''c'')^2 + \ldots$$

itself to be a minimum, or the persistence of the system in the rest state, is that the free motion of the individual points should lie closer than any of the other possible ones that might emerge.

The derivation of our principle from the two that were cited above comes about easily in the following way:

The force that acts upon the material point m is obviously composed, first of all, of the force that is coupled with the speed and direction at time t that takes a to c in the time dt and a second one that would lead from rest to c by way of cb in the same time if one considered the point to be free. The same thing will be true for the other points. From **d'Alembert**'s principle, the points $m, m', m'', \ldots$ must then be equilibrium under the effect of only the second forces along cb, $c'b'$, $c''b''$, … at the positions $c, c', c'', \ldots$ due to the constraints on the system.

From the principle of virtual velocities, equilibrium would demand that the sum of the products of each of the three factors (namely, each of the masses $m, m', m'', \ldots$ the lines cb, $c'b'$, $c''b''$, …, and any others that would project onto the latter, resp. due to the possible motions of that point that are compatible with the constraints on the system) would always have to equal zero, as one ordinarily expresses it (*), or rather, more correctly, by saying that each sum can never be positive.

(*) Remark by the author: The usual expression always assumes constraints that make the opposite of any possible motion equally possible, such as, e.g., that a point should remain on a well-defined surface, that the distance between two points should be unvarying, and the like. However, that is an unnecessary, and not always appropriate, restriction on nature. The outer surface of an impermeable body does not require that a material point that is found upon it must remain upon it, but merely prohibits it from appearing on the other side. For a tensed, inextensible, but flexible, string between two points, only an increase in the distance between two points is impossible, but not a decrease, etc. Why

Therefore, if γ, γ', γ'', … are positions that are different from c, c', c'', …, but still compatible with the constraints on the system, and θ, θ', θ'', … are the angles that $c\,\gamma$, $c'\gamma'$, $c''\gamma''$, … make with cb, $c'b'$, $c''b''$, …, then $\sum m \cdot cb \cdot c\gamma \cdot \cos\theta$ will always be either 0 or negative. Now, since:

$$\gamma b^2 = c\,b^2 + c\,\gamma^2 - 2\,cb \cdot c\gamma \cdot \cos\theta,$$

it will then be clear that:

$$\sum m \cdot \gamma b^2 - \sum m \cdot c\,b^2 = \sum m \cdot c\,\gamma^2 - 2 \sum m \cdot cb \cdot c\gamma \cdot \cos\theta$$

will always be positive as a result, so $\sum m \cdot \gamma b^2$ will always be greater than $\sum m \cdot c\,b^2$; i.e., that will be a minimum. Q. E. D.

It is very remarkable that although free motions cannot exist when constraints are imposed, by their very nature, they will be modified in the same way by the method of least squares, which relates to quantities that are necessarily coupled with each other by dependencies, as the calculating mathematician will confirm by experience. That analogy can be pursued even further, although I presently do not intend to do so.

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then would we not choose to express the law of virtual velocities in such a way that it encompasses *all* cases right from the beginning?

"Ueber das Princip des kleinsten Zwangs und die damit zusammenhängenden mechanischen Principe," Arch. Math. Phys. **6** (1845), 238-269.

On the principle of least constraint and the mechanical principle that is connected with it

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Translated by D. H. Delphenich

In the fourth volume of Crelle's Journal, pp. 232, **Gauss** presented a new general fundamental law of mechanics that was classified among the dynamical principles by the name above ("principle of least restraint") in an English textbook (**Earnshaw**, *Dynamics*, Cambridge, 1839). As far as I know, that is the only place in the literature where it was cited, but without examples, moreover, and the same can be said for **Gauss**. The purpose of this treatise is, after a few historical remarks on the treatment of the principle in the two aforementioned papers, to first test that principle in a pair of simplest-possible examples and to then undertake the general analytical treatment and to thus prove its connection with the remaining principles of mechanics.

§ 1.

With the expression that its author gave to it, the principle reads:

> "The motion of a system of material points that are always coupled to each other in some way, and whose motions are, at the same time, always constrained by external restrictions will take place at each moment *with the greatest possible coincidence with the free motion or with the smallest possible constraint*, in which the measure of constraint that the entire system suffers at every point in time is considered to be the sum of the products of the squares of the deviations of each point from its free motion with its mass."

Gauss and **Earnshaw** proved that law in different ways by reducing it to other mechanical principles.

I. – First of all, Gauss derived it from d'Alembert's principle, in conjunction with the principle of virtual velocities, it in the following way: For any point of the system let (Tab. IV, Fig. 1 [†]) *m* be its mass, let *A* be its location at the time *t*, let *B* be the location that it would assume after an infinitely-small time interval τ as a result of the forces that act upon it and the velocity that it would achieve at time *t* if it were completely free, let *C* be the actual location that would correspond to the time interval as a result of the system constraint, and finally, let *D* be any other location that is compatible with the system constraint, which one understands to be infinitely close to the points *A* and *C*. $\sum m \cdot \overline{BC}^2$ will then be a minimum when:

$$\sum m \cdot \overline{BD}^2 - \sum m \cdot \overline{BC}^2 > 0 .$$

However, if θ is the angle between *CB* and *CD* then one will have:

$$\overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2 - 2\overline{BC} \cdot \overline{CD} \cdot \cos\theta$$

for the triangle *BCD*, and as a result:

$$(a) \qquad \sum m \cdot \overline{BD}^2 - \sum m \cdot \overline{BC}^2 = \sum m \cdot \overline{CD}^2 - 2\sum m \cdot \overline{BC} \cdot \overline{CD} \cdot \cos\theta ,$$

but from the principle of virtual velocities, when applied to the equilibrium of lost forces that is required by d'Alembert's principle, one has:

$$(b) \qquad \sum m \cdot \overline{BC} \cdot \overline{CD} \cdot \cos\theta = 0,$$

and as a result:

$$\sum m \cdot (\overline{BD}^2 - \overline{BC}^2) = \sum m \cdot \overline{CD}^2 > 0 .$$

In fact, the product $m \cdot \overline{BC} \cdot \overline{CD} \cdot \cos\theta$ is the virtual moment that belongs to the material point *m*, insofar as $m \cdot \overline{BC}$ can be considered to be its lost force. In order to justify that, as well as to show the essential homogeneity of the equation (*a*) in regard to the infinitesimals, I shall permit myself to add the following remarks to the Gaussian proof:

The product $m \cdot \overline{BC}$ is, first of all, the product of the mass with a space or path that corresponds to the infinitely small time interval τ. However, one can replace the acceleration in it *p* that corresponds to a unit time with $\frac{1}{2} p \tau^2$, insofar as merely $\frac{1}{2}\tau^2$ will enter then as a common factor in the summation sign in (*b*). That is because even though the velocities *v* that are required at time *t* that enter into the expression of the principle above yield paths of the form $v\tau$ that will therefore not include the square of τ, the lost forces, or the paths *BC* that are substituted for them, will,

(†) Translator: The tables and figures referred to were not available to me at the time of translation.

however, not in fact depend upon those velocities *v*, which will be explained completely by the proof of the lost force in *BC*. To that end, (Tab. IV, Fig. 2), starting from the location *A* that corresponds to time *t*, let *AE* be the path that is traversed in the same time interval as a result of the newly-added force, while *AC*, as above, means the actual path: When one extends *AEC* to a parallelogram, $EC = AG$ will represent the actual component of the accelerating force *AF*, and when one extends *AGF*, $FG = AG$ will represent the lost component, and finally, the diagonal *AB* in the parallelogram at *AE* and *AF* will represent the path that is freely traversed in the time interval τ. However, that is, at the same, the diagonal in the parallelogram at *AC* and *AH*, and therefore the deviation *BC* from the free path that is parallel and equal to the lost force *AH*, which already illuminates the fact that *BC*, like *AF*, is a quantity of the form $\frac{1}{2}p\tau^2$. However, more directly, it shows the independence of the quantity *BC* (or *AH*) from the velocity that is required at time *t* thus: If AC' is equal and opposite to *AC* then *AH* will be the resultant of *AB* and AC'. However, if one decomposes them into their components, which are AE' and AG' for *AE* and *AF*, resp. (i.e., the quantities that are equal and opposite to *AE* and *AG*, resp.), then *AE* and AE' will cancel as components of *AH*. – Now, since $BC = \frac{1}{2}p\tau^2$, from the cited principle, $\sum m\cdot\overline{BC}\cdot\overline{CD}\cdot\cos\theta = \frac{1}{2}\tau^2\sum mp\cdot\overline{CD}\cdot\cos\theta$ will generally be zero. However, since *BC* is, at the same time, a second-order infinitesimal, *BD* and *CD* must also prove to be such things. Hence, let (*a*) be homogeneous, because otherwise the term in equation (*b*) would drop out as a higher-order infinitesimal, independently of the principle of virtual velocities. However, if *D* is a point of the same kind as *C* then one can repeat the previous construction for it when one replaces *C* with *D* everywhere, and starting from *A*, lets the given initial velocity *AE* agree with *AD*, just as it does as with *AC*, but lets any other accelerating force enter in place of *AF*. In that way, *BD* will prove to be a quantity of the form $\frac{1}{2}p'\tau^2$, where p' is any other acceleration that only corresponds to the conditions on the system that would lead *m* from *B* to *D*, instead of *C*, in the same time interval τ, and then *CD* will also be a quantity of the same form, since it is the resultant of *CB* and *BD*, namely, when λ is the opposite angle to *CD* in the triangle *BCD*, $\overline{CD}^2 = \frac{1}{4}\tau^4\ (p^2 + p'^2 - 2pp'\ \cos\lambda)$.

II. – **Earnshaw** sought a different type of proof by coupling the static and dynamic properties of the center of mass with d'Alembert's principle by a mechanical construction, to which I cannot, however, ascribe the least evidence for my complete understanding of it, but I can also get around that fact. Start from the fact the resultant of the lost forces – or, as is aptly expressed, the constraint forces (forces of restraint, restraining pressures) – that act upon *m* in the direction *BC*, so *BC* is the amount by which the resultant makes the particle *m* deviate in the infinitely small time interval τ: Thus, if one now imagines all material points of the system as being removed from their constraints and combined at any point β in space in the free state, and pushed away from it by pressures with the same magnitudes and directions as the constraint forces then any particle *m* will move from β to a point γ such that $\beta\gamma$ is equal and parallel to *BC*. Now, since the forces of constraint are such that they will produce equilibrium in the system, according to d'Alembert's principle, they can have no influence on the motion of the center of mass, according to the principle of the conservation of the center of mass, so the center of mass of the particles *m* will remain where it was found for the points γ. However, from a known property of the center of mass, $\sum m\,\overline{\beta\gamma}^2$ will be a minimum, and as a result $\sum m\,\overline{BC}^2$, as well; i.e., the value of that quantity will be smaller than when the constraint forces do not produce equilibrium, as they should according to d'Alembert's principle.

Now, this law of the center of mass is so deeply and subtly rooted in the nature of things that the sum of the products of each mass with the square of its distance from the center of mass must be a minimum: It seems to me that there is much to be done regarding the mechanical picture of constraint, due to the way that the law of conservation of the center of mass is applied. That is because there are such great demands imposed upon mathematical abstraction when one thinks of an arbitrary mass being concentrated at a point or likewise located along a line or on a surface that it occurs to me that the demand that several isolated material points (or masses that are thought of as concentrated into points) in a free (i.e., uncoupled) state such that each of them can move as if one could imagine that it is reduced to a point, or even more, that several material points of that type could define a system is quite unnatural, if not a *contradictio in adjecto*, and one must therefore first prove that β is the center of mass of that system before one can understand how it remains there. However, the fact that β is actually the center of mass of the material points that are found at the point γ follows from a purely static law that **Lagrange** cited in *Mécanique analytique* in section five of *Statics* (t. 1, page 107) as being due to **Leibnitz** and derived using his own formulas. The law consists of the fact that when several forces are in equilibrium at a point and lines are drawn from that point that represent those forces in magnitude and direction, the point in question can be the center of mass of just as many (as the number of forces present) equal masses that are attached to the endpoints of those lines, and one can then extend that to: If one represents each of the static forces P that are in equilibrium at a point by a product mp, the one factor of which m can be considered to be a mass, while the other one p can be considered to be an acceleration, or as a path that is traversed in a certain time interval, then each point will be the center of mass of the masses m that are placed at distances p from it, while those distances are taken in the direction of the given forces, and $\sum mp^2$ will then be a minimum, from a second law of statics that **Earnshaw** cited. I shall now intertwine the proof of the law that was thus posed, as well as the last one mentioned, with the justification for the English proof of our principle, which consists of the following steps:

1. As a result of the principle of the conservation of the center of mass, the constraint forces that are in equilibrium in the system due to d'Alembert's principle have no influence on the motion of the center of mass, but rather they must be in equilibrium at it, so at any point β in space at all (*). Therefore, if one sets $BC = \beta\gamma = \frac{1}{2}\,p\,\tau^2$, as in no. **I**, and if a, b, c are the angles that the line $\beta\gamma$ makes with the three rectangular axes of x, y, z, resp., then the equations:

(*) Or, more simply: Since the constraint forces produce equilibrium in the given system, from d'Alembert's principle, they must, in particular, cancel the advancing motion of the center of mass, and thus be in equilibrium at a point.

$$(\alpha) \qquad \sum mp\cos a = 0, \qquad \sum mp\cos b = 0, \qquad \sum mp\cos c = 0$$

will express the equilibrium of the constraint forces $m\cdot\overline{BC}$ at the point β.

2) Thus, if ξ, η, ζ are the coordinates of the point β and x, y, z are those of any point γ then:

$$\overline{\beta\gamma}^2 = \tfrac{1}{4}p^2\tau^4 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$$

and

$$\cos a = \frac{x-\xi}{\frac{1}{4}p^2\tau^4}, \qquad \cos b = \frac{y-\eta}{\frac{1}{4}p^2\tau^4}, \qquad \cos c = \frac{z-\zeta}{\frac{1}{4}p^2\tau^4}.$$

Therefore, from the foregoing equations (α):

$$(\beta) \qquad \sum m(x-\xi) = 0, \qquad \sum m(y-\eta) = 0, \qquad \sum m(z-\zeta) = 0,$$

and those will give:

$$\xi = \frac{\sum mx}{\sum m}, \qquad \eta = \frac{\sum my}{\sum m}, \qquad \zeta = \frac{\sum mz}{\sum m},$$

so the point β whose coordinates are ξ, η, ζ will be the center of mass of the material point m whose coordinates are x, y, z ; i.e., the masses m that are found at the points γ (*).

3) The first part of equations (β) are the partial derivatives of the functions $\frac{\tau^4}{4}\sum m\,p^2$ with respect to ξ, η, ζ, but taken with the opposite sign, and since the second part, namely, the one where one differentiates with respect to those variables a second time, reduces to the essentially positive quantity $\sum m$, while the others, in which one differentiates with respect to two different variables, all reduce to zero. Hence, all conditions are fulfilled that make that function – thus, $\sum m\cdot\overline{BC}^2$ – a minimum with respect to ξ, η, ζ ; i.e., it is smaller then when ξ, η, ζ do not have the values that correspond to the center of mass, so smaller than when the constraint forces $m\cdot\overline{BC}$ are not in equilibrium at a point, so not in equilibrium for the system, either (**).

(*) Now, since one can also set $P = mq$, $P' = mq'$, …, instead of $mp = P$, $m'p' = P'$, etc. (i.e., assume that all of the masses are equal), if n is the number of those masses (or forces P) then:

$$\xi = \tfrac{1}{2}\sum x, \qquad \eta = \tfrac{1}{2}\sum y, \qquad \zeta = \tfrac{1}{2}\sum z,$$

which is the form of this result that Lagrange derived in the cited place as an expression for Leibnitz's law.

(**) One can exhibit this law of the center of mass without differential calculus, namely, that if r_0 is the distance from a material point m to the center of mass then $\sum mr_0^2$ will be a minimum: Let $x_0 = x - \xi$, $y_0 = y - \eta$, $z_0 = z - \zeta$ be the coordinates of m relative to the center of mass, so $r_0^2 = x_0^2 + y_0^2 + z_0^2$, and likewise $r^2 = x^2 + y^2 + z^2$ and $\rho^2 = \xi^2 + \eta^2 + \zeta^2$, and one will have:

III. – If we now compare the two proofs then both of them start from the fundamental principle of dynamics – namely, the equilibrium of the constraint forces – but they differ in that **Gauss**'s proof brings about that equilibrium by means of the general condition of equilibrium, which consists of the vanishing of the virtual moments, while that of **Earnshaw** uses a partial condition that has merely the cancellation of the advancing motion as a consequence. Now, the relationship of our minimum to the one that gives the center of mass seems so intimate and fragile that this proof is in no way general and is initially suited merely to free systems, in which the forces of constraint must fulfill the condition that they are in equilibrium at a point, which will no longer be the case when the system includes a fixed point or a fixed axis, such that $\sum m \cdot \overline{BC}^2$ should reduce to $\sum m r_0^2$ (where r_0 is the distance from the mass to the center of mass) in all cases, so that must happen in the stated cases, in particular, or more generally, but then it could happen only by means of the principle of virtual velocities. Thus, one can assert that the English proof is merely a proof of the principle in one example (although it includes many other cases), as we will show in § **2** in another example of a general type, but that **Gauss**'s general proof must be carried out using the general formula of dynamics, namely, the coupling of d'Alembert's principle with that of virtual velocities. The consequences of the principle of least constraint are then exhibited in just enough generality that it can equally serve as the basis for the derivation of all the equations of motion of a given system, just as the principle of virtual velocities is applied to the equilibrium of lost forces, since the Gaussian proof explains directly that one can conversely arrive at that general formula of dynamics from the principle of least constraint.

Just as one can derive a static formula from any dynamic one by means of d'Alembert's theorem, and conversely, according to **Gauss**, equilibrium is only a special case of the general theorem into which merely the points *A* themselves enter, corresponding to the equilibrium position, instead of the points *C*, and $\sum m \cdot \overline{BA}^2$ is a minimum. In fact, that means the same thing as: Set everything in the general formula of dynamics that refers to the actual motion at the time t equal to zero in the case where the system of applied forces is itself in equilibrium. Furthermore, the proof can also be carried out in just the same way, since *D* corresponds to a virtual position of the material point *m*, such that *AD* is equivalent to *AB* relative to their magnitudes, so the triangle *BAD*, like *BCD* before it, will imply the relation:

$$(c) \qquad \sum m \cdot (\overline{BD}^2 - \overline{BA}^2) = \sum m \cdot \overline{AD}^2 - 2\sum m \cdot \overline{BA} \cdot \overline{AD} \cdot \cos\varphi ,$$

which is similar to (*a*), in which φ is the angle between the direction of the free motion *AB* or that of the force and that of the virtual path *AD*, and with:

$$\sum m(r^2 - r_0^2) = \sum m[(x_0+\xi)^2 - x_0^2 + (y+\eta)^2 - y_0^2 + (z+\zeta)^2 - z_0^2] = 2\xi \sum m x_0 + 2\eta \sum m y_0 + 2\zeta \sum m z_0 + \rho^2 \sum m .$$

As a result, since:

$$\sum m x_0 = \sum m y_0 = \sum m z_0 = 0, \qquad \text{one will have} \qquad \sum m(r^2 - r_0^2) = \rho^2 \sum m > 0,$$

which also might be the coordinate origin.

$$\sum m \cdot \overline{BA} \cdot \overline{AD} \cdot \cos\varphi = 0,$$

the expression:

$$\sum m \cdot \overline{BD}^2 - \sum m \cdot \overline{BA}^2$$

will reduce to the essentially positive quantity:

$$\sum m \cdot \overline{AD}^2 .$$

As is already suggested by the demand that *AD* should be equivalent to *AB* in magnitude, it should be remarked here that since $AB = \frac{1}{2} p \tau^2$, if *p* is, in turn, the accelerating force that acts upon *m* and τ is the infinitely small time interval then the virtual path *AD* must also be a second-order infinitesimal relative to τ, which is, in fact, also justified by the fact that the quantity *AD* can be thought of as the path that is traversed during the time interval τ when other accelerating forces *q* enter in place of the forces *p* that are in equilibrium at the point *A*. (Cf., moreover, § **2**, III, and § **3**, I.)

§ 2.

We shall now test the principle of least constraint directly in some of the simplest examples of motion, as well as equilibrium.

I. – In order to take the simplest possible case of the motion of a system, it is required that all points *A*, *B*, *C*, *D*, *E* for the material points fall along the same line. We therefore assume that two unequal masses *m*, *m′* are at the ends of a weightless and absolutely flexible and inextensible string that goes around a weightless pulley, and establish the arrangement of points *ABCDE* for the mass *m* and the corresponding one *A′B′C′D′E′* for the mass *m′* (Tab. IV, Fig. 3): Therefore, from the nature of the system, all distances from the remaining points to *E* on the one side and the distances to *E′*, on the other, will be equal and have the opposite senses, except for *EB* and *E′B′*, which will both have the sense of gravity. Now, one then has:

$$\overline{BC} = \overline{EB} - \overline{EC}, \qquad \overline{B'C'} = \overline{EB} + \overline{EC},$$

$$\overline{BD} = \overline{EB} - \overline{EC} - \overline{CD}, \qquad \overline{B'D'} = \overline{EB} + \overline{EC} + \overline{CD},$$

$$\overline{BD}^2 - \overline{BC}^2 = \overline{CD}^2 - 2 \cdot \overline{CD} \cdot (\overline{EB} - \overline{EC}),$$

$$\overline{B'D'}^2 - \overline{B'C'}^2 = \overline{CD}^2 + 2 \cdot \overline{CD} \cdot (\overline{EB} + \overline{EC}),$$

and as a result, one has the quantity:

$$\sum m \cdot (\overline{B'D'}^2 - \overline{B'C'}^2) = (m+m') \cdot \overline{CD}^2 + 2(m+m') \cdot \overline{CD} \cdot \overline{EC} - 2(m-m') \cdot \overline{CD} \cdot \overline{EB}$$
$$= (m+m') \cdot \left\{ \overline{CD}^2 + 2 \cdot \overline{CD} \cdot (\overline{EC} - \frac{m-m'}{m+m'} \cdot \overline{EB}) \right\}.$$

However, if, as usual, g is the acceleration of gravity and τ is a time interval that can have an arbitrary finite magnitude in our case, along with all of the foregoing lines, then:

$$\overline{EB} = \tfrac{1}{2} g\tau^2, \qquad \overline{EC} = \tfrac{1}{2} \frac{m-m'}{m+m'} g\tau^2,$$

so

$$\overline{EC} - \frac{m-m'}{m+m'} \cdot \overline{EB} = 0,$$

and as a result:

$$\sum m \cdot \overline{BD}^2 - \sum m \cdot \overline{BC}^2 = (m+m')\overline{CD}^2 = \sum m \cdot \overline{CD}^2 > 0,$$

which might also be arbitrary line CD, as long as one only takes it to have the direction of the string, and therefore:

$$m\overline{BC}^2 + m'\overline{B'C'}^2$$

will be a minimum.

If one introduces the values of EB, EC into the expression that is to be a minimum then when one makes $\frac{m-m'}{m+m'} = \mu$, one will have:

$$\overline{BC} = \tfrac{1}{2}(1-\mu)\, g\tau^2, \qquad \overline{B'C'} = \tfrac{1}{2}(1+\mu)\, g\tau^2,$$

and as a result:

$$\sum m \cdot \overline{BC}^2 = [m\,(1-\mu)^2 + m'(1+\mu)^2]\, \frac{g^2\tau^4}{4},$$

and one will obviously go from this to $\sum m \cdot \overline{BD}^2$ when one substitutes another acceleration $(\mu + \alpha)\, g$ for $\mu\, g$, in which α is an arbitrary (positive or negative) quantity, from which one gets:

$$\sum m \cdot \overline{BD}^2 - \sum m \cdot \overline{BC}^2 = (m+m')\frac{\alpha^2 g^2 \tau^4}{4},$$

i.e., $\sum m \cdot \overline{CD}^2$, so one has just $CD = \frac{1}{2}\,\alpha\, g\,\tau^2$. However, that also explains the fact that, in regard to proving the minimum by differential calculus, one must differentiate with respect to μ, and since the first two derivatives of the quantity above with respect to μ are:

$$[-m\,(1-\mu) + m'\,(1+\mu)]\,\frac{g^2\tau^4}{4} \qquad \text{and} \qquad (m+m')\,\frac{g^2\tau^4}{4}\,,$$

the first of which will be identically zero, from the value of μ, while the second one is essentially positive, the minimum will also be confirmed by this argument.

Finally, if one would like to apply the principle by first finding the acceleration of the system then let z, z' be the distances from the points A, A', resp., to the horizontal diameter of the pulley at time t, so because the condition on the system is $\Delta z = -\,\Delta z'$, corresponding to the time interval t, one will have:

$$EC = -\,E'C' = \frac{d^2 z}{dt^2}\cdot\frac{\tau^2}{2} + \ldots,$$

and as a result:

$$BC = -\left(g - \frac{d^2 z}{dt^2}\right)\frac{\tau^2}{2} - \ldots, \qquad B'C' = \left(g + \frac{d^2 z}{dt^2}\right)\frac{\tau^2}{2} - \ldots,$$

so the quantity that should be a minimum is:

$$\left\{ m\left(g - \frac{d^2 z}{dt^2}\right)^2 + m'\left(g + \frac{d^2 z}{dt^2}\right)^2 \right\}\frac{\tau^4}{4} + \ldots$$

Taking its first derivative with respect to the actual acceleration d^2z / dt^2 will then give:

$$\left\{ m\left(g - \frac{d^2 z}{dt^2}\right) + m'\left(g + \frac{d^2 z}{dt^2}\right) \right\}\frac{\tau^4}{4} + \ldots,$$

as before, and since that equation must be independent of the magnitude of the time interval τ, it must follow that:

$$m\left(g - \frac{d^2 z}{dt^2}\right) + m'\left(g + \frac{d^2 z}{dt^2}\right) = 0,$$

so:

$$\frac{d^2 z}{dt^2} = \frac{m - m'}{m + m'}\,g\,.$$

Since one has $d^2 z / dt^2 = 0$ immediately, and the same thing we be true for all of the following time derivatives, the expressions above will reduce to their first terms, and since the second derivative $(m + m') \tau^4 / 4$ is essentially positive, it will indicate a minimum.

II. – Furthermore, the principle can be applied to a single material point. There, as well, one has $m \cdot \overline{BC}^2$, so BC is a minimum, up to sign. That is self-explanatory for free motion, where $BC = 0$, but for constrained motion, one can generally understand it in a very simple way: For the motion of a point on a skew plane under the influence of gravity, e.g., in the direct geometric construction (Tab. IV, Fig. 4), BC is the normal to the skew plane, so it is the shortest line from B to the latter, and indeed for an arbitrary finite path or a finite time interval τ. However, that is obviously true for any motion of that kind, which might also be on a surface or a curve, for which the motion of the point is constrained, and which might come into play for forces when the time interval is generally taken to be infinitely small, because the lost force in that case is always a normal pressure on that curve or surface, so BC is normal, and therefore it is the shortest line from B to the curve or surface along which the compatible points D must also lie.

In order to also apply the differential equation for the minimum as a way of finding the acceleration to the simplest case of that kind here, namely, the one is which the prescribed path is rectilinear, let x, y be the coordinates of the point relative to two rectangular axes, where the y axis is taken to have the same sense as gravity, and let s be the path that traverses the skew plane at time t, so if α is the angle between the skew plane and the direction of gravity then $x = s \sin \alpha$, $y = s \cos \alpha$, and:

$$\overline{BC}^2 = \left[\frac{d^2x}{dt^2} \cdot \frac{\tau^2}{2} + \cdots\right]^2 + \left[\left(g - \frac{d^2x}{dt^2}\right) \cdot \frac{\tau^2}{2} - \cdots\right]^2$$

$$= \left[\left(\frac{d^2s}{dt^2}\right)^2 \sin^2\alpha + \left(g - \frac{d^2s}{dt^2}\cos^2\alpha\right)^2\right] \frac{\tau^4}{4} + \ldots,$$

from which, since the derivative with respect to $d^2 s / dt^2$ must vanish independently of τ, it will follow that:

$$\frac{d^2s}{dt^2} \sin^2\alpha + \left(g - \frac{d^2s}{dt^2}\cos\alpha\right)\cos\alpha = 0,$$

i.e.:

$$\frac{d^2s}{dt^2} = g \cos \alpha,$$

and since the higher derivatives of s with respect to t must vanish, and therefore those of x, y, as well, the second derivative will reduce to the positive quantity:

$$(\sin^2 \alpha + \cos^2 \alpha)\frac{\tau^4}{4}.$$

III. – Now, as far as equilibrium is concerned, for which the principle of least constraint gives a minimum to $\sum m \cdot \overline{AB}^2$, so one establishes that:

$$\sum m(\overline{BD}^2 - \overline{AB}^2) > 0$$

in the first of our examples (Tab. IV, Fig. 3), if we are to change nothing in the figure then we must consider E, E' to be the equilibrium locations, instead of A, A', resp.:

$$\overline{BE} = \overline{B'E'}, \quad \overline{ED} = \overline{E'D'},$$

$$\overline{BD} = \overline{BE} - \overline{ED}, \qquad \overline{B'D'} = \overline{BE} + \overline{ED},$$

and as a result:

$$\sum m \cdot (\overline{BD}^2 - \overline{BE}^2) = m\,[(\overline{BE} - \overline{ED})^2 - \overline{BE}^2] + m'[(\overline{BE} + \overline{ED})^2 - \overline{BE}^2]$$
$$= (m + m') \cdot \overline{ED}^2 - 2\,(m - m') \cdot \overline{ED} \cdot \overline{EB},$$

but in equilibrium one has $m = m'$, so as a result:

$$\sum m \cdot (\overline{BD}^2 - \overline{BE}^2) = 2m \cdot \overline{ED}^2 > 0.$$

A lever (Fig. IV, Tab. 5) will serve as a second example, for which the two masses m, m' will be in equilibrium when they are in the positions A, A', resp., with the lever arms a, a', resp., and the condition for that is $ma = m'a'$. Now, one has, in turn, $\overline{AB} = \overline{A'B'}$, and furthermore, if θ is the angle of rotation of the lever from the position AA' to the position DD' then:

$$AD = 2a \sin \tfrac{1}{2}\theta, \qquad A'D' = 2a' \sin \tfrac{1}{2}\theta,$$

so since the angles at A and A' in the triangles BAD, $B'A'D'$ are $\frac{1}{2}\theta$ and $180^{\circ} - \frac{1}{2}\theta$, resp.:

$$\overline{BD}^2 = \overline{AB}^2 + 4a^2 \sin^2 \tfrac{1}{2}\theta - 2a \cdot \overline{AB} \cdot \sin\theta,$$

$$\overline{B'D'}^2 = \overline{AB}^2 + 4a'^2 \sin^2 \tfrac{1}{2}\theta - 2a' \cdot \overline{AB} \cdot \sin\theta.$$

As a result:

$$\sum m \cdot (\overline{BD}^2 - \overline{BA}^2) = \sum m \cdot \overline{AD}^2 - 2\,(ma - m'a') \cdot \overline{AB} \cdot \sin\theta\,,$$

whose second term must vanish as a result of the equilibrium condition, but the first one will be:

$$\sum m \cdot \overline{AD}^2 = 4\,(ma^2 + m'a'^2)\,\sin^2 \tfrac{1}{2}\theta = 4ml\,\sin^2 \tfrac{1}{2}\theta$$

when one denotes the total length of the lever by l and the common rotational moment by μ, so:

$$\sum m \cdot (\overline{BD}^2 - \overline{BA}^2) = 4\mu\, l\, \sin^2 \tfrac{1}{2}\theta > 0.$$

In both examples, one has $AB = A'B' = \tfrac{1}{2} g\tau^2$, so the quantity to be minimized is:

$$\sum m \cdot \overline{AB}^2 = (m + m')\,\frac{g^2\tau^4}{4}\,,$$

in which one cannot overlook how and with respect to what it is to be differentiated if one is to confirm a minimum in that way under the assumption of an equilibrium condition or to derive the equilibrium condition when one assumes a minimum. It is only because that differentiation corresponds to the transition from BA to BD (during which, merely the point A changes, while B remains the same) that one must first introduce the coordinates of the point A in order to be able to complete the differentiation. Therefore, in the first example, let z_0, z_0' be the vertical ordinates of the equilibrium locations E, E', resp., and let z, z' be those of the points B, B', resp., so:

$$BE = z - z_0\,, \quad B'E' = z' - z_0'\,,$$

and as a result:

$$\sum m \cdot \overline{BE}^2 = m\,(z - z_0)^2 + m\,(z - z_0')^2,$$

and when one differentiates with respect to z_0, the first two derivatives will be:

$$-\,2m\,(z - z_0) - 2m\,(z - z_0')\,\frac{dz_0'}{dz_0}\,,$$

$$2m + 2m'\left(\frac{dz_0'}{dz_0}\right)^2 - 2m'(z' - z_0')\,\frac{d^2 z_0'}{dz_0^2}\,.$$

However, due to the condition on the system $z_0 + z_0' =$ const., it will follow that:

$$\frac{dz_0'}{dz_0} = -1, \qquad \frac{d^2 z_0'}{dz_0^2} = 0,$$

along with all the following derivatives. Therefore, when one now considers that $BE = B'E' = \frac{1}{2} g\tau^2$, the first one will reduce to $-(m - m')\, g\tau^2$ (i.e., to zero, when $m - m'$), and then the second will reduce to the positive quantity $4m$.

One refers the points A, A', B, B' in the second example to two rectangular axes (viz., the x and y axes) such that the origin lies at the fulcrum of the lever and the positive x axis makes an angle of α with the lever arm a in the equilibrium position, so it makes an angle of $180^\circ + \alpha$ with the other one a', from the condition on the system: Hence, if x_0, y_0 and x, y are the coordinates of m in the positions A, B, resp., and likewise x_0', y_0' and x', y' are those of m' in the positions A', B', resp., while AB, as well as $A'B'$ make the angles $90^\circ + \alpha$ and α with the positive x and y axes then:

$$\overline{AB} \cdot \sin\alpha = -(x - x_0), \qquad \overline{A'B'} \cdot \sin\alpha = -(x' - x_0'),$$

$$\overline{AB} \cdot \cos\alpha = y - y_0, \qquad \overline{A'B'} \cdot \cos\alpha = y' - y_0',$$

and

$$\sum m\,\overline{AB}^2 = m\,[(x - x_0)^2 + (y - y_0)^2] + m'\,[(x' - x_0')^2 + (y' - y_0')^2].$$

As a result, when one differentiates with respect to α, but only lets the coordinates that refer to the points A, A' vary, the first two derivatives will be:

$$-2m\left[(x - x_0)\frac{dx_0}{d\alpha} + (y - y_0)\frac{dy_0}{d\alpha}\right] - 2m'\left[(x' - x_0')\frac{dx_0'}{d\alpha} + (y' - y_0')\frac{dy_0'}{d\alpha}\right],$$

and

$$2m\left[\left(\frac{dx_0}{d\alpha}\right)^2 + \left(\frac{dy_0}{d\alpha}\right)^2\right] + 2m'\left[\left(\frac{dx_0'}{d\alpha}\right)^2 + \left(\frac{dy_0'}{d\alpha}\right)^2\right]$$

$$-2m\left[(x - x_0)\frac{d^2x_0}{d\alpha^2} + (y - y_0)\frac{d^2y_0}{d\alpha^2}\right] - 2m'\left[(x' - x_0')\frac{d^2x_0'}{d\alpha^2} + (y' - y_0')\frac{d^2y_0'}{d\alpha^2}\right].$$

Now, one has:

$$x_0 = a\cos\alpha, \qquad \frac{dx_0}{d\alpha} = -a\sin\alpha, \qquad \frac{d^2x_0}{d\alpha^2} = -a\cos\alpha,$$

$$y_0 = a\sin\alpha, \qquad \frac{dy_0}{d\alpha} = a\cos\alpha, \qquad \frac{d^2y_0}{d\alpha^2} = -a\sin\alpha,$$

$$x_0' = -a\cos\alpha, \qquad \frac{dx_0'}{d\alpha} = a'\sin\alpha, \qquad \frac{d^2x_0'}{d\alpha^2} = a'\cos\alpha,$$

$$y_0' = -a' \cos\alpha, \qquad \frac{dy_0'}{d\alpha} = -a'\cos\alpha, \qquad \frac{d^2y_0'}{d\alpha^2} = a'\sin\alpha,$$

which makes the foregoing expressions go to:

$$-2m\,a\,[(y-y_0)\cos\alpha-(x-x_0)\sin\alpha]+2m'a'\,[(y'-y_0^2)\cos\alpha-(x'-x_0')\sin\alpha]$$

and

$$2\,(ma^2+m'a'^2)+2ma\,[(y-y_0)\sin\alpha-(x-x_0)\cos\alpha]-2m'a'\,[(y'-y_0^2)\sin\alpha+(x'-x_0')\cos\alpha],$$

but:

$$(y-y_0)\cos\alpha-(x-x_0)\sin\alpha = AB,$$

$$(y'-y_0')\cos\alpha-(x'-x_0')\sin\alpha = A'B',$$

$$(y-y_0)\sin\alpha-(x-x_0)\cos\alpha = AB\sin 2\alpha,$$

$$(y'-y_0')\sin\alpha+(x'-x_0')\cos\alpha = A'B'\sin 2\alpha.$$

Hence, when one further recalls that after the differentiation is completed, one will have $AB = A'B' = \frac{1}{2}g\tau^2$, the first derivative will reduce to $-(ma-m'a')\,g\tau^2$, so to zero when $ma-m'a' = 0$, while the second one will reduce to:

$$2\,(ma^2+m'a'^2)+(ma-m'a')\,g\tau^2\sin 2\alpha,$$

so under the same condition, to the positive quantity $2\mu\,l$, when one, in turn, sets $ma = m'a' = \mu$, $a + a' = l$.

In this example of the lever, as well, the paths AB, AD, and the time interval τ can be assumed to be finite quantities, as in the foregoing cases of the pulley and the skew plane. Moreover, the difference between the cases is that whereas for the latter, in the application of the principle of virtual velocities, the projections onto the directions of the forces of the paths that the points of the system traverse under a displacement from its equilibrium position can be finite, that is not true for the lever, but it probably applies to the cases in which the virtual paths themselves can be taken instead of their projections, and they can then be finite quantities.

§ 3.

Since it results from the remarks in § **1**.III that in order to derive Gauss's principle from d'Alembert's in full generality, one must apply the equilibrium of lost forces that is established by the latter in the most general way (i.e., by means of the principle of virtual velocities), it shall now be shown how the general analytical expression for Gauss's theorem can be confirmed by the general formulas of dynamics that expresses the vanishing of the sum of the virtual moments of all lost forces. Moreover, that requires some prior discussion that we shall undertake in this paragraph.

I. – It is known that when one refers all points of the system to a rectangular coordinate system and all of the forces that act upon them to their projections onto the axes, d'Alembert's principle will be represented thus:

$$\sum m\left\{\left(X-\frac{d^2x}{dt^2}\right)\delta x+\left(Y-\frac{d^2y}{dt^2}\right)\delta y+\left(Z-\frac{d^2z}{dt^2}\right)\delta z\right\}=0,$$

which we will write as:

(1) $$\sum m\left[\left(X-\frac{d^2x}{dt^2}\right)\delta x\right]=0,$$

in which we abbreviate each of the terms in the expression that correspond to the three coordinates in succession by a single term that we enclose in square brackets. In that formula, x, y, z are the three coordinates of any one of the material points whose mass is m at the time t, so $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$ are its actual accelerations that correspond to the system constraint along the three axes, and X, Y, Z are the projections onto the axes of the accelerating forces that act upon them independently of the system constraints. Hence, $X-\frac{d^2x}{dt^2}$, $Y-\frac{d^2y}{dt^2}$, $Z-\frac{d^2z}{dt^2}$ are its lost forces, and finally, δx, δy, δz are the projections of its so-called virtual velocities onto the three axes, and thus onto the directions of the forces that are to be in equilibrium. We shall treat those quantities in the spirit of the calculus of functions in order to establish their meanings more precisely, since they are ordinarily considered to be infinitesimals when one appeals to the infinitesimal methods. To that end, if we return to the statement of the principle of virtual velocities, which requires the system to be subjected to a displacement from the equilibrium position that is generally infinitely small and compatible with its nature, and we project the paths that the individual points of the system traverse along the directions of the forces that should be in equilibrium at them, or project both of them onto a common direction, then the sum of the products of those two types of projection with the masses – i.e., the sum of the virtual moments – should vanish. Now, the projections onto the axes of the infinitely small paths or those of the infinitely small variations of the coordinates that correspond to the displacement are usually denoted by δx, etc., and considered to be the virtual velocities. However, we must distinguish between the virtual path and the virtual velocity, and for that reason, if Dx denotes an initially finite or infinitely small increment in x (since there also cases in which the virtual paths can be finite) then the next way of expressing the statement of the principle above can be represented by:

(2) $$\sum m\left[\left(X - \frac{d^2x}{dt^2}\right)Dx\right] = 0,$$

in which Dx is now a quantity of the same kind as Δx, except that Δx corresponds to the actual motion of the system during the time interval Δt when it is left to itself, while Dx corresponds to an arbitrary, but compatible, displacement, so to a virtual motion. However, one can convert:

$$\Delta x = \frac{dx}{dt}\Delta t + \frac{d^2x}{dt^2}\frac{\Delta t^2}{2} + \ldots$$

into Dx by replacing the time derivatives $\frac{dx}{dt}$, $\frac{d^2x}{dt^2}$, etc., that are determined by the actual motion with arbitrary functions of time, but with the restriction that they are constrained by the condition equations of the system, such that when one denotes those arbitrary quantities by δx, $\delta^2 x$, etc., as in the calculus of variations, one can set:

$$Dx = \delta x \cdot \Delta t + \delta^2 x \cdot \frac{\Delta t^2}{2} + \ldots,$$

in which Δt serves merely as an entirely arbitrary amount of advance that can be taken to be finite or infinitely small according to the situation, and thus determine the order of magnitude of Dx. That is entirely consistent with the spirit of the calculus of variations, where one goes from a given function $x = f\,t$ of t to the varied one by setting $x + Dx = \varphi\,(t, \varepsilon)$, if ε is a new variable that is absolutely independent and φ is an arbitrary function, except that one must have $\varphi\,(t, 0) = f\,t$, and when one develops $x + Dx$ in powers of ε, one will have:

$$x + Dx = \varphi_0 + \left(\frac{d\varphi}{d\varepsilon}\right)_0 \varepsilon + \left(\frac{d^2\varphi}{d\varepsilon^2}\right)_0 \frac{\varepsilon^2}{2} + \ldots = x + \varepsilon\,\delta x + \frac{\varepsilon^2}{2}\delta^2 x + \ldots$$

However, one can switch ε with Δt here, since ε is nothing but a completely independent quantity that can take on any degree of smallness, just like the time increment Δt, such that one will have $x + Dx = \varphi\,(t, \Delta t)$, while $x + \Delta x = f\,(t + \Delta t)$, and one can add that, since that varied function is always subject to the condition $x = \varphi\,(t, 0)$, in some situations, one can impose the further conditions that $\frac{dx}{dt} = \delta x$, $\frac{d^2x}{dt^2} = \delta^2 x$, etc., up to an arbitrary term at which Δx and Dx should coincide.

However, if one develops formula (2) in Δt using the values of Dx that were presented then one will get:

$$\sum m\left[\left(X - \frac{d^2x}{dt^2}\right)\delta x\right] + \tfrac{1}{2}\Delta t \sum m\left[\left(X - \frac{d^2x}{dt^2}\right)\delta^2 x\right] + \ldots = 0$$

after one drops the common factor Δt, and one must now distinguish between the cases in which the displacement of the system *must* be infinitely small and the ones in which it *can* be finite. In the former case, Δt is taken to be infinitely small, and the foregoing equation will reduce to its first term; i.e., to equation (1). In the latter case, where Δt must have only the degree of smallness that makes the sequence converge, because the equation must be true independently of that arbitrary quantity, it will decompose into an infinite set of equations, the first of which is (1), and which will have the general form:

$$\sum m\left[\left(X - \frac{d^2x}{dt^2}\right)\delta^n x\right] = 0, \tag{3}$$

in which n can be any positive whole number from unity onwards, and the agreement between all of those equations will be just the analytical indicator that the virtual paths can have a finite magnitude for a system in which that is true.

II. – The special equations of motion for the system into which formulas (1) decomposes can be represented by either introducing all of the condition equations of the system into formula (1) with undetermined factors and then treating all coordinates directly as if they were just as many independent variables or by assuming that the coordinates are reduced to the smallest number of independent variables by means of those condition equations, in which it is necessary in some situations to also appeal to the formulas of the coordinate transformation. In the first case, let $L = 0$, $L' = 0$, etc., be those condition equations – i.e., relations between the coordinates of the individual points of the system by which they are constrained during their entire motion – such that $dL / dt = 0$, etc., and likewise $\delta L = 0$, etc., and which can generally include time either merely directly, implicitly, or also explicitly, such that with the exception of the latter case, there are just as many equations $dL / dt = 0$, $dL = 0$, as there are:

$$\left[\frac{dL}{dx_1}\frac{dx_1}{dt}\right] + \left[\frac{dL}{dx_2}\frac{dx_2}{dt}\right] + \ldots = 0 \qquad \text{or} \qquad \sum\left[\frac{dL}{dx}\frac{dx}{dt}\right] = 0,$$

and

$$\left[\frac{dL}{dx_1}\delta x_1\right] + \left[\frac{dL}{dx_2}\delta x_2\right] + \ldots = 0 \qquad \text{or} \qquad \sum\left[\frac{dL}{dx}\delta x\right] = 0\,.$$

If $\lambda, \lambda', \lambda'', \ldots$ are the so-called elimination factors then the equation:

$$\sum m\left[\left(X - \frac{d^2x}{dt^2}\right)\delta x\right] + \lambda\,\delta L + \lambda'\,\delta L' + \ldots = 0, \tag{4}$$

which now enters in place of (1), will decompose into just as many equations as the number m of points that are present times three, and they will have the form:

$$
(5) \qquad \left\{
\begin{aligned}
m\left(X - \frac{d^2x}{dt^2}\right) + \lambda\frac{dL}{dx} + \lambda'\frac{dL'}{dx} + \cdots = 0,\\
m\left(Y - \frac{d^2y}{dt^2}\right) + \lambda\frac{dL}{dy} + \lambda'\frac{dL'}{dy} + \cdots = 0,\\
m\left(Z - \frac{d^2z}{dt^2}\right) + \lambda\frac{dL}{dz} + \lambda'\frac{dL'}{dz} + \cdots = 0,
\end{aligned}
\right.
$$

and the equations of motion of the system are the result of eliminating all quantities λ between those systems of that form.

In the second case, let ω, ψ, χ, etc., be the latter geometric independent variables, or just as many mutually independent functions of time, which one can set equal to the coordinates of the points of the system after one considers all of the conditions on it. Those coordinates are then, in turn, generally either purely geometric functions of those independent variables or coupled with them by relations in which time does not enter explicitly besides, such that one will have $x = f(\omega, \psi, \chi, \ldots)$, except in the latter case, and as a result:

$$
\frac{dx}{dt} = \frac{dx}{d\omega}\frac{d\omega}{dt} + \frac{dx}{d\psi}\frac{d\psi}{dt} + \ldots,
$$

$$
\frac{d^2x}{dt^2} = \frac{dx}{d\omega}\frac{d^2\omega}{dt^2} + \frac{d^2x}{d\omega^2}\left(\frac{d\omega}{dt}\right)^2 + 2\frac{d^2x}{d\omega\,d\psi}\frac{d\omega}{dt}\frac{d\psi}{dt} + \frac{dx}{d\psi}\frac{d^2\psi}{dt^2} + \frac{d^2x}{d\psi^2}\left(\frac{d\psi}{dt}\right)^2 + \ldots,
$$

$$
\begin{aligned}
\frac{d^3x}{dt^3} = {} & \frac{dx}{d\omega}\frac{d^3\omega}{dt^3} + \frac{d^3x}{d\omega^3}\left(\frac{d\omega}{dt}\right)^3 + 3\frac{d^2x}{d\omega^2}\frac{d\omega}{dt}\frac{d^2\omega}{dt^2}\\
& + 3\frac{d^2x}{d\omega\,d\psi}\left(\frac{d\psi}{dt}\frac{d^2\omega}{dt^2} + \frac{d\omega}{dt}\frac{d^2\psi}{dt^2}\right) + 3\frac{d^3x}{d\omega^2\,d\psi}\left(\frac{d\omega}{dt}\right)^2\frac{d\psi}{dt} + 3\frac{d^3x}{d\omega\,d\psi^2}\frac{d\omega}{dt}\left(\frac{d\psi}{dt}\right)^2\\
& + \frac{dx}{d\psi}\frac{d^3\psi}{dt^3} + \frac{d^3x}{d\psi^3}\left(\frac{d\psi}{dt}\right)^3 + 3\frac{d^2x}{d\psi^2}\frac{d\psi}{dt}\frac{d^2\psi}{dt^2} + \ldots,
\end{aligned}
$$

etc., and likewise:

$$
\delta x = \frac{dx}{d\omega}\delta\omega + \frac{dx}{d\psi}\delta\psi + \ldots,
$$

$$\delta^2 x = \frac{dx}{d\omega}\delta^2\omega + \frac{d^2x}{d\omega^2}\delta\omega^2 + 2\frac{d^2x}{d\omega\,d\psi}\delta\omega\,\delta\psi + \frac{dx}{d\psi}\delta^2\psi + \frac{d^2x}{d\psi^2}\delta\psi^2 + \ldots,$$

etc., and furthermore:

$$\Delta x = \frac{dx}{d\omega}\Delta\omega + \frac{dx}{d\psi}\Delta\psi + \cdots + \frac{d^2x}{d\omega^2}\frac{\Delta\omega^2}{2} + \frac{d^2x}{d\omega\,d\psi}\Delta\omega\,\Delta\psi + \frac{d^2x}{d\psi^2}\frac{\Delta\psi^2}{2} + \ldots,$$

$$Dx = \frac{dx}{d\omega}D\omega + \frac{dx}{d\psi}D\psi + \cdots + \frac{d^2x}{d\omega^2}\frac{D\omega^2}{2} + \frac{d^2x}{d\omega\,d\psi}D\omega\,D\psi + \frac{d^2x}{d\psi^2}\frac{D\psi^2}{2} + \ldots,$$

$$\Delta\omega = + \ldots, \qquad D\omega = \delta\omega\,\Delta t + \ldots, \text{ etc.}$$

Now, just as sometimes the coordinates of all points can enter into the condition equations $L = 0$, but sometimes just some of them, and sometimes just one of them, so can all of the variables, some of them, or all of the independent ones enter into the relations $x = f(\omega, \psi, \ldots)$. Furthermore, if one such independent variable ω is to befit several points of the system in common then that would mean that for each of those points, from any arbitrary time point t onwards, its change $\Delta\omega$ or $D\omega$ would have the same value, while the value that corresponds to the time point t would have the form $\alpha + \omega$, where α is a constant that has different values that are given by the system constraint for different points of the system that depend upon ω. However, in part in order to refer to the variable ω beforehand without distinguishing between all points of the system, and in part in order to also be able to deal with the special case in which if, e.g., ω is an angle then depending upon how many of them there are, $\Delta\omega$ would prove to be positive for the one point and negative for the others, one can establish more generally that for any point of the system, the value of ω at time t will have the form $\alpha + a\omega$, where a is a constant that is similar to α, namely, both of them can be zero, such that the relation between one coordinate x and the independent variables can generally represented by:

$$x = f(\alpha + a\omega, \beta + b\psi, \ldots),$$

instead of by $x = f(\omega, \psi, \ldots)$. Now, since the variations $\delta\omega$, $\delta^2\omega$, … of a geometric independent variable ω are considered to be completely arbitrary, mutually independent functions of time, when one introduces the value of δx into equation (1), it will decompose into just as many equations:

$$(6) \qquad \sum m\left[\left(X - \frac{d^2x}{dt^2}\right)\frac{dx}{d\omega}\right] = 0, \quad \sum m\left[\left(X - \frac{d^2x}{dt^2}\right)\frac{dx}{d\psi}\right] = 0, \quad \text{etc.,}$$

as there are independent variables ω, ψ, … They will be equations that all have order two with respect to the time derivatives of the independent variables, by means of the value of $\frac{d^2x}{dt^2}$, and will therefore suffice to determine the motion of the system.

III. – Of the two forms [viz., (5) and (6)] that one can give to the equations of motion for a system, it is the latter that is suitable for our purposes. Just as it can emerge from (1), by introducing independent variables, formula (3) will, in fact, decompose into equations whose general form is:

(7) $$\sum m\left[\left(X-\frac{d^2x}{dt^2}\right)\frac{d^nx}{d\omega^{n'}d\omega^{n''}\cdots}\right]=0,$$

in which n', n'', … are likewise positive whole numbers, but all of their values can be zero, and they are coupled by the condition $n' + n'' + \ldots = n$. However, since equations (6) already determine the motion completely, all of the equations in formula (7), except for that one, will include equations that are identical to it, which immediately demands that all higher derivatives of the coordinates with respect to ω, ψ, … will vanish or reproduce the first one; i.e., in general, that the coordinates will be either linear functions of the independent variables:

$$\Delta = A + a\omega + b\psi + c\chi + \ldots$$

or exponential functions of the form:

$$\Delta = K\, k^{\alpha\omega + \beta\psi + \gamma\chi + \ldots}$$

with the arbitrary base k, where K, A, a, b, c, … are constants relative to time that can vary from one point of the system to another, while α, β, γ, … are constants that are independent of the system constraints, like k. Hence, they would be the analytical conditions for the virtual paths to be finite. However, one will get even more coexisting equations in that case by successive differentiation of (7) with respect to t. The first one gives:

$$\sum m\left\{\left(X-\frac{d^2x}{dt^2}\right)\left(\frac{d^{n+1}x}{d\omega^{n'+1}d\psi^{n''}\cdots}\frac{d\omega}{dt}+\frac{d^{n+1}x}{d\omega^{n'}d\psi^{n''+1}\cdots}\frac{d\psi}{dt}+\cdots\right)+\frac{d}{dt}\left(X-\frac{d^2x}{dt^2}\right)\cdot\frac{d^nx}{d\omega^{n'}d\psi^{n''}\cdots}\right\}=0,$$

from which, since the first component of that expression will vanish due to (7), when the expressions $d\omega / dt$, $d\psi / dt$, … enter before the summation sign, one will have:

(8) $$\sum m\left[\frac{d}{dt}\left(X-\frac{d^2x}{dt^2}\right)\cdot\frac{d^nx}{d\omega^{n'}d\psi^{n''}\cdots}\right]=0\,.$$

A second one (i.e., differentiating the latter expression) will likewise lead to:

$$\sum m\left[\frac{d^2}{dt^2}\left(X-\frac{d^2x}{dt^2}\right)\cdot\frac{d^nx}{d\omega^{n'}d\psi^{n''}\cdots}\right]=0,$$

and when one proceeds in that way, one will arrive at the general formula:

(9) $$\sum m\left[\frac{d^i}{dt^i}\left(X-\frac{d^2x}{dt^2}\right)\cdot\frac{d^nx}{d\omega^{n'}d\psi^{n''}\cdots}\right]=0,$$

in which the foregoing, along with (7), are included for the values $i = 0, 1, 2$, resp., and in which i can take on all positive whole numbers from zero onwards. If the virtual path is not to be finite, so (7) will not be true, then the total time derivatives of equations (6) will indeed be valid, but they will not decompose into equations of the forms (8) and (9).

We finally infer the following result from this:

1) If we add equations (6) after multiplying by $d\omega / dt$, $d\psi / dt$, …, resp., then since we have restricted ourselves to the case in which x depends upon t only by way of the independent variables ω, ψ, …, we will get an equation:

(10) $$\sum m\left[\left(X-\frac{d^2x}{dt^2}\right)\frac{dx}{dt}\right]=0$$

that is entirely analogous to (6) and (1) in form, and whose integral is known to include the *vis viva* principle. One will likewise get it when one switches the variations or virtual velocities δx in (1) with the time derivatives – or actual velocities dx / dt. We can also say that we are switching the virtual path Dx in (2) with the actual path Δx, which is an admissible switch, as well as switching the relations between the coordinates and the independent variables or the condition equations of the system that do not include time explicitly, as **Lagrange** expressed it in analytical mechanics. Now, in the cases where Dx can be finite, the same thing must also be true for Δx, and one will then have as a consequence of:

(11) $$\sum m\left[\left(X-\frac{d^2x}{dt^2}\right)\Delta x\right]=0,$$

along (10), a series of equations with the general form:

(12) $$\sum m\left[\left(X-\frac{d^2x}{dt^2}\right)\frac{d^nx}{dt^n}\right]=0,$$

which is, in fact, justified by (7) when one introduces the expression for d^nx / dt^n in terms of the independent variables.

2) Since the quantities X and x include the increments ΔX, Δx during the time interval, from formula (1), one will have:

$$\sum m\left[\left(X+\Delta X-\frac{d^2(x+\Delta x)}{dt^2}\right)\delta(x+\Delta x)\right]=0 \tag{13}$$

[in which one can also write D for δ, which is analogous to formula (3)]. That is, one will have the equilibrium formula for the lost force at time $t + \Delta t$, and since:

$$\Delta X = \frac{dX}{dt}\Delta t+\frac{d^2X}{dt^2}\frac{\Delta t^2}{2}+\ldots,$$

$$\Delta x = \frac{dx}{dt}\Delta t+\frac{d^2x}{dt^2}\frac{\Delta t^2}{2}+\ldots,$$

and as a result:

$$\frac{d^2\Delta x}{dt^2} = \Delta\frac{d^2x}{dt^2} = \frac{d^3x}{dt^3}\Delta t+\frac{d^4x}{dt^4}\frac{\Delta t^2}{2}+\ldots,$$

$$\delta\,\Delta x = \Delta\delta x = \frac{d\delta x}{dt}\Delta t+\frac{d^2\delta x}{dt^2}\frac{\Delta t^2}{2}+\ldots,$$

$$X+\Delta X-\frac{d^2(x+\Delta x)}{dt^2} = X-\frac{d^3x}{dt^3}+\frac{dx}{dt}\left(X-\frac{d^2x}{dt^2}\right)\Delta t+\ldots,$$

the first part of (13) will define a development in Δt whose individual terms must vanish, due to the arbitrariness in the magnitude of the time interval, from which one will get a consequence of the equations:

$$\sum m\left[\left(X-\frac{d^2x}{dt^2}\right)\frac{d\delta x}{dt}+\frac{d}{dt}\left(X-\frac{d^2x}{dt^2}\right)\cdot\delta x\right]=0,$$

$$\sum m\left[\left(X-\frac{d^2x}{dt^2}\right)\frac{d^2\delta x}{dt^2}+2\frac{d}{dt}\left(X-\frac{d^2x}{dt^2}\right)\frac{d\delta x}{dt}+\frac{d^2}{dt^2}\left(X-\frac{d^2x}{dt^2}\right)\cdot\delta x\right]=0,$$

etc.,

and naturally these are nothing but the total derivatives of equations (1) with respect to time t. However, in the case where equation (7) is true, and as a result (9), they will, in turn, decompose into equations of the general form:

(14) $$\sum m\left[\frac{d^i}{dt^i}\left(X-\frac{d^2x}{dt^2}\right)\cdot\frac{d^{i'}\delta x}{dt^{i'}}\right]=0,$$

in which i and i' are positive whole numbers that can take on all values from zero onwards, such that equation (1) is contained in the latter equation for $i + i' = 0$, and for $i + i' = 1$, it will contain these two:

$$\sum m\left[\left(X-\frac{d^2x}{dt^2}\right)\cdot\frac{d\delta x}{dt}\right]=0,\qquad \sum m\left[\frac{d}{dt}\left(X-\frac{d^2x}{dt^2}\right)\cdot\delta x\right]=0,$$

which one can, in fact, justify immediately by means of (9) when one introduces the values of δx and $d\delta x / dt$; i.e.:

$$\frac{d\delta x}{dt}=\frac{dx}{d\omega}\frac{d\delta\omega}{dt}+\left(\frac{d^2x}{d\omega^2}\frac{d\omega}{dt}+\frac{d^2x}{d\omega\, d\psi}\frac{d\psi}{dt}+\cdots\right)\delta\omega+\ldots$$

Finally, since the equations that are included in (4) are also true then, it will likewise follow, just as (14) follows from (1), that:

(15) $$\sum m\left[\frac{d^i}{dt^i}\left(X-\frac{d^2x}{dt^2}\right)\cdot\frac{d^{i'}\delta^n x}{dt^{i'}}\right]=0,$$

which is justified in the same way that (14) is justified by (9).

§ 4.

It can now be shown that the principle of least constraint leads to the same equations as the formula for the equilibrium of the constraint forces, and indeed in such a way that in the cases where the principle of virtual velocities is true for finite displacements of the system, our principle will also be valid for a finite time interval, while in general (i.e., when the virtual path must be taken to be infinitely small), the time interval for which the free motion of the point of the system will be compared with the actual can also be taken to be infinitely small here.

I. – Let A (to preserve the terminology of § **1**) be the position of any material point m of the system whose rectangular coordinates at time t are x, y, z, and at the time $t + \Delta t$, it will go to the two corresponding positions C and B whose coordinates are $x + \Delta x$, $y + \Delta y$, $z + \Delta z$, and $x + \xi$, $y + \eta$, $z + \zeta$, resp., in which Δx, Δy, Δz, and likewise ξ, η, ζ, are functions of Δt then. If one sets:

$$BC = u \qquad \text{and} \qquad \sum mu^2 = U$$

then one will have:

$$u^2 = (\xi - \Delta x)^2 + (\eta - \Delta y)^2 + (\zeta - \Delta z)^2,$$

and the quantity U that is to be a minimum will be:

(*a*)
$$U = \sum m\left[(\xi - \Delta x)^2\right] .$$

Now, in order to develop U in Δt, and therefore as a function of Δt , one will set:

$$\xi = \xi_0 + \left(\frac{d\xi}{d\Delta t}\right)_0 \Delta t + \left(\frac{d^2\xi}{d\Delta t^2}\right)_0 \frac{\Delta t^2}{2} + \left(\frac{d^3\xi}{d\Delta t^3}\right)_0 \frac{\Delta t^3}{2\cdot 3} + \ldots,$$

but for $\Delta t = 0$, for time t, one will have:

$$\xi_0 = 0, \qquad \left(\frac{d\xi}{d\Delta t}\right)_0 = \frac{dx}{dt}, \qquad \left(\frac{d^2\xi}{d\Delta t^2}\right)_0 = X,$$

in which (as in § **3**) X, Y, Z are the components along the axes of the accelerating forces that act upon the material point at time t independently of the system constraints. Furthermore, one has:

$$\frac{d^2\xi}{d\Delta t^2} = X + \Delta X = X + \frac{dX}{dt}\Delta t + \frac{d^2X}{dt^2}\frac{\Delta t^2}{2} + \ldots$$

at the end of the time interval Δt, but also, with the foregoing expression for ξ:

$$\frac{d^2\xi}{d\Delta t^2} = \left(\frac{d^2\xi}{d\Delta t^2}\right)_0 + \left(\frac{d^3\xi}{d\Delta t^3}\right)_0 \Delta t + \left(\frac{d^4\xi}{d\Delta t^4}\right)_0 \frac{\Delta t^2}{2} + \ldots,$$

and as a result, upon comparing these two identical developments:

$$\left(\frac{d^3\xi}{d\Delta t^3}\right)_0 = \frac{dX}{dt}, \qquad \left(\frac{d^4\xi}{d\Delta t^4}\right)_0 = \frac{d^2X}{dt^2}, \qquad \text{etc.}$$

One will then have:

$$\xi = \frac{dx}{dt}\Delta t + X\frac{\Delta t^2}{2} + \frac{dX}{dt}\frac{\Delta t^3}{2\cdot 3} + \ldots$$

and

$$\Delta x = \frac{dx}{dt}\Delta t + \frac{d^2x}{dt^2}\frac{\Delta t^2}{2} + \frac{d^3x}{dt^3}\frac{\Delta t^3}{2\cdot 3} + \ldots,$$

and as a result:

$$\xi - \Delta x = \left(X - \frac{d^2x}{dt^2}\right)\frac{\Delta t^2}{2} + \frac{d}{dt}\left(X - \frac{d^2x}{dt^2}\right)\cdot\frac{\Delta t^3}{2\cdot 3} + \ldots,$$

so:

$$(b) \qquad U = \frac{\Delta t^4}{4}\sum m\left[\left(X - \frac{d^2x}{dt^2} + \frac{\Delta t}{3}\frac{d}{dt}\left(X - \frac{d^2x}{dt^2}\right) + \cdots\right)^2\right].$$

From the developments in § **3**, III, one also has:

$$\frac{d^2(\xi - \Delta x)}{d\Delta t^2} = X + \Delta X - \frac{d^2(x + \Delta x)}{dt^2},$$

so the formula (13) there can be represented in the form:

$$(16) \qquad \sum m\left[\frac{d^2(\xi - \Delta x)}{d\Delta t^2} \cdot D(x + \Delta x)\right] = 0$$

when one likewise replaces the virtual velocities with the virtual paths.

II. – Now, in order to show directly by analogy with the Gaussian proof that U is smaller than when one starts from A and takes a compatible point D (i.e., another point of the trajectory) at time $t + \Delta t$, instead of the actual location C, let $x + Dx$, $y + Dy$, $z + Dz$ be the coordinates of m at the position D, where Dx, Dy, Dz are arbitrary changes in the coordinates that are compatible with the state of the system at time t that will take one from A to D (hence, they are quantities of the type in § **3**) along the virtual path that is denoted in this way, so:

$$Dx = \delta x \cdot \Delta t + \delta^2 x \cdot \frac{\Delta t^2}{2} + \ldots,$$

but with the difference that, from the discussion in regard to the Gaussian proof in § **1**, the velocities dx / dt that are reached at the time t have ξ introduced into Dx, as in Δx ; i.e., one takes $\delta x = dx / dt$, and at the same time $\delta\omega = d\omega / dt$, etc. However, one will then have:

$$Dx - \Delta x = \left(\delta^2 x - \frac{d^2x}{dt^2}\right)\frac{\Delta t^2}{2} + \left(\delta^3 x - \frac{d^3x}{dt^3}\right)\frac{\Delta t^3}{2 \cdot 3} + \ldots,$$

and from the expressions for $\frac{d^2x}{dt^2}$, $\delta^2 x$, etc., in terms of the independent variables when one sets $\delta\omega = \frac{d\omega}{dt}$, $\delta\psi = \frac{d\psi}{dt}$, etc., in them:

$$\delta^2 x - \frac{d^2 x}{dt^2} = \frac{dx}{d\omega}\left(\delta^2\omega - \frac{d^2\omega}{dt^2}\right) + \frac{dx}{d\psi}\left(\delta^2\psi - \frac{d^2\psi}{dt^2}\right) + \ldots,$$

$$\delta^3 x - \frac{d^3 x}{dt^3} = \frac{dx}{d\omega}\left(\delta^3\omega - \frac{d^3\omega}{dt^3}\right) + \frac{dx}{d\psi}\left(\delta^3\psi - \frac{d^3\psi}{dt^3}\right) + \ldots$$
$$+\ 3\left(\frac{d^2 x}{d\omega^2}\frac{d\omega}{dt} + \frac{d^2 x}{d\omega\, d\psi}\frac{d\psi}{dt} + \cdots\right)\left(\delta^2\omega - \frac{d^2\omega}{dt^2}\right) + \ldots,$$

etc.

Now let:

$$BD = u', \qquad \sum m u'^2 = U',$$

so one can show that:

$$U' - U = \sum m (u'^2 - u^2) > 0.$$

However:

$$\begin{aligned} u'^2 &= (\xi - Dx)^2 + (\eta - Dy)^2 + (\zeta - Dz)^2 \\ &= [(\xi - \Delta x - (Dx - \Delta x))^2] \\ &= u^2 + [(Dx - \Delta x)^2] - 2\,[(\xi - \Delta x)\,(Dx - \Delta x)], \end{aligned}$$

and as a result:

(c) $$U' - U = \sum m[(Dx - \Delta x)^2] - 2\sum m[(\xi - \Delta x)(Dx - \Delta x)],$$

and one has $U' - U > 0$, so U will be a minimum when the second term in that expression vanishes, or when:

(17) $$\sum m[(\xi - \Delta x)(Dx - \Delta x)] = 0,$$

since $U' - U$ then comes down to the essentially positive expression:

$$\sum m[(Dx - \Delta x)^2] \qquad \text{or} \qquad \sum m \cdot \overline{CD}^2.$$

However, if one develops the first part of the equation (17) to be proved in terms of the expressions for $\xi - \Delta x$, $Dx - \Delta x$ in terms of Δt, and one puts all variations and time derivatives in front of the summation sign then one will find that each term is affected with a summation that has the form of the ones that define one side of equation (9). Thus, when the system is such that the virtual paths can have finite magnitudes, all of the individual terms in equation (17) will vanish due to equation (9), which will then be true, and those terms might also be Δt as long as only the first part of the series that it defines converges. That is, the principle of least constraint if true for finite time intervals Δt that can have any arbitrary magnitude, moreover, in the case where the coordinates are linear functions of the independent variables, because in the derivatives $\frac{d^n x}{d\omega^{n'}\,d\psi^{n''}\cdots}$, all of the terms except for the first one will vanish from n = 2 onward, so Δt can no longer fulfill any convergence conditions. However, if the system admits only infinitely small virtual paths, which is generally the case, then equation (9) will be true for only i = 0, n = 1, or will reduce to equations (6) due to the fact that only the first term in the development of equation (17) in Δt will vanish, so Δt must be taken to be infinitely small in order for that development to come down to the first term, or for (c) to come down to:

(d) $U' - U =$

$$= \frac{\Delta t^4}{4}\left\{\sum m\left[\left(\frac{dx}{d\omega}\left(\delta^2\omega - \frac{d^2\omega}{dt^2}\right) + \frac{dx}{d\psi}\left(\delta^2\psi - \frac{d^2\psi}{dt^2}\right) + \cdots\right)^2\right] - 2\left(\left(\delta^2\omega - \frac{d^2\omega}{dt^2}\right)\sum m\left(X - \frac{d^2\omega}{dt^2}\right)\frac{dx}{d\omega} + \cdots\right)\right\},$$

which likewise shows the homogeneity of that expression.

The development of $Dx - \Delta x$ above can also be done in such a way that one considers that quantity, and correspondingly $D\omega - \Delta\omega = h$, $D\psi - \Delta\psi = i$, etc., to be increments of Δx and $\Delta\omega$, $\Delta\psi$, … when one goes from the point C to the point D; one will then have:

$$Dx - \Delta x = \frac{d\Delta x}{d\Delta\omega}h + \frac{d\Delta x}{d\Delta\psi}i + \cdots + \frac{d^2 x}{d\Delta\omega^2}\frac{h^2}{2} + \frac{d^2\Delta x}{d\Delta\omega\, d\Delta\psi}hi + \frac{d^2 x}{d\Delta\psi^2}\frac{i^2}{2} + \cdots,$$

where:

$$h = D\omega - \Delta\omega = \left(\delta^2\omega - \frac{d^2\omega}{dt^2}\right)\frac{\Delta t^2}{2} + \ldots,$$

$$i = D\psi - \Delta\psi = \left(\delta^2\psi - \frac{d^2\psi}{dt^2}\right)\frac{\Delta t^2}{2} + \ldots,$$

etc.,

and with the expressions for Δx in terms of $\Delta\omega$, $\Delta\psi$, … that were given in § **3**, one will have:

$$\frac{d\Delta x}{d\Delta\omega} = \frac{dx}{d\omega} + \frac{d^2 x}{d\omega^2}\Delta\omega + \frac{d^2 x}{d\omega\, d\psi}\Delta\psi + \cdots = \frac{dx}{d\omega} + \left(\frac{d^2 x}{d\omega^2}\frac{d\omega}{dt} + \frac{d^2 x}{d\omega\, d\psi}\frac{d\psi}{dt} + \cdots\right)\Delta t + \cdots,$$

$$\frac{d^2\Delta x}{d\Delta\omega^2} = \frac{d^2 x}{d\omega^2} + \left(\frac{d^3 x}{d\omega^3}\frac{d\omega}{dt} + \frac{d^3 x}{d\omega^2\, d\psi}\frac{d\psi}{dt} + \cdots\right)\Delta t + \cdots,$$

$$\frac{d^2\Delta x}{d\Delta\omega\, d\Delta\psi} = \frac{d^2 x}{d\omega\, d\psi} + \left(\frac{d^3 x}{d\omega^2\, d\psi}\frac{d\omega}{dt} + \frac{d^3 x}{d\omega\, d\psi^2}\frac{d\psi}{dt} + \cdots\right)\Delta t + \cdots,$$

which are formulas that will used later. – Finally, one can suggest the transition from the point C to the point D by increments $D\,(x + \Delta x)$ in the coordinates $x + \Delta x$, such that the increment from D will have the form $x + \Delta x + D\,(x + \Delta x)$, where one next has:

$$D\,(x + \Delta x) = Dx + \Delta Dx = Dx + \frac{dDx}{dt} + 3\frac{d^2 Dx}{dt^2}\frac{\Delta t^2}{2} + \ldots,$$

as in formula (16), and:

$$Dx = \Delta t\;\delta x + \frac{\Delta t^2}{2}\delta^2 x + \ldots,$$

so

$$D\,(x + \Delta x) = \delta x\;\Delta t + \left(\delta^2 x + 2\frac{d\delta x}{dt}\right)\frac{\Delta t^2}{2} + \left(\delta^3 x + 3\frac{d\delta^2 x}{dt} + 3\frac{d^2\delta x}{dt^2}\right)\frac{\Delta t^3}{2\cdot 3} + \cdots,$$

such that this quantity will have the same scale as the square of Δt, and since one must also likewise take $\delta\omega = 0$, $\delta\psi = 0$, etc., one will have:

$$\delta^2 x = \frac{dx}{d\omega}\delta^2\omega + \frac{dx}{d\psi}\delta^2\psi + \cdots,$$

$$\delta^3 x = \frac{dx}{d\omega}\delta^3\omega + \frac{d^2 x}{d\psi^2}\delta^2\psi^2 + 2\frac{d^2 x}{d\omega\, d\psi}\delta^2\omega\,\delta^2\psi + \cdots,$$

etc.

If one then develops the equation that now enters in place of (17):

$$\sum m\,[(\xi - \Delta x)\,D\,(x + \Delta x)] = 0 \tag{18}$$

then it will be easy to see that, like the development of (15), it yields nothing but terms that will vanish due to equation (9) when it is true, whereas for an infinitely small Δt, instead of (d), one will now have:

$$(d')\qquad U' - U = \frac{\Delta t^4}{4}\left\{\sum m\left[\left(\frac{dx}{d\omega}\delta^2\omega + \frac{dx}{d\psi}\delta^2\psi + \cdots\right)^2\right] - 2\left(\delta^2\omega\sum m\left[\left(X - \frac{d^2 x}{dt^2}\right)\frac{dx}{d\omega}\right] + \cdots\right)\right\},$$

which is a homogeneous formula whose second part will vanish due to equations (6), and therefore, due to (1) (*).

III. – Now, in order to also treat the minimum directly by differential calculus, we can start with the form (*b*) for the *U* that is developed in Δt (as in § **2**, where we were led to that form by the nature of things) and differentiate with respect to the actual accelerations, and thus set equal to zero its first derivatives with respect to each of the independent accelerations (i.e., with respect to the second time derivatives of the independent variables), and thus obtain just as many equations as there are independent variables. When one reduces the development in Δt to its first term, which is affected with Δt^4, as in generally required, and sets:

$$\frac{d^2\omega}{dt^2} = \omega'', \qquad \frac{d^2\psi}{dt^2} = \psi'', \qquad \text{etc.,}$$

those equations will be:

$$\text{(19)} \qquad \sum m\left[\left(X - \frac{d^2x}{dt^2}\right)\frac{d}{d\omega''}\frac{d^2x}{dt^2}\right] = 0, \qquad \sum m\left[\left(X - \frac{d^2x}{dt^2}\right)\frac{d}{d\psi''}\frac{d^2x}{dt^2}\right] = 0, \qquad \text{etc.,}$$

i.e., equations (6), since due to the expression for $\frac{d^2x}{dt^2}$ in terms of the independent variables, one has:

$$\frac{d}{d\omega''}\frac{d^2x}{dt^2} = \frac{dx}{d\omega}, \qquad \frac{d}{d\psi''}\frac{d^2x}{dt^2} = \frac{dx}{d\psi}, \qquad \text{etc.}$$

However, we shall not pursue that aspect of the matter any further, since when ω', ω''', etc., represent the first, third, etc., time derivatives of ω in it, we will have:

(*) Naturally, the variations that correspond to the transition from *C* to *D* do not have the same meaning here that they had in formula (*d*), which referred to the transition from *A* to *D*. If one distinguishes between the two concepts by putting a prime on δ for now, then:

$$\delta'^2\omega = \delta^2\omega - \frac{d^2\omega}{dt^2}, \qquad \text{etc.,}$$

and correspondingly:

$$\delta'^2 x = \delta^2 x - \frac{d^2x}{dt^2}.$$

Above all, when one compares formulas (17) and (18), one must have:

$$D'(x + \Delta x) = Dx - \Delta x$$

identically, and a term-wise comparison of the developments of those two expressions in Δt will give the foregoing one, as well as all of the remaining relations between the variations δ and δ'.

$$\frac{d}{d\omega'}\frac{dx}{dt} = \frac{d}{d\omega''}\frac{d^2x}{dt^2} = \frac{d}{d\omega'''}\frac{d^3x}{dt^3} = \ldots = \frac{dx}{d\omega},$$

i.e., it will be equal to the first term in $\frac{d\Delta x}{d\Delta\omega}$, which points to a more general way of carrying out the differentiations that was already suggested in the foregoing, namely, with increments $\Delta\omega$, $\Delta\psi$, … in the independent variables that correspond to time increments Δt, since those differentiations generally correspond to the transition from the point C to D that was spoken of in no. II, in which we must revert to the first, undeveloped form (a) for the function U.

Since the point B, and therefore the quantities ξ, remain unchanged under that transition, the partial derivatives of first and second order of the function U with respect to $\Delta\omega$, $\Delta\psi$, … will be:

$$\frac{dU}{d\Delta\omega} = -2\sum m\left[(\xi-\Delta x)\frac{d\Delta x}{d\Delta\omega}\right], \qquad \frac{dU}{d\Delta\psi} = -2\sum m\left[(\xi-\Delta x)\frac{d\Delta x}{d\Delta\psi}\right],$$

etc.,

$$\frac{d^2U}{d\Delta\omega^2} = 2\sum m\left[\left(\frac{d\Delta x}{d\Delta\omega}\right)^2\right] - 2\sum m\left[(\xi-\Delta x)\frac{d^2\Delta x}{d\Delta\omega^2}\right],$$

$$\frac{d^2U}{d\Delta\psi^2} = 2\sum m\left[\left(\frac{d\Delta x}{d\Delta\psi}\right)^2\right] - 2\sum m\left[(\xi-\Delta x)\frac{d^2\Delta x}{d\Delta\psi^2}\right],$$

$$\frac{d^2U}{d\Delta\omega\, d\Delta\psi} = 2\sum m\left[\frac{d\Delta x}{d\Delta\omega}\frac{d\Delta x}{d\Delta\psi}\right] - 2\sum m\left[(\xi-\Delta x)\frac{d^2\Delta x}{d\Delta\omega\, d\Delta\psi}\right],$$

etc.,

and there are then three types of conditions for the minimum, namely:

$$(\alpha) \qquad \frac{dU}{d\Delta\omega} = 0, \qquad \frac{dU}{d\Delta\psi} = 0, \qquad \text{etc.},$$

$$(\beta) \qquad \frac{d^2U}{d\Delta\omega^2} > 0, \qquad \frac{d^2U}{d\Delta\psi^2} > 0, \qquad \text{etc.},$$

$$(\gamma) \qquad \frac{d^2U}{d\Delta\omega^2}\cdot\frac{d^2U}{d\Delta\psi^2} < \left(\frac{d^2U}{d\Delta\omega\, d\Delta\psi}\right)^2, \qquad \text{etc.}$$

They will be fulfilled when:

$$(20) \qquad 2\sum m\left[(\xi-\Delta x)\frac{d\Delta x}{d\Delta\omega}\right]=0, \quad 2\sum m\left[(\xi-\Delta x)\frac{d\Delta x}{d\Delta\psi}\right]=0, \qquad \text{etc.,}$$

$$(21) \qquad \sum m\left[(\xi-\Delta x)\frac{d^2\Delta x}{d\Delta\omega^2}\right]=0, \quad \sum m\left[(\xi-\Delta x)\frac{d^2\Delta x}{d\Delta\omega\, d\Delta\psi}\right]=0, \qquad \text{etc.,}$$

since the second derivatives of U will then reduce to their first terms, which are all essentially positive terms of the form $\frac{d^2U}{d\Delta\omega^2}$, as the conditions ($b$) would demand, and the presence of the conditions of the form (γ) is explained immediately by developing its components:

$$\frac{d^2U}{d\Delta\omega^2}\cdot\frac{d^2U}{d\Delta\psi^2}=4\sum m\left[\left(\frac{d\Delta x}{d\Delta\omega}\frac{d\Delta x}{d\Delta\psi}\right)^2\right]+4\sum m\left[\left(\frac{d\Delta x}{d\Delta\omega}\frac{d\Delta y}{d\Delta\psi}\right)^2+\left(\frac{d\Delta x}{d\Delta\psi}\frac{d\Delta y}{d\Delta\omega}\right)^2\right],$$

$$\left(\frac{d^2U}{d\Delta\omega\, d\Delta\psi}\right)^2=4\sum m\left[\left(\frac{d\Delta x}{d\Delta\omega}\frac{d\Delta x}{d\Delta\psi}\right)^2\right]+8\sum m\left[\frac{d\Delta x}{d\Delta\omega}\frac{d\Delta y}{d\Delta\psi}\frac{d\Delta x}{d\Delta\psi}\frac{d\Delta y}{d\Delta\omega}\right],$$

with which things will revert to the theorem that $a^2 + b^2 > 2ab$. However, if one develops the first part of equations (20) in Δt, while (21) are still to be proved, by means of the developments of the derivatives $\frac{d\Delta x}{d\Delta\omega}$, etc., and the quantities $\xi-\Delta x$, above then one will, in turn, find sums in all terms (after the associated splitting off of the factors that are independent of the summations) whose general form is contained in equation (9). Equations (20), (21) will then be true independently of the magnitude of the time interval Δt when all of those sums vanish as a result of equation (9). However, they are only true for $i = 0$, $n = 1$, so the time interval must be taken to be infinitely small, like the virtual path, and equations (20) will all be the same as equations (6), since the former equations will reduce to their first terms for that reason. By contrast, equations (21), whose first terms include the sums:

$$\sum m\left[\left(X-\frac{d^2x}{dt^2}\right)\frac{d^2x}{d\omega^2}\right], \qquad \sum m\left[\left(X-\frac{d^2x}{dt^2}\right)\frac{d^2x}{d\psi^2}\right], \qquad \text{etc.,}$$

will no longer be true, since one would then have to take $n = 2$ in formula (9), but that would also no longer be necessary for the fulfillment of conditions (β), (γ) in that case, because the second parts of the expressions for the second derivatives of U would cancel, due to the infinite smallness itself, since, e.g., $\frac{d^2U}{d\Delta\omega^2}$ would then reduce to:

$$\frac{d^2U}{d\Delta\omega^2} = 2\sum m\left[\left(\frac{dx}{d\omega}\right)^2\right] - \Delta t^2 \sum m\left[\left(X - \frac{d^2x}{dt^2}\right)\frac{d^2x}{d\psi^2}\right],$$

so to $2\sum m\left[\left(\frac{dx}{d\omega}\right)^2\right]$, since $\Delta t = 1 / \infty$, and likewise for the remaining ones.

§ 5.

Here, we shall consider the principle of least constraint in the case of equilibrium.

When the forces whose projections onto the axes are X, Y, Z are themselves in equilibrium in the system, the principle of virtual velocities will imply the equation:

$$(\alpha) \qquad \sum m\,[X\,Dx] = 0,$$

from which one will infer the formulas:

$$(\beta) \qquad \sum m\,[X\,\delta^n x] = 0, \qquad \sum m\left[X\,\frac{d^n x}{d\omega^{n'}\,d\psi^{n''}\cdots}\right] = 0, \qquad \sum m\left[\frac{d^i x}{dt^i}\,\frac{d^n x}{d\omega^{n'}\,d\psi^{n''}\cdots}\right] = 0,$$

when Dx can take on finite values, and one can obtain the latter from the foregoing by differentiating with respect to a variable t that all other ones can be considered to be a function of, or also (what amounts to the same thing) by differentiating with respect to the characteristic δ (except that one must then replace $\frac{d^i X}{dt^i}$ with $\delta^i X$), and that will generally come down to:

$$(\gamma) \qquad \sum m\,[X\,\delta x] = 0, \qquad \sum m\left[X\,\frac{dx}{d\omega}\right] = 0, \qquad \sum m\left[X\,\frac{dx}{d\psi}\right] = 0,$$

so the number of the latter equations will be equal to the number of geometric independent variables that are present, which will then determine the equilibrium position completely. It must now be shown to what extent those equations also express a minimum of the constraint in this case, which can be done in a manner that is completely similar to what was done in § **4**.

Let x, y, z be the coordinates of m in the equilibrium position A, and let $x + \xi$, $y + \eta$, $z + \zeta$, or x', y', z' be the coordinates in the position B where M arrives when it starts from A and moves freely from the equilibrium position under the influence of the forces X, Y, Z that act upon it, like all of the other points of the system. Finally, let $x + Dx$, $y + Dy$, $z + Dz$ be the coordinates of the position D to which A will arrive under an arbitrary displacement of points of the system that is compatible with the system constraints, or rather when one starts from A and perturbs the equilibrium during the time interval Δt during which it would freely come to B, and during which it would actually move under its coupling to the remaining points of the system (such that one can also set Dx, Dy, Dz equal to Δx, Δy, Δz). If one again sets:

$$AB = u, \qquad \sum m u^2 = U,$$

such that U is the function that should be a minimum when equations (β) or (γ) are true. Likewise, if the point D enters in place of A :

$$AB = u', \qquad \sum m u'^2 = U',$$

then

$$u^2 = \xi^2 + \eta^2 + \zeta^2,$$

$$u'^2 = [(\xi - Dx)^2] = u^2 + [Dx^2] - 2\,[\xi\, Dx].$$

Hence:

$$(a) \qquad U = \sum m[\xi^2], \qquad \text{or} \qquad U = \sum m[(x' - x)^2],$$

and

$$(b) \qquad \begin{cases} U' - U = \sum m[Dx^2] - 2\sum m[\xi\, Dx] > 0, \\ \text{i.e.,} \quad \sum m[\xi\, Dx] = 0, \end{cases}$$

will be the condition for the minimum of U. Alternatively, when one differentiates U in the second form with respect to the independent variables ω, ψ, …, where only the coordinates x of the point A will change, while the coordinates x' of the point B will remain unchanged (as in the example **2**.III) then one will have:

$$(c) \qquad \begin{cases} \dfrac{dU}{d\omega} = -2\sum m\left[\xi\,\dfrac{dx}{d\omega}\right] = 0, \quad \dfrac{dU}{d\psi} = -2\sum m\left[\xi\,\dfrac{dx}{d\psi}\right] = 0, \quad \text{etc.,} \\ \dfrac{d^2U}{d\omega^2} = 2\sum m\left[\left(\dfrac{dx}{d\omega}\right)^2\right] - 2\sum m\left[\xi\,\dfrac{d^2x}{d\omega^2}\right] > 0, \quad \text{etc.,} \\ \dfrac{d^2U}{d\omega^2}\cdot\dfrac{d^2U}{d\psi^2} > \left(\dfrac{d^2U}{d\omega\, d\psi}\right)^2, \quad \text{etc.,} \end{cases}$$

as the system of minimum conditions. Now, since $dx / dt = 0$, so one also has $\delta x = 0$, one will have:

$$\xi = X \cdot \frac{\Delta t^2}{2} + \frac{dX}{dt} \cdot \frac{\Delta t^3}{2 \cdot 3} + \ldots,$$

$$Dx = \delta^2 x \cdot \frac{\Delta t^2}{2} + \delta^3 x \cdot \frac{\Delta t^3}{2 \cdot 3} + \ldots$$

as the developments of those expressions, in which $\delta^2 x$, $\delta^3 x$, etc. have the same values, since one will also have $\delta\omega = \delta\psi = \ldots =$, as in the last development in § **4**.II, namely:

$$\delta^2 x = \frac{dx}{d\omega} \delta^2\omega + \frac{dx}{d\psi} \delta^2\psi + \ldots,$$

$$\delta^2 x = \frac{dx}{d\omega} \delta^3\omega + \frac{d^2 x}{d\omega^2} \delta^2\omega^2 + 2 \frac{d^2 x}{d\omega\, d\psi} \delta^2\omega \delta^2\psi + \ldots,$$

etc. However, those expressions explain the fact that whether Δt is taken to be finite or infinitely small, the minimum conditions in the one form (b) or the other (c) will be justified by the equilibrium conditions (β) or (γ).

"Ueber das Gauss'schen Grundsetz der Mechanik," Zeit. Math. Phys. **3** (1858), 197-223, 261-274.

On Gauss's fundamental law of mechanics

or

The principle of least constraint, as well as another new basic law of mechanics,
with an excursion into various situations that the mechanical principles apply to

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Translated by D. H. Delphenich

Contents

1. – Development of Gauss's law

In treatise no. 18 in Crelle's Journal for Mathematics, vol. **4**, pp. 232, our great mathematician **Gauss** enriched mechanics with a general fundamental law that should not be missing from any textbook on analytical mechanics, along with **d'Alembert**'s principle and the principle of virtual velocities, since it alone, without the aid of a second fundamental law, suffices completely to determine the motion and equilibrium of any system of bodies, and can thus be taken to be the foundation for all of mechanics. Even though the greater simplicity of Gauss's law, in principle, does not always bring about a greater *practical* simplicity or analytical brevity in the treatment of special problems, since in many cases, **d'Alembert**'s principle, in conjunction with that of virtual velocities, would be easier to implement, nonetheless, there are also cases in which Gauss's law would be more direct and convenient to employ. However, in addition, it reveals a most interesting property of every system of bodies that is found to be in *motion*, as well as every one that is in *equilibrium*, especially because it expresses a criterion for the laws of *motion* and *rest* with equal generality.

The fact that this law has not enjoyed a general acquaintance is perhaps based in the brevity of presentation that the inventor himself gave to it, by which the actual essence of that law and its relationship to the usual general fundamental laws of mechanics might not seem sufficiently clear to many. Therefore, it might be advisable to direct the attention of the mathematical public to that important law with some emphasis, and to that end, to explain the law itself somewhat more thoroughly and illustrate its application in some special cases.

However, in addition, we will take this opportunity to digress a bit further on the basic laws of mechanics and add something new to it.

Gauss defined his law, which one can rightly call ***Gauss's*** *principle*, or from its content, the *principle of least constraint*, as opposed to d'Alembert's principle, in words as follows:

The motion of a system of material points that are coupled to each other in whatever way, and whose motion is likewise constrained by whatever sort of restrictions, will take place at each moment with the greatest possible agreement with the free motion, or the ***least possible constraint****, in which one considers a measure of the constraint that the entire system experiences at each moment in time to be the sum of the products of the squares of the deflections of each point from its free motion with their masses.*

If one then has that (Table II, Fig. 1)(†):

$m, m', m'', \ldots$	are the masses of the material points
$a, a', a'', \ldots$	are their positions at time t,

(†) Translator: I have not been able to find the cited figures and tables.

$b, b', b'', \ldots$	are the locations that they would assume after the infinitely-small time interval *dt* as a result of the forces (that are *applied* to them) during that time interval and the velocities and directions that they would attain if they were all free,
$c, c', c'', \ldots$	are the locations that they would actually assume at time interval *dt*

then, from the principle above, of all of the locations that are compatible with the conditions on the system, the actual locations will be the ones for which the expression:

$$m\,(cb)^2 + m'\,(c'b')^2 + m''\,(c''b'')^2 + \ldots \tag{1}$$

is a minimum.

Equilibrium is obviously only a special case of the general law of motion, since in that case, the actual locations $c, c', c'', \ldots$ would coincide with the original ones $a, a', a'', \ldots$, as long as the equilibrium exists in the *rest* state, so for a system that is found in equilibrium, the expression:

$$m\,(ab)^2 + m'\,(a'b')^2 + m''\,(a''b'')^2 + \ldots \tag{2}$$

must be a minimum. It likewise follows from this that *the persistence of the system in the rest state lies closer to the free motion of the individual points than any possible state that might emerge from it.*

Gauss's law can be derived from d'Alembert's principle and that of virtual velocities as follows. Let (Table II, Fig. 2):

p	be the force that acts upon the material point *a*, which acts during the time interval *dt*, and if that point were completely free then it would go to *b* when one considers the velocity and direction that one achieves at time *t*.
q	be the force that acts upon the point *a* and is produced by the constraint on the system, as a result of which the point would deflect from the rest state from *b* to *c* like a completely-free mass during the time interval *dt*.
r	be the resultant of *p* and *q*, by whose action, the point *a* would actually go from *a* to *c* as a completely-free mass during the time interval *dt* when one considers the velocity and direction that are achieved at time *t*; hence, it is the so-called *effective* force on the point *a*.

Since the point *a* moves under the action of the force *p* and the constraints on the system as if it were free and merely affected with the force *r*, it would follow that if the force *r*, which acts in the opposite direction to the force *p* (so the force $-r$), were applied to *a*, in addition to *p* (so it would be subjected to the force $-q$ that is composed of *p* and $-r$, which would lead the completely-free point through the points in space of *cb* during the time interval *dt* under the remaining constraints on the system), then the system would be found in the *equilibrium* state. In fact, the forces $-q$, $-q'$, $-q''$, … represent the so-called *lost forces*, which must keep the system in equilibrium under the remaining constraints on the system, from **d'Alembert**'s principle.

If we apply the principle of virtual velocities in order to exhibit the condition equation for that equilibrium then we let γ, γ', γ'', … be the locations where the points a, a', a'', … might possibly arrive after the time dt, which are different from c, c', c'', …, but compatible with the conditions on the system. Now, obviously $c\gamma$, $c'\gamma'$, $c''\gamma''$, … are also the virtual motions that the points c, c', c'', … could assume under the constraints for the system that is found in equilibrium under the forces $-q$, $-q'$, $-q''$, …

If one drops a perpendicular $\gamma\beta$ from each of the points γ, γ', γ'', … (e.g., from γ) to cb then since the force $-q$ acts parallel to cb, $-q\,(c\beta)$ will be the virtual moment of that force. If one lets φ, φ', φ'', … denote the angles $bc\gamma$, $b'c'\gamma'$, $b''c''\gamma''$, … that $c\gamma$, $c'\gamma'$, $c''\gamma''$, … make with cb, $c'b'$, $c''b''$, …, resp., then $-q\,(c\gamma)\cos\varphi$, $-q'\,(c'\gamma')\cos\varphi'$, $-q''\,(c''\gamma'')\cos\varphi''$, … will be the virtual moments of the forces $-q$, $-q'$, $-q''$, …

Since the force $-q$ is such that it would push the mass m (which is thought to be completely *free*) from the rest state through the points of cb during the time dt, it will be proportional to the product $m\,(cb)$. If we then set the forces $-q$, $-q'$, $-q''$, … equal to the values $m\,(cb)$, $m'\,(c'b')$, $m''\,(c''b'')$, …, resp. (which are proportional to them), then their virtual moments will be:

$$m\,(cb)(c\gamma)\cos\varphi, \qquad m'\,(c'b')(c'\gamma')\cos\varphi', \qquad m''\,(c''b'')(c''\gamma'')\cos\varphi'', \ldots,$$

respectively.

From the principle of virtual velocities, the sum of those moments must be *equal to zero.* One will then have:

$$\sum m\,(cb)(c\gamma)\cos\varphi = 0. \tag{3}$$

Now, since:

$$(\gamma b)^2 = (cb)^2 + (c\gamma)^2 - 2\,(cb)(c\gamma)\cos\varphi$$

or

$$(cb)^2 = (\gamma b)^2 - (c\gamma)^2 + 2\,(cb)(c\gamma)\cos\varphi, \tag{4}$$

one will have:

$$\sum m\,(cb)^2 = \sum m\,(\gamma b)^2 - \sum m\,(c\gamma)^2 + 2\sum m\,(cb)(c\gamma)\cos\varphi.$$

It will then follow from equation (3) that:

$$\sum m\,(cb)^2 = \sum m\,(\gamma b)^2 - \sum m\,(c\gamma)^2. \tag{5}$$

The length cb is the *actual* deviation of the mass m from the *free motion*, while γb represents *any other possible* deviation. Now, since from equation (5), one has that $\sum m\,(cb)^2$ is always less than $\sum m\,(\gamma\, b)^2$, in that, one will find the proof of the *principle of least constraint* that was expressed above, namely, that the sum of the products of the actual deflections of the individual points from the free motion of the masses at those point must be a minimum; i.e., it must be smaller than the sum of the products of any other deflections of those masses that are possible under the conditions on the system.

For equilibrium in the rest state, equation (5) will become:

$$\sum m\,(ab)^2 = \sum m\,(\gamma\, b)^2 - \sum m\,(a\gamma)^2. \tag{5a}$$

2. – Explanation for Gauss's law.

The foregoing law requires some explanation, and for the sake of applying it in certain cases, a transformation of equation (5) or the wording of the principle that it expresses might be absolutely necessary.

When the force p that acts upon a free mass m is capable of endowing that mass with the velocity g in a unit of time, such that g represents the acceleration that the force p gives to the mass p, it is known that the following relationship exist:

$$p = m\, g. \tag{6}$$

The length s of the path that is that is traversed from the rest state at time t is:

$$s = \tfrac{1}{2}\, g\, t^2\,. \tag{7}$$

The path $ds = gt\; dt$ will be traveled in the time interval dt. For the first time interval that follows the rest state (so the one for which one has $t = 0$), that path length will be equal to zero, from the formula itself. However, that zero value for ds at $t = 0$ tells one only that for $t = 0$, the value of ds is no longer an infinitely-small quantity of *degree one* relative to dt, but one of higher degree. In fact, when one either sets t equal to dt directly in equation (7) or when one sets $t = 0$ in the value of the complete increment of s, so in:

$$\begin{aligned}\Delta s \;&=\; \frac{ds}{dt}\cdot dt + \frac{1}{1\cdot 2}\frac{d^2 s}{dt^2}\cdot dt^2 + \frac{1}{1\cdot 2\cdot 3}\frac{d^3 s}{dt^3}\cdot dt^3 + \cdots \\ &= gt\; dt + \tfrac{1}{2}\, g\; dt^2\,,\end{aligned}$$

one will get:

(8) $$\Delta s = \tfrac{1}{2}\, g\, dt^2$$

for the path that is traversed in the first time interval.

Now, if:

$-q, -q_1, -q_2$ are the forces that would push the free mass m at the material point a from c to γ during the time interval dt, and

$f, \quad f_1, \quad f_2$ are the accelerations that the forces $-q, -q_1, -q_2$, resp., endow the mass m with per unit time

then one will have:

(9) $$-q = m\, f, \qquad -q_1 = m\, f_1, \qquad -q_2 = m\, f_2,$$

(10) $$cb = \tfrac{1}{2} f\, t^2, \qquad \gamma b = \tfrac{1}{2} f_1\, t^2, \qquad c\gamma = \tfrac{1}{2} f_2\, t^2.$$

When one substitutes the values in (10), formulas (3) and (5) will assume the form:

(11) $$\sum m\, f\, f_2 \cos \varphi = 0,$$

(12) $$\sum m\, f^2 = \sum m\, f_1^2 - \sum m\, f_2^2,$$

and when one introduces the forces q, q_1, q_2 from equation (9), those equations will be converted into:

(13) $$\sum q\, f \cos \varphi = 0,$$

(14) $$\sum q\, f = \sum q_1\, f_1 - \sum q_2\, f_2\,.$$

If one so wishes, one can also write those formulas as:

(15) $$\sum q\, (c\gamma) \cos \varphi = 0,$$

(16) $$\sum q\, (cb) = \sum q\, (\gamma b) - \sum q\, (c\gamma)\,.$$

In the form of equation (12), the quantities that one treats in the principle of least constraint are freed from the consideration of *infinitely-small* paths. In that form, one deals with only *finite* values since the *measure of the constraint* for any material point now appears as the *product of its mass with the square of its acceleration due to the deflecting force.*

In the form of equation (14), the *deflecting forces q* themselves are introduced in place of the *masses m*. The *measure of the constraint* is now the *product of the deflecting force with its acceleration.*

In the case where a point has no mass at all, but only represents a geometric position in the system upon which the force p acts, eliminating the mass m of a point a by means of the formulas will take the form of an unacceptable necessity, because one would then have that the mass $m = 0$ for any point of that kind, but a finite force p would assign an infinitely-large acceleration to an infinitely-small mass, so the point b (Fig. 3) would be at an infinite distance. cb and $b\gamma$ (or also f and f_1) would become *infinitely large,* and in that way terms would arise in the formulas above would that take the form $0 \cdot \infty$ or ∞, which would make those formulas unusable. Only equation (13) would still remain useful in those cases.

As far as the remaining formulas are concerned, it is clear that it is not at all necessary to take the sum $\sum$ over *all* points of the system at once. One could also first take the sum S over a certain complex of points and then take the sum $\mathfrak{S}$ over the remaining points, such that one would then have $\sum = \mathrm{S} + \mathfrak{S}$. In that way, equation (3) would then become:

$$\text{(17)} \qquad \mathrm{S}\, m\, (cb)\, (c\gamma) \cos \varphi + \mathfrak{S}\, m\, (cb)\, (c\gamma) \cos \varphi = 0,$$

and when one splits $\sum m\, (cb)^2$ into $\mathrm{S}\, m\, (cb)^2 + \mathfrak{S}\, m\, (cb)^2$ in equation (5) and then substitutes the values from equation (4) that correspond to the partial sum $\mathfrak{S}\, m\, (cb)^2$ in it, equation (5) will become:

$$\text{(18)} \qquad \mathrm{S}\, m\, (cb)^2 + \mathfrak{S}\, m\, (cb)\, (c\gamma) \cos \varphi = \mathrm{S}\, m\, (cb)^2 - \mathrm{S}\, m\, (c\gamma)^2.$$

If one would like to eliminate the sum $\mathfrak{S}$ from equation (18) with the help of equation (17) then one would indeed obtain an equation with only the summation sign S, which would then refer to only an arbitrary part of the masses of the system. However, one would easily find that this equation was only a result of equation (4), so it is only a *geometric* relationship between the masses that enter into it, but not a *mechanical* one.

If one substitutes the value $m(cb) = \frac{1}{2}(-q)\, dt^2$ for the product $m\, (cb)$ in the summation sign $\mathfrak{S}$, by means of the relations (9) and (10), then that will yield:

$$\text{(19)} \qquad \mathrm{S}\, m\, (cb)\, (c\gamma) \cos \varphi + \tfrac{1}{2}(-q)\, dt^2\, \mathfrak{S}\, (-q)\, (c\gamma) \cos \varphi = 0,$$

$$\text{(20)} \qquad \mathrm{S}\, m\, (cb)^2 + dt^2\, \mathfrak{S}\, (-q)\, (c\gamma) \cos \varphi = \mathrm{S}\, m\, (\gamma b)^2 - \mathrm{S}\, m\, (c\gamma)^2,$$

instead of (17) and (18), resp.

Should the sign $\mathfrak{S}$ refer to only those points of the system that have no mass then one would have to observe that for each such point cb that is parallel to ab, the magnitude and direction of the *lost* force – q would have to be precisely equal to the *applied* force p that acts upon the point a (Fig. 2). One would then have:

$$(21)\qquad \mathrm{S}\, m\,(cb)\,(c\gamma)\cos\varphi + \tfrac{1}{2}dt^2\,\mathfrak{S}\,p\,(c\gamma)\cos\varphi = 0,$$

$$(22)\qquad \mathrm{S}\, m\,(cb)^2 + dt^2\,\mathfrak{S}\,p\,(c\gamma)\cos\varphi = \mathrm{S}\, m\,(\gamma b)^2 - \mathrm{S}\, m\,(c\gamma)^2,$$

under that assumption, and infinitely-large or indeterminate quantities would no longer enter in those formulas.

If one would also like to let the lost force – q appear under the S sign in place of the mass m [since that is true of equations (15) and (16)] then one would get:

$$(23)\qquad \mathrm{S}\,(-q)\,(cb)\,(c\gamma)\cos\varphi + \mathfrak{S}\,p\,(c\gamma)\cos\varphi = 0,$$

$$(24)\qquad \mathrm{S}\,(-q)(cb) + 2\,\mathfrak{S}\,p\,(c\gamma)\cos\varphi = \mathrm{S}\,(-q)(\gamma b) - \mathrm{S}\,(-q)(c\gamma)\,.$$

In order to fix the directions and the angle φ precisely, one must once more point out that – q is the *lost* force, which acts in the direction cb, so the *deflecting* force q would act in the directly *opposite* direction bc, but the angle $\varphi = bc\gamma$ lies between the direction cb of the *lost* force – q and the direction $c\gamma$, in which γ refers to any other displaced position of the point a that is possible under the constraints on the system.

As far as the case in which the system is found to be in *equilibrium* and *at rest* is concerned, the deflecting force q will be equal to – p or the lost force – $q = p$. In that case, equations (21) and (22) will assume the form:

$$(25)\qquad \mathrm{S}\, m\,(ab)\,(a\gamma)\cos\varphi + \tfrac{1}{2}dt^2\,\mathfrak{S}\,p\,(a\gamma)\cos\varphi = 0,$$

$$(26)\qquad \mathrm{S}\, m\,(ab)^2 + dt^2\,\mathfrak{S}\,p\,(a\gamma)\cos\varphi = \mathrm{S}\, m\,(\gamma b)^2 - \mathrm{S}\, m\,(a\gamma)^2,$$

resp., and equations (23) and (24) will assume the form:

$$(23)\qquad \sum p\,(cb)\,(a\gamma)\cos\varphi = 0,$$

$$(24)\qquad \mathrm{S}\,p\,(ab) + 2\,\mathfrak{S}\,p\,(a\gamma)\cos\varphi = \mathrm{S}\,p\,(\gamma b) - \mathrm{S}\,p\,(a\gamma)\,.$$

If the system were not precisely in the rest state, but in *uniform motion*, when it is in *equilibrium* then the point c would not in fact fall upon a, but at some other point a_1 to which the point a would be led during the time interval dt by the velocity, once it has been achieved. However, one would always have $q = -p$, and the foregoing equations (25) to (28) would remain valid under the assumption that ab represents the path that the point a would traverse in the time dt as a result of only the force p, but with no consideration given to the previously-achieved velocity, and that g refers to another location for the point a that likewise does not consider that velocity.

3. – Relationship between Gauss's law and d'Alembert's principle and the principle of virtual velocities.

Equation (4) expresses only a *geometric* relationship that prevails in the system, while equation (3) expresses a *mechanical* relationship. Now, since formula (5) is the result of simply combining (3) and (4), it will follow that in a strictly mechanical sense, formula (5) is *equivalent* to formula (3).

However, formula (3) will represent **d'Alembert**'s principle immediately (under which, the system will be in equilibrium with the lost forces $-q$) once one applies the *principle of virtual velocities*, whereas formula (5) is the complete expression for **Gauss**'s principle, in that it does not merely tell one that $\sum m\,(cb)^2$ is smaller than any possible $\sum m\,(\gamma b)^2$, and is thus a *minimum*, but at the same time, it shows *by how much* the former sum is smaller than the latter.

One sees from this that Gauss's principle includes d'Alembert's, in conjunction with the principle of virtual velocities, which are the *two* fundamental laws that define the basis for *statics* and *dynamics* in its usual presentation, and when they are taken together, one can deem that to be a *general* or *higher* principle of mechanics.

Gauss himself said, in the aforementioned treatise:

> "The special character of the principle of virtual velocities consists of the fact that it is a general formula for solving all static problems, and is therefore a replacement for all other principles without, however, immediately taking credit for the fact that it already seems plausible by itself, to the extent that it was only expressed. In *that* regard, the principle that I will present here seems to have the advantage. However, it also has a *second* one, namely, that it encompasses the law of motion and rest in exactly the same way in greatest generality."

The second advantage, namely, that Gauss's principle characterizes the state of motion and rest at once is sufficiently clear from the foregoing. However, the first advantage, namely, that this principle appears to be a *fundamental law* of mechanics, requires more explanation.

The general wording of Gauss's principle, namely, that the motion of a system at any moment proceeds with the greatest possible agreement with the free motion or *with the smallest possible constraint*, generally seems to be entirely plausible, and a proof would not be required. However, what is *constraint* in the strictly-scientific sense? How does one define the mathematical expression for that general concept? Obviously, for any material point a (Fig. 3), the constraint that leads it from the location b of its *free* motion to the location c of its *actual* motion in the infinitely-small time dt, must initially be proportional to the force q that pushed it away from the point b, and in addition, to the length of the path bc through which that mass was pushed, so it should be proportional to the product $q\,(bc)$, which represents the *work* done by the deflecting force. One can then take that product itself to be the constraint in question that the mass m of the point *a* experiences. Now, should the sum of the constraints that are exerted over the entire system be as small as possible then one would be led immediately to the condition that $\sum q\,(bc)$ must be a minimum.

If *m* denotes the mass of the point *a* then since the force *q* is proportional to the product $m\,(cb)$, from equations (9) and (10) one can also impose the demand that $\sum m\,(cb)^2$ must be a minimum, which constitutes the mathematical expression of Gauss's principle.

When one places Gauss's principle at the pinnacle of mechanics in that way, that will imply the remaining fundamental law – namely, the principle of virtual velocities – by the following argument:

Due to equation (4), on purely-geometric grounds, one has:

$$\sum m\,(cb)^2 = \sum m\,(\gamma b)^2 - \sum m\,(c\gamma)^2 + 2\sum m\,(cb)\,(c\gamma)\cos\varphi\,.$$

Now, since $\sum m\,(cb)^2$ is a minimum, so it is always smaller than $\sum m\,(\gamma\, b)^2$, the quantity $2\sum m\,(cb)\,(c\gamma)\cos\varphi$ will either be negative or smaller than $\sum m(c\gamma)^2$, when it is positive, which is a position that the point γ might also assume.

Now let $c\gamma$ (Table II, Fig. 4) be any infinitely-small motion that the point *c* is in a state to assume according to the constraints on the system, and let $c\gamma_1$ be the advance under the backwards motion that this point would adopt under the return from γ to *c*. That shows that any point γ_2 that lies between γ and γ_1 can be regarded as the endpoint of the motion $c\gamma_2$. Now, if $X\,dx$ is the analytical expression for *c g*, in which *X* is the function of any quantity *x* that has one and the same meaning for *all* points of the entire system, such that $X\,dx$, $X'\,dx$, $X''\,dx$, … refer to the material points *a*, *a′*, *a″*, …, resp., or *c*, *c′*, *c″*, …, resp., then $X\,dx$ will obviously represent *any* infinitely-small motion like $c\gamma_2$ that lies between γ and γ_1 when one merely replaces *dx* with the corresponding value, and it will be clear that the motions of *all* points of the system inside the infinitely-close limits γ and γ_1 *increase and decrease proportionally, as well as change signs simultaneously.*

Now, since:

$$2\sum m\,(cb)\,(c\gamma)\cos\varphi = 2\,dx\sum m\,(cb)\,X\cos\varphi$$

and

$$\sum m\,(c\gamma)^2 = dx^2\sum m\,X^2$$

from that relation, one sees immediately that for a suitable choice of *dx*, when the quantity $2\sum m\,(cb)\,(c\gamma)\cos\varphi$ is multiplied by the first power of *dx*, it will always be *positive*, and also always greater than the quantity $\sum m\,(c\gamma)^2$ times the second power of *dx* might be if it possessed any nonzero-values at all.

It follows from this, in general, that one must have:

$$\sum m\,(cb)\,(c\gamma)\cos\varphi = 0,$$

which implies equation (3).

Since m (cb) is thought to be proportional to the deflecting force $-q$, or also in the opposite direction (which one calls the lost force), the foregoing equation will go to:

$$\sum (-q)\,(c\gamma) \cos \varphi = 0 \qquad \text{or} \qquad \sum (-q)\,(c\beta) = 0.$$

However, equilibrium must exist under those lost forces $-q$, as **d'Alembert**'s principle itself would say, and proof would not be required. The foregoing formula then expresses a fundamental law of the forces that are found to be in equilibrium, and recognizes the *principle of virtual velocities* in it, which gets its foundation from Gauss's principle in that way.

4. – Simpler proof of the principle of virtual velocities.

In the above, Gauss's principle was derived from d'Alembert's and the principle of virtual velocities. However, it can also be shown how the principle of virtual velocities will follow when one assumes Gauss's principle.

As a rule, one would probably prefer the first path of development, partly because d'Alembert's principle and the principle of virtual velocities are implied by elementary intuitions and admit proofs that are free from objections, but partly because in most cases the last two principles admit an immediate and simpler application to given cases.

In regard to the latter, one must, in fact, observe that since a, a', a'', …, as well as c, c', c'', …, are positions that the masses m, m', m'', …, resp., can actually assume according to the constraints on the system (namely, the former, at the beginning, and the latter, at the end, of the time interval dt), when one applies d'Alembert's principle and the principle of virtual velocities, one can start immediately from the given positions a, a', a'', … of those masses at time t as the point of application of the lost forces $(-q)$, $(-q')$, $(-q'')$, …, whereas the application of Gauss's principle requires the consideration of the fictitious positions b, b', b'', … at which those masses would arrive at time interval dt if they were entirely free, as well as the positions c, c', c'', … at which they would actually arrive at that time, as well as the conditions under which $\sum m(cb)^2$ would become a minimum, or from equation (5), equal to $\sum m\,(\gamma b)^2 - \sum m(c\gamma)^2$, which are, as a rule, more cumbersome to develop than the conditions under which the sum of the virtual moments of the lost forces that act upon a, a', a'', … would be equal to zero.

However, one will always be able to derive *some* relationships from Gauss's principle more simply and directly than from the other two principles with a suitable handling of the formulas in question.

Meanwhile, if one starts with those other two principles then the simplest-possible proof of the principle of virtual velocities would be desirable. I shall then allow myself to communicate such a thing here.

The principle in question reads:

If the point of application of a system of forces that is found to be in equilibrium is displaced infinitely little, and indeed in a way that would be permitted by the constraints on the system, then sum of the products of the forces and the lengths of the paths that are traversed, which are parallel to the directions of those forces, will be equal to zero.

Therefore, if the point of application of any of the forces P describes the path δp under that motion in a direction that subtends the angle φ with the forward direction of the force P then:

(29) $$\sum P\,\delta p\cos\varphi = 0.$$

We next consider a *rigid* system of points, i.e., one in which all points are *rigidly* coupled to each other. At those points, let:

X, Y, Z be the components of the force P that acts upon the point a that are parallel to the three rectangular coordinate axes,

α, β, γ be the angles of inclination of the forward direction of P with respect to those axes,

$\alpha_1, \beta_1, \gamma_1$ be the angles of inclination of the forward direction along which the infinitely-small displacement δp of the point of application of P will result with respect to those axes, whereas

φ represents the angle of inclination of P with respect to δp,

x, y, z are the coordinates of the point of application a, and

$\delta x, \delta y, \delta z$ are the displacements of that point relative to the three axes or the projections of δp .

Any motion of a rigid body consists of a rectilinear *advance* and a *rotation* around any axis, as an entirely simple argument that requires no calculation will show (cf., my *Situationskalkul*, pp. 191). That advance can be resolved into three advancing motions that are parallel to three *given rectilinear axes*, and thus parallel to our coordinate axes, and the aforementioned rotation can be resolved into three rotations about those coordinate axes. Therefore, the equilibrium of the system will require that the forces P give it no ambition to move in the direction of any axis, so:

(30) $$\sum X = 0, \qquad \sum Y = 0, \qquad \sum Z = 0.$$

In addition, equilibrium will require that there is also no ambition to rotate about an axis, so the moment equations:

(31) $$\sum(xY - yX) = 0, \qquad \sum(yZ - zY) = 0, \qquad \sum(zX - xZ) = 0$$

must be valid.

If we focus our attention on a motion under which only a rotation about the z-axis takes place [so one for which the first of equations (31) is true], and we denote the infinitely-small positive angle of rotation from right to left by γ then, as we can easily infer from Fig. 5, the ordinate x will change by the quantity $an = -\varphi y$, and the ordinate y will change by the quantity $nm = \varphi x$. For a similar rotation around the x-axis through the angle ψ, the ordinate y will change by $-\psi x$ and the ordinate z, by ψy. Likewise, under a rotation about the y-axis through an angle of χ, the ordinate z will change by $-\chi x$ and the ordinate x will change by χz. As a result of all those three rotations:

the ordinate x will change by $\delta x = -\varphi y + \chi x$,
″ ″ y ″ ″ $\delta y = -\psi z + \varphi x$,
″ ″ z ″ ″ $\delta z = -\chi x + \psi y$.

If one now multiplies the first, second, and third of equations (31) by φ, ψ, χ, resp., and adds all three then that will give:

$$\sum [(-\varphi y + \chi z) X + (-\psi z + \varphi x) Y + (-\chi x + \psi y) Z] = 0$$

or

(32) $$\sum (X\,\delta x + Y\,\delta y + Z\,\delta z) = 0.$$

It is important to point out that this *one* equation does not just completely replace the *three foregoing ones*, which refer to the *rotation*, but also the *three* equations (30), which refer to the advance, because depending upon whether one sets δz or δy or δx equal to zero, one will get three equations for the relevant rotation about an axis, and depending upon whether one sets δz and δy or δx and δz or δx and δy equal to zero, one will get three equations for the relevant advance.

Equation (32) is then *necessary and sufficient* for equilibrium.

If one now sets:

$$X = P\cos\alpha, \qquad Y = P\cos\beta, \qquad Z = P\cos\gamma,$$

$$\delta x = \delta p\cos\alpha_1, \quad \delta x = \delta p\cos\beta_1, \quad \delta z = \delta p\cos\gamma_1$$

then one will have:

$$X\,\delta x + Y\,\delta y + Z\,\delta z = P\,\delta p\,(\cos\alpha\cos\alpha_1 + \cos\beta\cos\beta_1 + \cos\gamma\cos\gamma_1)$$

$$= P\,\delta p\cos\varphi.$$

With that, equations (32), which express the principle of virtual velocities (which was to be proved), will assume the simplest form:

(33) $$\sum P \;\delta p \cos \varphi = 0.$$

If one now connects that rigid system to a second, likewise rigid, system in such a way that it is not a rigid coupling that exists at the contact point, but a moving one, then the motion of the former system will be restricted by that in a certain way; i.e., certain displacements that were previously possible will now become impossible.

Obviously, the first system must be in equilibrium under all forces that act upon it when one counts among those forces not merely the P that are applied to it, but also those Q that the second system that is coupled to it will exert upon it at the contact point.

When one considers all forces P and Q, equation (33) will then be true for any remaining displacement of the first system, so obviously, also for any *possible* one. One will then have:

$$\sum P\,\delta p \cos\varphi + \sum Q\,\delta q \cos\psi = 0.$$

The same thing will be true for the second system, for which one will have:

$$\sum P'\,\delta p' \cos\varphi' + \sum Q'\,\delta q' \cos\psi' = 0$$

when one puts primes on all of its forces, for clarity. If one now observes that at each contact point between the two systems, the pressure on the one is equal to the counter-pressure on the other, so $Q' = Q$, and that for any possible displacement, the virtual motion of the point of application of Q' will be opposite to that of Q, so $\delta q' \cos \psi' = -\delta q \cos \psi$, then that will imply that $\sum Q' \delta q' \cos \psi' = -\sum Q \;\delta q \cos \psi$. If one then adds the two foregoing equations then all terms in Q will vanish, and only the form of equation (33) will remain, which P will now refer to all forces that are applied to the total moving system that is composed of the two individual ones, and the displacements are restricted to the ones that are still *possible* under the constraints on both systems.

In the same way, one can combine a third and fourth rigid system with previous ones by a constraint that is moving, but based upon immediate contact, without the principle of virtual velocities ceasing to be valid.

Finally, if one ponders the fact that from the foregoing consideration, it is irrelevant whether one or more of the systems considered have finite or infinitely-small dimensions (so the system would reduce to a material point in the latter case) then it would follow that the principle in question will remain applicable to any system at all, as well as couplings by rigid, flexible, extensible, compressible bodies, etc., because every non-rigid, finite body can be decomposed into infinitely-small parts that one can consider to be rigid.

At this point, I must remark that the proof of the principle of virtual velocities that **Moseley** gave in the book *The mechanical principles of engineering and architecture*, and which I also adapted in my own book that appeared with the title *Die mechanischen Principen der Ingenieurkunst und Architectur*, § **121**, pp. 170, is *incorrect*. Namely, that proof starts from the assumption that the components X, as well as Y and Z, that are parallel to the system are in equilibrium by themselves, so $\sum X \,\delta x = 0$, $\sum Y \,\delta y = 0$, $\sum Z \,\delta z = 0$, from which it would generally follow very simply that $\sum (X\,\delta x + Y\,\delta y + Z\,\delta z) = 0$. However, that assumption is inadmissible, since indeed the sum of the forces in each of the three parallel systems is equal to zero, so $\sum X = 0$, $\sum Y = 0$, $\sum Z = 0$, but by no means is the sum of the *moments* of them about any axis always equal to zero, much less will each of the three systems reduce to a *force-couple*, which *does not* represent *equilibrium*.

5. – Special remark by Gauss on the principle of virtual velocities.

In the oft-mentioned treatise, **Gauss** made the following remarks about the principle of virtual velocities:

> "…it is more correct to say that the sum of the virtual moments can *never be positive*, while one ordinarily says that it must be *equal to zero*, because the ordinary expression tacitly assumes that the opposite to any possible motion is likewise possible [or that the opposite of any impossible motion is likewise impossible], such as when a point is required to remain on a certain surface, when the distance between two points is required to be invariant, and the like. By itself, that is an unnecessary restriction. The outer surface of an impermeable body does not constrain a material point that is found on it to remain on it, but merely prohibits it from appearing on one side. A tensed inextensible, but flexible, string between two points makes only an increase in distance impossible, but not a decrease, etc. Why then would one not wish to express the law of virtual velocities in such a way that it would encompass all cases right from the outset?"

I believe that one could perhaps respond to that remark and question as follows: In general, fixed points, lines, and surfaces are often prohibited from executing *certain* motions, but will admit the direct *opposite ones*. Now, as long as one actually takes advantage of the fixed nature of such points, so an ambition to move in the impossible direction will prevail, there will exist forces on those points that take the form of the *resistance* of those points, which are quite necessary for the equilibrium of the system. However, whenever a displacement takes place in a possible direction that makes the resistance of a point inactive, the forces on the system that represent that resistance will *vanish*, which will generally have nothing but *positive* virtual moments for such a motion, when it persists, and will thus leave a *negative* sum for the moments of the remaining forces by its vanishing. However, the system will no longer remain the same under the vanishing of part of the original forces and will then leave the equilibrium state. Such motions will also never be suitable to determine the resistance of the fixed points, which plays the role of *external applied forces* entirely, and is necessary for equilibrium, and thus for developing the conditions for equilibrium *completely*. Accordingly, motions that lie directly opposite to the impossible ones will be likewise kept inadmissible if they have a change in the given system of forces as a consequence. The principle of virtual velocities will always require the *vanishing* of the sum of the virtual moments then when the *forces* and *resistances* that are necessary for the *complete* determination of equilibrium are applied to that principle.

For example, if a weight P (Table II, Fig. 6) lies on a fixed surface then it must feel a certain force P' of resistance. Above all, the whole criterion for a fixed body in a mechanical context consists of saying that it must be capable of experiencing the resistance that was just required. The impossibility of displacement is, in itself, a corollary to that, and in the spirit of the principle of virtual velocities, one can ignore it completely when one substitutes an external force for the resistance of a fixed barrier.

However, an *essential* condition for the foregoing system is that contact between both bodies must be preserved by the virtual displacement. When one performs a common displacement upwards or downwards through the path δp, that will lead to the formula $P\,\delta p - P'\,\delta p' = 0$, so $P = P'$. By contrast, if one would like to perform a *one-sided* motion of the weight upwards from a plane with no displacement in the plane then one would indeed get the negative value $-P\,\delta p$ for $\sum P\,\delta p$, as Gauss correctly remarked. Only the resistance P' of the plane would vanish then, while the entire system would change, and no more formulas would exist from which one could determine the forces that would be required for equilibrium.

A similar case occurs when a weight P hangs from a fixed point A on a flexible string, as in Fig. 7. That point must respond with the force of resistance $P' = P$. If one merely raises the fixed point then without displacing the fixed point then that resistance P' would vanish, and the forces that are required for the equilibrium of the system would no longer be present.

Therefore, in such cases, when one takes the *fixed nature* of certain points to be absolute, as was done here, and one would like to ignore the forces of resistance that those points produce completely, the principle of virtual velocities would give a likewise incomplete answer, which would consist of saying that the weight P can have *any arbitrary value*, which is indeed correct in itself, but one cannot recognize the essential fact that the fixed point must experience a resistance to the force of *that weight*.

Such a purely-extrinsic view of the concept of *constraints* on a system of material points with no rigorous consideration of the *resistance* of points that are fixed or restricted in their motions can, in turn, easily lead to entirely erroneous arguments. For instance, the case of *elastic* constraints belongs to them. When considered absolutely, such constraints allow arbitrary stretching, compression, and bending, from their mechanical properties. One cannot therefore deny that every motion of a point that is coupled with the remaining system by an elastic band will correspond to one of the conditions on the system, so it will be virtual. However, if one overlooks the intrinsic resistance that appears in that way and the other necessary changes in the forces that are applied to the system then one will get false results.

Having assumed that, let the weight P in Fig. 7 hang from the fixed point A by means of an elastic string, so a stretching of it is very probable, and thus a motion of the weight P directly downwards without the fixed point A simultaneously moving with it. However, such a virtual motion will produce a virtual moment $P\,\delta p$ that would be either equal to *zero*, *negative*, or rather decidedly *positive*. For that reason, the result is nonetheless false, because of the fact that stretching of the string cannot happen without overcoming the intrinsic elastic forces, and strictly speaking, without an increase in the weight P.

All of those considerations will lead us to the following rules that must be observed when one applies the principle of virtual velocities.

6. – Consideration of the variability of forces and the intrinsic resistance of a body to a virtual motion.

From the above, one can, with no further discussion, regard a virtual motion of the system to be one that is possible under the constraints on the system; i.e., under which, those constraints will not change, except for the geometrically-allowable absolute and relative motions, so ones under which no other effort is expended than the one that corresponds to the virtual moments of the system of externally-applied forces when the virtual moments of the internal resistance between two contact points of the system do not mutually cancel, and so, from equation (33), vanish by themselves.

That case will always occur when the constraints on the system are *independent* of the forces that are applied to it, such as, e.g., for a system that is composed of nothing but rigid bodies that can rotate about certain points or displace on their outer surfaces.

However, when, in the opposite case, the constraints depend upon the forces that act upon the system, or when the intended displacement of the system is possible at all only under the expenditure of certain quantities of work that are generated in the bonds of the system or when the system must be subjected to special external forces, that displacement can still be regarded as a virtual one *when one takes into account the requisite virtual moments that do not cancel by themselves.*

Hence, if (e.g., in Fig. 7) a weight P hangs from a fixed point A by means of an elastic string and one would like to displace the weight through δp downwards then one would find that a quantity of work would be required to displace it that has the value $W\,dp$, as it also would be from the law of elasticity for the string.

If the length p of the entire string increases by δp then the length of each element dx of the string will increase by $(\delta p / p)\,dx$. The expenditure of work that would be required by that increase in the length of the element dx (which is the differential of the work done by the tension P that acts upon the lower end of that element) will then have the value $-P(\delta p / p)$. The sum of those works over all elements of the string will then be:

$$-\int_0^p P\frac{\delta p}{p}\,dx\,.$$

Now, since P depends upon the length p, that tension will always be equal for all elements, so it will be independent of x, and one will then have:

$$-\int_0^p P\frac{\delta p}{p}\,dx = -\ P\frac{\delta p}{p}\int_0^p dx\ = -P\ \delta p\,.$$

The weight P will produce the virtual moment $P\ \delta p$ under the motion that we speak of. It will then realize equation (33) in the form of $P\ \delta p - P\ \delta p = 0$.

Something similar will occur for the system that is represented in Fig. 6 when one would like to displace the weight P horizontally through δx on the fixed surface, but one makes the assumption that *friction* exists between the weight and the surface, which has the magnitude $f P$. In that case, the displacement is, in fact, possible and allowable. However, it can be regarded as only a virtual one when one observes that it requires a force $f P$ in the horizontal direction that has not been given up to now, and which produces the positive virtual moment $f P\ \delta x$, and at the same time, that the resisting friction of the fixed surface must be overcome, which would, however, yield the opposite moment $- f P\ \delta x$, such that equation (33) will now be fulfilled in the form $f P\ \delta x - f P\ \delta x = 0$.

7. – Correct interpretation of the infinitely-small quantities in the principle of virtual velocities.

In the formula for the principle of virtual velocity, δp is an *infinitely-small* quantity, and therefore a quantity that continually strives to assume the value zero. Since equation (33) will first achieve complete validity for the limiting values of that quantity, but those limiting values are all zero, one can see from that fact itself that the ratios of all those infinitely-small quantities δp to one and the same independent infinitely-small quantity (which we would like to denote by δs) must be determined in such a way that one must get the values $\frac{\delta p}{\delta s}\delta s$, $\frac{\delta p'}{\delta s}\delta s$, $\frac{\delta p''}{\delta s}\delta s$, … for the δp, $\delta p'$, $\delta p''$, …, resp. in equation (33), so once one has divided all terms by the common factor δp, one will get the formula:

$$\sum P \frac{\delta p}{\delta s} \cos \varphi = 0, \tag{34}$$

which includes only *finite* quantities $\delta p / \delta s$.

If, according to the nature of the system, the quantities δp did not all depend upon one and the same basic quantity δs, but upon several such basic quantities in groups, then equation (33) would obviously already decompose into just as many special equations in that way, each term of which would contain the terms of one and the same group.

The aforementioned finite quantities $\delta p / \delta s$ will reduce to the *first differential coefficients* of the function p with respect to the independent variable s under the passage to their limiting values. It would then be entirely superfluous to determine the increment δp more precisely than its first differential, which has the form $A\ ds$, or to determine the quotient $\delta p / \delta s$ more precisely than the first finite term A, since the lower terms of second, third, and higher order are infinitely-small in comparison to them, and from the expression:

$$\frac{\delta p}{\delta s} = A + B\ ds + C\ ds^2 + \ldots,$$

they would all vanish when one passes to the limiting value.

That remark is especially important for those cases in which the first differential coefficient A is coincidentally *equal to zero* precisely, while the higher differential coefficients keep finite values. In such cases, if might seem on first glance as if the introduction of the next non-vanishing terms (so, e.g., the substitution $\delta p / \delta s = B\, ds$) would be necessary in order to determine the actual virtual moment of the force in question. That exchange becomes irrelevant by the foregoing remark that the term $B\, ds$ and all higher terms will be effectively equal to zero under the passage to the limiting values, for which only equation (33) will be true.

A practical case of the latter kind is represented, for example, in Fig. 8. In it, one assumes that a weight P is *affixed* to the deepest point of a circular hoop that can *roll* on a horizontal plane. For every small enough motion of the hoop, the weight P generally seems to always accomplish a certain amount of work, because it *lifts* somewhat, while the work done by the resistance P of the plane remains precisely equal to zero, such that equation (33) does not appear to be fulfilled here.

That error can be explained when one observes that for a rolling motion through the infinitely-small angle $\delta\alpha$, the vertical rise of the weight is $\delta p = r - r \cos \delta\alpha$, so, up to second-order terms, $\delta p = \frac{1}{2}(\delta\alpha)^2$, and as a result $\delta p / \delta s = \frac{1}{2}\delta\alpha$. When one passes to the limiting value, one would therefore not merely have that the first differential coefficient $dp / d\alpha$ is equal to zero, but also that the entire expression $\delta p / \delta \alpha$ is equal to zero.

8. – Correct interpretation of the infinitely-small quantities in Gauss's principle.

The infinitely-small quantities that enter into formulas (1), (5), etc., that relate to Gauss's principle of least constraint require much deeper attention. As one would learn from equations (10), the lines cb, γb, $c\gamma$ (Fig. 5) are infinitely-small quantities of order *two*, since they are multiplied by dt^2. Nonetheless, *in relation to each other*, they generally behave like *finite* quantities. However, in conjunction with the lines ab, ac, which are themselves *first*-order quantities, they are infinitely small. Nevertheless, with no further discussion, they can only be first neglected in comparison to those quantities in the end results, where only their ratios to relatively infinitely-large values should be considered. However, it can frequently happen that in the intermediate operations, relatively infinitely-large terms can cancel each other under addition and subtraction and that another relationship between the infinitely-small quantities might remain as a result, such as when one already neglects part of the latter terms prematurely in comparison to infinitely-larger ones.

In regard to that, we point out the following: In order to construct the point b to which the mass m of the point a would move in the time interval dt if it were completely free, and the point c to which it would actually move, let (Fig. 9, Table II):

v be the velocity of the mass m at the point a at time t in the direction $a\alpha$, so when one takes $a\alpha = v\, dt$, where α is the point at which the mass m would arrive without the influence of any force, merely as a result of the velocity that is achieved during the time element dt,

f, g, h be the velocities that the *applied* force p, the *deflecting* force q, and the *actual* force r, resp., are in a position to impart upon the mass m during a *unit of time*,

φ, ψ be the angles *pav* and *rav*, resp., that the *applied* force p and the *actual* force, resp., make with the direction of the velocity v or the path of the mass m at time t, where the angle is thought to be positive or negative according to whether those directions lie on one side or the other of the direction of v,

χ be the angle *par* between the applied and actual forces (which is then $\psi - \varphi$).

If one now makes *ab* parallel to p and equal to $\frac{1}{2} f\, dt^2$ then b will be the point at which the mass m would arrive during the time interval dt if it were completely free.

If one takes *bc* to be parallel to q and equal to $\frac{1}{2} g\, dt^2$ then c will be the point at which the mass actually arrives during that time.

One will also get the same point c when one takes *ac* to be parallel to r and equal to $\frac{1}{2} h\, dt^2$.

Since the mass m must describe a continuous curve for forces that act continuously, the smaller that one chooses the time interval dt to be, the more that the line *ac* will fall along the tangent *va* to that curve, and its length will become equal to $a\alpha + \alpha c \cdot \cos \psi$, such that this line, which represents the increment Δs in the path s that has been traversed at time, will then have the value:

$$\Delta s = v\, dt + \tfrac{1}{2} h \cos dt^2, \tag{35}$$

up to terms of dimension two. In that expression, the quantities of both dimensions are carefully kept long enough that one has to compare the line *ac* with similarly-constructed lines, such as *ab*.

It follows from equation (35) that:

$$\frac{\Delta s}{dt} = v + \tfrac{1}{2} h \cos \psi\, dt .$$

If one passes to limiting values then the second term on the right-hand side will vanish, and one will get the known formula $ds / dt = v$. However, the conclusion would become completely false that because the quantity v on the right-hand side of this formula is increased by $\frac{1}{2} h \cos \psi\, dt$, that increase will probably represent the increase that the velocity experiences during the time interval dt, so under the passage of the mass m from the point a to the point c, such that one can set:

$$\frac{\Delta s}{dt} = v + dv = v + \tfrac{1}{2} h \cos \psi\, dt ,$$

and therefore:

$$dv = \tfrac{1}{2} h \cos \psi\, dt \qquad \text{or} \qquad \frac{dv}{dt} = \tfrac{1}{2} h \cos \psi .$$

Moreover, the quotient $\Delta s / dt$ expresses nothing but the velocity that the mass would take on during the time dt or along the path *ac if it traversed that path with uniform velocity*, and $\frac{1}{2} h \cos \psi\, dt$ is the excess of that *fictitious* velocity over the one that prevails at the point a.

Since the motion of the mass *m* generally accelerates or decelerates, that fictitious velocity, which is, to some extent, the *mean* velocity that exists along the path *ac*, will differ essentially from the one that exists at time interval *dt*, and thus, upon the arrival at the point *c*. The latter velocity is:

$$v + dv = v + h \cos \psi \, dt,$$

which is its increase over the one that prevails at time *t*, namely, $dv = h \cos \psi \, dt$, so it will be *twice as large* as the aforementioned increase, because one has, in full generality:

$$s + \Delta s = s + \frac{ds}{dt} dt + \tfrac{1}{2} \frac{d^2 s}{dt^2} dt^2 + \ldots,$$

so

$$\Delta s = \frac{ds}{dt} dt + \tfrac{1}{2} \frac{d^2 s}{dt^2} dt^2 + \ldots,$$

or

(36) $$\Delta s = v \, dt + \tfrac{1}{2} \frac{dv}{dt} dt^2 + \ldots,$$

so a comparison of this formula with (35) will give:

$$\frac{dv}{dt} = h \cos \psi .$$

By contrast:

$$v + \Delta v = v + \frac{dv}{dt} dt + \tfrac{1}{2} \frac{d^2 v}{dt^2} dt^2 + \ldots,$$

and therefore:

$$(v + \Delta v) \, dt = v \, dt + \tfrac{1}{2} \frac{dv}{dt} dt^2 + \ldots$$

However, it would be incorrect for one to regard the line *ac*, which is actually equal to Δs, to be $(v + \Delta v) \, dt$, and accordingly, from equation (35), one would have:

$$v \, dt + \frac{dv}{dt} dt^2 = v \, dt + \tfrac{1}{2} h \cos \psi \, dt^2,$$

so one would like to set $dv / dt = \frac{1}{2} h \cos \psi$, since that says the same thing as assuming that the mass *m* traverses the path *ac* with the velocity $v + \Delta v$, although $v + \Delta v$ represents the velocity that the mass *s* would achieve at the endpoint *c* of that path.

In reality, from the equality of (36) and (35), the line *ac* has the value:

(38) $$ac = \Delta s = v\,dt + \tfrac{1}{2}\frac{dv}{dt}dt^2 = v\,dt + \tfrac{1}{2}h\cos\psi\,dt^2.$$

Furthermore, from the above and from (27):

(39) $$ac = \tfrac{1}{2}h\,dt^2 = \frac{1}{2\cos\psi}\frac{dv}{dt}dt^2\,.$$

The line *ab* is:

(40) $$ab = \tfrac{1}{2}f\,dt^2.$$

Thus, in the triangle $b\alpha c$, in which the angles are $b\alpha c = p\ a\ r = c$, one has that the square of the deflection *cb* is:

$$(cb)^2 = (\alpha b)^2 + (\alpha c)^2 - 2\,(\alpha\,b)\,(\alpha\,c)\cos\chi$$

or

(41) $$(cb)^2 = \tfrac{1}{4}d\,t^4\,(f^2 + h^2 - 2\,f\,h\cos\chi)$$

$$= \tfrac{1}{4}d\,t^4\left[f^2 + \left(\frac{1}{\cos\psi}\frac{dv}{dt}\right)^2 - 2\,f\,\frac{dv}{dt}\frac{\cos\chi}{\cos\psi}\right].$$

If one would like, then one can also set:

(42) $$(cb)^2 = \tfrac{1}{4}d\,t^4\,[(f\cos\varphi - h\cos\chi)^2 + (f\sin\varphi - h\sin\chi)^2]$$

$$= \tfrac{1}{4}d\,t^2\left[\left(f\cos\varphi - \frac{dv}{dt}\right)^2 + \left(f\sin\varphi - \frac{dv}{dt}\tan\psi\right)^2\right],$$

instead of (41).

9. – Transformation of Gauss's formula for the decomposition of the forces along three rectangular axes.

If one decomposes the force that is applied to the mass *m* into its components parallel to the rectangular axes, so if:

f, *g*, *h* are the velocities that those components that might be communicated to the mass *m*, whose coordinates are *x*, *y*, *z*, during a *unit time* at time *t*,

$\tfrac{1}{2}f\,dt^2$, $\tfrac{1}{2}g\,dt^2$, $\tfrac{1}{2}h\,dt^2$

are the distances in space through which the mass *m* in the rest state would be pushed by those forces during the time interval *dt*, and

u, v, w are the velocities parallel to the three axes that the mass *m* will actually possess at time *t*

then:

$$u = \frac{dx}{dt}, \qquad v = \frac{dy}{dt}, \qquad w = \frac{dz}{dt}.$$

When one develops the increments Δx, Δy, Δz up to second-order terms (*ac* in Fig. 10), the *actual* advance of the point *a* in the course of the time interval *dt* in the directions of the three axes will be:

$$\text{(43)} \qquad \begin{cases} \Delta x = \dfrac{dx}{dt}dt + \dfrac{1}{2}\dfrac{d^2x}{dt^2}dt^2 = u\,dt + \dfrac{1}{2}\dfrac{du}{dt}dt^2, \\ \Delta y = \dfrac{dy}{dt}dt + \dfrac{1}{2}\dfrac{d^2y}{dt^2}dt^2 = v\,dt + \dfrac{1}{2}\dfrac{dv}{dt}dt^2, \\ \Delta z = \dfrac{dz}{dt}dt + \dfrac{1}{2}\dfrac{d^2z}{dt^2}dt^2 = w\,dt + \dfrac{1}{2}\dfrac{dw}{dt}dt^2. \end{cases}$$

By contrast, the partial distances that the point *a* would traverse in the time *dt* if it were completely free (*ab* in Fig. 10) would be:

$$\text{(44)} \qquad \begin{cases} u\,dt + \tfrac{1}{2} f\,dt^2, \\ v\,dt + \tfrac{1}{2} g\,dt^2, \\ w\,dt + \tfrac{1}{2} h\,dt^2. \end{cases}$$

Therefore, the deflections into the direction of the three axes (*cb* = *ab* – *ac* in Fig. 10) would be:

$$\text{(45)} \qquad \begin{cases} \tfrac{1}{2} f\,dt^2 - \tfrac{1}{2}\dfrac{du}{dt}dt^2 = \tfrac{1}{2}dt^2\left(f - \dfrac{du}{dt}\right), \\ \tfrac{1}{2} g\,dt^2 - \tfrac{1}{2}\dfrac{dv}{dt}dt^2 = \tfrac{1}{2}dt^2\left(g - \dfrac{dv}{dt}\right), \\ \tfrac{1}{2} h\,dt^2 - \tfrac{1}{2}\dfrac{dw}{dt}dt^2 = \tfrac{1}{2}dt^2\left(h - \dfrac{dw}{dt}\right). \end{cases}$$

Since the square of the actual deflection is equal to the sum of the squares of the deflections along the three axes, Gauss's principle will require that the sum:

(46)
$$\sum m\left(f-\frac{du}{dt}\right)^2+\sum m\left(g-\frac{dv}{dt}\right)^2+\sum m\left(h-\frac{dw}{dt}\right)^2$$

should be a minimum.

As far as equation (5) is concerned, when δx, δy, δz denote the projections of any possible *displacement* $c\gamma$ (Fig. 3) of the point c, since the projection of any other possible *deflection* γb of the point b in the direction of the x-axis has the value $(cb) - \delta x = \frac{1}{2}dt^2\left(f-\frac{du}{dt}\right)-\delta x$, one will then have:

(47)
$$\tfrac{1}{4}dt^4\sum m\left(f-\frac{du}{dt}\right)^2+\tfrac{1}{4}dt^4\sum m\left(g-\frac{dv}{dt}\right)^2+\tfrac{1}{4}dt^4\sum m\left(h-\frac{dw}{dt}\right)^2$$

$$=\sum m\left[\tfrac{1}{2}dt^2\left(f-\frac{du}{dt}\right)-\delta x\right]^2+\sum m\left[\tfrac{1}{2}dt^2\left(g-\frac{dv}{dt}\right)-\delta y\right]^2+\sum m\left[\tfrac{1}{2}dt^2\left(h-\frac{dw}{dt}\right)-\delta z\right]^2$$

$$-\sum m(\delta x)^2-\sum m(\delta y)^2-\sum m(\delta z)^2 .$$

A development of the squares on the right-hand side leads directly to the known fundamental equation:

(48)
$$\sum m\left(f-\frac{du}{dt}\right)\delta x+\sum m\left(g-\frac{dv}{dt}\right)\delta y+\sum m\left(h-\frac{dw}{dt}\right)\delta z=0,$$

which enters in place of equation (3).

It should be remarked in that regard that when certain forces of the system act, not on *masses* m, but on *massless* points, and one denotes the components of those forces by X, Y, Z, one must convert the Σ sign in equation (47) into an S and add the sum:

$$dt^2\,\mathfrak{S}\,X\,\delta x+dt^2\,\mathfrak{S}\,Y\,\delta y+dt^2\,\mathfrak{S}\,Z\,\delta z$$

to the left-hand side, since the summation sign S refers to the *material* points, and the sign $\mathfrak{S}$ refers to the *massless* points.

10. – Application of Gauss's principle to the motion of a pendulum and the equilibrium of a lever.

In order to make the application of Gauss's principle more intuitive, we would like to consider the motion of two ponderous masses m, m' (Fig. 11) that are fixed at the endpoints a, a' of a lever that rotates about A. Let:

a, a' be the lever arms Aa, Aa', resp.,

φ be the angle BAa that the lever subtends with the horizontal at time t,

v be its angular velocity at that time,

g be the velocity that gravity communicates during the time interval,

p, p' $= mg$, mg', resp., be the weights of the masses m, m', resp.

Since the masses can move only along the circular lines in question, the force r that acts upon them will fall in the direction of the tangent $r\,a$ to that circle; the angle will then be $rap = BAa = \varphi$. The velocity of the mass m is $a\,v$. If one then takes $(a\alpha) = a\,v\,dt$ then that mass would arrive at α after the time element dt by means of its intrinsic velocity. If one makes the vertical $(\alpha b) = \frac{1}{2} g\,dt^2$ then b will be the point at which that mass would arrive during that time interval if it were completely free. Now, it actually arrives at c, such that the angle will be $\alpha Ac = d\varphi$ and one will have $d\varphi / dt = v$; therefore, let $(\alpha\,c) = x$.

If one puts primes on the quantities with the same symbols for the mass m' then one will get:

$$(a\,c) = (a\,\alpha) + x = a\,v\,dt + x,$$

$$(a'c') = (a'\alpha') + x' = a'v\,dt + x',$$

so

$$x' = \frac{a'}{a}\,x.$$

Furthermore, in the triangle bca, one has:

$$(bc)^2 = (\alpha\,b)^2 + (\alpha\,c)^2 - 2\,(\alpha\,b)\,(\alpha\,c)\cos(b\alpha\,c)$$

$$= \tfrac{1}{4} g^2\,dt^4 + x^2 - g\,x\cos\varphi\,dt^2.$$

Since the angle is $b'a'c' = \pi - \varphi$ here, one will get:

$$(b'c')^2 = \tfrac{1}{4} g^2 dt^4\,\frac{a'^2}{a^2}\,x^2 + \frac{a'}{a}\,g\,x\cos\varphi\,dt^2$$

for the triangle $b'\alpha'c'$. Thus:

$$\sum m\,(bc)^2 = \tfrac{1}{4}(m + m')\,g^2\,dt^4 + \frac{a^2 m + a'^2 m'}{a^2}\,x^2 - \frac{am - a'm'}{a}\,g\,x\cos\varphi\,dt^2. \tag{49}$$

In order for that sum to be minimum, as in Gauss's principle, we must set its differential with respect to x equal to zero. That will give:

(50) $$x = \frac{am - a'm'}{a^2 m + a'^2 m'} \frac{a\,g\cos\varphi}{2} dt^2 .$$

Since the velocity of the mass m at time t is equal to av [so $(ac) = x = \frac{1}{2} a \frac{dv}{dt} dt^2$], when one sets that expression equal to the foregoing one for x, one will get:

(51) $$\frac{dv}{dt} = \frac{am - a'm'}{a^2 m + a'^2 m'} g\cos\varphi ,$$

or also, since one has $\frac{dv}{dt} = \frac{d^2\varphi}{dt^2}$, one will get:

(52) $$\frac{1}{\cos\varphi} \frac{d^2\varphi}{dt^2} = \frac{am - a'm'}{a^2 m + a'^2 m'} g = \frac{ap - a'p'}{a^2 p + a'^2 p'} g$$

as the fundamental equation for the pendulum motion to be determined.

If one would like to introduce the angle φ as an independent variable and the angular velocity v as a dependent variable then, since $\frac{dv}{dt} = \frac{dv}{d\varphi}\frac{d\varphi}{dt} = v\frac{dv}{d\varphi}$, equation (51) will give:

$$v\,dv = \frac{ap - a'p'}{a^2 p + a'^2 p'} g\cos\varphi\, d\varphi ,$$

or upon integration, when the angular velocity is $v = v_0$ for $\varphi = 0$:

(53) $$\tfrac{1}{2}(v^2 - v_0^2) = \frac{ap - a'p'}{a^2 p + a'^2 p'} g \sin\varphi.$$

If one would like to exhibit the conditions for *equilibrium* of the masses m, m' or the weights p, p' on the lever aAa' then, from equation (50), $(ac) = x$ must be equal to *zero*. That will give the known relation:

(54) $$ap = a'p'.$$

11. – Application of Gauss's principle to the motion of a material point on a given surface or line.

The application of Gauss's principle takes an especially simple form for the motion of a material point on a given surface or line. We immediately direct our attention to the most general case of a given *surface*. In Fig. 12, let:

v be the velocity of the point a of the mass m at time t, and let

g be the velocity that the force that is applied to that mass (say, gravity) communicates to it in a unit time.

Now, one has $a\alpha = v\,dt$, and the line ab points in the direction of the applied force with a length that equals $\frac{1}{2} g\,dt^2$, so b is the location at which the mass m would arrive after the time dt if it were completely free, so c will be the location on the surface at which that mass would actually arrive, and it will then be the base point of the *normal bc that is dropped from b to the surface,* since that would be the *shortest* line that one could draw from b to the surface, and obviously that shortest line will satisfy the condition of Gauss's principle that $\sum m\,(bc)^2 = m\,(bc)^2$ is a minimum.

That property will suffice to develop all conditions for the motion of the given point. Namely, if:

φ denotes the angle $b\alpha\,n$ that the direction $\alpha\,b$ of the force makes with the normal $\alpha\,n$, which is an angle that is also equal to $\alpha\,b\,c$, by the infinite smallness of the figure that we speak of, and

ψ denotes the angle $c\,\alpha\,e$ by which the intersection $\alpha\,c$ of the normal plane $n\,\alpha\,b$ or $\alpha\,b\,c$ with the tangent plane is inclined from the direction $a\,\alpha\,e$ of the velocity of the mass m at time in question, then:

$$a\,\alpha = v\,dt, \qquad \alpha\,b = \tfrac{1}{2} g\,t^2, \qquad \alpha\,c = \tfrac{1}{2}\alpha\,b \sin\varphi = \tfrac{1}{2} g \sin\varphi\,dt^2,$$

and

$$ac = a\,\alpha + \alpha\,c \cdot \cos\psi = v\,dt + \tfrac{1}{2} g \sin\varphi\,\cos\psi\,dt^2.$$

Now, since one also has $ac = v\,dt + \frac{1}{2}\frac{dv}{dt}dt^2$, one has the fundamental equation:

$$\frac{dv}{dt} = g \sin\varphi \cos\psi\,.$$

12. – Application of Gauss's principle to the collision of inelastic bodies.

In order to apply Gauss's principle to the collision of inelastic bodies, let the velocity of the two masses m, m', which both move along a straight line, be equal to v, v', resp., before the impact and V after the impact. If no union of the masses occurred at the moment of collision, so there would be no constraint on the motion, then the two masses would move through the distances $v\,dt$, $v'\,dt$, resp., during the time interval dt if they were completely free. Under the conditions on the system (as a compound body), they would actually traverse the distance $V\,dt$. If $v < v'$ then the deflections will amount to $(V - v)\,dt$ and $(v' - V)\,dt$, resp. Therefore, the constraint is:

$$m\,(V - v)^2\,dt^2 + m'(v' - V)^2\,dt^2.$$

In order for that expression to be a minimum according to Gauss's principle, we set its differential with respect to V equal to zero. That will give the known relation:

$$V = \frac{mv + m'v'}{m + m'}\,. \tag{56}$$

(Conclusion in next issue)

13. – Processes that allow one to always consider the *actual* displacement of a system in an infinitely-small time interval to be a *virtual* one.

In Fig. 9 (Table II in the previous issue), α is the position that the material point a would occupy at the end of a time interval dt due to its intrinsic velocity at time t *if no forces at all acted upon it*. b is the position that it would occupy with that velocity under the effect of the force p applied to it *if it were completely free*. c is the position that it would occupy with that velocity under the action of the applied force p and the constraint on the system, so the one that it would *actually occupy* under the under the control of the effective force r. In addition, γ denotes any position besides c that the point in question *might possibly occupy as a result of a virtual displacement of the system*.

One understands a *virtual* displacement to be one that corresponds to the momentary constraint on the system that exists at time t. However, in the manner of presentation that is found in all textbooks on mechanics, the constraint itself is always regarded as completely unvarying during the displacement. If that constraint is to also depend upon *time t* then it must still be considered to be constant during the time interval dt under the virtual displacement. The infinitely-small path $c\gamma$ is therefore only the spatial variation of the point c that is allowed by the momentary constraints that exist on the system without one considering those variations that are produced by the way that the constraint might depends upon time t . In the determination of those variations, one must then treat time as constant when the law of dependency of the constraint on the system is to be given as a function of time t. Obviously, just the same thing is also true of the *forces* that act upon the system, as long as they are supposed to be functions of time t or the positions of the masses that they act upon, which are themselves functions of time t. Those forces must also be considered to be unvarying under the displacement during the time interval dt.

Accordingly, *in general*, the *actual* motion of the system during the time dt (hence, the displacement ca) cannot be regarded as a *virtual* one. Rather, that can happen only when the constraint on the system is *independent* of time t or the *forces* that act upon it. One result of that consideration, among other things, is that the principle of *vis viva* is only valid for those systems whose constraints do not depend upon time.

Obviously, one comes to that restriction of the virtual displacements by the tacitly-made assumption that among the forces p that act upon the material part of the system, the only motions that can be disregarded are the ones that are considered to be *externally-applied*, but *internal*, and which emerge as the *reaction of the couplings* in the system, so ones that are *themselves produced* as a result of the motion that the externally-applied forces bring about in some way. For the systems with completely *independent* or *unvarying* constraints (e.g., for the ones in which rigid, inelastic materials exist with fixed constraints, rotatable axes, completely-free isolated parts, and similar mechanisms, under which any change in the constraint is, in principle, absolutely *impossible*), the internal reactions within and between the couplings in the system will always be of the sort that for any possible displacement of the system, the *quantity of work* done by all of the reactions *will be equal to zero.* The moments of the internal forces in such system would always vanish then, no matter how one might displace the system. Therefore, that imaginary motion can also be regarded as a virtual one here. By contrast, for the systems with *variable* constraints (e.g., mechanisms with elastic couplings, with compressible or gaseous bodies, and the like, under which certain changes in the constraints are *possible*, as long as the required *forces are applied*), the quantity of work done by internal reactions in the couplings of the system will be *equal to zero* only for those displacements that produce no change in the constraint, but for other displacements under which special forces are developed, that work will have *a finite, positive or negative value*. Therefore, one restricts the field of virtual motions here to the ones that are independent of time t or to the ones under which the couplings in the system do not change, because only for those displacements will the moments of the internal forces vanish.

However, such a vanishing of the internal forces in the equation that expresses the principle of virtual velocities would have no particular use whatsoever, because it would be a big mistake to believe that one could avoid considering the internal forces (like elastic forces) completely in that way. That is by no means the case, because when the aforementioned equation is also free from internal forces under the popular restriction on virtual displacements, that will always make their consideration in isolation valid for the complete determination of the motion of the system.

That sheds light upon the fact that when one takes the internal forces (namely, elastic forces) that appear under a certain displacement into account, the concept of the *constraint* on a system can always be extended in such a way that the displacements that are *even possible under the action of forces* will seem to be *allowable* or *virtual*, and in that way, the difference above between the two types of constraints will vanish completely, and in addition, the arbitrary restriction on the virtual displacements will drop out for the latter type of systems. Along with those advantages of the generalization in principle, one also has the convenience of the fact that one will be led directly and necessarily to *all* requisite equations in the presentation of the fundamental equations for the motion of a system, so those former equations do not have to be extended by auxiliary considerations about the internal forces.

Under the latter assumption, one can also regard the *actual* displacement ac as a *virtual* one then *in all cases* (which is understood to mean systems with mutually-independent constraints). One will then have to consider the internal stresses to be overcome by the actual motion only in the case of a system with variable constraints.

However, the displacement $a\alpha$ will also be regarded in that way under *the same* conditions under which the *actual* displacement ac will seem to be virtual, since the latter arises from the assumption that the material point a will advance uniformly during the time interval dt with the velocity that it has gained at time t. However, the same state will also be attained due to the fact that one can assign the completely-allowable value of *zero* to the effective force r on each material point a that arises from all internal stresses under consideration, and in that way, the *actual* displacement ac will go directly to the $a\alpha$ that we imagined above.

14. – Explanatory example for the process that was just described.

An example might better explain the foregoing.

From Fig. 13 (Table II in the previous issue), let the material point a of mass m and weight n be coupled to the disc a' of mass m' and weight n' by a weightless string $a\alpha$ of length c. The system falls vertically downwards through the air, which makes air resistance act upon the disc a', which can be expressed by $k\,v'^2$, if v' denotes the velocity of the disc at time t, while v is that of the point a.

1) If the string aa' is *inextensible* then one will be dealing with an entirely-unvarying system that can be treated in the usual simple way. Namely, if one denotes the vertical abscissas of the points a and a' from any fixed point by x and x', resp., and the acceleration of gravity by g, and observes that $v' = v$, then one will have that:

$$\text{the lost force of the mass } a \text{ is} \qquad mg - m\frac{dv}{dt},$$

$$\text{"} \qquad \text{"} \qquad \text{"} \quad a' \text{ is} \qquad m'g - k\,v^2 - m'\frac{dv}{dt}.$$

Since those forces must be equilibrium, from d'Alembert's principle, the principle of virtual velocities will imply that:

$$\left(mg - m\frac{dv}{dt}\right)\delta x + \left(m'g - kv^2 - m'\frac{dv}{dt}\right)\delta x' = 0.$$

From the fixed constraint on the system, one has $x = x' + c$, so $\delta x = \delta x'$, and that illuminates the fact that one can also regard the *actual* motion during the time interval dt as a *virtual* one here, or $\delta x = v\,dt$, $\delta x' = v'\,dt = v\,dt$. The foregoing equation will always yield the relation:

$$(m + m')\,g - k\,v^2 - (m + m')\,\frac{dv}{dt} = 0,$$

from which the law of dependency between v and t can now be found by integration.

2). However, if one assumes that the string aa' is *extensible* then one will be dealing with a system whose constraint depends upon time t. Namely, c will be a function of time t in the equation $x = x' + c$ that represents that constraint.

From the usual prescriptions in the textbooks on mechanics, one would now proceed as follows: From the relation $x = x' + c$, one has:

$$\frac{dx}{dt} = \frac{dx'}{dt} + \frac{dc}{dt}, \qquad \text{i.e.,} \qquad v = v' + \frac{dc}{dt},$$

and furthermore:

$$\frac{dv}{dt} = \frac{dv'}{dt} + \frac{d^2c}{dt^2}.$$

One will then have that the lost forces are:

$$\text{at } a: \qquad Q = m\,g - m\frac{dv}{dt} = mg - m\frac{dv'}{dt} - m\frac{d^2c}{dt^2},$$

$$\text{at } a': \qquad Q' = m'g - k\,v'^2 - m'\frac{dv'}{dt}.$$

If one expresses the equilibrium of those forces by the principle of virtual velocities then one will get:

$$Q\,\delta x + Q'\,\delta x' = 0.$$

In order to determine δx and $\delta x'$ in that equation by the usual procedure, the constraint on the system during the time interval dt must be regarded as unvarying, so the quantity c in the relation $x = x' + c$ must be regarded as constant, and as a result, one must set $\delta x = \delta x'$, from which it will follow that:

$$Q + Q' = 0.$$

That illuminates the fact that in all such situations, it will be impossible for one to set the *virtual* displacements δx, $\delta x'$ equal to the *actual* ones during the time interval dt, since that would imply that $\delta x = v\,dt = v'dt + dc$ and $\delta x' = v'dt$, so:

$$Q\,(v'dt + dc) + Q'v'dt = 0,$$

or

$$Q + Q' = -\frac{1}{v'}\frac{dc}{dt},$$

which is an equation that contradicts the previously-found correct relation $Q + Q' = 0$.

One further sees that this ordinarily-applied process does not merely exclude the assumption that the *actual* motion is a *virtual* one, which seems so natural, but it also leaves the solution of the problem incomplete, as well, because it will imply only the single equation $Q + Q' = 0$, in addition to the relation $x = x' + c$ for the determination of the three unknown quantities v, v', c, which will assume the form:

$$(m + m')\, g - k\, v'^2 - (m + m')\frac{dv'}{dt} - m\frac{d^2c}{dt^2} = 0$$

under the requisite substitution. In order to get the missing third equation, one must now go further into a consideration of the internal forces on the system (viz., the stresses in and between the links).

To that end, if one defines the law of elasticity for the string aa' then the tension in it will be equal to zero in its original length a and it will increase in proportion to the increase in length under its extension. The length c at time t will require a force of tension that one can set equal to $(c - a)\, q$, in which q is a constant. Since that tension must obviously be equal to the lost force Q of the mass a, one will get the third equation:

$$(c - a)\, q = Q = mg - m\frac{dv'}{dt} - m\frac{d^2c}{dt^2},$$

for which, one can also take the equation:

$$(c - a)\, q = - Q' = - m' g + k\, v'^2 - m'\frac{dv'}{dt},$$

since $Q = - Q'$.

3) However, if one now generalizes the concept of a constraint on the system in the broadest way right from the start, such that each displacement that is possible from the physical nature of the system can be considered to be virtual, in which one considers the requisite internal forces that might appear, then not only will that unnatural restriction on virtual displacements go away, but all equations that are required for the determination of the phenomena of motion will vanish entirely in their own right.

If one then denotes the tension $(c - a)\, q$ in the string when its length is c by E then one will get the equation:

$$E = Q,$$

and the principle of virtual velocities will imply the equation:

$$Q\, \delta x + Q'\, \delta x' - E\, \delta c = 0,$$

when one observes that under the lengthening of the string by the length δc, the elastic forces that must be overcome will have the virtual moment $- E\, \delta c$. If one now sets $\delta x = \delta x' + \delta c$ then since $E = Q$, the foregoing equation will be converted into:

$$Q + Q' = 0,$$

which was also found before by means of the usual procedure.

15. – Magnitude of the constraint that is exerted upon a system.

We once more return to Fig. 9, in which b is the location to which the material point a would move during the time interval dt as a result of the velocity that it had attained at time t if it were completely *free*, c is the location to which it would *actually* move, α is the location to which it would move if it had advanced with the *uniform velocity* that it would have attained from the forces that act upon it, and finally, γ is any location that is allowed by the constraints to which that point would be *displaced*.

In no. **13**, we saw that $c\alpha$ can always be considered to be a *virtual* motion. If the constraint on the system is unvarying then no attention at all must be given to the internal forces. However, if the constraint is variable then it will only be necessary for one to consider the requisite internal forces that might appear during that motion and that exist within or between the links in the system; i.e., to treat them like the externally-applied forces p.

Under those assumptions, one can then put α in place of γ in equation (5). That will give the following expression for the total *constraint* $\sum m\,(bc)^2$ that is exerted upon the system:

$$\sum m\,(bc)^2 = \sum m\,(ab)^2 - \sum m\,(\alpha\, c)^2. \tag{57}$$

Since $m\,(bc)$, $m\,(ab)$, $m\,(\alpha\, c)$ are proportional to the forces q, p, r, resp., one can also write that equation as:

$$\sum q\,(bc)^2 = \sum p\,(ab)^2 - \sum r\,(\alpha\, c)^2. \tag{58}$$

That equation teaches us that the effect of the deflecting forces q, or the *constraint* that the system feels as a result of the coupling of its material parts at each moment in time, *is always equal to the difference between the effect that the forces p that are applied to it would provoke if all points were completely free and the effect that the effective forces r would actually provoke*.

Since that constraint is a minimum, from Gauss's principle, it will follow further that *the difference between the effect of the applied and effective forces is always as small as would be possible with the constraints that are given on the system*.

Nonetheless, as equation (57) teaches us, that difference will always be *positive, so as a result of the constraint on the individual material points of a system, there will always be a loss to the internal effect of the applied forces that is capable of being produced when all material points are completely free; however, that loss is always only as small as possible.*

In that, we must once more emphasize that when the constraint on the system is variable, along with the *applied* forces p, one must also include the internal forces – namely, the *elastic forces* – that might appear because of the motion of a to α in the links of the system. When one ignores those elastic forces, one would generally find that the losses that the *remaining external applied forces* suffer in many states of motion where the *overcoming* of the internal forces requires a certain effort will *increase*, but in many other states where the internal forces support the motion, it will *decrease,* and in the latter case there can, in turn, be a *gain* in mechanical work.

In addition, one must also point out here that when forces are present in the system that act upon *massless* points, not *massive* ones, equation (22) must be applied, which assumes the form:

$$\mathrm{S}\, q\, (c\, b)^2 + dt^2\, \mathfrak{S}\, p\, (c\alpha) \cos \varphi = \mathrm{S}\, p\, (\alpha\, b) - \mathrm{S}\, r\, (\alpha\, c)$$

or

$$\mathrm{S}\, q\, (c\, b)^2 = dt^2\, \mathfrak{S}\, p\, (\alpha\, c) \cos \varphi + \mathrm{S}\, p\, (\alpha\, b) - \mathrm{S}\, r\, (\alpha\, c)$$

here, where $c\alpha$ is regarded as a virtual displacement, and φ represents the angle $\alpha\, c\, b = raq$ between the forward directions of the forces r and q.

16. – A look back at the fundamental law of mechanics above and a comparison of it with the principle of least action of Maupertuis.

In the foregoing, we saw that one can take Gauss's principle to be the starting point of mechanics, as well as the principle of virtual velocities, in conjunction with d'Alembert's principle. Since each of those two foundations possesses the generality that is necessary if one is to develop the entire study of motion and equilibrium mathematically from it, one can already see from the outset, as **Gauss** also remarked in the aforementioned treatise, that when one has expressed the one, there can be no further essentially-new basic principle for mechanics that is not included in the former and could be derived from it from the nature of things. In fact, we have seen how both basic principles imply each other.

However, from **Gauss**'s further remark, it is not at all true that this new principle proves to be worthless due to that situation. Rather, it is always interesting and instructive to arrive at a new and advantageous viewpoint on the laws of nature if it happens that one can solve this or that problem more easily by means of it or if it reveals a special suitability.

In regard to the latter, we have weighed the two basic principles above against each other many times in the foregoing and found that the principle of virtual velocities, in conjunction with d'Alembert's principle, will permit a simpler or more convenient application in most cases, but that in many special cases, Gauss's principle will allow an immediately employment, and that the latter possesses greater simplicity, in addition, while the former must be composed of *two* laws in a sense, and that ultimately Gauss's principle, from its content, comes closer to the essence of a self-explanatory fundamental law that requires no proof than the principle of virtual velocities.

Up to now, no fundamental law with the same profundity has been expressed besides the foregoing, since the principle of least action that **Maupertuis** proposed carries only the character of a lemma, but can hardly make any claim to the title of a fundamental law. That is because from the statement of that law that was first given correctly by **Lagrange**, the sum over all material points in the system of the integrals of the products of the quantities of motion mv and the curve elements ds that are described between any two epochs in the motion – so the quantities $\int \sum mv\, ds$ – is a minimum (special cases in which that quantity can also be a maximum must be dealt with).

That integral sum is then smaller for the actual motion of the system than it would be if the material points that are pushed by those forces as a result of other constraints that would be necessary to reach the same endpoint of the motion were to follow other paths.

If one also must concede that it is obvious that the motion of a system in the manner that actually results would proceed in the easiest way then it would not be clear, with no further discussion, that the product of the quantity of motion and the path element would be the proper *measure* for that quantity that must be a minimum under such situations. Therefore, the law is very much in need of a proof. However, that law loses the property of a *fundamental law* entirely, since it is not completely general, but rather certain cases remain excluded in which the integral sum above can be a *maximum*.

17. – New fundamental law of mechanics.

From the viewpoint that was presented in the foregoing number, it would not be without interest to become acquainted with *a new, completely-general, fundamental law of mechanics* that takes the place of the other ones completely. Permit me to present it as follows:

From the coupling of material points upon which forces act into a system, those forces will indeed define a certain constraint, such that it will be prevented from performing the maximum of mechanical work that it would be capable of producing if all points were completely free. By itself, it must be regarded as something that is due to the nature of things or an immediate consequence of the constancy of matter and its forces that *the set of all works that the applied forces actually perform under the motion of the system will also appear completely – i.e., with no loss or gain*, since a loss or a gain in work must have a cause that might reveal itself to be nothing but an equivalent amount of work that appears.

That is what our new fundamental law consists of. It seems that it leaves nothing to be desired in *simplicity* and *evidence*, and that it can be aptly presented without proof as a fundamental law of mechanics, although if one feels that it would be desirable to analyze it and reduce it to the elementary theorems of statics and mechanics then one can also provide it with a special proof, as one will see shortly.

As far as the mathematical expression for that law is concerned, as before, in Fig. 14 (Table II of the previous issue), let α be the location that the material point a of the system would occupy as a result of the intrinsic velocity that it had attained at time t acting over the time interval dt, but with no forces at all acting upon it. Let p be the force that is *applied* to it, which would lead it from α to b during that time if it were completely free. Let r be the *effective* force, which would actually lead it from α to c, so it will then correspond to the actual motion ac when one recalls the intrinsic velocity that it already has, and finally, let m be the mass of the material point a.

For the sake of brevity, we symbolically let $\mathfrak{A}\ p\ a$ denote the work that a force p performs when its point of application traverses the straight path a, so it will have the expression $pa \cos \alpha$, in which α represents the angle of inclination between the forward direction of the force p and the path a, and the work that is *actually developed* during the motion in Fig. 14 by the *applied* force p during the time interval dt as it traverses the path ac will be equal to $\mathfrak{A}\ p\ (a\ c)$. By contrast, the *apparent* work done on the material point by the effective force is r is $\mathfrak{A}\ r\ (a\ c)$. Thus, from our fundamental law, we must have the equation:

$$\sum \mathfrak{A}\, r\ (a\ c) = \sum \mathfrak{A}\, p\ (a\ c)\ . \tag{59}$$

On simple geometric grounds, the work that is done by a force p when one traverses a broken path whose sides are a_1, a_2, a_3, … will be equal to the work that force does when it traverses the straight lines that connect the endpoints of the broken path; i.e., it is:

$$\mathfrak{A}\ p\ a_1 + \mathfrak{A}\ p\ a_2 + \mathfrak{A}\ p\ a_3 + \ldots = \mathfrak{A}\ p\ a,$$

in which the sum of the projections of the individual line segments a_1, a_2, a_3, … onto the direction of p equals the projection of the line a onto that direction.

It follows from this that the sum of the works done by the force p, as well as the force r, when they traverse the broken path $a\ \alpha\ c$ is equal to the work done by the force in question when it traverses the diagonal $a\ c$. One can also write:

$$\sum \mathfrak{A}\, r\ (a\ \alpha) + \sum \mathfrak{A}\, r\ (\alpha\ c) = \sum \mathfrak{A}\, r\ (a\ \alpha) + \sum \mathfrak{A}\, r\ (\alpha\ c), \tag{60}$$

instead of equation (59) then.

Now, $a\alpha$ is a motion that the point of the system might exhibit if it had a uniform velocity that was consistent with its constraint, but it was not acted upon by forces. Any *virtual displacement* of the system away from the location a can obviously be regarded as such a motion; i.e., one can think that when the virtual displacements over the time interval dt are replaced with uniform velocities, those velocities can be considered to be ones that the individual points of the system might possibly possess at time t. In this, as in no. **13**, it is generally assumed that should the constraint on the system depend upon time t or be variable then among the applied forces p, the ones that are considered to be required will be the ones by which that variability is required by the constraints.

That further illuminates the fact that, no matter how variable the line $a\alpha$ of the velocities of the points at time t that are consistent with the constraint on the system might be, the line $\alpha\ c$ will not depend upon those velocities at all, but will merely be required by the applied force p, or if one would prefer, the effective force r, when αc represents the direction that is given to the *deflection* of the material point at time t that is produced by only those forces. In order to make the validity of that assertion clearer, recall that no matter what the law of dependency between the line αc and the forces on the system might be, it can produce no other values for the deflection $\alpha\ c$, regardless of whether one determines that deflection from the point a or the point α, because no matter how variable the line $a\alpha$ might also be, that deflection will still be *infinitely small*, which has the consequence that the forces p on the system will have an effect on the point a that differs from the effect on the system at the point *a by only infinitely little;* i.e., when one passes to the limiting state in the sense of differential calculus.

There will be even more evidence for this theorem when one imagines that in the construction of the actual motion along the diagonal ac, it will not be the piece $a\alpha$ that is described with uniform velocity and then the deflection $\alpha\, c$, which might give the impression that the *later* component $\alpha\, c$ can possibly depend upon the *earlier* one $a\alpha$, but, from Fig. 15 (Table II of the previous issue), it would *first* describe the path $a\, c_1 = \alpha\, c$ that lies in the direction of the effective force r, *so the one that is merely required by the applied force p with no concern for any uniform velocity, and then the path* $c_1\, c = a\ \alpha$, which has a uniform velocity that is *given arbitrarily*.

In order to prevent all misunderstandings, we point out that *in a certain sense* the applied force p, and therefore also the effective force r and the line ac_1 can depend upon the velocity at time t, and therefore on the line $a\alpha$, such as, e.g., for motion in resistant media, in which the resistance of the medium varies with the direction and velocity of the moving mass. By itself, that fact is irrelevant for the present considerations, because we think of the applied forces p as being just the ones that correspond precisely to the *actual* motion at the end of time t. If those forces were not, in fact, also required by the velocities at time t, and therefore functions of that velocity, then we would assume *that they do not change when we substitute any other displacement for the virtual displacement* $a\alpha$.

Under those assumptions, the first terms on the left and right-hand sides of equation (60) will appear to be included among the arbitrarily-varying quantities that are given by the laws of constraint on the system, while the second terms are variable; i.e., quantities that are established by the nature of the system and its forces. On that basis already, and also when one considers that the first terms can be equal to zero by the permissible assumption that $a\alpha = 0$ itself, equation (60) will decompose into the following two separate equations:

(61) $$\sum \mathfrak{A}\, r\, (a\ \alpha) = \sum \mathfrak{A}\, p\, (a\ \alpha),$$

(62) $$\sum \mathfrak{A}\, r\, (\alpha\, c) = \sum \mathfrak{A}\, p\, (\alpha\, c)\ .$$

Since $a\alpha$ represents any arbitrary admissible virtual displacement of the point a, and obviously $\alpha\, c = a\, c_1$ is also such a displacement (for $a\alpha = 0$), equation (62) will be contained in equation (61), and thus superfluous.

Equation (61), as the immediate consequence of equation (59), which was given by our fundamental law, can indeed be likewise replaced with equation (59), but it would seem necessary to perform the foregoing derivation and emphasize the remarks that it provoked in order to show more clearly that in the expressions for the *works* done by the forces p and r, the path of the point of application of those forces would have to remain arbitrary within the limits of the *virtual displacements*. From equation (59), in which ac denotes the *actual* path of the point a, that is no more evident than the arbitrariness in that path in its resolution into *arbitrary* components $\alpha\, a$ and constant components $\alpha\, c$ proves to be, especially since one should not, with no further analysis, overlook that if that were true then any *virtual* motion could be regarded as an *actual* motion *that results from the governing forces p*, although there is no doubt that every virtual motion can be regarded as a motion $a\alpha$ that results *with uniform velocity, but without the action of the forces p*.

From that explanation, if one denotes any virtual displacement of the point a [so the line $a\alpha$ in equation (61) or the line $\alpha\, c$ in equation (59)] by δs then our basic equation will become:

$$\sum \mathfrak{A}\, r\, \delta s = \sum \mathfrak{A}\, p\, \delta s\,. \tag{63}$$

One sees that it can be easily reduced to the formula that represents d'Alembert's principle, with the help of the principle of virtual velocities, because if one denotes a coordinate line that is drawn parallel to the direction of the effective force r by ρ, then denotes the force r by $m\frac{d^2\rho}{dt^2}$, and further denotes the angle of inclination of r with respect to the virtual displacement δs of the point a by φ and the angle of inclination of p with respect to δs by ψ then the work done by the force r under that displacement will be equal to $m\frac{d^2\rho}{dt^2}\,\delta s\cos\varphi$, and the work done by the force p will be equal to $p\,\delta s\cos\psi$. In that way, equation (63) will become:

$$\sum m\frac{d^2\rho}{dt^2}\,\delta s\cos\varphi = \sum p\;\delta s\cos\psi. \tag{64}$$

If one would like to refer all quantities to a rectangular coordinate system, as one usually does, then that work done by any force would split into the works done by its components, and if X, Y, Z are the components of p and δx, δy, δz are the projections of the displacement δs onto the three axes then (from a derivation that was applied before in no. **4**):

$$\sum m\left(\frac{d^2x}{dt^2}\delta x+\frac{d^2y}{dt^2}\delta y+\frac{d^2z}{dt^2}\delta z\right) = \sum(X\,\delta x+Y\,\delta y+Z\,\delta z)\,,$$

which is an equation that is ordinarily presented in the form:

$$\sum\left[\left(X-m\frac{d^2x}{dt^2}\right)\delta x+\left(Y-m\frac{d^2y}{dt^2}\right)\delta y+\left(Z-m\frac{d^2z}{dt^2}\right)\delta z\right] = 0, \tag{65}$$

in order to express the equilibrium of the *lost* forces.

It hardly needs to be remarked that our fundamental law encompasses the state of *variable* motion, as well as that of *rest*, or even *equilibrium with uniform motion*, since only the effective forces r need to be set to zero for there to be equilibrium, which will make the entire left-hand side of our fundamental equation reduce to zero.

That further illuminates the fact that this, in itself very plausible, fundamental law possesses the advantage of greater simplicity over d'Alembert's, since the former first requires the assistance of the principle of virtual velocities in order to put the fundamental equations of motion into the form of a mathematical formula, and in addition, requires a detour through the concept of *lost* forces, to which end, certain forces must first be applied to the given system that do not exist in reality and only serve to produce a fictitious system with the so-called lost forces.

In addition, our fundamental law is applicable, with no further analysis, regardless of whether certain forces p in it act upon *material or massless* points, since one only has to set $m = 0$ for the massless points, which one cannot do in Gauss's law, since that would imply *infinite quantities* for massless points, which, as we showed in no. **2**, would make a conversion of the formula necessary, and to some extent the fundamental law itself would be annulled.

If one would like to assign a special name to the new law, for the sake of brevity of reference, then since the motion of the system completely realizes or brings to light the work done by the forces that are applied to it then the terminology of *the principle of the realization of work* might be suitable.

"Hamilton-Jacobische Theorie für Kräfte, deren Maass von der Bewegung der Körper abhangen," Abh. Kön. Ges. Wiss. **18** (1873), 3-54.

Hamilton-Jacobi theory of forces whose measure depends upon the motion of the body

By

Ernst Schering

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As is known, it was first in the year 1834 that Hamilton published a new method for treating some mechanical problems by reducing the determination of the motion to the integration of a first-order partial differential equation and in that way arrived at an especially simple form for the differential equations for the elements of a motion that was acted upon by so-called perturbing forces. **Jacobi** summarized the basic ideas of that theory in a simpler form and generalized the applicability of the method, and in that way established a complete reshaping of the approach to those problems in that broader context, to which **Richelot**, **Liouville**, **Bertrand**, **Donkin**, and **Lipschitz** have added new discoveries.

In the present pages, I will present that method in such a form that I will consider the starting point for the problem to be the introduction of other variables that one can base the known differential equations of motion upon, which are such that the equations between them take on a simple form that is analogous to the one that they possessed originally. The condition equations for such a substitution can be represented in an especially simpler form when one generally appeals to different types of differentiation for the complete differentiation with respect to time, which represents an actual motion, and the variation, which represents a virtual motion. It was in my academic lectures in the Summer semester of 1862 that I first communicated the theory of those canonical substitutions and their application to the integration of the equations of motion for the effects of forces whose measure depends upon not only the mutual positions of the bodies, but also upon their changes in position, as well as the properties of the general equations for the variations of elements that are presented in Article IX, and then the equations that prove to be special cases of the ones that were found by **Lagrange**, **Poisson**, **Hamilton**, and **Jacobi**.

In addition to those investigations, the following pages include a derivation of **Hamilton**'s equations from **Gauss**'s principle of least constraint. Another treatise will address the proof of the existence of a normal form for any canonical substitution in terms of only partially-given substitutions and the differential determinants of the canonical variables.

I. – Principle of least constraint.

Among the various fundamental laws of mechanics, **Gauss**'s principle of least constraint possesses several advantages. It takes exactly the same form for motion that it does for rest and for those conditions and restrictions on motion that might or might not possibly oppose any motion. It also suffices completely to determine the motion in all spaces in which the square of the element of length is represented by a homogeneous expression of degree two in the coordinate differentials that correspond to the element of length.

Gauss expressed his principle in the following form (v. 5 of his *Werke*):

> "The motion of a system of material points that are coupled in whatever way and whose motion is, at the same time, constrained by whatever sort of external restrictions will occur at each moment with the greatest possible agreement with the free motion, or the least possible constraint, when one considers the measure of that constraint that the entire system experiences at each point in time to be the sum of the products of the squares of the deviation of that point from its free motion with its mass."

The application of that fundamental law to the determination of the motion of bodies of stated kind then requires knowledge of the motion of an isolated free mass particle. The laws that are true for that come from the nature of the bodies and the effect of the forces that are present, so they are essentially physical. The assumptions that are most generally valid and most closely connected with the usual concept of force are the following two:

A free, isolated moving mass particle on which forces act that moves along a shortest line in space with unvarying velocity will describe equally-large path segments in equally-large time intervals.

A free, isolated moving particle with mass m that momentarily has no motion, but is under the influence of a force R will begin to move in the direction of the force R with an acceleration that is equal to R / m, so it will cover a path of $\frac{1}{2}(R / m)\, dt^2$ *in that direction during the next time element dt.*

Those two laws, in themselves, still do not determine the motion of a free mass particle under the assumption of an initial motion and the simultaneous effect of one or more forces, but those cases can be resolved with the assistance of the principle of least constraint in its most general interpretation. In the determination of the motion of a system, when one adds that principle, one will also be justified in replacing any given group of free motions with any other fictitious motions and the given conditions and restrictions for the motion of the system with other conceivable conditions such that the conditions will collectively continue to exist, and the motions of the total system that result from those fictitious free motions will be the same as the motions of the total system that result from that group of free motions. One of the most fruitful types of application of that process consists of imagining that the individual mass-particles m are once more decomposed into small mass-particles m_0, m_1, … such that $m = m_0 + m_1 + …$, and one can add the new condition to the existing ones that m_0, m_1, … must remain inseparably coupled to each other. Any free motion that is immanent in the mass-particle m can then be replaced with an arbitrary well-defined free motion that can be ascribed to the particle – for example, m_0 .

Let the position of a point in space through which the motion goes be given by the values of the mutually-independent variables $x_1, x_2, …, x_h, …$ Let the shortest lines be drawn from the point $x_1, x_2, …, x_h, …$ to the point $x_1 + dx_1$, $x_2 + dx_2$, …, $x_h + dx_h$, … and to the point $x_1 + \delta x_1$, $x_2 + \delta x_2$, …, $x_h + \delta x_h$, …, and then construct the shortest line from the latter point to the first line, or by extension, the point at which it meets the latter. The shortest line that is drawn from the point x to that point of intersection is called the *projection* of the line that is drawn from x to $x + \delta x$ onto the line that is drawn from x to $x + dx$ and will considered to be positive when the projection and that line lie on the same side and negative when they lie on opposite sides. The product of the length of the projection times the length of that line will be denoted by $\mathfrak{D}$ and set equal to:

$$\sum_{h,k} X_{hk}\, dx_h\, \delta x_k ,$$

in which the X_{hk} depend upon the nature of the space and the chosen coordinates $x_1, x_2, …$ and will generally satisfy the conditions that $X_{hk} = X_{kh}$ and that they are functions of $x_1, x_2, …$ alone, but not $dx_1, dx_2, …, \delta x_1, \delta x_2, …$, and in which the summation $\sum$ is further extended over as many values 1, 2, 3, … of the indices h and k as the space has dimensions. If the point $x + dx$ coincides with $x + \delta x$ then that expression will go to:

$$\sum_{h,k} X_{hk}\, dx_h\, dx_k ,$$

which will be denoted by $\mathfrak{T}$ and shall mean the square of the length of the line that is drawn from the point x to $x + dx$, so it will always take on a positive value for arbitrary dx.

The length of a line whose points are given by the values of the $x_1, x_2, …, x_h, …$ as functions of one independent variable is equal to:

$$\int \sqrt{\sum_{hk} X_{hk}\, dx_h\, dx_k} ,$$

so that integral must become a minimum for a shortest line that goes through two fixed points that correspond to the constant limiting values of the integral. If the variation δ denotes an arbitrary change in the functions $x_1, x_2, …, x_h, …$ then a relation must exist between those variables for a shortest line such that $\delta\sqrt{\mathfrak{T}}$ will reduce to a complete differential d. Now when one takes the δ differentiation in the expression $\mathfrak{D}$ above to have the same sense as this variation, one will have:

$$\delta\sqrt{\mathfrak{T}} - d\,\frac{\mathfrak{D}}{\sqrt{\mathfrak{T}}} = \frac{\frac{1}{2}\delta\mathfrak{T} - d\mathfrak{D}}{\sqrt{\mathfrak{T}}} + \frac{\mathfrak{D}}{\mathfrak{T}}\,d\sqrt{\mathfrak{T}}$$

$$= \frac{1}{2\sqrt{\mathfrak{T}}}\sum_{h,k}\delta X_{hk}\cdot dx_h\cdot dx_k - \frac{1}{\sqrt{\mathfrak{T}}}\sum_{h,k} d(X_{hk}\,dx_h)\cdot\delta x_k + \frac{d\sqrt{\mathfrak{T}}}{\mathfrak{T}}\sum_{h,k} X_{hk}\,dx_h\,\delta x_k\,,$$

so that expression, which depends upon the δx, and no longer on its differentials $d\delta x$, and differs from $\delta\sqrt{\mathfrak{T}}$ by only a total differential, namely, $d\,\frac{\mathfrak{D}}{\sqrt{\mathfrak{T}}}$, must vanish for a shortest line.

If that is the path of freely-moving mass-particle, and one considers time to be the only independent variable in the d differentiation, while its differential dt is constant, then $\sqrt{\mathfrak{T}}$ will be equal to the velocity times dt, and $d\sqrt{\mathfrak{T}}$ will be equal to the acceleration times dt^2, so when the mass-particle moves freely without the influence of forces, from the fundamental law, one must have $d\sqrt{\mathfrak{T}} = 0$, and as a result of the equation above, one must then also have:

$$\tfrac{1}{2}\delta\mathfrak{T} - d\,\mathfrak{D} = 0$$

for any arbitrary system of values for the δx_1, δx_2, …, δx_h, …, and among others, for $\delta x_1 = d\,x_1$, $\delta x_k = d\,x_k$, $\delta x_h = d\,x_h$, as well, such that foregoing equation $d\sqrt{\mathfrak{T}} = 0$ will again arise as a special case.

From the fundamental law, one can consider the motion of a mass-particle m that starts from rest and is provoked by a force R during the first time-element dt to initially coincide with the shortest line that is drawn from the point x to $x + dx$ when it has the same direction as the force R. It would emerge easily from the meaning of the notations that we have chosen here that this condition can be represented analytically by saying that $\mathfrak{D}/\sqrt{\mathfrak{T}}$ shall denote the length of the projection of a so-called virtual motion from the point x_1, x_2, x_3, … to an arbitrary infinitely-close point $x_1 + \delta x_1$, $x_2 + \delta x_2$, $x_3 + \delta x_3$, … in the direction of the force, so the motion that one cares to call the virtual motion δr is performed by the mass-particle m in the direction of the force R. If one again takes time t to be the independent variable of the differentiation d and dt to be constant, moreover, then $\sqrt{\mathfrak{T}}$ will mean the product of the velocity times dt, so from the fundamental law, it will be zero at the onset of the motion $t = t_0$, which can only happen when the derivatives $\frac{dx_1}{dt}$, $\frac{dx_2}{dt}$, … vanish at that time-point. Under the same assumptions, $d\sqrt{\mathfrak{T}}$ will be the product of the acceleration times dt^2, so from the fundamental law, it will be equal to the value of $(R / m)\,dt^2$. If one multiplies the general equation for a shortest line above by $\sqrt{\mathfrak{T}}$ then one will get:

$$\tfrac{1}{2}\delta\mathfrak{T} - d\,\mathfrak{D} + \frac{\mathfrak{D}}{\sqrt{\mathfrak{T}}}\,d\sqrt{\mathfrak{T}} = -\sum_{h,k} X_{hk}\, ddx_h \cdot \delta x_k + \delta r\,\frac{R}{m}\,dt^2 = 0,$$

$$d\sqrt{\mathfrak{T}} = \frac{R}{m}\,dt^2,\ \mathfrak{T} = 0, \qquad \frac{dx_1}{dt} = 0,\ \frac{dx_2}{dt} = 0,\ \ldots,\ \frac{\mathfrak{D}}{\sqrt{\mathfrak{T}}} = \delta r, \qquad \text{for } t = t_0$$

and arbitrary δx_1, δx_2, … as the equations that determine the motion that starts from the state of rest at time $t = t_0$ and is provoked by the force R acting on the freely-moving mass-particle.

We now turn to the investigation of an arbitrary system of mass-particles and denote the coordinates of the isolated mass-particle m by x_1, x_2, x_3, … and consider the differentiation d to be with respect to dt, and indeed the change in the quantities that would actually arise as a consequence of motion. Any mass-particle m might possess an intrinsic motion by means of which, it would move from the point x_h to the point:

$$x_h + d_0\,x_h + \tfrac{1}{2}\,d_0\,d_0\,x_h + \frac{1}{1\cdot 2\cdot 3}\,d_0^3\,x_h + \ldots$$

if it were free at that time-point t and did not feel the effect of any force, and for which one has:

$$\tfrac{1}{2}\delta\mathfrak{T}_0 - d_0\,\mathfrak{D}_0 = 0$$

for a δx_h that is now arbitrary, namely, when $\mathfrak{T}_0$ and $\mathfrak{D}_0$ denote the same expressions that arise when the differentiation d is taken to mean d_0, and indeed is again taken to be arbitrarily different for the different mass-particles m. A special group of forces R_i, R_n, … acts upon each mass-particle m that would move it to the point:

$$x_h + d_i\,x_h + d_i\,d_i\,x_h + \frac{1}{1\cdot 2\cdot 3}\,d_i^3 x_h + \ldots$$

or

$$x_h + d_n\,x_h + d_n\,d_n\,x_h + \frac{1}{1\cdot 2\cdot 3}\,d_n^3 x_h + \ldots,$$

etc., in which:

$$\tfrac{1}{2}\delta\mathfrak{T}_i - d_i\,\mathfrak{D}_i + \frac{\mathfrak{D}_i}{\sqrt{\mathfrak{T}_i}}\,d_i\sqrt{\mathfrak{T}_i} = -\sum_{h,k} X_{hk}\,d_i d_i x_h \cdot \delta x_k + \delta r_i\,\frac{R_i}{m}\,dt^2 = 0,$$

$$\tfrac{1}{2}\delta\mathfrak{T}_n - d_n\,\mathfrak{D}_n + \frac{\mathfrak{D}_n}{\sqrt{\mathfrak{T}_n}}\,d_n\sqrt{\mathfrak{T}_n} = -\sum_{h,k} X_{hk}\,d_n\,d_n\,x_h \cdot \delta x_k + \delta r_n\,\frac{R_n}{m}\,dt^2 = 0,$$

when those forces act individually on m, and the latter is instantaneously in the rest state, but freely moving, and corresponding statements will be true for the remaining forces.

Any mass-particle m will be decomposed into smaller mass-particles m_0, m_1, … arbitrarily for each individual m, such that one must add to the original conditions the new one that the m_0, m_1, … that make up the components of a mass m must be rigidly coupled to each other. We would like to assume that the quantities $(m_i\ m_0)_h$ are determined in such a way that it would make no difference on the total motion whether the mass-point m possessed an intrinsic motion that would be given to the aforementioned point if it were free and exposed to no forces, or whether the components m_i , m_n , … possessed no intrinsic motion, but the component m_0 possessed one such that if it moved freely then it would arrive at the point:

$$x_h + (m, m_0)_h \left\{ d_0\, x_h + \tfrac{1}{2} d_0\, d_0\, x_h + \frac{1}{1\cdot 2\cdot 3} d_0^{\,3} x_h + \cdots \right\},$$

and furthermore, $(m, m_i)_h$, $(m, m_n)_h$, … might be determined to be such that the effects of the forces R_i , R_n on the masses could be replaced with forces that act upon the individual components m_i , m_n , … alone, such that it would arrive at the point:

$$x_h + (m, m_i)_h \left\{ d_i\, x_h + \tfrac{1}{2} d_i\, d_i\, x_h + \frac{1}{1\cdot 2\cdot 3} d_i^{\,3} x_h + \cdots \right\}$$

and the point

$$x_h + (m, m_n)_h \left\{ d_n\, x_h + \tfrac{1}{2} d_n\, d_n\, x_h + \frac{1}{1\cdot 2\cdot 3} d_n^{\,3} x_h + \cdots \right\}$$

under free motion.

The motion that is actually performed then takes each component m_0, m_i, m_n of the mass m from the point x_h to:

$$x_h + d\, x_h + \tfrac{1}{2} d\, dx_h + \frac{1}{1\cdot 2\cdot 3} d^{\,3} x_h + \ldots,$$

so we can then represent any other position of the system that is compatible with the internal coupling of the masses and with the conditions and external restrictions in such a way that the mass m, and therefore each of its components m_0, m_i, m_n, …, will assume the position:

$$x_h + \delta x_h + dx_h + \tfrac{1}{2} d\, dx_h + \frac{1}{1\cdot 2\cdot 3} d^{\,3} x_h + \ldots,$$

such that because δ and d mean infinitely-small changes, the position:

$$x_h + \delta x_h$$

will also be compatible with the conditions for the mass m. Those differentials of the coordinates that correspond to any possible deviation from the free motion of the particle will then be:

$$(m, m_0)_h\ (d_0\, x_h + \tfrac{1}{2} d_0\, d_0\, x_h + \cdots) - (\delta x_h + dx_h + \tfrac{1}{2} d\, dx_h + \cdots)$$

for m_0, and:

$$(m, m_i)_h \ (d_i\,x_h + \tfrac{1}{2} d_i\,d_i\,x_h + \cdots) - (\delta x_h + dx_h + \tfrac{1}{2} d\,dx_h + \cdots)$$

for m_i, and so forth, so from the assumption that was made for space and those coordinates, the square of that deviation will be:

$$\sum_{h,k} X_{hk}\,\{(m, m_0)_h \ (d_0\,x_h + \tfrac{1}{2} d_0\,d_0\,x_h + \cdots) - (\delta x_h + dx_h + \tfrac{1}{2} d\,dx_h + \cdots)\}$$
$$\times \{(m, m_0)_k \ (d_0\,x_k + \tfrac{1}{2} d_0\,d_0\,x_k + \cdots) - (\delta x_k + dx_k + \tfrac{1}{2} d\,dx_k + \cdots)\}\,\}$$

for m_0, and:

$$\sum_{h,k} X_{hk}\,\{(m, m_i)_h \ (d_i\,x_h + \tfrac{1}{2} d_i\,d_i\,x_h + \cdots) - (\delta x_h + dx_h + \tfrac{1}{2} d\,dx_h + \cdots)\}$$
$$\times \{(m, m_i)_k \ (d_i\,x_k + \tfrac{1}{2} d_i\,d_i\,x_k + \cdots) - (\delta x_k + dx_k + \tfrac{1}{2} d\,dx_k + \cdots)\}\,\}$$

for m_i, so the measure of the constraint for that motion is equal to:

$$\sum_m [m_0 \sum_{h,k} X_{hk}\,\{(m, m_0)_h \ (d_0\,x_h + \tfrac{1}{2} d_0\,d_0\,x_h + \cdots) - (\delta x_h + dx_h + \tfrac{1}{2} d\,dx_h + \cdots)\}$$
$$\times \{(m, m_0)_k \ (d_0\,x_k + \tfrac{1}{2} d_0\,d_0\,x_k + \cdots) - (\delta x_k + dx_k + \tfrac{1}{2} d\,dx_k + \cdots)\}\,\}$$

$$+\ m_i \sum_{h,k} X_{hk}\,\{(m, m_i)_h \ (d_i\,x_h + \tfrac{1}{2} d_i\,d_i\,x_h + \cdots) - (\delta x_h + dx_h + \tfrac{1}{2} d\,dx_h + \cdots)\}$$
$$\times \{(m, m_i)_k \ (d_i\,x_k + \tfrac{1}{2} d_i\,d_i\,x_k + \cdots) - (\delta x_k + dx_k + \tfrac{1}{2} d\,dx_k + \cdots)\}\,\}$$
$$+ \text{ etc.}]$$

From **Gauss**’s principle, the dx_h and ddx_h, … are to be determined in such a way that among all possible values of the δx_h, this expression will assume its smallest value for $\delta x_1 = 0$, $\delta x_2 = 0$, …, $\delta x_h = 0$, so when one develops that expression in powers of δx, the sum of the linear terms that this yields, namely:

$$-2\sum_m \sum_{h,k} X_{hk}\,\{m_0\,(m, m_0)_h \ (d_0\,x_h + \tfrac{1}{2} d_0\,d_0\,x_h + \cdots)$$
$$+ m_i\,(m, m_i)_k \ (d_i\,x_k + \tfrac{1}{2} d_i\,d_i\,x_k + \cdots) - m\,(dx_k + \tfrac{1}{2} d\,dx_k + \cdots)\}\delta x_h\,\},$$

in which $m_0 + m_i + m_n + \ldots$ is replaced with m, will never be negative.

One obtains the still-unknown $(m, m_0)_h$, $(m, m_i)_h$, … by considering the fact that if the mass-point m, which consists of m_0, m_i, m_n, …, were free and acted upon by no forces then it would move to:

$$x_h + dx_h + \tfrac{1}{2} d\,dx_h + \frac{1}{1\cdot 2\cdot 3} d^3 x_h + \ldots = x_h + d_0\,x_h + \tfrac{1}{2} d_0\,d_0\,x_h + \frac{1}{1\cdot 2\cdot 3} d_0^3 x_h + \ldots,$$

and if m_0, along with its rigidly-coupled components, moves freely from there then it must also arrive at:

$$x_h + (m, m_0)\,(d_0\, x_h + \tfrac{1}{2} d_0\, d_0\, x_h + \frac{1}{1\cdot 2\cdot 3} d_0^3 x_h + \ldots),$$

while none of the remaining components possess free motions, so one must have:

$$d_i\, x_h + \tfrac{1}{2} d_i\, d_i\, x_h + \frac{1}{1\cdot 2\cdot 3} d_i^3 x_h + \ldots = 0 \qquad \text{for all } h\,.$$

From the principle of least constraint:

$$-2\sum_{h,k} X_{hk}\ \{m_0\,(m, m_0)_h\ (d_0\, x_h + \tfrac{1}{2} d_0\, d_0\, x_h + \cdots) - m\,(dx_k + \tfrac{1}{2} d\, dx_k + \cdots)\}\delta x_h$$

cannot become negative for any system of values $\pm\,\delta x_1$, …, $\pm\,\delta x_1$, …, so it must be equal to zero. The factor of $X_{hk}\ \delta x_k$ in it is only a special value of δx_k , so it must be equal to zero, since the sum is proportional to the square of a length-element in space for that special case, so when one refers to the equation above, that will imply the relation:

$$\begin{aligned} m_0\,(m, m_0)_h\ (d_0\, x_h + \tfrac{1}{2} d_0\, d_0\, x_h + \cdots) &= m\,(dx_k + \tfrac{1}{2} d\, dx_k + \cdots) \\ &= m\,(d_0\, x_h + \tfrac{1}{2} d_0\, d_0\, x_h + \cdots)\,, \end{aligned}$$

so one will have:

$$(m, m_0)_h = \frac{m}{m_0}\,.$$

The same argument will imply that:

$$(m, m_i)_h = \frac{m}{m_i}\,, \qquad (m, m_n)_h = \frac{m}{m_n}\,, \ \ldots$$

When one substitutes those values, the sum of the terms that are linear in δx in the measure of the constraint will assume the form:

$$\begin{aligned} -2\sum_m m \sum_{h,k} X_{hk}\ \{\,(d_0\, x_h + \tfrac{1}{2} d_0\, d_0\, x_h + \cdots) + (d_i\, x_h + \tfrac{1}{2} d_i\, d_i\, x_h + \cdots) + \ldots \\ - (dx_k + \tfrac{1}{2} d\, dx_k + \cdots)\}\delta x_h\,. \end{aligned}$$

One still has $\dfrac{d_i\, x_h}{dt} = 0$, $\dfrac{d_n\, x_h}{dt} = 0$ for all indices h in that expression. If there are no internal couplings of the masses and external restrictions on the motion that would give rise to a discontinuity in the magnitude or direction of motion then one will have:

$$d_0\, x_h = dx_h$$

for all indices h and all mass-particles m.

The part of the measure of the constraint that is linear in δx will then reduce to:

$$-2\sum_m m \sum_{h,k} X_{hk}\left(\tfrac{1}{2} d_0\, d_0 x_h + \tfrac{1}{2} d_i\, d_i x_h + \tfrac{1}{2} d_n\, d_n x_h + \cdots - \tfrac{1}{2} d\, dx_h\right)\delta x_k\,,$$

or when one considers that, from the above, one has:

$$\tfrac{1}{2}\delta\mathfrak{T} - d_0\mathfrak{D}_0 = \tfrac{1}{2}\sum_{h,k}\delta X_{hk} d_0 x_h \cdot d_0 x_h - \sum_{h,k} d_0 X_{hk}\cdot d_0 x_h \cdot \delta x_k - \sum_{h,k} X_{hk} d_0\, d_0 x_h \cdot \delta x_k = 0,$$

$$\tfrac{1}{2}\delta\mathfrak{T}_i - d_i\mathfrak{D}_i + \frac{\mathfrak{D}_i}{\sqrt{\mathfrak{T}_i}} d_i\sqrt{\mathfrak{T}_i} = -\sum_{h,k} X_{hk} d_i\, d_i x_h \cdot \delta x_k + \delta r_i \frac{R_i}{m} dt^2 = 0,$$

$$\tfrac{1}{2}\delta\mathfrak{T}_n - d_n\mathfrak{D}_n + \frac{\mathfrak{D}_n}{\sqrt{\mathfrak{T}_n}} d_n\sqrt{\mathfrak{T}_n} = -\sum_{h,k} X_{hk} d_n\, d_n x_h \cdot \delta x_k + \delta r_n \frac{R_n}{m} dt^2 = 0,$$

....

$$\begin{aligned}\tfrac{1}{2}\delta\mathfrak{T} - d\mathfrak{D} &= \tfrac{1}{2}\sum_{h,k}\delta X_{hk} dx_h \cdot dx_h - \sum_{h,k} dX_{hk}\cdot dx_h \cdot \delta x_k - \sum_{h,k} X_{hk} d\, dx_h \cdot \delta x_k \\ &= \tfrac{1}{2}\sum_{h,k}\delta X_{hk} d_0 x_h \cdot d_0 x_h - \sum_{h,k} d_0 X_{hk}\cdot d_0 x_h \cdot \delta x_k - \sum_{h,k} X_{hk} d\, dx_h \cdot \delta x_k\,,\end{aligned}$$

it will reduce to:

$$\sum_m m\left(\tfrac{1}{2}\delta\mathfrak{T} - d\mathfrak{D}\right) - \sum_i R_i\, \delta r_i\, dt^2$$

or

$$-\delta\sum_m \tfrac{1}{2} m \sum_{h,k} X_{hk} dx_h \cdot dx_h + d\sum_m n \sum_{h,k} X_{hk} dx_h \cdot \delta x_k - \sum_i R_i\, \delta r_i\, dt^2\,,$$

which is an expression that must never be negative then for a continuous motion of the mass-particle m, namely, if x_h are the coordinates of m at time t, while:

$$x_h + dx_h + \tfrac{1}{2} d\, dx_h + \frac{1}{1\cdot 2\cdot 3} d^3 x_h + \ldots$$

are the coordinates of m at time $t + dt$, and:

$$x_h + \delta x_h$$

mean the coordinates of m that determine a position of that mass-particle that is possible under its given internal couplings and the external conditions and restrictions on the motion of the system. The R_i mean all of the forces that act upon the mass-particles, and the δr_i mean the virtual motions that the points of application of R_i would describe in the direction of those forces for the virtual motions $\delta x_1, \delta x_2, \ldots, \delta x_h, \ldots$

For a space with the property that under an n-fold extension, the length element in it can be represented by the ν^{th} root of an irreducible expression that is homogeneous of degree ν in the differentials of the n coordinates, namely:

$$\sqrt[\nu]{\mathfrak{T}} = \sqrt[\nu]{\sum_h X_{h_1 h_2 \cdots h_\nu} dx_{h_1} \cdot dx_{h_2} \cdots dx_{h_\nu}} ,$$

the way that one determines the motion will have its closest analogy with the one that was just considered when one denotes the actual forces by R_i and lets $\mathfrak{D}$ denote the ν-fold sum:

$$\sum_h X_{h_1 h_2 \cdots h_\nu} \delta x_{h_1} \cdot dx_{h_2} \cdots dx_{h_\nu} ,$$

which extends over all indices $h_1, \ldots, h_\nu$ that are taken from the sequence 1, 2, 3, …, n, and one does not let the expression:

$$-\sum_m m\left(\tfrac{1}{\nu}\,\delta\mathfrak{T} - d\mathfrak{D}\right) - \sum_i R_i\,\delta r_i\,dt^\nu \tag{1}$$

become negative for any virtual motion $\delta x_1, \delta x_2, \ldots, \delta x_n$ of the mass-particle m that is compatible with the given restrictions. Here as well, the free motion of a mass-particle m on which no forces act would take place with equal velocity along a shortest line, because one has:

$$\delta\sqrt[\nu]{\mathfrak{T}} - d(\mathfrak{D}\cdot\mathfrak{T}^{1/\nu-1}) = \left(\tfrac{1}{\nu}\,\delta\mathfrak{T} - d\mathfrak{D}\right)\mathfrak{T}^{1/\nu-1} + (\nu-1)\mathfrak{D}\cdot\mathfrak{T}^{-1}\cdot d\sqrt[\nu]{\mathfrak{T}} .$$

If the conditions on the motion that are given are such that the opposite of any possible motion is also possible then the expression (1) above, which contains opposite signs in the two cases and can never be negative, must be equal to zero.

II. – Force function.

The first two terms in the expression [1] above are related to each other in such a way that the first term, taken with the opposite sign, $\frac{1}{\nu}\delta\mathfrak{T}$, contains the complete δ variation that appears in the second term $d\,\mathfrak{D}$ after one performs the suggested differentiation, and conversely, the second term contains the complete d differentiation that appears in the first term (when taken with the opposite sign) after one performs the δ variation. Each of the two terms is already determined by the other one with that rule for forming the terms.

If one denotes all coordinates x_h of all mass-particles m by $\xi_1, \xi_2, \ldots, \xi_l, \ldots$, and one sets:

$$\frac{dx_h}{dt} = x_h', \qquad \frac{d\xi_l}{dt} = \xi_l',$$

in general, and sets the quantity that **Leibnitz** called the *vis viva* for the case $\nu = 2$ equal to:

$$\sum_m m \sum_h X_{h_1 h_2 \cdots h_\nu} x_{h_1}' \, x_{h_2}' \cdots x_{h_\nu}' = \nu \, T$$

then the basic equation will be:

$$-\delta T + \frac{d}{dt}\sum_l \frac{\vartheta T}{\vartheta \xi_l'} \delta\xi_l - \sum_i R_i \, \delta p_i = 0,$$

when the partial differentiation ϑ of a function considers the quantities ξ and ξ_l' to be mutually independent, and the ξ_l in the sum have been set equal to all coordinates of all mass-particles m in succession.

The basic equation for motion will then take on an especially simple form when the last term $\sum_i R_i \, \delta r_i \, dt^\nu$ can also be represented in the form of the difference between a total variation and a total differential. As **Lagrange** first pointed out, for most of the forces in nature, $\sum_i R_i \, \delta r_i$ is the total variation of a function that depends upon only the coordinates of the mass-particle m, and not on its state of motion, so either the variation of that function will not contain any differential or it will be set equal to zero.

Gauss first considered forces whose measure depends upon not only the position of the mass-particle m, but also on its state of motion. For our further investigations, we would like to assume that this dependency is such that:

$$\sum_i R_i \, \delta r_i$$

is the difference between a total variation and a total derivative with respect to time. If the total variation is:

$$= \delta V$$

then the total derivative must be:

$$\frac{d}{dt}\left\{\sum_l \frac{\vartheta V}{\vartheta \xi_l'} \delta\xi_l + \sum_l \frac{\vartheta V}{\vartheta \xi_l''} \delta\xi_l' + \cdots\right\},$$

in which $\xi_l'' = \dfrac{dd\xi_l}{dt}$, etc. The quantity V might be called the "potential" for the given forces under the motion of a system, as a generalization of the name that **Gauss** introduced, or the "force function," as a generalization of **Hamilton**'s terminology. We would like to restrict our examination to the case in which V contains no derivatives that are higher than the first ξ_l', such that we will then have:

$$\sum_i R_i\,\delta r_i = \delta V - \frac{d}{dt}\sum_l \frac{\theta V}{\theta\xi_l'}\,\delta\xi_l\,,$$

and the fundamental equation (1) of motion will assume the form:

$$-\,\delta(T+V) + \frac{d}{dt}\sum_l \frac{\vartheta(T+V)}{\vartheta\xi_l'}\,\delta\xi_l = 0. \tag{2}$$

The expression:

$$\sum_l \frac{\vartheta(T+V)}{\vartheta\xi_l'}\,\delta\xi_l$$

in this equation possesses the property that its value will remain unchanged, which can also be based on coordinates ξ that are fixed or moving in space and dependent or independent of each other.

Namely, if $q_1, q_2, \ldots, q_\lambda, \ldots$ denote mutually-independent variables then one must be able to represent $\ldots, \xi_l, \ldots$ as functions of t and the q, so one must have:

$$\frac{d\xi_l}{dt} = \frac{\partial\xi_l}{\partial t} + \sum_h \frac{\partial\xi_l}{\partial q_h}\frac{dq_h}{dt} \qquad \text{or} \qquad \xi_l' = \frac{\partial\xi_l}{\partial t} + \sum_h \frac{\partial\xi_l}{\partial q_h}\,q_h'\,,$$

in which ∂ denotes the partial differentiations with respect to t and the q, and $\dfrac{\partial\xi_l}{\partial t}$, $\dfrac{\partial\xi_l}{\partial q_h}$ are independent of all $\ldots,\ q_h', \ldots$, such that one will have:

$$\frac{\partial\xi_l'}{\partial q_h'} = \frac{\partial\xi_l}{\partial q_h}\,,$$

in general, and in that way one will have:

$$\sum_l \frac{\vartheta(T+V)}{\vartheta\xi_l'}\,\delta\xi_l = \sum_l \frac{\vartheta(T+V)}{\vartheta\xi_l'}\sum_h \frac{\partial\xi_l}{\partial q_h}\,\delta q_h = \sum_l \frac{\vartheta(T+V)}{\vartheta\xi_l'}\sum_h \frac{\partial\xi_l'}{\partial q_h'}\,\delta q_h = \sum_l \frac{\vartheta(T+V)}{\vartheta q_l'}\,\delta q_l\,,$$

as will be proved.

If we now set:

$$\frac{\vartheta(T+V)}{\vartheta q_l'} = p_l\,, \tag{3}$$

following the path that was first taken by **Lagrange**, then equation (1) will become:

$$\begin{aligned} 0 &= -\,\delta\,(T+V) + \frac{d}{dt}\sum_l \frac{\vartheta(T+V)}{\vartheta q_l'}\,\delta q_l \\ &= -\sum \frac{\vartheta(T+V)}{\vartheta q_l}\,\delta q_l - \sum \frac{\vartheta(T+V)}{\vartheta q_l'}\,\delta q_l' + \sum \frac{d}{dt}\left[\frac{\vartheta(T+V)}{\vartheta q_l'}\right]\cdot\delta q_l + \sum \frac{\vartheta(T+V)}{\vartheta q_l'}\,\frac{d\,\delta q_l}{dt} \\ &= \sum\left\{\frac{\vartheta(T+V)}{\vartheta q_l} + \frac{d}{dt}\left[\frac{\vartheta(T+V)}{\vartheta q_l'}\right]\right\}\delta q_l = \sum\left\{\frac{dp_l}{dt} - \left[\frac{\vartheta(T+V)}{\vartheta q_l}\right]\right\}\delta q_l\,, \end{aligned} \tag{4}$$

in which the summations extend over all values 1, 2, 3, …, n of the only index l that appears in the expression, and in which we shall denote the number of variable quantities q by n from now on.

III. – General differentials.

The study of many remarkable properties of the function $T + V$ will be simplified considerably when one introduces the concept of a general differential D, in the sense that it represents any sort of changes in a function and the quantities that enter into it that are required by the form of that function such that when the integral equations for the function and its argument that the given differential equations satisfy are added, the integration constants must also be subjected to that general differentiation.

The variation δ that was used up to now, which means an arbitrary virtual motion, is a more general differentiation then the so-called complete differentiation with respect to time t, but of the general differentiations, it encompasses only the ones for which the coordinates experience an infinitely-small change that is compatible with the given conditions.

After one introduces the quantities q, which determine the position of the system of moving masses at the time t, and might be called the coordinates in the general sense for that reason, the function $T + V$, which will be initially given as a function of t, …, q_l , …, q_l' , so when we once more denote partial differentiation with respect to those quantities by ϑ, the general differential will become:

$$D\,(T+V) = \frac{\vartheta(T+V)}{\vartheta t}\,Dt + \sum \frac{\vartheta(T+V)}{\vartheta q_l}\,Dq_l + \sum \frac{\vartheta(T+V)}{\vartheta q_l'}\,Dq_l'\,,$$

or when one recalls the differential equations (3) for the p and the equation of motion (4) that was just found:

$$D\,(T+V) = \frac{\vartheta(T+V)}{\vartheta t}Dt + \sum p_l'\,Dq_l + \sum p_l\,Dq_l',$$

in which the Dq_l' and Dt mean completely-independent differentials, while $Dq_1, Dq_2, \ldots, Dq_n$ must satisfy the restrictions that are given for the motion.

The two differentiations D and d that enter here are mutually independent in their sequence, so then can be switched, and in that way the last equation will imply that:

$$D\,(T+V) = \frac{\vartheta(T+V)}{\vartheta t}Dt + \frac{d}{dt}\sum p_l\,Dq_l\,.$$

If one takes the general differentiation D in this to have the special sense of complete differentiation d with respect to t, and one divides the equation that arises in that way by the factor dt, which is constant for the complete differentiation d with respect to time t then one will have:

$$\frac{d(T+V)}{dt} = \frac{\vartheta(T+V)}{\vartheta t} + \frac{d}{dt}\sum p_l\,Dq_l',$$

and when one substitutes the value of the partial derivative of $T + V$ with respect to t that this yields, the general equation will go to:

$$D\,(T+V) = \frac{d}{dt}\{(T+V-\sum p_l\,q_l')Dt + \sum p_l\,Dq_l\} \tag{5}$$

or

$$D\,(T+V) = \frac{d}{dt}(T+V-\sum p_l\,q_l')Dt + \sum p_l'\,Dq_l + \sum p_l\,Dq_l', \tag{6}$$

and under special assumptions on the general differentiations $Dt, \ldots, Dq_l, \ldots, Dq_l', \ldots,$ the defining equations above (3) for the p will yield the equations of motion [4] and the value of $\vartheta(T + V) / \vartheta t$ that was found before.

If one subtracts the corresponding sides of the identity equation:

$$D\sum p_l\,q_l' = \sum q_l'\,Dp_l + \sum p_l\,Dq_l'$$

from the two sides of the previous equation then that will give:

$$D(T+V-\sum p_l\,q_l') = \frac{d}{dt}(T+V-\sum p_l\,q_l')\cdot Dt + \sum p_l\,Dq_l' - \sum q_l'\,Dp_l,$$

or, when one sets:

$$-T-V+\sum p_l\,q_l' = -(T+V) + \sum q_l'\,\frac{\vartheta(T+V)}{\vartheta q_l'} = H \tag{7}$$

and

$$\frac{dH}{dt} = H',$$

to abbreviate, that will give:

$$DH = H' Dt - \sum p_l' Dq_l + \sum q_l' Dp_l \,. \tag{8}$$

For the case in which the variables q are mutually independent in regard to the internal couplings and the given external restrictions, when one thinks of the quantities q' in the expression above for H, which **Jacobi** called the **Hamiltonian** function, as being determined by $t, \ldots, q_l, \ldots, p_l, \ldots$ with the help of the defining equation (3) for the p, and one denotes the partial differentiation with respect to the latter variables by ∂, those equations will contain the following equations that **Hamilton** presented:

$$\frac{\partial H}{\partial p_l} = q_l' = \frac{dq_l}{dt},$$

$$-\frac{\partial H}{\partial q_l} = p_l' = \frac{dp_l}{dt} = \frac{\vartheta(T+V)}{\vartheta q_l}, \tag{8*}$$

$$\frac{\partial H}{\partial t} = H' = \frac{dH}{dt} = -\frac{\vartheta(T+V)}{\vartheta t}$$

as special cases under special assumptions that pertain to how they are defined here.

The general differential of $T + V$ is represented by a complete derivative with respect to t in (6) above, but if one now restricts the meaning of that general differentiation to that of a variation then that will imply the generalized **Hamilton** theorem:

$$0 = \delta \int (T+V)\, dt = \delta \int \left(\sum p_l \frac{dq_l}{dt} - H \right) dt \,,$$

namely, when the values of the quantities at the limits of this **Hamiltonian** integral are assumed to be unvarying. Upon performing the variation, one will get:

$$0 = \delta \int (T+V)\, dt = \int \frac{d}{dt} \left(\sum \frac{\vartheta(T+V)}{\vartheta q_l'} \delta q_l \right) dt + \int \sum \left\{ \frac{\vartheta(T+V)}{\vartheta q_l'} - \frac{d}{dt} \left[\frac{\vartheta(T+V)}{\vartheta q_l'} \right] \right\} \delta q_l \, dt \,,$$

such that the equations of motion that were exhibited before will once more follow from the condition of the vanishing of the variation.

In his "Untersuchung eines Problem der Variationsrechnung, in welchem das Problem der Mechanik enhalten ist" Borchardt's Journal, Bd. 74, **Lipschitz** took that generalization of **Hamilton**'s theorem to be the basis for determining the motion when the motion takes place under the influence of forces that depend upon the position, and not the evolution, of the system and possess a force function V, and when space is further thought of as being constructed in such a way that its element of length is represented by the ν^{th} root of a homogeneous expression of degree ν in the coordinate differentials.

If follows from the equation $\frac{dH}{dt} = \frac{\theta(T+V)}{\theta t}$ that when $T + V$ does not contain the quantity t explicitly along with the quantities q and q':

$$\sum p_l\, q_l' - (T + V) = H = \text{const.}$$

will be an integral of equations (8*) for the motion of the system, and that will define a generalization of the principle of the conservation of *vis viva* that **Johann Bernoulli** first found.

If the q are fixed coordinates in space then T will not contain time t explicitly, so in that case, one needs only for the potential V to not contain time t explicitly in order for the integral above to be valid.

If the potential V is independent of the motion (so it does not contain q') then …, q_l, … will be fixed coordinates in space, and when one applies **Euler**'s theorem to T as a homogeneous function of degree ν in the quantities q', one will get:

$$\sum_l p_l\, q_l' = \sum_l q_l' \frac{\vartheta(T+V)}{\vartheta q_l'} = \sum_l q_l' \frac{\vartheta T}{\vartheta q_l'} = \nu\, T.$$

If the potential V does not contain time t explicitly either, so H = constant, then one will have:

$$\int \sum (p_l\, q_l' - H)\, dt = \int \nu\, T\, dt - H \int dt$$
$$= \int \sum m_i\, v_i^{\nu} dt - H \int dt = \int \sum m_i\, v_i^{\nu-1} ds_i - H \int dt,$$

when ds_i or $v_i\, dt$ means the path that is traversed by the mass-particle m_i during the time dt. Since the variation of the first term in this equation vanishes, from the aforementioned generalized **Hamilton** theorem, the variation of $\int \sum m_i\, v_i^{\nu-1} ds_i$ must also become zero with the aid of the integral equation H = const., as **Maupertuis**'s principle of least action would require for $\nu = 2$.

Under the assumptions that were made here, and for $\nu = 2$, one will also get the principle of the conservation of *vis viva*:

$$\text{const.} = H = \sum p_l\, q_l' - T - V = (n-1)\, T - V = T - V.$$

One can add two more systems of differential equations to the two systems that were exhibited above. If one subtracts equation (6), after introducing the function H in equation (7), namely:

$$D\,(T+V)=-\sum p_l'\,Dq_l+\sum p_l\,Dq_l'+\sum q_l'\,Dp_l+\sum q_l\,Dp_l',$$

from the identity equation:

$$D\frac{d}{dt}\sum p_l\,q_l=\sum p_l'\,Dq_l+\sum p_l\,Dq_l'+\sum q_l'\,Dp_l+\sum q_l\,Dp_l'$$

then that will give:

$$D\left(\frac{d}{dt}\sum p_l\,q_l-T-V\right)=H'\,Dt+\sum q_l'\,Dp_l+\sum q_l\,Dp_l',$$

so $\frac{d}{dt}\sum p_l\,q_l-T-V$ will be represented as a function of the variables $t, p_1, \ldots, p_n, p_1', \ldots, p_n'$, and its partial derivatives with respect to those variables will be equal to $H', q_1', \ldots, q_n', q_1, \ldots, q_n$, respectively.

If one subtracts the same equation (6) from the identity equation:

$$D\sum p_l'\,q_l=\sum p_l'\,Dq_l+\sum q_l\,Dp_l'$$

then that will give:

$$D\left(\sum p_l'\,q_l-T-V\right)=H'\cdot Dt+\sum q_l\,Dp_l'-\sum p_l\,Dq_l',$$

so $\sum p_l'\,q_l-T-V$ will then be represented as a function of the variables $t, p_1', \ldots, p_n', q_1', \ldots, q_n'$, and its partial derivatives with respect to those variables will be equal to:

$$H, q_1, \ldots, q_n, -p_1, \ldots, -p_n\,.$$

IV. – Substitution function. Integration. Perturbation theory.

The especially simple form of the differential equations that are presented by a mechanical problem comes from the fact that a suitable system of variables $\ldots, p_l, \ldots$ was introduced for a system of independent coordinates $\ldots, q_l, \ldots$, and indeed the original $\ldots, q_l, \ldots$ can be chosen entirely arbitrarily, so there will always be associated $\ldots, p_l \ldots$ However, systems of associated variables can also be found in an even more general way that have the property that they give that simple form to the differential equations, and for that reason, **Jacobi** gave them the name of *canonical variables*. In fact, the equation:

$$D\,(T+V)=\frac{d}{dt}\left\{\left(T+V-\sum p_l\frac{dq_l}{dt}\right)Dt+\sum p_l\,Dq_l\right\},$$

which includes all of the remaining ones, shows that if φ and ψ are to define a new system of independent canonical variables, instead of the p and q, then it would only be necessary for the function $T + V$ to either be the same function, but expressed in terms of t, φ, ψ after replacing the p and q with the φ and ψ in that equation, or set equal to a new function. We can give that new function the form $T + V - S$, in which S remains to be determined more precisely, and we will then get:

$$D\,(T + V - S') = \frac{d}{dt}\{(T + V - S' - \sum \varphi_l \frac{d\psi_l}{dt}) Dt + \sum \varphi_l\, D\psi_l\}\,,$$

and after subtracting that equation from the foregoing, we will get:

$$DS' = \frac{d}{dt}\{(S' + \sum \varphi_l \frac{d\psi_l}{dt} - \sum p_l \frac{dq_l}{dt}) Dt - \sum \varphi_l\, D\psi_l + \sum p_l\, Dq_l\}\,.$$

Should that equation for the substitution of canonical variables φ_l, ψ_l for the p_l, q_l be true in general (that is, independently of the special equations for a certain mechanical problem), then since the one side is a complete derivative with respect to time t, the other one DS', and therefore S', must also be so. There must then be a function S that fulfills the equations:

$$\frac{dS}{dt} = S',$$

$$DS = \left(\frac{dS}{dt} + \sum \varphi_l \frac{D\psi_l}{dt} - \sum p_l \frac{Dq_l}{dt}\right) Dt - \sum \varphi_l\, D\psi_l + \sum p_l\, Dq_l\,, \tag{9}$$

$$DS = -E\, Dt + \sum p_l\, Dq_l - \sum \varphi_l\, D\psi_l\,,$$

in which one sets:

$$E = \sum p_l \frac{dq_l}{dt} - \sum \varphi_l \frac{d\psi_l}{dt} - \frac{dS}{dt}\,. \tag{10}$$

Conversely, if equation (9) is satisfied for arbitrary functions S and E then the variables ψ and φ that were introduced will be a canonical system, since equation (10) will follow from (9) as a special case of D differentiation, and the fundamental equation that was exhibited above for …, ψ_l, …, φ_l, … will arise from both of the fundamental equations (6) for …, q_l, …, p_l, …, which can also be presented in the form:

$$D\,(T + V - S') = \frac{d}{dt}(T + V - S' - \sum \varphi_l\, \psi_l') \cdot Dt + \sum \varphi_l'\, D\psi_l + \sum \varphi_l\, D\psi_l'$$

or

$$D(T + V - S' - \sum \varphi_l\, \psi_l') = \frac{d}{dt}(T + V - S' - \sum \varphi_l\, \psi_l') \cdot Dt + \sum \varphi_l'\, D\psi_l - \sum \psi_l'\, D\varphi_l$$

$$= -D(H-E) = (H'-E')\,Dt - \sum \psi_l'\,D\varphi_l + \sum \varphi_l'\,D\psi_l\,. \tag{11}$$

It follows from the first of those two equations that when $T + V - S'$ is regarded as a function of t, ψ_l', ψ_l, its partial derivatives with respect to those quantities will be equal to $-H' + E'$, φ_l, φ_l'. If one considers $T + V - S' - \sum \varphi_l\,\psi_l'$ or $-H + E$ to be a function of t, φ_l, ψ_l, and denotes the partial derivatives with respect to those variables by ϑ then one will obtain from the second (11) of those two equations that:

$$\begin{aligned} \frac{\vartheta(H-E)}{\vartheta\varphi_l} &= \psi_l' = \frac{d\psi_l}{dt}, \\ -\frac{\vartheta(H-E)}{\vartheta\psi_l} &= \varphi_l' = \frac{d\varphi_l}{dt}, \\ \frac{\vartheta(H-E)}{\vartheta t} &= H' - E' = \frac{d(H-E)}{dt}. \end{aligned} \tag{12}$$

The fundamental equation of motion, the equation of substitution, and the equation of motion that is transformed in that way agree in form in such a way that the general relations that exist between the quantities $q_1, \ldots, q_n$, $p_1, \ldots, p_n$, $\psi_1, \ldots, \psi_n$, $\varphi_1, \ldots, \varphi_n$ alone, and which will be developed more thoroughly in the following articles, will also exist between the quantities $q_1, \ldots, q_n$, $p_1', \ldots, p_n'$, $q_1', \ldots, q_n'$, $-p_1, \ldots, -p_n$, and likewise between $q_1, \ldots, q_n$, $p_1', \ldots, p_n'$, $p_1, \ldots, p_n$, $q_1', \ldots, q_n'$, and furthermore between $\psi_1, \ldots, \psi_n$, $\varphi_1', \ldots, \varphi_n'$, $\varphi_1, \ldots, \varphi_n$, $\psi_1', \ldots, \psi_n'$, and so forth.

The general equation of substitution includes the special case in which the quantities ψ, like the q, have the meaning of coordinates in such a form that S, as well as S', will be zero, and that furthermore the quantities q will be given as functions of t and the ψ, and indeed in such a way that they will be independent of how the ψ are represented as functions of t and q, and that finally, the quantities E and φ are determined by the equation of substitution.

Another very general, and especially important, type of substitution is the one for which the relations between the two systems of variables can be represented in such a way that the p will become functions of the quantities t, q, ψ. All remaining quantities can also be determined by the latter then when one substitutes the expressions that are obtained for p_l. Due to the importance of that kind of representation of the various variables, we would like to introduce a special symbol for the partial derivatives with respect to t, q, ψ, namely, δ, since that differentiation will include the variation that was considered above as a special case. The general equation of substitution will then give:

$$\frac{\delta S}{\delta q_l} = p_l\,, \qquad \frac{\delta S}{\delta \psi_l} = -\varphi_l\,, \qquad \frac{\delta S}{\delta t} = -E = \frac{dS}{dt} - \sum \varphi_l \frac{d\psi_l}{dt} + \sum p_l \frac{dq_l}{dt}\,,$$

and it is clear from this how *when the* $p_1, \ldots, p_n$ *are given as functions of* $t, q_1, \ldots, q_n, \psi_1, \ldots, \psi_n$ *in such a way that they can be the partial derivatives of one and the same function with respect to* $q_1, \ldots, q_n$, *the remaining variables can then be determined as a canonical system of variables* $\psi_1, \ldots, \psi_n, \varphi_1, \ldots, \varphi_n$.

If $H - E$ is independent of one or more, or even all, of the quantities ψ and φ under such a substitution then it will follow from equations (12) for the partial derivatives of $H - E$ that the quantities that correspond to $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n$ and are provided with the same index will be integration constants in each case. If $H - E$ were zero or also only independent of $\psi_1, \ldots, \psi_n, \varphi_1, \ldots, \varphi_n$, then the latter would all be integration constants and define a complete system of integrals of the differential equations:

$$\frac{\partial H}{\partial p_l} = \frac{dq_l}{dt},$$

$$-\frac{\partial H}{\partial q_l} = \frac{dp_l}{dt}.$$

The problem of integrating these equations completely can also be expressed in such a form that the quantities $-H, p_1, \ldots, p_n$ are represented as functions of $t, q_1, \ldots, q_n$, and a number of quantities $\psi_1, \ldots, \psi_n$ that is equal to the number of q such that they can be the partial derivatives of a single function, and indeed the partial derivatives with respect to $t, q_1, \ldots, q_n$, respectively. The multi-termed quadrature:

$$\int \left(\sum p_l \, Dq_l - H \, Dt\right),$$

whose lower limits are absolute constants or depend upon the quantities ψ, which are regarded as constant for only the integration, will then yield a substitution function S whose partial derivatives with respect to ψ, together with the ψ, will define a complete system of integrals of the given differential equations.

A special form of that solution consists of representing the quantities p as functions of the q and an equal number of quantities ψ such that they can be the partial derivatives of a common function, as before, and at the same time, H can reduce to a function of t and ψ alone, so the multi-term quadrature:

$$\int \left(\sum p_l \, Dq_l - H \, Dt\right)$$

will then give the same sort of substitution function that it did before.

The problem can also be expressed in the form that **Hamilton** and **Jacobi** employed: In the given equation:

$$H = \text{funct.}\,(t, q_1, \ldots, q_l, \ldots, q_n, p_1, \ldots, p_l, \ldots, p_n),$$

one substitutes:

$$-H = \frac{\delta W}{\delta t}, \qquad p_l = \frac{\delta W}{\delta q_l}$$

and converts it into a partial differential equation:

$$0 = \frac{\delta W}{\delta t} + \text{funct.}\left(t, q_1, \ldots, q_n, \frac{\delta W}{\delta q_1}, \ldots, \frac{\delta W}{\delta q_n}\right),$$

whose general integral W is a function of the quantities t, q_1, …, q_n that depends upon one additive constant and m other integration constants ψ_1, …, ψ_n. That function W will then be a substitution function like S, and the remaining integrals of the equations of motion will come about when one sets $\delta W / \delta\psi_l$ = const.

In the study that is being carried out here, the force function V can depend upon the quantities $\frac{dq_1}{dt}, \ldots, \frac{dq_n}{dt}$ in an arbitrary way, and therefore so can $T + V$, as well as $H = -\frac{\delta W}{\delta t}$ can depend upon $p_l = \frac{\delta W}{\delta q_l}$ in an arbitrary way, so the following general developments will be directly applicable to any first-order partial differential equation when one further observes that, following **Jacobi**, one can reduce a differential equation that includes not only the independent variables and the partial derivatives of the desired function W^*, but also the function W^* itself, to a differential equation that includes the function W without differentiations by the substitution:

$$W = \tau\, W^*, \qquad \text{so} \qquad W^* = \frac{\delta W}{\delta \tau}, \qquad \frac{\delta W^*}{\delta t} = \frac{1}{\tau}\frac{\delta W}{\delta t}, \qquad \frac{\delta W^*}{\delta q_l} = \frac{1}{\tau}\frac{\delta W}{\delta q_l}.$$

If the principle of conservation of *vis viva* is valid then H will be a constant, and the partial differential equation:

$$0 = -H + \text{funct.}\left(q_1, \ldots, q_n, \frac{\delta W}{\delta q_1}, \ldots, \frac{\delta W}{\delta q_n}\right)$$

will give a general integral that takes the form of a function W that depends upon an additive constant and $n - 1$ other constants ψ_1, …, ψ_{n-1}, and:

$$W - H \cdot t$$

will be a substitution function S, in which H appears in place of ψ or a function of ψ_1, …, ψ_{n-1}, ψ_n.

The first form of the problem that was given here, which coincides with the complete integration of $2n$ equations:

$$\frac{\partial H}{\partial p_l} = \frac{dq_l}{dt},$$

$$-\frac{\partial H}{\partial q_l} = \frac{dp_l}{dt},$$

then includes the entirely-special case, which is nonetheless capable of many applications, in which each of the quantities p_k is, in a sense, to be determined as a function of the q_k (if that is even possible) that are equipped with the same index, and a system of n quantities $\psi_1, \ldots, \psi_n$, in such a way that the function H will become independent of the q when one substitutes those expressions for the p. The integrals in:

$$\int p_1\,Dq_1 + \cdots + \int p_n\,Dq_n - \int H\,Dt = S$$

will then become simple quadratures for unvarying $\psi_1, \ldots, \psi_n$, and when one takes the integrals of the functions of the ψ with fixed limits, S will become a substitution function, and $\psi_1, \ldots, \psi_n$, $\frac{\delta S}{\delta \psi_1}, \ldots, \frac{\delta S}{\delta \psi_n}$ will define a complete system of integration constants for the given differential equations.

In that form, one can determine the motion of a free mass-particle, which can be inferred directly from **Newton**'s laws for one or two fixed mass-particles, or also one that is constrained to remain on an ellipsoidal surface without the action of forces when one introduces ellipsoidal coordinates as independent variables, as **Jacobi** did.

The **Hamilton-Jacobi** form of perturbation theory is obtained from the canonical substitution in the following way: If H denotes the **Hamiltonian** function (7) for the completely-mechanical problem (8^*), so when one includes the so-called perturbing forces, while E is the **Hamiltonian** function for the motion that would arise if the perturbing forces were not present, and furthermore $\psi_1, \ldots, \psi_n, \varphi_1, \ldots, \varphi_n$ are the canonical integrals for the latter problem, so for the $2n$ equations:

$$\frac{\partial E}{\partial p_l} = \frac{\vartheta q_l}{\vartheta t},$$

$$-\frac{\partial E}{\partial q_l} = \frac{\vartheta p_l}{\vartheta t},$$

and finally, if:

$$S = \int\left(\sum p_l \frac{\vartheta q_l}{\vartheta t} - E\right)dt$$

is the associated **Hamiltonian** integral, so:

$$DS = -E\,Dt + \sum p_l\,Dq_l - \sum \varphi_l\,D\psi_l,$$

then, as in equations (12), the elements ψ and φ that are altered by the perturbing forces will be determined by means of the $2n$ differential equations:

$$\frac{\vartheta(H-E)}{\vartheta\varphi_l} = \frac{d\psi_l}{dt},$$

$$-\frac{\vartheta(H-E)}{\vartheta\psi_l} = \frac{d\varphi_l}{dt},$$

in which $H - E$ is thought of as representing a function of $t, \psi_1, \ldots, \psi_n, \varphi_1, \ldots, \varphi_n$.

V. – Forces whose measure depends upon motion.

In the year 1835, **Gauss** (as would emerge from his handwritten notes that were published in volume five of his complete works, which I edited) was the first to think of determining forces that would depend upon not only the mutual position of the interacting bodies, but also upon the motion itself. His investigations, which were directed along those lines on many occasions, had the goal of explaining forces such as the ones that appear in the phenomena associated with galvanic currents. Under the assumption that the interactions between the galvanic currents and its carrier would be such that every force that acts upon the current would be transmitted to the carrier, and that furthermore the two forces that act in opposite directions upon two different types of electrical particle at the same place would provoke galvanic currents whose intensity is just as large throughout the entire linear current conductor and is proportional to the sum of the two forces, in my prize essay "Zur mathematischen Theorie electrischer Ströme" in the year 1857, I was the first to prove rigorously how the electrodynamical and electromotive laws that were discovered by **Ampère**, **Faraday**, **Lenz**, and **Franz Neumann** could be explained by the sort of forces that **Gauss** examined. Unfortunately, **Gauss**'s handwritten notes were still not available to me at that time, since otherwise I would have been spared some investigations, although a proof of the lemma of the coincidence of the potential for the interaction between galvanic currents with the potential for the interaction between magnetic surfaces, which I gave in that essay, was still not found in **Gauss**, but only the proof of the coincidence between the force components that were parallel to the coordinate axes (**Gauss**'s *Werke*, Bd. V, pp. 624).

The very incisive investigations that were most recently carried out by **Helmholtz** into the nature of electrodynamical forces have shown that when one does not determine the interaction between the electrical bodies and their carriers completely (which is what has been done up to now), the assumption that there are forces that depend upon the motion must lead to phenomena that contradict our conception of the nature of the forces that provoke the motions.

At this point, I would like to determine the forces that depend upon the motion only in regard to the fact that their analytical treatment agrees with the treatment of the forces that depend upon the mutual positions of the bodies that act upon each other as much as possible. The principle of action and reaction is directly applicable then. The principles of the conservation of the motion of the center of mass and the conservation of the areal velocity will then persist when the force between two mass-particles is proportional to the mass, its direction lies along the connecting line between the two masses or its extension, and the magnitude of the force depends upon only the distance between the two masses, moreover, so when the distance between two mass-particles with intensities ε', ε'' is r, the sum of the virtual moments of the two reciprocal forces that are exerted upon the mass-particles will be represented by:

$$\varepsilon' \varepsilon'' F \left(r, \frac{dr}{dt}, \frac{ddr}{dt^2}, \frac{d^2 r}{dt}, \dots \right) \delta r \,.$$

In the derivation of the equation of motion above, I showed that the simplicity of its form was essentially based upon the fact that the virtual moments of the forces can be represented as the sum of a total variation of a function and the total derivative with respect to time of a sum of functions, multiplied by the variation of the coordinates. If that simple form for the equation of motion for the forces that are considered here is to remain valid then:

$$\varepsilon' \varepsilon'' F \left(r, \frac{dr}{dt}, \frac{ddr}{dt^2}, \frac{d^2 r}{dt}, \dots \right) \delta r$$

$$= \delta V \left(r, \frac{dr}{dt}, \frac{ddr}{dt^2}, \frac{d^2 r}{dt}, \dots \right) + \frac{d}{dt} \{ V_1 \left(r, \frac{dr}{dt}, \frac{ddr}{dt^2}, \frac{d^2 r}{dt}, \dots \right) \} \delta r$$

$$= \frac{\partial V}{\partial r} \delta r + \frac{\partial V}{\partial \frac{dr}{dt}} \cdot \delta \frac{dr}{dt} + \frac{\partial V}{\partial \frac{ddr}{dt^2}} \cdot \delta \frac{ddr}{dt^2} + \dots + \frac{\partial V_1}{\partial r} \frac{dr}{dt} \cdot \delta r + \frac{\partial V_1}{\partial \frac{dr}{dt^2}} \frac{ddr}{dt^2} \cdot \delta r + \dots + V_1 \frac{d \delta r}{dt} + \dots$$

can be an identity, and therefore:

$$\varepsilon' \varepsilon'' F \cdot \delta r = \frac{\partial V}{\partial r} \delta r + \frac{\partial V_1}{\partial t} \frac{dr}{dt} \delta r + \frac{\partial V_1}{\partial \frac{dr}{dt}} \frac{ddr}{dt^2} \delta r + \dots,$$

$$0 = \frac{\partial V}{\partial \frac{dr}{dt}} \delta \frac{dr}{dt} + V_1 \delta \frac{dr}{dt} \,,$$

$$0 = \frac{\partial V}{\partial \frac{ddr}{dt}} \delta \frac{ddr}{dt^2} \,,$$

$$0 = \frac{\partial V}{\partial \frac{d^2 r}{dt^2}} \delta \frac{d^2 r}{dt^2},$$

so one has:

$$V = \text{function } (r, dr / dt),$$

$$V_1 = -\frac{\partial V}{\partial \frac{dr}{dt}},$$

$$\varepsilon' \varepsilon'' F\left(r, \frac{dr}{dt}, \frac{ddr}{dt^2}\right) = \frac{\partial V}{\partial r} - \frac{\partial}{\partial r}\left(\frac{\partial V}{\partial \frac{dr}{dt}}\right) \cdot \frac{dr}{dt} - \frac{\partial}{\partial \frac{dr}{dt}}\left(\frac{\partial V}{\partial \frac{dr}{dt}}\right) \cdot \frac{ddr}{dt^2},$$

$$\varepsilon' \varepsilon'' F\left(r, \frac{dr}{dt}, \frac{ddr}{dt^2}\right) \delta r = \frac{\partial V}{\partial r} - \frac{d}{dt}\left\{\frac{\partial V}{\partial \frac{dr}{dt}} \delta r\right\}.$$

If, for example:

$$V = V_0 + \sum_n V_n \cdot \left(\frac{dr}{dt}\right)^n,$$

in which V_0 and V_n are independent of dr / dt, then one will have:

$$\varepsilon' \varepsilon'' F\left(r, \frac{dr}{dt}, \frac{ddr}{dt^2}\right) \delta r = \delta \{V_0 + \sum_n V_n \cdot \left(\frac{dr}{dt}\right)^n\} - \frac{d}{dt}\left\{\sum n V_n \left(\frac{dr}{dt}\right)^{n-1} \cdot \delta r\right\},$$

$$\varepsilon' \varepsilon'' F\left(r, \frac{dr}{dt}, \frac{ddr}{dt^2}\right) = \frac{\partial V_0}{\partial r} - \sum_n (n-1) \frac{\partial V_n}{\partial r}\left(\frac{dr}{dt}\right)^n - \sum_n n(n-1) V_n \left(\frac{dr}{dt}\right)^n \frac{ddr}{dt^2}\},$$

and for $n = 2$ and constant values of $r\,V_0$ and $r\,V_n$, that will imply the law that **W. Weber** published in the year 1852.

VI. – Two free mass particles.

In order to keep in mind the complete determination of the motion under the action of forces that depend upon the motion of the bodies, I would like to work through two simply-soluble problems using the special method that was given in Art. IV, and first consider two mass-particles that move in an ν-fold extended flat space.

If:

$m, x_1, \ldots, x_\nu$ are the inertial mass and rectangular rectilinear coordinates of a mass-point, and

$M, X_1, \ldots, X_\nu$ are the corresponding things for the other point

then the distance r between the two points will satisfy:

$$rr = \sum_{\lambda=1}^{\nu} (x_\lambda - X_\lambda)^2 ,$$

and by assumption, the force function V depends upon only m, M, r, and dr / dt. The total *vis viva* will be:

$$2T = m \sum_{\lambda=1}^{\nu} x'_\lambda x'_\lambda + M \sum_{\lambda=1}^{\nu} X'_\lambda X'_\lambda .$$

If we set:

$$m + M = L^{-2}, \quad \frac{m+M}{m\,M} = N^2,$$

to abbreviate, and introduce the quantities $q_1, \ldots, q_{2n}$ by the equations:

$$m\,x_1 = m\,L\,q_{\nu+1} + \frac{1}{N} q_1 \cos q_2 ,$$

$$m\,x_\lambda = m\,L\,q_{\nu+\lambda} + \frac{1}{N} q_1 \sin q_2 \cos q_3 \ldots \sin q_\lambda \cos q_{\lambda+1} \qquad \text{for } 1 < \lambda < \nu$$

$$m\,x_\nu = m\,L\,q_{2\nu} + \frac{1}{N} q_1 \sin q_2 \cos q_3 \ldots \sin q_{\nu-1} \sin q_\nu ,$$

$$M\,X_1 = M\,L\,q_{\nu+1} - \frac{1}{N} q_1 \cos q_2 ,$$

$$M\,X_\lambda = M\,L\,q_{\nu+\lambda} - \frac{1}{N} q_1 \sin q_2 \cos q_3 \ldots \sin q_\lambda \cos q_{\lambda+1} \qquad \text{for } 1 < \lambda < \nu,$$

$$M\,X_\nu = M\,L\,q_{2\nu} - \frac{1}{N} q_1 \sin q_2 \cos q_3 \ldots \sin q_{\nu-1} \sin q_\nu ,$$

then we will have $r = N\,q_1$, and the total *vis viva* will be:

$$2T = q_1' q_1' + \sum_{\lambda=2}^{\nu} \left(q_1 \sin q_2 \cdots \sin q_{\lambda-1} q_\lambda' \right)^2 + \sum_{\nu=\nu+1}^{2\nu} q_\mu' q_\mu' ,$$

so

$$p_1 = \frac{\vartheta(T+V)}{\vartheta q_1'} = q_1' + \frac{\vartheta V}{\vartheta q_1'} ,$$

$$p_\lambda = \frac{\vartheta(T+V)}{\vartheta q_\lambda'} = (q_1 \sin q_2, \ldots, \sin q_{\lambda-1})^2 q_\lambda' \qquad \text{for } 1 < \lambda \leq \nu,$$

$$p_\mu = \frac{\vartheta(T+V)}{\vartheta q_\mu'} = q_\mu' \qquad \text{for } \nu + 1 \leq \mu \leq 2\nu,$$

and therefore:

$$H = \sum_{l=1}^{2\nu} p_l q_l' - T - V = T - V + q_1' \frac{\vartheta V}{\vartheta q_1'} ,$$

$$= - V + q_1' \frac{\vartheta V}{\vartheta q_1'} + \tfrac{1}{2} q_1' q_1' + \tfrac{1}{2} \sum_{\lambda=2}^{\nu} (q_1 \sin q_2 \ldots \sin q_{\lambda-1})^{-2} p_\lambda p_\lambda + \tfrac{1}{2} \sum_{\mu=\nu+1}^{2\nu} p_\mu p_\mu .$$

If we set:

$$\tfrac{1}{2} p_\mu p_\mu = \psi_\mu \qquad \text{for } \nu \leq \mu \leq 2\nu$$

$$\tfrac{1}{2} p_\lambda p_\lambda + \psi_{\lambda+1} \csc q_\lambda^2 = \psi_\lambda \qquad \text{for } 1 \leq \lambda \leq \nu$$

$$- V + q_1' \frac{\vartheta V}{\vartheta q_1'} + \tfrac{1}{2} q_1' q_1' + \frac{\psi_2}{q_1 q_1} = \psi_1 ,$$

in analogy with the **Jacobi** process, then:

$$H = \psi_1 + \sum_{\mu=\nu+1}^{2\nu} \psi_\mu ,$$

and when we represent the quantity p_1 in the equation:

$$S = - \psi_1 t - \sum_{\mu=\nu+1}^{2\nu} \psi_\mu t + \int p_1 dq_1 + \sum_{\lambda=2}^{\nu-1} \int (2\psi_\lambda - 2\psi_{\lambda+1} \csc q_\lambda^2)^{1/2} dq_\lambda + \sum_{\mu=\nu}^{2\nu} q^\mu \sqrt{2\psi_\mu}$$

as a function of q_1 and the ψ with the help of the defining equation for ψ_1, all of the integrals in that equation will become quadratures whose upper limits are once more q_1, q_λ for constant ψ.

The **Hamiltonian** function H can then be represented in terms of the mutually-independent quantities ψ alone, so the differential expression $\sum p_l \, Dq_l$ will then become a complete differential for unvarying ψ by that substitution, and the functions that are determined by the equations above and are set to:

$$\psi_l = \text{const.}, \qquad \frac{\delta S}{\delta \psi_l'} = -\,\varphi_l = \text{const.}$$

for all indices $l = 1, 2, 3, \ldots, 2n$ will be the $4n$ integral equations by which one determines the motion of the free, mutually-interacting, mass-particles m and M in ν-fold extended space according to the law of the force function V.

For the special case in which the force function has the simple form:

$$V = V_0 + V_1 \frac{dr}{dt} + V_2 \frac{dr^2}{dt^2},$$

and V_0, V_1, V_2 are functions of only r, one will have:

$$p_1 = NV_1 + (2 + N\,N\,V^2)^{1/2} \left(V_0 - \frac{\psi_2}{q_1\,q_1} + \psi_1 \right)^{1/2}.$$

VII. – Two mass particles in multiply-extended Gaussian and Riemannian spaces.

If one finds that a mass-particle is fixed at the coordinate origin and the radius vector r is drawn from that point to the moving point, the shortest lines are drawn from its midpoint to the ν mutually-rectangular coordinate axes that are composed of shortest lines, and one measures out the segments $\xi_1, \xi_2, \ldots, \xi_\nu$ along those axes from the coordinate origin in well-defined directions, measured positively, then from my investigations into the multiply-extended Gaussian and Riemannian spaces in the Nachrichten von der Königlichen Gesellschaft der Wissenschaftern zu Göttingen 1873 January, no. 2, Lehrsatz IV, one will have:

$$\sin \tfrac{1}{2} i\, r^2 = \frac{\sum \tan i\, \xi_\mu^2}{1 + \sum \tan i\, \xi_\mu^2},$$

and the square of the element of length will be equal to:

$$\frac{4}{ii} \frac{\sum (d \tan i\, \xi_\mu)^2}{(1 + \sum \tan i\, \xi_\mu^2)^2},$$

namely, when the summations are extended over $\mu = 1, 2, 3, \ldots, \nu$, and i means the reciprocal value of the absolute unit of length for a Riemannian or homogeneous finite space, while it means the reciprocal value of the absolute unit of length, multiplied by $\sqrt{-1}$ for Gaussian or infinite space.

If one now sets:

$$\tan i\,\xi_1 = \tan \tfrac{1}{2} i\,q_1 \cos q_2 \,,$$

$$\tan i\,\xi_2 = \tan \tfrac{1}{2} i\,q_1 \sin q_2 \cos q_3 \,,$$

$$\tan i\,\xi_\mu = \tan \tfrac{1}{2} i\,q_1 \sin q_2 \sin q_3 \ldots \sin q_\mu \cos q_{\mu+1} \quad \text{for } \mu < \nu,$$

$$\tan i\,\xi_{\nu-1} = \tan \tfrac{1}{2} i\,q_1 \sin q_2 \sin q_3 \ldots \sin q_{\nu-1} \cos q_\nu \,,$$

$$\tan i\,\xi_\nu = \tan \tfrac{1}{2} i\,q_1 \sin q_2 \sin q_3 \ldots \sin q_{\nu-1} \sin q_\nu$$

then one will have:

$$\sum_{\mu=1}^{\nu} \tan i\,\xi_\mu^2 = \tan \tfrac{1}{2} i\,q_1^2 \,, \qquad q_1 = r,$$

and when one assumes that the mass of the moving particles is unity, the *vis viva* will be equal to:

$$\begin{aligned} 2T = {} & q_1'\,q_1' + \frac{1}{i\,i}\sin i\,q_1^2\; q_2'\,q_2' + \frac{1}{i\,i}\sin i\,q_1^2 \cdot \sin q_2^2 \cdot q_2'\,q_2' \\ & + \frac{1}{i\,i}\sin i\,q_1^2 \cdot \sin q_2^2 \cdot \sin q_3^2 \cdot \sin q_{\mu-1}^2 \cdot q_\mu'\,q_\mu' \\ & + \frac{1}{i\,i}\sin i\,q_1^2 \cdot \sin q_2^2 \cdot \sin q_3^2 \cdot \sin q_{\nu-1}^2 \cdot q_\nu'\,q_\nu' \,, \end{aligned}$$

so:

$$p_1 = \frac{\vartheta(T+V)}{\vartheta q_1'} = \frac{\vartheta V(q_1, q_1')}{\vartheta q_1'} + q_1' \,,$$

$$p_2 = \frac{\vartheta(T+V)}{\vartheta q_2'} = \frac{1}{ii}\sin i\,q_1^2 \cdot q_2' \,,$$

$$p_\mu = \frac{\vartheta(T+V)}{\vartheta q_\mu'} = \frac{1}{ii}\sin i\,q_1^2 \cdot \sin q_2^2 \cdot \sin q_3^2 \cdots \sin q_{\mu-1}^2 \cdot q_\mu' \,, \qquad \text{for } 1 < \mu \leq \nu,$$

and therefore:

$$H = \sum_{l=1}^{n} p_l\, q_l' - T - V = T - V + q_1' \frac{\vartheta V}{\vartheta q_1'}$$

$$= - V + q_1' \frac{\vartheta V}{\vartheta q_1'} + \tfrac{1}{2} q_1' q_1' + \sum_{\mu=2}^{\nu} \tfrac{1}{2} ii \csc i\, q_1^2 \cdot \csc i\, q_2^2 \cdot \csc i\, q_3^2 \cdots \csc i\, q_{\mu-1}^2 \cdot p_\mu\; p_\mu\,.$$

The substitution:

$$\tfrac{1}{2} p_\nu\, p_\nu = \psi_\nu\,,$$

$$\tfrac{1}{2} p_{\nu-1}\, p_{\nu-1} + \psi_\nu \csc\; q_{\nu-1}^{\;2} = \psi_{\nu-1}\,,$$

$$\tfrac{1}{2} p_\lambda\, p_\lambda + \psi_{\lambda+1} \csc\; q_\lambda^{\;2} = \psi_\lambda \qquad \text{for } 1 < \lambda < \nu,$$

$$- V + q_1' \frac{\vartheta V}{\vartheta q_1'} + \tfrac{1}{2} q_1' q_1' + ii\; \psi_2 \csc\, i\, q_1^2 = \psi_1$$

yields:

$$H = \psi_1\,,$$

and for constant ψ:

$$DS = - H\, Dt + \sum_{l=1}^{n} p_l\, Dq_l\,.$$

The substitution function is:

$$S = -\psi_1\, t + \int p_1\, Dq_1 + \sum_{\mu=2}^{\nu-1} \int (2\psi_\mu - 2\psi_{\mu+1} \csc q_\mu^2)^{1/2} dq_\mu + q_\nu \sqrt{2\psi_\nu}\,,$$

since p_1 will be a function of q_1 and the quantities ψ alone, when one consults the equation for ψ_1. The upper limits of the integrals are q_μ.

The motion of a free mass-particle in a homogeneous ν-fold extended space when a force-function $V(r, dr/dt)$ acts according to a fixed law is then determined completely by the equations:

$$\psi = \text{const.}, \qquad \frac{\delta S}{\delta \psi_l} = -\varphi_l = \text{const.},$$

in which l means the indices 1, 2, 3, …, ν in succession.

VIII. – General differential equations for the substitutions.

In the theory of general perturbations, the perturbation formulas that **Lagrange** and **Poisson** found assume an important position. They relate to the variations of those quantities – viz., the so-called "elements" – that would be integration constants for the unperturbed motion. As **Jacobi** pointed out, they take on especially simple values for the canonical integration constants that **Hamilton** employed.

Those relations, along with the new equations that **Hamilton** and **Jacobi** added to them, are obtained very simply from the substitution equation (9) that was given above:

$$DS = \sum p_l \, Dq_l - \sum \varphi_l \, D\psi_l - E \, Dt \, .$$

If one differentiates this using a general differentiation Δ, which is nonetheless independent of the *D* differentiation, then that will give:

$$\Delta DS = \sum p_l \, \Delta Dq_l - \sum \varphi_l \, \Delta D\psi_l - E \, \Delta Dt + \sum \Delta p_l \, Dq_l - \sum \Delta\varphi_l \, D\psi_l - \Delta E \, Dt \, .$$

However, if one imagines that the general differential Δ was used in the first equation then:

$$\Delta S = \sum p_l \, \Delta q_l - \sum \varphi_l \, \Delta \psi_l - E \, \Delta t \, ,$$

and if one then differentiates by *D* then that will imply:

$$D\Delta S = \sum p_l \, \Delta Dq_l - \sum \varphi_l \, D\Delta \psi_l - E \, \Delta Dt + \sum Dp_l \, \Delta q_l - \sum D\varphi_l \, \Delta \psi_l - DE \, \Delta t \, .$$

The two differentiations *D* and Δ are independent of each other, so the sequence in which they are performed will have no influence on the value, and when one subtracts the two second-order differential equations from each other, one will get the equation:

$$\sum (Dq_l \, \Delta p_l - \Delta q_l \, Dp_l) = \sum (D\psi_l \Delta\varphi_l - \Delta \psi_l \, D\varphi_l) + Dt \cdot \Delta E - \Delta t \cdot DE \, , \tag{13}$$

or, when one calls the expression $Dq_l \, \Delta p_l - \Delta q_l \, Dp_l$ a *differential determinant* of the function-pair q_l and p_l, one can express that in words:

If the $q_1, \ldots, q_n$ and $p_1, \ldots, p_n$ define a system of canonical variables then in order for the quantities $\psi_1, \ldots, \psi_n$ and $\varphi_1, \ldots, \varphi_n$ that are introduced by the substitution equations to also define a system of canonical variables, in general, it is necessary and sufficient that the sums of the general two-parameter differential determinants of all associated pairs q_l and p_l should differ from the sums of the ψ_n and φ_l that are formed in the same way by only the two-parameter differential determinant of the variables t and any function E.

That theorem will also be true when one restricts the concept of general differentiation in such a way that the time *t* remains unchanged. The two sums of the differential determinants will be equal to each other, and there will always be a function E that fulfills the conditions for that complete lemma.

We will prove that the differential equation (13) is also sufficient for the quantities φ and ψ to stay a system of canonical variables by distinguishing six cases:

1. The p and q are given as functions of the q, ψ, and t. Then let:

$$\chi_l = p_l, \quad \chi_{n+l} = -\varphi_l, \quad \chi_{2n+1} = \chi_m = -E,$$
$$x_l = q_l, \quad x_{n+l} = \psi_l, \quad x_{2n+1} = x_m = t.$$

2. If the q and φ are given as functions of the p, ψ, and t then let:

$$\chi_l = -q_l, \quad \chi_{n+l} = -\varphi_l, \quad \chi_{2n+1} = \chi_m = -E,$$
$$x_l = p_l, \quad x_{n+l} = \psi_l, \quad x_{2n+1} = x_m = t.$$

3. If the p and ψ are given as functions of the q, φ, and t then let:

$$\chi_l = p_l, \quad \chi_{n+l} = \psi_l, \quad \chi_{2n+1} = \chi_m = -E,$$
$$x_l = q_l, \quad x_{n+l} = \varphi_l, \quad x_{2n+1} = x_m = t.$$

4. If the q and ψ are given as functions of the p, φ, and t then let:

$$\chi_l = -q_l, \quad \chi_{n+l} = \psi_l, \quad \chi_{2n+1} = \chi_m = -E,$$
$$x_l = p_l, \quad x_{n+l} = \varphi_l, \quad x_{2n+1} = x_m = t.$$

5. If the q and p are given as functions of the ψ, φ, and t then let:

$$\chi_l = \sum_h p_h \frac{\vartheta q_h}{\vartheta \psi_l} - q_l, \quad \chi_{n+l} = \sum_h p_h \frac{\vartheta q_h}{\vartheta \varphi_l}, \quad \chi_{2n+1} = \chi_m = \sum_h p_h \frac{\vartheta q_h}{\vartheta t} - E,$$
$$x_l = p_l, \quad x_{n+l} = \varphi_l, \quad x_{2n+1} = x_m = t.$$

6. If the ψ and φ are given as functions of the q, p, and t then let:

$$\chi_l = q_l - \sum_h \varphi_h \frac{\vartheta \psi_h}{\vartheta q_l}, \quad \chi_{n+l} = -\sum_h \varphi_h \frac{\vartheta \psi_h}{\vartheta p_l}, \quad \chi_{2n+1} = \chi_m = -\sum_h \varphi_h \frac{\vartheta \psi_h}{\vartheta t} - E,$$
$$x_l = p_l, \quad x_{n+l} = \varphi_l, \quad x_{2n+1} = x_m = t.$$

In all cases, the condition equation (13) goes to one of the form:

$$\sum_{k=l}^{m} (Dx_k \, \Delta\chi_k - \Delta x_k \, D\chi_k) = 0,$$

so if one now takes the $\xi_1, \ldots, \xi_m$ to be any functions of the x that do not make the expression $\xi_1 \chi_1 + \xi_2 \chi_2 + \ldots + \xi_m \chi_m$ vanish, and one imagines that the equations:

$$\frac{dy}{\sum_{k=1}^{m} \xi_k \chi_k} = \frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \ldots = \frac{dx_m}{\xi_m}$$

are integrated completely then m integration constants $y_1, y_2, \ldots, y_m$ will appear in that way, one of which – say, y_m – is coupled with y by addition, and the variables x can be considered to be functions of the quantities $y, y_1, y_2, \ldots, y_m$. That will then imply:

$$\chi_1 \frac{\partial x_1}{\partial y} + \chi_2 \frac{\partial x_2}{\partial y} + \cdots + \chi_m \frac{\partial x_m}{\partial y} = \chi_1 \frac{\partial x_1}{\partial y_m} + \chi_2 \frac{\partial x_2}{\partial y_m} + \cdots + \chi_m \frac{\partial x_m}{\partial y_m} = 1,$$

so for a general differentiation D:

$$\chi_1\, Dx_1 + \chi_2\, Dx_2 + \ldots + \chi\, Dx_m = D\,(y + y_m) + Y_1\, Dy_2 + \ldots + Y_{m-1}\, Dy_{m-1}\,,$$

in which $Y_1, \ldots, Y_{m-1}$ are functions of $y, y_1, y_2, \ldots, y_m$ that must fulfill the equation:

$$\sum_{k=l}^{m-1} (Dy_k\, \Delta Y_k - \Delta y_k\, DY_k) = 0$$

between the χ and x. In the special case in which all quantities y are constant for the D differentiation, except for y_l, where $1 \leq l \leq m - 1$, and all quantities y are constant for the Δ differentiations, with the exception of y, in one case, and then y_m, the equation will become:

$$Dy_l \cdot \frac{\partial Y_l}{\partial y} \Delta y = 0 \qquad\qquad Dy_l \cdot \frac{\partial Y_l}{\partial y_m} \Delta y_m = 0,$$

so for every index l between 1 and m – 1, Y_l will be independent of y and y_m. Therefore:

$$Y_1\, Dy_1 + Y_2\, Dy_2 + \ldots + Y_{m-1}\, Dy_{m-1}$$

will be a differential expression with only m – 1 independent variables, and the coefficients $Y_1, \ldots, Y_{m-1}$, along with their independent variables $y_1, \ldots, y_{m-1}$, will satisfy the corresponding condition as the coefficients $\chi_1, \ldots, \chi_m$ in the linear expression with the m independent variables x. The differential expression with m – 1 terms can then be decomposed once more by the same process into a differential and a linear differential expression with m – 2 independent variables, and with a corresponding condition. By carrying out that process, one will then arrive at a representation of the linear expression as the differential of a single function:

$$\chi_1\, Dx_1 + \chi_2\, Dx_2 + \ldots + \chi_m\, Dy_m = Dw.$$

If we denote the functions that arise each time in the six cases that were distinguished above by $w_1, w_2, \ldots, w_6$, respectively, in the application of that theorem to our investigations then we will have:

$$\sum p_l\, Dq_l - \sum \varphi_l\, D\psi_l - E\, DT = Dw_1\,,$$

$$-\sum q_l\, Dp_l - \sum \varphi_l\, D\psi_l - E\, DT = Dw_2\,,$$

$$\sum p_l\, Dq_l + \sum \psi_l\, D\varphi_l - E\, DT = Dw_3\,,$$

$$-\sum q_l\, Dp_l + \sum \psi_l\, D\varphi_l - E\, DT = Dw_4\,,$$

$$\sum_l \left(\sum_h p_h \frac{\partial q_h}{\partial \psi_l} - \varphi_l \right) D\psi_l + \sum_l \sum_h p_l \frac{\partial q_h}{\partial \varphi_l} D\varphi_l + \left(\sum_h p_h \frac{\partial q_h}{\partial t} - E \right) Dt = Dw_5\,,$$

$$\sum_l \left(p_l - \sum_h \varphi_l \frac{\partial \psi_h}{\partial q_l} \right) D q_l - \sum_l \sum_h \varphi_h \frac{\partial \psi_h}{\partial p_l} D\, p_l - \left(\sum_h \varphi_h \frac{\partial \psi_h}{\partial t} + E \right) Dt = Dw_6\,,$$

or with the assistance of the identity equations:

$$D\sum p_l\, q_l = \sum p_l\, Dq_l + \sum q_l\, Dp_l\,, \qquad D\sum \varphi_l\, \psi_l = \sum \varphi_l\, D\psi_l + \sum \psi_l\, D\varphi_l\,,$$

when one adds them together, the partial differentials:

$$\sum p_l\, Dq_l - \sum \varphi_l\, D\psi_l - E\, dt = Dw_1 = D\left(w_2 + \sum p_l\, q_l\right) = D\left(w_3 - \sum \varphi_l\, \psi_l\right)$$
$$= D\left(w_4 + \sum p_l\, q_l - \sum \varphi_l\, \psi_l\right) = Dw_5 = Dw_5$$

will exist in all cases, so one will have a substitution function S which couples the two systems of variables (namely, the q, p and the ψ), φ in such a way that when the one is a canonical system, the other one will also be so.

If one restricts the definitions of the general differentials D and Δ in such a way that one leaves time t unchanged then for a canonical substitution:

$$\sum (Dp_l\, \Delta p_l - \Delta q_l\, Dp_l) = \sum (D\psi_l\, \Delta\varphi_l - \Delta\psi_l\, D\varphi_l)\,, \qquad Dt = 0, \qquad \Delta t = 0.$$

That form satisfies the condition equation that would make the substitution a canonical one. In fact, if one assumes that $Dt = 0$, $\Delta t = 0$ in the foregoing proof then that will yield the result that there exist functions w_1 , w_2 , …, w_6 that satisfy the equations that were found before under the assumption that $Dt = 0$ and will then be completely independent of E. If one then sets:

$$E = -\frac{\delta w_1}{\delta t}, \qquad E = -\frac{\partial_2 w_2}{\partial_2 t}, \qquad E = -\frac{\partial_3 w_3}{\partial_3 t}, \qquad E = -\frac{\partial_4 w_4}{\partial_4 t},$$

$$E = -\frac{\vartheta w_5}{\vartheta t} + \sum p_h \frac{\vartheta q_h}{\vartheta t}, \qquad E = -\frac{\partial w_6}{\partial t} - \sum \varphi_h \frac{\partial \psi_h}{\partial t},$$

in which the partial differentiations δ, ∂_2 , ∂_3 , ∂_4 , ϑ, ∂ refer to those systems of variables that are considered to be mutually independent and by which the remaining quantities in each of the six cases are represented as functions, then S will be determined in the same way as before.

IX. – Jacobi's perturbation formulas.

The general differential equation (13):

$$\sum (Dq_l\, \Delta p_l - \Delta q_l\, Dp_l) = \sum (D\psi_l\, \Delta \varphi_l - \Delta \psi_l\, D\varphi_l) + Dt\, \Delta E - \Delta t\, DE,$$

when one performs the differentiations in the special senses:

$$\text{all } Dq = 0, \qquad Dp_l = 0 \qquad \text{for} \qquad l \neq h, \quad Dt = 0,$$

$$\Delta \psi_l = 0 \qquad \text{for } l \neq h, \qquad \text{all} \qquad \Delta \psi_l = 0, \Delta t = 0,$$

will become:

$$-\frac{\vartheta q_h}{\vartheta \psi_k} \Delta \psi_k \cdot Dp_h = -\Delta \psi_k \frac{\vartheta \varphi_k}{\vartheta p_h} Dp_h,$$

so one will have:

$$\frac{\vartheta q_h}{\vartheta \psi_k} = \frac{\vartheta \varphi_k}{\vartheta p_h}.$$

If one imposes various special assumptions on the differentiations in such a way that one assumes that all of the quantities p, q, t, except one, are unvarying under the D differentiations, and likewise all of the ψ, φ, t, except one, are unvarying under the Δ differentiations, then one will get the new system of equations:

$$\frac{\vartheta q_h}{\vartheta \psi_k} = \frac{\partial \varphi_k}{\partial p_h}, \qquad \frac{\vartheta q_h}{\vartheta \varphi_k} = -\frac{\partial \psi_k}{\partial p_h}, \qquad \frac{\vartheta q_h}{\vartheta t} = \frac{\partial E}{\partial p_h},$$

$$\frac{\vartheta p_h}{\vartheta \psi_k} = -\frac{\partial \varphi_k}{\partial q_h}, \qquad \frac{\vartheta p_h}{\vartheta \varphi_k} = \frac{\partial \psi_k}{\partial q_h}, \qquad \frac{\vartheta p_h}{\vartheta t} = -\frac{\partial E}{\partial q_h}, \tag{14}$$

$$\frac{\vartheta E}{\vartheta \psi_k} = \frac{\partial \varphi_k}{\partial t}, \qquad \frac{\vartheta E}{\vartheta \varphi_k} = -\frac{\partial \psi_k}{\partial t}, \qquad \frac{\vartheta E}{\vartheta t} = \frac{\partial E}{\partial t}$$

that **Jacobi** exhibited, which are valid for all indices h and k. In order to give a common form to those various systems, we would like to introduce the notations:

$$q_{-\nu} = p_\nu, \qquad q_{+0} = E, \qquad q_{-0} = t,$$

$$\psi_{-\nu} = \varphi_\nu, \qquad \psi_{+0} = E, \qquad \psi_{-0} = t,$$

$$[h] = +1 \quad \text{for } h \geq \pm 0, \qquad [h] = -1 \quad \text{for } h < 0,$$

so the common form will become:

$$[h]\frac{\vartheta q_h}{\vartheta \psi_{-k}} = [-k]\frac{\vartheta \psi_k}{\vartheta q_{-h}}, \qquad h = +0, \pm 1, \pm 2, \ldots, \pm n, \; k = +0, \pm 1, \pm 2, \ldots, \pm n. \tag{14*}$$

Conversely, one also has the theorem that when the **Jacobi** equations are fulfilled, that substitution of the quantities q, p with the ψ, φ will be canonical, because when one performs the summation over the stated values of h and k, one will have:

$$\sum_h \sum_k \{[h]\frac{\vartheta q_h}{\vartheta \psi_{-k}} - [-k]\frac{\vartheta \psi_k}{\vartheta q_{-h}}\}\Delta q_{-h}\, D\psi_{-k} = \sum_h [h]\, Dq_h\, \Delta q_{-h} - \sum_k [-k]\, \Delta \psi_k\, D\psi_{-k}$$

identically, so the two sides of this equation will be zero, with which the differential equation (13), which is true for the canonical substitution in general, will arise when one reintroduces the original notations.

If the function E is not given then one needs to assume only that $Dt = 0 = \Delta t$ in the development that was just carried out. The differential equation that then arises will not contain the function E, and it can be determined in the way that was done in article VIII.

X. – Poisson's perturbation formulas.

If q, p can be represented as function of ψ, φ, t, and conversely, ψ, φ can also be represented as functions of q, p, t, and Φ denotes a function of the $4n + 1$ quantities q, p, ψ, φ, t, and Ψ is a function of Φ, then one will have:

$$\frac{\vartheta\Psi}{\vartheta\Phi} = \frac{\partial\Psi}{\partial\Phi} + \sum \frac{\partial\Psi}{\partial q_l}\frac{\vartheta q_l}{\vartheta\Phi} + \sum \frac{\partial\Psi}{\partial p_l}\frac{\vartheta p_l}{\vartheta\Phi},$$

$$\frac{\partial\Psi}{\partial\Phi} = \frac{\vartheta\Psi}{\vartheta\Phi} + \sum \frac{\vartheta\Psi}{\vartheta\psi_l}\frac{\partial\psi_l}{\partial\Phi} + \sum \frac{\vartheta\Psi}{\vartheta\varphi_l}\frac{\partial\varphi_l}{\partial\Phi}$$

identically when the summations are extended over the indices $l = 1, 2, 3, \dots, n$. If one takes the Ψ and Φ in these equations to be any two of the quantities φ, ψ, t, in succession, and replaces the $\frac{\vartheta q_l}{\vartheta\Phi}$ and $\frac{\vartheta p_l}{\vartheta\Phi}$ with the analogous derivatives then one will get the following conditions for a canonical substitution:

$$\begin{aligned}
&\sum_l \left(\frac{\partial\psi_h}{\partial q_l}\frac{\partial\psi_k}{\partial p_l} - \frac{\partial\psi_h}{\partial p_l}\frac{\partial\psi_k}{\partial q_l}\right) = 0,\\
&\sum_l \left(\frac{\partial\psi_h}{\partial q_l}\frac{\partial\varphi_k}{\partial p_l} - \frac{\partial\psi_h}{\partial p_l}\frac{\partial\varphi_k}{\partial q_l}\right) = \begin{cases} 0 & h \neq k, \\ 1 & h = k, \end{cases},\\
&\sum_l \left(\frac{\partial\varphi_h}{\partial q_l}\frac{\partial\varphi_k}{\partial p_l} - \frac{\partial\varphi_h}{\partial p_l}\frac{\partial\varphi_k}{\partial q_l}\right) = 0,\\
&\sum_l \left(\frac{\partial E}{\partial q_l}\frac{\partial\psi_k}{\partial p_l} - \frac{\partial E}{\partial p_l}\frac{\partial\psi_k}{\partial q_l}\right) = \frac{\partial\psi_h}{\partial t},\\
&\sum_l \left(\frac{\partial E}{\partial q_l}\frac{\partial\varphi_k}{\partial p_l} - \frac{\partial E}{\partial p_l}\frac{\partial\varphi_k}{\partial q_l}\right) = \frac{\partial\varphi_h}{\partial t}
\end{aligned} \tag{15}$$

for $l = 1, 2, 3, \dots, n$.

If we employ the same notations as in the previous article and use $\left(\frac{\partial\psi_h}{\partial\psi_{-k}}\right)$ and $\left(\frac{\partial\psi_h}{\partial q_\lambda}\right)$ to mean that:

$$\left(\frac{\partial\psi_h}{\partial\psi_{-k}}\right) = 1 \qquad \text{for } h = -k,$$

$$\left(\frac{\partial\psi_h}{\partial\psi_{-k}}\right) = \frac{\partial\psi_h}{\partial t} \qquad \text{for } h = +0,$$

but in all other cases:

$$\left(\frac{\partial\psi_h}{\partial\psi_{-k}}\right) = 0,$$

and that:

$$\left(\frac{\partial\psi_h}{\partial q_\lambda}\right) = 1 \qquad \text{for } h = \lambda = +0,$$

but in all other cases:

$$\left(\frac{\partial\psi_h}{\partial q_\lambda}\right) = \frac{\partial\psi_h}{\partial q_\lambda},$$

and we set $[\lambda] = +1$ for a positive λ, while $[\lambda] = -1$ for a negative value of λ, and $[+0] = [-0] = +1$, then we can give the five systems of equations above the common form:

$$[k]\left(\frac{\partial\psi_h}{\partial\psi_{-k}}\right) - \sum_{\lambda=+0}^{\mp n}[-\lambda]\left(\frac{\partial\psi_h}{\partial q_\lambda}\right)\left(\frac{\partial\psi_k}{\partial q_{-\lambda}}\right) = 0, \tag{15*}$$

and on the other hand, it will follow that this equation remains valid for all systems of values ± 0, ± 1, ± 2, …, $\pm n$ of the h and k, with the exception of $h = -k = -0$.

Poisson was the first to exhibit differential expressions of the type that appear above in the summations in (15) that relate to l in his "Mémoire sur la variation des constantes arbitraires dans les questions de Mécanique," 16 October 1809, Journal de l'École polytechnique, Cah. 15.

If one excludes the system of values $h = +0$ and $h = -k = -0$ then the term for $l = +0$ will always vanish in the summation, and equation (15*) will assume the simpler form:

$$[k]\left(\frac{\partial\psi_h}{\partial\psi_{-k}}\right) + \sum_{\lambda=\pm 1}^{\mp n}[\lambda]\frac{\partial\psi_h}{\partial q_\lambda}\frac{\partial\psi_k}{\partial q_{-\lambda}} = 0.$$

If equations (15) or (15*) are fulfilled then conversely the substitution will be a canonical one, because when one lets:

$$\left(\frac{\vartheta q_\nu}{\vartheta\psi_h}\right) = 1 \quad \text{for } \nu = h = +0,$$

but:

$$\left(\frac{\vartheta q_\nu}{\vartheta\psi_h}\right) = \frac{\vartheta q_\nu}{\vartheta\psi_h}$$

for all other systems of values for ν and h in the expression:

$$\sum_{h,k,\nu}[k][\nu]\left(\frac{\vartheta q_\nu}{\vartheta\psi_{+k}}\right)Dq_{-\nu}\,D\psi_{-k}\cdot\left\{[k]\left(\frac{\partial\psi_{+h}}{\partial\psi_{-k}}\right) - \sum_{\lambda=+0}^{\pm n}[-\lambda]\left(\frac{\partial\psi_h}{\partial q_\lambda}\right)\left(\frac{\partial\psi_k}{\partial q_{-\lambda}}\right)\right\},$$

then if one deals with the individual cases in which the bracketed terms have a different sense from the derivatives separately and then performs the summations over h for the values $\pm 0, \pm 1, \pm 2, \dots, \pm n$, and the summations over λ, ν, k for the values $\pm 0, \pm 1, \pm 2, \dots, \pm n$, except for the combination $h = -k = -0$, then that will imply:

$$- \sum_{\nu} [\nu] \Delta q_{\nu} \, D q_{-\nu} + \sum_{k} [-k] \, D\psi_k \, \Delta \psi_{-k} .$$

That expression must then become zero and in that way, once more imply the differential equation (13) that is true for a canonical substitution. If the function E is not known then one needs only to set $Dt = \Delta t = 0$ in that development and exclude the indices ± 0, and the equations that include E will not enter into the calculations then, and that function will first be determined from the substitution S that was calculated before in article VIII.

XI. – Lagrange's perturbation formulas.

If one takes the differentiations D and Δ in the general differential equation (13) for the canonical substitution to have the special meaning that any two of the quantities $\psi_1, \dots, \psi_n, \varphi_1, \dots, \varphi_n$, and t vary independently, but the remaining ones can be considered to be unvarying, then one will get:

$$\sum_l \left(\frac{\vartheta q_l}{\vartheta \psi_h} \frac{\vartheta p_l}{\vartheta \psi_k} - \frac{\vartheta p_l}{\vartheta \psi_h} \frac{\vartheta q_l}{\vartheta \psi_k} \right) = 0,$$

$$\sum_l \left(\frac{\vartheta q_l}{\vartheta \psi_h} \frac{\vartheta p_l}{\vartheta \varphi_k} - \frac{\vartheta p_l}{\vartheta \psi_h} \frac{\vartheta q_l}{\vartheta \varphi_k} \right) = \begin{cases} 0 & h \neq k \\ 1 & h = k \end{cases},$$

$$\sum_l \left(\frac{\vartheta q_l}{\vartheta \varphi_h} \frac{\vartheta p_l}{\vartheta \varphi_k} - \frac{\vartheta p_l}{\vartheta \varphi_h} \frac{\vartheta q_l}{\vartheta \varphi_k} \right) = 0, \tag{16}$$

$$\sum_l \left(\frac{\vartheta q_l}{\vartheta t} \frac{\vartheta p_l}{\vartheta \psi_h} - \frac{\vartheta p_l}{\vartheta t} \frac{\vartheta q_l}{\vartheta \psi_h} \right) = \frac{\vartheta E}{\vartheta \psi_h},$$

$$\sum_l \left(\frac{\vartheta q_l}{\vartheta t} \frac{\vartheta p_l}{\vartheta \varphi_h} - \frac{\vartheta p_l}{\vartheta t} \frac{\vartheta q_l}{\vartheta \varphi_h} \right) = \frac{\vartheta E}{\vartheta \psi_h}.$$

Conversely, those five systems of equations characterize that substitution as a canonical one, because when one multiplies those equations by:

$$D\psi_h \, \Delta\psi_k ,$$

$$\begin{aligned}&D\psi_h\,\Delta\varphi_k - \Delta\psi_h\,D\varphi_k\,,\\ -&D\psi_h\,\Delta\varphi_k\,,\\ &Dt\,\Delta\psi_h - \Delta t\,D\psi_h\,,\\ &Dt\,\Delta\varphi_h - \Delta t\,D\varphi_h\,,\end{aligned}$$

respectively, and then sums over all indices, adds the equations obtained together and assembles the sums of partial differentials, one will again get the general differential equation (13) that is true for the canonical substitution.

The first three systems also satisfy the equations (16) that would make the substitution canonical, as one will find when one assumes that $Dt = 0 = \Delta t$ in the foregoing investigation and determines the functions S and E as in article VIII.

If one applies the general differential equation (13) to the case in which $\psi_1, \ldots, \psi_n, \varphi_1, \ldots, \varphi_n$ are integration constants and represents them in terms of functions of any other $2n$ integration constants $c_1, c_2, \ldots, c_{2n}$, and then takes the differentiations D and Δ to mean that only c_μ varies for D and only c_ν varies for Δ, while the remaining c and t remain unchanged, then when one multiplies both sides of the general differential equation (13) by the product of $Dc_\mu\,\Delta c_\nu$ with a function of the integration constants, one will get **Lagrange**'s theorem:

$$\sum_l \left(\frac{dq_l}{dc_\mu}\frac{dp_l}{dc_\nu} - \frac{dp_l}{dc_\mu}\frac{dq_l}{dc_\nu}\right) = \text{const.}$$

XII. – Hamilton's perturbation formulas.

If the quantities p and φ can be represented as functions of the q, ψ, and t then one can take:

$$\sum (Dq_l\,\Delta p_l - \Delta q_l\,Dp_l) = \sum (D\psi_l\,\Delta\varphi_l - \Delta\psi_l\,D\varphi_l) + Dt\,\Delta E - \Delta t\,DE,$$

$$\begin{array}{lllll} Dq_l = 0 & \text{for } l \neq h, & \text{all} & D\psi = 0, & Dt = 0,\\ \Delta q_l = 0 & \text{for } l \neq k, & \text{all} & \Delta\psi = 0, & \Delta t = 0\end{array}$$

in the general equations, which will make:

$$Dq_h \cdot \frac{\delta p_h}{\delta q_k}\Delta q_k - \Delta q_h \cdot \frac{\delta p_k}{\delta q_h} Dq_k = 0,$$

so that will imply:

$$\frac{\delta p_h}{\delta q_k} = \frac{\delta p_k}{\delta q_h},$$

if the partial derivatives with respect to the variables q, ψ, and t are again denoted by δ.

If one sets:

$$Dq_l = 0 \quad \text{for } l \neq h, \quad \text{all} \quad D\psi = 0, \quad Dt = 0,$$
$$\Delta\psi_l = 0 \quad \text{for } l \neq k, \quad \text{all} \quad \Delta q = 0, \quad \Delta t = 0$$

then equation (13) will go to:

$$Dq_h \cdot \frac{\delta p_h}{\delta \psi_k} \Delta \psi_k = -\Delta \psi_k \cdot \frac{\delta \varphi_k}{\delta q_h} Dq_k ,$$

so:

$$\frac{\delta p_h}{\delta \psi_k} = -\frac{\delta \varphi_k}{\delta q_h} .$$

If one sets:

$$Dq_l = 0 \quad \text{for } l \neq h, \quad \text{all} \quad D\psi = 0, \quad Dt = 0,$$
$$\text{all} \quad \Delta q = 0 \quad \text{all} \quad \Delta\psi = 0$$

then the general equation will imply that:

$$Dq_h \cdot \frac{\delta p_h}{\delta t} \Delta t = -\Delta t \cdot \frac{\delta E}{\delta q_h} Dq_h ,$$

so

$$\frac{\delta p_h}{\delta t} = -\frac{\delta E}{\delta q_h} .$$

If one carries out the examination of all permissible special assumptions of that kind for the D and Δ then one will get the five systems of equation that Hamilton presented under special assumptions:

$$\begin{gathered}\frac{\delta p_h}{\delta q_k} = \frac{\delta p_k}{\delta q_h}, \quad \frac{\delta p_h}{\delta \psi_k} = -\frac{\delta \varphi_k}{\delta q_h}, \quad \frac{\delta \varphi_h}{\delta \psi_k} = \frac{\delta \varphi_k}{\delta \psi_h}, \\ \frac{\delta p_h}{\delta t} = -\frac{\delta E}{\delta q_h}, \quad \frac{\delta \varphi_h}{\delta t} = \frac{\delta E}{\delta \psi_h},\end{gathered} \tag{17}$$

which are true for all indices of h and k.

However, if conversely those equations are satisfied for an arbitrary function E then, as before, it will follow that the assumed representation of the q and p as functions of the ψ, φ, and t will then define a canonical substitution, so those equations will be the known condition equations for the existence of a function S whose partial derivatives with respect to $q_1, \ldots, q_n$, $\psi_1, \ldots, \psi_n$, and t are equal to $p_1, \ldots, p_n$, $-\varphi_1, \ldots, -\varphi_n$, and $-E$.

If we set:

$$Q_\nu = q, \qquad Q_{-\nu} = \psi_\nu , \qquad Q_0 = t,$$

$$P_\nu = p_\nu , \qquad P_{-\nu} = \varphi_\nu , \qquad P_0 = E$$

for a positive ν then the five systems of **Hamilton** equation can be written in the common form:

$$[-h]\frac{\delta P_h}{\delta Q_k} = [-k]\frac{\delta P_k}{\delta Q_h} \qquad \text{for } h \text{ and } k \text{ equal to } 0, \pm 1, \pm 1, \ldots, \pm n . \tag{17*}$$

If we multiply the two sides of that equation by DQ_k and ΔQ_h and sum over all values of h and k then we will get:

$$\sum [-h]\, DP_h\, \Delta Q_h = \sum [-k]\, DQ_k\, \Delta P_k ,$$

which is once more the general differential equation for a canonical substitution.

The five systems of equations above are complete, in the sense that arbitrarily many of the functions $p_1, \ldots, p_n, \varphi_1, \ldots, \varphi_n, E$ that are expressed in terms of $q_1, \ldots, q_n, \psi_1, \ldots, \psi_n$, and t can be given such that only the equations between those given functions that are valid for that system are fulfilled, and the remaining functions can then be determined in such a way that they collectively define a canonical substitution.

In fact, in the last equation, one needs only to assume that those Dq_h and Δq_h, $D\psi_\lambda$ and $\Delta\psi_\lambda$ are equal to zero for which the respective p_h and φ_λ that are provided with the same index are not given. Likewise, Dt and Δt are set equal to zero when E is not given, so the p_h and φ_λ, and perhaps E, as well, that are not given will not enter into that equation, and for just the given ones:

$$p_1, p_2, \ldots, p_m, \varphi_1, \ldots, \varphi_\mu, \qquad \text{and possible } E,$$

one will get the equation:

$$0 = \sum_{l=1}^{m} (Dq_l\, \Delta p_l - \Delta q_l Dp_l) - \sum_{\lambda=1}^{\mu} (D\psi_\lambda\, \Delta\varphi_\lambda - \Delta\psi_\lambda D\varphi_\lambda) + Dt \cdot \Delta E - \Delta t \cdot DE,$$

and from article VIII, no. 1, that is the condition for the expression:

$$\sum_{l=1}^{m} p_l\, Dq_l - \sum_{\lambda=1}^{\mu} \varphi_\lambda D\psi_\lambda - E\, Dt$$

for constant $q_{m+1}, \ldots, q_n$, $\psi_{\mu+1}, \ldots, y_n$ to be the complete differential DS^* of a function S^* whose partial derivatives are:

$$\frac{\delta S^*}{\delta q_{m+1}} = p_{m+1}, \ldots, \frac{\delta S^*}{\delta q_n} = p_n, \quad \frac{\delta S^*}{\delta \psi_{\mu+1}} = -\varphi_{\mu+1}, \ldots, \quad \frac{\delta S^*}{\delta \psi_n} = -\varphi_n ,$$

and to set:

$$\frac{\delta S^*}{\delta t} = -E$$

when E is not given.

XIII. – New differential equations for the canonical substitution.

Three different systems of independent variables come under consideration in the **Jacobi** and **Hamilton** differential equations: first of all, the quantities q, p, t, then ψ, φ, t, and finally the q, ψ, t ; we have denoted the three different corresponding differentiations by ∂, ϑ, and δ. Now, even more groupings of the independent variables are required for many investigations.

If we set:

$$\begin{array}{llll} p_\nu = q_{-\nu}\,, & E = \psi_{+0} & \text{or} & E = q_{+0}\,, \\ \varphi_\nu = \psi_{-\nu}\,, & t = \psi_{-0} & \text{or} & t = q_{-0}\,, \end{array}$$

for ease of understanding, then we would like to imagine choosing $2n$ of the quantities $q_{\pm 1}, \ldots, q_{\pm n}$, $\psi_{\pm 1}, \ldots, \psi_{\pm n}$, and one of the q_{-0} , ψ_{-0} as a system of $2n + 1$ independent variables, and denote them with:

$$q_{h_1}, \ldots, q_{h_\nu}, \psi_{k_1}, \ldots, \psi_{k_\mu},$$

while their partial derivatives are denoted by $\mathfrak{d}$, such that one will then have:

$$\frac{\partial P}{\partial q_l} = \frac{\mathfrak{d} P}{\mathfrak{d} q_l} + \sum_k \frac{\mathfrak{d} P}{\mathfrak{d} \psi_k} \frac{\partial \psi_k}{\partial q_l},$$

$$\frac{\vartheta P}{\vartheta \psi_l} = \frac{\mathfrak{d} P}{\mathfrak{d} \psi_l} + \sum_h \frac{\mathfrak{d} P}{\mathfrak{d} q_h} \frac{\partial q_h}{\partial \psi_l},$$

$$\frac{\mathfrak{d} P}{\mathfrak{d} q_l} = \sum_l \frac{\vartheta P}{\vartheta \psi_l} \frac{\partial \psi_l}{\partial q_h},$$

$$\frac{\mathfrak{d} P}{\mathfrak{d} \psi_l} = \sum_l \frac{\partial P}{\partial q_l} \frac{\mathfrak{d} q_l}{\mathfrak{d} \psi_k}$$

identically for every function P, in which the summations over h and k are extended over all q_h and y_k that appear as independent variables, and the summation over l is extended over all values $-0, \pm 1, \pm 2, \ldots, \pm n$.

From the first of the two formulas, the equation:

$$\sum_h \sum_k \frac{\eth\Phi}{\eth q_h} \frac{\eth P}{\eth \psi_k} \left([-k] \frac{\vartheta q_h}{\vartheta \psi_{-k}} - [h] \frac{\vartheta \psi_k}{\vartheta \psi_{-h}} \right) = 0,$$

which follows immediately from the **Jacobi** equations (14), art. IX, will go to:

$$\sum_h [h] \frac{\eth\Phi}{\eth q_h} \left(\frac{\eth P}{\eth q_{-h}} - \frac{\eth P}{\eth q_{-h}} \right) + \sum_k [-k] \frac{\eth P}{\eth \psi_k} \left(\frac{\vartheta\Phi}{\vartheta \psi_{-k}} - \frac{\eth\Phi}{\eth \psi_{-k}} \right) = 0. \tag{18}$$

That equation includes **Jacobi**'s equation as a special case when one takes the $\eth$ differentiation to mean that the independent variables are, among others, e.g., q_l and ψ_λ, but not q_{-l} and $\psi_{-\lambda}$, and that one then sets $\Phi = q_l$, $P = \psi_\lambda$. Equation (18) goes to the second **Hamilton** equation (17) when one refers the $\eth$ differentiation to the independent variables $q_1, \ldots, q_n$, $\psi_1, \ldots, \psi_n$, t, and sets $P = p_l$, $\Phi = \varphi_\lambda$. With the help of the equation that is obtained in that way, equation (18) above also implies the first **Hamilton** equation when one sets $P = p_l$, $\Phi = p_\lambda$, as well as the third when one sets $P = \varphi_l$, $\Phi = \varphi_\lambda$, and one can also derive the fourth equation directly when one refers the $\eth$ differentiation to the quantities E, $\psi_1, \ldots, \psi_n$, $q_1, \ldots, q_n$ as the independent variables, and one sets:

$$\Phi = t, \quad P = p_l = q_{-\lambda}, \quad \psi_0 = E, \qquad \psi_{-0} = t$$

in equation (18) above; one would then have:

$$0 = -\frac{\eth t}{\eth q_l} + \frac{\eth p_l}{\eth E} = \frac{\eth t}{\eth E} \frac{\delta E}{\delta q_l} + \frac{\delta p_l}{\delta t} \frac{\eth t}{\eth E}.$$

One gets the fifth **Hamilton** equation in an analogous way.

We would not like to examine the general form for the equation in the case where E proves to be the independent variables in (18) here.

What is remarkable about the general relation (18) above is that it will also arise from the expression above for those ψ that appear to be independent under the $\eth$ differentiation and those ψ that enter into Φ when one performs the summation over l in:

$$\sum_k \sum_l \left(\frac{\vartheta\Phi}{\vartheta \psi_l} - \frac{\eth\Phi}{\eth \psi_l} \right) \frac{\eth P}{\eth \psi_k} \left\{ [-k] \left(\frac{\partial \psi_l}{\partial \psi_{-k}} \right) + \sum_h [-h] \left(\frac{\partial \psi_l}{\partial q_h} \right) \left(\frac{\partial \psi_k}{\partial q_{-h}} \right) \right\}$$

over the ψ_l that come under consideration. In regard to that, from the general equation (18) above, one can choose the quantities:

$$q_1, q_2, \ldots, q_n, \psi_1, \psi_2, \ldots, \psi_i, p_{i+1}, p_{i+2}, \ldots, p_n$$

to be independent for the ∂ differentiation, and set $P = p_l$, Φ = funct. $(\psi_1, \dots, \psi_n) = f$, and derive the equation:

$$0 = -\frac{\partial f}{\partial q_\lambda} + \sum_{h=i+1}^{n}\left(\frac{\partial f}{\partial q_h}\frac{\partial p_\lambda}{\partial p_h} - \frac{\partial f}{\partial p_h}\frac{\partial p_\lambda}{\partial q_h}\right),$$

which is valid for every $\lambda \leq i$, that **Jacobi** presented in his treatise "Nova methodus, aequationes differentiales partiales primi ordinis inter numerum variabilium quemcunque propositas integrandi," Borchardt's Journal, Bd. 60, as a special case of (18).

"Ueber einen algebraischen Typus der Bedingungen eines bewegten Massensystems," J. f. reine u. angew. Math. **66** (1866), 363-374.

On an algebraic type of condition for a system of moving masses

(By **R. Lipschitz** in Bonn)

Translated by D. H. Delphenich

1.

If a system of material of points is acted upon by a system of forces for which a force function exists and is subject to a system of condition equations that do not depend upon time then, as is known, there is always an integral into whose expression the individual nature of the constraints does not enter, namely, the *vis viva* integral. However, as far as I know, up to now it has not been noticed that one can derive a second result from the laws of motion that is independent of the nature of the constraints, as long as a certain restriction of a general sort is imposed upon them. If one calls the points of the system $P_1, P_2, \ldots, P_n$, respectively, and further lets $(B_1, B_2, \ldots, B_n)$ denote the position that the point P_α assumes at the location B_α, and denotes a certain distinguished position by $(A_1, A_2, \ldots, A_n)$, and one then denotes the coordinates of the locations B_α and A_α in a rectangular coordinate system by $x_\alpha, y_\alpha, z_\alpha$ and $a_\alpha, b_\alpha, c_\alpha$, resp., and finally defines the reigning constraints by the equations:

$$(1) \qquad \Phi_1 = 0, \qquad \Phi_2 = 0, \ \ldots, \quad \Phi_l = 0$$

then the restriction that was spoken of can be expressed by saying the $\Phi_1, \Phi_2, \ldots, \Phi_l$ should be homogeneous algebraic functions of the $3n$ coordinate differences:

$$x_\alpha - a_\alpha, \qquad y_\alpha - b_\alpha, \qquad z_\alpha - c_\alpha .$$

Now, $(B_1, B_2, \ldots, B_n)$ means the position that the mass-system will assume during its motion at time t. If one then forms the product of the mass m_α that belongs to each point P_α with the square of the distance $A_\alpha B_\alpha$ and takes the sum of those products over the n points, which might be equal to $2G$, then one will have the theorem that second differential quotient with respect to time t of the sum $2G$ will have a value that does not depend upon the nature of the individual constraints, but only on the position of the moving mass-system. I will prove that theorem, and with its help, I will present the conditions for the stability of the motion for certain problems of motion.

In regard to the general nature of a homogeneous algebraic function W of given elements, one can mention that since its degree is equal to a fraction $\mathfrak{p} / \mathfrak{q}$ whose numerator and denominator are whole numbers, its power $W^{\mathfrak{q}}$ will be a homogeneous function of degree $\mathfrak{p}$, and for that reason, it must always satisfy an algebraic equation:

$$E\, W^{\mathfrak{q}s} + E_1\, W^{\mathfrak{q}(s-1)} + \ldots + E_s = 0,$$

whose coefficients $E, E_1, \ldots, E_s$ are rational entire homogeneous functions of the elements with degrees $\mathfrak{e}$, $\mathfrak{e} + \mathfrak{p}$, …, $\mathfrak{e} + \mathfrak{p}s$, resp. Among the $\mathfrak{q}s$ roots of the equation in W, two roots will have equal values if and only if the discriminant Δ of the equation vanishes. For that reason, if one prescribes continuous changes for which the function Δ does not go through the value zero then the corresponding changes in either of the roots can ensue without becoming double-valued. Therefore, under certain assumptions, those equations whose discriminant Δ has the property that its sign does not change for any real system of elements and vanishes only when all elements vanish simultaneously will have the advantage that each of their real roots can be regarded as a real, continuous, and single-valued function of the elements for all combinations of real values of the elements. It is tacitly assumed that we shall speak of only algebraic functions with that property in what follows.

The geometric meaning of the given type of condition equations is easy to recognize: By assumption, each function Φ_β (where the index β runs from 1 to l) has the property that whenever one substitutes the expressions $a_\alpha + p\,(x_\alpha - a_\alpha)$, $b_\alpha + p\,(y_\alpha - b_\alpha)$, $c_\alpha + p\,(z_\alpha - c_\alpha)$ for the variables a_α, b_α, c_α, respectively, it will go to the expression $p^{\mathfrak{g}\beta}\,\Phi_\beta$ for each p, when the degree of Φ_β is equal to $\mathfrak{g}_\beta$. Therefore, as long as the l equations $\Phi_\beta = 0$ are fulfilled for some position $(B_1, B_2, \ldots, B_n)$, they will also be satisfied for each position $(B_1', B_2', \ldots. B_n')$ that relates to the former in such a way that the three locations A_α, B_α, B_α' of each point P_α lie along a straight line, and that the location B_α' divides the line segment $A_\alpha\, B_\alpha$ in a ratio that is the same for all n lines.

In order to prove the stated property of the function G, if the force function is called U then the differential equations for the motion of the mass-system might be written as follows:

$$(2)\qquad \left\{\begin{aligned}
m_\alpha \frac{d^2 x_\alpha}{dt^2} &= \frac{\partial U}{\partial x_\alpha} + \lambda_1 \frac{\partial \Phi_1}{\partial x_\alpha} + \cdots + \lambda_l \frac{\partial \Phi_l}{\partial x_\alpha}, \\
m_\alpha \frac{d^2 y_\alpha}{dt^2} &= \frac{\partial U}{\partial y_\alpha} + \lambda_1 \frac{\partial \Phi_1}{\partial y_\alpha} + \cdots + \lambda_l \frac{\partial \Phi_l}{\partial z_\alpha}, \\
m_\alpha \frac{d^2 z_\alpha}{dt^2} &= \frac{\partial U}{\partial z_\alpha} + \lambda_1 \frac{\partial \Phi_1}{\partial y_\alpha} + \cdots + \lambda_l \frac{\partial \Phi_l}{\partial z_\alpha},
\end{aligned}\right.$$

when one uses the *Lagrange* method, which applies l undetermined multipliers $\lambda_1, \lambda_2, \ldots, \lambda_l$. I multiply those three equations by the factors $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$, in turn, add them, and take the sum of the aggregate over the n points of the system, which will produce the following equation:

$$\text{(3)}\quad \left\{\begin{aligned} &\sum_\alpha m_\alpha \left[(x_\alpha - a_\alpha)\frac{d^2 x_\alpha}{dt^2} + (y_\alpha - b_\alpha)\frac{d^2 y_\alpha}{dt^2} + (z_\alpha - c_\alpha)\frac{d^2 z_\alpha}{dt^2} \right] \\ = &\sum_\alpha \left[(x_\alpha - a_\alpha)\frac{\partial U}{\partial x_\alpha} + (y_\alpha - b_\alpha)\frac{\partial U}{\partial y_\alpha} + (z_\alpha - c_\alpha)\frac{\partial U}{\partial z_\alpha} \right] \\ &+\sum_\alpha \sum_\beta \lambda_\beta \left[(x_\alpha - a_\alpha)\frac{\partial \Phi_\beta}{\partial x_\alpha} + (y_\alpha - b_\alpha)\frac{\partial \Phi_\beta}{\partial y_\alpha} + (z_\alpha - c_\alpha)\frac{\partial \Phi_\beta}{\partial z_\alpha} \right]. \end{aligned}\right.$$

However, the basic property of homogeneous functions yields the equation:

$$\text{(4)}\quad \sum_\alpha \left[(x_\alpha - a_\alpha)\frac{\partial \Phi_\beta}{\partial x_\alpha} + (y_\alpha - b_\alpha)\frac{\partial \Phi_\beta}{\partial y_\alpha} + (z_\alpha - c_\alpha)\frac{\partial \Phi_\beta}{\partial x_\alpha} \right] = \mathfrak{g}_\beta\, \Phi_\beta ,$$

whose right-hand side vanishes, due to equations (1). Thus, the double sum on the right-hand side of (3) will take on the value zero, and when one converts the left-hand side of that equation by means of relations that correspond to the three coordinates and take the form:

$$\frac{d^2 (x_\alpha - a_\alpha)^2}{dt^2} = 2\left(\frac{dx_\alpha}{dt}\right)^2 + 2\,(x_\alpha - a_\alpha)\frac{d^2 x_\alpha}{dt^2},$$

one will get:

$$\text{(5)}\quad \left\{\begin{aligned} &\tfrac{1}{2}\sum_\alpha m_\alpha \left[\frac{d^2 (x_\alpha - a_\alpha)^2}{dt^2} + \frac{d^2 (y_\alpha - b_\alpha)^2}{dt^2} + \frac{d^2 (z_\alpha - c_\alpha)^2}{dt^2} \right] - \sum_\alpha m_\alpha \left[\left(\frac{dx_\alpha}{dt}\right)^2 + \left(\frac{dy_\alpha}{dt}\right)^2 + \left(\frac{dz_\alpha}{dt}\right)^2 \right] \\ &\qquad = \sum_\alpha \left[(x_\alpha - a_\alpha)\frac{\partial U}{\partial x_\alpha} + (y_\alpha - b_\alpha)\frac{\partial U}{\partial y_\alpha} + (z_\alpha - c_\alpha)\frac{\partial U}{\partial z_\alpha} \right]. \end{aligned}\right.$$

The sum whose general term is the product of the mass m_α with the square of the distance $A_\alpha B_\alpha$ was previously set to:

$$\sum_\alpha m_\alpha \left[(x_\alpha - a_\alpha)^2 + (y_\alpha - b_\alpha)^2 + (z_\alpha - c_\alpha)^2 \right] = 2G,$$

but now the sum of the *vis viva* is assumed to be:

$$\sum_\alpha m_\alpha \left[\left(\frac{dx_\alpha}{dt}\right)^2 + \left(\frac{dy_\alpha}{dt}\right)^2 + \left(\frac{dz_\alpha}{dt}\right)^2 \right] = 2T,$$

and equation (5) will be converted into the following one:

(6) $$\frac{d^2G}{dt^2} - 2T = \sum_\alpha \left[(x_\alpha - a_\alpha)\frac{\partial U}{\partial x_\alpha} + (y_\alpha - b_\alpha)\frac{\partial U}{\partial y_\alpha} + (z_\alpha - c_\alpha)\frac{\partial U}{\partial z_\alpha}\right].$$

One gets the *vis viva* integral from this in the following form:

(7) $$T - U = T(0) - U(0),$$

in which $T(0)$ and $U(0)$ mean the values of the functions T and U that arise when one substitutes the values of the quantities in the expressions x_α, y_α, z_α, dx_α / dt, dy_α / dt, dz_α / dt that are given at a certain moment in time t. The elimination of the function T from equations (6) and (7) then yields the result:

(8) $$\frac{d^2G}{dt^2} = 2U + \sum_\alpha \left[(x_\alpha - a_\alpha)\frac{\partial U}{\partial x_\alpha} + (y_\alpha - b_\alpha)\frac{\partial U}{\partial y_\alpha} + (z_\alpha - c_\alpha)\frac{\partial U}{\partial z_\alpha}\right] + 2T(0) - 2U(0),$$

and according to the assertion above, that will represent the second differential quotient with respect to time of the quantity G as a pure function of the position of the moving mass-system.

Equations (6) and (8) will assume an even simpler form when the force function U, like the functions Φ_β, is a homogeneous function of the elements $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$. If one then denotes the order of the function U by $\mathfrak{k}$ and writes U instead of Φ_β and $\mathfrak{k}$ instead of $\mathfrak{g}_\beta$ in equations (4) then the foregoing relation will lead to the new equations:

(6*) $$\frac{d^2G}{dt^2} - 2T = \mathfrak{k}\, U,$$

(8*) $$\frac{d^2G}{dt^2} = (2 + \mathfrak{k})\, U + 2T(0) - 2U(0).$$

Under the assumption that the motion of the mass-system is free of conditions, the considerations that were just presented will coincide essentially with the ones that *Jacobi* developed in Bd. XVII, page 120 of this journal and on page 21 of his *Vorlesungen über Dynamik*. However, there, the points that were called $A_1, A_2, \ldots, A_n$ here were combined into one, and that point was chosen to be the origin of the rectangular coordinates. For the application that *Jacobi* gave to a system of masses that were under the influence of only a mutual attraction, one has the special circumstance that the choice of that coordinate origin is irrelevant for the problem, and for that reason the second differential quotient with respect to time t of the function G will always take on the same value for it, at which one might also fix the point $A_1, A_2, \ldots, A_n$.

2.

When one assumes that all of the Φ_β are entire rational functions of degree one of the elements $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$, but the force function U is an entire rational function of degree two of those elements, one will have the general assumption that will pertain as soon as the mass-system describes small oscillations about the position $(A_1, A_2, \ldots, A_n)$ (*). From the theory of that problem, one knows the necessary and sufficient conditions for completely arbitrary initial positions and velocities of the mass-point to never exceed certain fixed values for the distance from the mass-point to the position $(A_1, A_2, \ldots, A_n)$ or the velocity of the mass-point at any time, respectively, so that the mass-system would not be at rest in any other position besides $(A_1, A_2, \ldots, A_n)$. However, it is a far-reaching property of the algebraic type of condition equations that were characterized above that the question of the stability of the motion can be resolved just as simply as it can in the case of small oscillations when they apply, and at the same the force function U is a homogeneous algebraic function of the elements $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$ that is equal to zero whenever all elements likewise vanish and always increases to infinity whenever an element exceeds its corresponding limit. Namely, the associated answer can be summarized as follows: When all of the functions Φ_β, as well as the force function U, are homogeneous algebraic functions of the elements $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$, and the function U will be equal to zero whenever all of its elements go to the value zero and will grow to infinity whenever any element increases beyond measure such that the degree of the function is equal to a positive number, the necessary and sufficient condition for neither the distance from a mass-point to the position $(A_1, A_2, \ldots, A_n)$ nor the velocity of a mass-point at any time to exceed certain fixed limits when one is given completely arbitrary initial positions and velocities and for it to be impossible that the mass-system is at rest anywhere but the position $(A_1, A_2, \ldots, A_n)$ will then consist of saying that the force function U must take on a finite negative value for all finite, real systems of those elements that that are compatible with the conditions $\Phi_\beta = 0$, except for the system $x_\alpha - a_\alpha = 0$, $y_\alpha - b_\alpha = 0$, $z_\alpha - c_\alpha = 0$. The justification for that criterion for the stability of motion shall follow directly.

It will first be shown that when the motion of the mass-system preserves the prescribed stability character, the function U in question must have the given character. To that end, I eliminate the function U from equation (6*), which is true by assumption, and from the *vis viva* integral (7) and obtain the equation:

$$(2+\mathfrak{k})T = \frac{d^2G}{dt^2} + \mathfrak{k}[T(0) - U(0)]\,. \tag{9}$$

The two sides of it, which represent the motion at every moment as completely-determined functions of time t, can be multiplied by the elements dt and integrated from the value $t = \sigma$ to the value $t = \tau$. That gives the result:

$$(2+\mathfrak{k})\int_\sigma^\tau T\,dt = \left(\frac{dG}{dt}\right)_\sigma^\tau + \mathfrak{k}[T(0) - U(0)](\tau - \sigma), \tag{10}$$

where the function dG / dt has the meaning of:

(*) *Lagrange*, *Mécanique analytique*, Part Two, section VI.

(11) $$\frac{dG}{dt} = \sum_\alpha m_\alpha \left[(x_\alpha - a_\alpha)\frac{dx_\alpha}{dt} + (y_\alpha - b_\alpha)\frac{dy_\alpha}{dt} + (z_\alpha - c_\alpha)\frac{dz_\alpha}{dt} \right],$$

and the expression $\left(\frac{dG}{dt}\right)_\sigma^\tau$ means the difference of the two values of dG / dt that correspond to the substitutions $t = \sigma$ and $t = \tau$. Now, when none of the quantities $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$, dx_α / dt, dy_α / dt, dz_α / dt exceeds a fixed limit at any time during the motion, by assumption, the same thing will also be true for the expression dG / dt, and it is obvious that the ratio $\frac{\left(\frac{dG}{dt}\right)_\sigma^\tau}{\tau - \sigma}$ must have the value zero as a limit when the quantity τ increases beyond measure, but the quantity σ remains fixed. Equation (10) will now be divided by the expression $\mathfrak{k}\,(\tau - \sigma)$, and one can increase the value of τ beyond any number, while the value σ is fixed, and the following equation will arise in the limiting case:

(12) $$\frac{2+\mathfrak{k}}{\mathfrak{k}} \lim \frac{1}{\tau - \sigma}\int_\sigma^\tau T\,dt = T(0) - U(0)\,.$$

By its nature, the sum of the *vis vivas* T can never be equal to a negative value, but its value at any time is contained between fixed limits under the existing assumptions and can vanish continually only in the case where one continually has $x_\alpha - a_\alpha = 0$, $y_\alpha - b_\alpha = 0$, $z_\alpha - c_\alpha = 0$. For that reason, as long as the system $x_\alpha - a_\alpha = 0$, $y_\alpha - b_\alpha = 0$, $z_\alpha - c_\alpha = 0$ is not continually true, the expression $\lim \frac{1}{\tau - \sigma}\int_\sigma^\tau T\,dt$ will always have a finite positive value, and the ratio $\frac{2+\mathfrak{k}}{\mathfrak{k}}$ will always be a positive quantity in any event as a consequence of the hypothesis. If one would now like to assume that the function U assumes either a positive or vanishing or infinitely large value for a system of finite real values x_α, y_α, z_α that satisfy the equations $\Phi_\beta = 0$ and is different from the system $x_\alpha = a_\alpha$, $y_\alpha = b_\alpha$, $z_\alpha = c_\alpha$ then one would need only to choose the relevant values of x_α, y_α, z_α to be the coordinates of the corresponding point P_α for the moment in time $t = t_0$ (which is allowed by hypothesis) in order to imply a contradiction. On the given grounds, the left-hand side of equation (12) will then have a finite positive value, but when the value of $U(0)$ is negative, vanishing, or infinitely large for the system of values x_α, y_α, z_α in question, from the hypothesis, one can, in each case, assign the velocities of the mass-points that correspond to the time-point $t = t_0$, and as a result the value of $T(0)$, such that the right-hand side of equation (12) becomes negative or vanishing or infinitely large, respectively. The contradiction that emerges then necessarily proves that the function U will take on a finite negative value for all finite real systems of elements that are compatible with the equations $\Phi_\beta = 0$, except for the system $x_\alpha - a_\alpha = 0$, $y_\alpha - b_\alpha = 0$, $z_\alpha - c_\alpha = 0$.

Once that point is reached, there is no difficulty to proving that when the force function U possesses the prescribed property, equilibrium will be impossible for the system except for the position $(A_1, A_2, \ldots, A_n)$, and that the distances from the mass-points to that position, as well as their velocities, is contained within fixed limits. First of all, if it is to be possible for equilibrium to exist at a position of the system that is different from (A_1, A_2, …, A_n) then one would have T = 0, and for that reason, due to the equation (12), one would have U (0) = 0, which is contrary to assumption. The validity of the second part of the assertion is a consequence of the *vis viva* integral:

$$T - U = T\,(0) - U\,(0)\,.$$

From the assumption that was made, the function U can assume only finite negative values for all finite systems of values for x_α, y_α, z_α that are compatible with the equations $\Phi_\beta = 0$, excluding the system $x_\alpha - a_\alpha = 0$, $y_\alpha - b_\alpha = 0$, $z_\alpha - c_\alpha = 0$, but for the present system $U = 0$. Therefore, the expression $T - U$ is an aggregate of the two functions T and $- U$ that remains finite and never negative for all allowable systems of finite values x_α, y_α, z_α, dx_α / dt, dy_α / dt, dz_α / dt, and the same thing will then be true for the special value of the aggregate T (0) – U (0) . Now, by its very nature, the function T has the property of increasing when one of the quantities dx_α / dt, dy_α / dt, dz_α / dt increases beyond measure, and according to the general assumption that was made above, the function U has the property that it likewise increases without end when one of the quantities $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$ increases beyond measure. Therefore, under the assumption that prevails above, the values $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$, dx_α / dt, dy_α / dt, dz_α / dt will be coupled with certain finite limits that are not exceeded at any time by the equation $T - U = T\,(0) - U\,(0)$, and that is what was asserted. The criterion that was proposed for the stability of the motion of the problem of motion in question is thus established completely.

3.

From the viewpoint that has been reached, one can also be consider problems of motion for which the functions Φ_β, as well as the function U, are homogeneous algebraic functions of the elements $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$, and in addition the function U has the property that it is positive for all finite systems of elements that are compatible with the conditions $\Phi_\beta = 0$, it will vanish only when an element increases to infinity, and will be infinitely large if and only if all elements vanish simultaneously, but behave in such a way that the product of the function U and the function G always converges to zero when all elements decrease. The number $\mathfrak{k}$, which denotes the degree of the function U will then lie between the limits 0 and – 2, exclusive of them. Under those assumptions, it can be proved that for a finite value of the function G, the mass-system cannot remain in equilibrium, that the constant T (0) – U (0) must have a finite negative value when the functions G and T remains smaller than certain finite limits at any time in the motion (*), and that the function G cannot exceed a finite limit when that constant has a finite negative value.

The validity of the first and second part of the assertion follows from the fact that as long as the functions G and T cannot increase, equation (12) will be in force. If T is continually equal to zero then, as was also remarked above, U (0) would likewise have to be equal to zero, and that

(*) Cf., the places cited above in *Jacobi*.

would be incompatible with the assumption that the function G has a finite value. However, in the case of an actual motion, the constant $T(0) - U(0)$ will be equal to the product of a finite negative value $\frac{2+\mathfrak{k}}{\mathfrak{k}}$ with the finite positive value $\lim \frac{1}{\tau - \sigma}\int_\sigma^\tau T\,dt$. The third part of the assertion will be resolved by the *vis viva* equation of $T - U = T(0) - U(0)$, because if the function G is to exceed all limits for a negative value of the constant $T(0) - U(0)$ then, by assumption, the function U must approach zero, so the never-negative function T would be equal to the negative constant $T(0) - U(0)$, which would be impossible.

Due to the equation $T - U = T(0) - U(0)$, the function T can become infinite only when the function U likewise increases to infinity, and by assumption, that can once more happen only when all of the elements $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$ approach zero. If one then multiplies the *vis viva* equation times the function G and observes that, by assumption, the product GU will always have zero for a limit when all elements vanish then one will see that under the relationships that exist, the function T can only be infinitely large in the case when the mass-system goes to the position $(A_1, A_2, \ldots, A_n)$ at the same time and the product GT converges to zero. For that reason, the function T can never grow to infinity whenever some situation obstructs the process of the product GT converging to zero when all of the elements $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$ decrease.

One can now give a simple condition for such a situation to arise, as long as the functions Φ_β and U are subject to a further restriction. Any function V that includes the elements $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$ only in the $n(n+1)/2$ homogeneous combinations:

$$(x_\alpha - a_\alpha)(x_{\alpha'} - a_{\alpha'}) + (y_\alpha - b_\alpha)(y_{\alpha'} - b_{\alpha'}) + (z_\alpha - c_\alpha)(z_{\alpha'} - c_{\alpha'}),$$

where α and α' run through all values from 1 to n, obviously satisfies the three partial differential equations:

$$\text{(13)} \qquad \begin{cases} \sum_\alpha \left[\frac{\partial V}{\partial y_\alpha}(z_\alpha - c_\alpha) - \frac{\partial V}{\partial z_\alpha}(y_\alpha - b_\alpha)\right] = 0, \\ \sum_\alpha \left[\frac{\partial V}{\partial z_\alpha}(x_\alpha - a_\alpha) - \frac{\partial V}{\partial x_\alpha}(z_\alpha - c_\alpha)\right] = 0, \\ \sum_\alpha \left[\frac{\partial V}{\partial x_\alpha}(y_\alpha - b_\alpha) - \frac{\partial V}{\partial y_\alpha}(x_\alpha - a_\alpha)\right] = 0. \end{cases}$$

Therefore, if the functions Φ_β and U include the elements $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$ only in the $n(n+1)/2$ homogeneous combinations:

$$(x_\alpha - a_\alpha)(x_{\alpha'} - a_{\alpha'}) + (y_\alpha - b_\alpha)(y_{\alpha'} - b_{\alpha'}) + (z_\alpha - c_\alpha)(z_{\alpha'} - c_{\alpha'})$$

then they will satisfy the three partial differential equations for V, and one can derive the following equation from the differential equations of motion (2):

$$(14)\qquad \begin{cases} \sum_\alpha m_\alpha \left[\dfrac{d^2 y_\alpha}{dt^2}(z_\alpha - c_\alpha) - \dfrac{d^2 z_\alpha}{dt^2}(y_\alpha - b_\alpha)\right] = 0, \\ \sum_\alpha m_\alpha \left[\dfrac{d^2 z_\alpha}{dt^2}(x_\alpha - a_\alpha) - \dfrac{d^2 x_\alpha}{dt^2}(z_\alpha - c_\alpha)\right] = 0, \\ \sum_\alpha m_\alpha \left[\dfrac{d^2 x_\alpha}{dt^2}(y_\alpha - b_\alpha) - \dfrac{d^2 y_\alpha}{dt^2}(x_\alpha - a_\alpha)\right] = 0, \end{cases}$$

which immediately implies the three integrals:

$$(15)\qquad \begin{cases} m_\alpha \left[\dfrac{dy_\alpha}{dt}(z_\alpha - c_\alpha) - \dfrac{dz_\alpha}{dt}(y_\alpha - b_\alpha)\right] = \mathfrak{A}, \\ m_\alpha \left[\dfrac{dz_\alpha}{dt}(x_\alpha - a_\alpha) - \dfrac{dx_\alpha}{dt}(z_\alpha - c_\alpha)\right] = \mathfrak{B}, \\ m_\alpha \left[\dfrac{dx_\alpha}{dt}(y_\alpha - b_\alpha) - \dfrac{dy_\alpha}{dt}(x_\alpha - a_\alpha)\right] = \mathfrak{C}. \end{cases}$$

They will go to the integrals that are known by the name of the area theorems when the n points $A_1, A_2, \ldots, A_n$ are united into one point. I will now prove the fact that when the sum of the squares of the three new integration constants $\mathfrak{A}^2 + \mathfrak{B}^2 + \mathfrak{C}^2$ is not equal to zero, the product GT cannot vanish when all of the elements $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$ decrease, and that as a result the function T will never grow to infinity at any time under the given relationships.

For the sake of brevity, one might use the following notations:

$$(16)\qquad \begin{cases} \dfrac{dy_\alpha}{dt}(z_\alpha - c_\alpha) - \dfrac{dz_\alpha}{dt}(y_\alpha - b_\alpha) = p_\alpha, \\ \dfrac{dz_\alpha}{dt}(x_\alpha - a_\alpha) - \dfrac{dx_\alpha}{dt}(z_\alpha - c_\alpha) = q_\alpha, \\ \dfrac{dx_\alpha}{dt}(y_\alpha - b_\alpha) - \dfrac{dy_\alpha}{dt}(x_\alpha - a_\alpha) = r_\alpha, \\ (x_\alpha - a_\alpha)^2 + (y_\alpha - b_\alpha)^2 + (z_\alpha - c_\alpha)^2 = s_\alpha^2, \\ \left(\dfrac{dx_\alpha}{dt}\right)^2 + \left(\dfrac{dy_\alpha}{dt}\right)^2 + \left(\dfrac{dz_\alpha}{dt}\right)^2 = v_\alpha^2, \\ \left(\sum_\alpha m_\alpha p_\alpha\right)^2 + \left(\sum_\alpha m_\alpha q_\alpha\right)^2 + \left(\sum_\alpha m_\alpha r_\alpha\right)^2 = D^2, \end{cases}$$

which will make:

$$\sum_\alpha m_\alpha\, s_\alpha^2 = 2G, \qquad \sum_\alpha m_\alpha\, v_\alpha^2 = 2T\,.$$

Now, when the symbol α runs through the values from 1 to n, as before, while the symbols α and α' in the double summation represent only pairs of unequal numbers from the sequence from 1 to n, one will have the equation:

$$(17)\quad \left\{ \begin{aligned} 4GT - D^2 = \; & \sum_\alpha m_\alpha^2 (s_\alpha^2 v_\alpha^2 - p_\alpha^2 - q_\alpha^2 - r_\alpha^2) \\ & + \sum_\alpha \sum_{\alpha''} m_\alpha m_{\alpha'} (s_\alpha^2 v_{\alpha'}^2 + s_{\alpha'}^2 v_\alpha^2 - 2 p_\alpha p_{\alpha'} - 2 q_\alpha q_{\alpha'} - 2 r_\alpha r_{\alpha'}). \end{aligned} \right.$$

Since one generally has the inequality:

$$(p_\alpha p_{\alpha'} + q_\alpha q_{\alpha'} + r_\alpha r_{\alpha'})^2 \leq (p_\alpha^2 + q_\alpha^2 + r_\alpha^2)(p_{\alpha'}^2 + q_{\alpha'}^2 + r_{\alpha'}^2),$$

and the known relation between the quantities p_α, q_α, r_α that results from the way that they were defined:

$$(18)\qquad p_\alpha^2 + q_\alpha^2 + r_\alpha^2 = s_\alpha^2 v_\alpha^2 - s_\alpha^2 \left(\frac{ds_\alpha}{dt} \right)^2,$$

one will have the consequences:

$$s_\alpha^2 v_\alpha^2 - p_\alpha^2 - q_\alpha^2 - r_\alpha^2 = s_\alpha^2 \left(\frac{ds_\alpha}{dt} \right)^2 \geq 0,$$

$$(p_\alpha p_{\alpha'} + q_\alpha q_{\alpha'} + r_\alpha r_{\alpha'})^2 < s_\alpha^2 v_\alpha^2 s_{\alpha'}^2 v_{\alpha'}^2,$$

$$s_\alpha^2 v_{\alpha'}^2 + s_{\alpha'}^2 v_\alpha^2 - 2 p_\alpha p_{\alpha'} - 2 q_\alpha q_{\alpha'} - 2 r_\alpha r_{\alpha'} \geq 0 .$$

Hence, the combination $4GT - D^2$ is equal to an aggregate of nothing but non-negative quantities. As a result of the three integrals (15), D^2 will be equal to a constant, and that will explain the fact that when it has a non-zero value, the product GT cannot vanish when all of the elements $x_\alpha - a_\alpha$, $y_\alpha - b_\alpha$, $z_\alpha - c_\alpha$ decrease, either; however, that was to be proved.

When the time $t = t_0$ corresponds to a system of finite values x_α, y_α, z_α that satisfies the conditions $\Phi_\beta = 0$ and is different from the system $x_\alpha = a_\alpha$, $y_\alpha = b_\alpha$, $z_\alpha = c_\alpha$, but entirely arbitrary, by assumption, the associated value of $U(0)$ will be finite and positive. For that reason, the choice of velocity components dx_α / dt, dy_α / dt, dz_α / dt for the time $t = t_0$ can just as well be arranged to make the constant $T(0) - U(0)$ take on a negative value and to make the constant $\mathfrak{A}^2 + \mathfrak{B}^2 + \mathfrak{C}^2$ take on a non-zero one, as to make the constant $T(0) - U(0)$ take on a positive value. One reaches the conclusion from this that when the position of the mass-system under the prevailing relationships is given arbitrarily, but such that the function G has a non-zero value, whether the motion does or does not have the character of stability will depend upon simply the associated values of the velocity components.

Bonn, 16 October 1866.

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Theorems at the interface between mechanics and geometry

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Translated by D. H. Delphenich

Since mechanics considers bodies in a state of motion, it must take advantage of geometry in order to establish its foundations and must always exploit it in order to achieve its goals.

However, it was first by the ongoing construction of mechanics and geometry that attention was directed to questions whose exploration both sciences have an equal interest in, and which depend upon precisely the same algorithms when they are expressed analytically. The development of a concept that touches upon the interface of mechanics and geometry can be found in the treatise “Über einen algebraischen Typus der Bedingungsgleichungen eines bewegten Massensystems” (Borchardt’s Journal for Mathematics, vol. 66, pp. 363). One imagines a system of material points whose masses might be called m_1, m_2, …, m_q, respectively. The position of each individual mass m_e will be referred to the rectangular coordinates x_e, y_e, z_e. For a certain arrangement of the system, the coordinates of the individual points will take on the well-defined values $x_e = a_e$, $y_e = b_e$, $z_e = c_e$. The concept in question can then be defined to be the *sum over all e points of the products of those masses m_e with the squares of the distance from the position* (x_e, y_e, z_e) *to the position* (a_e, b_e, c_e) :

$$2G = \sum_e m_e[(x_e - a_e)^2 + (y_e - b_e)^2 + (z_e - c_e)^2]. \tag{1}$$

The assumption that the system of material points m_e moves in space free from the influence of accelerating forces and with no restricting conditions leads to a new definition of that concept. The uniform advance of each point along a straight line will necessarily follow from the requirement that the first variation of the associated *integral of least action:*

$$R = \int \sqrt{\sum_e m_e (dx_e^2 + dy_e^2 + dz_e^2)}, \tag{2}$$

must vanish at the initial and final positions of the individual points. Now when the coordinates (a_e, b_e, c_e) determine the initial position of the mass m_e and the coordinates (x_e, y_e, z_e) determine the final position that will easily imply the equation:

$$(3) \qquad R^2 = 2G.$$

The function $2G$ *is then equal to the square of the associated integral of least action for a motion of the system from the point* m_e *from the position* (a_e, b_e, c_e) *to the position* (x_e, y_e, z_e), *respectively*. On the other hand, the representation of the integral of least action by the equation $R = \sqrt{2G}$ coincides with the representation that **Hamilton** introduced as the *characteristic function of the mechanical problem in question.*

In order to derive the relationship of the function $2G$ to a more general motion of the system of points m_e, it will be assumed that the system is under the influence of accelerating forces for which there exists a force function U and that it is subject to a series of condition equations:

$$(4) \qquad \Phi_1 = \text{const.}, \quad \Phi_2 = \text{const.}, \quad \ldots, \quad \Phi_l = \text{const.},$$

in which the l functions contain only the coordinates of the q individual points and not time t. In that case, the principle that goes back to **Hamilton** prescribes that the first variation of the integral:

$$(5) \qquad \int \left\{ \tfrac{1}{2} \sum_e m_e \left[\left(\frac{dx_e}{dt}\right)^2 + \left(\frac{dy_e}{dt}\right)^2 + \left(\frac{dz_e}{dt}\right)^2 \right] + U + \lambda_1 \Phi_1 + \cdots + \lambda_l \Phi_l \right\} dt$$

must be made equal to zero, while one draws upon the l equations (4) for fixed initial and final values of the $3q$ coordinates. The multipliers $\lambda_1, \ldots, \lambda_l$ to be determined prove to be pure functions of time t. Due to the rules of the calculus of variations, that problem will produce the equation:

$$(6) \qquad \sum_e m_e \left(\frac{d^2 x_e}{dt^2} \delta x_e + \frac{d^2 y_e}{dt^2} \delta y_e + \frac{d^2 z_e}{dt^2} \delta z_e \right) = \delta U + \lambda_1 \, \delta\Phi_1 + \lambda_2 \, \delta\Phi_2 + \ldots \lambda_l \, \delta\Phi_l ,$$

which must be fulfilled independently of the $3q$ variations δx_ε, δy_ε, δz_ε, and in that way, conclude the system of differential equations of the mechanical problem in its own right. The variations δx_ε, δy_ε, δz_ε in equation (6) can be replaced with final differences $x_\varepsilon - a_\varepsilon$, $y_\varepsilon - b_\varepsilon$, $z_\varepsilon - c_\varepsilon$, respectively (*). The left-hand side will then go to an expression that is connected with the function G by the *characteristic relation:*

$$(7) \qquad \sum_e m_e \left[\frac{d^2 x_e}{dt^2}(x_e - a_e) + \frac{d^2 y_e}{dt^2}(y_e - b_e) + \frac{d^2 z_e}{dt^2}(z_e - c_e) \right]$$
$$= \frac{d^2 G}{dt^2} - \sum_e m_e \left[\left(\frac{dx_e}{dt}\right)^2 + \left(\frac{dy_e}{dt}\right)^2 + \left(\frac{dz_e}{dt}\right)^2 \right],$$

(*) **Jacobi**, *Vorlesungen über Dynamik*, 4th Lecture, pp. 21.

and on the right-hand side, the variation δU will be converted into the expression:

$$(8) \qquad \sum_e m_e \left[\frac{\partial U}{\partial x_e}(x_e - a_e) + \frac{\partial U}{\partial y_e}(y_e - b_e) + \frac{\partial U}{\partial z_e}(z_e - c_e) \right],$$

while the variation $\delta \Phi_\gamma$ is converted into an expression that is formed analogously. That implies the equation:

$$(9) \qquad \frac{d^2 G}{dt^2} - \sum_e m_e \left[\left(\frac{dx_e}{dt}\right)^2 + \left(\frac{dy_e}{dt}\right)^2 + \left(\frac{dz_e}{dt}\right)^2 \right]$$

$$= \sum_e m_e \left[\frac{\partial U}{\partial x_e}(x_e - a_e) + \frac{\partial U}{\partial y_e}(y_e - b_e) + \frac{\partial U}{\partial z_e}(z_e - c_e) \right]$$

$$+ \sum_\gamma \sum_e \lambda_\gamma \left[\frac{\partial \Phi_\gamma}{\partial x_e}(x_e - a_e) + \frac{\partial \Phi_\gamma}{\partial y_e}(y_e - b_e) + \frac{\partial \Phi_\gamma}{\partial z_e}(z_e - c_e) \right].$$

Under the assumptions that $\Phi_1, \Phi_2, \ldots, \Phi_l$ *are homogeneous functions of the combinations* $x_e - a_e$, $y_e - b_e$, $z_e - c_e$ *and that the constant values that are prescribed in* (4) *are all zero, the double sum on the right-hand side of the last equation will vanish*, and that equation will be identical to equation (6) in the cited treatise.

If one would like to draw upon those cases of the motion of a system of material points for which the condition equations (4) are true, but the components along the *x*, *y*, *z* axes of the forces X_e, Y_e, Z_e that act upon the individual mass points m_e cannot be derived from a force function, then it is known that the differential equations of the problem can be summarized in *one equation* that will emerge from equation (6) above when one substitutes the expression:

$$(10) \qquad \sum_e (X_e\, \delta x_e + Z_e\, \delta y_e + Z_e\, \delta z_e)$$

for the variation δU (*). For that reason, all of the conclusions that we inferred will remain in full force, as long as one uses the expression:

$$(8^*) \qquad \sum_e [X_e\,(x_e - a_e) + Z_e\,(y_e - b_e) + Z_e\,(z_e - c_e)]$$

in place of the expression (8) above.

The existence of a force function for the applied forces is not assumed in the treatise by **Clausius**: "Über einen auf die Wärme anwendbaren mechanische Satz," (Sitzungsberichte der Niederrheinishcn Gesellschaft in Bonn, June 1870, and Comptes rendu de l'Academie des

(*) **Lagrange**, *Mécanique analytique*, Part two, Section IV, arts. 10 and 11.

Sciences Paris, T. LXX, 20 June 1870), or in the treatise of **Ivan Villarceaux** "Sur un nouveau théorème de mécanique générale" (C. R. Acad. Sci., t. LXXV, no. 5, 29 July 1872). The relationship between those two treatises, which start from purely-mechanical considerations and apply the results that are found to the *mechanical theory of heat*, and the paper that was just cited, along with the earlier works of **Jacobi** in regard to it in (Crelle's Journal, Bd. 17, pp. 97), and *Vorlesungen über Dynamik*, pp. 21, are discussed in the Bulletin des sciences mathématiques et astronomiques, which is edited by **G. Darboux** and **J. Houel**, tome III, November 1872, pp. 349. Another paper by **Clausius** belongs with them, namely "Über die Beziehungen zwischen den bei Centralbewegungen vorkommenden charakteristischen Grösse," (Nachrichten v. d. K. G. d. Wiss. zu Göttingen, 25 December 1872, pp. 600). **De Gasparis** gave applications of the function that was just denoted by $2G$ to the *theory of attraction* in "Lettre sur un nouveau théorème de mécanique, communiquée par M. Ivan Villarceaux" (C. R. Acad. Sci. tome LXXV, no. 9, August 1872), along with **S. Newcomb** "Note sur un théorème de mécanique céleste" (C. R. Acad. Sci., tome LXXV, no. 26, 23 December 1872). **F. Lucas** gave applications to the *theory of small oscillations* in "Théorèmes généraux sur l'équilibre et le mouvement des systèmes matérials" (C. R. Acad. Sci., tome LXXV, no. 23, 2 December 1872), as well as a report by **de Saint-Venant** and "Partage de la force vive, due à un mouvement vibratoire composé, en celles qui seraient dues aux mouvement pendulaires simples et isochrones composants, de diverses périodes et amplitudes. Partage du travail dû au même mouvement composé, entre deux instants quelconques, en ceux qui seraient dus aux mouvements composants" (C. R. Acad. Sci., *ibid.*)

The oft-cited article "Über einen algebraischen Typus der Bedingungsgleichungen eines bewegten Massensystems" contains applications to two types of mechanical problems, where one type is concerned with *small oscillations of a system of material points*, while the other is concerned with *the attraction of a material point to a fixed center.* I think that I will treat some further applications of the function $2G$ to the problems of theoretical physics on a later occasion.

The present study mainly has to do with the assumption that the system of points m_e is not driven by any accelerating forces and is subject to *only one condition equation.* Accordingly, the force function U will be equal to zero in the problem that was just referred to, and the l conditions (4) will reduce to the *one:*

$$\Phi_1 = \text{const.} \tag{4*}$$

As long as *only one mass point* m_1 *is present*, that condition will mean that the mass-point cannot leave the *surface* Φ_1 = const., and that the function $2G$ will be equal to the product of the mass m_1 with the square of the distance from the position (x_1, y_1, z_1) to the position (a_1, b_1, c_1). If one now imagines that the position (x_1, y_1, z_1) on the surface Φ_1 = const. is given arbitrarily at a certain time t, along with the advance of the point on the surface during the next time-element dt, then the element of the path of the point in question will be associated with a certain normal section of the surface, and the point (a_1, b_1, c_1) can be determined as the *center of curvature for that normal section*. As a result of this, the function $2G$ can be connected with the associated *radius of curvature* ρ by the equation:

$$2G = m_1\,\rho^2. \tag{11}$$

When the equation (6) above corresponds to the simple assumption that was spoken of, the expression $\lambda_1\, \delta\Phi_1$ that appears on the right-hand side will represent *the moment of the pressure that the motion of the point* m_1 *exerts upon the surface* Φ_1 = const. The known relation between the value of the pressure and the radius of curvature ρ can then be expressed as follows by means of the relation (11):

$$\frac{-1}{\sqrt{2G}} = \frac{\lambda_1\sqrt{\left(\frac{\partial\Phi_1}{\partial x_1}\right)^2 + \left(\frac{\partial\Phi_1}{\partial y_1}\right)^2 + \left(\frac{\partial\Phi_1}{\partial z_1}\right)^2}}{\sqrt{m_1^3}\left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dy_1}{dt}\right)^2 + \left(\frac{dz_1}{dt}\right)^2\right]}. \tag{12}$$

It will now be shown how one can determine a system of q associated points (a_e, b_e, c_e) from a system of q points (x_e, y_e, z_e) by the demand that the first and second complete differentials of the function G in question must differ from the first and second complete differentials, respectively, of the given function Φ_1 only by a finite factor. The system of q points thus-defined will represent a generalization of the concept of the center of curvature of a surface. There is then a complete analogy to equations (2) above that exists between the corresponding values of the function $2G$ and the values λ_1 that enter into equation (6), for which the function U equals zero and the number λ is assumed to be equal to unity. It will generalize the previously-cited theorem that when a point that is free of the influence of an accelerating force moves on a given surface, the reciprocal value of the radius of curvature is proportional to the pressure that acts upon the surface.

After I have established that fact, I will discuss the place that the associated concepts of mechanics and geometry assume from the standpoint that leads to the investigations that were published in Borchardt's Journal für Mathematik (Bd. 70, pp. 71-102 and Bd. 72, pp. 1-56) with the title "Untersuchungen in Betreff der ganzen homogenen Functionen von *n* Differentialen" and in (*ibid*., Bd. 74, pp. 116-149) with the title "Entwicklung einiger Eigenschaften der quadratischen Formen von *n* Differentialen," and in (*ibid*., Bd. 74, pp. 116-149) with the title "Untersuchung eines Problems der Variationsrechnung, in welchem das Problem der Mechanik enthalten ist." Some parts of the theory that have been separate up to now will be connected with each other by that consideration and will make one aware of a new confirmation of the fact that the assumptions of mechanics and geometry that are, in fact, valid are distinguished from some other closely-related assumptions in a characteristic way.

1.

When one juxtaposes the first complete differential of the function G and that of the function Φ_1 :

$$dG = \sum_e m_e\,[(x_e - a_e)\,dx_e + (y_e - b_e)\,dy_e + (z_e - c_e)\,dz_e] \tag{13}$$

(14) $$d\Phi_1 = \sum_e \left(\frac{\partial \Phi_1}{\partial x_e} dx_e + \frac{\partial \Phi_1}{\partial y_e} dy_e + \frac{\partial \Phi_1}{\partial z_e} dz_e \right),$$

and demands that the differential dG should be equal to the differential $d\Phi_1$, up to a finite factor for given values of the $3q$ variables x_e, y_e, z_e *independently of the values of the differentials* dx_e, dy_e, dz_e, one can then determine the combinations:

$$m_e (x_e - a_e), \quad m_e (y_e - b_e), \quad m_e (z_e - c_e)$$

for the $3q$ quantities a_e, b_e, c_e, which must be equal to the partial differential quotients:

$$\frac{\partial \Phi_1}{\partial x_e}, \quad \frac{\partial \Phi_1}{\partial y_e}, \quad \frac{\partial \Phi_1}{\partial z_e},$$

respectively, up to a factor that coincides throughout. By means of the expression:

(15) $$(1, 1) = \sum_e \frac{1}{m_e} \left[\left(\frac{\partial \Phi_1}{\partial x_e} \right)^2 + \left(\frac{\partial \Phi_1}{\partial y_e} \right)^2 + \left(\frac{\partial \Phi_1}{\partial z_e} \right)^2 \right],$$

that state of affairs can be represented by the equations:

(16) $$\begin{cases} \dfrac{m_e (x_e - a_e)}{\sqrt{2G}} = \dfrac{\dfrac{\partial \Phi_1}{\partial x_e}}{\sqrt{(1,1)}}, \\[2ex] \dfrac{m_e (y_e - b_e)}{\sqrt{2G}} = \dfrac{\dfrac{\partial \Phi_1}{\partial y_e}}{\sqrt{(1,1)}}, \\[2ex] \dfrac{m_e (z_e - c_e)}{\sqrt{2G}} = \dfrac{\dfrac{\partial \Phi_1}{\partial z_e}}{\sqrt{(1,1)}}. \end{cases}$$

At the same time, one has the relation between the differentials dG and $d\Phi_1$ that has the prescribed behavior:

(17) $$\frac{dG}{\sqrt{2G}} = \frac{d\Phi_1}{\sqrt{(1,1)}}.$$

We shall now define the second complete differentials of the function G and the function Φ_1 :

$$d^2 G = \sum_e m_e\,[(x_e - a_e)\,d^2x_e + (y_e - b_e)\,d^2y_e + (z_e - c_e)\,d^2z_e] + \sum_e m_e (dx_e^2 + dy_e^2 + dz_e^2)\,, \tag{18}$$

$$d^2\Phi_1 = \sum_e \left(\frac{\partial \Phi_1}{\partial x_e} d^2 x_e + \frac{\partial \Phi_1}{\partial y_e} d^2 y_e + \frac{\partial \Phi_1}{\partial z_e} d^2 z_e \right) + \sum_{e,e'} \frac{\partial^2 \Phi_1}{\partial x_e\,\partial x_{e'}} dx_e\,dx_{e'} + \sum_{e,e'} \frac{\partial^2 \Phi_1}{\partial x_e\,\partial y_{e'}} dx_e\,dy_{e'} + \cdots + \sum_{e,e'} \frac{\partial^2 \Phi_1}{\partial z_e\,\partial z_{e'}} dz_e\,dz_{e'}\,, \tag{19}$$

and then express the requirement that the differential d^2G must be equal to the differential $d^2\Phi_1$, up to a finite factor, *for arbitrarily-varying values of the second differentials* d^2x_e, d^2y_e, d^2z_e, *and arbitrary, but fixed, values of the first differentials* dx_e, dy_e, dz_e. Equation (18) will then be converted into the *characteristic relation* (7) after dividing by dt^2. Equations (16) have the consequences that the first component of d^2G and first component of $d^2\Phi_1$, which include the second differentials, have the desired character and that the former of them has the same relationship to the latter that the expression $\sqrt{2G}$ has to the expression $\sqrt{(1,1)}$. For that reason, the equation:

$$\frac{d^2 G}{\sqrt{2G}} = \frac{d^2 \Phi_1}{\sqrt{(1,1)}} \tag{20}$$

must be true.

In order for our requirement to be fulfilled, it is necessary and sufficient that the second components of d^2G and the second component of $d^2\Phi_1$, which are equal to quadratic forms in the $3q$ differentials dx_e, dy_e, dz_e, should likewise have that relationship. The equation thus arises:

$$\frac{\sum_e m_e\,(dx_e^2 + dy_e^2 + dz_e^2)}{\sqrt{2G}} = \frac{\sum_{e,e'} \frac{\partial^2 \Phi_1}{\partial x_e\,\partial x_{e'}} dx_e\,dx_{e'} + \frac{\partial^2 \Phi_1}{\partial x_e\,\partial y_{e'}} dx_e\,dy_{e'} + \cdots + \frac{\partial^2 \Phi_1}{\partial z_e\,\partial z_{e'}} dz_e\,dz_{e'}}{\sqrt{(1,1)}}\,. \tag{21}$$

The $3q$ *quantities* a_e, b_e, c_e *will be determined completely by that equation, in conjunction with equations* (16), *for given values of the variables* x_e, y_e, z_e, *and the differentials* dx_e, dy_e, dz_e, *so the system of values* (a_e, b_e, c_e) *that emerges will represent a generalization of the center of curvature.*

In the mechanical problem for which the force function U vanishes, and *only one* condition equation (4^*) is given, the equation (6) above will become the following one:

(22) $$\sum_e m_e \left(\frac{d^2 x_e}{dt^2} \delta x_e + \frac{d^2 y_e}{dt^2} \delta y_e + \frac{d^2 z_e}{dt^2} \delta z_e \right) = \lambda_1 \, \delta \Phi_1 \,.$$

Therefore, the expression $\lambda_1 \, \delta \Phi_1$ represents *the sum of the moments of all pressures* that are produced by the condition equation (4^*). One can infer the following consequences from that equation:

(23) $$\frac{d\Phi_1}{dt} = 0, \qquad \frac{d^2\Phi_1}{dt^2} = 0 \,.$$

The choice of the differential quotients $\frac{dx_e}{dt}$, $\frac{dy_e}{dt}$, $\frac{dz_e}{dt}$ will be restricted by the first of those systems. One can employ the second one to represent the expression λ_1 . When one divides the second differential $d^2\Phi_1$ that is written out explicitly in (19) by the quantity dt^2, the quantities $\frac{d^2 x_e}{dt^2}$, $\frac{d^2 y_e}{dt^2}$, $\frac{d^2 z_e}{dt^2}$ will contain the factors $\frac{\partial \Phi_1}{\partial x_e}$, $\frac{\partial \Phi_1}{\partial y_e}$, $\frac{\partial \Phi_1}{\partial z_e}$, while the corresponding quantities on the left-hand side of (22) will exhibit the factors $m_e \, \delta x_e$, $m_e \, \delta y_e$, $m_e \, \delta z_e$. As soon as those expressions are replaced with the expressions $\frac{\partial \Phi_1}{\partial x_e}$, $\frac{\partial \Phi_1}{\partial y_e}$, $\frac{\partial \Phi_1}{\partial z_e}$ in (22), the components in question will coincide, and at the same time, the expression:

$$\delta \Phi_1 = \sum_e \left(\frac{\partial \Phi_1}{\partial x_e} \delta x_e + \frac{\partial \Phi_1}{\partial y_e} \delta y_e + \frac{\partial \Phi_1}{\partial z_e} \delta z_e \right)$$

will be converted into the expression:

$$\sum_e \frac{1}{m_e} \left[\left(\frac{\partial \Phi_1}{\partial x_e} \right)^2 + \left(\frac{\partial \Phi_1}{\partial y_e} \right)^2 + \left(\frac{\partial \Phi_1}{\partial z_e} \right)^2 \right] = (1, 1) \,.$$

In that way, the determination of the quantity λ_1 will follow from (19) and (22):

(24) $$\lambda_1 \, (1, 1) = \sum \left(\frac{d^2 x_e}{dt^2} \frac{\partial \Phi_1}{\partial x_e} + \frac{d^2 y_e}{dt^2} \frac{\partial \Phi_1}{\partial y_e} + \frac{d^2 z_e}{dt^2} \frac{\partial \Phi_1}{\partial z_e} \right) - \frac{d^2 \Phi_1}{dt^2} \,,$$

which one can also give the form:

(24^*) $$\lambda_1 \, (1, 1) = \sum_{e,e'} \frac{\partial^2 \Phi_1}{\partial x_e \, \partial x_{e'}} \frac{dx_e}{dt} \frac{dx_{e'}}{dt} - \sum_{e,e'} \frac{\partial^2 \Phi_1}{\partial x_e \, \partial y_{e'}} \frac{dx_e}{dt} \frac{dy_{e'}}{dt} - \ldots - \sum_{e,e'} \frac{\partial^2 \Phi_1}{\partial z_e \, \partial z_{e'}} \frac{dz_e}{dt} \frac{dz_{e'}}{dt}$$

by another application of (19). A comparison of that result with the one in (21) will then produce the equation:

$$\frac{-1}{\sqrt{2G}} = \frac{\lambda_1\sqrt{(1,1)}}{\sum_e\left[\left(\frac{dx_e}{dt}\right)^2+\left(\frac{dy_e}{dt}\right)^2+\left(\frac{dz_e}{dt}\right)^2\right]}. \tag{25}$$

That includes equation (12) *within it, and describes the connection between the function* $\sqrt{2G}$ *and the function* $\lambda_1\sqrt{(1,1)}$ *, the first of which represents a generalization of the radius of curvature, while the second one represents a generalization of the concept of pressure.*

It can be noted in passing that when one chooses the function G itself, instead of the function Φ_1, in the variational problem that leads to equation (22), the relevant system of differential equations will belong to the category that was integrated completely in Journal f. Math., Bd. 72, pp. 38. At the same time, it will follow from what was done there that the form:

$$\sum m_e(dx_e^2+dy_e^2+dz_e^2),$$

which consists of $3q = n$ positive squares, will go to a form in $(n - 1)$ differentials when one variable is eliminated by using the equation $G = 1 / 2\alpha$, in place of (4^*), and that form will represent the square of the line element for the manifold of $(n - 1)$ remaining variables and with constant positive curvature α.

2.

From the ideas that were presented earlier, the concepts in mechanics and geometry that were just spoken of can be extended as follows: Let $x_\mathfrak{a}$ be a system of n variable quantities, in which the symbol $\mathfrak{a}$, like $\mathfrak{b}$, $\mathfrak{c}$, …, as well later on, runs through the numbers 1, 2, …, n. Let $f(dx)$ mean an essentially-positive form of degree p in the differentials $dx_\mathfrak{a}$, in which the coefficients depend upon the variables $x_\mathfrak{a}$ arbitrarily. Let the determinant of the second derivatives $\frac{\partial^2 f(dx)}{\partial dx_\mathfrak{a}\,\partial dx_\mathfrak{b}}$ be not equal to zero identically, and let U and $\Phi_1, \Phi_2, \ldots, \Phi_l$ be functions of the only variables $x_\mathfrak{a}$. One now demands that the variables $x_\mathfrak{a}$ should be made to depend upon an independent variable t in such a way that the first variation of the integral:

$$\int\left[f\left(\frac{dx}{dt}\right)+U+\lambda_1\Phi_1+\lambda_2\Phi_2+\cdots+\lambda_l\Phi_l\right]dt \tag{26}$$

will vanish for fixed initial and final values of the variables $x_\mathfrak{a}$, while the l equations:

(27) $$\Phi_\alpha = \text{const.}$$

must be fulfilled. That problem will be converted into the variational problem for the integral (5) when the n variables $x_\mathfrak{a}$ go to the $3q$ coordinates x_e , y_e , z_e , and the form $f(dx)$ goes to the form $\frac{1}{2}\sum m_e(dx_e^2 + dy_e^2 + dz_e^2)$. The functions U, Φ_1, …, Φ_l, and the multipliers to be determined λ_1 , λ_2 , …, λ_l are denoted in the same way in both cases.

We shall next consider the integral (26) under the assumptions that no conditions (27) are present, and that the function U is equal to zero. The demand that was just expressed will coincide with the other demand that the first variation of the integral:

(28) $$R = \int \sqrt{p\, f(dx)}$$

must be zero (*). The integration values $x_\mathfrak{a}$ that satisfy that requirement and are determined by the conditions that they must satisfy the equations $x_\mathfrak{a} = x_\mathfrak{a}(0)$ and $x'_\mathfrak{a} = x'_\mathfrak{a}(0)$ for a value $t = t_0$, in which the addition of a prime suggests differentiation with respect to the variable t and the constants $x_\mathfrak{a}(0)$ and $x'_\mathfrak{a}(0)$ are given arbitrarily, will be *functions of only the quantities* $x_\mathfrak{a}(0)$ *and the combinations* (**):

(29) $$x'_\mathfrak{a}(0)\,(t - t_0) = u_\mathfrak{a}$$

in this case. When the quantities $x_\mathfrak{a}(0)$ are constant and the combinations $u_\mathfrak{a}$ are variable, the latter will represent a *system of normal variables for the form* $f(dx)$ (***). The associated value of the integral R, when extended from the system $x_\mathfrak{a}(0)$ to the system $x_\mathfrak{a}$, will then be expressed by the equation (†):

(30) $$R^p = p\, f_0(u).$$

$f_0(u)$ emerges from the form $f(dx)$, when the relevant values $x_\mathfrak{a}(0)$ are substituted for the variables $x_\mathfrak{a}$, and the relevant values $u_\mathfrak{b}$ are substituted for the differentials $dx_\mathfrak{b}$. When one introduces the variables $u_\mathfrak{b}$ into the form $f(dx)$ that will yield the transformation equation:

(31) $$f(dx) = \varphi(du).$$

(*) Journal f. Mathematik, Bd. 74, pp. 120, *et seq.*
(**) Journal f. Mathematik, Bd. 70, pp. 86, *et seq.*
(***) Journal f. Mathematik, Bd. 72, pp. 1, *et seq.*
(†) Journal f. Mathematik, Bd. 74, pp. 126.

The resulting form of degree p in the differentials du_α , $\varphi\,(du)$ will be called a *normal type* for the form $f\,(dx)$.

When one introduces the normal variables u_α into the functions U, Φ_1, …, Φ_l will convert the more general integral to be varied (26) into the integral:

$$(26^*)\qquad \int\left[\varphi\left(\frac{dx}{dt}\right)+U+\lambda_1\Phi_1+\lambda_2\Phi_2+\cdots+\lambda_l\Phi_l\right]dt\,.$$

Under the aforementioned special assumption that the coordinates x_e , y_e , z_e should enter in place of the variables x_α , the form:

$$\tfrac{1}{2}\sum m_e(dx_e^2+dy_e^2+dz_e^2)$$

will appear in place of $f\,(dx)$, and we imagine that the coordinates a_e , b_e , c_e will enter in place of the initial values $x_\alpha\,(0)$. Now, since the variational problem for the integral (2) will enter into the variational problem for the integral (28), and since the variational problem for the integral (2) will be solved by the advance of each mass-point m_e along a straight line with uniform velocity, and therefore by the equations:

$$x_e = a_e + x_e'(0)\,(t - t_0),$$
$$y_e = b_e + y_e'(0)\,(t - t_0),$$
$$z_e = c_e + z_e'(0)\,(t - t_0),$$

under the prevailing relationships, the normal variables u_α *will be nothing but the coordinate differences:*

$$x_e - a_e\,,\qquad y_e - b_e\,,\qquad z_e - c_e\,.$$

For that reason, the normal type $\varphi\,(dx)$ *will be equal to the given form:*

$$\tfrac{1}{2}\sum m_e(dx_e^2+dy_e^2+dz_e^2)$$

identically, and the function $2\,f_0\,(u)$ *will coincide with the function:*

$$2G = \sum m_e\,[(x_e - a_e)^2 + (y_e - b_e)^2 + (z_e - c_e)^2]\,.$$

The variational problem for the integral (26*) will generally imply the equation:

$$(32)\qquad \sum_\alpha\left[\frac{d}{dt}\frac{\partial\varphi(u')}{\partial u_\alpha'}-\frac{\partial\varphi(u')}{\partial u_\alpha}\right]\delta u_\alpha = \delta U + \lambda_1\,\delta\Phi_1 + \ldots + \lambda_l\,\delta\Phi_l\,,$$

which must be satisfied independently of the values of the variations $\delta u_\mathfrak{a}$. *We replace the variations $\delta u_\mathfrak{a}$ with the normal variables themselves $u_\mathfrak{a}$, respectively, and obtain the equation:*

$$(33) \qquad \sum_\mathfrak{a}\left[\frac{d}{dt}\frac{\partial\varphi(u')}{\partial u'_\mathfrak{a}} - \frac{\partial\varphi(u')}{\partial u_\mathfrak{a}}\right]u_\mathfrak{a} = \sum_\mathfrak{a}\frac{\partial U}{\partial u_\mathfrak{a}}u_\mathfrak{a} + \sum_{\mathfrak{a},\gamma}\lambda_\gamma\frac{\partial\Phi_\gamma}{\partial u_\mathfrak{a}}u_\mathfrak{a} .$$

Everything now comes down to showing how the expression that is found on the left-hand side of this equation is connected with the function $f_0(u)$. If one once more introduce the differentials $du_\mathfrak{a}$ in place of the differential quotients $u'_\mathfrak{a}$ in the normal type $\varphi(u')$ then one will define the identity relation:

$$(34) \qquad \sum_\mathfrak{a} d\frac{\partial\varphi(du)}{\partial\, du_\mathfrak{a}}\delta u_\mathfrak{a} = d\sum_\mathfrak{a}\left[\frac{\partial\varphi(du)}{\partial\, du_\mathfrak{a}}\delta u_\mathfrak{a} - \frac{\partial\varphi(\delta u)}{\partial\,\delta u_\mathfrak{a}}du_\mathfrak{a}\right]$$

$$+ \; d\sum_\mathfrak{a}\frac{\partial\varphi(\delta u)}{\partial\,\delta u_\mathfrak{a}}du_\mathfrak{a} - \sum_\mathfrak{a}\frac{\partial\varphi(du)}{\partial\, du_\mathfrak{a}}\,d\,\delta u_\mathfrak{a} ,$$

which is valid for all systems $du_\mathfrak{a}$ and $\delta u_\mathfrak{a}$. Now, the complete differential $df_0(u)$ can be represented in the following way, in which the substitution of the quantities $u_\mathfrak{a}$ for the $\delta u_\mathfrak{a}$ is suggested by enclosing the expression in question with square brackets (*):

$$(35) \qquad df_0(u) = \sum_\mathfrak{a}\left[\frac{\partial\varphi(\delta u)}{\partial\,\delta u_\mathfrak{a}}\right]du_\mathfrak{a} \; .$$

The fact that $\varphi(du)$ is a homogeneous function of degree p in the differentials $du_\mathfrak{a}$ further implies that:

$$(36) \qquad p\;\varphi(du) = \sum_\mathfrak{a}\frac{\partial\varphi(du)}{\partial\, du_\mathfrak{a}}\,du_\mathfrak{a} .$$

When one substitutes $u_\mathfrak{a}$ for $\delta u_\mathfrak{a}$ in (34), that will yield the equation:

$$(37) \qquad \sum_\mathfrak{a} d\frac{\partial\varphi(du)}{\partial\, du_\mathfrak{a}}u_\mathfrak{a} = d\left\{\sum_\mathfrak{a}\left[\frac{\partial\varphi(du)}{\partial\, du_\mathfrak{a}}\delta u_\mathfrak{a} - \frac{\partial\varphi(\delta u)}{\partial\,\delta u_\mathfrak{a}}du_\mathfrak{a}\right]\right\} - d^2 f_0(u) - p\;\varphi(du),$$

which reveals the desired connection completely. As long as the number $p = 2$, the expression:

(*) Journal f. Mathematik, Bd. 72, pp. 8.

$$\sum_{\mathfrak{a}}\left[\frac{\partial\varphi(du)}{\partial\,du_{\mathfrak{a}}}\,\delta u_{\mathfrak{a}}-\frac{\partial\varphi(\delta u)}{\partial\,\delta u_{\mathfrak{a}}}\,du_{\mathfrak{a}}\right]$$

will be equal to zero, from a basic property of quadratic forms. *Therefore, under the assumption that* $p = 2$, *one will have the characteristic relation:*

$$\sum_{\mathfrak{a}} d\,\frac{\partial\varphi(du)}{\partial\,du_{\mathfrak{a}}}\,u_{\mathfrak{a}} = d^{2} f_0(u) - 2\,\varphi(du)\,. \tag{38}$$

From now on, we will assume that this assumption has been made. *Hence, equation* (33) *will take on the definitive form:*

$$\frac{d^2 f_0(u)}{dt^2} - 2\varphi\left(\frac{du}{dt}\right) = \sum_{\mathfrak{a}} \frac{\partial}{\partial u_{\mathfrak{a}}}\left[\varphi\left(\frac{du}{dt}\right)+U\right]u_{\mathfrak{a}} + \sum_{\mathfrak{a},\gamma}\lambda_\gamma\,\frac{\partial\Phi_\gamma}{\partial u_{\mathfrak{a}}}\,u_{\mathfrak{a}}\,, \tag{39}$$

when we apply (38). *The equation* (9) *above is included in this equation as a special case*, and it can be linked with considerations that are similar to the ones that were made in regard to that equation.

3.

In the case where the function U is equal to zero and one again sets the number $p = 2$, the variational problem of the integral (26) will become the one that was presented in (Journal f. Mathematik, Bd. 71, pp. 275). In agreement with the notations that are used there, let:

$$f(dx) = \tfrac{1}{2}\sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\,dx_{\mathfrak{a}}\,dx_{\mathfrak{b}}\,, \tag{40}$$

and further:

$$|\,a_{\mathfrak{a},\,\mathfrak{b}}\,| = \Delta, \qquad \frac{\partial\Delta}{\partial a_{\mathfrak{a},\mathfrak{b}}} = A_{\mathfrak{a},\,\mathfrak{b}}\,. \tag{41}$$

The functions $y_1, y_2, \ldots, y_l$, which were set equal to constant in that article, presently correspond to the functions $\Phi_1, \Phi_2, \ldots, \Phi_l$. The multipliers $\lambda_1, \lambda_2, \ldots, \lambda_l$ will be denoted by the same signs. We shall ponder the assumption that *only one* function Φ_1 is present. The vanishing of the first variation of the given integral (26) will then yield the equation:

$$\sum_{\mathfrak{a}}\left(\frac{d}{dt}\,\frac{\partial f(x')}{\partial x'_{\mathfrak{a}}} - \frac{\partial f(x')}{\partial x_{\mathfrak{a}}}\right)\delta x_{\mathfrak{a}} = \lambda_1\;\delta\Phi_1\,. \tag{42}$$

In order to determine the expression λ_1 by means of the equations:

$$(43)\qquad \begin{cases} \dfrac{d\Phi_1}{dt} = \sum_{\mathfrak{c}} \dfrac{\partial \Phi_1}{\partial x_{\mathfrak{c}}} \dfrac{dx_{\mathfrak{c}}}{dt} = 0, \\ \dfrac{d^2\Phi_1}{dt^2} = \sum_{\mathfrak{c}} \dfrac{\partial \Phi_1}{\partial x_{\mathfrak{c}}} \dfrac{d^2 x_{\mathfrak{c}}}{dt^2} + \sum_{\mathfrak{a},\mathfrak{b}} \dfrac{\partial^2 \Phi_1}{\partial x_{\mathfrak{a}}\,\partial x_{\mathfrak{b}}} \dfrac{dx_{\mathfrak{a}}}{dt} \dfrac{dx_{\mathfrak{b}}}{dt} = 0, \end{cases}$$

which will be true from now on, one can introduce expressions in place of the variations $\delta x_{\mathfrak{a}}$ in (42) such that the factor of $\dfrac{d^2 x_{\mathfrak{c}}}{dt^2}$ on the left-hand side of that equation coincides with $\dfrac{\partial \Phi_1}{\partial x_{\mathfrak{c}}}$, which is the corresponding factor in the expression for $\dfrac{d^2\Phi_1}{dt^2}$. Since the coefficient $a_{\mathfrak{a},\mathfrak{c}}$ of the form $2f(dx)$ appears on the left-hand side of (42) as a factor of the combination $\dfrac{d^2 x_{\mathfrak{a}}}{dt^2}\,\delta x_{\mathfrak{a}}$, the variation $\delta x_{\mathfrak{a}}$ must be replaced with the expression:

$$(44)\qquad \sum_{\mathfrak{k}} \frac{A_{\mathfrak{a},\mathfrak{k}}}{\Delta} \frac{\partial \Phi_1}{\partial x_{\mathfrak{k}}}$$

for the given purpose. With that substitution, the complete variation $\delta\Phi_1 = \sum_{\mathfrak{a}} \dfrac{\partial \Phi_1}{\partial x_{\mathfrak{a}}}\,\delta x_{\mathfrak{a}}$ will go to the expression:

$$(45)\qquad \sum_{\mathfrak{a},\mathfrak{c}} \frac{A_{\mathfrak{a},\mathfrak{c}}}{\Delta} \frac{\partial \Phi_1}{\partial x_{\mathfrak{a}}} \frac{\partial \Phi_1}{\partial x_{\mathfrak{c}}} = (1, 1),$$

and that will imply the following representation for λ_1 :

$$(46)\qquad \lambda_1\,(1, 1) = \sum_{\mathfrak{a}} \left(\frac{d}{dt} \frac{\partial f(x')}{\partial x'_{\mathfrak{a}}} - \frac{\partial f(x')}{\partial x_{\mathfrak{a}}} \right) \sum_{\mathfrak{k}} \frac{A_{\mathfrak{a},\mathfrak{k}}}{\Delta} \frac{\partial \Phi_1}{\partial x_{\mathfrak{a}}} - \frac{d^2\Phi_1}{dt^2},$$

which coincides with the representation that was given in the cited reference. At this point, it should be emphasized that when an arbitrary system of new independent variables is introduced in place of the system of variables $x_{\mathfrak{a}}$ in the relevant variational problem, the expression (1, 1), as well as the expression λ_1, will go to corresponding expressions that are constructed from the new elements (*). Therefore, if the previously-defined system of *normal variables* $u_{\mathfrak{a}}$ is introduced and one appeals to the notations:

$$(47)\qquad \left\{\begin{aligned} &\varphi(du) = \tfrac{1}{2}\sum_{\mathfrak{a},\mathfrak{b}} p_{\mathfrak{a},\mathfrak{b}}\, du_{\mathfrak{a}}\, du_{\mathfrak{b}}, \\ &|\, p_{\mathfrak{a},\mathfrak{b}} \,| = \Pi, \quad \frac{\partial \Pi}{\partial p_{\mathfrak{a},\mathfrak{b}}} = P_{\mathfrak{a},\mathfrak{b}}, \end{aligned}\right.$$

for the *normal type* $\varphi(du)$, then (45) and (46) will imply the new equations:

$$(48)\qquad \sum_{\mathfrak{a},\mathfrak{k}} \frac{P_{\mathfrak{a},\mathfrak{k}}}{\Pi} \frac{\partial \Phi_1}{\partial u_{\mathfrak{a}}} \frac{\partial \Phi_1}{\partial u_{\mathfrak{k}}} = (1, 1),$$

$$(49)\qquad \lambda_1\,(1, 1) = \sum_{\mathfrak{a}} \left(\frac{d}{dt} \frac{\partial \varphi(u')}{\partial u'_{\mathfrak{a}}} - \frac{\partial \varphi(u')}{\partial u_{\mathfrak{a}}} \right) \sum_{\mathfrak{k}} \frac{P_{\mathfrak{a},\mathfrak{k}}}{\Pi} \frac{\partial \Phi_1}{\partial u_{\mathfrak{a}}} - \frac{d^2 \Phi_1}{dt^2}.$$

The associated system of quantities $x_{\mathfrak{a}}(0)$ has a decisive meaning for the system of normal variables $u_{\mathfrak{a}}$. From the definition that is given in (29), all of the normal variables $u_{\mathfrak{a}}$ will vanish whenever the system of values $x_{\mathfrak{a}}$ satisfies the equations $x_{\mathfrak{a}} = x_{\mathfrak{a}}(0)$. Moreover, a manifold of first order that starts from the system of values $u_{\mathfrak{a}} = 0$, and for which the ratios of the variables $u_{\mathfrak{a}}$ to each other remain unchanged, will satisfy the variational problem of the integral (28). The fact that, under the assumptions of article **1**, this first-order manifold represents nothing but the advance of each mass-point m_e from the position (a_e, b_e, c_e) along a straight line with uniform velocity has been mentioned numerous times. Just as the system of values a_e, b_e, c_e was determined before, we will now determine the system of values $x_{\mathfrak{a}}(0)$ by certain requirements, and indeed the explicit expression of the variables $x_{\mathfrak{a}}$ in terms of the variables $u_{\mathfrak{a}}$ and the fixed values $x_{\mathfrak{a}}(0)$ will not be required for that.

The first of those requirements points to the fact that the differential $df_0(u)$ must be equal to the differential $\delta\Phi_1$ for arbitrary values of the differentials $du_{\mathfrak{a}}$, up to a finite factor. Therefore, due to the relation (35), the quantities $\left[\frac{\partial \varphi(\delta u)}{\partial\, \delta u_{\mathfrak{a}}}\right]$ must have just the same relationship to the corresponding quantities $\frac{\partial \Phi_1}{\partial u_{\mathfrak{a}}}$. Since, from (47), one has:

(*) When the function Φ_1, which is to be set to a constant, is replaced with a function of that function, which is to be set to a constant, the product $\lambda_1\sqrt{(1,1)}$, which represents the generalization of the concept of pressure, will still remain invariant.

$$\frac{\partial \varphi(\delta u)}{\partial\, \delta u_{\mathfrak{a}}} = p_{\mathfrak{a},1}\ \delta u_1 + p_{\mathfrak{a},2}\ \delta u_2 + \ldots + p_{\mathfrak{a},\mathfrak{n}}\ \delta u_{\mathfrak{n}}\,,$$

the quantities $[\delta u_{\mathfrak{a}}] = u_{\mathfrak{a}}$ must have same relationship to the combination $\sum_{\mathfrak{k}} \frac{P_{\mathfrak{a},\mathfrak{k}}}{\Pi} \frac{\partial \Phi_1}{\partial u_{\mathfrak{a}}}$, and the expression $\sqrt{[2\varphi(\delta u)]}$, which is equal to the expression $\sqrt{2 f_0(u)}$ (*), must have the some relationship to the expression $\sqrt{(1,1)}$, which is defined by equation (48). On those grounds, the equations:

(50)
$$\frac{u_{\mathfrak{a}}}{\sqrt{2 f_0(u)}} = \frac{\sum_{\mathfrak{k}} \frac{P_{\mathfrak{a},\mathfrak{k}}}{\Pi} \frac{\partial \Phi_1}{\partial u_{\mathfrak{a}}}}{\sqrt{(1,1)}}$$

and the equation:

(51)
$$\frac{df_0(u)}{\sqrt{2 f_0(u)}} = \frac{d\Phi_1}{\sqrt{(1,1)}}$$

must be true. The second requirement, under which, the second differentials $d^2 u_{\mathfrak{a}}$ must vary independently, but the first differentials are regarded as fixed, will be represented by the equation:

(52)
$$\frac{d^2 f_0(u) - \sum_{\mathfrak{a}} \frac{\partial \varphi(du)}{\partial u_{\mathfrak{a}}} u_{\mathfrak{a}}}{\sqrt{2 f_0(u)}} = \frac{d^2 \Phi_1}{\sqrt{(1,1)}}\,.$$

The same equation will be satisfied by the second differentials since equation (51) is true. *The characteristic relation* (38), *which was appealed to for the intended conversion of* (33) *into* (39), *will also effect the intended conversion of* (52) *into the equation:*

(53)
$$\frac{\sum_{\mathfrak{a}} d \frac{\partial \varphi(du)}{\partial\, du_{\mathfrak{a}}} u_{\mathfrak{a}} - \sum_{\mathfrak{a}} \frac{\partial \varphi(du)}{\partial u_{\mathfrak{a}}} u_{\mathfrak{a}} + 2\varphi(du)}{\sqrt{2 f_0(u)}} = \frac{d^2 \Phi_1}{\sqrt{(1,1)}}\,.$$

An application of (50) will lead to the representation:

(54)
$$\frac{2\varphi(du)}{\sqrt{2 f_0(u)}} = -\sum_{\mathfrak{a}} \left(d \frac{\partial \varphi(du)}{\partial\, du_{\mathfrak{a}}} - \frac{\partial \varphi(du)}{\partial u_{\mathfrak{a}}} \right) \frac{\sum_{\mathfrak{k}} \frac{P_{\mathfrak{a},\mathfrak{k}}}{\Pi} \frac{\partial \Phi_1}{\partial u_{\mathfrak{k}}}}{\sqrt{(1,1)}} + \frac{d^2 \Phi_1}{\sqrt{(1,1)}}\,.$$

(*) Journal f. Mathematik, Bd. 72, pp. 7, formula (13).

Now, since the right-hand side of this equation, when divided by dt^2, coincides with the right-hand side of (49), *up to sign, that will yield the result that:*

$$\frac{-1}{\sqrt{2\,f_0(u)}} = \frac{\lambda_1\sqrt{(1,1)}}{2\varphi(u')}, \tag{55}$$

which includes equation (25) *as a special case.*

4.

The n quantities x_α (0) that belong to the system of normal variables are determined indirectly by equation (51), which represents a system of (n – 1) independent equations, due to the independence of the differentials du_α and equation (55) that we just obtained. In order to get a direct determination, we turn our attention on the aforementioned first-order manifold that solved the variational problem for the integral (28) and extended from the system of values $u_\alpha = 0$ to the given system of values u_α that satisfies the equation Φ_1 = const. When that first-order manifold is referred to the variables x_α , it will extend from the system of values x_α (0) to the system of values x_α that satisfies the equations Φ_1 = const. and corresponds to the system of values u_α . Since $p = 2$, the associated values of the integral R in terms of normal variables u_α will be expressed by the equation:

$$R = \sqrt{2\,f_0(u)}\,. \tag{56}$$

When the variables take on the increments Dx_α as one advances along the first-order manifold that was spoken of, the expression $\dfrac{df_0(u)}{\sqrt{2\,f_0(u)}}$ will admit the following representation in terms of the variables x_α (*):

$$\frac{df_0(u)}{\sqrt{2\,f_0(u)}} = \frac{\sum_\alpha \frac{\partial f(Dx)}{\partial Dx_\alpha}dx_\alpha}{\sqrt{2\,f(Dx)}}\,. \tag{57}$$

The combination (1, 1) is expressed in terms of the variables x_α by equation (45). *We can therefore replace* (51) *with the equation:*

(*) Journal f. Mathematatik, Bd. 74, pp. 128.

$$(58) \qquad \frac{\sum_\alpha \frac{\partial f(Dx)}{\partial Dx_\alpha} dx_\alpha}{\sqrt{2\,f(Dx)}} = \frac{d\Phi_1}{\sqrt{(1,1)}}.$$

The ratios of the differentials Dx_α will be determined by them; i.e., *the final element of the indicated first-order manifold will be determined in such a way that the final element* Dx_α *is normal to the manifold of order* $(n - 1)$ Φ_1 = const. *when one recalls the form* $2f(Dx)$, which is discussed in (Journal f. Math., Bd. 74, pp. 144).

The form $2\varphi(u')$ in equation (55) can also be replaced by the form $2\,f(x')$ by means of (31), and will represent a combination in the system of values x_α and dx_α / dt by means of equations (45) and (46). *That will yield the equation:*

$$(59) \qquad \frac{-1}{R} = \frac{\lambda_1\sqrt{(1,1)}}{2\,f(x')},$$

which will imply the value of the integral R in terms of the system of values x_α *and* dx_α / dt.

Equations (58) and (59) then determine the system of values $x_\alpha(0)$ by the conditions that the first-order manifold that starts from it and makes the first variation of the integral (28) vanish will emerge from a given system of values x_α that belongs to the manifold of order $(n - 1)$, while the final element Dx_α will be normal to that manifold of order $(n - 1)$ relative to the form $2f(Dx)$, and that the associated integral R must assume the prescribed value. Conversely, if one imagines that the first-order manifold that is spoken of starts from the system of values x_α then its evolution will be determined completely by that system and the element Dx_α, and the prescribed value of the integral R will ultimately determine the system of values $x_\alpha(0)$ in question (*). The possibility of that determination is assumed in that. One easily recognizes that equations (58) and (59) have the property that when a new system of independent variables are introduced in place of the variables x_α and also when the function Φ_1 that is to be set to a constant is replaced with a function of that function, those equations will go to *equations that are formed analogously from the new elements. The given determination is therefore completely independent of the choice of the form of the function* Φ_1. As long as only one mass-point is assumed in the considerations of article 1, the indicated first-order manifold will be the straight line that starts from the point (x_1, y_1, z_1) and points normally to the surface Φ_1 = const., and which cuts out the length of the radius of curvature that is determined from the point (x_1, y_1, z_1) to the center of curvature (a_1, b_1, c_1).

When one regards the quantities x_α to be fixed and the quantities dx_α / dt to be variable and, from (43), restricted by only the equation:

(*) Journal f. Mathematik, Bd. 74, pp. 130, *et seq.*

$$\frac{d\Phi_1}{dt} = 0,$$

and when one raises the question of *what system of values* $dx_\mathfrak{a} / dt$ *for the first differential in the expression that is defined in* (59) *will make* $1 / R$ *vanish*, one will have expressed *a maximum-minimum problem*, which emerges from *the general maximum-minimum problem* that was presented in (Journal f. Math., Bd. 71, pp. 277) and was discussed under the assumption that $l = 1$. The results that were published in that reference for that problem are therefore applicable to the present problem with no further discussion. That sheds light upon the fact that the indicated problem will become the *problem of largest and smallest radii of curvature* in the aforementioned simplest case of article 1.

When one compares the more general results that were just found with the more specialized ones that were presented earlier, it must emerge that equations (39) and (52) contain one of the expressions:

$$\sum_\mathfrak{a} \frac{\partial\varphi\left(\frac{du}{dt}\right)}{\partial u_\mathfrak{a}} u_\mathfrak{a} \qquad \text{and} \qquad \sum_\mathfrak{a} \frac{\partial\varphi(du)}{\partial u_\mathfrak{a}} u_\mathfrak{a}\,,$$

respectively, which do not enter into the corresponding equations (9) and (20). As was emphasized, *the form with constant coefficients* $\frac{1}{2}\sum_e m_e (dx_e^2 + dy_e^2 + dz_e^2)$ appears in the latter equations in place of the form $f(dx)$, so the *normal variables* $u_\mathfrak{a}$ will go to *the differences* $(x_e - a_e)$, $(y_e - b_e)$, $(z_e - c_e)$, *the normal type* $\varphi(du)$ will coincide with the form $\frac{1}{2}\sum_e m_e (dx_e^2 + dy_e^2 + dz_e^2)$ *itself*, and *the function* $f_0(u)$ will become *the function G*. Under those circumstances, the *normal type* $\varphi(du)$ *will be a form with constant coefficients*, and whenever $\varphi(du)$ becomes a form with constant coefficients, the expression $\sum_\mathfrak{a} \frac{\partial\varphi(du)}{\partial u_\mathfrak{a}} u_\mathfrak{a}$ must obviously vanish. However, it is also proved in (Journal f. Math., Bd. 70, pp. 92, *et seq.*) that *when the form* $f(dx)$ *can be transformed into a form with constant coefficients, the normal type* $\varphi(du)$ *will represent such a form, and that the expression* $\sum_\mathfrak{a} \frac{\partial\varphi(du)}{\partial u_\mathfrak{a}} u_\mathfrak{a}$ *can vanish only when the form* $f(dx)$ *can be transformed into a form with constant coefficients.* Namely, the left-hand side of the equation that was denoted by (59) on (*loc. cit*., pp. 94) will go to the expression $\sum_\mathfrak{a} \frac{\partial\varphi(\delta u)}{\partial u_\mathfrak{a}} u_\mathfrak{a}$ when it is multiplied by $(t - t_0)$. *The necessary and sufficient condition for the vanishing of the expression* $\sum_\mathfrak{a} \frac{\partial\varphi(du)}{\partial u_\mathfrak{a}} u_\mathfrak{a}$ *then consists of saying that* *the form f* (dx) *can be transformed into a form with constant coefficients* (*). *Since it was initially demanded that the quadratic form f* (dx) *must be essentially-positive and have a non-vanishing determinant, the normal type* φ (du) *must have the same property, and when the form f* (dx) *can be transformed into a form with constant coefficients, that normal type must necessarily equal an aggregate of squares of n differentials*. On those grounds, the assumptions that are actually true in mechanics, which were founded in article 1, represent the most general situation under which an essentially-positive quadratic form f (dx) is compatible with the vanishing of the expression $\sum_{\mathfrak{a}} \frac{\partial \varphi(du)}{\partial u_{\mathfrak{a}}} u_{\mathfrak{a}}$.

When one lets the variables $x_{\mathfrak{a}}$ coincide with the combinations $\sqrt{m_e}\,(x_e - a_e)$, $\sqrt{m_e}\,(y_e - b_e)$, $\sqrt{m_e}\,(z_e - c_e)$, the form $\frac{1}{2}\sum_e m_e(dx_e^2 + dy_e^2 + dz_e^2)$ will coincide with the form $\frac{1}{2}\sum_{\mathfrak{a}} dx_{\mathfrak{a}}^2$. It was already set down in (Journal f. Math., Bd. 71, pp. 284) how, under the assumption that f (dx) = $\frac{1}{2}\sum_{\mathfrak{a}} dx_{\mathfrak{a}}^2$, the theory of the function $\lambda_1\sqrt{(1,1)}$ is very closely connected with the *extension of the theory of curvature* that **Kronecker** gave (Monatsbericht der Berliner Akademie, August 1869). In fact, the quantity that **Kronecker** called ρ coincides with the one that was called $\sqrt{2G}$ or R above, and the extension that we gave in article 1 for the concept of center of curvature differs from the one that **Kronecker** developed only by its connection with the mechanical representation and the choice of the steps that would lead to that objective. Moreover, the *maximum-minimum problem* that was suggested corresponds precisely to the one that **Kronecker** treated in the aforementioned place. However, in order to explain *why admission to those investigations is even possible when one starts in mechanics and geometry, and why the results of mechanics that are contained in equations* (9) *and* (39) *and the results of geometry that are based upon equations* (20) *and* (52) *can depend upon the same algorithms*, I would like to recall something that **Gauss** said in the paper "Beiträge zur Theorie der algebraischen Gleichungen" in regard to the manner by which one proves fundamental theorems about algebraic equations that nonetheless takes on a much broader sense. He said:

> "However, at its basis, the actual content of all argumentation belongs to a higher realm in which one studies general, abstract quantities that are independent of spatial ones, and in which one addresses those combinations of quantities that are connected with continuity, which is a realm that has been explored only slightly at this time, and in which one also cannot move without having a language that is borrowed from spatial structures."

Bonn, 16 February 1873.

(*) A *direct criterion* for the form f (dx) to have that character is presented and proved in Journal f. Math., Bd. 70, pp. 94, *et seq.*

"Beitrag zu der Theorie der Krümmung," J. reine angew. Math. **81** (1876), 230-242.

Contribution to the theory of curvature

(By Herrn **R. Lipschitz** in Bonn)

Translated by D. H. Delphenich

The theorem that under the bending of a surface, the set of the reciprocal values of the two radii of principal curvature will change, but the product of the reciprocal values of the two radii of principal curvature – or the *Gaussian curvature* – will remain unchanged has allowed me to pose a corresponding question in the context of the generalization of the theory of curvature that I published in the treatises "Entwicklung einiger Eigenschaften der quadratischen Formen von n Differentialen," Bd. 71, pp. 274 and pp. 288 of this Journal. There, for the manifold of n variables $x_{\mathfrak{a}}$, where the symbol $\mathfrak{a}$ (as well as $\mathfrak{b}$, $\mathfrak{c}$ in what follows) varies from 1 to n, the square of the line element is assumed to be equal to an essentially positive quadratic form $2\, f\, (dx)$ of the n differentials $x_{\mathfrak{a}}$ whose coefficients are arbitrary functions of the variables $x_{\mathfrak{a}}$, and the system of equations that is true for those variables y_1 = const., y_2 = const., …, y_l = const. will determine a manifold of order $n - l$. Instead of considering the reciprocal radius of curvature of the normal section of a surface, one can consider the moment of the pressure that a material point on the surface experiences as long as it must move on it free from accelerating forces. That mechanical concept will be extended to the sum of the moments of all pressures when the motion of a system of material points is subject to a series of condition equations, but no accelerating forces. As long as one regards the coordinates of all material points as the variables $x_{\mathfrak{a}}$ of a manifold of order n, the sum of the *vis vivas* as the square of the line element $2\, f\, (dx)$ of the manifold of order n, divided by the square of the time element dt, and the condition equations that exist between the coordinates of the material points as the aforementioned equations:

$$y_1 = \text{const.}, \qquad y_2 = \text{const.}, \qquad \ldots, \qquad y_n = \text{const.},$$

the sum of all moments of all pressures will be consistent with the suggested intuition, and will be represented by the expression:

$$\lambda_1\, \delta y_1 + \lambda_2\, \delta y_2 + \ldots + \lambda_l\, \delta y_l$$

which was referred to by (14) in Bd. 71, pp. 277 of this Journal. It must be covariant in the form $f\,(dx)$ and the system functions that are set to constants, since a mechanical concept cannot depend upon the notation that was introduced. If one forms the square of that expression and replaces the products of any two variations $\delta y_\alpha\ \delta y_\beta$, where α and β mean all numbers from 1 to l, with the expressions (α, β) that were defined in *loc. cit.* then that will produce the expression:

$$\sum_{\alpha,\beta} \lambda_\alpha\, \lambda_\beta\, (\alpha, \beta),$$

which is covariant in the same way as in the basic theorems that were developed there. That expression, which I was led to consider on another occasion, represents the sum of the squares of the pressures that are present (always divided by the mass of the point in question), so it is the quantity that is supposed to be a minimum in the *principle of least constraint* that **Gauss** introduced, in the associated sense of the term. In the event that only *one* condition function y_1 exists, and the number l is accordingly equal to unity, the expression will define a complete square, and its square root:

$$\lambda_1 \sqrt{(1,1)}$$

will be equal to minus the reciprocal value of a quantity ρ that represents a generalization of the radius of curvature of a surface to the relevant manifold of order $n - 1$. In regard to that concept, allow me to refer to the paper "Sätze aus dem Grenzgebiet der Mechanik und der Geometrie," Bd. 6, pp. 417 of **C. Neumann**'s mathematical annals. In Bd. 71, pp. 282 of this journal, the aforementioned quantity $\lambda_1\sqrt{(1,1)}$ appeared as a quadratic form in the differential quotients $x'_\mathfrak{a}$, which are coupled by the equation $2f(x') = 1$. Now, the differential quotients $x'_\mathfrak{a}$ in λ_1 might be replaced with the differentials $dx_\mathfrak{a}$, which are free of the corresponding condition, and that will imply the equation for $-1/\rho$:

$$-\frac{1}{\rho} = \frac{\lambda_1 \sqrt{(1,1)}}{2\,f\,(dx)}.$$

At the same time, the equation $D(\omega) = 0$, which was denoted by (18) in Bd. 71, pp. 278 of this journal, will give the value of the $n - 1$ radius of principal curvature with the substitution $\omega = -1/\rho$. When one develops the function $D(\omega)$, which is represented by formula (30) in *loc. cit.* for the case that we speak of, in powers of ω:

$$D(\omega) = D_0\, \omega^{n-1} + D_1\, \omega^{n-2} + \ldots + D_{n-1},$$

the expression D_{n-1} / D_0 will then represent the product of the reciprocal values of the $n - 1$ radii of principal curvature. The even-order coefficients and the products of any two odd-order coefficients in the function $D(\omega) / D_0$ are covariant in the form $f(dx)$ and the function y_1 that is set equal to a constant. As long as the form $2\,f(dx)$ is equal to the special form $\sum_\mathfrak{a} dx_\mathfrak{a}^2$, or can be transformed into that form, the quantity ρ will be converted into that generalization of the radius of curvature, and the expression D_{n-1} / D_0 will be converted into that generalization of the curvature that **Kronecker** had presented in the Monatsbericht der Berliner Akademie on August 1869.

The aforementioned assumption shall be made from now on, and the quadratic form in the n – 1 differentials, into which the form $2\,f\,(dx) = \sum_{\mathfrak{a}} dx_{\mathfrak{a}}^2$ will go under the introduction of n – 1 independent variables y_σ (where σ, as well as τ, ranges through the sequence of numbers from 2 to n) might be denoted by $2\bar{g}\,(dy) = \sum_{\sigma,\tau} e_{\sigma,\tau}\, dy_\sigma\, dy_\tau$. It means the square of the line element for the manifold of order n – 1 that is determined by the equation y_1 = const. in the original manifold of order n of the $x_{\mathfrak{a}}$, and a transformation of it by means of the introduction of a new system of n – 1 independent variables corresponds to a bending of a surface under which the square of the line element of the surface does not change. Thus, the question that was posed to begin with becomes the question of how the coefficients of the function $D\,(\omega) / D_0$ will behave under a transformation of the form $2\bar{g}\,(dy)$. If follows from the basic theorems that were established that the even-order coefficients and the products of any two odd-order coefficients, as long as they can be expressed in terms of only the quantities $e_{\sigma,\tau}$ and their partial derivatives with respect to the y_σ, will likewise be invariants of the form $\bar{g}\,(dy)$. On pp. 293 of *loc. cit.*, I have verified that all of the *even-order coefficients* $D_2 / D_0, D_4 / D_0, \ldots$ possess that property under the given conditions, and from a remark that I made there, *as long as n is an even number*, one will also find the coefficient D_{n-1} / D_0 among them, which is the aforementioned generalization of curvature. An examination of the behavior of the *odd-order coefficients* – to which the coefficient D_{n-1} / D_0 will belong *when n is an even number* – will give the following theorem:

All combinations $D_{2r+1}\, D_{2s+1} / D_0^2$ *for which the sum of the indices* $2r + 2s + 2$ *is greater than or equal to* $n + 1$ *are invariants of the form* $\bar{g}\,(dy)$.

Upon introducing the n – 1 variables y_σ, the quantity $\lambda_1\sqrt{(1,1)}$ that was defined just now will go to a quadratic form in the differentials dy_σ, and it will be called:

$$2\bar{\mu}\,(dy) = \sum_{\sigma,\tau} \mu_{\sigma,\tau}\, dy_\sigma\, dy_\tau \;.$$

One will then have:

$$-\frac{1}{\rho} = \frac{\bar{\mu}\,(dy)}{\bar{g}\,(dy)},$$

and by means of a transformation that was performed on pp. 285, *loc. cit.,* the aforementioned function $D\,(\omega)$ will be equal to the product of a factor that is independent of ω (whose value is determined on pp. 32, Bd. 76, of this journal and denoted by Δ / E) with the determinant:

$$|-\mu_{\sigma,\tau} + \omega\, e_{\sigma,\tau}\,| \;.$$

Thus, the coefficients $D_1 / D_0, D_2 / D_0, \ldots$ can be constructed by means of a known algorithm. Now the elements $\mu_{\sigma,\tau}$, as a result of their laws of construction, depend upon other determining data, such as the quantities $e_{\sigma,\tau}$ and their partial derivatives with respect to the y_σ , whereas all of the second-order partial determinants that are obtained from the $\mu_{\sigma,\tau}$ are represented in terms of the

$e_{\sigma,\tau}$ and their first and second-order partial derivatives with respect to the y_σ. Namely, if one denotes (as in Bd. 71, pp. 292 of this journal) a certain form that is covariant in $\bar{g}\,(dy)$ and whose coefficients are composed from the quantities $e_{\sigma,\tau}$ and their partial derivatives with respect to the y_σ by:

$$\bar{\Omega}(dy, \overset{2}{dy}, \overset{1}{dy}, \overset{3}{dy}) = \sum_{\sigma,\tau,\sigma',\tau'} (\sigma, \tau, \sigma', \tau')(dy_\sigma\, \overset{2}{dy}_{\sigma'} - \overset{2}{dy}_\sigma\, dy_{\sigma'})(\overset{1}{dy}_\tau\, \overset{3}{dy}_{\tau'} - \overset{3}{dy}_\tau\, \overset{1}{dy}_{\tau'})$$

then *under the prevailing condition that the form* $2\bar{g}\,(dy)$ *arises from a form with constant coefficients* $\sum_{\mathfrak{a}} dx_{\mathfrak{a}}^2$ *by adding the equation* y_1 = const, one will have the equation:

$$-\tfrac{1}{2}\bar{\Omega}(dy, \overset{2}{dy}, \overset{1}{dy}, \overset{3}{dy}) = \sum_{\sigma,\tau,\sigma',\tau'} (\mu_{\sigma,\tau}\, \mu_{\sigma',\tau'} - \mu_{\sigma,\tau'}\, \mu_{\sigma',\tau})(dy_\sigma\, \overset{2}{dy}_{\sigma'} - \overset{2}{dy}_\sigma\, dy_{\sigma'})(\overset{1}{dy}_\tau\, \overset{3}{dy}_{\tau'} - \overset{3}{dy}_\tau\, \overset{1}{dy}_{\tau'}),$$

which was denoted by (14) in *loc. cit.* It is then by means of the equations:

$$-\tfrac{1}{2}(\sigma, \sigma', \tau, \tau') = \mu_{\sigma,\tau}\, \mu_{\sigma',\tau'} - \mu_{\sigma,\tau'}\, \mu_{\sigma',\tau}$$

that every coupling of the coefficients D_1 / D_0, D_2 / D_0, … that can be expressed in terms of only the second-order partial determinants that are obtained from the $\mu_{\sigma,\tau}$ can be represented in the given way by means of the quantities $e_{\sigma,\tau}$. From a known theorem, any partial determinant of even order from an arbitrary system of elements can be expressed completely and rationally in terms of second-order partial determinants. The given property of the even-order coefficients D_2 / D_0, D_4 / D_0, … follows immediately from that. In order to investigate the product of two odd-order coefficients $D_{2r+1}\, D_{2s+1} / D_0^2$, I remark that in the suggested representation, the denominator will be equal to the square of the determinant $|\, e_{\sigma,\tau}\,|$, and the numerator will be equal to the product of two combinations, the first of which is a linear function of the partial determinants of order $2r + 1$ of the elements $\mu_{\sigma,\tau}$, while the second one is a linear function of the partial determinants of order $2s + 1$. Let P denote any partial determinant of order $2r + 1$, and let Q denote any of those partial determinants of order $2s + 1$. The combination $D_{2r+1}\, D_{2s+1} / D_0^2$ can then be expressed by means of the second-order partial determinants that we spoke of, provided that all of the products PQ possess that property. However, as will be shown, that is always the case, as long as the sum of the two order-numbers $2r + 2s + 2$ is greater than or equal to $n + 1$.

The first index σ in the elements $\mu_{\sigma,\tau}$ of which the determinant P consists runs through a sequence of $2r + 1$ different numbers; the first index s of the elements $\mu_{\sigma,\tau}$ of which the determinant Q consists run through a sequence of $2s + 1$ different numbers. Now at least two numbers in those two sequences of numbers must coincide when $2r + 2s + 2 \geq n + 1$, because when the two sequences either differ in all numbers of have only one number in common, either $2r + 2s + 2 \geq n + 1$ or $2r + 2s + 2 \geq n$, respectively, must appear, while only the $n - 1$ different indices 2, 3, …, n are present to begin with. I shall now introduce two numbers that occur in the two sequences and assume, for simplicity of notation, that they are the numbers 2 and 3. Furthermore, the second index τ in the determinant P might run through the numbers φ, while it will run through the numbers ψ in the determinant Q. Now, if the partial determinant $\partial P / \partial \mu_{2,\varphi}$ of order $2r$ were equal to $P_{2,\varphi}$ and the partial determinant of order $2s$ were set $\partial Q / \partial \mu_{3,\psi} = Q_{3,\psi}$ then the following equations would be true:

$$\sum_{\varphi} \mu_{2,\varphi} P_{2,\varphi} = P, \quad \sum_{\psi} \mu_{2,\psi} Q_{3,\psi} = 0,$$
$$\sum_{\varphi} \mu_{3,\varphi} P_{2,\varphi} = 0, \quad \sum_{\psi} \mu_{3,\psi} Q_{3,\psi} = Q,$$

and that would imply the following representation of the product PQ :

$$\sum_{\varphi,\psi} (\mu_{2,\varphi}\, \mu_{3,\psi} - \mu_{3,\varphi}\, \mu_{2,\psi}) P_{2,\varphi}\, Q_{3,\psi} = PQ .$$

That implies that there is a complete and rational representation in terms of second-order partial determinants in itself, because that factor $\mu_{2,\varphi}\, \mu_{3,\psi} - \mu_{3,\varphi}\, \mu_{2,\psi}$ is itself such a determinant, and the factors $P_{2,\varphi}$ and $Q_{3,\psi}$, as second-order partial determinants, can be expressed completely and rationally in terms of such determinants, from the cited theorem. A corresponding representation of the combination $D_{2r+1}\, D_{2s+1} / D_0^2$ will follow from that as long as the sum $2r + 2s + 2 \geq n + 1$. As was asserted, under the prevailing conditions, that combination can then be represented in terms of the quantities $e_{\sigma,\tau}$ and their first and second-order partial derivatives with respect to the y_σ, and will then be an invariant of the form $\bar{g}\,(dy)$.

From the rule that was just proved, as long as n means one of the *two* extra (*übertreffend*) even numbers, the square of the aforementioned generalization of the curvature D_{n-1}^2 / D_0^2 will be an invariant of the form $\bar{g}\,(dy)$. In that way, one will resolve the desired decision that was posed at the conclusion of the cited article in Bd. 71, pp. 295 of this journal, and it can be expressed as the theorem:

The generalization of the curvature D_{n-1} / D_0 is an invariant of the form $\bar{g}\,(dy)$ for odd n and the square root of an invariant of the form $\bar{g}\,(dy)$ for each of the two extra even n.

For the value $n = 4$, the second-order partial determinants that are constructed from the $\mu_{\sigma,\tau}$ will coincide with the adjoint elements of the form $2\,\bar{\mu}\,(dy)$, and the property of the square D_3^2 / D_0^2 that was proved will be based upon the theorem that the square of the determinant $|\,\mu_{\sigma,\tau}\,|$ is equal to the determinant of the adjoint elements of the form $2\,\bar{\mu}\,(dy)$. A representation of the square D_3^2 / D_0^2 from which that property can be read off immediately is included in the abstract from a very remarkable paper by **F. Souvorof** that the author made known in **Darboux**'s Bulletin, t. 4, pp. 181, and indeed that representation is found in the equation that is notated by (III_{a^*}) there.

It is not difficult to arrange the invariants that one finds for a given n clearly. The inequalities:

$$2r + 1 \leq n - 1, \qquad 2s + 1 \leq n - 1, \qquad 2r + 2s + 2 \geq n + 1$$

that are true for the indices in the combination $D_{2r+1}\, D_{2s+1} / D_0^2$ explain the fact that each index must be equal to at least *three*, and can be equal to at most the largest odd number β that is not greater than $n - 1$. Now when one lets the first index $2r + 1$ run through all odd numbers from β to *three* and demands that the second index $2s + 1$ cannot be greater than the first one, that will produce the following schema, which includes any allowable combination once and only once:

$$\begin{array}{c}
\dfrac{D_\beta^2}{D_0^2}, \quad \dfrac{D_\beta\, D_{\beta-2}}{D_0^2}, \quad \dfrac{D_\beta\, D_{\beta-4}}{D_0^2}, \quad \ldots, \quad \dfrac{D_\beta\, D_7}{D_0^2}, \quad \dfrac{D_\beta\, D_5}{D_0^2}, \quad \dfrac{D_\beta\, D_3}{D_0^2}, \\
\dfrac{D_{\beta-2}^2}{D_0^2}, \quad \dfrac{D_{\beta-2}\, D_{\beta-4}}{D_0^2}, \quad \ldots, \quad \dfrac{D_{\beta-2}\, D_7}{D_0^2}, \quad \dfrac{D_{\beta-2}\, D_5}{D_0^2}, \\
\dfrac{D_{\beta-4}^2}{D_0^2}, \quad \ldots, \quad \dfrac{D_{\beta-4}\, D_7}{D_0^2}, \\
\ldots
\end{array}$$

Any row includes two combinations less than the previous one, and the last row consists of the one combination $D_{(\beta+3)/2}^2 / D_0^2$ or the two combinations $D_{(\beta+5)/2}^2 / D_0^2$ and $D_{(\beta+5)/2}\, D_{(\beta+1)/2} / D_0^2$, according to whether $(\beta + 3) / 2$ or $(\beta + 5) / 2$, resp., is an odd number. The first or second case will occur according to whether the number of combinations in the first row is odd or even, respectively. Thus, the number of combinations that appear will be equal to $(\beta + 1)^2 / 16$ in the first case and equal to $(\beta - 1)(\beta + 3) / 16$ in the second. However, when one introduces the remainder that one gets from dividing the number n by *four* and sets the number n equal to one of the forms:

$$4k, \quad 4k + 1, \quad 4k + 2, \quad 4k + 3,$$

one will obtain the corresponding values for the odd number β:

$$4k - 1, \quad 4k - 1, \quad 4k + 1, \quad 4k + 1,$$

and the number $(\beta + 3) / 2$ will be an odd number for the first two remainders, while the number $(\beta + 5) / 2$ will be an odd number for the last two.

The invariants that were found yield a remarkable determination of the odd-order coefficients of the function $D(\omega) / D_0$; as expected, the coefficient D_1 / D_0, which includes the quantities $\mu_{\sigma,\tau}$ linearly, does not come into play in that. If the invariant D_β^2 / D_0^2 has a non-zero value then the coefficient D_β / D_0 will be obtained by extracting a square root and will satisfy the first row of the schema that was previously exhibited in order to represent all remaining odd-order coefficients $D_{\beta-2} / D_0$, $D_{\beta-4} / D_0$, …, D_3 / D_0 rationally in terms of those invariants and the aforementioned square root. If one further ponders the fact that all even-order coefficients D_2 / D_0, D_4 / D_0, … are invariants then that will imply the result that as long as the invariant D_β^2 / D_0^2 does not vanish, all of the coefficients of the function $D(\omega) / D_0$, with the exception of the coefficient D_1 / D_0, are invariants or can be expressed rationally in terms of invariants with the help of the square root of the invariant D_β^2 / D_0^2. As long as the invariant D_β^2 / D_0^2 is equal to zero, while the invariant $D_{\beta-2}^2 / D_0^2$ is non-zero, all invariants in the first row of the schema that was defined will vanish, the coefficient $D_{\beta-2} / D_0$ will be determined by extracting a square root, and the second row can be expressed rationally in terms of the coefficients $D_{\beta-4} / D_0, \ldots, D_5 / D_0$ by means of that square root and the remaining invariants. In that way, the uppermost row of the schema, whose terms do not all vanish, will always serve to represent the odd-order coefficients of the function $D(\omega) / D_0$.

An explicit expression for the form $\bar{\mu}(dy)$ was not required to achieve the stated goal. I have applied the coefficients of those forms $2g(dy) = \sum_{\mathfrak{a},\mathfrak{b}} e_{\mathfrak{a},\mathfrak{b}}\, dy_{\mathfrak{a}}\, dy_{\mathfrak{b}}$ into which the form $2f(dx)$ is transformed by introducing the n variables y_1 and y_σ . With the assumption that is presently valid (namely, that $2f(dx) = \sum_{\mathfrak{a}} dx_{\mathfrak{a}}^2$), the form (dy) can be represented with the help of the partial derivatives of the y_σ with respect to the $x_{\mathfrak{a}}$ as follows: Formulas (31) on pp. 283 and (32) on pp. 284, *loc. cit.*, yield the expression for the function $\lambda_1\sqrt{(1,1)}$:

$$\lambda_1 \sqrt{(1,1)} = \frac{-1}{\sqrt{(1,1)}} \sum_{\mathfrak{a},\mathfrak{b}} \frac{\partial^2 y_1}{\partial x_{\mathfrak{a}}\, \partial x_{\mathfrak{b}}}\, dx_{\mathfrak{a}}\, dx_{\mathfrak{b}}\,,$$

$$(1,1) = \sum_{\mathfrak{a}} \left(\frac{\partial y_1}{\partial x_{\mathfrak{a}}}\right)^2,$$

in which the $x'_{\mathfrak{a}}$ have been replaced with the $dx_{\mathfrak{a}}$. The component:

$$\sum_{\mathfrak{a},\mathfrak{b}} \frac{\partial^2 y_1}{\partial x_{\mathfrak{a}}\, \partial x_{\mathfrak{b}}}\, dx_{\mathfrak{a}}\, dx_{\mathfrak{b}} = d^2 y_1 - \sum_{\mathfrak{c}} \frac{\partial y_1}{\partial x_{\mathfrak{c}}}\, d^2 x_{\mathfrak{c}}$$

will assume the form:

$$-\sum_{\mathfrak{c}} \frac{\partial y_1}{\partial x_{\mathfrak{c}}} \sum_{\mathfrak{a},\mathfrak{b}} \frac{\partial^2 x_{\mathfrak{c}}}{\partial y_{\mathfrak{a}}\, \partial y_{\mathfrak{b}}}\, dy_{\mathfrak{a}}\, dy_{\mathfrak{b}}$$

when one introduces the n new variables y_1 and y_σ . Now, if the determinant $\sum \pm \frac{\partial x_1}{\partial y_1} \cdots \frac{\partial x_n}{\partial y_n}$ is denoted by C and the partial determinant $\partial C / \partial \left(\frac{\partial x_{\mathfrak{c}}}{\partial y_1}\right)$ by $C_{\mathfrak{c}}$ then it is known that $\frac{\partial y_1}{\partial x_{\mathfrak{c}}} = \frac{C_{\mathfrak{c}}}{C}$, and therefore the function $\lambda_1\sqrt{(1,1)}$ will be converted into:

$$\lambda_1 \sqrt{(1,1)} = \frac{\sum_{\mathfrak{c}} C_{\mathfrak{c}} \sum_{\mathfrak{a},\mathfrak{b}} \frac{\partial^2 x_{\mathfrak{c}}}{\partial y_{\mathfrak{a}}\, \partial y_{\mathfrak{b}}}\, dy_{\mathfrak{a}}\, dy_{\mathfrak{b}}}{\sqrt{\sum_{\mathfrak{c}} C_{\mathfrak{c}}^2}}.$$

Since the function $\lambda_1\sqrt{(1,1)}$ will go to the form $2\bar{\mu}(dy)$ when the variables $x_{\mathfrak{a}}$ that satisfy the equation y_1 = const. are expressed in terms of the independent variables y_σ , that will produce the desired transformation of that form in such a way that the differential dy_1 in the foregoing expression will be set equal to zero, and one will have:

$$2\bar{\mu}(dy) = \frac{\sum_{\mathfrak{c}} C_{\mathfrak{c}} \sum_{\sigma,\tau} \frac{\partial^2 x_{\mathfrak{c}}}{\partial y_\sigma \, \partial y_\tau} dy_\sigma \, dy_\tau}{\sqrt{\sum_{\mathfrak{c}} C_{\mathfrak{c}}^2}}.$$

Thus, the quantities $\mu_{\sigma,\tau}$ will then be determined from:

$$\mu_{\sigma,\tau} = \frac{\sum_{\mathfrak{c}} C_{\mathfrak{c}} \frac{\partial^2 x_{\mathfrak{c}}}{\partial y_\sigma \, \partial y_\tau}}{\sqrt{\sum_{\mathfrak{c}} C_{\mathfrak{c}}^2}}.$$

At the same time, the corresponding representation exists for the form $2\bar{g}$ (dy):

$$2\bar{g}(dy) = \sum_{\sigma,\tau} \sum_{\mathfrak{a}} \frac{\partial x_{\mathfrak{a}}}{\partial y_\sigma} \frac{\partial x_{\mathfrak{a}}}{\partial y_\sigma} dy_\sigma \, dy_\tau .$$

For $n = 3, 4$, the use of those expressions yields the representation of the *Gaussian curvature* that was published in art. 10 of his *disquisitions generales circa superficies curvas*. Moreover, it will give the same representation of the coefficients D_{n-1} / D_0 as the suggested *generalization of Gaussian curvature* that **R. Beez** had developed in a treatise "Über das Krümmungsmass von Mannigfaltigkeiten höherer Ordnung" that was published in Bd. 7, pp. 387 of **C. Neumann**'s mathematical annals (*).

I must still emphasize the essential difference that exists between the generalization that was discussed in the foregoing and another generalization of **Gaussian** curvature. The basic property of **Gaussian** curvature that it is independent of any bending of the surface in question is also true of the shortest lines that can be drawn on the surface. Starting from that, **Riemann**, in his investigations into the foundations of geometry, replaced the square of the line element of a surface

(*) In another treatise "Zur Theorie des Krümmungsmasses von Mannigfaltigkeiten höherer Ordnung" in Jahrgang 20, Heft 6, pp. 424 of **Schlömilch**'s Zeitschrift für Mathematik und Physik, the same author derived, by direct calculation, the same equations that I have previously presented in Bd. 71, pp. 283, of this Journal and represented in symbols as $-\frac{1}{2}(\sigma, \sigma', \tau, \tau') = \mu_{\sigma,\tau}\, \mu_{\sigma',\tau'} - \mu_{\sigma,\tau'}\, \mu_{\sigma',\tau}$, and he suggested that I had overlooked the relations between the quantities $(\sigma, \sigma', \tau, \tau')$ that follow from those equations, although precisely those relations are mentioned explicitly in Bd. 71, pp. 295. On the other hand, **Beez** failed to recognize that those relations can be proved only when the equations that they are inferred from are valid, and that the equations in question are true only as long as the form $\bar{g}(dy)$ to which the quadrilinear form:

$$\bar{\Omega}(dy, \overset{2}{dy}, \overset{1}{dy}, \overset{3}{dy})$$

belongs arises from a form with constant coefficients $\frac{1}{2}\sum_{\mathfrak{a}} dx_{\mathfrak{a}}^2$ by adding the one equation y_1 = const. In the same treatise, the assertion is made that the generalization of the curvature D_{n-1} / D_0 can be represented in terms of the quantities $e_{\sigma,\tau}$ and their partial derivatives with respect to the y_σ only for $n = 3$, while it is not representable for larger values of n, with the single exception of the case in which the variables $x_{\mathfrak{a}}$ are rational functions of degree one and two in the variables y_σ. That assertion, whose incorrectness was already obvious for odd values of the number n, from the property of the expression D_{n-1} / D_0 that was quoted in Bd. 71, pp. 295, of this journal, will also be contradicted for even values of the number n by the theorem that was presented above in regard to that expression.

with an arbitrary essentially positive quadratic form in arbitrarily many differentials, and focused his attention on the first-order manifolds that would correspond to the shortest lines that could be drawn on a surface. If that quadratic form is denoted by $2\bar{g}\,(dy) = \sum_{\sigma,\tau} e_{\sigma,\tau}\, dy_\sigma\, dy_\tau$ then they will be the first-order manifolds for which the first variation of the integral $\int \sqrt{2\bar{g}\,(dy)}$ vanishes. It later became possible to represent that concept (for which, we have **Riemann** to thank) without adding those first-order manifolds, and it then took the form (*) (with the notations that were used above) of the quotient of the two covariants that are associated with the form $\bar{g}\,(dy)$:

$$\frac{-\frac{1}{2}\bar{\Omega}(dy, \overset{2}{d}y, \overset{1}{d}y, \overset{3}{d}y)}{\sum\limits_{\sigma,\tau,\sigma',\tau'} (e_{\sigma,\tau}\, e_{\sigma',\tau'} - e_{\sigma,\tau'} e_{\sigma',\tau})(dy_\sigma\, \overset{2}{d}y_\tau - dy_\tau\, \overset{2}{d}y_{\tau'})}\,.$$

For the original case of a surface, where one has $n - 1 = 2$, that quotient of two covariants will always be independent of the differentials that enter into it and equal to an invariant – namely, the **Gaussian** curvature itself. For an arbitrary value of the number $n - 1$, the quotient will vanish if and only if the form $2\bar{g}\,(dy)$ can be transformed into a form with constant coefficients. Moreover, it will be constant if and only if the form $2\bar{g}\,(dy)$ can be transformed into a certain special form that allows transformations into itself. Hence, **Riemann**'s generalization preserves the three properties of the **Gaussian** curvature that it does not change under a bending of the surface in question, that it will vanish whenever the surface can be developed into a plane, and it will be constant whenever the subsets of the surface can be displaced into each other.

The extension of the curvature of a surface that was treated previously refers to a manifold of order $n - 1$ that emerges from a manifold of order n by adding one equation. The square of the line element of the manifold of order n was initially assumed to be equal to an arbitrary quadratic form $2\,f\,(dx)$, and the expression D_{n-1} / D_0 correspondingly represented the product of the reciprocal values of the $n - 1$ radius of principal curvature. Later, it was assumed that the form $2\,f\,(dx)$ was equal to the special form $\sum_\mathfrak{a} dx_\mathfrak{a}^2$ or can be transformed into it, and only under that assumption was the theorem above proved that the expression D_{n-1} / D_0 is an invariant for odd n and the square root of an invariant of the form $\bar{g}\,(dy)$ for even n. Under the same assumption, the expression D_2 / D_0 will represent an invariant of the form $\bar{g}\,(dy)$ for any n. It is easy to see that the expression means the aggregate of the $(n - 2)(n - 3) / 2$ pairs of reciprocal values of the $n - 1$ radii of principal curvature, and will then likewise become the **Gaussian** curvature for $n = 3$. At the same time, it defines an intermediate term between the generalization of the radius of curvature that we speak of and **Riemann**'s generalization of the curvature, because when the previously-given quotient of two covariants is independent of the differentials for an arbitrary value of the number $n - 1$, and is converted into an invariant of the form $\bar{g}\,(dy)$, that invariant (**) will be equal to the invariant D_2 / D_0 divided by the number of pairs $(n - 2)(n - 3) / 2$, under the oft-cited condition that $2\,f\,(dx) = \sum_\mathfrak{a} dx_\mathfrak{a}^2$.

Bonn, 16 December 1875.

(*) Bd. 74, pp. 24, of this journal and **Darboux**'s Bulletin, t. 4, pp. 150.
(**) Bd. 71, pp. 294 and Bd. 72, pp. 33, of this journal.

“Bemerkungen zu dem Princip des kleinstes Zwanges,” J. reine angew. Math. **82** (1877), 316-342.

Remarks on the principle of least constraint

By **R. Lipschitz** in Bonn

Translated by D. H. Delphenich

1.

The collected mathematical works of **Riemann**, whose publication defines an enduring contribution by **R. Dedekind** and **H. Weber**, contains his response (written in Latin) to a problem that the Paris Academy posed in the year 1858 in regard to a problem in the distribution of heat, which was an article that was extended most remarkably to **Riemann**’s paper “Über die Hypothesen, welche der Geometrie zu Grunde liegen.” In the second part of the aforementioned article, **Riemann** developed the conditions for a given quadratic form in n differentials whose coefficients are arbitrary functions of the n variables in question to be transformable into a form with constant coefficients, and in it, he summarized the conditions for all of the coefficients of a certain form [which is denoted by (II) on page 382 in the cited location] that is quadratic in two systems of differentials and covariant to the given form to vanish. That criterion can be combined with the one that I derived for that question in my treatise “Untersuchungen in Betreff der ganzen homogenen Functionen von n Differentialen” (vol. 70 of this journal, page 71). The form Ψ, which was defined on page 84 of that reference by four systems of linear forms, which were also equal to the form that **Christoffel** denoted by G_4 in the same volume of the journal on page 58, will go to **Riemann**’s aforementioned form (II) as long as two plus two of the associated systems of differentials to the former system are set equal to each other, and from page 94 of the cited volume, the *necessary* and *sufficient condition* for the given quadratic form in n differentials to be convertible into a quadratic form with constant coefficients consists of the identical vanishing of the associated quadrilinear form Ψ. With the help of the aforementioned form (II), **Riemann** exhibited an analytical expression (III) for the concept of the curvature in a manifold of n^{th} order in that reference under which the given quadratic form in n differentials represented the square of the line element, but the expression (III) emerged from the analytical expression for that concept that was given in the treatise: “Fortgesetze Untersuchungen in Betreff der ganzen homogenen Functionen von n Differentialen,” vol. 72 of this journal, page 1, and especially page 24 (which was the quantity k_0 in that treatise), and it coincided completely with the expression that was cited in vol. 4 of **Darboux**’s Bulletin on page 150 and reiterated in vol. **81** of this journal on page 241.

Riemann communicated yet another way of representing the form that he denoted by (II), once the law of defining its coefficients is given, in which different types of variation signs were used, and three second-order equations in the variations were prescribed. That curious algorithm is explained by a fact that has been known to me for some years now. The form (II), which is covariant to the given quadratic form in n differentials, can in fact be regarded as the aggregate of two covariants, one of which equals **Riemann**'s second expression that is in question, while the other one will vanish, due to the equations that **Riemann** indicated. Now the essential components of the latter covariant make it the same covariant that the principle of least constraint required to be a minimum, and that will define the subject of the present article.

The result that was just quoted can be concluded from an equation that appeared as (37) in the second-cited treatise in vol. 72 of this journal on page 16. As in that location, let the given quadratic form in n differentials $dx_{\mathfrak{a}}$, whose coefficients are arbitrary functions of the n variables $x_{\mathfrak{a}}$, and in which the indices $\mathfrak{a}$, $\mathfrak{b}$, … go from 1 to n, be the following one:

$$f(dx) = \tfrac{1}{2}\sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, dx_{\mathfrak{a}}\, dx_{\mathfrak{b}}\,. \tag{1}$$

Let the bilinear form that is derived from it be:

$$f(dx, \overset{1}{dx}) = \tfrac{1}{2}\sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, dx_{\mathfrak{a}}\, \overset{1}{dx}_{\mathfrak{b}}\,. \tag{2}$$

One further has the equation:

$$-\,\delta f(dx, \overset{1}{dx}) + df(\delta x, \overset{1}{dx}) + \overset{1}{df}\,(\delta x, dx) = \sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, d\,\overset{1}{dx}_{\mathfrak{b}}\,\delta x_{\mathfrak{a}} + \sum_{\mathfrak{a}} f_{\mathfrak{a}}(dx, \overset{1}{dx})\,\delta x_{\mathfrak{a}} \tag{3}$$

for the unrestricted application of the three variation symbols. $f_{\mathfrak{a}}(dx, \overset{1}{dx})$ is then a form that is bilinear in the differentials $dx_{\mathfrak{a}}$ and $\overset{1}{dx}_{\mathfrak{b}}$ that has this developed form:

$$f_{\mathfrak{a}}(dx, \overset{1}{dx}) = \tfrac{1}{2}\sum_{\mathfrak{a},\mathfrak{g},\mathfrak{b}} f_{\mathfrak{a},\mathfrak{g},\mathfrak{b}}\, dx_{\mathfrak{g}}\, \overset{1}{dx}_{\mathfrak{b}} = \tfrac{1}{2}\sum_{\mathfrak{g},\mathfrak{b}}\left(\frac{\partial a_{\mathfrak{a},\mathfrak{g}}}{\partial x_{\mathfrak{b}}} + \frac{\partial a_{\mathfrak{a},\mathfrak{b}}}{\partial x_{\mathfrak{g}}} - \frac{\partial a_{\mathfrak{g},\mathfrak{b}}}{\partial x_{\mathfrak{a}}}\right) dx_{\mathfrak{g}}\, \overset{1}{dx}_{\mathfrak{b}}\,. \tag{4}$$

In addition, let:

$$\sum_{\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, d\,\overset{1}{dx}_{\mathfrak{b}} + f_{\mathfrak{a}}(dx, \overset{1}{dx}) = \Psi_{\mathfrak{a}}(dx, \overset{1}{dx}), \tag{5}$$

so

$$-\,\delta f(dx, \overset{1}{dx}) + df(\delta x, \overset{1}{dx}) + \overset{1}{df}\,(\delta x, dx) = \sum_{\mathfrak{b}} \Psi_{\mathfrak{a}}(dx, \overset{1}{dx})\,\delta x_{\mathfrak{a}}\,. \tag{6}$$

If one forms the same linear expressions in the coefficients $a_{\mathfrak{a},1}, a_{\mathfrak{a},2}, \ldots, a_{\mathfrak{a},n}$ that one finds in $\Psi_{\mathfrak{a}}(dx, \overset{1}{dx})$, i.e.:

(7) $$\sum_{\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}} [d\, \overset{1}{dx}_{\mathfrak{b}} + \xi_{\mathfrak{a}}(dx, \overset{1}{dx})] = \Psi_{\mathfrak{a}}(dx, \overset{1}{dx}) ,$$

then the combinations $\xi_{\mathfrak{a}}(dx, \overset{1}{dx})$ will be defined by:

(8) $$\xi_{\mathfrak{a}}(dx, \overset{1}{dx}) = \sum_{\mathfrak{c}} \frac{A_{\mathfrak{b},\mathfrak{c}}}{\Delta} f_{\mathfrak{c}}(dx, \overset{1}{dx}) ,$$

with the help of the non-zero determinant $| a_{\mathfrak{a},\mathfrak{b}} | = \Delta$ and the adjoint element $\partial\Delta / \partial a_{\mathfrak{a},\mathfrak{b}} = A_{\mathfrak{a},\mathfrak{b}}$. The aforementioned form $\Psi(\overset{1}{dx}, \overset{1}{\delta x}, dx, \delta x)$, which is linear in the four systems of differentials $\overset{1}{dx}_{\mathfrak{a}}$, $\overset{1}{\delta x}_{\mathfrak{a}}$, $dx_{\mathfrak{g}}$, $\delta x_{\mathfrak{b}}$ and covariant to the given form $f(dx)$, will then be expressed in terms of the likewise-cited equation (37) as:

(9) $$\begin{cases} \tfrac{1}{2}\Psi(\overset{1}{dx}, \overset{1}{\delta x}, dx, \delta x) = \sum_{\mathfrak{b}} [d\Psi_{\mathfrak{b}}(\delta x, \overset{1}{dx})\, \overset{1}{\delta x}_{\mathfrak{b}} - \Psi_{\mathfrak{b}}(\delta x, \overset{1}{dx})\, \xi_{\mathfrak{b}}(dx, \overset{1}{\delta x})] \\ \qquad\qquad - \sum_{\mathfrak{b}} [\delta\Psi_{\mathfrak{b}}(dx, \overset{1}{dx})\, \overset{1}{\delta x}_{\mathfrak{b}} - \Psi_{\mathfrak{b}}(dx, \overset{1}{dx})\, \xi_{\mathfrak{b}}(\delta x, \overset{1}{\delta x})]. \end{cases}$$

One now obtains the desired conversion when one replaces the expression $\xi_{\mathfrak{a}}(dx, \overset{1}{\delta x})$ with the combination of the expressions $-d\, \overset{1}{\delta x}_{\mathfrak{b}}$ and $d\, \overset{1}{\delta x}_{\mathfrak{b}} + \xi_{\mathfrak{b}}(dx, \overset{1}{\delta x})$ and replaces the expression $\xi_{\mathfrak{a}}(dx, \overset{1}{\delta x})$ with the combination of the corresponding expressions $-\delta\, \overset{1}{\delta x}_{\mathfrak{b}}$ and $\delta\, \overset{1}{\delta x}_{\mathfrak{b}} + \xi_{\mathfrak{b}}(\delta x, \overset{1}{\delta x})$. In that way, the form $\tfrac{1}{2}\Psi(\overset{1}{dx}, \overset{1}{\delta x}, dx, \delta x)$ will be equal to the aggregate of the combination:

(10) $$\sum_{\mathfrak{b}} [d\Psi_{\mathfrak{b}}(\delta x, \overset{1}{dx})\, \overset{1}{\delta x}_{\mathfrak{b}} + \Psi_{\mathfrak{b}}(\delta x, \overset{1}{dx})\, d\, \overset{1}{\delta x}_{\mathfrak{b}}] - \sum_{\mathfrak{b}} [\delta\Psi_{\mathfrak{b}}(dx, \overset{1}{dx})\, \overset{1}{\delta x}_{\mathfrak{b}} + \Psi_{\mathfrak{b}}(dx, \overset{1}{dx})\, \delta\overset{1}{\delta x}_{\mathfrak{b}}]$$

and the combination:

(11) $$-\sum_{\mathfrak{b}} \Psi_{\mathfrak{b}}(\delta x, \overset{1}{dx})[d\, \overset{1}{\delta x}_{\mathfrak{b}} + \xi_{\mathfrak{b}}(dx, \overset{1}{\delta x})] + \sum_{\mathfrak{b}} \Psi_{\mathfrak{b}}(dx, \overset{1}{dx})[\delta\, \overset{1}{\delta x}_{\mathfrak{b}} + \xi_{\mathfrak{b}}(\delta x, \overset{1}{\delta x})] .$$

The combination (10) is obviously equal to the complete variation $d\sum_{\mathfrak{b}} \Psi_{\mathfrak{b}}(\delta x, \overset{1}{dx})\, \overset{1}{\delta x}_{\mathfrak{b}}$, minus the complete variation $\delta\sum_{\mathfrak{b}} \Psi_{\mathfrak{b}}(dx, \overset{1}{dx})\, \overset{1}{\delta x}_{\mathfrak{b}}$. However, the sums to be varied can be represented as aggregates of first variations by means of formula (6). Thus, the combination (10) will appear to be the aggregate of second variations:

$$d \overset{1}{d} f \, (\delta x, \overset{1}{\delta} x) + \delta \overset{1}{\delta} f \, (dx, \overset{1}{d} x) - d \overset{1}{\delta} f \, (dx, \overset{1}{d} x) - \delta \overset{1}{d} f \, (dx, \overset{1}{\delta} x). \tag{12}$$

The expressions $d \overset{1}{d} x_{\mathfrak{b}} + \xi_{\mathfrak{b}} (dx, \overset{1}{d} x)$ can be represented as follows:

$$d \overset{1}{d} x_{\mathfrak{b}} + \xi_{\mathfrak{b}} (dx, \overset{1}{d} x) = \sum_{\mathfrak{c}} \frac{A_{\mathfrak{b},\mathfrak{c}}}{\Delta} \Psi_{\mathfrak{c}} (dx, \overset{1}{d} x), \tag{7*}$$

so the combination (11) will assume the form:

$$\sum_{\mathfrak{b},\mathfrak{c}} \frac{A_{\mathfrak{b},\mathfrak{c}}}{\Delta} [\Psi_{\mathfrak{b}} (dx, \overset{1}{d} x) \Psi_{\mathfrak{c}} (\delta x, \overset{1}{\delta} x) - \Psi_{\mathfrak{b}} (\delta x, \overset{1}{d} x) \Psi_{\mathfrak{c}} (dx, \overset{1}{\delta} x)] \ . \tag{13}$$

The combination (12), as well as the combination (13), is a covariant of the form $f(dx)$, and the basis for that is given in the cited location (vol. 72 of this journal, pages 16 and 17). *One therefore has the theorem that one-half the value of the form* $\Psi(\overset{1}{d} x, \overset{1}{\delta} x, dx, \delta x)$ *equals the aggregate of the covariant* (12) *and the covariant* (13). Both covariants contain first and second variations of the variables, but all of the second variations will cancel when one forms their aggregate, and all that will remain are the first variations.

In order to make the transition to **Riemann**'s formulas, the variation symbol $\overset{1}{d}$ must now be set equal to the variation symbol d, and the symbol $\overset{1}{\delta}$ must be set equal to the symbol δ. The covariant (12) will then be converted into the expression:

$$\tfrac{1}{2} dd \sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}} \, \delta x_{\mathfrak{a}} \, \delta x_{\mathfrak{b}} - d\delta \sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}} \, dx_{\mathfrak{a}} \, \delta x_{\mathfrak{b}} + \tfrac{1}{2} \delta\delta \sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}} \, dx_{\mathfrak{a}} \, dx_{\mathfrak{b}} \tag{14}$$

by the complete representation of the quadratic and bilinear forms, and the covariant (13) will be converted into the expression:

$$\sum_{\mathfrak{b},\mathfrak{c}} \frac{A_{\mathfrak{b},\mathfrak{c}}}{\Delta} [\Psi_{\mathfrak{b}} (dx, dx) \, \Psi_{\mathfrak{c}} (\delta x, \delta x) - \Psi_{\mathfrak{b}} (dx, \delta x) \, \Psi_{\mathfrak{c}} (dx, \delta x)]. \tag{15}$$

At the same time, as a result of the theorem that was proved, the aggregate of the two expressions (14) and (15) will be equal to one-half the value of the form $\Psi(dx, \delta x, dx, \delta x)$. In the investigations that are connected with the present ones, the square of the line element for the manifold of n variables $x_{\mathfrak{a}}$ was denoted by $2f(dx) = \sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}} \, dx_{\mathfrak{a}} \, dx_{\mathfrak{b}}$ in vol. 72 of this journal, page 24. Therefore, that quadratic form will have the same meaning as **Riemann**'s form $\sum b_{\mathfrak{i},\mathfrak{i}'} \, ds_{\mathfrak{i}} \, ds_{\mathfrak{i}'}$, and at the same time, the form Ψ (dx, δx, dx, δx) will correspond to the first representation of **Riemann**'s form (II). One thus recognizes that one-half the value of **Riemann**'s form (II) is equal to the aggregate of the covariants (14) and (15). Furthermore, with the notation that was introduced, the expression (14) above will be equal to one-half the expression that one will find on page 381 of **Riemann**'s paper in the fifth line from the bottom, and will define the second representation of his form (II). Now, **Riemann**'s three equations that appear at the bottom of that page say that the combinations $\Psi_{\mathfrak{b}}$ (dx, δx), $\Psi_{\mathfrak{b}}$ (dx, dx), $\Psi_{\mathfrak{b}}$ (δx, δx) should be taken to be zero, since on the basis of equation (6) above, the left-hand sides of the three equations will coincide with the three expressions:

$$-2\sum_{\mathfrak{a}} \Psi_{\mathfrak{a}}(dx, \delta x)\, \overset{1}{\delta x}_{\mathfrak{a}}\ ,$$

$$-2\sum_{\mathfrak{a}} \Psi_{\mathfrak{a}}(dx, dx)\, \overset{1}{\delta x}_{\mathfrak{a}}\ ,$$

$$-2\sum_{\mathfrak{a}} \Psi_{\mathfrak{a}}(\delta x, \delta x)\, \overset{1}{\delta x}_{\mathfrak{a}}\ ,$$

respectively, so those sums must vanish independently of the n variations $\overset{1}{\delta x}_{\mathfrak{a}}$, and that can happen only if the combinations that were spoken of themselves vanish. That confirms that the covariant (15) will be equal to zero as a result of the three equations that **Riemann** exhibited. Now, since one-half the value of the form Ψ (dx, δx, dx, δx) is equal to the aggregate of the covariants (14) and (15), under the stated assumption, the covariant (14) will yield a representation of one-half the value of the form Ψ (dx, δx, dx, δx) in its own right. As was mentioned before, the form Ψ (dx, δx, dx, δx) is equal to **Riemann**'s form (II), and the covariant (14) is equal to one-half the value of **Riemann**'s second representation of the form (II). We have then derived **Riemann**'s second way of representing his form (II) from the property of the form $\Psi(\overset{1}{dx}, \overset{1}{\delta x}, dx, \delta x)$ that was proved just now that it equals an aggregate of two covariants.

2.

We shall now address the development of the connection between the covariant (15) of the previous article and the expression that must be a minimum under the principle of least constraint. However, there is a certain complication that must be overcome. As is known, **Gauss** expressed his principle in words, but not analytical symbols, in vol. 4 of this journal (page 232), and then used synthetic considerations to reduce it to **d'Alembert**'s principle and the principle of virtual velocities. Thus, **Gauss** himself lacked an analytical formulation for his own principle, and that is all the more regrettable, since the words that **Gauss** used in his formulation admitted more than one interpretation at one point. Namely, he did not establish from the outset what sense he was imparting to the expression "the free motion of a point." In order to shed some light upon the question, we imagine that the mass-points of the system that is in motion are referred to a system of rectangular coordinates. For the first mass-point, they might be z_1, z_2, z_3, for the second one z_4,

z_5, z_6, …, and for the last mass-point, they might be z_{n-2}, z_{n-1}, z_n. The mass of the first point will be denoted by $m_1 = m_2 = m_3$, the mass of the second point $m_4 = m_5 = m_6$, … Let the components of the applied forces, when decomposed along those three axes, be Z_1, Z_2, Z_3, respectively, for the first point, Z_4, Z_5, Z_6, for the second point, … Let the system of points be subject to a sequence of condition equations Φ_1 = const., Φ_2 = const., …, $\Phi_{\mathfrak{l}}$ = const., which depend upon only the coordinates and contain neither time t nor the derivatives of the coordinates with respect to time. Now, there can be no doubt that the given values of the coordinates of all mass-points at a moment in time t must satisfy the $\mathfrak{l}$ condition equations:

$$\Phi_1 = \text{const.}, \quad \Phi_2 = \text{const.}, \quad \ldots, \Phi_{\mathfrak{l}} = \text{const.}$$

By contrast, as far as the components of the velocities of the individual mass-points relative to the rectangular coordinate system are concerned, there exist two possibilities: Either the components of the velocity are chosen such that they are found to agree with those $\mathfrak{l}$ condition equations and satisfy the $\mathfrak{l}$ equations:

$$\frac{d\Phi_1}{dt} = 0, \qquad \frac{d\Phi_2}{dt} = 0, \qquad \ldots, \qquad \frac{d\Phi_{\mathfrak{l}}}{dt} = 0,$$

which follow from them, or they are chosen in such a way that they contradict them. The expression "free motion of a point" that **Gauss** used is consistent with both of those assumptions. Therefore, in order to ascertain the true content of the principle of least constraint, nothing seems to remain but to formulate it analytically under each of the two assumptions using **Gauss**'s words and examine whether the principle leads to a correct representation of the problem of motion in both cases. That exercise would show that the principle is valid only for the first assumption.

Under the first assumption that was pointed out, the coordinates of the individual mass-points, as well as their first derivatives with respect to time t, must be considered to be given at the moment in time t. By contrast, the second derivatives of the coordinates with respect to time t are considered to be unknown and must be determined by precisely that principle of least constraint. If τ means a small increment in time t then the rectangular coordinates of a mass-point that belongs to a moving system (for example, the first mass-point) at the time $t + \tau$ will assume the values:

$$z_1 + \frac{dz_1}{dt}\tau + \frac{1}{2}\frac{d^2z_1}{dt^2}\tau^2,$$

$$z_2 + \frac{dz_2}{dt}\tau + \frac{1}{2}\frac{d^2z_2}{dt^2}\tau^2,$$

$$z_3 + \frac{dz_3}{dt}\tau + \frac{1}{2}\frac{d^2z_3}{dt^2}\tau^2,$$

respectively, for a well-defined actual motion that is described with a precision that goes up to order τ^2. By contrast, the coordinates of that point at the same time when the motion that results from the influence of the given applied force at that point proves to be free would be:

$$z_1 + \frac{dz_1}{dt}\tau + \frac{1}{2}\frac{Z_1}{m_1}\tau^2,$$
$$z_2 + \frac{dz_2}{dt}\tau + \frac{1}{2}\frac{Z_2}{m_2}\tau^2,$$
$$z_3 + \frac{dz_3}{dt}\tau + \frac{1}{2}\frac{Z_3}{m_3}\tau^2,$$

respectively. Therefore, the square of the deviation of the first point from its free motion will be measured by the sum of the squares of the corresponding coordinate differences, and will have the expression:

$$\left[\left(\frac{d^2z_1}{dt^2} - \frac{Z_1}{m_1}\right)^2 + \left(\frac{d^2z_2}{dt^2} - \frac{Z_2}{m_2}\right)^2 + \left(\frac{d^2z_3}{dt^2} - \frac{Z_3}{m_3}\right)^2\right]\frac{\tau^4}{4}.$$

From the rule that **Gauss** gave, that will be multiplied by the mass of the point in question, which we have called $m_1 = m_2 = m_3$, and the sum of the products that are defined in the same way for all of points of the system will then represent the expression that must be a minimum. That sum will be equal to the product of the factor $\tau^4 / 4$, which is considered to be unvarying, with the combination:

$$(1) \qquad \sum_{\mathfrak{a}} m_{\mathfrak{a}}\left(\frac{d^2z_{\mathfrak{a}}}{dt^2} - \frac{Z_{\mathfrak{a}}}{m_{\mathfrak{a}}}\right)^2,$$

in which the symbol $\mathfrak{a}$ runs through the sequence of numbers from 1 to n, as in art. **1**. The principle of least constraint can then be expressed by saying that for the given values of $z_{\mathfrak{a}}$ and $\frac{dz_{\mathfrak{a}}}{dt}$, the quantities $\frac{d^2z_{\mathfrak{a}}}{dt^2}$ can be determined in such a way that the combination (1) will become a minimum.

In order to address that problem, above all, one must ponder the equations that the desired quantities $\frac{d^2z_{\mathfrak{a}}}{dt^2}$ must satisfy. If any of the functions $\Phi_1, \Phi_2, \ldots, \Phi_{\mathfrak{l}}$ is denoted by $\Phi_{\mathfrak{a}}$ then what will follow first from each condition equation $\Phi_{\mathfrak{a}} = \text{const.}$ is the aforementioned equation:

$$(2) \qquad \frac{d\Phi_\alpha}{dt} = \sum_{\mathfrak{a}} \frac{\partial \Phi_\alpha}{\partial z_{\mathfrak{a}}}\frac{dz_{\mathfrak{a}}}{dt} = 0,$$

which is fulfilled by the first derivatives of the coordinates, and then secondly, the equation:

$$\frac{d^2\Phi_\alpha}{dt^2} = \sum_{\mathfrak{a}} \frac{\partial \Phi_\alpha}{\partial z_{\mathfrak{a}}} \frac{d^2 z_{\mathfrak{a}}}{dt^2} + \sum_{\mathfrak{a},\mathfrak{b}} \frac{\partial^2 \Phi_\alpha}{\partial z_{\mathfrak{a}} \partial z_{\mathfrak{b}}} \frac{dz_{\mathfrak{a}}}{dt} \frac{dz_{\mathfrak{b}}}{dt} = 0, \tag{3}$$

which the second derivatives of the coordinates must fulfill. However, since the values of $z_{\mathfrak{a}}$ and $\frac{dz_{\mathfrak{a}}}{dt}$ are fixed by the prevailing relations, the $\mathfrak{l}$ equations (3) express only the idea that the aggregate $\sum_{\mathfrak{a}} \frac{\partial \Phi_\alpha}{\partial z_{\mathfrak{a}}} \frac{d^2 z_{\mathfrak{a}}}{dt^2}$ that appears in them must have unvarying values. When one applies the undetermined multipliers $\lambda_1, \lambda_2, \ldots, \lambda_{\mathfrak{l}}$, the minimum problem to be solved will lead to the n equations:

$$m_{\mathfrak{a}} \left(\frac{d^2 z_{\mathfrak{a}}}{dt^2} - \frac{Z_{\mathfrak{a}}}{m_{\mathfrak{a}}} \right) = \lambda_1 \frac{\partial \Phi_1}{\partial z_{\mathfrak{a}}} + \lambda_2 \frac{\partial \Phi_2}{\partial z_{\mathfrak{a}}} + \cdots + \lambda_{\mathfrak{l}} \frac{\partial \Phi_{\mathfrak{l}}}{\partial z_{\mathfrak{a}}}, \tag{4}$$

from the well-known rules. However, these are nothing but the differential equations of the problem of motion that was posed. The principle of least constraint is therefore justified for the first assumption that was made.

The second assumption can be characterized by saying that the given velocity components *do not* correspond to the equations (2). The values of the velocity components might be called ζ_1, ζ_2, ζ_3 for the first point, ζ_4, ζ_5, ζ_6 for the second point, etc. Therefore, values of the first derivatives of the coordinates for the motion of the individual points cannot, in fact, prove to be equal to the given values, and on those grounds, the first, as well as the second, derivatives of the coordinates with respect to time must now be regarded as unknowns. For that reason, the rectangular coordinates of the first mass-point at time $t + \tau$ will have the previously-exhibited expressions for the actual motion that is to be determined, up to a precision that goes to order τ^2. By contrast, the coordinates in question of that point will have the following expressions:

$$z_1 + \zeta_1 \tau + \frac{1}{2} \frac{Z_1}{m_1} \tau^2,$$

$$z_2 + \zeta_2 \tau + \frac{1}{2} \frac{Z_2}{m_2} \tau^2,$$

$$z_3 + \zeta_3 \tau + \frac{1}{2} \frac{Z_3}{m_3} \tau^2$$

for the free motion that is now supposed to result with the given velocity components and under the influence of the associated applied force. The square of the deviation of the first point from its motion will then equal the sum of the squares of the coordinate differences:

$$\left[\left(\frac{dz_1}{dt}-\zeta_1\right)\tau+\frac{1}{2}\left(\frac{d^2z_1}{dt^2}-\frac{Z_1}{m_1}\right)\tau^2\right]^2+\left[\left(\frac{dz_2}{dt}-\zeta_2\right)\tau+\frac{1}{2}\left(\frac{d^2z_2}{dt^2}-\frac{Z_2}{m_1}\right)\tau^2\right]^2$$
$$+\left[\left(\frac{dz_3}{dt}-\zeta_3\right)\tau+\frac{1}{2}\left(\frac{d^2z_3}{dt^2}-\frac{Z_3}{m_3}\right)\tau^2\right]^2.$$

When one multiplies that by the mass $m_1 = m_2 = m_3$ of the point in question, deals with all points of the system similarly, and takes the sum of the resulting expressions, what one will get is the combination to be minimized:

$$(5) \qquad \sum_{\mathfrak{a}} m_{\mathfrak{a}}\left[\left(\frac{dz_{\mathfrak{a}}}{dt}-\zeta_{\mathfrak{a}}\right)+\frac{1}{2}\left(\frac{d^2z_{\mathfrak{a}}}{dt^2}-\frac{Z_{\mathfrak{a}}}{m_{\mathfrak{a}}}\right)\tau\right]^2,$$

multiplied by the factor τ^2.

The conception of the principle of least constraint that is assumed in that is the basis for the generalization of that principle that **Schering** presented in the essay "**Hamilton-Jacobische** Theorie für Kräfte, deren Mass von der Bewegung der Körper abhängt," in volume XVIII of the Abh. de K. G. d. Wiss. zu Göttingen. In order to obtain the expression (5) above from **Schering**'s formulas, the more general assumptions that he made in them must be replaced with the simpler assumptions for a problem that actually arises in mechanics. With that simplification, the specifics of **Schering**'s conception of the principle of least constraint will emerge quite clearly, and one can make a more confident decision about the justification for that concept.

The problem that was posed of minimizing the expression (5) above differs from the minimum problem that one solves for the expression (1) by the fact that the values of $\frac{d^2z_{\mathfrak{a}}}{dt^2}$, as well as the values of $\frac{dz_{\mathfrak{a}}}{dt}$, must be determined for the former, while only the values of $\frac{d^2z_{\mathfrak{a}}}{dt^2}$ must be determined for the latter. For that reason, the $\mathfrak{l}$ condition equations (2) and the $\mathfrak{l}$ condition equations (3) must be considered in such a way that the $\frac{d^2z_{\mathfrak{a}}}{dt^2}$ and the $\frac{dz_{\mathfrak{a}}}{dt}$ prove to be variable. Therefore, if one introduces $\mathfrak{l}$ multipliers ρ_α, as well as $\mathfrak{l}$ multipliers that might be denoted by $\tau\,\sigma_\alpha$, then the requirement in question might be expressed by saying that the expression:

$$(6)\quad \begin{cases} \sum_{\mathfrak{a}} m_{\mathfrak{a}} \left[\left(\frac{dz_{\mathfrak{a}}}{dt} - \zeta_{\mathfrak{a}} \right) + \frac{1}{2} \left(\frac{d^2 z_{\mathfrak{a}}}{dt^2} - \frac{Z_{\mathfrak{a}}}{m_{\mathfrak{a}}} \right) \tau \right]^2 \\ \qquad - 2 \sum_{\alpha} \rho_{\alpha} \sum_{\mathfrak{a}} \frac{\partial \Phi_{\alpha}}{\partial z_{\mathfrak{a}}} \frac{dz_{\mathfrak{a}}}{dt} - \tau \sum_{\alpha} \sigma_{\alpha} \left(\sum_{\mathfrak{a}} \frac{\partial \Phi_{\alpha}}{\partial z_{\mathfrak{a}}} \frac{dz_{\mathfrak{a}}}{dt} + \sum_{\mathfrak{a},\mathfrak{b}} \frac{\partial^2 \Phi_{\alpha}}{\partial z_{\mathfrak{a}} \partial z_{\mathfrak{b}}} \frac{dz_{\mathfrak{a}}}{dt} \frac{dz_{\mathfrak{b}}}{dt} \right) \end{cases}$$

must be a minimum.

From the well-known rules, that will yield the equations:

$$(7)\quad \begin{cases} m_{\mathfrak{a}} \left(\frac{dz_{\mathfrak{a}}}{dt} - \zeta_{\mathfrak{a}} \right) + \frac{m_{\mathfrak{a}}}{2} \left(\frac{d^2 z_{\mathfrak{a}}}{dt^2} - \frac{Z_{\mathfrak{a}}}{m_{\mathfrak{a}}} \right) \tau = \sum_{\alpha} \rho_{\alpha} \frac{\partial \Phi_{\alpha}}{\partial z_{\mathfrak{a}}} + \tau \sum_{\alpha} \sigma_{\alpha} \sum_{\mathfrak{b}} \frac{\partial^2 \Phi_{\alpha}}{\partial z_{\mathfrak{a}} \partial z_{\mathfrak{b}}} \frac{dz_{\mathfrak{b}}}{dt}, \\ m_{\mathfrak{a}} \left(\frac{dz_{\mathfrak{a}}}{dt} - \zeta_{\mathfrak{a}} \right) + \frac{m_{\mathfrak{a}}}{2} \left(\frac{d^2 z_{\mathfrak{a}}}{dt^2} - \frac{Z_{\mathfrak{a}}}{m_{\mathfrak{a}}} \right) \tau = \sum_{\alpha} \sigma_{\alpha} \frac{\partial \Phi_{\alpha}}{\partial z_{\mathfrak{a}}}. \end{cases}$$

The fact that these equations do not embody the nature of the corresponding mechanical problem can be seen with no further discussion. When one subtracts the two equations, it will follow that:

$$\sum_{\alpha} (\rho_{\alpha} - \sigma_{\alpha}) \frac{\partial \Phi_{\alpha}}{\partial z_{\mathfrak{a}}} + \tau \sum_{\alpha} \sigma_{\alpha} \sum_{\mathfrak{b}} \frac{\partial^2 \Phi_{\alpha}}{\partial z_{\mathfrak{a}} \partial z_{\mathfrak{b}}} \frac{dz_{\mathfrak{b}}}{dt} = 0,$$

and in the case where $\mathfrak{l} = 1$, in which *only one* condition equation Φ_1 = const. is given, that will demand that the quotient:

$$\frac{1}{\frac{\partial \Phi_1}{\partial z_{\mathfrak{a}}}} \sum_{\mathfrak{b}} \frac{\partial^2 \Phi_1}{\partial z_{\mathfrak{a}} \partial z_{\mathfrak{b}}} \frac{dz_{\mathfrak{b}}}{dt}$$

must have the same value for each index $\mathfrak{a}$. However, such a prescription is entirely foreign to theoretical mechanics. On those grounds, the principle of least constraint cannot be applied when **Gauss**'s words are formulated under the assumption that we called the second one.

Earlier I said that in the expression that **Gauss** gave to his principle, the meaning of the words "free motion of a point" *could not be established from the outset.* However, that opinion was based upon the fact that **Gauss** could only recognize the meaning that those words should have from the *proof* that he carried out. **Gauss** based his proof on the principle of virtual velocities, and that depended upon whether the virtual displacements of the points could all be regarded as small quantities of the same order under the principle of virtual velocities. If that were allowed then those quantities, which Gauss called $c\gamma$, $c'\gamma'$, $c''\gamma''$, … in his proof, would all have to be small quantities of the same order. However, that assumption will hold true only under our first assumption, where the curve that a point of the system of masses that is considered to be in motion describes and the curve that point of the fictitious free system would describe would have the same tangent, while the convention that relates to the second assumption would not hold true unconditionally. For myself, I do not doubt that the principle of virtual velocities must include the convention that was referred to intrinsically. However, due to the nature of the principle of virtual velocities, since a rigorous proof of the necessity of that condition probably cannot be achieved, I have preferred an analytical discussion that is completely convincing, as far as I can see.

At the same time, the consideration that was just discussed shows the way by which the principle of least constraint must be modified in order to deduce results that are acknowledged in theoretical mechanics for those values of the velocity components that are compatible with the condition equations of the problem. It suffices that for a first application of the principle, the coordinates of the mass-points that relate to the time-point $t + \tau$ should be taken with a precision that goes up to only order τ. The second derivatives of the coordinates will not enter into consideration of the actual motion that results, nor will the given applied forces come into play for the consideration of the fictitious free motion, and the first derivatives of the motion that actually results will be determined by the requirement that the expression:

$$\sum_{\mathfrak{a}} m_{\mathfrak{a}} \left(\frac{dz_{\mathfrak{a}}}{dt} - \zeta_{\mathfrak{a}} \right)^2 \tag{8}$$

must be a minimum. The second requirement can then be satisfied when one includes the values $\frac{dz_{\mathfrak{a}}}{dt}$ that are obtained in that way, namely, that the expression (1) can be minimized by a choice of the second derivatives $\frac{d^2 z_{\mathfrak{a}}}{dt^2}$. Meanwhile, I cannot suppress a remark in regard to that subject that does not refer to the treatment of the mechanical problem in question as much as it does to the essence of it. If one assumes that the velocity components of a system of moving mass-points contradict the governing condition equations at some time-point and that after a vanishingly-small time τ has elapsed the individual points of the system will assume velocities that satisfy the condition equations, and under which the motion will proceed according to the given applied forces, then the conversion of the given velocity components into the velocity components that are actually maintained can take place only in such a way that the sum of the *vis vivas* that are impressed upon the system inside of the vanishingly-small time τ will experience a loss. However, while the formulas of theoretical mechanics can represent such a process, the approximation to the true state of affairs must be much closer than the one that is attained in those mechanical problems whose representation does not assume a momentary violation of continuity and momentary loss of *vis viva*.

3.

Once it has been emphasized that the principle of least constraint should refer to those quantities that were referred to in (1) of the previous article, their expressions should be ascertained under the assumption that an arbitrary system of n independent variables must be introduced in place of the rectangular coordinates $z_\mathfrak{a}$ for the mass-points of the system in motion. It is known that the transformation of the system of differential equations (4) in article **2** depends upon only the fact that the quadratic form $\frac{1}{2}\sum_\mathfrak{a} m_\mathfrak{a}\, dz_\mathfrak{a}^2$ in the n differentials $dz_\mathfrak{a}$ and the expression $\sum_\mathfrak{a} Z_\mathfrak{a}\, dz_\mathfrak{a}$ that is defined by the components of the applied forces must be represented in the new variables. Consistent with the notation in article **1**, one will have:

$$\frac{1}{2}\sum_\mathfrak{a} m_\mathfrak{a}\, dz_\mathfrak{a}^2 = \frac{1}{2}\sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, dx_\mathfrak{a}\, dx_\mathfrak{b} = f(dx), \tag{1}$$

and furthermore:

$$\sum_\mathfrak{a} Z_\mathfrak{a}\, dz_\mathfrak{a} = \sum_\mathfrak{a} X_\mathfrak{a}\, dx_\mathfrak{a}\,. \tag{2}$$

The $\mathfrak{l}$ functions Φ_α of the coordinates $z_\mathfrak{a}$ are converted into functions of the variables $x_\mathfrak{a}$, and equations (2) and (3) of the previous article will be replaced by the equations:

$$\frac{d\Phi_\alpha}{dt} = \sum_\mathfrak{a} \frac{\partial\Phi_\alpha}{\partial x_\mathfrak{a}}\frac{dx_\mathfrak{a}}{dt} = 0, \tag{3}$$

$$\frac{d^2\Phi_\alpha}{dt^2} = \sum_\mathfrak{a} \frac{\partial\Phi_\alpha}{\partial x_\mathfrak{a}}\frac{d^2x_\mathfrak{a}}{dt^2} + \sum_{\mathfrak{a},\mathfrak{b}} \frac{\partial^2\Phi_\alpha}{\partial x_\mathfrak{a}\,\partial x_\mathfrak{b}}\frac{dx_\mathfrak{a}}{dt}\frac{dx_\mathfrak{b}}{dt}\,. \tag{4}$$

The following equations will enter in place of equations (4) of the previous article, which are likewise constructed with the undetermined multipliers $\lambda_1, \lambda_2, \ldots, \lambda_\mathfrak{l}$, and in which the notations for differentials that were defined in article **1** are adapted to differential quotients:

$$\sum_\mathfrak{b} a_{\mathfrak{a},\mathfrak{b}}\left[\frac{d^2x_\mathfrak{b}}{dt^2} + \xi_\mathfrak{b}\left(\frac{dx}{dt}, \frac{dx}{dt}\right)\right] - X_\mathfrak{a} = \lambda_1\frac{\partial\Phi_1}{\partial x_\mathfrak{a}} + \cdots + \lambda_\mathfrak{l}\frac{\partial\Phi_\mathfrak{l}}{\partial x_\mathfrak{a}}\,. \tag{5}$$

By means of equation (7) of article **1**, they can also assume the form:

$$\Psi_\mathfrak{a}\left(\frac{dx}{dt}, \frac{dx}{dt}\right) - X_\mathfrak{a} = \lambda_1\frac{\partial\Phi_1}{\partial x_\mathfrak{a}} + \cdots + \lambda_\mathfrak{l}\frac{\partial\Phi_\mathfrak{l}}{\partial x_\mathfrak{a}}\,. \tag{6}$$

Now, it follows from equation (6) of article **1** that the sum $\sum_\mathfrak{a} \Psi_\mathfrak{a}\left(\frac{dx}{dt}, \frac{dx}{dt}\right)\delta x_\mathfrak{a}$, and from the nature of the quantities $X_\mathfrak{a}$, the sum $\sum_\mathfrak{a} X_\mathfrak{a}\,\delta x_\mathfrak{a}$, will go over to an analogously-constructed expression when one introduces another arbitrary system of variables; that is, the expression is covariant to the form $f(dx)$ and the current problem. Therefore, the sum:

(7) $$\sum_{\mathfrak{a}} \left[\Psi_{\mathfrak{a}} \left(\frac{dx}{dt}, \frac{dx}{dt} \right) - X_{\mathfrak{a}} \right] \delta x_{\mathfrak{a}}$$

will also have that property. The form $f(dx)$ might now go over to the form $g(dy) = \frac{1}{2} \sum_{\mathfrak{k},\mathfrak{l}} e_{\mathfrak{k},\mathfrak{l}}\, dy_{\mathfrak{k}}\, dy_{\mathfrak{l}}$ by the introduction of another arbitrary system of variables $y_{\mathfrak{l}}$, and one further lets the determinant be $| e_{\mathfrak{k},\mathfrak{l}} | = E$ and the adjoint element by $\partial E / \partial e_{\mathfrak{k},\mathfrak{l}} = E_{\mathfrak{k},\mathfrak{l}}$. When the linear expressions:

$$a_{\mathfrak{a},1}\, dx_1 + a_{\mathfrak{a},2}\, dx_2 + \ldots + a_{\mathfrak{a},n}\, dx_n = p_{\mathfrak{a}}$$

and

$$e_{\mathfrak{k},1}\, dy_1 + e_{\mathfrak{k},2}\, dy_2 + \ldots + e_{\mathfrak{k},n}\, dy_n = q_{\mathfrak{k}}$$

are introduced into the forms in question, each of the forms will be converted into its adjoint form as follows:

$$2 f(dx) = \sum_{\mathfrak{a},\mathfrak{b}} \frac{A_{\mathfrak{a},\mathfrak{b}}}{\Delta} p_{\mathfrak{a}}\, p_{\mathfrak{b}}, \qquad 2 g(dy) = \sum_{\mathfrak{k},\mathfrak{l}} \frac{E_{\mathfrak{k},\mathfrak{l}}}{E} q_{\mathfrak{k}}\, q_{\mathfrak{l}},$$

and likewise, the equation:

(8) $$\sum_{\mathfrak{a},\mathfrak{b}} \frac{A_{\mathfrak{a},\mathfrak{b}}}{\Delta} p_{\mathfrak{a}}\, p_{\mathfrak{b}} = \sum_{\mathfrak{k},\mathfrak{l}} \frac{E_{\mathfrak{k},\mathfrak{l}}}{E} q_{\mathfrak{k}}\, q_{\mathfrak{l}}$$

must be true, as a result of the equation $f(dx) = g(dy)$. As soon as one applies another system of independent variations $dx_{\mathfrak{a}}$ and the corresponding system of variations $dy_{\mathfrak{l}}$, one will also get the equation:

$$\sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, dx_{\mathfrak{a}}\, \delta x_{\mathfrak{b}} = \sum_{\mathfrak{k},\mathfrak{l}} e_{\mathfrak{k},\mathfrak{l}}\, dy_{\mathfrak{k}}\, \delta y_{\mathfrak{l}}$$

from the transformation that was performed, which can be replaced with the equation:

(9) $$\sum_{\mathfrak{a}} p_{\mathfrak{a}}\, \delta x_{\mathfrak{a}} = \sum_{\mathfrak{k}} q_{\mathfrak{k}}\, \delta y_{\mathfrak{k}}$$

by means of the quantities $p_{\mathfrak{a}}$ and $q_{\mathfrak{k}}$. That equation exhibits the linear dependency that exists between the quantities $p_{\mathfrak{a}}$ and $q_{\mathfrak{k}}$, and it illuminates the fact that as long as equation (9) between two systems of quantities $p_{\mathfrak{a}}$ and $q_{\mathfrak{k}}$ is fulfilled, equation (8) must follow.

The fact that the sum (7) above is covariant to the form $f(dx)$ in our mechanical problem means that the expressions $\Psi_{\mathfrak{a}} \left(\frac{dx}{dt}, \frac{dx}{dt} \right) - X_{\mathfrak{a}}$ have the same relationship to the corresponding expressions that are formed with a new system of variables that is prescribed between the quantities $p_{\mathfrak{a}}$ and $q_{\mathfrak{k}}$ in (9). Therefore, the expression that is found on the left-hand side of (8):

$$\sum_{\mathfrak{a},\mathfrak{b}} \frac{A_{\mathfrak{a},\mathfrak{b}}}{\Delta} \left[\Psi_{\mathfrak{a}} \left(\frac{dx}{dt}, \frac{dx}{dt} \right) - X_{\mathfrak{a}} \right] \left[\Psi_{\mathfrak{b}} \left(\frac{dx}{dt}, \frac{dx}{dt} \right) - X_{\mathfrak{b}} \right] \tag{10}$$

must be equal to the analogously-constructed expression when one introduces a system of new variables. *The expression* (10) *is then covariant to the form f* (dx) *and the mechanical problem in question.*

One will likewise get the form that the combination (10) assumes from that property as soon as one again introduces the rectilinear coordinates $z_{\mathfrak{a}}$ in place of the variables $x_{\mathfrak{a}}$. Due to equation (1), for the case when $\mathfrak{a}$ and $\mathfrak{b}$ are different from each other, the adjoint elements $A_{\mathfrak{a}\mathfrak{b}} / \Delta$ must be replaced with $1 / m_{\mathfrak{a}}$, which are, by contrast, zero for the case of $\mathfrak{a} = \mathfrak{b}$. From (6), $\Psi_{\mathfrak{a}} \left(\frac{dx}{dt}, \frac{dx}{dt} \right)$ will go to the expression $m_{\mathfrak{a}} \frac{d^2 z_{\mathfrak{a}}}{dt^2}$, and from (2), the quantities $X_{\mathfrak{a}}$ will go to the quantities $Z_{\mathfrak{a}}$. The expression:

$$\sum_{\mathfrak{a}} \frac{1}{m_{\mathfrak{a}}} \left(m_{\mathfrak{a}} \frac{d^2 z_{\mathfrak{a}}}{dt^2} - Z_{\mathfrak{a}} \right)^2 \tag{11}$$

will then arise, which is equal to the expression (1) in article **2** identically. *The quantity that is minimized by the principle of least constraint will then be represented by the covariant* (10) *of the given mechanical problem.* A comparison of the covariant (10) with the covariant (13) in article **1** will show that their structures agree completely. Both covariants are based upon the form that is adjoint to the form $2f$ (dx). The covariant (13) in article **1** is equal to a difference of two values of the adjoint form, one of which is constructed from the variables $\Psi_{\mathfrak{b}}$ $(dx, \overset{1}{dx})$ and $\Psi_{\mathfrak{k}}$ $(\delta x, \overset{1}{\delta x})$, while the other is constructed from the system of variables $\Psi_{\mathfrak{b}}$ $(\delta x, \overset{1}{dx})$ and $\Psi_{\mathfrak{k}}$ $(dx, \overset{1}{\delta x})$. The covariant (10) above is equal to a value of the adjoint form in which *only one* system of variables appears that is represented by the difference $\Psi_{\mathfrak{a}} \left(\frac{dx}{dt}, \frac{dx}{dt} \right) - X_{\mathfrak{a}}$.

One can further convert the covariant (10) by consulting the differential equations (6) when one replaces $\Psi_{\mathfrak{a}} \left(\frac{dx}{dt}, \frac{dx}{dt} \right) - X_{\mathfrak{a}}$ with the sum $\sum_{\alpha} \lambda_{\alpha} \frac{\partial \Phi_{\alpha}}{\partial x_{\mathfrak{a}}}$ and replaces $\Psi_{\mathfrak{b}} \left(\frac{dx}{dt}, \frac{dx}{dt} \right) - X_{\mathfrak{b}}$ with the sum $\sum_{\beta} \lambda_{\beta} \frac{\partial \Phi_{\beta}}{\partial x_{\mathfrak{b}}}$, which amounts to the same thing. One will then get the expression:

$$\sum_{\mathfrak{a},\mathfrak{b}} \sum_{\alpha,\beta} \frac{A_{\mathfrak{a},\mathfrak{b}}}{\Delta} \lambda_{\alpha} \frac{\partial \Phi_{\alpha}}{\partial x_{\mathfrak{a}}} \lambda_{\beta} \frac{\partial \Phi_{\beta}}{\partial x_{\mathfrak{b}}} . \tag{12}$$

If one now introduces the schema that was given on page 277 of vol. 71 of this journal:

(13) $$\sum_{\mathfrak{a},\mathfrak{b}} \frac{A_{\mathfrak{a},\mathfrak{b}}}{\Delta} \frac{\partial \Phi_\alpha}{\partial x_{\mathfrak{a}}} \frac{\partial \Phi_\beta}{\partial x_{\mathfrak{b}}} = (\alpha, \beta)$$

then the covariant (10) will be equal to the following double sum, in which the indices α and β go from 1 to $\mathfrak{l}$:

(14) $$\sum_{\alpha,\beta} \lambda_\alpha \, \lambda_\beta (\alpha, \beta) \, .$$

I cited this expression for the quantity that must be minimized by the principle of least constraint under the assumption that the system of mass-points in motion is subject to condition equations, but no accelerating forces, in vol. 81 of this journal on page 231.

4.

When one extends the problem in mechanics in such a way that for each mass-point, the square of the line element in space is equal to an arbitrary essentially-positive quadratic form in the coordinate differentials, such that consistent with that, the sum of the *vis vivas* of all mass-points of the system in motion will be equal to an essentially-positive quadratic form $2f\left(\frac{dx}{dt}\right)$ in the differential quotients of the coordinates $x_{\mathfrak{a}}$ with respect to time *t*, and such that the expression $\sum_{\mathfrak{a}} X_{\mathfrak{a}} \, \delta x_{\mathfrak{a}}$ and the $\mathfrak{l}$ condition equations $\Phi_{\mathfrak{a}}$ = const. take on a corresponding meaning, one will obtain a system of differential equations by means of the fundamental theorems that were developed in the treatise "Untersuchung eines Problem der Variationsrechnung, in welchem das Problem der Mechanik enthalten ist" (vol. 74 of this journal, page 116, *et seq*.) that has exactly the same form as the system of differential equations (6) of the previous article. An expression that is constructed under the cited assumptions from the given quadratic form $2f\,(dx)$ and the sum $\sum_{\mathfrak{a}} X_{\mathfrak{a}} \, \delta x_{\mathfrak{a}}$ by the prescription that was given for constructing the combination (10) must, on the same grounds, be a covariant relative to the extended mechanical problem, and that covariant must be considered to be the extension of that concept that was represented by the expression (10) for the original mechanical problem. That also easily shows that when one considers the values $x_{\mathfrak{a}}$ and $dx_{\mathfrak{a}} / dt$ to be given and imposes the demand that the values of $d^2 x_{\mathfrak{a}} / dt^2$ must be determined in such a way that the covariant that was spoken of becomes a minimum, a system of equations will arise that coincides with the system of differential equations of the extended mechanical problem in question. However, that embodies the associated extension of the principle of least constraint.

The treatise that is found in volume 74 of this journal refers to a further extension of the mechanical problem that replaces the line element for each mass-point in space with the p^{th} root of an essentially-positive form of degree *p* in the coordinate differentials, and the *vis viva* of each mass-point is measured by multiplying the mass of the point by the p^{th} power of the line element and dividing by the p^{th} power of the time-element. The p^{th} part of the sum of the *vis vivas* of all mass-points of a moving system is then equal to an essentially-positive form $f\,(dx\,/\,dt)$ of degree p in the differential quotients with respect to time t of the coordinates $x_{\mathfrak{a}}$ of all points, a function U of the variables $x_{\mathfrak{a}}$ that represents the force function, and $\mathfrak{l}$ condition equations $\Phi_{\mathfrak{a}}$ = const., and the analogy with **Hamilton**'s variational problem will lead to the requirement that the first variation of the integral:

$$\int_{t_0}^{t_1}\left[f\left(\frac{dx}{dt}\right)+U\right]dt \tag{1}$$

must be made to vanish. In the cited place, it was assumed that the variables in the problem were chosen in such a way that they fulfilled the given condition equations. By contrast, the problem is formulated precisely as above in my treatise "Sätze aus dem Grenzgebiet der Mechanik und Geometrie" in vol. VI of **Clebsch** and **Neumann**'s Mathematischen Annalen on page 416. There, the function U that appears under the integral sign in the integral (1) was added to the expression $\lambda_1\,\Phi_1 + \lambda_2\,\Phi_2 + \ldots + \lambda_{\mathfrak{l}}\,\Phi_{\mathfrak{l}}$ that is formed with the undetermined multipliers. The system of differential equations that is associated with the variational problem will then read as follows:

$$\frac{d}{dt}\left[\frac{\partial f\left(\frac{dx}{dt}\right)}{\partial\frac{dx_{\mathfrak{a}}}{dt}}\right]-\frac{\partial f\left(\frac{dx}{dt}\right)}{\partial x_{\mathfrak{a}}} = \frac{\partial U}{\partial x_{\mathfrak{a}}}+\lambda_1\frac{\partial\Phi_1}{\partial x_{\mathfrak{a}}}+\cdots+\lambda_{\mathfrak{l}}\frac{\partial\Phi_{\mathfrak{l}}}{\partial x_{\mathfrak{a}}}\,. \tag{2}$$

Now, that shows that the principle of least constraint is sufficiently robust that it will also be valid in this domain. When one separates the terms in the left-hand side of (2) that contain the second differential quotients $d^2\,x_{\mathfrak{a}}\,/\,dt^2$ from the terms in which only the first differential quotients occur, the following expression will arise:

$$\sum_{\mathfrak{b}}\frac{\partial^2 f\left(\frac{dx}{dt}\right)}{\partial\frac{dx_{\mathfrak{a}}}{dt}\,\partial\frac{dx_{\mathfrak{b}}}{dt}}\frac{d^2x_{\mathfrak{b}}}{dt^2}+f_{\mathfrak{a}}\left(\frac{dx}{dt}\right), \tag{3}$$

in which $f_{\mathfrak{a}}\left(\frac{dx}{dt}\right)$ means a homogeneous function of degree p in the $dx_{\mathfrak{a}}\,/\,dt$. That representation is taken from vol. 70 of this journal on page 76, and, at the same time, formula (10) there says that the sum:

$$\sum_{\mathfrak{a}} \left[\sum_{\mathfrak{b}} \frac{\partial^2 f\left(\dfrac{dx}{dt}\right)}{\partial \dfrac{dx_{\mathfrak{a}}}{dt}\, \partial \dfrac{dx_{\mathfrak{b}}}{dt}} \frac{d^2 x_{\mathfrak{b}}}{dt^2} + f_{\mathfrak{a}}\left(\frac{dx}{dt}\right) \right] \delta x_{\mathfrak{a}} \tag{4}$$

is covariant with the form $f\,(dx)$. Similarly, the sum:

$$\sum_{\mathfrak{a}} \frac{\partial U}{\partial x_{\mathfrak{a}}}\, \delta x_{\mathfrak{a}} \tag{5}$$

has a value that is independent of the chosen system of variables $x_{\mathfrak{a}}$. As long as the system of differential equations (2) must be convertible into the system of differential equations (5) of the previous article in such a way that the arbitrary functions $X_{\mathfrak{a}}$ will enter in place of the $\partial U / \partial x_{\mathfrak{a}}$, one must maintain the condition that the sum $\sum_{\mathfrak{a}} X_{\mathfrak{a}}\, \delta x_{\mathfrak{a}}$ is likewise independent of the chosen system of variables $x_{\mathfrak{a}}$ in order for the system of differential equations to be meaningful independently of it, respectively. For the sake of brevity, I will introduce the notation:

$$\frac{\partial^2 f\left(\dfrac{dx}{dt}\right)}{\partial \dfrac{dx_{\mathfrak{a}}}{dt}\, \partial \dfrac{dx_{\mathfrak{b}}}{dt}} = a_{\mathfrak{a},\mathfrak{b}} \tag{6}$$

for the second derivatives of the form $f\left(\dfrac{dx}{dt}\right)$. As long as $p = 2$, the expressions $a_{\mathfrak{a},\mathfrak{b}}$ will coincide with the coefficients of the $2f\left(\dfrac{dx}{dt}\right)$ form and take on their previous meaning accordingly. However, for a form of degree p when $p > 2$, they will be equal to forms of degree $p - 2$ in the elements $dx_{\mathfrak{a}} / dt$. Furthermore, let the determinant be $|\, a_{\mathfrak{a},\mathfrak{b}} \,| = \Delta$ and let the adjoint element be $\partial\Delta / \partial a_{\mathfrak{a},\mathfrak{b}} = A_{\mathfrak{a},\mathfrak{b}}$.

If the form $f\,(dx)$ goes to the form $g\,(dy)$ when one substitutes a new system of arbitrary variables $y_{\mathfrak{l}}$, and the variations $\delta x_{\mathfrak{a}}$ again correspond to the variations $\delta y_{\mathfrak{l}}$, then a basic algebraic property of homogeneous functions will imply the equation:

$$\sum_{\mathfrak{a},\mathfrak{b}} \frac{\partial^2 f\,(dx)}{\partial\, dx_{\mathfrak{a}}\, \partial\, dx_{\mathfrak{b}}}\, \delta x_{\mathfrak{a}}\, \delta x_{\mathfrak{b}} = \sum_{\mathfrak{k},\mathfrak{l}} \frac{\partial^2 g\,(dy)}{\partial\, dy_{\mathfrak{k}}\, \partial\, dy_{\mathfrak{l}}}\, \delta y_{\mathfrak{k}}\, \delta y_{\mathfrak{l}}, \tag{7}$$

and by means of the notation (6) and the corresponding notation:

$$\frac{\partial^2 g\left(\frac{dy}{dt}\right)}{\partial \frac{dy_{\mathfrak{k}}}{dt} \partial \frac{dy_{\mathfrak{l}}}{dt}} = e_{\mathfrak{k},\mathfrak{l}}$$

that will lead to the following equation:

(7)
$$\sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, \delta x_{\mathfrak{a}}\, \delta x_{\mathfrak{b}} = \sum_{\mathfrak{k},\mathfrak{l}} e_{\mathfrak{k},\mathfrak{l}}\, \delta x_{\mathfrak{k}}\, \delta x_{\mathfrak{l}}\,.$$

One can now conclude from this, in the way that was developed in the previous article, that as long as the determinant $| e_{\mathfrak{k},\mathfrak{l}} |$ equals E and the adjoint element $\partial E / \partial e_{\mathfrak{k},\mathfrak{l}}$ is set to $E_{\mathfrak{k},\mathfrak{l}}$, and when the equation:

$$\sum_{\mathfrak{a}} p_{\mathfrak{a}}\, \delta x_{\mathfrak{a}} = \sum_{\mathfrak{k}} q_{\mathfrak{k}}\, \delta y_{\mathfrak{k}}$$

is true for the arbitrary variations $\delta x_{\mathfrak{a}}$ and $\delta y_{\mathfrak{k}}$ that correspond to the two systems of quantities $p_{\mathfrak{a}}$ and $q_{\mathfrak{k}}$, the equation:

$$\sum_{\mathfrak{a},\mathfrak{b}} \frac{A_{\mathfrak{a},\mathfrak{b}}}{\Delta}\, p_{\mathfrak{a}}\, p_{\mathfrak{b}} = \sum_{\mathfrak{k},\mathfrak{l}} \frac{E_{\mathfrak{k},\mathfrak{l}}}{E}\, q_{\mathfrak{k}}\, q_{\mathfrak{l}}$$

will exist. Therefore, since the sums (4) and (5) in our problem are covariant, from the remarks that were made, and the same thing will also be true for the sum $\sum_{\mathfrak{a}} X_{\mathfrak{a}}\, \delta x_{\mathfrak{a}}$ that replaces (5), the expression:

(8)
$$\sum_{\mathfrak{a},\mathfrak{b}} \frac{A_{\mathfrak{a},\mathfrak{b}}}{\Delta}\left[\sum_{\mathfrak{c}} a_{\mathfrak{a},\mathfrak{c}}\frac{d^2 x_{\mathfrak{c}}}{dt^2} + f_{\mathfrak{a}}\left(\frac{dx}{dt}\right) - \frac{\partial U}{\partial x_{\mathfrak{a}}}\right]\left[\sum_{\mathfrak{b}} a_{\mathfrak{b},\mathfrak{b}}\frac{d^2 x_{\mathfrak{b}}}{dt^2} + f_{\mathfrak{b}}\left(\frac{dx}{dt}\right) - \frac{\partial U}{\partial x_{\mathfrak{b}}}\right]$$

will be *a covariant for the variational problem that was posed*, and the expression:

(9)
$$\sum_{\mathfrak{a},\mathfrak{b}} \frac{A_{\mathfrak{a},\mathfrak{b}}}{\Delta}\left[\sum_{\mathfrak{c}} a_{\mathfrak{a},\mathfrak{c}}\frac{d^2 x_{\mathfrak{c}}}{dt^2} + f_{\mathfrak{a}}\left(\frac{dx}{dt}\right) - X_{\mathfrak{a}}\right]\left[\sum_{\mathfrak{b}} a_{\mathfrak{b},\mathfrak{b}}\frac{d^2 x_{\mathfrak{b}}}{dt^2} + f_{\mathfrak{b}}\left(\frac{dx}{dt}\right) - X_{\mathfrak{b}}\right]$$

will be *a covariant for the system of differential equations that arises from the system* (2) *when one substitutes* $X_{\mathfrak{a}}$ *for* $\partial U / \partial x_{\mathfrak{a}}$, namely:

(10)
$$\sum_{\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\frac{d^2 x_{\mathfrak{b}}}{dt^2} + f_{\mathfrak{a}}\left(\frac{dx}{dt}\right) = X_{\mathfrak{a}} + \lambda_1\frac{\partial \Phi_1}{\partial x_{\mathfrak{a}}} + \cdots + \lambda_{\mathfrak{l}}\frac{\partial \Phi_{\mathfrak{l}}}{\partial x_{\mathfrak{a}}}\ .$$

As long as the form $f(dx)$ is a quadratic form and the assumptions that are actually true for mechanics are accepted, the covariant (9) will be converted into the covariant (10) of article **3**, and as the general rules of minimization problems will show, the former will have the common property with the latter that the associated system of differential equations (10) will emerge from the requirement that for given values of $x_\mathfrak{a}$ and $dx_\mathfrak{a} / dt$, the value of (9) will be minimized when one determines the values of the $d^2 x_\mathfrak{a} / dt^2$. In fact, as a result of the $\mathfrak{l}$ condition equations $\Phi_\mathfrak{a}$ = const., the equations that correspond to equations (3) and (4) of article **3** will also be definitive here:

$$\frac{d\Phi_\mathfrak{a}}{dt} = \sum_\mathfrak{a} \frac{\partial \Phi_\alpha}{\partial x_\mathfrak{a}} \frac{dx_\mathfrak{a}}{dt} = 0,$$

$$\frac{d^2\Phi_\mathfrak{a}}{dt^2} = \sum_\mathfrak{a} \frac{\partial \Phi_\alpha}{\partial x_\mathfrak{a}} \frac{d^2 x_\mathfrak{a}}{dt^2} + \sum_{\mathfrak{a},\mathfrak{b}} \frac{\partial^2 \Phi_\alpha}{\partial x_\mathfrak{a}\, \partial x_\mathfrak{b}} \frac{dx_\mathfrak{a}}{dt} \frac{dx_\mathfrak{b}}{dt} = 0.$$

Moreover, since the quantities $x_\mathfrak{a}$ and $dx_\mathfrak{a} / dt$ can be considered to be unvarying in the minimum problem, the combinations $a_{\mathfrak{a},\mathfrak{b}}$ will also figure in a form of degree p, where $p > 2$, only as unvarying quantities, and for the same reason, the sums $\sum_\mathfrak{a} \frac{\partial \Phi_\alpha}{\partial x_\mathfrak{a}} \frac{d^2 x_\mathfrak{a}}{dt^2}$ will have unvarying values, as before. That is the basis for the assertion that was made that the principle of least constraint can be adapted to that extension of the problem in mechanics that is included in the system of differential equations (10) by the use of the covariant (9).

5.

The study by **Schering** that was cited in article **2** refers to the assumption that the square of the line element in space is equal to an essentially-positive quadratic form in the coordinate differentials and pursues the objective of arriving at the system of differential equations in that domain that would be obtained from the corresponding generalization of **Hamilton**'s variational problem by means of an extension of the concept of force and an extension of the principle of least constraint. Under the intended application of the principle of least constraint, **Schering**'s deduction is connected precisely with the expressions that **Gauss** chose, but it also raises the aforementioned objection yet again. That deduction must satisfy the requirement that it represents no other concept than the one that was included in the original train of thought for the case in which the square of the line element in space can be represented as an aggregate of squares of three differentials, so the space in question must be **Euclidian** space itself. However, for the case of **Euclidian** space, the coordinates that were used in **Schering**'s deduction, which were not assigned any special properties, would differ by a small amount from the general coordinates by which a point in **Euclidian** space is determined. In **Schering**'s way of looking at things, the square of the deviation of a point from its free motion would be equal to the square of the distance between two points whose coordinates differ from each other by quantities that are not all of only first order. Now, that sheds some light upon the fact that when a point in **Euclidian** space is referred to rectangular coordinates z_1 , z_2 , z_3 , the square of the line element in space will have the expression:

$$(1) \qquad dz_1^2 + dz_2^2 + dz_3^2 ,$$

and at the same time, that the square of the distance between two arbitrary points $z_1^{(1)}$, $z_2^{(1)}$, $z_3^{(1)}$ and $z_1^{(2)}$, $z_2^{(2)}$, $z_3^{(2)}$ will have the expression:

$$(2) \qquad (z_1^{(2)} - z_1^{(1)})^2 + (z_2^{(2)} - z_2^{(1)})^2 + (z_3^{(2)} - z_3^{(1)})^2 .$$

By contrast, as soon as the same point z_1 , z_2 , z_3 in space is referred to arbitrary coordinates x_1 , x_2 , x_3 , and the square of the line element (1) goes over to the form:

$$(3) \qquad a_{11}\,dx_1^2 + a_{22}\,dx_2^2 + a_{33}\,dx_3^2 + 2a_{23}\,dx_2\,dx_3 + 2a_{31}\,dx_3\,dx_1 + 2a_{12}\,dx_1\,dx_2 ,$$

it cannot be asserted that the square of the distance between the points $z_1^{(1)}$, $z_2^{(1)}$, $z_3^{(1)}$ and $z_1^{(2)}$, $z_2^{(2)}$, $z_3^{(2)}$ will generally be expressed correctly when one forms the expression:

$$(4) \qquad \left\{ \begin{aligned} & a_{11}\,(x_1^{(2)} - x_1^{(1)})^2 + a_{22}\,(x_2^{(2)} - x_2^{(1)})^2 + a_{33}(x_3^{(2)} - x_3^{(1)})^2 + 2a_{23}(x_2^{(2)} - x_2^{(1)})(x_3^{(2)} - x_3^{(1)}) \\ & \qquad + 2a_{31}(x_3^{(2)} - x_3^{(1)})(x_1^{(2)} - x_1^{(1)}) + 2a_{12}\,(x_1^{(2)} - x_1^{(1)})(x_2^{(2)} - x_2^{(1)}) \end{aligned} \right.$$

from the associated coordinates $x_1^{(1)}$, $x_2^{(1)}$, $x_3^{(1)}$ and $x_1^{(2)}$, $x_2^{(2)}$, $x_3^{(2)}$ of those points. The transformation from rectangular coordinates to polar coordinates already suffices to show that this process is inadmissible. However, it was precisely that process that **Schering** appealed to on page 11 of his paper, where he wished to express the square of the deviation of a possible motion of a point from the free motion. There, **Schering** called the coordinate differences differentials, in general. However, the expressions that he gave for the coordinate differences as aggregates of terms that were of first and second order in the element of time and the points that would follow from those terms in the formulas alluded to terms of even higher order. In order for the process to also be valid for the terms of first and second order, one would generally have to be able to draw the conclusion from the equality of the expressions (1) and (3) that when the differences $z_\mathfrak{a}^{(2)} - z_\mathfrak{a}^{(1)}$ are replaced with the aggregate $dz_\mathfrak{a} + \frac{1}{2} d^2 z_\mathfrak{a}$ for the values $\mathfrak{a}$ = 1, 2, 3, and at the same time, the differences $x_\mathfrak{a}^{(2)} - x_\mathfrak{a}^{(1)}$ are replaced with the corresponding aggregate $dx_\mathfrak{a} + \frac{1}{2} d^2 x_\mathfrak{a}$, the expression (2) will be equal to the expression (4). Thus, as a result of setting terms of the same higher order equal to each other, the equation:

$$\sum_{\mathfrak{a}} dz_\mathfrak{a}\, d^2 z_\mathfrak{a} = \sum_{\mathfrak{a},\mathfrak{b}} a_{\mathfrak{a},\mathfrak{b}}\, dx_\mathfrak{a}\, d^2 x_\mathfrak{b}$$

must exist, which would also be incorrect in the cited example of the transformation from rectangular coordinates to polar coordinates.

Schering's work aroused the desire in me to see how the concept of a force that acts upon a point in that domain would carry over to the one that pertained to the variational problem of the integral that was denoted by (1) in the previous article. It will now be assumed in that problem that only a single point of unit mass moves freely. The line element for the point that is referred to by the coordinates x_α will then have the expression $\sqrt[p]{p\,f(dx)}$, and the requirement that the first variation of the integral:

$$\int \sqrt[p]{p\,f(dx)} \tag{5}$$

must vanish will determine the first-order manifold that corresponds to the shortest line in the relevant space for the coordinates x_α. When one thinks of the variables x_α as independent of a variable t, the integral (5) can take on the form:

$$\int_{t_0}^{t_1} \sqrt[p]{p\,f\left(\frac{dx}{dt}\right)}\,dx\,, \tag{5*}$$

from which the differential drops out. Therefore, it still remains completely undetermined how the variables x_α should depend upon the variable t in the variational problem for the integral (5*). The associated system of differential equations that is given on page 124 of vol. 74 of this journal reads:

$$\frac{d}{dt}\left[\frac{\partial \sqrt[p]{p\,f\left(\frac{dx}{dt}\right)}}{\partial \frac{dx_\alpha}{dt}}\right] - \frac{\partial \sqrt[p]{p\,f\left(\frac{dx}{dt}\right)}}{\partial x_\alpha} = 0. \tag{6}$$

The quantities x_α will be determined completely by it when initial values x_α (0) and the $n - 1$ ratios of the initial differentials dx_α (0) are given for the value $t = t_0$, and we assume that the integration was carried out for that data. The value of the integral (5*), which might be called r, accordingly represents the length of the shortest line that goes from the point x_α (0) to the point x_α, or the distance between the point x_α and the point x_α (0), and when it is represented as a pure function of the system of values x_α (0) and x_α, it must satisfy the equation that is included in (7^b) in the cited location:

$$(7) \qquad \delta r = \frac{1}{\left[p\, f\left(\frac{dx}{dt}\right)\right]^{\frac{p-1}{p}}} \sum_{\alpha} \frac{\partial f\left(\frac{dx}{dt}\right)}{\partial \frac{dx_\alpha}{dt}} \delta x_\alpha - \frac{1}{\left[p\, f_0\left(\frac{dx(0)}{dt}\right)\right]^{\frac{p-1}{p}}} \sum_{\alpha} \frac{\partial f_0\left(\frac{dx(0)}{dt}\right)}{\partial \frac{dx_\alpha(0)}{dt}} \delta x_\alpha(0)\,.$$

The addition of the symbol 0 to the form *f* (*dx*) means the substitution of the quantities $x_\alpha(0)$ for the corresponding x_α in the coefficients of the form. The differential *dt* can also appear in the foregoing equation (7) only formally and will cancel by means of the homogeneity of the form *f* (*dx*).

We shall now consider the variational problem of the integral:

$$(8) \qquad \int_{t_0}^{t_1} \left[f\left(\frac{dx}{dt}\right) + U \right] dt\,,$$

under the assumption that the force function *U* is a pure function of the function *r* that is referred to a fixed system $x_\alpha(0)$ and the moving system x_α ; it will be called *P* (*r*). That problem defines a generalization of the problem for the free motion of a point in which the given force function is a pure function of the distance between the moving point and a fixed point. That problem was solved under the assumption that the form *f* (*dx*) is a quadratic form that belongs to a certain genus of forms in the treatise "Extension of the planet problem to a space of *n* dimensions and constant integral curvature" (Quarterly Journal of Mathematics, no. 48, pp. 349), and indeed under the condition that the values of x_α and dx_α / dt are given arbitrarily for a time-point $t = t_1$. **Schering** solved the same problem for a space of *n* dimensions and constant curvature by a somewhat more extended assumption in regard to the force function on page 35 of the treatise that was cited above. In the present discussion, the degree *p* of the form *f* (*dx*) can be arbitrary, although it was established that the variables x_α should assume the values $x_\alpha = x_\alpha(0)$ for the time-point $t = t_0$, which was chosen for the definition of the quantity *r*. We will then focus our attention on the free motion of a point that moves under the influence of a force function *P* (*r*), and whose motion begins from that fixed point, and whose distance to that point is measured by *r*. The differential equations of the variational problem that was just posed can be obtained from the differential equations (2) of the previous article when one drops the condition functions and replaces *U* with *P* (*r*). They will then be these:

$$(9) \qquad \frac{d}{dt}\left[\frac{\partial f\left(\frac{dx}{dt}\right)}{\partial \frac{dx_\alpha}{dt}}\right] - \frac{\partial f\left(\frac{dx}{dt}\right)}{\partial x_\alpha} = \frac{\partial r}{\partial x_\alpha} \frac{dP(r)}{dr}\,.$$

It can now be verified that under the assumption in question, by which the equations $x_\alpha = x_\alpha(0)$ must be true for $t = t_0$, those differential equations for the variables x_α will prescribe the same first-order manifold that is determined by the system (6). That is, a point that is under the influence of the force function P (r) must move along a shortest line from the point $x_{\mathfrak{a}}$ (0). One can give the system (6) above the following form:

$$(10) \qquad \frac{d}{dt}\left[\frac{\partial\, f\left(\frac{dx}{dt}\right)}{\partial \frac{dx_{\mathfrak{a}}}{dt}}\right] - \frac{\partial\, f\left(\frac{dx}{dt}\right)}{\partial x_{\mathfrak{a}}} = \frac{1}{\left[p\, f\left(\frac{dx}{dt}\right)\right]^{\frac{p-1}{p}}} \frac{\partial f\left(\frac{dx}{dt}\right)}{\partial \frac{dx_{\mathfrak{a}}}{dt}} \frac{d}{dt}\left[p\, f\left(\frac{dx}{dt}\right)\right]^{\frac{p-1}{p}}$$

by means of a conversion that was performed in (3^{b*}) on page 124 of vol. 74 of this journal. Due to equation (7), the partial differential quotient of the function r that is expressed in terms of the quantities $x_{\mathfrak{b}}$ and $x_{\mathfrak{b}}$ (0) with respect to the individual values $x_{\mathfrak{a}}$ will have the expression:

$$(11) \qquad \frac{\partial r}{\partial x_{\mathfrak{a}}} = \frac{1}{\left[p\, f\left(\frac{dx}{dt}\right)\right]^{\frac{p-1}{p}}} \frac{\partial f\left(\frac{dx}{dt}\right)}{\partial \frac{dx_{\mathfrak{a}}}{dt}}.$$

Therefore, the system (10) will be converted into this one:

$$(12) \qquad \frac{d}{dt}\left[\frac{\partial\, f\left(\frac{dx}{dt}\right)}{\partial \frac{dx_{\mathfrak{a}}}{dt}}\right] - \frac{\partial\, f\left(\frac{dx}{dt}\right)}{\partial x_{\mathfrak{a}}} = \frac{\partial r}{\partial x_{\mathfrak{a}}} \frac{d}{dt}\left[p\, f\left(\frac{dx}{dt}\right)\right]^{\frac{p-1}{p}}.$$

Since r is the value of the integral (5^*), one will have the equation:

$$(13) \qquad \frac{dr}{dt} = \left[p\, f\left(\frac{dx}{dt}\right)\right]^{\frac{1}{p}}.$$

For that reason, one will also have the following expression for the system (12):

$$(14) \qquad \frac{d}{dt}\left[\frac{\partial\, f\left(\frac{dx}{dt}\right)}{\partial \frac{dx_{\mathfrak{a}}}{dt}}\right] - \frac{\partial\, f\left(\frac{dx}{dt}\right)}{\partial x_{\mathfrak{a}}} = \frac{\partial r}{\partial x_{\mathfrak{a}}} \frac{d\left(\frac{dr}{dt}\right)^{p-1}}{dt}.$$

The left-hand side of equation (14) is identical to the left-hand side of equation (9) for every value of the index $\mathfrak{a}$, and similarly the factor that is found on the right-hand side will coincide with $\partial r / \partial x_{\mathfrak{a}}$. Now since the system (14) determined only the first-order manifold for the variables $x_{\mathfrak{a}}$, the dependency of the individual variables on the variable t will, however, remain undetermined, so it is possible to arrange that dependency in such a way that the system (6) coincides with the system (9), and that will come about when one assumes that the equation:

$$\frac{d}{dt}\left(\frac{dr}{dt}\right)^{p-1} = \frac{dP(r)}{dr} \tag{15}$$

is valid. The desired integration of the system (9), in which the equations $x_{\mathfrak{a}} = x_{\mathfrak{a}}(0)$ must be true for $t = t_0$ and the initial values $\frac{dx_{\mathfrak{a}}}{dt} = \frac{dx_{\mathfrak{a}}(0)}{dt}$ should be proportional to the corresponding differentials that were chosen in the integration of the system (6) or (14), will then yield the first-order manifold for the variables $x_{\mathfrak{a}}$ that is predicted by the system (14), as was stated, while equation (15) determined the dependency of the path length r, which follows the shortest line, on time t. Here, the differential quotient of the function $P(r)$ with respect to the quantity r will take over the role of the force that acts upon the point of unit mass. Equation (15) says that under the motion that is spoken of, which results in the shortest line that starts from the point $x_{\mathfrak{a}}(0)$, the differential quotient with respect to time of the $(p-1)^{\text{th}}$ power of the first differential quotient of the path length r with respect to time will be equal to the given quantity $dP(r) / dr$. However, for the value $p = 2$, that rule will go over to the rule that under the motion in question, the second differential quotient with respect to time of the path length r must be equal to the given quantity $dP(r) / dr$. For the sake of simplicity of expression, I have assumed that the initial values $x_{\mathfrak{a}} = x_{\mathfrak{a}}(0)$ were prescribed in the integration of the system (9), and that the initial values $\frac{dx_{\mathfrak{a}}}{dt} = \frac{dx_{\mathfrak{a}}(0)}{dt}$ were given to be proportional to the values of the initial differentials that were chosen in the integration of the system (6). However, the reduction of the system (9) to the system (6) can be accomplished in the same way when one demands that for the system (9) at an arbitrary time-point $t = t_1$, only those values of $x_{\mathfrak{a}} = x_{\mathfrak{a}}(1)$ and $\frac{dx_{\mathfrak{a}}}{dt} = \frac{dx_{\mathfrak{a}}(1)}{dt}$ should be valid that are taken from the first-order manifold that is determined by the integration of the system (6) that was performed. That is, in a different language, the motion of point that is under the influence of the force function $P(r)$ when it begins from an arbitrary point of a shortest line that goes through the fixed point $x_{\mathfrak{a}}(0)$, and indeed in the direction of that shortest line, will always remain on that shortest line and obey equation (15).

When equation (15) is multiplied by dr/ dt, it will take on the form:

$$(p-1)\left(\frac{dr}{dt}\right)^{p-1}\frac{d^2r}{dt^2} = \frac{dP(r)}{dr}.$$

That equation admits the undetermined integration:

(16) $$\frac{p-1}{p}\left(\frac{dr}{dt}\right)^{p} = P(r) + H,$$

in which H means an arbitrary constant whose value is determined by the given initial values. Equation (16) will go to the equation:

(17) $$(p-1)\, f\left(\frac{dr}{dt}\right) = P(r) + H,$$

by means of (13). For the present problem that is nothing but the equation that was denoted by (5^a) on page 123 of vol. 74 of this journal, and the integral will then represent the *vis viva*. Equation (16) will imply the equation:

(18) $$dt = \frac{dr}{\sqrt[p]{\dfrac{p}{p-1}[P(r)+H]}},$$

which will yield the dependency of the path length r on time t by performing a quadrature and inverting the resulting equation.

Bonn, 13 November 1876.

"Das Princip der kleinsten Arbeit der verlorenen Kräfte als ein allgemeines Princip der Mechanik," Zeit. Math. und Phys. **24** (1879), 206-220.

The principle of least work done by lost forces as a general principle of mechanics

By

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See Tab. III, Fig. 1 (†)

Translated by D. H. Delphenich

§ 1. – Before we begin the task of setting down what we understand the principle of least work done by lost forces to mean and deriving the equations of motion for a system of material points from it, we would like to prove a theorem from pure analysis.

Let:

$$X\,\delta x + Y\,\delta y + Z\,\delta z + \ldots \tag{1}$$

be a function that is homogeneous and linear in the infinitely small quantities δx, δy, δz, …, where X, Y, Z, … are functions of x, y, z, … In addition, let:

$$A\,\delta x + B\,\delta y + C\,\delta z + \ldots, \qquad A_1\,\delta x + B_1\,\delta y + C_1\,\delta z + \ldots, \tag{2}$$

and

$$U\,\delta x + V\,\delta y + W\,\delta z + \ldots, \qquad U_1\,\delta x + V_1\,\delta y + W_1\,\delta z + \ldots \tag{3}$$

be given functions of δx, δy, δz, …, in which the coefficients of δx, δy, δz, … are functions of x, y, z, … .

(†) Translator: The table and figure were not available to me at the time of translation.

We would like to see what conditions the coefficients $X, Y, Z, \ldots$ of the functions (1) and the coefficients $A, B, \ldots, A_1, B_1, \ldots, U, V, \ldots, U_1, V_1, \ldots$ of the functions (2) and (3) must fulfill in order for the function (1) to not assume any positive values for those infinitely small $\delta x, \delta y, \delta z, \ldots$ that make the functions (2) positive or equal to zero and the function (3) equal to zero.

If we assume that the number of functions (2) and (3) is smaller than that of the variables $x, y, z, \ldots$ then we shall add just as many arbitrary homogeneous linear functions $\delta\Omega, \delta\Omega_1, \ldots$ of the $\delta x, \delta y, \delta z, \ldots$ as it takes to make the number of functions (2) and (3), along with the arbitrary $\delta\Omega$, $\delta\Omega_1, \ldots$, equal to the number of infinitely small $\delta x, \delta y, \delta z, \ldots$

Since the functions $\delta\Omega, \delta\Omega_1, \ldots$ are arbitrary and different from (2) and (3), they can be either positive, negative, or zero for those values of $\delta x, \delta y, \delta z, \ldots$ that make the functions (2) positive or equal to zero, and the functions (3) equal to zero.

No matter what the coefficients of the functions (1), (2), and (3) might also be, we can always set:

$$
\begin{aligned}
X\,\delta x + Y\,\delta y + Z\,\delta z + \ldots &+ \lambda\ (A\ \delta x + B\ \delta y + C\ \delta z + \ldots) \\
&+ \lambda_1 (A_1\ \delta x + B_1\ \delta y + C_1\ \delta z + \ldots) + \ldots \\
&+ \mu\ (U\ \delta x + V\ \delta y + W\ \delta z + \ldots) \\
&+ \lambda_1 (U_1\ \delta x + V_1\ \delta y + W_1\ \delta z + \ldots) + \ldots \\
&+ \omega\,\delta\Omega + \omega_1\,\delta\Omega_1 + \ldots = 0
\end{aligned}
\tag{4}
$$

for all arbitrary values of $\delta x, \delta y, \delta z, \ldots$, in which we understand $\lambda, \lambda_1, \ldots, \mu, \mu_1, \ldots, \omega, \omega_1, \ldots$ to mean undetermined quantities. In fact, since $\delta x, \delta y, \delta z, \ldots$ are always completely arbitrary and λ, $\lambda_1, \ldots, \mu, \mu_1, \ldots, \omega, \omega_1, \ldots$ remain undetermined, we will have to set the coefficients that enter into equation (4) next to the latter equal to zero. In that way, one will get just as many equations as quantities $\lambda, \lambda_1, \ldots, \mu, \mu_1, \ldots, \omega, \omega_1, \ldots$, and from those equations, we can ascertain the values of $\lambda, \lambda_1, \ldots, \mu, \mu_1, \ldots, \omega, \omega_1, \ldots$ that equation (4) makes possible for all arbitrary $\delta x, \delta y, \delta z, \ldots$

We would now like to present equation (4) in the form:

$$
\begin{aligned}
X\,\delta x + Y\,\delta y + \ldots \quad &= -\lambda\,(A\ \delta x + B\ \delta y + \ldots) - \lambda_1\,(A_1\ \delta x + B_1\ \delta y + \ldots) + \ldots \\
&\quad - \mu\,(U\ \delta x + V\ \delta y + \ldots) - \mu_1\,(U_1\ \delta x + V_1\ \delta y + \ldots) + \ldots \\
&\quad - \omega\,\delta\Omega - \omega_1\,\delta\Omega_1 - \ldots
\end{aligned}
\tag{5}
$$

In order for the first part of this equation to not assume positive values for any values of the finitely small quantities that make the functions (2) positive or zero and the functions (3) equal to zero, one must be able to determine the quantities $\delta x, \delta y, \delta z, \ldots$ as being equal to zero. Otherwise, one would be able to choose only those values for the infinitely small $\delta x, \delta y, \delta z, \ldots$ that make the functions (2) equal to zero, and the function (1) would be equal to a sum of arbitrary functions $\delta\Omega$, $\delta\Omega_1, \ldots$ that could also prove to be positive, but the latter consequence would contradict the basic assumption. As a result, one must have:

$$
\begin{aligned}
X\,\delta x + Y\,\delta y + \ldots = &-\lambda\,(A\ \ \delta x + B\ \ \delta y + \ldots) \\
&-\lambda_1\,(A_1\ \delta x + B_1\ \delta y + \ldots) + \ldots
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
&- \mu\,(U \quad \delta x + V \;\; \delta y + \ldots) \\
&- \mu_1\,(U_1\, \delta x + V_1\, \delta y + \ldots) + \ldots = 0
\end{aligned}
$$

for all arbitrary values of δx, δy, δz, …, or what amounts to the same thing:

$$
(7) \qquad \left\{
\begin{array}{l}
X + \lambda A + \lambda_1 A_1 + \cdots + \mu U + \mu_1 U_1 + \cdots = 0, \\
Y + \lambda B + \lambda_1 B_1 + \cdots + \mu V + \mu_1 V_1 + \cdots = 0, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{array}
\right.
$$

which are the equations into which the expression (6) decomposes. Furthermore, it is clear that the coefficients λ, λ_1, … must be determined to be positive from equations (7), in order to make the first part of equation (5) assume no positive values for those values of the infinitely-small quantities that make the functions (2) positive or equal to zero and the functions (3) equal zero. Otherwise, the first part of equation (5) would prove to have negative values λ if one were to one choose values for δx, δy, δz, … that made the function:

$$
A\,\delta x + B\,\delta y + C\,\delta z + \ldots
$$

positive and the other functions (2) and (3) equal to zero.

We have then found the following conditions that must be fulfilled in order for the function (1) to assume no positive values for those values of δx, δy, δz, … that make the functions (2) positive or equal to zero and the functions (3) equal to zero.

1. Equation (6) must be valid for all arbitrary values of δx, δy, δz, …; i.e., when the function (1) is added to the functions (2) and (3), each of which is multiplied by an undetermined factor, that sum must be equal to zero for all arbitrary values of δx, δy, δz, …

2. The factors λ, λ_1, … must be positive. That condition is not only *necessary*, but also *sufficient*, since the function (1) will assume no positive values when it is fulfilled for all values of δx, δy, δz, … that make the functions (2) positive or equal to zero and the functions (3) equal to zero.

Since δx, δy, δz, … are arbitrary, their coefficients in equation (6) must be set equal to zero. One will then get just as many equations (7) from that as there are infinitesimals δx, δy, δz, ... If one eliminates the undetermined factors λ, λ_1, … μ, μ_1, … from those equations then one will get the expressions that represent the stated mutual dependency of the coefficients of the functions (1), (2), and (3) upon each other.

One must combine the equations that are obtained in that way with the inequalities that should express the idea that the factors that are ascertained from equations (7) must always remain positive, no matter what numerical values they might also assume.

§ 2. – We shall now go on to the explanation of what we understand the principle of least work done by lost forces to mean and the derivation of the equations of motion for a system of material points from it. In order to be able to ascertain the motion of a system of material points, one must first determine the latter; i.e., one must show what displacements are possible for the system considered. The forces that act upon the material points must then be given. It is known that a system of material points is determined analytically by means of the following conditions: Displacements will be possible that make certain linear functions of the displacements equal to zero or equal to zero and positive. In the latter case, the aforementioned functions will change signs only when one goes from possible displacements to impossible ones. The system whose motion is to be ascertained consists of n material points whose masses are $m_1, m_2, \ldots, m_i, \ldots, m_n$, and whose coordinates at the end of time t are:

$$(x_1, y_1, z_1), \qquad (x_2, y_2, z_2), \quad \ldots, \quad (x_i, y_i, z_i), \quad \ldots, \quad (x_n, y_n, z_n) .$$

Let those displacements of the system:

$$(x_1, y_1, z_1), \qquad (x_2, y_2, z_2), \quad \ldots, \quad (x_i, y_i, z_i), \quad \ldots, \quad (x_n, y_n, z_n)$$

be possible that make the linear functions of them:

$$(8) \qquad \left\{ \begin{array}{l} \sum\left\{\dfrac{\partial L'}{\partial x_i}\delta x_i + \dfrac{\partial L'}{\partial y_i}\delta y_i + \dfrac{\partial L'}{\partial z_i}\delta z_i\right\} + \dfrac{\partial L'}{\partial t}\delta t, \\ \sum\left\{\dfrac{\partial L''}{\partial x_i}\delta x_i + \dfrac{\partial L''}{\partial y_i}\delta y_i + \dfrac{\partial L''}{\partial z_i}\delta z_i\right\} + \dfrac{\partial L''}{\partial t}\delta t, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{array} \right.$$

positive or equal to zero, and make the functions:

$$(9) \qquad \left\{ \begin{array}{l} \sum\left\{\dfrac{\partial M'}{\partial x_i}\delta x_i + \dfrac{\partial M'}{\partial y_i}\delta y_i + \dfrac{\partial M'}{\partial z_i}\delta z_i\right\} + \dfrac{\partial M'}{\partial t}\delta t, \\ \sum\left\{\dfrac{\partial M''}{\partial x_i}\delta x_i + \dfrac{\partial M''}{\partial y_i}\delta y_i + \dfrac{\partial M''}{\partial z_i}\delta z_i\right\} + \dfrac{\partial M''}{\partial t}\delta t, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{array} \right.$$

equal to zero. In the cited expressions, x_i, y_i, z_i mean the coordinates of a point m_i of the system, and the i are assigned all whole number values from 1 to n.

We further assume that forces $F_1, F_2, \ldots, F_i, \ldots, F_n$ act upon the material points of the system that is determined by means of the conditions on the possible displacements (8) and (9):

$$(X_1, Y_1, Z_1), \quad (X_2, Y_2, Z_2), \quad \ldots, \quad (X_i, Y_i, Z_i), \quad \ldots, \quad (X_n, Y_n, Z_n).$$

If one knows the conditions on the possible displacements of a system and the forces that act upon its material points then one will find the conditions (equations) on an actual displacement of the system during an infinitely small time interval ∂t, and in that way, ascertain the motion of the system. We would like to say that it is characteristic of a differential that it represents those changes in the coordinates that take place as a result of actual displacements of the material point.

Since the actual displacements:

$$(\partial x_1, \partial y_1, \partial z_1), \quad (\partial x_2, \partial y_2, \partial z_2), \quad \ldots, \quad (\partial x_i, \partial y_i, \partial z_i), \quad \ldots, \quad (\partial x_n, \partial y_n, \partial z_n)$$

of the material points of a system will also belong to its possible displacements then, they must make the functions (8) and (9) equal to zero in the case where the constraints to which those functions refer actual exist as such. The constraints to which the functions (8) refer exclude one part of space for the masses that move in it and leave them free to move in the other part. For the former part of space, the functions (8) will be negative, while they will be positive for the latter. The constraint only comes into effect when it obstructs the transition of a material point from one part of space to the other. As a result, the actual displacements of the system will proceed to the boundary that separates the two spaces from each other, since otherwise the displacements would be independent of the constraints. Now, as far as the functions (9) are concerned, it is clear that the actual displacements that are possible in them are the ones that make the functions (9) equal to zero. We conclude from this that an actual displacement that makes the functions (8) and (9) equal to zero can be determined by means of the following equations:

$$(10) \quad \left\{ \begin{array}{l} \sum\left\{\frac{\partial L'}{\partial x_i}\delta x_i + \frac{\partial L'}{\partial y_i}\delta y_i + \frac{\partial L'}{\partial z_i}\delta z_i\right\} + \frac{\partial L'}{\partial t}\delta t, \\ \sum\left\{\frac{\partial L''}{\partial x_i}\delta x_i + \frac{\partial L''}{\partial y_i}\delta y_i + \frac{\partial L''}{\partial z_i}\delta z_i\right\} + \frac{\partial L''}{\partial t}\delta t, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{array} \right.$$

$$(11) \quad \left\{ \begin{array}{l} \sum\left\{\frac{\partial M'}{\partial x_i}\delta x_i + \frac{\partial M'}{\partial y_i}\delta y_i + \frac{\partial M'}{\partial z_i}\delta z_i\right\} + \frac{\partial M'}{\partial t}\delta t, \\ \sum\left\{\frac{\partial M''}{\partial x_i}\delta x_i + \frac{\partial M''}{\partial y_i}\delta y_i + \frac{\partial M''}{\partial z_i}\delta z_i\right\} + \frac{\partial M''}{\partial t}\delta t, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array} \right.$$

§ 3. – Since the number of equations (10) and (11) is always smaller than the number of coordinates that determine the position of the material points of a system, the actual displacements must be determined by some other conditions.

Let O_i be the position of a material point of the system at the end of time t, let $\overline{O_i A_i}$ represent the direction of the velocity v_i that the point m_i has at the end of time t . $\overline{O_i A_i}$ will then likewise represent the displacement that the point m_i would have as a result of the velocity that it has acquired during the infinitely small time interval ∂t .

Let $\overline{O_i B_i}$ be the displacement that the point m_i would experience during the time ∂t under the action of the force F_i if it did not acquire any velocity and were entirely free. The diagonal line $\overline{O_i C_i}$ of the parallelogram that is constructed from $\overline{O_i A_i}$ and $\overline{O_i B_i}$ would represent the displacement of the material point m_i in the event that the latter were free at time t and moved with the previously acquired velocity v_i under the action of the force F_i . However, since the point in question is not free, it cannot displace along $\overline{O_i C_i}$ and complete an actual displacement $\overline{O_i E_i}$. We would like to decompose the force F_i that acts along the direction $\overline{A_i C_i}$ into two others that act along the directions $\overline{A_i E_i}$ and $\overline{E_i C_i}$, respectively, and denote them by J_i and P_i , resp. The first of those forces will produce a displacement $A_i E_i$ that has the actual displacement of the material point as a consequence when it is coupled with the displacement $O_i A_i$ that takes place as a result of the acquired velocity as if that point moved freely. If one carries out that decomposition for all points of the system then one will see that the forces J_1 , J_2 , …, J_i , …, J_n will produce *actual* displacements as if each of the points were free.

As far as the forces F_1 , F_2 , …, F_i , …, F_n are concerned, *they cannot seek to generate a displacement that will have a possible displacement as a consequence when it is coupled with an actual one.* The latter condition serves to exhibit the missing equations in the actual displacements. However, we would first like to focus our attention more upon those displacements that *do give possible displacements when they are coupled with the actual ones.* Let:

$$(\Delta x_1 , \Delta y_1 , \Delta z_1), \quad (\Delta x_2 , \Delta y_2 , \Delta z_2), \quad \ldots, (\Delta x_i , \Delta y_i , \Delta z_i), \quad \ldots, (\Delta x_n , \Delta y_n , \Delta z_n) \tag{12}$$

be an arbitrary displacement of a system. If it is coupled with the actual displacement then that will yield the displacements:

$$\begin{gathered}(\Delta x_1 + \partial x_1 , \Delta y_1 + \partial y_1 , \Delta z_1 + \partial z_1), \quad (\Delta x_2 + \partial x_2 , \Delta y_2 + \partial y_2 , \Delta z_2 + \partial z_2), \\ \ldots, (\Delta x_i + \partial x_i , \Delta y_i + \partial y_i , \Delta z_i + \partial z_i), \\ \ldots, (\Delta x_n + \partial x_n , \Delta y_n + \partial y_n , \Delta z_n + \partial z_n) .\end{gathered} \tag{13}$$

If the displacement (12), in conjunction with the actual one, has a possible displacement as a consequence then $\Delta x_i + \partial x_i$, $\Delta y_i + \partial y_i$, $\Delta z_i + \partial z_i$ must make the functions:

$$(14)\quad \begin{cases} \sum\left\{\frac{\partial L'}{\partial x_i}\delta x_i + \frac{\partial L'}{\partial y_i}\delta y_i + \frac{\partial L'}{\partial z_i}\delta z_i\right\} \\ +\sum\left\{\frac{\partial L'}{\partial x_i}\partial x_i + \frac{\partial L'}{\partial y_i}\partial y_i + \frac{\partial L'}{\partial z_i}\partial z_i\right\} + \frac{\partial L'}{\partial t}\cdot\partial t, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \end{cases}$$

positive or equal to zero when they are substituted in the functions (8) and (9) in place of δx_i, δy_i, δz_i, and make the functions:

$$(15)\quad \begin{cases} \sum\left\{\frac{\partial M'}{\partial x_i}\delta x_i + \frac{\partial M'}{\partial y_i}\delta y_i + \frac{\partial M'}{\partial z_i}\delta z_i\right\} \\ +\sum\left\{\frac{\partial M'}{\partial x_i}\partial x_i + \frac{\partial M'}{\partial y_i}\partial y_i + \frac{\partial M'}{\partial z_i}\partial z_i\right\} + \frac{\partial M'}{\partial t}\cdot\partial t, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \end{cases}$$

equal to zero.

However, as a result of equations (10) and (11), the expressions (14) and (15) will go to the following homogeneous linear functions:

$$(16)\qquad \sum\left\{\frac{\partial L'}{\partial x_i}\Delta x_i + \frac{\partial L'}{\partial y_i}\Delta y_i + \frac{\partial L'}{\partial z_i}\Delta z_i\right\},\quad \sum\left\{\frac{\partial L''}{\partial x_i}\Delta x_i + \frac{\partial L''}{\partial y_i}\Delta y_i + \frac{\partial L''}{\partial z_i}\Delta z_i\right\},\dots$$

and

$$(17)\qquad \sum\left\{\frac{\partial M'}{\partial x_i}\Delta x_i + \frac{\partial M'}{\partial y_i}\Delta y_i + \frac{\partial M'}{\partial z_i}\Delta z_i\right\},\quad \sum\left\{\frac{\partial M''}{\partial x_i}\Delta x_i + \frac{\partial M''}{\partial y_i}\Delta y_i + \frac{\partial M''}{\partial z_i}\Delta z_i\right\},\dots$$

It is clear from this that all displacements (12) that have *possible* displacements as a consequence in conjunction with actual ones will make the homogeneous linear functions (16) positive or equal to zero and the functions (17) equal to zero. One also sees that the expressions (16) and (17) represent changes in the functions L', L'', …, M', M'', … that are *independent* of the changes in time.

Let $\overline{E_i D_i}$ be the aforementioned displacement Δs_i of a material point m_i and let Δx_i, Δy_i, Δz_i be its projections onto the coordinates. From the triangle $D_i\, E_i\, C_i$, one gets:

$$\overline{D_i C_i}^2 = \overline{D_i E_i}^2 + \overline{E_i C_i}^2 - 2\overline{D_i E_i}\cdot\overline{E_i C_i}\cdot\cos(\overline{D_i E_i}, \overline{E_i C_i}),$$

in which $(\overline{D_iE_i}, \overline{E_iC_i})$ denotes the angle between the displacement and the direction of the lost force P_i . If one multiplies those equations by m_i and sums over all points of the system then one will get:

$$\sum m_i \cdot \overline{D_iC_i}^2 - \sum m_i \cdot \overline{E_iC_i}^2 = \sum m_i \cdot \overline{D_iE_i}^2 - 2\sum m_i \cdot \overline{D_iE_i} \cdot \overline{E_iC_i} \cdot \cos(\overline{D_iE_i}, \overline{E_iC_i}). \tag{18}$$

In the second term on the right-hand side of the last equation, $\overline{E_iC_i}$ denotes the displacement that the lost force P_i would generate for the free motion of the material point in question.

However, that displacement will be determined by the equation:

$$\overline{E_iC_i} = \frac{P_i}{m_i} \cdot \frac{\partial t^2}{2}. \tag{19}$$

If $\overline{E_iC_i}$ were replaced with its value then one would get:

$$2\sum m_i \cdot \overline{D_iE_i} \cdot \overline{E_iC_i} \cdot \cos(\overline{D_iE_i}, \overline{E_iC_i}) = \partial t^2 \cdot \sum P_i \cdot \overline{D_iE_i} \cdot \cos(P_i, \overline{D_iE_i}).$$

However, since the lost forces P_1, P_2, …, P_i , …, P_n cannot generate displacements that would have possible displacements as a consequence in conjunction with the actual ones, and since forces in general have no ambition to generate displacements relative to which the total moment can assume no positive values, we conclude that the total moment of the lost force:

$$\sum P_i \cdot \overline{D_iE_i} \cdot \cos(P_i, \overline{E_iC_i})$$

and as a result, the second term on the right-hand side of equations (18):

$$2\sum m_i \cdot \overline{D_iE_i} \cdot \overline{E_iC_i} \cdot \cos(\overline{D_iE_i}, \overline{E_iC_i})$$

cannot assume positive values and that the right-hand sides of the aforementioned equations must always remain positive. In that way, one will come to the conclusion that during the motion of a system of material points, one will have:

$$\sum m_i \cdot \overline{E_iC_i}^2 < \sum m_i \cdot \overline{D_iC_i}^2.$$

That inequality includes the so-called **Gaussian** *principle of least constraint*, although **Gauss** only proved it in the case where the conditions on the system were independent of time. As **Gauss** said:

> “The new principle is the following one: The motion of a system of material points that are coupled with each other in some way, and whose motions are, at the same time, constrained by whatever sort of external restrictions, will take place at each moment with the greatest possible agreement with the free motion or under the smallest possible constraint, when one considers a measure of the constraint

that the entire system suffers at each time point to be the sum of the products of the square of the deviation of each point from its free motion with its mass."

(**Gauss**, *Werke*, Bd. V, pp. 26)

However, one can consider the inequality (20) from a different angle that expresses its mechanical meaning more precisely. Namely, the expression:

$$\sum m_i \cdot \overline{E_i C_i}^2 ,$$

which assumes its smallest value for the actual displacements, can be transformed into the form:

$$\frac{\partial t^2}{2} \cdot \sum P_i \cdot \overline{E_i C_i}$$

using equation (19), or also into:

$$\frac{\partial t^2}{2} \cdot \sum P_i \, p_i$$

when one sets $E_i C_i = p_i$. However, since P_i denotes the lost force and p_i denotes the displacement that the force would impart to the material point under free motion, $P_i p_i$ will represent the work that the lost forces would do under free motion. The meaning of the expression (20) is clear from that: *The work that is done by lost forces under the motion of a system of material points will have its smallest value for free motion. The infinitely small increase in that work done will remain positive for any displacement that has a possible displacement as a consequence when it is combined with the actual one.*

One then sees that **Gauss**'s *principle of least constraint* can be called the *principle of least work done by lost forces*, which will bring one into much closer agreement with the present viewpoint on natural phenomena.

§ 4. – We would now like to derive the equations of the actual displacements from the principle of least work done by lost forces.

Let $\Delta \sum (P_i p_i)$ denote the change in the lost work that would be produced by the displacements that generate possible displacements when combined with the actual ones. From the principle of least work done by lost forces:

$$\Delta \sum (P_i \, p_i) \tag{21}$$

will always be positive then. If one replaces P_i with its magnitude in equation (19) then one will get:

$$\Delta \sum (P_i \, p_i) = \frac{2}{\partial t^2} \cdot \Delta \sum m_i \, p_i^2 = \frac{2}{\partial t^2} \cdot \sum m_i \cdot \Delta p_i^2 .$$

The coordinates x_i, y_i, z_i determine the position of the material point m_i at the time t. One lets a_i, b_i, c_i denote the coordinates of m_i at time $t + \partial t$ in the case where the material point in question moves entirely freely during the time interval ∂t, and lets ξ_i, ψ_i, ζ_i denote the coordinates of that point for the case of the actual displacement of it that takes place. One will then have:

$$p_i^2 = (\xi_i - a_i)^2 + (\eta_i - b_i)^2 + (\zeta_i - c_i)^2 .$$

One lets $\Delta\xi_i$, $\Delta\psi_i$, $\Delta\zeta_i$ denote the projections onto the coordinate axes of $D_i E_i$; i.e., the changes that the coordinates ξ_i, ψ_i, ζ_i experience under those infinitely small displacements of the system that have a possible displacement as a consequence when they are combined with the actual one. If one draws a line $O_i G_i$ through the point O_i whose coordinates are x_i, y_i, z_i that is parallel and equal in length to $E_i D_i$ then one will see that:

$$\Delta\xi_i = \Delta x_i , \qquad \Delta\eta_i = \Delta y_i , \qquad \Delta\zeta_i = \Delta z_i ,$$

$$\Delta p_i^2 = 2\{(\xi_i - a_i) \cdot \Delta x_i + (\eta_i - b_i) \cdot \Delta y_i + (\zeta_i - c_i) \cdot \Delta z_i \},$$

$$\text{(22)} \qquad \Delta \sum (P_i\, p_i) = \frac{4}{\partial t^2} \sum m_i\{(\xi_i - a_i) \cdot \Delta x_i + (\eta_i - b_i) \cdot \Delta y_i + (\zeta_i - c_i) \cdot \Delta z_i\}.$$

From the *principle of the least work done by lost forces*, the right-hand side of the equation above must remain positive for any displacement that has a possible displacement as a consequence when it is combined with the actual one, or what amounts to the same thing, the function:

$$\text{(23)} \qquad \frac{2}{\partial t^2} \sum m_i\{(a_i - \xi_i) \cdot \Delta x_i + (b_i - \eta_i) \cdot \Delta y_i + (c_i - \zeta_i) \cdot \Delta z_i\},$$

which is homogeneous and linear in the displacements, must not assume any positive values for those displacements that make the functions (16) positive and equal to zero and the functions (17) equal to zero.

In order to fulfill those conditions, it will suffice that when the function (23) is added to the functions (16) and (17), each of which is multiplied by a suitable factor, that sum will remain equal to zero for all arbitrary displacements of the system:

$$\begin{aligned} &\frac{2}{\partial t^2} \sum m_i\{(a_i - \xi_i) \cdot \Delta x_i + (b_i - \eta_i) \cdot \Delta y_i + (c_i - \zeta_i) \cdot \Delta z_i\} \\ &\quad + \lambda' \sum \left\{ \frac{\partial L'}{\partial x_i} \cdot \Delta x_i + \frac{\partial L'}{\partial y_i} \cdot \Delta y_i + \frac{\partial L'}{\partial z_i} \cdot \Delta z_i \right\} + \cdots \\ &\quad + \mu' \sum \left\{ \frac{\partial M'}{\partial x_i} \cdot \Delta x_i + \frac{\partial M'}{\partial y_i} \cdot \Delta y_i + \frac{\partial M'}{\partial z_i} \cdot \Delta z_i \right\} + \cdots = 0, \end{aligned}$$

where λ', λ'', …, μ', μ'', … are the aforementioned factors, of which λ', λ'', … always remain positive. Since Δx_i , Δy_i , Δz_i are arbitrary, the equation above will decompose into the following one:

$$(24) \qquad \begin{cases} \dfrac{2m_i}{\partial t^2}(a_i - \xi_i) + \lambda' \dfrac{\partial L'}{\partial x_i} + \cdots + \mu' \dfrac{\partial M'}{\partial x_i} + \cdots = 0, \\ \dfrac{2m_i}{\partial t^2}(b_i - \eta_i) + \lambda' \dfrac{\partial L'}{\partial y_i} + \cdots + \mu' \dfrac{\partial M'}{\partial y_i} + \cdots = 0, \\ \dfrac{2m_i}{\partial t^2}(c_i - \zeta_i) + \lambda' \dfrac{\partial L'}{\partial z_i} + \cdots + \mu' \dfrac{\partial M'}{\partial z_i} + \cdots = 0, \end{cases}$$

in which i is set to all whole numbers from 1 to n . From the theory of the motion of a material point, one has:

$$a_i = x_i + v_i \cos(v_i, x) \cdot \partial t + \frac{X_i}{m_i} \cdot \frac{\partial t^2}{2},$$

$$b_i = y_i + v_i \cos(v_i, y) \cdot \partial t + \frac{Y_i}{m_i} \cdot \frac{\partial t^2}{2},$$

$$c_i = z_i + v_i \cos(v_i, z) \cdot \partial t + \frac{Z_i}{m_i} \cdot \frac{\partial t^2}{2}.$$

The actual position of the material point m_i whose coordinates at time t are x_i , y_i , z_i will be determined by the equations:

$$\xi_i = x_i + \frac{\partial x_i}{\partial t} + \frac{\partial^2 x_i}{\partial t^2} \cdot \frac{\partial t^2}{2},$$

$$\eta_i = y_i + \frac{\partial y_i}{\partial t} + \frac{\partial^2 y_i}{\partial t^2} \cdot \frac{\partial t^2}{2},$$

$$\zeta_i = z_i + \frac{\partial z_i}{\partial t} + \frac{\partial^2 z_i}{\partial t^2} \cdot \frac{\partial t^2}{2}$$

at time $t + \partial t$. If one introduces the expression for a_i , b_i , c_i , ξ_i , η_i , ζ_i thus obtained into equations (24) and remarks that:

$$v_i \cos(v_i, x) = \frac{\partial x_i}{\partial t}, \quad v_i \cos(v_i, y) = \frac{\partial y_i}{\partial t}, \quad v_i \cos(v_i, z) = \frac{\partial z_i}{\partial t}$$

then one will get the equations of motion:

$$(25)\qquad \begin{cases} X_i - m_i \dfrac{\partial^2 x_i}{\partial t^2} + \lambda' \cdot \dfrac{\partial L'}{\partial x_i} + \cdots + \mu' \dfrac{\partial M'}{\partial x_i} + \cdots = 0, \\ Y_i - m_i \dfrac{\partial^2 y_i}{\partial t^2} + \lambda' \cdot \dfrac{\partial L'}{\partial y_i} + \cdots + \mu' \dfrac{\partial M'}{\partial y_i} + \cdots = 0, \\ Z_i - m_i \dfrac{\partial^2 z_i}{\partial t^2} + \lambda' \cdot \dfrac{\partial L'}{\partial z_i} + \cdots + \mu' \dfrac{\partial M'}{\partial z_i} + \cdots = 0, \end{cases}$$

whose number is known to amount to $3n$. Those equations, in conjunction with the ones for the actual displacements (10) and (11), determine the $3n$ coordinates of the of the material point and the factors $\lambda', \lambda'', \ldots, \mu', \mu'', \ldots$

§ 5. – From what was said above, it is easy to show the connection that exists between the *principle of least work done by lost forces* and the *principle of virtual displacements, in conjunction with d'Alembert's principle.*

If one couples the principle of virtual displacements with that of **d'Alembert** and extends it to the case in which the conditions of a system depend upon time then it is known that the principle expresses the idea that the lost forces cannot generate displacements of the system that have a possible one as a result of being combined with the actual one. However, in order for that to be true, it is necessary that the total moment of the lost forces:

$$(26)\qquad \sum P_i \cdot \cos(P_i, \Delta s_i) \cdot \Delta s_i$$

cannot assume a positive value for the displacement $\Delta s_1, \Delta s_2, \ldots, \Delta s_i, \ldots, \Delta s_n$ that might be possible when combined with the actual one. Now, if one decomposes the lost force P_i into two others: viz., the actual force F_i and the force $J_i = m_i \dfrac{\partial^2 z_i}{\partial t^2}$ that generates the actual motion of the material point and is taken with the opposite sign:

$$P_i \cdot \cos(P_i, x) = X_i - m_i \frac{\partial^2 x_i}{\partial t^2},$$

$$P_i \cdot \cos(P_i, y) = Y_i - m_i \frac{\partial^2 y_i}{\partial t^2},$$

$$P_i \cdot \cos(P_i, z) = Z_i - m_i \frac{\partial^2 z_i}{\partial t^2},$$

$$\Delta x_i = \Delta s_i \cdot \cos(\Delta s_i, x), \qquad \Delta y_i = \Delta s_i \cdot \cos(\Delta s_i, y), \qquad \Delta z_i = \Delta s_i \cdot \cos(\Delta s_i, z),$$

$$\cos(P_i, \Delta s_i) = \cos(P_i, x) \cdot \cos(\Delta s_i, x) + \cos(P_i, y) \cdot \cos(\Delta s_i, y) + \cos(P_i, z) \cdot \cos(\Delta s_i, z),$$

then the total moment of the lost forces (26) will assume the form:

$$\sum\left\{\left(X_i - m_i\frac{\partial^2 x_i}{\partial t^2}\right)\cdot\Delta x_i + \left(Y_i - m_i\frac{\partial^2 y_i}{\partial t^2}\right)\cdot\Delta y_i + \left(Z_i - m_i\frac{\partial^2 z_i}{\partial t^2}\right)\cdot\Delta z_i\right\}.$$

That function, which is linear in the displacements, cannot assume any positive value for those displacements that are possible when combined with the actual one; i.e., ones that make the functions (16) positive and equal to zero and the functions (17) equal to zero. However, from the lemma that was discussed at the beginning, it is necessary and sufficient that the equation:

$$\begin{aligned}\sum\left\{\left(X_i - m_i\frac{\partial^2 x_i}{\partial t^2}\right)\cdot\Delta x_i + \left(Y_i - m_i\frac{\partial^2 y_i}{\partial t^2}\right)\cdot\Delta y_i + \left(Z_i - m_i\frac{\partial^2 z_i}{\partial t^2}\right)\cdot\Delta z_i\right\}& \\ + \lambda'\sum\left\{\frac{\partial L'}{\partial x_i}\cdot\Delta x_i + \frac{\partial L'}{\partial y_i}\cdot\Delta y_i + \frac{\partial L'}{\partial z_i}\cdot\Delta z_i\right\} + \ldots& \\ + \mu'\sum\left\{\frac{\partial M'}{\partial x_i}\cdot\Delta x_i + \frac{\partial M'}{\partial y_i}\cdot\Delta y_i + \frac{\partial M'}{\partial z_i}\cdot\Delta z_i\right\} + \ldots = 0&\end{aligned}$$

should be valid for all arbitrary displacements of the system, and therefore λ', λ'', … must be positive. The equation above decomposed into equations (25). If one introduces the values of a_i, b_i, c_i, ξ_i, η_i, ζ_i into equation (22) then one will get:

$$-\Delta\sum(P_i\, p_i) = 2\sum\left\{\left(X_i - m_i\frac{\partial^2 x_i}{\partial t^2}\right)\cdot\Delta x_i + \left(Y_i - m_i\frac{\partial^2 y_i}{\partial t^2}\right)\cdot\Delta y_i + \left(Z_i - m_i\frac{\partial^2 z_i}{\partial t^2}\right)\cdot\Delta z_i\right\},$$

or

$$-\Delta\sum(P_i\, p_i) = 2\sum P_i\cdot\Delta s_i\cdot\cos(P_i, \Delta s_i)\,.$$

That equation expresses the *connection* between the *principle of virtual velocities, in conjunction with* ***d'Alembert****'s principle and that of the least work done by lost forces.* From the latter principle, the increase in the lost work is always positive, while the principle of virtual displacements, in conjunction with **d'Alembert**'s, expresses the idea that the total moment of the lost forces:

$$\sum P_i\cdot\Delta s_i\cdot\cos(P_i, \Delta s_i)$$

does not assume any positive value relative to those displacements, as above.

§ 6. – We shall now move on to the *principle of least action*, about which an English mathematician expressed the following opinion: *The Principle of Least Action, in the form commonly given, is a meaningless proposition*.

One lets T denote the *vis viva* of a system of material points:

$$T = \tfrac{1}{2}\sum m_i \left(\frac{\partial x_i^2 + \partial y_i^2 + \partial z_i^2}{\partial t^2} \right)$$

and sets:

$$\sum (X_i\, \Delta x_i + Y_i\, \Delta y_i + Z_i\, \Delta z_i) = \Delta U,$$

in which U can generally be a function of time, since Δ refers to the coordinate changes that are independent of time. If one changes T by means of the symbol Δ then one will get, in succession:

$$\begin{aligned}
\Delta T &= \sum m_i \left(\frac{\partial x_i}{\partial t^2} \cdot \Delta\, \partial x_i + \frac{\partial y_i}{\partial t^2} \cdot \Delta\, \partial y_i + \frac{\partial z_i}{\partial t^2} \cdot \Delta\, \partial z_i \right) \\
&= \sum m_i \left(\frac{\partial x_i}{\partial t^2} \cdot \partial \Delta\, x_i + \frac{\partial y_i}{\partial t^2} \cdot \partial \Delta\, y_i + \frac{\partial z_i}{\partial t^2} \cdot \partial \Delta\, z_i \right) \\
&= \frac{\partial}{\partial t} \sum m_i \left(\frac{\partial x_i}{\partial t} \cdot \Delta\, x_i + \frac{\partial y_i}{\partial t} \cdot \Delta\, y_i + \frac{\partial z_i}{\partial t} \cdot \Delta\, z_i \right) \\
&\quad - \sum m_i \left(\frac{\partial^2 x_i}{\partial t^2} \cdot \Delta\, x_i + \frac{\partial^2 y_i}{\partial t^2} \cdot \Delta\, y_i + \frac{\partial^2 z_i}{\partial t^2} \cdot \Delta\, z_i \right),
\end{aligned}$$

from which it will follow that:

$$\begin{aligned}
&\sum m_i \left(\frac{\partial^2 x_i}{\partial t^2} \cdot \Delta\, x_i + \frac{\partial^2 y_i}{\partial t^2} \cdot \Delta\, y_i + \frac{\partial^2 z_i}{\partial t^2} \cdot \Delta\, z_i \right) \\
&= \frac{\partial}{\partial t} \sum m_i \left(\frac{\partial x_i}{\partial t} \cdot \Delta\, x_i + \frac{\partial y_i}{\partial t} \cdot \Delta\, y_i + \frac{\partial z_i}{\partial t} \cdot \Delta\, z_i \right) - \Delta T\,.
\end{aligned} \tag{28}$$

As a result of this, equation (27) will assume the form:

$$-\Delta \sum (P_i\; p_i) = 2 \left\{ \Delta U + \Delta T - \frac{\partial}{\partial t} \sum m_i \left(\frac{\partial x_i}{\partial t} \cdot \Delta\, x_i + \frac{\partial y_i}{\partial t} \cdot \Delta\, y_i + \frac{\partial z_i}{\partial t} \cdot \Delta\, z_i \right) \right\}.$$

If one multiplies that equation, which is always considered to be positive, by ∂t and integrates both sides of it between the limits $t = t_0$ and $t = t_1$, which correspond to two well-defined positions of the system in space, such that one will have:

$$\Delta x_i = 0, \qquad \Delta y_i = 0, \qquad \Delta z_i = 0$$

at the limits, then one will get:

$$-\int_{t_0}^{t_1} \Delta \sum (P_i\, p_i) \cdot \partial t \;=\; 2\int_{t_0}^{t_1} (\Delta U + \Delta T) \cdot \partial t \,.$$

However, since Δ refers to a time-independent change, one can put the equation above into the form:

$$-\int_{t_0}^{t_1} \Delta \sum (P_i\, p_i) \cdot \partial t \;=\; 2 \cdot \Delta \int_{t_0}^{t_1} (U + T) \cdot \partial t \,.$$

That equation expresses the connection between the so-called principle of least action and that of least lost work and explains the dynamical meaning of the expression in the right-hand side of the equation above. Since $\Delta\,(P_i\, p_i)$ cannot assume a positive value for the displacement that is possible when it is combined with the actual one, we will see that:

$$\Delta \int_{t_0}^{t_1} (U + T) \cdot \partial t$$

cannot assume positive values for that displacement. In that sense, one can call the principle of least action, more precisely, *the principle of greatest action*, although neither of the two statements can be regarded to be strictly rigorous, and that can cause some confusion as to their meaning.

If one considers the expression:

$$\int_{t_0}^{t_1} (U + T) \cdot \partial t$$

by itself, which will experience negative increments for a displacement that is possible in conjunction with the actual one, then one will be in a position to derive the equations of motion of a system of material points regularly even in the case where the conditions depend upon time and are expressed by means of equations and inequalities. In fact, from what was said above:

$$\Delta \int_{t_0}^{t_1} (U + T) \cdot \partial t \qquad \text{or} \qquad \int_{t_0}^{t_1} (\Delta U + \Delta T) \cdot \partial t$$

cannot assume positive values for the displacements that make the functions (16) positive and equal to zero and the functions (17) equal to zero. If one introduces the value of ΔU into the integral above, namely:

$$\sum (X_i\, \Delta x_i + Y_i\, \Delta y_i + Z_i\, \Delta z_i)\,,$$

and the expression for ΔT in equation (28) then one will get:

(29) $$\int_{t_0}^{t_1}\left\{\sum\left[\left(X_i - m_i\frac{\partial^2 x_i}{\partial t^2}\right)\cdot\Delta x_i + \left(Y_i - m_i\frac{\partial^2 y_i}{\partial t^2}\right)\cdot\Delta y_i + \left(Z_i - m_i\frac{\partial^2 z_i}{\partial t^2}\right)\cdot\Delta z_i\right]\right\}\partial t\,,$$

since the sum:

$$\sum m_i\left(\frac{\partial x_i}{\partial t}\cdot\Delta x_i + \frac{\partial y_i}{\partial t}\cdot\Delta y_i + \frac{\partial z_i}{\partial t}\cdot\Delta z_i\right)$$

will be zero at the limits of the interval. In order for the expression (29) to not assume positive values for the displacements that make the functions (16) positive or equal to zero and the functions (17) equal to zero, it is necessary and sufficient that the equation:

$$\begin{aligned}\int_{t_0}^{t_1=1}\Bigg\{\sum&\left[\left(X_i - m_i\frac{\partial^2 x_i}{\partial t^2}\right)\cdot\Delta x_i + \left(Y_i - m_i\frac{\partial^2 y_i}{\partial t^2}\right)\cdot\Delta y_i + \left(Z_i - m_i\frac{\partial^2 z_i}{\partial t^2}\right)\cdot\Delta z_i\right]\partial t\\ &+ \lambda'\sum\left\{\frac{\partial L'}{\partial x_i}\cdot\Delta x_i + \frac{\partial L'}{\partial y_i}\cdot\Delta y_i + \frac{\partial L'}{\partial z_i}\cdot\Delta z_i\right\} + \ldots\\ &+ \mu'\sum\left[\frac{\partial M'}{\partial x_i}\cdot\Delta x_i + \frac{\partial M'}{\partial y_i}\cdot\Delta y_i + \frac{\partial M'}{\partial z_i}\cdot\Delta z_i\right] + \cdots\Bigg\} + \ldots = 0\end{aligned}$$

should be true for all arbitrary displacements, in which λ', λ'', …, μ', μ'', … are undetermined factors, the first of which λ', λ'', … always remain positive. Due to the arbitrariness of Δx_i , Δy_i, Δz_i the equation above will decompose into $3n$ equations (25). If the expression (29) could assume only *positive* values for the displacements that make the functions (16) positive and equal to zero and the functions (17) equal to zero then we would reach the false conclusion that the factors λ', λ'', … would have to be negative.

"Ueber die Anwendung des Princips des kleinsten Zwanges auf die Elektrodynamik," Sitz. Kön. Bay. Akad. Wiss. **24** (1894), 219-230.

On the application of the principle of least constraint to electrodynamics

By **A. Wassmuth** in Graz

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A point *m* in a system of particles might move from *a* to *b* in time τ if it were free, while its *actual* motion is represented by the line segment *ac*. It is known that the principle of least constraint that **Gauss** expressed will then say that $\sum m \cdot (bc)^2$ must be a minimum, or that one must always have $\sum m \cdot (bc)^2 < \sum m \cdot (bd)^2$, when *ad* is a virtual motion (i.e., one that is consistent with the conditions on the system). If *x*, *y*, *z* are the coordinates of the point *m* on which the forces *m X*, *m Y*, *m Z* should act then any coordinate *x* will go to:

$$x + \frac{dx}{dt}\tau + \frac{1}{2}\frac{d^2x}{dt^2}\tau^2$$

in a very small time τ under the actual motion, and:

$$x + \frac{dx}{dt}\tau + \frac{1}{2}X\,\tau^2$$

for the free motion, such that the square of the deviation $(bc)^2$ or the square of the coordinate differences will be equal to $\frac{\tau^4}{4}\left[\left(\frac{d^2x}{dt^2} - X\right)^2 + \cdots\right]$. One then has an expression *Z* that shall be called the *constraint* (from the German *Zwang*) and that takes the form:

$$Z = \sum m\left[\left(\frac{d^2x}{dt^2} - X\right)^2 + \left(\frac{d^2y}{dt^2} - Y\right)^2 + \left(\frac{d^2z}{dt^2} - Z\right)^2\right],$$

in which the summation extends over all particles, and that is the function that must be minimized in regard to the various accelerations $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$, …, which shall be written briefly as $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, … If one differentiates the condition equations for the system $\varphi_1 = 0$, $\varphi_2 = 0$, … twice with respect to time then, as would emerge from the derivation above ([1]), one must currently regard the coordinates *x* and their first differential quotients as *given*. The equation $\frac{d^2\varphi}{dt^2} = 0$ expresses only that the $\frac{\partial\varphi}{\partial t}\ddot{x} + \ldots$ must possess *unvarying* values. For given values of *x* and dx / dt, the $\ddot{x}$ shall then be determined in such a way that *Z* will be a minimum. One will then get the known equations:

$$m(\ddot{x} - X) + \lambda_1 \frac{\partial\varphi_1}{\partial x} + \lambda_2 \frac{\partial\varphi_2}{\partial x} + \ldots = 0, \ldots$$

Now, *n* mutually-independent variables $p_1, p_2, p_3, \ldots, p_n$ will be introduced in place of the coordinates $x, y, z, \ldots$ such that the virtual work will be equal to $P_1\,\delta p_1 + P_2\,\delta p_2 + \ldots + P_n\,\delta p_n$ and the *vis viva* will be equal to $T = \frac{1}{2}\sum_{\kappa,\lambda} a_{\kappa\lambda}\,\dot{p}_\kappa\,\dot{p}_\lambda$, in which the P_μ and the $a_{\kappa\lambda} = a_{\lambda\kappa}$ depend upon only the coordinates, and the Greek symbols will go from 1 to *n*, as they always will from now on.

Then, as **Lipschitz** showed (*loc. cit.*, pp. 330), the constraint *Z* will be expressed by:

$$Z = \sum_{\mu,\nu} \frac{A_{\mu\nu}}{\Delta} \left\{ a_{1\mu}\,\ddot{p}_1 + a_{2\mu}\,\ddot{p}_2 + \cdots + \begin{bmatrix} 11 \\ \mu \end{bmatrix} \dot{p}_1\,\dot{p}_1 + \begin{bmatrix} 12 \\ \mu \end{bmatrix} \dot{p}_1\,\dot{p}_2 + \cdots - P_\mu \right\}$$

$$\times \left\{ a_{1\nu}\,\ddot{p}_1 + a_{2\nu}\,\ddot{p}_2 + \cdots + \begin{bmatrix} 11 \\ \nu \end{bmatrix} \dot{p}_1\,\dot{p}_1 + \begin{bmatrix} 12 \\ \nu \end{bmatrix} \dot{p}_1\,\dot{p}_2 + \cdots - P_\nu \right\},$$

in which Δ represents the determinant of the $a_{\kappa\lambda}$ and $A_{\mu\nu}$ represents the adjoint:

$$\left(A_{\mu\nu} = \frac{\partial\Delta}{\partial a_{\mu\nu}} \right),$$

and one sets:

$$\begin{bmatrix} \kappa\,\lambda \\ \mu \end{bmatrix} = \frac{1}{2}\left[\frac{\partial a_{\kappa\mu}}{\partial p_\lambda} + \frac{\partial a_{\lambda\mu}}{\partial p_\kappa} - \frac{\partial a_{\kappa\lambda}}{\partial p_\mu} \right].$$

The expression for *Z* will become clearer and more suited to (some) physical problems when the *vis viva T* is introduced. Namely, if one sets:

([1]) **Lipschitz**, Borch. Journ., Bd. 82**, pp. 316** (**Rausenberger**, *Mechanik I*, pp. 166). **Gibbs**, Supplement IV, pp. 319.

$$T_\mu = \frac{d}{dt}\frac{\partial T}{\partial \dot{p}_\mu} - \frac{\partial T}{\partial p_\mu}$$

then ([1]) one will also have:

$$T_\mu = a_{1\mu}\,\ddot{p}_1 + a_{2\mu}\,\ddot{p}_2 + \cdots + \begin{bmatrix} 11 \\ \mu \end{bmatrix} \dot{p}_1\,\dot{p}_1 + \begin{bmatrix} 12 \\ \mu \end{bmatrix} \dot{p}_1\,\dot{p}_2 + \cdots,$$

and one will have:

$$Z = \sum_{\mu,\nu} \frac{A_{\mu\nu}}{\Delta}\,[T_\mu - P_\mu]\,[T_\nu - P_\nu], \tag{I}$$

i.e.:

$$Z = \frac{1}{\Delta}\left\{ \begin{aligned} & A_{11}(T_1 - P_1)^2 + A_{22}(T_2 - P_2)^2 + \cdots \\ & \quad + 2A_{12}(T_1 - P_1)(T_2 - P_2) + \cdots \\ & \qquad + 2A_{23}(T_2 - P_2)(T_3 - P_3) + \cdots \end{aligned} \right\}.$$

Since this expression for the constraint Z would seem to be new, it would not be irrelevant to show that one comes to the Lagrange equations when one addresses the minimum condition for Z. Therefore, as a result of the remark above, one must regard the quantities p_1 and $\dot{p}_1$ as *given* or *fixed* in the differentiation of Z with respect to $\ddot{p}_1$, and make use of the relation:

$$\frac{\partial T}{\partial \ddot{p}_\kappa} = a_{\mu\kappa} = a_{\kappa\mu}\,.$$

One then gets:

$$\frac{\partial Z}{\partial \ddot{p}_1} = \sum_{\mu,\nu} \frac{a_{1\mu}\,A_{\mu\nu}}{\Delta}(T_\nu - P_\nu) + \sum_{\mu,\nu} \frac{a_{1\nu}\,A_{\mu\nu}}{\Delta}(T_\mu - P_\mu),$$

or, since those sums are equal to each other:

$$\frac{\partial Z}{\partial \ddot{p}_1} = \frac{2}{\Delta}\sum_{\mu,\nu} a_{1\mu}\,A_{\mu\nu}\,(T_\nu - P_\nu)\,,$$

or

$$\frac{2}{\Delta}\sum_{\nu}(T_\nu - P_\nu)\sum_{\mu} a_{1\mu}\,A_{\mu\nu} = \frac{2}{\Delta}\sum_{\nu}(T_\nu - P_\nu)[a_{11}\,A_{1\nu} + a_{12}\,A_{2\nu} + \cdots]\,.$$

Now, from a property of the determinant that $a_{11}\,A_{1\nu} + a_{12}\,A_{2\nu} + \ldots = \Delta$ or zero according to whether $\nu = 1$ or $\nu > 1$, so it would follow from $\frac{\partial Z}{\partial \ddot{p}_1} = 0$ that one must also have $T_1 - P_1 = 0$; i.e., Lagrange's equation:

([1]) Cf., e.g., **Rayleigh**, *Sound*, pp. 111 (in the German translation). **Stäckel**, Borch. Jour. **107**, pp. 322.

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{p}_1} - \frac{\partial T}{\partial p_1} = P_1 \,.$$

One can find even more expressions for the constraint Z in a way that is entirely similar to the way that one exhibits *auxiliary forms* ([1]) for Lagrange's fundamental equation.

It is important for the applications to note that the constraint Z can be represented in such a way that *one* acceleration – e.g., $\ddot{p}_1$ – will appear in it detached from the remaining ones. One has: $Z \cdot \Delta = \frac{1}{2}(L_1\,\ddot{p}_1^2 + 2M_1\,\ddot{p}_1 + N_1)$, in which L_1, M_1, N_1 *do not* include $\ddot{p}_1$, and L_1 and M_1 can be easily found from the equation $\frac{\partial Z}{\partial \ddot{p}_1} = 0$.

If the virtual work $\sum_\mu P_\mu\,\delta p_\mu$ *and the vis viva* $T = \frac{1}{2}\sum_{\kappa,\lambda} a_{\kappa\lambda}\,\dot{p}_\kappa\,\dot{p}_\lambda$ *are given for a physical problem then the constraint on the system can be determined by means of equation* (I). *The minimum property of Z expresses a law for the system that is certainly new in many cases and entirely distinct from the fact that other, perhaps already known, laws would follow from Z by actual differentiation.*

That is how, e.g., **Boltzmann** has derived Maxwell's equations for electricity in a wonderfully simple way from Lagrange's equations in volume I of his lectures, and thus supported it by mechanical processes. It illuminated the fact that one could also start from the principle of least constraint in the form of equation (I) above as the main theorem, and once the virtual work and *vis viva* were given, one would have to arrive at Lagrange's equations by appealing to the minimum condition, along with Maxwell's equations – in which one now proceeds entirely as **Boltzmann** did. Although it might also become very hard to simplify **Boltzmann**'s classical methods (especially in Part 2 of his lectures) any more in that way, nonetheless, the condition Z = minimum will, after all, express a *newly-recognized* truth.

As an example, one might consider the case of two cyclic coordinates $\dot{p}_1 = l_1'$ and $\dot{p}_2 = l_2'$ (the slowly-varying parameters k will be ignored temporarily). Here, one has:

$$T = \tfrac{1}{2}a_{11}\,\dot{p}_1^2 + \tfrac{1}{2}a_{22}\,\dot{p}_2^2 + \tfrac{1}{2}a_{12}\,\dot{p}_1\,\dot{p}_2 = \frac{A}{2}l_1'^2 + \frac{B}{2}l_2'^2 + C\,l_1' l_2' ,$$

when one introduces Boltzmann's notation. It will then follow that:

$$a_{11} = A, \qquad a_{22} = B, \qquad a_{12} = C,$$

$$\Delta = \begin{vmatrix} A & C \\ C & B \end{vmatrix} = AB - C^2,$$

([1]) **Weinstein**, Wied. Ann. **15**. **Budde**, *Mechanik I*, pp. 397.

$$A_{11} = B, \qquad A_{12} = -C, \qquad A_{22} = A,$$

$$T_1 = \frac{d}{dt}(A l_1' + B l_2'), \quad T_2 = \frac{d}{dt}(B l_2' + C l_1').$$

In addition, when there is friction or viscosity, the dissipation function (*loc. cit.*, pp. 108 and 109) that Rayleigh exhibited might be denoted by:

$$F = \tfrac{1}{2}\sum(\kappa_1 \dot{x}_1^2 + \cdots) = \tfrac{1}{2} b_{11} \dot{p}_1^2 + \tfrac{1}{2} b_{22} \dot{p}_2^2 + \tfrac{1}{2} b_{12} \dot{p}_1 \dot{p}_2 .$$

As is known, one then adds $-\dfrac{\partial F}{\partial \dot{p}_\mu}$ to the forces $P_\mu = L_\mu$. It is also clear that $b_{12} = 0$, on intrinsic grounds.

Ultimately, the constraint Z will then be expressed by:

$$Z \cdot (A B - C^2) = B\left[\frac{d}{dt}(A l_1' + C l_2') - L_1 - b_{11} l_1'\right]^2$$

$$-2C\left[\frac{d}{dt}(A l_1' + C l_2') - L_1 - b_{11} l_1'\right]\left[\frac{d}{dt}(A l_2' + C l_1') - L_2 - b_{11} l_2'\right]$$

$$+ A\left[\frac{d}{dt}(A l_2' + C l_1') - L_2 - b_{22} l_2'\right]^2,$$

and Z must be a minimum, in such a way that one will have $\dfrac{\partial Z}{\partial l_1'} = 0$ and $\dfrac{\partial Z}{\partial l_2'} = 0$. In this (Boltzmann I, pps. 34 and 35), l_1' and l_2' represent the current strengths in the two conductors, b_{11} and b_{22} are their resistances, L_1 and L_2 are the electromotive forces in them, A and B are the coefficients of self-induction and, and C is the mutual induction. If condensers (*loc. cit.*, I, pp. 35) are also included then terms of the form $d_1\, l_1$ and $d_2\, l_2$ will also enter into the brackets.

The condition Z = minimum then expresses a *basic electrodynamical law* and implies the theory of self-induction and mutual induction for current fluctuations that are not too fast. If one would also like to include ponderomotive forces then one would have to introduce a slowly-varying parameter k along with l_1 and l_2 as a third variable, exhibit the general expression for Z, and construct the equation $\dfrac{\partial Z}{\partial \ddot{k}} = 0$, in which k and $\dot{k}$ are assumed to be constant. Only *afterwards* does one take $\dot{k} = 0$ and $\ddot{k} = 0$ and obtain, as Boltzmann did, the relation:

$$K = -\frac{\partial T}{\partial k} = -\frac{l_1'^2}{2}\frac{\partial A}{\partial k} - \frac{l_2'^2}{2}\frac{\partial B}{\partial k} - l_1' l_2' \frac{\partial C}{\partial k}.$$

Acoustics also serves as another field of applications for the principle above. Frequently, only purely-quadratic terms with constant coefficients appear in the expression for the *vis viva* in that context, which makes the equation for the constraint take an even simpler form;

If one introduces the abbreviation: $T_\mu - P_\mu = Q_\mu$ then the constraint Z will be given by:

$$\begin{aligned} Z \cdot D &= \sum_{\mu,\nu} A_{\mu\nu}(T_\mu - P_\mu)(T_\nu - P_\nu) = \sum_{\mu,\nu} Q_\mu\, Q_\nu \\ &= A_{11}\, Q_1^2 + 2\, A_{12}\, Q_1\, Q_2 + \ldots + 2\, A_{1n}\, Q_1\, Q_n \\ &\quad + A_{22}\, Q_2^2 + \ldots \qquad + 2\, A_{2n}\, Q_2\, Q_n \\ &\quad + A_{nn}\, Q_n^2 . \end{aligned}$$

The condition $\dfrac{\partial Z}{\partial \ddot{p}_\rho} = 0$ can be replaced with $\dfrac{\partial Z}{\partial Q_\rho} = 0$. Namely, one has:

$$\frac{\partial Z}{\partial \ddot{p}_\rho} = \frac{\partial Z}{\partial Q_1}\frac{\partial Q_1}{\partial \ddot{p}_\rho} + \cdots + \frac{\partial Z}{\partial Q_n}\frac{\partial Q_n}{\partial \ddot{p}_\rho},$$

or, since:

$$\frac{\partial Q_\nu}{\partial \ddot{p}_\rho} = \frac{\partial T_\nu}{\partial \ddot{p}_\rho} = a_{\nu\rho},$$

one will have:

$$\frac{\partial Z}{\partial \ddot{p}_\rho} = a_{1\rho}\frac{\partial Z}{\partial Q_1} + a_{2\rho}\frac{\partial Z}{\partial Q_2} + \cdots + a_{n\rho}\frac{\partial Z}{\partial Q_n} \qquad (\rho = 1, \ldots, n).$$

Since the determinant $D = |\, a_{\mu\nu} \,|$ does not vanish, it will generally follow from these n equations that:

$$\frac{\partial Z}{\partial Q_\nu} = 0.$$

If one actually differentiates the expression above then that will yield the n equations:

$$\frac{1}{2}\frac{\partial Z}{\partial Q_\nu} = A_{1\nu}\, Q_1 + A_{2\nu}\, Q_2 + \ldots + A_{\nu n}\, Q_n = 0 \qquad (\nu = 1, \ldots, n),$$

and since the determinant $|\, A_{\mu\nu} \,| = D^{n-1}$ can never be zero, that will, in turn, imply the Lagrange equations: $Q_1 = 0, \ldots, Q_n = 0$.

If the force P has a potential U, such that $P_\mu = -\,\partial U / \partial p_\mu$, and the conditions do not include time t explicitly then one will have, on the one hand, the *vis viva* in the form:

$$2T = \frac{\partial T}{\partial \dot{p}_1}\dot{p}_1 + \frac{\partial T}{\partial \dot{p}_2}\dot{p}_2 + \ldots$$

or

$$2\frac{dT}{dt} = \dot{p}_1 \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}_1}\right) + \ddot{p}_1 \frac{\partial T}{\partial \dot{p}_1} + \ldots,$$

whereas, on the other hand, since T is also a function of p_1, $\dot{p}_1$, …, one will have:

$$\frac{dT}{dt} = \dot{p}_1 \frac{\partial T}{\partial p_1} + \ddot{p}_1 \frac{\partial T}{\partial \dot{p}_1} + \ldots$$

Since:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}_\mu}\right) - \frac{\partial T}{\partial p_\mu} = T_\mu,$$

it will follow upon subtraction that:

$$\frac{dT}{dt} = \dot{p}_1 T_1 + \dot{p}_2 T_2 + \ldots,$$

to which, one adds:

$$\frac{dU}{dt} = \frac{\partial U}{\partial p_1}\dot{p}_1 + \ldots = -\,\dot{p}_1 P_1 - \dot{p}_2 P_2 + \ldots,$$

and since $T_\mu - P_\mu = Q_\mu$, one will ultimately arrive at the equation:

$$\frac{d(T+U)}{dt} = \dot{p}_1 Q_1 + \dot{p}_2 Q_2 + \cdots + \dot{p}_n Q_n = R.$$

One sees that for $Q_1 = 0$, …, $Q_n = 0$, one will also have $R = 0$; i.e., $T + U$ must be equal to a constant, or that the principle of the conservation of energy must also hold, from the Lagrange equations that were found above. Both of them can be obtained *simultaneously* when one eliminates one of the quantities Q – e.g., Q_1 – with the help of the relation $R = \dot{p}_1 Q_1 + \ldots$ and presents the minimum conditions: $\frac{\partial Z}{\partial Q_1} = 0$, $\frac{\partial Z}{\partial Q_n} = 0$ afterwards. It is preferable to use the determinant form for the constraint Z in that. For example, for $n = 3$, one has:

$$-Z \cdot D = \begin{vmatrix} 0 & Q_1 & Q_2 & Q_3 \\ Q_1 & a_{11} & a_{12} & a_{13} \\ Q_2 & a_{21} & a_{22} & a_{23} \\ Q_3 & a_{31} & a_{32} & a_{33} \end{vmatrix} = \frac{1}{\dot{p}_1^2} \begin{vmatrix} 0 & R & Q_2 & Q_3 \\ R & b_{11} & b_{12} & b_{13} \\ Q_2 & b_{21} & a_{22} & a_{23} \\ Q_3 & b_{31} & a_{32} & a_{33} \end{vmatrix},$$

which will make:

$$b_{11} = (a_{11}\,\dot{p}_1 + a_{21}\,\dot{p}_2 + a_{31}\,\dot{p}_3) + \ldots = \frac{\partial T}{\partial \dot{p}_1}\,\dot{p}_1 + \ldots = 2T,$$

$$b_{21} = b_{12} = a_{21}\,\dot{p}_1 + a_{22}\,\dot{p}_2 + a_{23}\,\dot{p}_3 = \frac{\partial T}{\partial \dot{p}_2},$$

$$b_{31} = b_{13} = a_{31}\,\dot{p}_1 + a_{32}\,\dot{p}_2 + a_{33}\,\dot{p}_3 = \frac{\partial T}{\partial \dot{p}_3}.$$

Applying the minimum conditions will yield:

$$A_{11}\,R + Q_2\,[\dot{p}_1\,A_{12} - \dot{p}_2\,A_{11}] + Q_3\,[\dot{p}_1\,A_{13} - \dot{p}_3\,A_{11}] = 0,$$

$$A_{12}\,R + Q_2\,[\dot{p}_1\,A_{22} - \dot{p}_2\,A_{12}] + Q_3\,[\dot{p}_1\,A_{23} - \dot{p}_3\,A_{12}] = 0,$$

$$A_{13}\,R + Q_2\,[\dot{p}_1\,A_{23} - \dot{p}_2\,A_{13}] + Q_3\,[\dot{p}_1\,A_{33} - \dot{p}_3\,A_{13}] = 0,$$

or, since the determinant of this system, namely, $\dot{p}_1^2 \left| A_{\mu\nu} \right| = \dot{p}_1^2 \cdot D$ never vanishes, $R = 0$ and $Q_2 = 0$, $Q_3 = 0$; i.e., one has the law of energy and Lagrange's equations.

If one eliminates, say Q_1, from the general equation:

$$A_{1\nu}\,Q_1 + A_{2\nu}\,Q_2 + \ldots + A_{n\nu}\,Q_n = 0$$

by using the relation:

$$R = \dot{p}_1\,Q_1 + \ldots$$

then it will likewise follow naturally that:

$$R = 0, \quad Q_2 = 0, \ldots\, Q_n = 0.$$

Addendum

Concerning linear current branches.

If $p_1, \ldots, p_k$ are, in turn, cyclic coordinates, $T = \frac{1}{2}a_{11}\,\dot{p}_1^2 + \frac{1}{2}a_{22}\,\dot{p}_2^2 + \ldots + \frac{1}{2}a_{12}\,\dot{p}_1\,\dot{p}_2 + \ldots$ is the *vis viva*, and $F = \frac{1}{2}b_{11}\,\dot{p}_1^2 + \frac{1}{2}b_{22}\,\dot{p}_2^2 + \ldots + \frac{1}{2}b_{12}\,\dot{p}_1\,\dot{p}_2 + \ldots$ is the dissipation function that **Lord Rayleigh** introduced then the force $-\frac{\partial F}{\partial \dot{p}_\mu} = -(b_{1\mu}\,\dot{p}_1 + b_{2\mu}\,\dot{p}_2 + \ldots)$ will be added to any force P_μ, and that will yield the principle of least constraint, namely, that:

$$Z \cdot D = \sum A_{\mu\nu}\,Q_\mu\,Q_\nu = A_{11}\,Q_1^2 + A_{22}\,Q_2^2 + 2A_{12}\,Q_1\,Q_2 + \ldots \tag{1}$$

must be a minimum for any Q. Therefore, one has:

$$Q_\mu = \frac{d}{dt}\left[a_{1\mu}\,\dot{p}_1 + a_{2\mu}\,\dot{p}_2 + \cdots\right] - P_\mu + \left[b_{1\mu}\,\dot{p}_1 + b_{2\mu}\,\dot{p}_2 + \cdots\right], \tag{2}$$

$$D = |\,a_{\kappa\lambda}\,|, \qquad A_{\kappa\lambda} = \frac{\partial D}{\partial a_{\kappa\lambda}}.$$

If one goes over to electrodynamics and sets: $\dot{p}_1 = J_1$, $\dot{p}_2 = J_2$, …, as well as:

$$Q_\mu = \frac{d}{dt}\left[a_{1\mu}\,J_1 + a_{2\mu}\,J_2 + \cdots\right] - P_\mu + \left[b_{1\mu}\,J_1 + b_{2\mu}\,J_2 + \cdots\right], \tag{3}$$

then the *minimum property* (1) *will imply a property of a linear current branch*. In it, a_{11}, a_{22}, a_{33}, … are the coefficients of the self-inductions of the first, second loops, a_{12}, a_{13}, a_{23}, … are the coefficients of mutual induction, and P_μ is the constant electromotive force. Furthermore, b_{11} is the resistance of the entire first loop, b_{22} is that of the entire second loop, etc, and b_{12} as is the resistance of that piece of the conductor that is common to loops 1 and 2. b_{12} is positive when J_1 and J_2 have the same directions and negative when they have opposite directions. Applying the minimum condition will imply that $Q_1 = 0$, i.e.:

$$P_1 = b_{11}\,J_1 + b_{12}\,J_2 + \ldots + b_{1n}\,J_n + \frac{d}{dt}\,[a_{11}\,J_1 + a_{12}\,J_2 + \ldots], \quad \ldots \tag{4}$$

Those are (in a somewhat generalized form) the equations that **H. von Helmholtz** presented in 1851 (*Abhandlungen* I, pp. 435) for the induction in linear current branches, which one must think of as being decomposed into the smallest-possible number of simple loops.

For the work done by the retarding forces, one gets:

$$\frac{\partial F}{\partial \dot{p}_1} dp_1 + \ldots = \left[\frac{\partial F}{\partial \dot{p}_1} d\dot{p}_1 + \cdots \right] dt = 2F\, dt = [b_{11} J_1^2 + b_{12} J_2^2 + \cdots + 2 b_{12} J_1 J_2 + \cdots],$$

up to sign; i.e., the Joule heat, and all of the summands in the bracket are positive.

Now assume that $a_{12} = a_{23} = \ldots = 0$; $a_{11} = a_{22} = a_{23} = \ldots = a$, which is a case is not too difficult to realize experimentally. One will then have:

$$Q_\mu = a \frac{d}{dt} [J_1 + J_2 + \ldots] - P_\mu + b_{1\mu} J_1 + b_{2\mu} J_2 + \ldots,$$

and one must have that $Z \cdot a = Q_1^2 + Q_2^2 + \ldots$ is a minimum when a is taken to be small enough. For lim $a = 0$, the current strengths will be independent of time, thus constant, and it will follow from the condition $Z^1 = Q_1^2 + Q_2^2 + \ldots =$ minimum. $Q_\mu = b_{1\mu} J_1 + \ldots - P_\mu = 0$.

Since the determinant of the b does not vanish, the equation $\frac{\partial Z^1}{\partial Q_\mu} = 0$ can be replaced with $\frac{\partial Z^1}{\partial J_\mu} = 0$. For constant currents, one then has to minimize:

$$Z^1 = \sum_\mu [(b_{1\mu} J_1 + \cdots) - P_\mu]^2$$

for every J.

Remark: One can get an oft-mentioned minimum property of constant currents from the well-known equation:

$$\frac{d(T+U)}{dt} = -2F \qquad \text{or} \qquad -\left[\frac{dU}{dt} + F + \frac{dT}{dt} \right] = F, \tag{1}$$

in which $U = P_1 p_1 + \ldots$ represents the potential of the constant force, and as above:

$$T = \tfrac{1}{2} a_{11} \dot{p}_1^2 + \ldots = \tfrac{1}{2} a_{11} J_1^2 + \ldots, \qquad F = \tfrac{1}{2} b_{11} \dot{p}_1^2 + \ldots = \tfrac{1}{2} b_{11} J_1^2 + \ldots$$

If the current strengths J_1, J_2, which are initially zero, attain their full strengths J_1', J_2', … for $t = \infty$ (since $\frac{dT}{dt} = \frac{\partial T}{\partial J_1} \frac{dJ_1}{dt} + \ldots$), and also $\frac{dT}{dt} = 0$, then the F that is found on the right-hand side of (I), which consists of nothing but positive terms, will attain its *greatest* value. Therefore, the negative left-hand side of (I); i.e.:

$$F + \frac{dU}{dt} = \tfrac{1}{2} b_{11} J_1^2 + \ldots - (P_1 J_1 + P_2 J_2 + \ldots),$$

must represent a minimum for any J.

"Über die Transformation des Zwanges in allgemeine Coordinaten," Sitz. kaiserliche Akad. Wiss. math.-phys. Klasse, Abt. II.a **104** (1895), 281-285.

On the transformation of the constraint into general coordinates

By

Prof. Dr. **A. Wassmuth**

(Presented at the session on 21 March 1895)

Translated by D. H. Delphenich

The principle of least constraint that **Gauss** ([1]) gave in 1829, which he himself called a new general basic law of mechanics, has suffered a peculiar fate in its further development. Although the importance of that principle was recognized many times ([2]), the recent textbooks on mechanics do not go very far beyond the presentation of it that was given originally. The basis for that might lie in the fact that **Gauss** himself gave no "analytical" formulation of his principle. Indeed, in 1858 **Scheffler** ([3]) had already given the expression for Z, that is, the function that is to be differentiated in order to make it a minimum, and found that:

$$Z = \sum m\left[\left(x'' - \frac{X}{m}\right)^2 + \left(y'' - \frac{Y}{m}\right)^2 + \left(z'' - \frac{Z}{m}\right)^2\right],$$

in which the sum is extended over all mass-points m with the accelerations x'', y'', z'', and the force components X, Y, Z. However, **Lipschitz** first clearly showed in 1877 ([4]) that one must regard the accelerations x'', y'', z'' as variable, but the coordinates x, y, z and the velocities x', y', z' as constant. **Lipschitz** ([5]) also introduced general variables into Z in place of the rectangular coordinates (that is, ones that fulfill the condition equations identically) when he exhibited a certain covariant that must be minimized in the principle of least constraint. A study of that very significant work demands that one has acquired the knowledge of two works by the same author on his investigations in regard to homogeneous functions of n differentials ([6]), which would involve considerable effort, time, and prior knowledge. That is the only way to explain the striking fact that the physical literature made no further use of the ideas. The excessively brief derivation of

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([1]) **Gauss**, *Werke* V, Crelle IV.
([2]) **Lagrange**, *Mécan. anal.* II, pp. 350, Note by Bertrand.
([3]) **Scheffler**, Zeit. f. Math. und Phys. III.
([4]) **Lipschitz**, Borch. Journ., Bd. **82**, pp. 323.
([5]) *Loc. cit.*, pp. 330.
([6]) Bd. **70** and **72** of this journal.

the transformation equations that is given below, which is based completely upon physical foundations, would therefore not be unwelcome.

The problem at hand is to introduce the variables $p_1, p_2, \ldots, p_k$, which fulfill the condition equations identically, into the expression for the constraint:

$$Z = \sum_{i=1}^{n} Z_i = \sum_{i=1}^{n} m_i \left[\left(x_i'' - \frac{X_i}{m_i} \right)^2 + \left(y_i'' - \frac{Y_i}{m_i} \right)^2 + \left(z_i'' - \frac{Z_i}{m_i} \right)^2 \right],$$

instead of the rectangular coordinates x_i, y_i, z_i of the n points, in which one might set $x_i = f_1^i(p_1, \ldots, p_k)$, $y_i = f_2^i(p_1, \ldots, p_k)$, and $z_i = f_3^i(p_1, \ldots, p_k)$. If one sets $\partial x_i / \partial p_\mu = f_1^i$, $\partial y_i / \partial p_\mu = f_2^i$, $\partial z_i / \partial p_\mu = f_3^i$, etc., to abbreviate then the variations will be:

$$\delta x_i = f_{11}^i \cdot \delta p_1 + f_{12}^i \cdot \delta p_2 + \cdots + f_{1k}^i \cdot \delta p_k, \quad \delta y_i = \cdots, \quad \delta z_i = \cdots,$$

and analogously (but only when the conditions do not contain time explicitly) the velocities will be:

$$x_i' = f_{11}^i p_1' + \cdots + f_{1k}^i \cdot p_k',$$

etc., and the *vis viva* will be:

$$L = \sum_{i=1}^{n} L_i = \tfrac{1}{2} \sum a_{\mu\nu} p_\mu' p_\nu' \quad (\mu, \nu = 1, 2, \ldots, k).$$

If one now considers the variation of a function H (it coincides with energy in the case of a force function) that is given by:

$$\delta H = \sum_{i=1}^{n} [(m_i x_i'' - X_i)\, \delta x_i + \cdots]$$

then one will get:

$$\delta H = \sum_{\mu=1}^{k} \delta p_\mu \sum_{i=1}^{n} [(m_i x_i'' - X_i)\, f_{1\mu}^i + \cdots]$$

when one introduces the values for δx_i, δy_i, δz_i above and inverts the order of summation. Now, it is known that when $P_{i,\mu} = X_i f_{1\mu} + Y_i f_{2\mu} + Z_i f_{3\mu}$, one has the identity:

$$(m_i x_i'' - X_i)\, f_{1\mu}^i + (m_i y_i'' - Y_i)\, f_{2\mu}^i + (m_i z_i'' - Z_i)\, f_{3\mu}^i = \frac{d}{dt}\frac{\partial L}{\partial p_\mu'} - \frac{\partial L}{\partial p_\mu} - P_{i\mu} = Q_{i\mu},$$

such that when one sets:

$$\sum_{i=1}^{n} Q_{i\mu} = \frac{d}{dt}\frac{\partial L}{\partial p_\mu'} - \frac{\partial L}{\partial p_\mu} - \sum_{i=1}^{n} P_{i\mu} = Q_\mu,$$

and performs the summation, the known equation:

$$\delta H = \sum_{\mu=1}^{k} \delta p_\mu \cdot Q_\mu = Q_1\ \delta p_1 + \ldots + Q_k\ \delta p_k$$

will result.

Now, one can arrive at yet *another* expression for δH when one introduces the constraint Z. The possibility rests upon the fact that one has:

$$f_{1\mu}^{i} = \frac{\partial x_i}{\partial p_\mu} = \frac{\partial x_i'}{\partial p_\mu'}, \quad \text{which is also} = \frac{\partial x_i''}{\partial p_\mu''},$$

which is why:

$$\delta H = \sum_{\mu=1}^{k} \delta p_\mu \cdot \sum_{i=1}^{n} [(m_i\, x_i'' - X_i)\, f_{1\mu}^{i} + \cdots] = \sum_{\mu=1}^{k} \delta p_\mu \cdot \sum_{i=1}^{n} \left[(m_i\, x_i'' - X_i) \frac{\partial x_i''}{\partial p_\mu''} + \cdots \right]$$

$$= \tfrac{1}{2} \sum_{\mu=1}^{k} \delta p_\mu \cdot \sum_{i=1}^{n} \frac{1}{m_i} \frac{\partial}{\partial p_\mu''} \left[(m_i\, x_i'' - X_i)^2 + \cdots \right],$$

or it would follow that:

$$\delta H = Q_1\, \delta p_1 + \cdots + Q_k\, \delta p_k = \tfrac{1}{2} \sum_{\mu=1}^{k} \delta p_\mu \cdot \frac{\partial Z}{\partial p_\mu''} = \frac{1}{2} \frac{\partial Z}{\partial p_1''} \delta p_1 + \cdots + \frac{1}{2} \frac{\partial Z}{\partial p_k''} \delta p_k .$$

It follows from this that: "If $\delta H = 0$ (i.e., if **d'Alembert**'s principle is true) then due to the independence of the δp, that would imply: $\partial Z / \partial p_1'' = 0, \ldots, \partial Z / \partial p_k'' = 0$ (i.e., **Gauss**'s principle), and conversely, the former would follow from the latter; viz., both principles are completely equivalent." Moreover, that implies that:

$$Q_r = \frac{1}{2} \frac{\partial Z}{\partial p_r''}.$$

Now, it is known that ([1]):

$$\frac{\partial Q_\rho}{\partial p_r''} = a_{\rho r} = a_{r\rho},$$

and therefore:

$$\frac{\partial Z}{\partial p_r''} = \frac{\partial Z}{\partial Q_1} \frac{\partial Q_1}{\partial p_r''} + \cdots = a_{1r} \frac{\partial Z}{\partial Q_1} + \cdots + a_{kr} \frac{\partial Z}{\partial Q_k} = 2 Q_r .$$

([1]) **Wassmuth**, "Über die Anwendung des Principes des kleinsten Zwanges auf die Elektrodynamik," Wied. Ann. **54**, pp. 166 [or Sitz. d. kais. Bayer. Akad. (1894), pp. 226 and 222.]

If one sets $r = 1, 2, \ldots, k$ in this in succession then one will get k linear equations whose determinant $D = (a_{\mu\nu})$ does not vanish. If $A_{\mu\nu} = \partial D / \partial a_{\mu\nu}$ is one of the sub-determinants then that will give the solution:

$$\frac{1}{2}\frac{\partial Z}{\partial Q_1} = \frac{1}{D}[A_{11}\,Q_1 + A_{12}\,Q_2 + \cdots + A_{1k}\,Q_k],$$

etc., from which, one will conclude that the constraint is:

$$Z = \frac{1}{D}\sum A_{\mu\nu}\,Q_\mu\,Q_\nu + \varphi(p_1, p_1', p_2, p_2', \ldots).$$

The function φ, which contains only the p and their *first* differential quotients with respect to time, must enter into that, since only the p'' were regarded as variable in the previous differentiation. *The transformation of the constraint Z into general coordinates* is effected by that formula, *to the extent that* ***Gauss****'s principle demands.* The (here unnecessary) determination of φ does not introduce any complications, either.

Now, as far as the importance of **Gauss**'s principle is concerned, one might recall ([1]) that when the virtual work and the *vis viva* are given for a physical problem, the minimum property of the constraint Z expresses a *law for the system.* That will become all the more valid when strives to give a series of descriptions of theories by mechanical analogies, as **W. Voigt** called them. I have already given an application of that to electrodynamics, and likewise thermodynamics.

Since the constraint Z unites *all* of **Lagrange**'s equations: $Q_1 = 0, \ldots, Q_k = 0$, **Gauss**'s principle will prefer those equations when one tries precisely to *unite* them; e.g., in the case of a string that one thinks of as constructed from discrete points.

The value of the principle as a fundamental law is probably best indicated by the fact that **Hertz** ([2]) built all of his mechanics upon that principle and the law of inertia.

([1]) **Wassmuth**, *loc. cit.*, 167.

([2]) **Hertz**, *Principien der Mechanik*, page 185, no. 391.

"Über die analytische Darstellung des Zwanges eines materiellen Systemes in allgemeinen Coordinaten,"Monats. f. Math. u. Physik **7** (1896), 27-33.

On the analytical representation of the constraint on a material system in general coordinates

By **M. Radaković** in Graz

Translated by D. H. Delphenich

The subject of the following pages is defined by the transformation of the expression for the constraint on a material system from its analytical representation by means of rectangular coordinates into one that uses general coordinates. That problem was solved for the first time by **Lipschitz** ("Bemerkungen zu dem Princip des kleinsten Zwanges," Borch. Journ., Bd. 82, pp. 316) and in recent times in a very elegant way by **Wassmuth** (Sitzber. kais. Akad. Wiss. Wien, Bd. CIV, Abt. II.a). Nonetheless, the specification of a method of proof that deviates from the path to solution that was employed in cited papers would not be lacking in theoretical interest.

Let a system of n mass-points be given. The position of each individual point might be determined by specifying its three coordinates relative to a rectangular coordinate system, where the ν^{th} mass-point will be denoted $x_{3\nu-2}$, $x_{3\nu-1}$, $x_{3\nu}$ when one fixes a definite sequence of axes. In a corresponding way, the components of the resultant of all forces that act upon the mass-point relative to the three coordinates axes will then be represented by the symbols $X_{3\nu-2}$, $X_{3\nu-1}$, $X_{3\nu}$ while the equivalent symbols $m_{3\nu-2} = m_{3\nu-1} = m_{3\nu}$ will be chosen for the mass of the point in question in order to achieve unity in our notation. The system of mass-points considered might be assumed to be such that its connections can be described by specifying $(3n - k)$ equations:

$$f_i\,(x_1, x_2, \ldots, x_{3n}) = 0 \qquad (i = 1, 2, \ldots, 3n - k) \tag{1}$$

between the $3n$ coordinates of the system points analytically, in which it will be assumed that these equations do not contain time explicitly, for the sake of greater simplicity.

Under those assumptions, one can also determine the position of the individual system points by specifying k independent variables p_1, p_2, …, p_k that are connected with the rectangular coordinates by $3n$ equations:

$$x_\mu = \varphi_\mu\,(p_1, p_2, \ldots, p_k) \qquad (\mu = 1, 2, \ldots, 3n)\ . \tag{2}$$

The state of the system in question at a particular moment in time can be regarded as being given when one specifies the position and velocity of the individual mass-points, such that the values of the coordinates x_μ and their first derivatives $\dot{x}_\nu$ can be regarded as having known values for the moment in time considered. The accelerations of the system points are restricted from the outset by the constraint on the system in such a way that they cannot assume any arbitrary value b_ν, but only ones for which the equations:

(I) $$\sum_{\kappa=1}^{3n} \frac{\partial f_i}{\partial x_\kappa} b_\kappa = - \sum_{\lambda=1}^{3n} \sum_{\mu=1}^{3n} \frac{\partial^2 f_i}{\partial x_\lambda \, \partial x_\nu} \dot{x}_\lambda \, \dot{x}_\nu \qquad (i = 1, 2, \ldots, 3n - k)$$

are satisfied. The right-hand sides of those equations can be regarded as constant, since the given values of those quantities at the moment in time considered can be substituted in them for the coordinates and their first derivatives with respect to time. With the chosen system connections between the *possible* accelerations b_ν, the accelerations that *actually* occur for a given force configuration can be selected using Gauss's principle using the condition that the function:

$$Z = \sum_{\tau=1}^{3n} m_\tau \left\{ b_\kappa - \frac{X_\tau}{m_\tau} \right\}^2$$

must be a minimum for all possible values of the accelerations that might actually occur.

For the purpose of representing that minimum condition with the use of general coordinates, it is important to first derive a lemma. If one denotes the accelerations that actually occur by g_ν then one can decompose any acceleration b_ν into a sum:

(3) $$b_\nu = g_\nu + z_\nu \, .$$

Physically, that corresponds to the decomposition of any possible motion of the system for a well-defined initial position and a given velocity configuration into the actually-occurring motion plus a possible motion for the same initial positions of all points when they are all at rest, under the assumptions that were made. One sees that this argument corresponds to the fact that the supplementary accelerations z_ν cannot be chosen arbitrarily. Moreover, they must satisfy the conditions (I), in which all of the initial velocities are set equal to zero, such that the accelerations z_ν are chosen to correspond with the $(3n - k)$ equations:

(II) $$\sum_{\kappa=1}^{3n} \frac{\partial f_i}{\partial x_\kappa} z_\kappa = 0 \qquad (i = 1, 2, \ldots, 3n - k) \, .$$

If one substitutes the decomposition of the possible accelerations b_ν that was encountered above into the expression for the constraint then one will put it into the form:

$$Z = \sum_{\tau=1}^{3n} m_\tau \left\{ g_\kappa - \frac{X_\tau}{m_\tau} + z_\tau \right\}^2$$

$$= \sum_{\tau=1}^{3n} m_\tau \left\{ g_\kappa - \frac{X_\tau}{m_\tau} \right\}^2 + 2\sum_{\tau=1}^{3n} m_\tau \left\{ g_\kappa - \frac{X_\tau}{m_\tau} \right\} z_\tau + \sum_{\tau=1}^{3n} m_\tau z_\tau^2 .$$

The required condition that this function Z should be a minimum for all possible choices of the accelerations for the choice $b_\nu = g_\nu$ can also be replaced with the condition that the expression:

$$2\sum_{\tau=1}^{3n} m_\tau \left\{ g_\kappa - \frac{X_\tau}{m_\tau} \right\} z_\tau + \sum_{\tau=1}^{3n} m_\tau z_\tau^2 > 0$$

for all admissible [i.e., compatible with the conditions (II)] choices of the supplementary accelerations z_τ. That condition can be expressed in yet another form. Namely, if z'_τ means any system of supplementary accelerations z_τ that are compatible with the equations (II) then $\rho z'_\tau$ will also be such a thing, in which the numerical factor ρ can be chosen arbitrarily. Hence, the inequality:

$$2\sum_{\tau=1}^{3n} m_\tau \left\{ g_\kappa - \frac{X_\tau}{m_\tau} \right\} z'_\tau + \rho\sum_{\tau=1}^{3n} m_\tau z'^2_\tau > 0$$

must then be fulfilled for every positive value of ρ. However, that is possible only when the sum:

$$S = 2\sum_{\tau=1}^{3n} m_\tau \left\{ g_\kappa - \frac{X_\tau}{m_\tau} \right\} z'_\tau$$

is either zero or positive.

Nevertheless, since any choice z'_τ of the supplementary accelerations can also be associated with the choice of $z_\tau = -z'_\tau$, for which the sum considered can change sign under the assumption that it is positive, the expression S can only be zero.

The condition that is contained in Gauss's principle that the constraint on the system should be a minimum for the motion that actually occurs can then be replaced with the condition that the sum will be:

$$\text{(III)} \qquad \sum_{\tau=1}^{3n} m_\tau \left\{ g_\kappa - \frac{X_\tau}{m_\tau} \right\} z_\tau = 0,$$

and in fact for all choices of the numbers z_τ that correspond to the conditions (II).

It then seems that the **d'Alembert**'s principle is the expression for not only the sufficient, but also necessary, condition for the constraint on the system to be a minimum for the actually-occurring motion compared to all possible motions. However, since **d'Alembert**'s principle can also be regarded as a combination of **Lagrange**'s differential equations of order 1 or 2, it is also permissible to regard the fulfillment of the latter equations as the necessary and sufficient condition for the function Z to have the minimum property that is required by the principle of least constraint.

With the use of equation (III) that was just developed, the expression for the constraint on the system will go over to the form:

$$Z = \sum_{\tau=1}^{3n} m_\tau \left\{ g_\kappa - \frac{X_\tau}{m_\tau} \right\}^2 + \sum_{\tau=1}^{3n} m_\tau \, z_\tau^2 \, .$$

The problem of the transformation of that expression into general coordinates will then further reduce to the transformation of the sum:

$$Z' = \sum_{\tau=1}^{3n} m_\tau \, z_\tau^2 \, ,$$

since the sum:

$$\varphi = \sum_{\tau=1}^{3n} m_\tau \left\{ g_\kappa - \frac{X_\tau}{m_\tau} \right\}^2 ,$$

which exhibits the actually-occurring minimum value of the constraint, necessarily plays the role of a constant in the search for the minimum of the function Z.

The transformation of the sum Z' can be performed in the following form. The κ arbitrarily-chosen numbers β_κ ($\kappa = 1, 2, \ldots, k$) might represent arbitrary accelerations of the coordinates p_κ. Any choice of those numbers β_κ then belongs to a choice of accelerations b_ν, in which the reciprocal relationships are given by the $3n$ equations:

$$\text{(IV)} \qquad b_\nu = \sum_{\kappa=1}^{k} \frac{\partial \varphi_\nu}{\partial p_\kappa} \beta_\kappa + \sum_{\lambda=1}^{k} \sum_{\mu=1}^{k} \frac{\partial^2 \varphi_\nu}{\partial p_\lambda \, \partial p_\mu} \dot{p}_\lambda \, \dot{p}_\mu \qquad (\nu = 1, 2, \ldots, 3n),$$

such that all possible choices of the numbers b_ν also seem to be given by all arbitrary choices of the numbers β_κ. Naturally, the quantities p_κ and $\dot{p}_\kappa$ in equations (IV) are to be assigned vales that correspond to the initial values x_ν and $\dot{x}_\nu$. Now, one can once more decompose the accelerations β_κ into sums:

$$\text{(4)} \qquad \beta_\kappa = \gamma_\kappa + \zeta_\kappa ,$$

in which the numbers γ_κ mean the accelerations of the general coordinates that correspond to the actual motions, while the numbers ζ_κ are supplementary accelerations of the p_κ that one is completely free to choose. Since that decomposition of the accelerations β_κ once more corresponds to the same decomposition of any possible motion of the system into the actually-occurring motion plus a possible motion with the same initial position of all points that can be performed when all of then are completely at rest, which was the case for the decomposition of the accelerations b_ν into the sum $g_\nu + z_\nu$, all of the choices of supplementary accelerations z_ν that are compatible with the conditions (II) must represent all conceivable choices of the values ζ_κ that correspond to equations (IV), when one only sets all of the velocities $\dot{p}_\kappa$ equal to zero in those equations. One will then get the $3n$ equations:

$$\text{(V)} \qquad z_\nu = \sum_{\kappa=1}^{k} \frac{\partial \varphi_\nu}{\partial p_\kappa} \zeta_\kappa \qquad (\nu = 1, 2, \ldots, 3n)$$

between the values z_ν and ζ_κ .

It is easy to replace the supplementary accelerations ζ_κ with other variables. If L means the *vis viva* of the system, and P_κ means the general force components, then it is known that the theorem that the expressions:

$$Q_\kappa = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_\kappa}\right) - \frac{\partial L}{\partial p_\kappa} - P_\kappa \qquad (\kappa = 1, 2, \ldots, k)$$

should vanish for the actual motion will follow from equations (III). The expressions Q_κ contain the accelerations in the first power, and indeed:

$$\text{(VI)} \qquad Q_\kappa = \sum_{\lambda=1}^{k} a_{\lambda,\kappa}\, \beta_\lambda - F_\kappa \qquad (\kappa = 1, 2, \ldots, k),$$

and the F_κ in those equations mean well-defined functions of the coordinates p_κ and their first derivatives $\dot{p}_\kappa$, as well as the general force components P_κ , while the factors $a_{\lambda,\kappa}$ are defined by the equations:

$$a_{\lambda,\kappa} = \sum_{\nu=1}^{3n} m_\nu \frac{\partial \varphi_\nu}{\partial p_\lambda} \frac{\partial \varphi_\nu}{\partial p_\kappa} \qquad \begin{pmatrix} \lambda = 1, 2, \ldots, k \\ \kappa = 1, 2, \ldots, k \end{pmatrix} .$$

If one replaces the accelerations β_λ in the right-hand sides of equations (VI) with the sums $\gamma_\lambda + \zeta_\lambda$, which are equivalent to them, and further observes that the expressions Q_κ must vanish as a result of equations (III) when the possible accelerations β_λ are replaced with the ones that actually occur then one will get the k linear relations:

$$Q_\kappa = \sum_{\lambda=1}^{k} a_{\lambda,\kappa}\, \zeta_\lambda \qquad (\kappa = 1, 2, \ldots, k)$$

between the expressions Q_κ and the supplementary ζ_κ , from which one can conclude the relations:

$$\text{(VII)} \qquad \zeta_\kappa = \frac{1}{D} \sum_{\lambda=1}^{k} A_{\lambda,\kappa}\, Q_\lambda \qquad (\kappa = 1, 2, \ldots, k) .$$

In them, D means the determinant of the k^2 elements $a_{\lambda,\kappa}$, and $A_{\lambda,\kappa}$ means the adjoint of the element $a_{\lambda,\kappa}$. Equations (V) and (VII) imply the equations:

(VIII) $$z_\nu = \frac{1}{D}\sum_{\kappa=1}^{k}\sum_{\lambda=1}^{k}\frac{\partial\varphi_\nu}{\partial p_\kappa}A_{\lambda,\kappa}\,Q_\lambda \qquad (\nu = 1, 2, \ldots, 3n),$$

which define the relations between the supplementary accelerations z_ν of the rectangular coordinates and the expressions Q_λ. Now, with the help of equations (VIII), one will be in a position to exhibit the expression for the function Z' with the use of general coordinates. To that end, one merely has to replace the quantities z_ν^2 in Z' with the Q_λ and obtain:

$$Z' = \frac{1}{D^2}\cdot\sum_{\kappa=1}^{k}\sum_{\lambda=1}^{k}Q_\kappa\,Q_\lambda\cdot\sum_{\rho=1}^{k}\sum_{\sigma=1}^{k}A_{\rho\kappa}A_{\rho\lambda}\sum_{\nu=1}^{3n}m_\nu\frac{\partial\varphi_\nu}{\partial p_\rho}\frac{\partial\varphi_\nu}{\partial p_\sigma}.$$

If one observes the defining equations of the quantities $a_{\lambda\kappa}$ and the relation:

$$\sum_{\rho=1}^{k}\sum_{\sigma=1}^{k}A_{\rho\kappa}A_{\rho\lambda}\,a_{\rho\sigma} = D\cdot A_{\lambda,\kappa},$$

which ensues from the property of the determinant that the sum $\sum_{\rho=1}^{k}A_{\rho,\kappa}a_{\rho,\sigma}$ is either zero or D according to whether σ is or is not different from κ, resp., then one will arrive at the formula:

$$Z' = \frac{1}{D}\sum_{\kappa=1}^{k}\sum_{\lambda=1}^{k}A_{\kappa,\lambda}\,Q_\kappa\,Q_\lambda\ .$$

That will then imply the theorem that the expression for the constraint on a material system with the use of general coordinates can be written in the form:

$$Z = \frac{1}{D}\sum_{\kappa=1}^{k}\sum_{\lambda=1}^{k}A_{\kappa,\lambda}\,Q_\kappa\,Q_\lambda + \varphi,$$

in which the function:

$$\varphi = \sum_{\nu=1}^{3n}m_\nu\left\{g_\nu - \frac{X_\nu}{m_\nu}\right\}^2$$

means the value of the constraint for the motion that actually occurs, which plays the role of a constant in the search for the minimum value of the function Z. As one can see from the representation of Z' in rectangular coordinates itself, its smallest value is zero, and that can happen only when all $z_\nu = 0$, or when all Q_λ vanish.

"Ueber die Aufstellung der Differentialgleichungen der Bewegung für reibungslose Punktsysteme, die Bedingungsungleichungen unterworfen sind," Ber. der Verh. Kön. Sächs. Ges. Wiss. Leipzig **51** (1899), 224-244.

On the differential equations of motion for frictionless point-systems that are subject to condition inequalities

By **A. Mayer**

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Translated by D. H. Delphenich

A system of material points that is coupled with each other in any way, and at the same time can be subject to external restrictions, moves under the influence of given forces.

If the driving forces, couplings, and restrictions on the system, as well as the masses of its points, are given, and one knows, moreover, the *initial state* of the system – i.e., one knows the *position* and *velocity* that each system-point possesses at a given *initial moment* – then it is assumed in so doing that, other than the positions of the system-points, the given forces depend upon at most only the velocities and time, and that all possible *friction* can be neglected, and the *motion of the system* will be defined *uniquely* in a mechanical context. However, in order to also be able to carry out *calculations*, one must find the *differential equations* of the motion thus-defined; i.e., when one refers everything in space to a fixed rectangular system of axes and suggests the complete differentiations with respect to time by primes, one must know how to express the *accelerations* x'', y'', z'' that the system-points would achieve at an arbitrary moment t in the moment in terms of the *coordinates* x, y, z and the velocities x', y', z' of the points at the same moments.

As is known, that basic problem in dynamics can be solved very easily as long as the couplings and restrictions on the system are expressed by *condition equations* between the coordinates of its points or also between them and time. However, the problem will become much more difficult when the system is subordinate to the constraint of *condition inequalities*, e.g., when its points are coupled, not by rigid lines, but by inextensible strings whose weights can be neglected.

As far as I know, only **Ostragradsky** has addressed the latter case up to now, first in "Considérations générales sur les moments des forces," but then, above all, in the treatise "Sur les déplacements instantanées des systèmes assujettis à des conditions variables," which were papers that were submitted to the St. Petersburg Academy in 1834 and 1838 ([1]). However, in that second treatise, **Ostragradsky** arrived at two different systems of inequalities that the solution to the

([1]) The latter seems to remain unknown to **Jacobi**. However, he cited the result of the former in a Berlin lecture, which unfortunately remained unconsidered in the publication of **Jacobi**'s *Vorlesungen über Dynamik*, although **Scheibner** has presented it in an excellent fashion.

problem must fulfill, and was content to assert that the one system could be replaced with the other one without in any way establishing that replaceability, which is entirely essential to the validity of the result. On first glance, his argument generally seems so natural and convincing that I have long believed that the gap in the method of proof can be eliminated in a purely mathematical way. However, from a correspondence that I had with **E. Study** on that question in 1889, I learned that it is not merely an oversight that is included in "Déplacements," but an actual fallacy. **Study** was then, above all, the first person we have to thank for the correct solution to the problem, which shall be summarized here, at least in principle. One will obtain it most simply and clearly when one starts from ***Gauss****'s principle of least constraint*, which also reduces to **Ostragradsky**'s concept of the equilibrium of lost forces.

§ 1. – The method of solution and the impossibility of a direct solution.

Let $m_1, m_2, \ldots, m_n$ be the masses of the system-points, and in general, at the moment t in the motion. Let x_i, y_i, z_i be the coordinates of the points m_i, and let X_i, Y_i, Z_i be the components of the driving forces that act upon them, which shall be given, single-valued functions of time, and the coordinates and velocities of the points in all of what follows. Finally, let the couplings and restrictions of the system be defined by *inequalities:*

$$f_1 \leq 0, \quad f_2 \leq 0, \ldots, \tag{1}$$

in which $f_1, f_2, \ldots$ denote given single-valued functions of the coordinates that might possibly include time itself, as well. I assume that these functions, along with their first, second, and third partial differential quotients, will always remain continuous for all of the motions of the system that come under consideration.

The positions and velocities of all points are known at a given moment t. One next asks *what accelerations can those system points assume at that instant?*

The system allows only those motions for which the coordinates of its points continually satisfy the conditions (1). Any of those conditions might be represented by:

$$f \leq 0.$$

For any motion of the system, the coordinates x_i, y_i, z_i of its points m_i will be functions of time with continuous differential quotients. If we consider those coordinates to be such functions and let t go to $t + dt$, in order to focus on the following moment, then when the system condition $f \leq 0$ is developed in powers of dt, it will be converted into:

$$f + f'\,dt + f''\frac{dt^2}{2} + r\,dt^3 \leq 0, \tag{a}$$

in which $r\,dt^3$ denotes the remaining terms in the **Taylor** development.

Besides t, that development of f includes only the *coordinates* of the system-point, along with its *velocity* f', and finally, its *acceleration* f''.

If we now think of the coordinates x_i, y_i, z_i and the velocities x_i', y_i', z_i' as being set equal to the values (which are known, by assumption) that they possess at the moment t then f can only be < 0 or $= 0$, since otherwise the known *positions* of the points at time t would not indeed be possible positions of them.

If f is only < 0, *but not* $= 0$ then for a sufficiently-small dt, the condition (a) will already be fulfilled by itself, and the system will not be instantaneously restricted in any way.

However, if $f = 0$ then, after dividing by the positive quantity dt, the condition (a) will reduce to:

$$f' + f''\frac{dt}{2} + r\,dt^2 \leq 0, \tag{b}$$

and that condition can only be fulfilled for an arbitrarily-small dt when its first term is already itself ≤ 0.

Hence, as long as the position of the system at the moment t satisfies the assumption $f = 0$, the *velocities* of its points must already fulfill the condition:

$$f' \leq 0$$

by themselves.

However, if f' *is only* < 0 *and not* $= 0$ at the moment t for which the *positions* and *velocities* of the point are known then for a sufficiently-small dt, the condition (b) will once more be fulfilled by itself, and therefore the system will not be assigned any sort of restrictions instantaneously.

Rather, such a restriction will first occur *when one also has* $f' = 0$. The condition (b) will then reduce to:

$$f'' + 2r\,dt \leq 0$$

and can then exist for arbitrarily-small dt only when one already has:

$$f'' \leq 0\,. \tag{c}$$

Therefore, of the system conditions (1), the *accelerations* of a point are not restricted in any way at the moment t in question by:

1. All of the ones that are not fulfilled as *equations*, but only as *inequalities*, as well as:

2. All of the ones that exist precisely as equations whose complete first derivatives are however $\neq 0$, but only < 0, for the known position and velocity of the point at the moment t.

After dropping all of those *instantaneously ineffective* conditions, let:

$$f_1 \leq 0,\ f_2 \leq 0,\ \ldots,\ f_r \leq 0 \tag{2}$$

be the remaining system conditions (1). For them, the equations:

(3) $$f_1 = 0, \quad f_2 = 0, \ldots, f_r = 0$$

and

(4) $$f_1' = 0, \quad f_2' = 0, \ldots, \quad f_r' = 0$$

will then be fulfilled by the known state of the system, *and at that moment, the system will admit all accelerations that are compatible with the conditions:*

(5) $$f_1'' \leq 0, \quad f_2'' \leq 0, \ldots, \quad f_r'' \leq 0.$$

Among all of those instantaneously-possible accelerations, one must also include the *unknown true accelerations* that the points will attain at the moment t for the actual motion of the system. In order to find them, we turn to the aid of just the *principle of least constraint.*

The point m_i that is found *at the position* $a_i \equiv x_i, y_i, z_i$ at the moment t and already possesses the velocities x_i', y_i', z_i' is acted upon by the given accelerating forces:

$$\frac{X_i}{m_i}, \frac{Y_i}{m_i}, \frac{Z_i}{m_i}.$$

If the point were *free* then it would arrive *at a position* b_i with the coordinates:

$$\xi_i = x_i + x_i' \, dt + \frac{X_i}{m_i} \frac{dt^2}{2},$$

$$\eta_i = y_i + y_i' \, dt + \frac{Y_i}{m_i} \frac{dt^2}{2},$$

$$\zeta_i = z_i + z_i' \, dt + \frac{Z_i}{m_i} \frac{dt^2}{2}$$

at the next infinitely-small time-element dt. However, due to its coupling to the system, it will not actually reach that location, but rather, in the time dt, *it will come to another location* c_i, and *when we understand* x_i'', y_i'', z_i'' *to mean the unknown true accelerations of the point at the moment* t, it will possess the coordinates:

$$\bar{\xi}_i = x_i + x_i' \, dt + x_i'' \frac{dt^2}{2},$$

$$\bar{\eta}_i = y_i + y_i' \, dt + y_i'' \frac{dt^2}{2},$$

$$\bar{\zeta}_i = z_i + z_i' \, dt + z_i'' \frac{dt^2}{2}.$$

Now, from the principle of least constraint, the *actual positions* c_i of the system-points at time $t + dt$ are characterized by the fact that, among all positions c_i to which they can be brought from the position a_i in the time interval dt considered by any possible motion of the system of points m_i, they are the ones for which the sum:

$$\sum_{i=1}^{n} m_i \, \overline{c_i b_i}^2 \equiv \sum_{i=1}^{n} m_i \{(\xi_i - \bar{\xi}_i)^2 + (\eta_i - \bar{\eta}_i)^2 + (\zeta_i - \bar{\zeta}_i)^2\}$$

attains the least-possible value.

If one replaces the coordinates of the locations b_i and c_i in this with their values then one will see immediately that this principle comes down to:

For the desired true accelerations x_i'', y_i'', z_i'' *of the system-point* m_i *at the moment* t, *one must have that the sum:*

(6) $$\sum_{i=1}^{n} m_i \left\{ \left(\frac{X_i}{m_i} - x_i'' \right)^2 + \left(\frac{Y_i}{m_i} - y_i'' \right)^2 + \left(\frac{Z_i}{m_i} - z_i'' \right)^2 \right\} = \text{min};$$

i.e., it must be smaller than it is for all other accelerations that are allowed for points of the system instantaneously.

If we understand:

$$x_i'' + \delta x_i'', \qquad y_i'' + \delta y_i'', \qquad z_i'' + \delta z_i''$$

to mean any other instantaneously-possible accelerations of the system-points m_i that likewise deviate only slightly from the *unknown*, true accelerations of the system, and when we observe the identities:

$$\frac{\partial f''}{\partial x_i''} \equiv \frac{\partial f}{\partial x_i}, \ldots$$

and introduce the notation:

(7) $$\delta f'' \equiv \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \delta x_i'' + \frac{\partial f}{\partial y_i} \delta y_i'' + \frac{\partial f}{\partial z_i} \delta z_i'' \right),$$

the conditions (5), which instantaneously restrict the accelerations of the system exclusively, will also imply only the conditions:

$$(8) \qquad f_1'' + \delta f_1'' \leq 0, \quad f_2'' + \delta f_2'' \leq 0, \ \ldots, \ f_r'' + \delta f_r'' \leq 0$$

for the variations of the accelerations. From (6), we will then get the following requirement for the determination of the true accelerations x_i'', y_i'', z_i'' that *one must have*:

$$(9) \qquad \sum_{i=1}^{n} \{(X_i - m_i\, x_i'')\,\delta x_i'' + (Y_i - m_i\, y_i'')\,\delta y_i'' + (Z_i - m_i\, z_i'')\,\delta z_i''\} \leq 0$$

for all sufficiently-small values of the variations $\delta x_i''$, $\delta y_i''$, $\delta z_i''$ *that the conditions* (8) *fulfill.*

However, when the unknown, true accelerations x_i'', y_i'', z_i'' make:

$$f'' \neq 0, \text{ but only} < 0,$$

the condition:

$$f'' + \delta f'' \leq 0$$

will in no way restrict the variations of the acceleration, since they will then be fulfilled by themselves for all arbitrary, but sufficiently-small, values of those variations.

Rather, a restriction will first occur when the true acceleration satisfies the equation:

$$f'' = 0,$$

which will make that condition reduce to:

$$\delta f'' \leq 0 .$$

How can one nonetheless recognize which of the two derivatives f'' at the given moment t during the actual motion of the system possesses the value zero and which of them is only < 0?

On first glance, it would seem that this *cardinal question* must necessarily obviate the entire investigation. Indeed, we do not know the true acceleration itself at all, so how can we decide whether $f'' = 0$ or only < 0 for it, and yet we have to know that in order to even be able to calculate the true acceleration.

In general, **Ostragradsky** helped us avoid that complication by an argument that, in fact, initially seems very enlightening for the consideration of surfaces. Namely, he simply argued that [1]: One replaces the unknown accelerations x_i'', y_i'', z_i'' of the system-points with the values:

$$\frac{X_i}{m_i}, \frac{Y_i}{m_i}, \frac{Z_i}{m_i}$$

[1] Déplacements, § **8**.

that those accelerations would possess if the points were free and then examines which of the conditions (5) are fulfilled by that replacement and which are not. Only the latter conditions will restrict the accelerations instantaneously, while the former will express obstacles that do not presently stand in the way of the motion and will therefore also not come under consideration.

However, the argument is correct only in the case of a single condition (5). For more than one condition, as the discussion of the case $r = 2$ in **§ 3** will show clearly, it can very well happen that a condition that poses no obstacle to the *free* motion of the point in time will define a real restriction for the *actual* motion of the system as a result of other conditions, and likewise a condition can, conversely, instantaneously limit the *free* motion, but pose no obstacle to the *actual* motion. In fact, **Ostragradsky** also came to *two systems of inequalities* by his argument, about which, he said (but totally forgot to prove): "We will see that the one of them is always fulfilled at the same time as the other, and that one can therefore replace the one with the other." However, that is contradicted by the fact that, in reality, the one can be fulfilled without the other one being fulfilled.

This much is then clear in any case, that it is impossible to answer the basic question above *directly*. Therefore, nothing else remains but to try to make that decision in an *indirect* way. To that end, we will first have to establish *how the true accelerations of the system-point would be determined if we had already solved our cardinal question*, and we must then see *whether we cannot find some criterion from which we could then recognize whether the values that were found for the accelerations are correct or false*.

§ 2. – Indirect solution of the problem.

From the foregoing, I shall now assume that we already know that *during the unknown, true motion of our system*, which is instantaneously subordinate to only the r conditions (5), *the k equations:*

$$(10) \qquad f_g'' \equiv \sum_{i=1}^{n} \left(\frac{\partial f_g}{\partial x_i} x_i'' + \frac{\partial f_g}{\partial y_i} y_i'' + \frac{\partial f_g}{\partial z_i} z_i'' \right) + F_g = 0, \quad (g = 1, 2, \ldots, k)$$

are valid, while the r – k expressions:

$$(11) \qquad f_\gamma'' \equiv \sum_{i=1}^{n} \left(\frac{\partial f_\gamma}{\partial x_i} x_i'' + \frac{\partial f_\gamma}{\partial y_i} y_i'' + \frac{\partial f_\gamma}{\partial z_i} z_i'' \right) + F_\gamma, \qquad (\gamma = k + 1, 2, \ldots, r)$$

all possess negative values. Obviously, the F_g and F_γ denote the sums of all terms in the complete second differential quotients f_g'' and f_γ'' that are free of the second differential quotients of the coordinates, and k can be any number from the sequence 0, 1, …, r.

One next asks: *What values will this assumption (which is still entirely arbitrary, for the moment, mind you) yield for the true accelerations of the system-point at time t* ?

I shall make that assumption even more precise by establishing that for a given state of motion of the system at the moment t, none of equations (10) should be a mere consequence of the remaining ones. Indeed, such an equation in (10) would not contribute to the determination of the unknown accelerations and would therefore be simply dropped. I shall then assume that *the k equations* (10) *determine k of the* 3*n unknowns* x_i'', y_i'', z_i'' *in terms of the remaining* 3*n* – *k*.

With those assumptions, the variations of the acceleration are instantaneously subject to only the *k* conditions:

$$\delta f_g'' \equiv \sum_{i=1}^{n} \left(\frac{\partial f_g}{\partial x_i} \delta x_i'' + \frac{\partial f_g}{\partial y_i} \delta y_i'' + \frac{\partial f_g}{\partial z_i} \delta z_i'' \right) \leq 0 \qquad (g = 1, 2, \ldots, k), \tag{12}$$

and the unknown, true accelerations x_i'', y_i'', z_i'' must then satisfy the demand (9) for all values of their variations $\delta x_i''$, $\delta y_i''$, $\delta z_i''$ that fulfill those *k* conditions.

In particular, one must then have:

$$\sum_{i=1}^{n} \{ (X_i - m_i\, x_i'')\, \delta x_i'' + (Y_i - m_i\, y_i'')\, \delta y_i'' + (Z_i - m_i\, z_i'')\, \delta z_i'' \} = 0$$

for all $\delta x_i''$, $\delta y_i''$, $\delta z_i''$ that satisfy the *k* equations:

$$\delta f_h'' = 0, \qquad (h = 1, 2, \ldots, k), \tag{12$'$}$$

and from our assumption on the nature of equations (10), equations (12′) will determine *k* of those variations as functions of the remaining 3*n* – *k*.

If one multiplies the last equations by the temporarily undetermined factors – λ_h and then adds them to the foregoing conditions then one will get the equation:

$$\sum_{i=1}^{n} \{ (X_i - m_i\, x_i'')\, \delta x_i'' + (Y_i - m_i\, y_i'')\, \delta y_i'' + (Z_i - m_i\, z_i'')\, \delta z_i'' \} - \sum_{h=1}^{k} \lambda_h\, \delta f_h' = 0. \tag{13}$$

Now, one can always determine the multipliers λ_1, λ_2, …, λ_k in such a way that the coefficients of the *k* variations in that equation that are expressed in terms of the other ones from equations (12′) will be zero, and all that will remain then in equation (13) will be completely-arbitrary variations. Their coefficients must also vanish then, and one will thus get the following 3*n* equations from (13):

(14)
$$\begin{cases} X_i - m_i\, x_i'' = \sum_{h=1}^{k} \lambda_h \dfrac{\partial f_h}{\partial x_i}, \\ Y_i - m_i\, z_i'' = \sum_{h=1}^{k} \lambda_h \dfrac{\partial f_h}{\partial y_i}, \qquad (i = 1, 2, \dots, n) \\ Z_i - m_i\, z_i'' = \sum_{h=1}^{k} \lambda_h \dfrac{\partial f_h}{\partial z_i}. \end{cases}$$

However, by assumption, the components X_i , Y_i , Z_i of the driving force that acts at the point m_i are given, single-valued functions of time t, the coordinates, and the velocity of the system-point, so they are functions whose values at the moment t will be, at the same time, determined completely by the state of motion of the system. Equations (14) then express the $3n$ unknowns x_i'' , y_i'' , z_i'' in terms of the k multipliers uniquely.

If one further sets, in general:

(15)
$$\Phi_{lh} \equiv \sum_{i=1}^{n} \frac{1}{m_i}\left(\frac{\partial f_l}{\partial x_i}\frac{\partial f_h}{\partial x_i} + \frac{\partial f_l}{\partial y_i}\frac{\partial f_h}{\partial y_i} + \frac{\partial f_l}{\partial z_i}\frac{\partial f_h}{\partial z_i}\right),$$

such that $\Phi_{lh} \equiv \Phi_{hl}$ and each $\Phi_{ll} > 0$, and if:

(16)
$$\Phi_l^0 \equiv \sum_{i=1}^{n} \frac{1}{m_i}\left(X_i \frac{\partial f_l}{\partial x_i} + Y_i \frac{\partial f_l}{\partial y_i} + Z_i \frac{\partial f_l}{\partial z_i}\right) + F_l$$

denotes the values that the complete second derivative of f_l would assume for the *free* motion of the point, so Φ_{lh} and Φ_l^0 will also be quantities whose instantaneous values are known completely, then substituting the values of the x_i'' , y_i'' , z_i'' in equations (13) in the k equations (10) will yield the following k linear equations for the determination of the multipliers λ_h themselves:

(17)
$$\sum_{h=1}^{k} \Phi_{gh}\, \lambda_h = \Phi_g^0 \qquad (g = 1, 2, ..., k)\,.$$

One also sees directly that these k equations will actually determine the k unknown λ_1 , λ_2, …, λ_k , so their determinant:

(18)
$$\Delta_k \equiv \sum \pm \Phi_{11}\, \Phi_{22} \cdots \Phi_{kk}$$

will not be zero.

Namely, due to equations (14), and from (15), the sum (6) will obtain the value:

$$2\Omega \equiv \sum_{g=1}^{k}\sum_{h=1}^{k}\Phi_{gh}\,\lambda_g\,\lambda_h\,.$$

However, that sum is always positive and will vanish only when all 3*n* differences:

$$\frac{X_i}{m_i} - x_i'',\qquad \frac{Y_i}{m_i} - y_i'',\qquad \frac{Z_i}{m_i} - z_i''$$

vanish simultaneously. Moreover, among the 3*n* equations (14), there will already be *k* of them whose derivatives determine the *k* multipliers λ_h as linear, homogeneous functions of *k* of those differences.

The values of those 3*n* differences that are implied by equations (14) can then vanish simultaneously only when all $\lambda_h = 0$. Therefore, the value of 2Ω that the sum (6) assumes because of equations (14) will be a positive-definite form of the variables $\lambda_1, \lambda_2, \ldots, \lambda_k$, and that is known to involve, *eo ipso,* the fact that Δ_k *is non-zero,* and indeed *will necessarily be positive*, which will be used in the following §.

The *k* linear equations (17) then, in fact, determine their *k* unknowns λ_h, and when one substitutes the solutions in equations (14), one will get the 3*n* unknowns x_i'', y_i'', z_i'' themselves uniquely and expressed in terms of nothing but quantities whose values are known at time *t*.

With that, the question that was posed at the beginning of this § is answered, and one sees *that it always admits only one solution.*

However, we must now ask: Were the assumptions that we started with in the foregoing in a completely arbitrary way correct or not?

Certain criteria can be obtained for them, as well.

In fact, we must impose *even more conditions* upon our assumptions, as well as in the principle of least constraint, than the ones that we have fulfilled by way of equations (14) and (17).

Namely, for one thing, of the *r* conditions (5) that restrict the motion of our system at the moment *t*, up to now only the first *k* were satisfied, and indeed, they were satisfied as a result of the *k* equations (10). If our assumptions were applicable then the last $r - k$ conditions (5) would be fulfilled by our solution automatically.

However, by substituting the values of the x_i'', y_i'', z_i'' in equations (14), from (15), (16), and (17), the $r - k$ expressions (11) would take on the values:

$$(19)\qquad f_\gamma'' = \Phi_\gamma^0 - \sum_{h=1}^{k}\Phi_{\gamma h}\,\lambda_h \equiv \Phi_\gamma^k \qquad (\gamma = k+1, \ldots, r),$$

in which Φ_γ^k shall denote the value that the linear function of the λ_h on the left-hand side would assume if one were to substitute the solutions of equations (17). These values of Φ_γ^k are also once more merely functions of time, the coordinates and the velocities, and as such, can likewise be determined completely at the moment *t*.

Our assumptions can therefore next be true in any case only when every $\Phi_\gamma^k \leq 0$. *By contrast, whenever any* $\Phi_\gamma^k > 0$ *that assumption would be false*, and the last $r - k$ conditions (5) would instantaneously prevent the motion that we have calculated from taking place.

Furthermore, the formulas (14) will convert equation (13) into an identity that is valid for all arbitrary values of the variations $\delta x_i''$, $\delta y_i''$, $\delta z_i''$, and will thus reduce our original requirement (9) to:

$$\sum_{h=1}^{k} \lambda_h \, \delta f_h'' \leq 0.$$

In addition, this condition must then be fulfilled for all values of the variations that fulfill the k conditions (12), so the ones that make each:

$$\delta f_h'' \leq 0.$$

However, it is necessary and sufficient for this to be true that no $\lambda_h < 0$. The principle of least constraint then appends the following k conditions to our equations (14) and (17):

$$\lambda_h > 0 \qquad (h = 1, 2, \ldots, k) \tag{20}$$

in which the > sign should not exclude equality.

Our assumptions will, in turn, be false whenever equations (17) *imply a negative value for any* λ_h, and once more the system cannot be instantaneously capable of performing the motion that is determined by equations (14) and (17).

Conversely, however, *when the accelerations that are calculated from* (14) *and* (17) *are not associated with negative* λ_h *or positive* Φ_γ^k, one can conclude that *they will give the true instantaneous accelerations of the system points* m_i .

Strictly speaking, in order to leave no doubt in regard to this conclusion, one must obviously first show that *it is only by that one way of fulfilling the r instantaneous system conditions* (5) *that one will arrive at values for the* x_i'', y_i'', z_i'' *that fulfill not only those conditions, but also all of the demands of the principle of least constraint.* The proof of that will be provided in the following § for the two simplest cases of $r = 1$ and $r = 2$. By contrast, its implementation can present very great difficulties for arbitrary r. On the other hand, for that reason, one might also regard it as obvious that two different systems of accelerations with the given properties cannot exist, since if they did exist, there would exist no means whatsoever of establishing which of the two systems is the correct one, because whether or not the sum (6) is an actual minimum is entirely irrelevant for mechanics, and anyway, due to the conditions (5) and as a result of equations (10) and (14), that sum will, in fact, actually attain a smallest value whenever it leads to either a negative λ_h *or positive* Φ_γ^k.

§ 3. – Proof of the uniqueness of the solution in the two simplest cases $r = 1$ and $r = 2$.

Although, on the grounds that were just cited, one can probably consider the proof of the uniqueness of the solution to be a much-too-tedious point of rigor, on the other hand, it is still important to recall the different steps that one must take from the foregoing in order to ascertain the true instantaneous motion in the various possible cases. Therefore, the following complete discussion of the two simplest cases $r = 1$ and $r = 2$ might not be superfluous in its own right.

Therefore, first let $r = 1$, so *only one condition* (5) *is present*, such that the number k of equations (10) can possess only the values 0 and 1. At the moment t considered, the accelerations are now subordinate to only one condition:

$$f_1'' \leq 0. \tag{5′}$$

In order to find the true instantaneous accelerations then, one must first calculate the value Φ_1^0 that f_1'' assumes for the accelerations:

$$x_i'' = \frac{X_i}{m_i}, \qquad y_i'' = \frac{Y_i}{m_i}, \qquad z_i'' = \frac{Z_i}{m_i} \tag{10}$$

that the points m_i would possess instantaneously for *free* motion.

If that yields $\Phi_1^0 \leq 0$ then there would be nothing present instantaneously that could prevent the point from exhibiting that free motion, and that will also occur necessarily.

By contrast, if $\Phi_1^0 > 0$ then the condition (5′) will not allow the free motion to take place, and the true accelerations must then satisfy the equation:

$$f_1'' = 0. \tag{10′}$$

From (14) and (17), one can then determine those accelerations from the equations:

$$\left\{\begin{array}{c} m_i\, x_i'' = X_i - \lambda_1 \dfrac{\partial f_1}{\partial x_i}, \quad m_i\, y_i'' = Y_i - \lambda_1 \dfrac{\partial f_1}{\partial y_i}, \quad m_i\, z_i'' = Z_i - \lambda_1 \dfrac{\partial f_1}{\partial z_i}, \\ \Phi_{11}\lambda_1 = \Phi_1^0, \end{array}\right. \tag{1}$$

and due to that fact that $\Phi_{11} > 0$ and the assumption $\Phi_1^0 > 0$, those equations will, in fact, also imply that $\lambda_1 > 0$.

By contrast, in the case of $\Phi_1^0 < 0$, they would imply that $\lambda_1 < 0$, and that would directly characterize the accelerations (1) as incorrect, while for $\Phi_1^0 = 0$, one would also have $\lambda_1 = 0$, and the *constrained motion* (1) would coincide with the *free* motion (0).

Now let $r = 2$, such that the two conditions to be fulfilled at the moment t are:

$$(5'') \qquad f_1'' \leq 0, \qquad f_2'' \leq 0,$$

and k can assume the values 0, 1, 2.

We then first calculate the values Φ_1^0 and Φ_2^0 that f_1'' and f_2'' assume for the free accelerations (0).

If both values are ≤ 0 then neither of the two conditions (5′) will pose an obstacle to the free motion, which will already be the true instantaneous motion then.

By contrast, if Φ_1^0 and Φ_2^0 are not both ≤ 0 *the let:*

$$\Phi_1^0 > 0$$

in any case. The first condition (5″) will oppose the free motion then, and we must once more examine the motion (1), which satisfies equation (10′), and only that equation. For that motion, we will have:

$$(1) \qquad f_2'' = \Phi_2^0 - \Phi_{21}\,\lambda_1 = \Phi_2^0 - \Phi_1^0 \frac{\Phi_{12}}{\Phi_{11}} \equiv \Phi_2^1 .$$

Therefore, if $\Phi_2^1 \leq 0$ then nothing prevents the system from exhibiting the motion (1), and that will therefore also occur presently.

By contrast, if $\Phi_2^1 > 0$ then the second equation (5″) will also prevent the acceleration (1) from taking place, and one will then have to discuss *the motion* (2) that would occur if only that second condition were in effect at the moment considered.

However, that motion (2) will itself be once more different according to whether one has:

$$\Phi_2^0 \leq 0 \qquad \text{or} \qquad > 0 .$$

In the first case, the motion (2) is nothing but the free motion (0), and therefore already impossible, due to the first condition (5″).

With our present assumption that:

$$\Phi_1^0 > 0, \qquad \Phi_2^0 \leq 0, \qquad \Phi_2^1 > 0,$$

the true accelerations must necessarily satisfy the two conditions:

$$(10'') \qquad f_1'' = 0, \qquad f_2'' = 0$$

then. From (14) and (17), the accelerations can now be determined from the equations:

(1.2)
$$\begin{cases} m_i\, x_i'' = X_i - \lambda_1 \dfrac{\partial f_1}{\partial x_i} - \lambda_2 \dfrac{\partial f_2}{\partial x_i}, \\ m_i\, y_i'' = Y_i - \lambda_1 \dfrac{\partial f_1}{\partial y_i} - \lambda_2 \dfrac{\partial f_2}{\partial y_i}, \\ m_i\, z_i'' = Z_i - \lambda_1 \dfrac{\partial f_1}{\partial z_i} - \lambda_2 \dfrac{\partial f_2}{\partial z_i}, \end{cases}$$

and

(1.2′)
$$\begin{cases} \Phi_{11}\,\lambda_1 + \Phi_{12}\,\lambda_2 = \Phi_1^0, \\ \Phi_{21}\,\lambda_1 + \Phi_{22}\,\lambda_2 = \Phi_2^0. \end{cases}$$

In fact, if one sets:

$$\Delta_2 \equiv \Phi_{11}\,\Phi_{22} - \Phi_{12}\,\Phi_{21},$$

from (18), then since $\Phi_{21} \equiv \Phi_{12}$, the last two equations will give:

(1.2″)
$$\begin{cases} \Delta_2\,\lambda_1 = \quad \Phi_1^0\,\Phi_{22} - \Phi_2^0\,\Phi_{12}, \\ \Delta_2\,\lambda_2 = -\Phi_1^0\,\Phi_{12} + \Phi_2^0\Phi_{11}. \end{cases}$$

However, from (1)′, the last equation can be written:

$$\Delta_2\,\lambda_2 = \Phi_{11}\,\Phi_2^1,$$

and therefore, since $\Delta_2 > 0$, $\Phi_{11} > 0$, along with $\Phi_2^1 > 0$, it will likewise also imply that $\lambda_2 > 0$.

It follows, moreover, from our assumptions:

$$\Phi_1^0 > 0, \qquad \Phi_2^0 \leq 0, \qquad \Phi_2^1 \equiv \Phi_2^0 - \Phi_1^0\,\frac{\Phi_{12}}{\Phi_{11}} > 0$$

that Φ_{12} must necessarily be < 0. Due to the fact that $\Phi_{22} > 0$, the first equation (1.2″) will then tell us that we also have $\lambda_1 > 0$.

One sees here how, in agreement with the objection that was made against **Ostrogradsky**'s argument at the conclusion of § **1**, although the system condition $f_2'' \leq 0$ in itself presents no obstacle to the *free* motion, nevertheless, due to the assumption $\Phi_2^0 \leq 0$, the *actual* motion will be subject to the restriction that $f_2'' = 0$, as a result of the other condition that $f_1'' \leq 0$.

By contrast, if $\Phi_2^0 > 0$ then the accelerations (2) can be determined from the equations:

$$(2) \qquad \begin{cases} m_i\, x_i'' = X_i - \lambda_2 \dfrac{\partial f_2}{\partial x_i}, \quad m_i\, y_i'' = Y_i - \lambda_2 \dfrac{\partial f_2}{\partial y_i}, \quad m_i\, z_i'' = Z_i - \lambda_2 \dfrac{\partial f_2}{\partial z_i}, \\ \Phi_{22}\lambda_2 = \Phi_2^0, \end{cases}$$

and one will have:

$$(2)' \qquad f_1'' = \Phi_1^0 - \Phi_{12}\,\lambda_2 = \Phi_1^0 - \Phi_2^0 \frac{\Phi_{12}}{\Phi_{22}} \equiv \Phi_1^2$$

for them.

Therefore, if $\Phi_1^2 \le 0$, or the case:

$$\Phi_1^0 > 0, \qquad \Phi_2^0 > 0, \qquad \Phi_2^1 > 0, \qquad \Phi_1^2 \le 0$$

occurs, then the accelerations (2), in which $\lambda_2 > 0$, will already be the true accelerations.

Since $\Phi_1^0 > 0$, now, as opposed to before, the condition $f_1'' \le 0$ will be a real obstacle to the *free* motion, and therefore it will imply no restriction on the *actual* motion.

By contrast, if $\Phi_1^2 > 0$ then one will have the inequalities:

$$\Phi_1^0 > 0, \qquad \Phi_2^0 > 0, \qquad \Phi_2^1 > 0, \qquad \Phi_1^2 > 0,$$

so the first condition (5″) will allow the accelerations (2) to take place just as little as it will the second of the accelerations (1). The actual accelerations must then once more necessarily satisfy equations (1, 2), and since, from (1′) and (2′), equations (1.2″) can be written:

$$\Delta_2\,\lambda_1 = \Phi_{22}\,\Phi_1^2, \qquad \Delta_2\,\lambda_2 = \Phi_{11}\,\Phi_2^1,$$

λ_1 and λ_2 will, in fact, both be < 0 now, as well.

With that, the case of $r = 2$ is also dealt with, and one sees quite clearly how among the various possible motions of the system, ultimately there will always be *just one of them that is determined completely* to be the one that represents the true instantaneous motion.

However, at the same time, the foregoing argument also illuminates the fact that for larger values of r, in some situations, a good number of detailed investigations might be necessary before one is fortunate enough to discover which of equations (10) and (14) determines the true instantaneous motion of the system.

§ 4. – Determining the motion of the system during an arbitrary finite time interval.

However, once one has found those equations, one can also try to pursue the further motion of the system during an arbitrary finite time interval with the help of them ([1]).

To that end, one must integrate the system of $3n$ second-order differential equations in the $3n$ coordinates x_i , y_i , z_i , and time t that equations (14) go to upon substituting the values of λ from equations (17), which is an integration in which one must also appeal to the $2k$ equations:

$$f'_g = 0 \qquad \text{and} \qquad f_g = 0 \qquad (g = 1, 2, \ldots, k).$$

In other words, one must look for the motion that the system would execute if it were subject to the k condition equations $f_g = 0$ and no further restrictions. The integration constants are determined from the known initial state of the system at time t, which, by assumption, will fulfill the $2(r - k)$ equations:

$$f'_\gamma = 0 \qquad \text{and} \qquad f_\gamma = 0 \qquad (\gamma = k + 1, 2, \ldots, r),$$

in addition to those $2k$ equations. If one has succeeded in calculating the coordinates of the system-point, and therefore the multipliers λ, as well, in that way then if the system is subordinate to only the k conditions:

$$f_1 \leq 0, \quad \ldots, f_k \leq 0$$

from the outset (which assumes that $r = k$, in particular), those functions will represent the *true* motion of the system *whenever none of the multipliers λ are negative.*

By contrast, if those k conditions define only a part of the system conditions (1) then the fact that the multipliers remain positive alone will still not suffice to ensure that the calculated motion is the true system motion, since one or more of the other system conditions that do not restrict the motion of the system at the initial moment t can come into play in some situations that would prevent the system from proceeding with its initial motion.

In order to get information about that, so to decide whether any of the system conditions that are in effect might meanwhile make the persistence of the motion that is determined by equations (10) and (14) impossible, by substituting the functions of time that are obtained for the coordinates and multipliers, one must further examine:

First of all, whether the $r - k$ quantities Φ_γ^k that are defined by equations (19) and (17) are all ≤ 0 at the initial moment t (viz., the assumption that $f_\gamma = 0$, $f'_\gamma = 0$), and

([1]) Déplacements, § 13.

Secondly, whether the functions f_σ in the original system conditions $f_\sigma \leq 0$ either possess only negative values (the case of $f_\sigma < 0$) or go straight through zero to negative values (the case of $f_\sigma = 0$, $f'_\sigma = 0$) at time t, since none of them are positive.

The motion that is calculated from (10) *and* (14) *will coincide with the true motion whenever either one of all those functions or any of the multipliers changes signs.*

By contrast, as soon as one or more of the aforementioned quantities changes its sign by going through zero at a certain moment t_1 , that coincidence will persist from that moment on, and one must then once more address the problem from the beginning with those initial values of the coordinates and velocities that the calculated motion implies for $t = t_1$.

However, when such a later deviation of the calculated motion of the system from the true one is not predicted by merely the sign change of any multiplier λ, the tool that was obtained in the foregoing might become *inadequate*; i.e., by itself, it would not succeed in solving the new problem for the new values that the accelerations of the system points achieve at the moment $t = t_1$. Namely, whenever the point that has moved according to any system condition $f_\sigma \leq 0$ up to now such that one continues to have $f_\sigma < 0$, and with a velocity for which f'_σ is not exactly zero, but possesses a finite positive value, gets into a position in which $f_\sigma = 0$ (so whenever, e.g., a connecting string that was loose up to now suddenly tensed violently or a system-point impinges upon a rigid wall), a *shock* to the system will always arise. The velocities of the old motion will then suddenly come into conflict with the system conditions and must first be regulated in such a way that those conditions are once more obeyed, and those regularized shock velocities will be the initial velocities of the new motion. One then sees how it can become necessary to appeal to the *theory of shocks* in order to determine the system motion during a finite time interval from time to time ([1]).

Finally, it is self-explanatory that when the constraints and restrictions on the point-system in question are defined by not only inequalities, but partly by *inequalities* and partly by *condition equations*, only the condition equations on the system:

$$\varphi_\rho = 0 \qquad\qquad (\rho = 1, 2, \ldots)$$

which are always valid, will have to be appended to those condition equations:

$$f_g \leq 0 \qquad\qquad (g = 1, 2, \ldots, k)$$

that actually restrict the mobility of the system at the moment t considered. In that way, terms of the form:

$$\mu_1 \frac{\partial \varphi_1}{\partial x_i} + \mu_2 \frac{\partial \varphi_2}{\partial x_i} + \cdots,$$

([1]) Cf., the following article.

$$\mu_1 \frac{\partial \varphi_1}{\partial y_i} + \mu_2 \frac{\partial \varphi_2}{\partial y_i} + \cdots,$$

$$\mu_1 \frac{\partial \varphi_1}{\partial z_i} + \mu_2 \frac{\partial \varphi_2}{\partial z_i} + \cdots$$

will occur in the right-hand sides of equations (14), and equations (17) will be altered accordingly. However, the new multipliers μ_1, μ_2, … can have arbitrary signs now and will no longer be subject to the condition that they must be positive, unlike the λ.

"Zur Regulierung der Stösse in reibungslosen Punktsystemen, die dem Zwange von Bedingungsungleichungen unterliegen," Ber. Verh. Kön. Sächs. Ges. Wiss. Leipzig **51** (1899), 245-264.

On the regularization of collisions in frictionless point-systems that are constrained by condition inequalities

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Translated by D. H. Delphenich

In recent times, the important problem of calculating the velocities that exist in a frictionless system of material points when one or more points suffer the same simultaneous collision has been treated only under the assumption that the system was subject to nothing but *condition equations*, and that the *duration* of the time during which those equations were valid was likewise known. One can also treat the problem in independent determining parts of the system, which is what APPELL did very beautifully and clearly in connection with some investigations of NIVEN and ROUTH in the note "Sur l'emploi des équations de LAGRANGE dans la théorie du choc et des percussions" ([1]). By contrast, to my knowledge, no one since OSTRAGRADSKY's "Mémoire sur la théorie générale de la percussion" ([2]) has further addressed the very interesting case in which the system is subject to the constraint of *condition inequalities*, and one does not know at all from the outset whether the system conditions that are fulfilled as *equations* at the moment of collision do or do not continue in the same form after the collision, and as significant as the results included in that treatise were, that still leaves the peculiar fact that the fundamental question of the duration of those equations still remains entirely untouched. Therefore, the following attempt to solve the problem for a point-system that is constrained by *condition inequalities* seems entirely justified. The solution is based upon the same conclusions that first led STUDY to the correct explanation and which also led to the presentation of the differential equations of motion for point systems of the kind considered ([3]), and indeed the argument is, in principle, actually simpler than the latter, but questions will arise in the present problem (cf., § **2**) that did not present themselves at all in the earlier work and which therefore also require a new handling.

([1]) Journal de Mathématiques (1896), 5-20. Cf., also APPELL, *Traité de mécanique rationelle*, t. II, 500-503.
([2]) Mémoires de l'Académie Impér. de Sci. de Saint-Petersbourg, sixth series, Sci. math. et phys., t. VI, 269-303.
([3]) Cf., the foregoing article (in this Journal).

§ 1. – Given external impulsive forces.

A system of n material points with masses $m_1, m_2, \ldots, m_n$ is in motion under the action of given forces. Let x_i, y_i, z_i be the coordinates of the point m_i at time t when referred to fixed rectangular axes, and let:

$$(1) \qquad f_1 \leq 0, \quad f_2 \leq 0, \ldots$$

be the analytical expresses for the constraints and restrictions on the system. The left-hand sides of the conditions (1) are then single-valued functions of the coordinates of the system points and possibly time t, as well. I assume that those functions, along with their first and second partial differential quotients, are continuous at all times and at all positions of the system that come under consideration.

Although the system has moved completely unperturbed up to now, at the moment t, one or more points of it might suddenly be subjected to *impacts*, and therefore to very strong forces whose duration is, however, only exceptionally short, such that one can ignore them in all of the changes in position of the system that result from them, or one can neglect the *duration* of the impacts.

Let the positions of the points at the beginning of the impact be known, and let the velocities $\alpha_i, \beta_i, \gamma_i$ *that each system point* m_i *has attained at the end of the impact when it was free at the beginning of it be given.*

As a result of the constraints and restriction on the system, those impact velocities $\alpha_i, \beta_i, \gamma_i$ do not actually come about, but must be regularized in such a way that they satisfy the conditions on the system. When one neglects not only the duration of the impact, but also any *friction* that might develop due to, say, the restrictions on the system, one will be dealing with *the calculation of the velocities that have actually been produced in the system at the end of the impact*, and indeed that problem shall be solved here on the basis of *Gauss's principle of least constraint* because it leads to the solution in the clearest and most natural way, in my opinion.

Since the duration of the impact is to be neglected, the point m_i will possess the same coordinates x_i, y_i, z_i immediately after the impact that it had at the moment t. If it were free then it would arrive at a position B_i whose coordinates are:

$$x_i + \alpha_i\, dt, \qquad y_i + \beta_i\, dt, \qquad z_i + \gamma_i\, dt$$

after the subsequent infinitely-small time interval dt. However, in reality, the point does not have the velocities $\alpha_i, \beta_i, \gamma_i$ at the end of the impact, but rather it has arrived at the *still-unknown velocities* x_i', y_i', z_i'. Therefore, at the moment $t + dt$, it does not reach the position B_i, but another position C_i whose coordinates are:

$$x_i + x_i'\, dt, \qquad y_i + y_i'\, dt, \qquad z_i + z_i'\, dt.$$

Now, from the principle of least constraint, among all of the positions C_i to which the points m_i can arrive during the time interval considered without violating the conditions on the system, their actual locations will be distinguished by the fact that:

$$\sum_{i=1}^{n} m_i \overline{C_i B_i}^2$$

must be a minimum for them, and with the coordinates of the points B_i and C_i, that requirement will reduce to this one:

$$(2) \qquad \sum_{i=1}^{n} m_i \left\{(\alpha_i - x_i')^2 + (\beta_i - y_i')^2 + (\gamma_i - z_i')^2\right\} = \text{min.} ;$$

i.e., the sum on the left must be smaller for the true final velocities x_i', y_i', z_i' than it is for all velocities that the points of the system can assume at the end of the impact, and therefore at the moment t, since one has neglected the duration of the impact ([1]).

Therefore, in order to ascertain the true regularized impact velocities x_i', y_i', z_i', first and foremost, it is important to find *those restrictions* that the system conditions impose upon the velocities of its points at that moment.

Now, by assumption, the system might exhibit only those motions for which the coordinates of its points will continually satisfy the conditions (1).

Any one of those conditions might be represented by:

$$f \leq 0.$$

Corresponding to any such possible motion, if one considers the coordinates x_i, y_i, z_i of each system point m_i to be continuous functions of time t, lets t go to $t + dt$, and develops the condition considered in powers of dt then it will go to:

$$(\text{a}) \qquad f + f' dt + r\, dt^2 \leq 0,$$

where f' is the complete differential quotient of the function f with respect to time t, and $r\, dt^2$ denotes the remainder term in the TAYLOR development.

The value that the first term f possesses at the moment t is known, since it contains only time and the coordinates, and therefore quantities values are known at that moment, by assumption. By contrast, the first differential quotients of the coordinates – or the velocities x_i', y_i', z_i' of the system points – also enter into f'.

([1]) Or, in words: *If a system of material points is suddenly subjected to impacts that, in conjunction with the velocities that the individual points have attained, make them strive to attain given velocities, then those velocities can be regularized in such a way that the vis viva will be a minimum as a result of the conditions and restrictions on the system's lost velocities.*

Furthermore, as a result of the system condition $f \leq 0$, the known value of f can only be < 0 or = 0. Therefore, if it is not precisely = 0 then for a sufficiently small dt, the condition (a) will already be fulfilled by itself and will not restrict the velocities in any way momentarily.

By contrast, if $f = 0$ then one can divide the condition (a) by the positive quantity dt and thus reduce it to:

$$f' + r\,dt \leq 0 .$$

However, that condition can be fulfilled for arbitrarily small dt only when one already has:

$$f' \leq 0 .$$

Of the system conditions (1), only the ones that exist as *equations* precisely will restrict the velocities of the system points at the moment t. Let them be the *r conditions:*

$$f_1 \leq 0, \qquad f_2 \leq 0, \qquad \ldots, \qquad f_r \leq 0.$$

I then assume that the known position of the system at the moment t corresponds to the r equations:

$$f_1 = 0, \qquad f_2 = 0, \qquad \ldots, \qquad f_r = 0,$$

while perhaps all of the remaining system conditions (1) exist momentarily with only the upper sign.

The system points m_i at that moment are then allowed to have all velocities $x_i', \; y_i', \; z_i'$ that the conditions:

$$f_1' \leq 0, \qquad f_2' \leq 0, \qquad \ldots, \qquad f_r' \leq 0 \tag{3}$$

will tolerate. If we now once more restrict the notations $x_i', \; y_i', \; z_i'$ to the desired true velocities of the system points m_i at the end of the impact then we will, on the other hand, understand:

$$x_i' + \delta x_i', \qquad y_i' + \delta y_i', \qquad z_i' + \delta z_i'$$

to mean any other velocities of the system points that are possible at the same time and likewise deviate only slightly from the unknown true velocities, and on the grounds of the identities:

$$\frac{\partial f'}{\partial x_i'} \equiv \frac{\partial f}{\partial x_i}, \; \ldots,$$

if one introduces the abbreviation:

$$\delta f' \equiv \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} x_i' + \frac{\partial f}{\partial y_i} y_i' + \frac{\partial f}{\partial z_i} z_i' \right) \tag{4}$$

then the conditions (3) will imply the following *r* conditions on the variations $\delta x_i'$, $\delta y_i'$, $\delta z_i'$, of the velocities:

$$f_1' + \delta f_1' \leq 0, \quad f_2' + \delta f_2' \leq 0, \quad \ldots, \quad f_r' + \delta f_r' \leq 0,$$

and from (2), one must have:

$$\sum_{i=1}^{n} m_i \{(\alpha_i - x_i')\delta x_i' + (\beta_i - y_i')\delta y_i' + (\gamma_i - z_i')\delta z_i'\} \leq 0 \tag{5}$$

for all sufficiently small values of those variations fulfill those *r* conditions.

However, if:

$$f' \neq 0, \text{ but only} < 0$$

for the unknown true final velocities x_i', y_i', z_i' then the condition:

$$f' + \delta f' \leq 0$$

will not restrict the velocities in any way, since it is then fulfilled by itself for all arbitrary, but sufficiently small, values of the variations.

Such a restriction will first come into play, moreover, when the actual final velocities satisfy the equation:

$$f' = 0,$$

which will reduce our condition to:

$$\delta f' \leq 0 .$$

However, the true final velocities are still yet-to-be-found. For the time being, they are still completely unknown, and we can then by no means decide *directly* which of the derivatives f' are = 0 for the true velocities at the moment considered *t* and which of them are < 0.

Hence, nothing else remains but to attempt to solve the problem by an *indirect* path, and indeed we can obviously proceed only as follows: We first assume, in a purely arbitrary way, that the unknown regularized impact velocities fulfill some of the conditions (3) as equations, and the others as only inequalities, determine the values of our unknowns from that arbitrary assumption, and thereafter look around for criteria that will show us whether the calculated values of the velocities will or will not be correct.

We then assume now that the unknown true final velocities x_i', y_i', z_i' satisfy, say, the ρ equations:

$$f_\lambda' \equiv \frac{\partial f_\lambda}{\partial t} + \sum_{i=1}^{n}\left(\frac{\partial f_\lambda}{\partial x_i} x_i' + \frac{\partial f_\lambda}{\partial y_i} y_i' + \frac{\partial f_\lambda}{\partial z_i} z_i'\right) = 0 \qquad (\lambda = 1, 2, \ldots, \rho) \tag{6}$$

and the $r - \rho$ inequalities:

(6′) $$f'_\lambda \equiv \frac{\partial f_\lambda}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial f_\lambda}{\partial x_i} x'_i + \frac{\partial f_\lambda}{\partial y_i} y'_i + \frac{\partial f_\lambda}{\partial z_i} z'_i \right) < 0 \qquad (\mu = \rho + 1, 2, \ldots, r),$$

in which ρ can be any one of the numbers 0, 1, …, r, and then ask *what values of the desired velocities does that assumption imply?*

I shall make that more specific with the further assumption that none of the ρ equations (6) should be a mere consequence of the remaining ones in the known momentary position of the system. Otherwise, they would indeed contribute nothing to the determination of the unknowns x'_i, y'_i, z'_i, and could therefore be simply dropped. I shall then assume *that the ρ equations* (6) *determine ρ of the unknowns x'_i, y'_i, z'_i in terms of the* $3n - \rho$ *remaining ones.*

With those assumptions, the variations of the velocities are momentarily subject to only the ρ conditions:

(7) $$\delta f'_\lambda \equiv \sum_{i=1}^{n} \left(\frac{\partial f_\lambda}{\partial x_i} \delta x'_i + \frac{\partial f_\lambda}{\partial y_i} \delta y'_i + \frac{\partial f_\lambda}{\partial z_i} \delta z'_i \right) \leq 0 \qquad (\lambda = 1, 2, \ldots, \rho),$$

and the desired velocities must then satisfy the requirement (5) for all values of their variations that are compatible with those ρ conditions.

In particular, one must then have:

$$\sum_{i=1}^{n} m_i \left\{ (\alpha_i - x'_i)\, \delta x'_i + (\beta_i - y'_i)\, \delta y'_i + (\gamma_i - z'_i)\, \delta z'_i \right\} = 0$$

as long as one subjects the $3n$ variations $\delta x'_i$, $\delta y'_i$, $\delta z'_i$ to the ρ equations:

(7′) $$\delta f'_\lambda = 0 \qquad (\lambda = 1, 2, \ldots, \rho),$$

and from the assumption that was introduced in regard to equations (6), those equations will determine ρ of the variations as functions of the remaining $3n - \rho$.

If one now multiplies the latter equations by the temporarily undetermined factors $-l_\lambda$ and then adds them to the previous equation then one will get the equation:

(8) $$\sum_{i=1}^{n} m_i \left\{ (\alpha_i - x'_i)\, \delta x'_i + (\beta_i - y'_i)\, \delta y'_i + (\gamma_i - z'_i)\, \delta z'_i \right\} = \sum_{\lambda=1}^{\rho} l_\lambda\, \delta f'_\lambda .$$

However, one can determine the multipliers $l_1, l_2, \ldots, l_r$ in such a way that the coefficients of those of the ρ variations $\delta x'_i$, $\delta y'_i$, $\delta z'_i$ that can be expressed in terms of the remaining ones by the ρ equations (7) will be equal to each other on the left and right. After dropping those equal terms, equation (8) will contain only entirely arbitrary variations. Therefore, the coefficients of the $3n$ –

ρ variations that remain on both sides of it must be equal to each other. In that way, one will arrive at the $3n$ equations:

$$(9) \qquad \left\{ \begin{aligned} m_i(\alpha_i - x_i') &= \sum_{\lambda=1}^{\rho} l_\lambda \frac{\partial f_\lambda}{\partial x_i}, \\ m_i(\beta_i - y_i') &= \sum_{\lambda=1}^{\rho} l_\lambda \frac{\partial f_\lambda}{\partial y_i}, \\ m_i(\gamma_i - z_i') &= \sum_{\lambda=1}^{\rho} l_\lambda \frac{\partial f_\lambda}{\partial z_i}, \end{aligned} \right. \qquad (i = 1, 2, \ldots, n)$$

However, one still has the ρ equations (6) themselves, moreover, and it is easy to see that under our assumption, the determinant of the $3n + \rho$ linear equations (6) and (9) in the $3n + \rho$ unknowns:

$$x_i', \qquad y_i', \qquad z_i', \qquad l_\lambda$$

cannot be zero.

Namely, if it were = 0 then one would be able to satisfy the $\rho + 3n$ homogeneous linear equations (7′) and:

$$(9') \qquad \left\{ \begin{aligned} m_i\,\delta x_i' + \sum_{\lambda=1}^{\rho} l_\lambda \frac{\partial f_\lambda}{\partial x_i} &= 0, \\ m_i\,\delta y_i' + \sum_{\lambda=1}^{\rho} l_\lambda \frac{\partial f_\lambda}{\partial y_i} &= 0, \\ m_i\,\delta z_i' + \sum_{\lambda=1}^{\rho} l_\lambda \frac{\partial f_\lambda}{\partial z_i} &= 0 \end{aligned} \right.$$

with values of the $3n + \rho$ variables $\delta x_i'$, $\delta y_i'$, $\delta z_i'$, l_λ that do not all vanish.

However, from (7′), it follows from equations (9′) upon multiplying by $\delta x_i'$, $\delta y_i'$, $\delta z_i'$, and adding them that:

$$\sum_{i=1}^{n} m_i(\delta x_i'^2 + \delta y_i'^2 + \delta z_i'^2) = 0.$$

Equations (7′) and (9′) then require that all $\delta x_i'$, $\delta y_i'$, $\delta z_i'$ will be = 0, and as a result of our assumption in regard to equations (6) to (9′), that will also imply the vanishing of all multipliers l_λ.

The determinant of equations (6) to (9) is then, in fact, ≠ 0, and those equations will then determine their $3n + \rho$ unknowns x_i', y_i', z_i', l_λ uniquely in terms of the given impact velocities α_i, β_i, γ_i, the coordinates x_i, y_i, z_i, and possibly the time t, so in terms of nothing but quantities whose values will be known completely at the moment t.

Our assumption has then yielded a single completely-determined system of values for those velocities x_i', y_i', z_i' *that the points of the system can attain at the end of the impact.*

However, that assumption cannot by any means be established *a priori*. *Thus, whether or not the values of the velocities that are obtained from it are also the true regularized impact velocities is still entirely questionable.*

Meanwhile, our assumption itself, as well as the principle of least constraint, contains *more conditions* than the ones that are fulfilled already.

Namely, of the r conditions (3) that restrict the velocities of the system points at the moment t, up to now, we have only satisfied the first ρ, and indeed satisfied them by means of equations (6). Hence, if our assumption were correct then, above all, the last $r - \rho$ conditions (3) would have to be fulfilled by the values of the x_i', y_i', z_i' thus-obtained by themselves; i.e., *of the uniquely determined values that the expressions* (6′) *take on by substituting the solutions of equations* (6) *and* (9), *none of them can be* > 0. Therefore, if one of those values were found to be > 0 then our assumption would be false, and the calculated values of the velocities would not be their true values.

Moreover, the formulas (9) convert equation (8) into an identity and then reduce our original demand (5) to:

$$\sum_{\lambda=1}^{\rho} l_\lambda \, \delta f_\lambda' \leq 0.$$

That condition must also be fulfilled then for all variations $\delta x_i'$, $\delta y_i'$, $\delta z_i'$ that fulfill the ρ conditions (7), and that is identical to the condition that *none of the multipliers can satisfy:*

$$l_\lambda < 0.$$

The principle of least constraint then adds the ρ conditions:

$$(10) \qquad l_\lambda > 0 \qquad (\lambda = 1, 2, \ldots, \rho)$$

to our equations (6) and (9), in which the > sign should not exclude equality.

Thus, whenever the solution to equations (6) and (9) for the unknowns x_i', y_i', z_i', l_λ yields a *negative* value for some l_λ, our assumption will not, in turn, correspond to reality, and once more the calculated values of the velocities cannot be the correct ones.

Since those two criteria have only a *negative* nature, they will generally tell us nothing immediately except that *conversely, whenever the solution of equations* (6) *and* (9) *does not contain a negative l nor do any of the derivatives* (6′) *provide a positive value, that solution will also certainly represent the true velocities of the system points at the end of the impact.* In order to prove that, strictly speaking, one would first have to show that *one could not also arrive at other systems of values for the* x_i', y_i', z_i' *that likewise satisfy all of those conditions, as well as all of the demands of the principle of least constraint with any other decomposition of the conditions* (3). However, if one ignores the fact that one can actually carry out this proof only in the two simplest cases $r = 1$ and $r = 2$ ([1]), and very appreciable complications seem to present themselves for larger values of r, then one might, on the other hand, probably regard it as obvious for that reason that two different systems of velocities with the required behavior cannot exist, because if they did exist then there would be no means of deciding which of the two was the correct one. Namely, for the sake of dynamics, it is only essential that the demand (5) should be fulfilled, but otherwise it is entirely irrelevant whether the sum (2) is an actual minimum or not, and in addition, whether the value that the sum takes on by means of equations (6) and (9) is also, in fact, the smallest of all of the values that it might assume under the conditions (3) and the assumptions (6), as long as it does not imply that some $l < 0$ and some $f'_\mu > 0$.

In all cases, one can state with certainty that our method must surely yield the true velocities of the system points at the end of the impact in all examples in which it leads to only a single solution. However, one likewise sees that for larger values of r, very many detailed investigations might be necessary until one arrives at those assumptions (6) that ultimately fulfill all requirements.

In the foregoing, we have sought the *total final velocities* that the system points have attained at the end of the impacts. However, instead of that, one might also wish to know the *changes in velocity* that the points of the system will suffer during the impacts.

If one lets u_i, v_i, w_i *denote the velocities that the system points* m_i *possess immediately before the impacts, and lets* Δu_i, Δv_i, Δw_i *denote changes in velocity that the impacts actually confer to those points* then one will have:

$$\Delta u_i = x'_i - u_i\,, \quad \Delta v_i = y'_i - v_i\,, \quad \Delta w_i = y'_i - w_i\,, \tag{11}$$

when one denotes the regularized impact velocities by x'_i, y'_i, z'_i, as always. On the other hand, the velocities α_i, β_i, γ_i that the points m_i would have attained at the end of the impacts if they were free at the beginning of it will be the resultants of the velocities u_i, v_i, w_i and the velocities a_i, b_i, c_i *that the impacts would impart to the free points* m_i *when starting from rest.* One will then have:

$$\alpha_i = u_i + a_i\,, \quad \beta_i = v_i + b_i\,, \quad \gamma_i = w_i + c_i\,, \tag{12}$$

such that in order to know the values of the α_i, β_i, γ_i, the individual values of the u_i, v_i, w_i and the a_i, b_i, c_i must be given.

Now, it follows from (11) and (12) that:

$$\alpha_i - x'_i = a_i - \Delta u_i\,, \quad \ldots$$

When one generally sets:

$$F_\nu \equiv \frac{\partial f_\nu}{\partial t} + \sum_{i=1}^{n}\left(\frac{\partial f_\nu}{\partial x_i}u_i + \frac{\partial f_\nu}{\partial y_i}v_i + \frac{\partial f_\nu}{\partial z_i}w_i\right), \tag{13}$$

([1]) See pp. 237 of this volume.

from (6) and (9), one will then have the following equations for the calculation of the changes in velocity Δu_i , Δv_i , Δw_i :

$$(14) \qquad f'_\lambda \equiv \sum_{i=1}^{n}\left(\frac{\partial f_\nu}{\partial x_i}\Delta u_i + \frac{\partial f_\nu}{\partial y_i}\Delta v_i + \frac{\partial f_\nu}{\partial z_i}\Delta w_i\right) + F_\lambda = 0 \qquad (\lambda = 1, 2, \ldots, \rho)$$

$$(15) \qquad \begin{cases} m_i(a_i - \Delta u_i) = \sum_{\lambda=1}^{\rho} l_\lambda \dfrac{\partial f_\lambda}{\partial x_i}, \\ m_i(b_i - \Delta v_i) = \sum_{\lambda=1}^{\rho} l_\lambda \dfrac{\partial f_\lambda}{\partial y_i}, \\ m_i(c_i - \Delta w_i) = \sum_{\lambda=1}^{\rho} l_\lambda \dfrac{\partial f_\lambda}{\partial z_i}. \end{cases} \qquad (i = 1, 2, \ldots, n)$$

One will then get the desired changes in velocity when one solves the $\rho + 3n$ equations (14) and (15) for the $3n + \rho$ unknowns:

$$\Delta u_i , \Delta v_i , \Delta w_i , l_\lambda ,$$

and in order for those solutions to represent the correct changes in velocity, none of the l can be < 0, and when they are substituted, no of the $r - \rho$ expressions:

$$(14') \qquad f'_\mu \equiv \sum_{i=1}^{n}\left(\frac{\partial f_\mu}{\partial x_i}\Delta u_i + \frac{\partial f_\mu}{\partial y_i}\Delta v_i + \frac{\partial f_\mu}{\partial z_i}\Delta w_i\right) + F_\lambda \qquad (\mu = \rho + 1, \ldots, r)$$

will be > 0.

In the foregoing, it was assumed that the point-system considered was subject to *condition inequalities* exclusively. If *condition equations:*

$$\varphi_1 = 0, \qquad \varphi_2 = 0, \ldots$$

that its points should satisfy at every time are also prescribed, moreover, then it should be obvious that all that will change is that along with equations (6) and the equations:

$$\varphi'_1 = 0, \qquad \varphi'_2 = 0, \ldots,$$

and therefore equations (14), as well, one must add the ones that arise by introducing the Δu_i , Δv_i , Δw_i in place of the x'_i, y'_i, z'_i in the ones that were just written down. In that way, terms of the form:

$$p_1 \frac{\partial \varphi_1}{\partial x_i} + p_2 \frac{\partial \varphi_2}{\partial x_i} + \ldots,$$

$$p_1 \frac{\partial \varphi_1}{\partial y_i} + p_2 \frac{\partial \varphi_2}{\partial y_i} + \ldots,$$
$$p_1 \frac{\partial \varphi_1}{\partial z_i} + p_2 \frac{\partial \varphi_2}{\partial z_i} + \ldots$$

will appear on the right-hand sides of equations (9) and (15), but whose multipliers p_1, p_2, … are no longer subordinate to the conditions $p > 0$, but might be positive, as well as negative.

§ 2. – Internal collisions.

Up to now, we have always tacitly assumed that the collisions were produced by actual *external impacts*. In that way of looking at things, in the absence of impacts, the system would continue to move unperturbed, and the velocities u_i, v_i, w_i of the system points would already be compatible with the momentary system conditions (3) immediately before the impact. Due to the condition $f_\nu \leq 0$ and the assumptions that $f_\nu = 0$ at the moment t, from (13), one will also have $F_\nu \leq 0$ in each case then, and indeed one will have $F_\nu < 0$ when the constraint or restriction $f_\nu = 0$ proves to be rigorously true, but = 0 when it also persists for unperturbed motion.

However, the *sudden changes in velocity in the system also* arise *without any special impacts* in such a way that once the system has moved for a long enough time that one or more of its conditions $f_\lambda \leq 0$ consists of only an inequality, at the moment t, it will arrive at a position in which those conditions will be fulfilled as equalities. In that way, the functions f_λ in question will increase, so their complete differential quotients must also satisfy $f'_\lambda \geq 0$ at the moment t, and the new position can only be attained only with velocities u_i, v_i, w_i for the system points for which the associated $F_\lambda \geq 0$.

Therefore, whenever one of the expressions $F_\lambda > 0$, *an impact will occur in the system*, and the possibility of such impacts, which are most simply illustrated by loose connecting threads that are suddenly tensed violently or the impinging of system points on rigid surfaces, shows quite clearly that one cannot attach the constraints and restrictions on the system to any *insurmountable obstacle* without contradicting the continuity of the changes in velocity beyond resolution.

Namely, whenever $f_\lambda = 0$, the system condition $f_\lambda \leq 0$ will forbid any increase in the function f_λ, and will also demand that $f'_\lambda \leq 0$ then. However, if $F_\lambda > 0$ then the moment at which the equation $f_\lambda = 0$ is established will be when the value of f'_λ just becomes > 0. Now, the velocities of the system points cannot unexpectedly jump from values for which f'_λ possesses a finite positive value to values that will make $f'_\lambda \leq 0$. It must then be necessary (if also only minimally and during an exceptionally short time) to overcome the obstacle of the condition $f_\lambda \leq 0$ and for f_λ to increase until the exceptionally turbulent, but also continuously varying, velocities have been regularized in such a way that the equation $f'_\lambda = 0$ has been established, *at which point, the impact considered will have reached its conclusion* ([1]).

In that argument, it should not at all be said that the equation $f_\lambda = 0$ must now necessarily persist after the impact. Rather, as the second example in the following § will show, in some situations, that equation might be in force only during the impact itself, but once more cease to apply afterwards, just as it did beforehand.

For the sake of brevity, I would like to call the impacts that arise within the system itself *internal collisions*, in contrast to the ones that are produced by actual impulsive forces, although naturally they can arise from sudden tensions in connecting threads, as well as from collisions with external obstacles. Indeed, the latter must also be included among the system conditions as long as the system falls within their scope.

What are the values of the velocities α_i, β_i, γ_i that such internal collisions would impart to the points m_i that are already equipped with the velocities u_i, v_i, w_i when they were free immediately before them and obviously also remain free afterward? Now, by assumption, the impacts now originate from only the fact that one or more constraints (restrictions, resp.) $f_\lambda = 0$ suddenly come about that did not exist before. Hence, if the points were free and remained free then they would experience no impacts at all, and the impact velocities α_i, β_i, γ_i would then reduce to simply the velocities u_i, v_i, w_i that the points possessed before the equations $f_\lambda = 0$ became valid in the system. From (12), that coincidence of the α_i, β_i, γ_i with the u_i, v_i, w_i will simultaneously show immediately that the velocities a_i, b_i, c_i will all have the value zero, moreover.

Since the values of the α_i, β_i, γ_i, as well as the a_i, b_i, c_i, are also known for internal collisions then, we can also seek to determine the final velocities or changes in velocity from the previous types that are produced by such collisions in the system. To that end, among equations (6), we must now absorb those of the equations $f'_\lambda = 0$ that correspond to the equations $f_\lambda = 0$ that are established at the moment t when F_λ has positive values, and once more construct equations (9) or (15) with the equations (6) that are obtained in that way, and set ([2]):

([1]) OSTRAGRADSKY, pp. 287. Naturally, the process does not play out in the same way when the system constraint or restriction $f_\lambda \leq 0$ is *elastic*. The turbulent changes in velocity will then endure beyond the moment at which the equation $f'_\lambda = 0$ has been established, and at the end of the impact one will already have $f'_\lambda < 0$. In that case, one will then no longer have any right to subject the desired velocities to the equation $f'_\lambda = 0$, and one will then also no longer have the sufficient number of equations for determining those unknowns, but one must (as in the collision of moving bodies) appeal to the *theory of elasticity* for the missing equations. However, in that way, the entire problem will lose its purely *mechanical* character, and will become a *physical* problem, and in a certain sense, the same thing will also be true whenever any sort of *friction* should be considered.

([2]) For internal collisions, from (16), the general formula (2) will go to:

$$\sum_{i=1}^{n} m_i \left\{ (x'_i - u_i)^2 + (y'_i - v_i)^2 + (z'_i - w_i)^2 \right\} = \text{min.},$$

and one will then get the theorem:

Any sudden appearance of new constraints or restrictions in a system of material points will change the velocities of the points in such a way that the vis viva of the changes in velocities that come about will be a minimum.

See pp. 216 of this volume.

$$(16) \qquad \begin{cases} \alpha_i = u_i, \quad \beta_i = v_i, \quad \gamma_i = w_i \quad \text{or} \\ a_i = 0, \quad b_i = 0, \quad c_i = 0 \end{cases}$$

in them everywhere. Moreover, the further conditions that were cited in § **1** must be fulfilled if the velocities or changes in velocity that are calculated in that way are to be the correct ones.

However, if sudden changes of velocity in the system are produced, in particular, in such a way that one imposes entirely new condition equations $f_\lambda = 0$ at the moment t, such as, e.g., when system points are suddenly obliged (or rather seized and forced) to move in a given way, then the conditions $l_\lambda > 0$ will, in turn, drop away for the corresponding multipliers l_λ ([1]).

§ 3. – Examples.

In order to explain the method that was deduced, allow me to pursue it in two simple examples.

I. *Two material points m_1 and m_2 are linked by an inextensible string, and when the string is tensed, they move from the side $f_1 < 0$ of a fixed surface:*

$$f_1(x, y, z) = 0$$

until the point m_1 impinges upon it at the moment t. What changes in velocity will the two points experience then?

If we call the length of the connecting thread L then we will have the following equations here:

$$f_1(x_1, y_1, z_1) = 0,$$

$$f_2 \equiv (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - L^2 = 0,$$

$$F_2 \equiv (x_1 - x_2)(u_1 - u_2) + (y_1 - y_2)(v_1 - v_2) + (z_1 - z_2)(w_1 - w_2) = 0,$$

while preserving the previous notations; at the same time:

$$f_1(x_2, y_2, z_2) \neq 0,$$

$$F_1 \equiv \frac{\partial f_1}{\partial x_1} u_1 + \frac{\partial f_1}{\partial y_1} v_1 + \frac{\partial f_1}{\partial z_1} w_1 > 0,$$

([1]) Should impacts from outside the system points also be simultaneously exerted at the moment of the internal collisions, then one would obviously preserve the a_i, b_i, c_i in (15) and give the α_i, β_i, γ_i the values (12), in which, as before, one understands the a_i, b_i, c_i to mean the velocities that the external impulsive forces would impart upon the free points m_i from a state of rest.

while the system conditions themselves are:

$$f_1\,(x_1, y_1, z_1) \le 0, \qquad f_2 \le 0\;.$$

If we next assume that *the string still remains tensed after the collision* then we will get the following eight equations for the determination of the changes in velocity of the two points from (14), (15), and (16):

$$f_1' \equiv \frac{\partial f_1}{\partial x_1}\Delta u_1 + \frac{\partial f_1}{\partial y_1}\Delta v_1 + \frac{\partial f_1}{\partial z_1}\Delta w_1 + F_1 = 0,$$

$$f_2' \equiv (x_1 - x_2)(\Delta u_1 - \Delta u_2) + (y_1 - y_2)(\Delta v_1 - \Delta v_2) + (z_1 - z_2)(\Delta w_1 - \Delta w_2) = 0,$$

$$m_1\,\Delta u_1 + l_1\frac{\partial f_1}{\partial x_1} + l_2\,(x_1 - x_2) = 0, \ldots$$

$$m_2\,\Delta u_2 - l_2\,(x_1 - x_2) = 0, \ldots,$$

and when we substitute the values of the Δu, Δv, Δw from the last six equations into the first two and employ the abbreviations:

$$\text{(A)} \qquad \left\{\begin{aligned} B_1 &\equiv +\sqrt{\left(\frac{\partial f_1}{\partial x_1}\right)^2 + \left(\frac{\partial f_1}{\partial y_1}\right)^2 + \left(\frac{\partial f_1}{\partial z_1}\right)^2}, \\ B_{12} &\equiv (x_1 - x_2)\frac{\partial f_1}{\partial x_1} + (y_1 - y_2)\frac{\partial f_1}{\partial y_1} + (z_1 - z_2)\frac{\partial f_1}{\partial z_1}, \end{aligned}\right.$$

they will come down to these two:

$$\text{(B)} \qquad \left\{\begin{aligned} B_1^2\,l_1 + B_{12}\,l_2 &= m_1\,F_1, \\ m_2\,B_{12}\,l_1 + (m_1 + m_2)L^2\,l_2 &= 0. \end{aligned}\right.$$

However, if one lets λ_1, μ_1, ν_1 denote the direction cosines of the normal to the surface $f_1 = 0$ that points upwards from the side $f_1 > 0$ at the point x_1, y_1, z_1, and lets φ_1 denote the angle that the normal makes with the line $\overrightarrow{m_2\,m_1}$ then one will have:

$$\text{(C)} \qquad \frac{1}{B_1}\frac{\partial f_1}{\partial x_1} = \lambda_1, \quad \frac{1}{B_1}\frac{\partial f_1}{\partial y_1} = \mu_1, \quad \frac{1}{B_1}\frac{\partial f_1}{\partial z_1} = \nu_1,$$

and as a result:

$$B_{12} = L\,B_1 \cos\varphi_1\;.$$

The two equations (B) then show that l_1 and l_2 cannot both be > 0 whenever φ_1 is an acute angle. On the other hand, the line $\overrightarrow{m_2\, m_1}$ meets the surface $f_1 = 0$ on the side $f_1 < 0$. If one ignore the latter case then equations (B) will require that one must have $l < 0$, in any event, and that will prove that *the string cannot remain tensed after the collision* ([1]).

The actual changes in velocity cannot satisfy the equation $f_2' = 0$ then. We must drop that equation, and then set $l_2 = 0$, so we will get only the equations:

$$f_1' = 0,$$

$$\text{(D)} \qquad \begin{cases} m_1\,\Delta u_1 + l_1 \dfrac{\partial f_1}{\partial x_1} = 0, & \cdots \\ m_2\,\Delta u_2 \qquad\qquad = 0, & \cdots \end{cases}$$

for the true values of the $\Delta u, \Delta v, \Delta w$. In fact, from (A), they imply that:

$$B_1^2\, l_1 = m_1\, F_1 > 0\,,$$

from which, (C) will imply that:

$$\Delta u_1 = -\frac{F_1}{B_1}\lambda_1\,, \qquad \Delta v_1 = -\frac{F_1}{B_1}\mu_1\,, \qquad \Delta w_1 = -\frac{F_1}{B_1}\nu_1\,,$$

and due to the fact that $\Delta u_2 = \Delta v_2 = \Delta w_2 = 0$, one will have:

$$f_2' = -\frac{L\,F_1}{B_1}\cos\varphi_1 < 0$$

with those values, so they fulfill all of the requirements that the true changes in velocity are subject to and are, at the same time, the unique solutions to our problem.

II. *One finds a rigid circular ring in a state of forced rectilinear translational motion on the fixed horizontal xy-plane, such that its center advances with constant positive velocity* ϖ *along the x-axis. At the moment t, the ring collides with a point of mass m that rests upon the xy-plane. What velocity does the point attain by the collision with the ring when the plane and ring are assumed to be completely smooth?*

The condition equation:

([1]) Moreover, it will remain tensed only in the case $\varphi_1 = \pi / 2$ or $B_{12} = 0$, where equations (B) will imply that $l_1 > 0$, $l_2 = 0$, as well as in the case $F_1 = 0$, in which $l_1 = l_2 = 0$, and there will be no collision at all.

(a) $$2\,f\,(x, y, z) \equiv (x - \varpi\, t)^2 + y^2 - r^2 = 0$$

between the coordinates x, y of the point and the time t is suddenly established at the moment t, but $2f \leq 0$ or $-2f \leq 0$ according to whether the ring meets the point *on the inside* or *on the outside*, resp.

In order to calculate the initial velocities x', y' that the collision of the ring and the point brings about, since the point was at rest beforehand, we will then get the following equations from (9) and (16):

$$f' \equiv (x - \varpi\, t)\,(x' - \varpi) + y\,y' = 0, \qquad -m\,x' = l\,(x - \varpi\, t), \qquad -m\,y' = l\,y$$

in the *first* case, and when we consider (a), they will imply that:

$$l\,r^2 = -m\,\varpi\,(x - \varpi\, t)\,.$$

Therefore (in agreement with the impact condition $F_\lambda > 0$ in § **2**, which reduces to $\partial f / \partial t > 0$ here), one must have:

$$x - \varpi\, t < 0$$

at the moment t of the collision, which should be obvious *a priori*, since otherwise no collision would even happen. If equation (a) is fulfilled by the substitutions:

(b) $$x - \varpi\, t = r \cos \varphi, \qquad y = r \sin \varphi$$

then one will get the following values for the initial velocities of m :

(c) $$x' = \varpi \cos^2 \varphi, \qquad y' = \varpi \cos \varphi \sin \varphi\,.$$

By contrast, in the *second* case, as a result of the condition $-2f \leq 0$, one will have:

$$m\,x' = l\,(x - \varpi\, t), \qquad m\,y' = l\,y,$$

and one will get:

$$l\,r^2 = +m\,\varpi\,(x - \varpi\, t)\,.$$

Hence, the condition $l > 0$ now demands that:

$$x - \varpi\, t > 0,$$

which should be, in turn, obvious, and one again comes back to the same initial velocities (c).

By contrast, the further motion is entirely different in both cases.

Namely, if the point remains on the advancing circular ring then its motion will obey the differential equations:

$$m\,x'' = \lambda\,(x - \varpi\,t), \qquad -\,m\,y'' = \lambda\,y,$$

and therefore, one must have either $\lambda > 0$ or $\lambda < 0$ according to whether the motion proceeds on the *inner* or *outer* side of the ring, resp. ([1]) However, in conjunction with the equation:

$$(x - \varpi\,t)\,x'' + y\,y'' + (x - \varpi\,t)^2 + y'^2 = 0,$$

which follows from (a), that will yield the differential equations:

$$\lambda\,r^2 = m\{(x - \varpi\,t)^2 + y'^2\}.$$

Therefore, λ can never be negative and can vanish only for $x' = \varpi$, $y' = 0$. Thus, whereas the point will remain on the ring as long it meets the ring from the *inside*, by contrast, it will always collide with the ring when it meets it from the *outside*, except when $\varphi = 0$; i.e., when the point is found along the x-axis itself. In the former case, the point will traverse the inner circular ring with the constant angular velocity:

$$\varphi' = \frac{\varpi \sin\varphi}{r},$$

while in the latter case, it will separate from the ring with the constant velocity:

$$V = \varpi \cos\varphi$$

in the direction of the line that goes through the point and the center of the circle at the moment of impact without being overtaken by the ring again. Only when the point lies on the x-axis itself will the same motion occur in both cases, namely, the point will be at rest relative to the ring.

([1]) Cf., pp. 236 of the volume.

"Ueber die Bewegung eines Punktsystems bei Bedingungsungleichungen," Nach. Ges. Wiss. Göttingen, Math.-phys. Klasse **3** (1899), 306-310.

On the motion of a point-system under condition inequalities

By

E. Zermelo

Submitted by **D. Hilbert** through the presiding secretary at the session on 3 February

Translated by D. H. Delphenich

In his article on 3 July of this year, **A. Mayer** [Sitz. kön. Sächs. Ges. Wiss. Leipzig (1899), 224-244], in reference to an older paper of **Ostragradski** (1834), treated the problem of exhibiting the differential equations of motion for a frictionless point-system that is subject to condition inequalities. There, he gave a method for finding the accelerations of all points when their instantaneous positions and velocities are given in each such case with the help of **Gauss**'s "principle of least constraint." However, his method was only an indirect one: One must always distinguish between a number of cases according to whether their introduction will change the final formulas. Nonetheless, which of those cases is actually realized can generally be first decided from the result, such that, as a rule, one must first test a series of Ansätze before one can find one of them that satisfies all conditions. Meanwhile, in that process, one must prove that such a solution always exists and is unique in order for one to convince oneself that Gauss's principle also suffices to determine the actual motion. That proof of uniqueness, which **A. Mayer** gave for only the simplest cases in which no more than two condition equations come under consideration (*loc. cit.*, pp. 237), is the subject of my present communication, and is achieved very easily on the basis of the remark (whose importance was brought to my attention by Herrn Prof. Hilbert) that the auxiliary conditions in the minimum problem are all *linear* in the unknowns – viz., the accelerations.

We denote the coordinates of all points of the system successively by $x_1, x_2, \ldots, x_n$, the associated masses by $m_1, m_2, \ldots, m_n$, and the corresponding components of the external force, which shall depend upon only the coordinates x_i, by $X_1, X_2, \ldots, X_n$. Gauss's principle of least constraint then demands that for any given system (x_i, x'_i) of the coordinates and velocities, the expression:

(1) $$P \equiv \sum_{i=1}^{n} m_i \left(x''_i - \frac{X_i}{m_i} \right)^2 \equiv P(x''_i)$$

will be made a minimum for the correct choice of accelerations x''_i , while the coordinates x_i satisfy a number of inequalities:

(2) $$f_1(x_i) \le 0, \qquad f_2(x_i) \le 0, \qquad \dots$$

Now, as Mayer showed (*loc. cit.*), for a given state of motion (x_i, x'_i), we need to consider only those r inequalities $f_\mu \le 0$ for which not only the functions f_μ themselves vanish at the given moment, but also their first total derivatives with respect to time f'_μ, so one has the $2r$ equations:

(3) $$f_\mu(x_i) = 0, \quad f'_\mu(x_i, x'_i) = 0 \qquad (\mu = 1, 2, \dots, r),$$

and conditions (2), if they are to remain generally valid, then demand that one must also have:

(4) $$f''_\mu(x''_i) \equiv f''_\mu(x_i, x'_i, x''_i) \equiv \sum_{i=1}^{n} \frac{\partial f_\mu}{\partial x_i} x''_i + F_\mu(x_i, x'_i) \le 0\,.$$

If we are given a system of quantities (x_i, x'_i) that satisfies the conditions (3) then the unknowns x''_i shall make the expression P a minimum under the auxiliary conditions (4).

From Mayer's method, the determination of the minimum comes about as follows: We assume that for *some* of the functions f_μ, one might have the equal sign in the conditions, while one has the inequality sign for the remaining ones, such that we have:

(4.a) $$f''_\mu(x''_i) = 0 \qquad (\mu = 1, 2, \dots, k),$$

(4.b) $$f''_\mu(x''_i) < 0 \qquad (\mu = k+1, k+2, \dots, r).$$

In the case of a minimum, the accelerations must then have the form:

(5) $$m_i\, x''_i = X_i - \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial x_i} \qquad (i = 1, 2, \dots, n),$$

and the k factors λ_μ can be determined uniquely from the k equations (4.a), which are *linear* in the x''_i, and also in the λ_μ, after the substitution (5), assuming that their determinant Δ_k does not vanish (cf., Mayer, *loc. cit.*, pp. 234). Now those values λ_μ must themselves all be ≤ 0, when the assumption that was made about the distribution of the functions f_μ over the classes (4.a) and (4.b) is supposed to be correct (Mayer, pp. 236), and they must satisfy the $r - k$ conditions (4.b) by means of the substitution (5), in addition. However, if such a solution has been found then it will be the only one, and the problem will be solved completely. As I will show, the expression P will then, in fact, assume a *smaller* value than it does for *all other* systems of values x''_i that satisfy the conditions (4), which excludes the possibility of further secondary minima, and the *uniqueness* of the solution will then be proved.

Namely, if one replaces each x''_i with $x''_i + \omega_i$ in the expression P then one will get:

$$P(x''_i + \omega_i) \equiv \sum_{i=1}^{n} m_i \left(x''_i - \frac{X_i}{m_i} \right)^2 + 2 \sum_{i=1}^{n} m_i\, \omega_i \left(x''_i - \frac{X_i}{m_i} \right) + \sum_{i=1}^{n} m_i\, \omega_i^2$$

(6)
$$\equiv P(x_i'') + 2\,P_1(x_i'', \omega_i) + P_2(\omega_i)\,.$$

However, on the basis of (5), one will have:

$$P_1(x_i'', \omega_i) = -\sum_{i=1}^{n} m_i\, \omega_i \sum_{\mu=1}^{k} \frac{\lambda_\mu}{m_i} \frac{\partial f_\mu}{\partial x_i}$$
$$= -\sum_{\mu=1}^{n} \lambda_\mu \left[f_\mu''(x_i + \omega_i) - f_\mu''(x_i) \right] \geq 0,$$

since, due to (4):

$$f_\mu''(x_i'' + \omega_i) = f_\mu''(x_i) + \sum_{i=1}^{n} \omega_i \frac{\partial f_\mu}{\partial x_i} \leq 0,$$

and from (4.a), one should have $f''_\mu(x''_i) = 0$ for $\mu \leq k$, while all other λ_μ are positive. Therefore:

$$P(x''_i + \omega_i) \geq P(x''_i) + P_2(\omega_i) > P(x''_i)\,,$$

as asserted, as long as not all quantities ω_i vanish simultaneously.

I shall add a second proof of the existence of a *single* minimum that admits an especially simple geometric interpretation when we first use the linear substitution:

$$\sqrt{m_i}\, x_i'' - \frac{X_i}{\sqrt{m_i}} = \xi_i \qquad (i = 1, 2, \ldots, n)$$

to put the expression P into the form:

(1′)
$$P \equiv \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2 \equiv P(\xi_i),$$

while the conditions (4) will go to the following ones:

(4′)
$$\varphi_\mu(\xi_i) \equiv a_{\mu 1}\, \xi_1 + a_{\mu 2}\, \xi_2 + \ldots + a_{\mu n}\, \xi_n \leq 0 \qquad (\mu = 1, 2, \ldots, r)\,.$$

Here, the r functions φ_μ are once more *linear* in the new unknowns ξ_i , so they represent a system of $(n - 1)$-dimensional planes $\varphi_\mu = 0$ that cut out a subset R' in the n-dimensional space of coordinates ξ_i that satisfies the conditions (4′). The problem is now to make either the quadratic form $P = \rho^2$ or the *distance* ρ *from the coordinate origin to a point* (ξ_i) *in the subset* R' a minimum under the conditions (4′). R' must possess *at least one* shortest distance ρ_0 from the point O in any event, since the "lower bound" of a continuous function in a continuous region will also always represent a minimum at the same time. One must therefore show only that this minimum $P_0 = \rho_0^2$ will be assumed at *only one* location in R', and that will follow simply from the fact that the subset R' is *simply connected and everywhere convex*. Namely, if two points (ξ'_i) and (ξ''_i) lie on the same side of the boundary plane, such that they satisfy one of the conditions (4′), then the same thing will be true of all points that lie between them along their connecting line, and the same thing will also be true for all of those r conditions. Thus, if both points A', A'' belong to the region R' then then that connecting line $A'A''$ will also lie completely in its interior, and when both of them have the *same* distance $\rho' = \rho''$ from O, *all* points between A' and A'' will have a *smaller* distance, and ρ' cannot be at the shortest one. The same thing is even more true when the distance ρ'' to A'' is already shorter than the distance to A', and there will then be points along the aforementioned connecting line that are *arbitrarily close* to A' and have distances $< \rho'$. There can then be no *secondary* minima either, but only a single absolute minimum $P_0 = \rho_0^2$ that will be assumed at only a single location.

Analytically, that proof takes the following form: If two systems of values (ξ'_i) and (ξ''_i) satisfy the conditions (4′) then one will also have:

$$\xi_i = (1-\vartheta)\,\xi'_i + \vartheta\,\xi''_i \qquad (i = 1, 2, \ldots, n;\ 0 < \vartheta < 1),$$
$$\varphi_\mu(\xi_i) = (1-\vartheta)\,\varphi_\mu(\xi'_i) + \vartheta\,\varphi_\mu(\xi''_i) \leq 0$$

for both systems.

Furthermore, one will then have:

$$P(\xi_i) = (1-\vartheta)^2\, P(\xi'_i) + \vartheta\, P(\xi''_i) + 2\vartheta(1-\vartheta) P_1(\xi'_i, \xi''_i),$$

where $P_1(\xi'_i, \xi''_i)$, as in (6), is a bilinear form (viz., the polar form of P), so when one assumes that:

$$\rho''^2 = P(\xi''_i) \leq P(\xi'_i) = \rho'^2,$$

one will have:

$$P(\xi_i) = P(\xi'_i) - \vartheta[P(\xi_i) - P(\xi''_i)] - \vartheta(1-\vartheta)[P(\xi'_i) + P(\xi''_i) - 2P_1(\xi'_i, \xi''_i)] < P(\xi'_i),$$

since:

$$P(\xi'_i) + P(\xi''_i) - 2P_1(\xi'_i, \xi''_i) \equiv P(\xi'_i - \xi'')$$

is positive, as a definite quadratic form in the n variables $\xi'_i - \xi''_i$.

"Sur une forme générales des équations de la dynamique et sur le principe de Gauss," J. reine angew. Math. **122** (1900), 205-208.

On a general form of the equations of dynamics and Gauss's principle

(By **P. Appell** at St. Germain-en-Laye)

Translated by D. H. Delphenich

We published a paper "Sur une forme générales des équations de la dynamique" on pp. 310 in v. 121 of this journal. We now ask permission to present two complementary remarks in regard to that subject about **Gauss**'s principle of least constraint, one of which is of a mathematical order, while the other is of a bibliographic order.

1.

The **Lagrange** equations are applicable when the constraints on a system without friction can be expressed in finite terms, and when one employs parameters that are true coordinates. Suppose, to simplify, that there exists a force function U. One can then write the equations of motion once one knows the expressions for one-half the *vis viva* T and U as functions of the independent parameters.

On the contrary, if the constraints cannot all be expressed by relations in finite terms then one can no longer apply **Lagrange**'s equations. In order to write out the equations of motion, it suffices to know U and the function $S = \frac{1}{2}\sum m J^2$, which is composed from the accelerations in the same way that T is composed from the velocities. But is that necessary?

Might there not exist equations of motion that are more general than **Lagrange**'s that are applicable to all cases and require only that one must know the two functions T and U in order to write them down? We shall show that such equations do not exist. In order to do that, we shall indicate two different systems in which the functions T and U are identically the same, although the equations of motion are not the same.

First system: Imagine a ponderous solid that fulfills the following conditions:

1. The solid is bounded by a sharp edge that has the form of a circle K of radius a.

2. The center of gravity G of the body is situated at the center of the circle K.

3. The ellipsoid of inertia that relates to the center of gravity G is an ellipsoid of revolution around the perpendicular Gz to the plane of the circle.

Now suppose that the solid body, thus-constructed, is subject to rolling without slipping on a fixed horizontal plane and that it touches the circular edge K.

Let $G\alpha$ be the ascending vertical that is drawn through G, take the Gy axis to be the perpendicular to the plane αGz and the Gx to be the perpendicular to the plane yGz. Gy is then a horizontal to the plane of the circle K, and Gx is a line of greatest slope to that plane that ends at the point where the circle touches the fixed plane. Let Θ denote the angle between Gz and the ascending vertical $G\alpha$, and let ψ be the angle between Gy and a fixed horizontal. Those two angles determine the orientation of the trihedron $Gxyz$. In order for fix the position of the solid body with respect to the trihedron $Gayz$, it will suffice to know the angle φ that a radius of the circle K that is invariably coupled with the body makes with the axis Gy. The instantaneous rotation ω of the body is then the resultant of the rotation of the trihedron and a rotation $d\varphi/dt = \varphi'$ around Gz. The components p, q, r of ω are then:

$$p = -\psi' \sin\Theta, \qquad q = \Theta', \qquad r = \psi' \cos\Theta + \varphi'.$$

On the other hand, the condition that the circle K is rolling shows that the square of the velocity of the center of gravity G will be $a^2(q^2 + r^2)$. By definition, if one takes the mass of the body to be unity and lets A and C denote the moments of inertia about Gx and Gy, respectively, then one will have:

$$2T = a^2(q^2 + r^2) + A(p^2 + q^2) + C r^2,$$

so, one will have:

$$\text{(1)} \qquad \begin{cases} 2T = A\psi'^2 \sin^2\Theta + (A + a^2)\Theta'^2 + (C + a^2)(\psi'\cos\Theta + \varphi')^2 \\ U = -ga\sin\Theta \end{cases}$$

as the defining expressions for the functions T and U.

Second system: Let a second ponderable body have the same form, the same radius a, and the same mass as before. Imagine that the distribution of the mass is different, in such a way that if one lets A_1 and C_1 denote the moments of inertia that are analogous to A and C, resp., then one will have:

$$A_1 = A, \qquad C_1 = C + a^2.$$

Subject the body to the following two constraints: The body touches a fixed horizontal plane P_1 on which it slides without friction at the circular edge K. The center of gravity G of the body slides without friction on a fixed vertical circumference whose radius is a and whose center O is in the fixed plane P_1.

In order to express those constraints, we take the same moving axes $Gxyz$ and the same notations as above. Let x_1, y_1, z_1 denote the absolute coordinates of the point G with respect to the two axes Ox_1 and Oy_1 in the plane P_1 and an ascending vertical Oz_1. One can suppose that the fixed vertical circumference that is described by G is in the plane x_1Oz_1. One will then have:

$$\textit{First constraint:} \quad z_1 = a \sin \Theta,$$

$$\textit{Second one:} \qquad y_1 = 0, \qquad x_1^2 + y_1^2 = a^2,$$

so one obviously has:

$$x_1 = a \cos \Theta.$$

Under those conditions, one has:

$$2T_1 = x_1'^2 + y_1'^2 + z_1'^2 + A_1 (p^2 + q^2) + C_1 r^2,$$

or, from the values of x_1, y_1, z_1, A_1, and C_1:

$$(2) \qquad \begin{cases} 2T_1 = A\psi'^2 \sin^2\Theta + (A + a^2)\,\Theta'^2 + (C + a^2)(\psi' \cos\Theta + \varphi')^2, \\ U_1 = -ga \sin\Theta. \end{cases}$$

One sees that the functions T and T_1, U and U_1 are identical. Meanwhile, the equations of motion are different since **Lagrange**'s equations apply to the second system and not to the first. That is what we would like to show.

One can point out that of the three equations of motion, two of them can be put into the same form in the two systems. Indeed, the integral of the *vis viva* is obviously the same for both of them. Moreover, as **Slesser** has already shown in an article in the Quarterly Journal of Mathematics (1873), one has the right to write down the **Lagrange** equation that relates to Θ for the first system, which one can obviously do for the second one. However, the third equations are different for the two motions: For the second system, one has the integral $r = r_0$, which does not exist for the first one.

It is obvious that the difference between the two motions will appear immediately when one forms the two functions S and S_1 by applying the formulas in our preceding paper. (See also Journal de Mathématiques pures et appliqués, first fascicle, 1900.)

2.

Bibliographic notes. At the end of the preceding paper, we gave some very quick and very incomplete indications in regard to the analytical statement of **Gauss**'s principle. **A. Mayer** of Leipzig has been most helpful in providing the following historical and bibliographic information: The analytical statement of **Gauss**'s principle was indicated already by **Jacobi** in a lecture that is no longer in print. It was given, independently of **Jacobi**, by **Scheffler** (Volume III of Schlömilch's Zeitschrift, pp. 197). It was found to be reproduced in **Mach** (*Die Mechanik in ihrer Entstehung historisch-kritisch dargestellt*, Laipzig, 1883), in **Hertz**, which we have cited, and in **Boltzmann** (*Vorlesungen über die Principe der Mechanik*, Leipzig, 1897). Finally, **J. Willard**

Gibbs, in a beautiful paper "On the fundamental formulae of Dynamics" (American Journal of Mathematics, vol. II, 1879), gave the analytical statement of **Gauss**'s principle and some applications to various problems, and notably to the question of the rotation of solid bodies.

"Das Restglied bei der Transformation des Zwanges in allgemeine Coordinaten," Sitz. kaiserliche Akad. Wiss. math.-phys. Klasse, Abt. II.a **110** (1901), 387-413.

The remainder term for the transformation of the constraint into general coordinates

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It was only in recent times that the meaning of **Gauss**'s principle of least constraint first came to light, once **Gibbs** showed that under some circumstances the principle accomplished more than that of virtual displacements, and **Hertz** and **Boltzmann** assigned a leading role to **Gauss**'s principle in their representations of mechanics.

If one considers n points whose rectangular coordinates are $x_{3\mu-2}$, $x_{3\mu-1}$, $x_{3\mu}$, and $m_{3\mu-2} = m_{3\mu-1} = m_{3\mu}$ is the mass of the μ^{th} point, while $X_{3\mu-2}$, $X_{3\mu-1}$, $X_{3\mu}$ are the components of the external forces that act upon that point then the principle of least constraint says that the expression:

$$Z = \sum_{i=1}^{3n} m_i \left[\ddot{x}_i - \frac{X_i}{m_i} \right]^2$$

must be a minimum for all of the accelerations $\ddot{x}_i$ that are compatible with the conditions when *only* the accelerations $\ddot{x}_i$ change, while the velocities $\dot{x}_i$ and the coordinates x_i can be regarded as *constant*. If one sets $m\,\ddot{x}_i - X_i = y_i$, to abbreviate, then the expression for Z will become:

$$Z = \sum_{i=1}^{3n} \frac{1}{m_i} y_i^2 .$$

For the applications to many physical problems, it is preferable to introduce the so-called general coordinates (which are briefly called the parameters $p_1, p_2, \ldots, p_k$ here), which fulfill the $(3n - k)$ condition equations that constrain the system identically, in place of the rectangular ones. The $3n$ rectangular coordinates x_i appear as functions of those k parameters of the form $x_i = f_i\,[p_1, p_2, \ldots, p_k]$, and should there be further differential quotients with respect to the p, then they will be indicated by more indices. One would then have, e.g., $f_{123} = \frac{\partial^2 f_1}{\partial p_2\, \partial p_3}$, $f_{23} = \frac{\partial f_1}{\partial p_3}$, etc. As **Lipschitz** ([1]) showed, then **Wassmuth** ([2]) and **Radakovich** ([3]) in various ways, the expression for Z will then be converted into:

$$Z = Z' + \Phi,$$

in which Z' represents a function of all accelerations $\ddot{p}_1, \ddot{p}_2, \ldots, \ddot{p}_k$ of the form that was ascertained before, while Φ does not include *any* of those accelerations, and has still not been determined up to now.

The quantity Z' has the value:

$$Z' = \frac{1}{D} \sum_{\nu=1}^{k} \sum_{\mu=1}^{k} A_{\nu\mu}\, Q_\nu\, Q_\mu .$$

The meaning of the various symbols is illuminated as follows: If one calls the *vis viva* L then one will have:

$$2L = \sum_{i=1}^{3n} m_i\, \dot{x}_i^2 = \sum_{\nu=1}^{k} \sum_{\mu=1}^{k} a_{\nu\mu}\, \dot{p}_\nu\, \dot{p}_\mu ,$$

when one replaces $\ddot{x}_i$ with the values that are obtained from $x_i = f_i\,(p_1, \ldots, p_k)$ upon differentiating with respect to time; one will find that:

$$a_{\nu\mu} = \sum_{i=1}^{3n} m_i\, f_{i\nu}\, f_{i\mu} ,$$

and one now lets D denote the determinant of the a, which is known to not vanish identically, such that $D = |\, a_{\kappa\lambda} \,|$, with the subdeterminant $A_{\kappa\lambda}$. Furthermore, one has:

$$Q_\mu = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_\mu}\right) - \frac{\partial L}{\partial p_\mu} - \sum_{i=1}^{3n} X_i\, f_{i\mu} = \sum_{\kappa=1}^{k} a_{\kappa\mu}\, \ddot{p}_\kappa + \sum_{\kappa=1}^{k} \sum_{\lambda=1}^{k} \begin{bmatrix} \kappa\, \lambda \\ \mu \end{bmatrix} \dot{p}_\kappa \dot{p}_\lambda - P_\mu ,$$

in which $P_\mu = \sum_{i=1}^{3n} X_i\, f_{i\mu}$, and one defines the symbol:

$$\begin{bmatrix} \kappa\, \lambda \\ \mu \end{bmatrix} = \frac{1}{2}\left[\frac{\partial a_{\kappa\mu}}{\partial p_\lambda} + \frac{\partial a_{\lambda\mu}}{\partial p_\kappa} - \frac{\partial a_{\kappa\lambda}}{\partial p_\mu}\right].$$

The problem shall now be solved of *likewise determining function Φ, which is unknown up to now, and of which one knows that it does not include the accelerations $\ddot{p}_1, \ddot{p}_2, \ldots, \ddot{p}_\mu$.*

([1]) **Lipschitz**, Borch. Journ., Bd. **82**, pp. 323.
([2]) **Wassmuth**, These Sitzungsberichte, Abt. II.a, Bd. **104** (1895).
([3]) **Radakovic**, Wiener Monatshefte f. Math. u. Physik, **7** (1896).

Now, as far as the uses of such a determination of Φ are concerned, one can probably object that this function will drop our *completely*, as soon as one applies the principle to practical cases – i.e., differentiates Z ($Z' + \Phi$, resp.) with respect to the accelerations $\ddot{p}_\mu$ or the quantities Q_μ – by which, one will get the Lagrange equations:

$$Q_1 = 0, \qquad Q_2 = 0, \ldots, \qquad Q_k = 0 .$$

On the other hand, one cannot deny that Z gives the expression Z' only *incompletely*, and that there can be physical problems in which it might of interest to us to know the *complete* function that should assume a minimal value. Now, as far as the solution to the problem is concerned, the relation:

$$\Phi = Z - Z'$$

will yield a very general one, where everything on the right-hand side is known from the formulas above, and the accelerations on the right can be taken to be $\ddot{p}_1 = \ddot{p}_2 = \ldots = \ddot{p}_k = 0$, *a priori*, since those accelerations must cancel unconditionally in $Z - Z'$. The expression for Φ that is obtained in that way is, however, neither clear not useful for physical applications. Both conditions correspond to a different form that I will obtain when I represent Φ *in a different way as a sum of squares.*

Namely, as I have already proved before, when one takes $y_i = m\,\ddot{x}_i - X_i$, as above, one can also set Q_μ equal to:

$$Q_\mu = \sum_{i=1}^{3n} y_i\, f_{i\mu}.$$

That important formula is obtained from the known identity:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_\mu}\right) - \frac{\partial L}{\partial p_\mu} = \sum_{i=1}^{3n}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i}\right]\frac{\partial x_i}{\partial p_\mu},$$

among others, in which one sets:

$$L = \tfrac{1}{2}\sum_{i=1}^{3n} m_i\, \dot{x}_i^2, \quad \frac{\partial L}{\partial x_i} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = m_i\, \ddot{x}_i, \quad \frac{\partial x_i}{\partial p_\mu} = f_{i\mu},$$

such that:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_\mu}\right) - \frac{\partial L}{\partial p_\mu} = \sum_{i=1}^{3n} m_i\, \ddot{x}_i\, f_{i\mu},$$

and since:

$$P_\mu = X_1 \frac{\partial x_1}{\partial p_\mu} + \cdots = \sum_{i=1}^{3n} X_i\, f_{i\mu},$$

upon subtraction, one will get:

$$Q_\mu = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_\mu}\right) - \frac{\partial L}{\partial p_\mu} - P_\mu = \sum_{i=1}^{3n} y_i\; f_{i\mu} = y_1\; f_{1\mu} + y_2\; f_{2\mu} + \cdots + y_{3n}\; f_{3n,\mu}\,.$$

The Q_μ are then *linear* functions of the quantities y_i , and therefore:

$$\Phi = \sum_{i=1}^{3n} \frac{y_i^2}{m_i} - \frac{1}{D}\sum_{\nu=1}^{k}\sum_{\mu=1}^{k} A_{\nu\mu}\, Q_\nu\, Q_\mu = \sum_{i=1}^{3n} \frac{y_i^2}{m_i} - \frac{1}{D}\sum_{\nu=1}^{k}\sum_{\mu=1}^{k} A_{\nu\mu} \cdot \sum_{r=1}^{3n} y_r\, f_{r\nu} \cdot \sum_{s=1}^{3n} y_s\, f_{s\mu}$$

appears to be a *quadratic form* in the y_i .

Now, any acceleration $\ddot{x}_i$, and therefore y_i, as well, depends upon the accelerations $\ddot{p}_\nu$ linearly by way of the formula:

$$\ddot{x}_i = \sum_{\nu=1}^{k} f_{i\nu}\; \ddot{p}_\nu + \sum_{\nu=1}^{k}\sum_{\mu=1}^{k} f_{i\nu\mu}\; \dot{p}_\nu\; \dot{p}_\mu\,.$$

Now if Φ, as a homogeneous quadratic function of the y, is *not* supposed to contain the $\ddot{p}_\nu$ then one must construct *certain expressions in y that do not contain any of the accelerations* $\ddot{p}_\nu$ *and which appear in quadratic form in* Φ.

We then have a sequence of such y that must be combined into a linear expression such that the accelerations $\ddot{p}_i$ can be eliminated from it, and we will then suspect that the squares or products of every two such expressions will enter into Φ . For the investigation of that, it is advisable to proceed from the simpler cases to the more general ones and to determine the number of mutually independent quantities in each special case.

Two points and one parameter

Two points with the masses $m_1 = m_2 = m_3$ and $m_4 = m_5 = m_6$ have the coordinates x_1, x_2, x_3 and x_4, x_5, x_6, respectively, and one has:

$$x_1 = f_1\,(p_1), \qquad x_2 = f_2\,(p_1), \qquad \ldots, \qquad x_6 = f_6\,(p_1),$$

where p_1 represents a general coordinate (or parameter). One then gets the components of the velocity from that:

$$\dot{x}_1 = f_{11}\cdot\dot{p}_1, \quad \dot{x}_2 = f_{21}\cdot\dot{p}_1, \quad \ldots, \quad \dot{x}_6 = f_{61}\cdot\dot{p}_1,$$

as well as the accelerations:

$$\ddot{x}_1 = f_{11}\cdot\ddot{p}_1 + f_{111}\,\dot{p}_1^2, \quad \ddot{x}_2 = f_{21}\cdot\ddot{p}_1 + f_{211}\,\dot{p}_1^2, \quad \ldots, \quad \ddot{x}_6 = f_{61}\cdot\ddot{p}_1 ++ f_{611}\,\dot{p}_1^2\,.$$

If one multiplies the latter expressions by the associated masses $m_1, m_2, \ldots, m_6$ and sets $m\,\ddot{x}_i - X_i = y_i$, as above, then that will produce the six equations:

$$\begin{aligned} y_1 &= m_1\, f_{11} \cdot \ddot{p}_1 + m_1\, f_{111}\, \dot{p}_1^2 - X_1, \\ y_2 &= m_2\, f_{21} \cdot \ddot{p}_1 + m_2\, f_{211}\, \dot{p}_1^2 - X_2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_6 &= m_6\, f_{61} \cdot \ddot{p}_1 + m_6\, f_{611}\, \dot{p}_1^2 - X_6\,. \end{aligned}$$

Eliminating $\ddot{p}_1$ from each *two* of those equations will produce fifteen combinations that are *free* of the acceleration $\ddot{p}_1$. For example, combining the second and fourth equation will give the result that the determinant:

$$\begin{vmatrix} y_2 - (m_2\, f_{211}\, \dot{p}_1^2 - X_2) & m_2\, f_{21} \\ y_4 - (m_4\, f_{411}\, \dot{p}_1^2 - X_4) & m_4\, f_{41} \end{vmatrix} = 0,$$

and therefore, the expression:

$$u_{24} = \begin{vmatrix} y_2 & m_2\, f_{21} \\ y_4 & m_4\, f_{41} \end{vmatrix} = y_2 \cdot m_4\, f_{41} - y_4 \cdot m_2\, f_{21}$$

will be *free* of $\ddot{p}_1$. Now, since Φ is a homogeneous, quadratic form in the $y_1, y_2, \ldots, y_6$ that *cannot* include the acceleration $\ddot{p}_1$, one might expect that Φ will include the fifteen quantities $u_{12}, u_{13}, u_{14}, \ldots, u_{56}$. Namely, it can be *proved* that one has:

$$D \cdot \Phi = a_{11} \cdot \Phi = \frac{1}{m_1\, m_2} u_{12}^2 + \frac{1}{m_1\, m_3} u_{13}^2 + \cdots + \frac{1}{m_5\, m_6} u_{56}^2\,, \tag{II}$$

and therefore, Φ will be a sum of squares. Namely, one has:

$$D = a_{11} = m_1\, f_{11}^2 + m_2\, f_{21}^2 + \cdots + m_6\, f_{61}^2$$

and

$$a_{11} \cdot \Phi = a_{11} \left| \frac{y_1^2}{m_1} + \cdots + \frac{y_6^2}{m_6} \right| - [y_1\, f_{11} + y_2\, f_{21} + \cdots + y_6\, f_{61}]^2. \tag{I}$$

Next, the terms with y_1 or y_2 will yield:

$$\begin{aligned} &\frac{y_1^2}{m_1} a_{11} + \frac{y_2^2}{m_2} a_{11} - y_1^2\, f_{11} + y_2^2\, f_{21}^2 - 2 y_1\, y_2\, f_{21} \\ &\qquad = \left(\frac{y_1^2}{m_1} + \frac{y_2^2}{m_2} \right) (m_3\, f_{31}^2 + m_4\, f_{41}^2 + m_5\, f_{51}^2 + m_6\, f_{61}^2) + \frac{1}{m_1\, m_2} u_{12}\,. \end{aligned}$$

If one further adds the terms with y_3^2, $2y_1\, y_3$, $2y_2\, y_3$ then that will give:

$$\frac{y_3^2}{m_3} a_{11} - y_3^2 f_{31}^2 - 2y_1\, y_3\, f_{11}\, f_{31} - 2y_2\, y_3\, f_{21}\, f_{31}$$

$$= \frac{y_3^2}{m_3} [m_1 f_{11}^2 + m_2 f_{21}^2 + m_4 f_{41}^2 + m_5 f_{51}^2 + m_6 f_{61}^2] - 2y_1\, y_3\, f_{11}\, f_{31} - 2y_2\, y_3\, f_{21}\, f_{31} \,,$$

and the sum of the terms that were considered up to now will then be:

$$\left[\frac{y_1^2}{m_1} + \frac{y_2^2}{m_2} + \frac{y_3^2}{m_3}\right] [m_4 f_{41}^2 + m_5 f_{51}^2 + m_6 f_{61}^2] + \frac{1}{m_1\, m_2} u_{12}^2 + \frac{1}{m_1\, m_3} u_{13}^2 + \frac{1}{m_2\, m_3} u_{23}^2 \,.$$

If one further considers the terms with y_4^2, $2y_1\, y_4$, $2y_2\, y_4$, $2y_3\, y_4$ then that will give:

$$\frac{y_4^2}{m_4} a_{11} + y_4^2\, f_{41}^2 - 2y_1\, y_4\, f_{11}\, f_{41} - 2y_2\, y_4\, f_{21}\, f_{41} - 2y_3\, y_4\, f_{31}\, f_{41}$$

$$= \frac{y_4^2}{m_4} [m_1\, f_{11}^2 + m_2\, f_{21}^2 + m_3\, f_{31}^2 + m_5\, f_{51}^2 + m_6\, f_{61}^2] - 2y_1\, y_4\, f_{11}\, f_{41} - 2y_2\, y_4\, f_{21}\, f_{41} - 2y_3\, y_4\, f_{31}\, f_{41} \,,$$

and the sum of the terms that were singled out up to now will be:

$$\left[\frac{y_1^2}{m_1} + \frac{y_2^2}{m_2} + \frac{y_3^2}{m_3} + \frac{y_4^2}{m_4}\right] [m_5 f_{51}^2 + m_6 f_{61}^2] + \frac{1}{m_1\, m_2} u_{12}^2 + \frac{1}{m_1\, m_3} u_{13}^2 + \frac{1}{m_2\, m_3} u_{23}^2$$

$$+ \frac{1}{m_1\, m_4} u_{14}^2 + \frac{1}{m_2\, m_4} u_{24}^2 + \frac{1}{m_3\, m_4} u_{34}^2 \,.$$

If one now adds the terms with y_5^2, $2y_1\, y_5$, $2y_2\, y_5$, $2y_3\, y_5$, $2y_4\, y_5$, i.e.:

$$\frac{y_5^2}{m_5} [m_1\, f_{11}^2 + m_2\, f_{21}^2 + m_3\, f_{31}^2 + m_4\, f_{41}^2 + m_6\, f_{61}^2]$$

$$- 2y_1\, y_5\, f_{11}\, f_{51} - 2y_2\, y_5\, f_{21}\, f_{51} - 2y_3\, y_5\, f_{31}\, f_{51} - 2y_4\, y_5\, f_{41}\, f_{51} \,,$$

then the sum will become:

$$\left[\frac{y_1^2}{m_1} + \frac{y_2^2}{m_2} + \frac{y_3^2}{m_3} + \frac{y_4^2}{m_4} + \frac{y_5^2}{m_5}\right] \cdot m_6\, f_{61}^2 + \frac{1}{m_1\, m_2} u_{12}^2 + \cdots + \frac{1}{m_3\, m_4} u_{34}^2 + \frac{1}{m_1\, m_5} u_{15}^2 + \cdots + \frac{1}{m_4\, m_5} u_{45}^2 \,.$$

Finally, the terms that still remain in $a_{11}\, \Phi$ will be:

$$\frac{y_6^2}{m_6} [m_1\, f_{11}^2 + \cdots + m_5\, f_{51}^2] - 2y_1\, y_6\, f_{11}\, f_{61} - 2y_2\, y_6\, f_{21}\, f_{61} - 2y_3\, y_6\, f_{31}\, f_{61} - 2y_4\, y_6\, f_{41}\, f_{61} - 2y_5\, y_6\, f_{51}\, f_{61} \,,$$

and when they are added, that will yield:

$$a_{11}\cdot\Phi = \frac{1}{m_1\,m_2}u_{12}^2 + \frac{1}{m_1\,m_3}u_{13}^2 + \frac{1}{m_2\,m_3}u_{23}^2 + \frac{1}{m_1\,m_4}u_{14}^2$$
$$+\frac{1}{m_2\,m_4}u_{24}^2 + \frac{1}{m_3\,m_4}u_{34}^2 + \frac{1}{m_1\,m_5}u_{15}^2 + \frac{1}{m_2\,m_5}u_{25}^2$$
$$+\frac{1}{m_3\,m_5}u_{35}^2 + \frac{1}{m_4\,m_5}u_{45}^2 + \frac{1}{m_1\,m_6}u_{16}^2 + \frac{1}{m_2\,m_6}u_{26}^2$$
$$+\frac{1}{m_3\,m_6}u_{36}^2 + \frac{1}{m_4\,m_6}u_{46}^2 + \frac{1}{m_5\,m_6}u_{56}^2,$$

i.e. the form (II). (The method of proof can clearly be extended to more than six quantities y, as long as one assumes a single p_1 .)

The essence of this conversion consists of the fact that the quadratic form in six variables y in question can be represented as the sum of squares of more than six linear functions of the y_i that are naturally no longer independent of each other and can be chosen in such a way that they no longer contain the accelerations $\ddot{p}$.

One linear relation exists between any three u with the indices $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$; e.g., from their definitions, u_{12}, u_{23}, u_{31} will yield:

$$u_{12} = y_1\cdot m_2\,f_{21} - y_2\cdot m_1\,f_{11} = -\,u_{13} = y_3\cdot m_1\,f_{11} - y_1\cdot m_3\,f_{31}\,,$$

$$u_{23} = y_2\cdot m_3\,f_{31} - y_3\cdot m_2\,f_{21}\,,$$

such that the three quantities u_{12}, u_{23}, u_{31} of the combination 123 are then coupled by the relation:

$$u_{12}\cdot m_3\,f_{31} + u_{23}\cdot m_1\,f_{11} + u_{31}\cdot m_2\,f_{21} = 0\,,$$

and perhaps u_{23} is expressible in terms of u_{12} and u_{13} . Similarly, a consideration of the combination 124 will imply that u_{24} is expressible in terms of u_{12} and u_{14} . If one then forms *all* combinations of three elements in that way then one will convince oneself that of all the fifteen quantities u_{12}, …, u_{56}, only *five* of them – say, u_{12}, u_{13}, u_{14}, u_{15}, and u_{16} – can be regarded as independent of each other.

That agrees with the knowledge that Φ is independent of the acceleration $\ddot{p}_1$, and therefore $\partial\Phi / \partial\ddot{p}_1 = 0$; i.e., since $y_i = m_i\,f_{i1}\,\ddot{p}_1 + m_i\,f_{i11}\,\dot{p}_1^2 - X_i$, one must have:

$$m_1\ f_{11}\frac{\partial\Phi}{\partial y_1} + m_2\ f_{21}\frac{\partial\Phi}{\partial y_2} + \cdots + m_6\ f_{61}\frac{\partial\Phi}{\partial y_6} = 0.$$

There then exists a linear relation between the partial differential quotients of the quadratic form Φ with respect to the six quantities y, and therefore, from a known theorem, Φ must be expressible in terms of *five* mutually-independent new variables $z_1, \ldots, z_5$ that depend upon $y_1, \ldots, y_6$ linearly.

One can also convince oneself of the validity of equation (II) by differentiating (I), as well as (II), twice with respect to the y and verifying the equality of the differential quotients that are obtained in that way. One will obtain from (I) that:

$$\tfrac{1}{2} a_{11} \frac{\partial \Phi}{\partial y_1} = a_{11} \frac{y_1}{m_1} - [y_1 \, f_{11} + y_2 \, f_{21} + \cdots + y_6 \, f_{61}] \cdot f_{11}$$

and

$$\tfrac{1}{2} a_{11} \frac{\partial^2 \Phi}{\partial y_1^2} = \frac{a_{11}}{m_1} - f_{11}^2 = \frac{1}{m_1} [m_2 \, f_{21}^2 + m_3 \, f_{31}^2 + \cdots + m_6 \, f_{61}^2].$$

Similarly, (II) implies that:

$$\tfrac{1}{2} a_{11} \frac{\partial \Phi}{\partial y_1} = \frac{1}{m_1} [m_2 \, f_{21} + m_3 \, f_{31} + \cdots + m_6 \, f_{61}]$$

and

$$\tfrac{1}{2} a_{11} \frac{\partial^2 \Phi}{\partial y_1^2} = \frac{1}{m_1} [m_2 \, f_{21}^2 + m_3 \, f_{31}^2 + \cdots + m_6 \, f_{61}^2] ,$$

as before.

In a similar way, it will follow from (I) that:

$$\tfrac{1}{2} a_{11} \frac{\partial^2 \Phi}{\partial y_1 \, \partial y_2} = - f_{21} f_{11} ,$$

and from (II) that:

$$\tfrac{1}{2} a_{11} \frac{\partial^2 \Phi}{\partial y_1 \, \partial y_2} = \frac{1}{m_1} \, f_{21} \cdot \frac{\partial u_{12}}{\partial y_2} = - f_{21} f_{11} ,$$

which proves the agreement.

If one has *just one* point $m_1 = m_2 = m_3 = m$ then:

$$a_{11} = m[\, f_{11}^2 + f_{21}^2 + f_{31}^2],$$

$$a_{11} \cdot \Phi = [y_1 \, f_{21} - y_2 \, f_{11}]^2 + [y_2 \, f_{31} - y_3 \, f_{21}]^2 + [y_3 \, f_{11} - y_1 \, f_{31}]^2,$$

and when $x_3 = 0$ (i.e., the point moves in the $x_1 \, x_2$-plane):

$$a_{11} \cdot \Phi = [y_1 \, f_{21} - y_2 \, f_{11}]^2 .$$

If one sets, e.g., $p_1 = s$, which is the arc-length that a point that moves along a curved line in the plane will describe, then:

$$p_1 = \frac{ds}{dt} = v,$$
$$-\frac{1}{\rho} = \frac{dx_1}{ds}\frac{d^2x_2}{ds^2} - \frac{dx_2}{ds}\frac{d^2x_1}{ds^2} = f_{11}\, f_{211} - f_{21}\, f_{111}, \quad a_{11} = m,$$
$$m\Phi = \left[\frac{mv^2}{\rho} - R\cos(R, N)\right]^2,$$

in which ρ is the radius of curvature, v is the total velocity, R is the resultant of X_1 and X_2, and $\sphericalangle R, N$ represents the angle between R and the normal N. Dr. **Wilhelm** found that intuitive result in a different way in a dissertation that was submitted to this institution.

Two points and two parameters

Let two points with masses $m_1 = m_2 = m_3$ and $m_4 = m_5 = m_6$ be given with the corresponding coordinates:

$$x_1 = f_1\,(p_1, p_2), \qquad x_2 = f_2\,(p_1, p_2), \qquad \ldots, \qquad x_6 = f_6\,(p_1, p_2),$$

where p_1 and p_2 represent general coordinates or parameters. The velocity components of the first point will then be:

$$\dot{x}_1 = f_{11}\,\dot{p}_1 + f_{12}\,\dot{p}_2,$$
$$\dot{x}_2 = f_{21}\,\dot{p}_1 + f_{22}\,\dot{p}_2,$$
$$\dot{x}_3 = f_{31}\,\dot{p}_1 + f_{32}\,\dot{p}_2,$$

while those of the second point will be:

$$\dot{x}_4 = f_{41}\,\dot{p}_1 + f_{42}\,\dot{p}_2,$$
$$\dot{x}_5 = f_{51}\,\dot{p}_1 + f_{52}\,\dot{p}_2,$$
$$\dot{x}_6 = f_{61}\,\dot{p}_1 + f_{62}\,\dot{p}_2.$$

One derives the acceleration components from that from another differentiation with respect to time:

$$\begin{aligned}
\ddot{x}_1 &= f_{11}\,\ddot{p}_1 + f_{12}\,\ddot{p}_2 + f_{111}\,\dot{p}_1^2 + 2 f_{112}\,\dot{p}_1\,\dot{p}_2 + f_{122}\,\dot{p}_2^2,\\
\ddot{x}_2 &= f_{21}\,\ddot{p}_1 + f_{22}\,\ddot{p}_2 + f_{211}\,\dot{p}_1^2 + 2 f_{212}\,\dot{p}_1\,\dot{p}_2 + f_{222}\,\dot{p}_2^2,\\
\ddot{x}_3 &= f_{31}\,\ddot{p}_1 + f_{32}\,\ddot{p}_2 + f_{311}\,\dot{p}_1^2 + 2 f_{312}\,\dot{p}_1\,\dot{p}_2 + f_{322}\,\dot{p}_2^2,\\
\\
\ddot{x}_4 &= f_{41}\,\ddot{p}_1 + f_{42}\,\ddot{p}_2 + f_{411}\,\dot{p}_1^2 + 2 f_{412}\,\dot{p}_1\,\dot{p}_2 + f_{422}\,\dot{p}_2^2,\\
\ddot{x}_5 &= f_{51}\,\ddot{p}_1 + f_{52}\,\ddot{p}_2 + f_{511}\,\dot{p}_1^2 + 2 f_{512}\,\dot{p}_1\,\dot{p}_2 + f_{522}\,\dot{p}_2^2,\\
\ddot{x}_6 &= f_{61}\,\ddot{p}_1 + f_{62}\,\ddot{p}_2 + f_{611}\,\dot{p}_1^2 + 2 f_{612}\,\dot{p}_1\,\dot{p}_2 + f_{622}\,\dot{p}_2^2,
\end{aligned} \tag{1}$$

since, e.g., $f_{412} = f_{421}$, etc.

If those equations are multiplied by the masses $m_1, m_2, \ldots, m_6$, in turn, then one will easily see that there are $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = 20$ linear combinations of the quantities $m_1\,\ddot{x}_1, \ldots, m_6\,\ddot{x}_6$ that are *free* of $\ddot{p}_1$ and $\ddot{p}_2$, namely, the quantities $\ddot{p}_1$ and $\ddot{p}_2$ can be eliminated from any three of equations (1). If one considers, e.g., the first, fourth, and fifth of those equations and sets:

$$m_1\,\ddot{x}_1 - X_1 = y_1\,, \qquad \ldots, \qquad m_5\,\ddot{x}_5 - X_5 = y_5\,, \qquad \text{etc.},$$

as before, and sets:

$$m_1\,[f_{111}\,\dot{p}_1^2 + 2\,f_{112}\,\dot{p}_1\,\dot{p}_2 + f_{122}\,\dot{p}_2^2] - X_1 = -\,\varepsilon_1\,,$$
$$m_4\,[f_{411}\,\dot{p}_1^2 + 2\,f_{412}\,\dot{p}_1\,\dot{p}_2 + f_{422}\,\dot{p}_2^2] - X_4 = -\,\varepsilon_4\,,$$
$$m_5\,[f_{511}\,\dot{p}_1^2 + 2\,f_{512}\,\dot{p}_1\,\dot{p}_2 + f_{522}\,\dot{p}_2^2] - X_5 = -\,\varepsilon_5\,,$$

to abbreviate, then it will follow from the first, fourth, and fifth equation that the determinant is:

$$\begin{vmatrix} y_1 + \varepsilon_1 & m_1\,f_{11} & m_1\,f_{12} \\ y_4 + \varepsilon_4 & m_4\,f_{41} & m_4\,f_{42} \\ y_5 + \varepsilon_5 & m_5\,f_{51} & m_5\,f_{52} \end{vmatrix} = 0\,,$$

or that the expression:

$$u_{145} = \begin{vmatrix} y_1 & m_1\,f_{11} & m_1\,f_{12} \\ y_4 & m_4\,f_{41} & m_4\,f_{42} \\ y_5 & m_5\,f_{51} & m_5\,f_{52} \end{vmatrix}$$

must be free of $\ddot{p}_1$ and $\ddot{p}_2$. With the use of the notation:

$$F_{\kappa\lambda} = -F_{\lambda\kappa} = \begin{vmatrix} f_{\kappa 1} & f_{\kappa 2} \\ f_{\lambda 1} & f_{\lambda 2} \end{vmatrix}$$

developing the elements of the first column will give the relation:

$$u_{145} = y_1 \cdot m_4\,m_5\,F_{45} - y_4 \cdot m_1\,m_5\,F_{15} + y_5 \cdot m_1\,m_4\,F_{14}$$

or

$$\frac{1}{m_1\,m_4\,m_5} \cdot u_{145} = \frac{y_1}{m_1} \cdot F_{45} - \frac{y_4}{m_4} \cdot F_{15} + \frac{y_5}{m_5} \cdot F_{14}\,;$$

analogously, one would have, e.g.:

$$u_{123} = y_1 \cdot m_2\,m_3\,F_{23} - y_2 \cdot m_1\,m_3\,F_{13} + y_3 \cdot m_1\,m_2\,F_{12}\,, \qquad \text{etc.}$$

As mentioned before, all of those quantities:

$$u_{\lambda\mu\nu} = y_\lambda \cdot m_\mu\,m_\nu\,F_{\mu\nu} - y_\mu \cdot m_\lambda\,m_\nu\,F_{\lambda\nu} + y_\nu \cdot m_\lambda\,m_\mu\,F_{\lambda\mu}$$

have the property that they are independent of the values of $\ddot{p}_1$ and $\ddot{p}_2$, which is why one might suspect that they will appear in the remainder term Φ (naturally in quadratic form). That is actually the case. Namely, if we temporarily do not assume that the masses m_1, m_2, m_3 and m_4, m_5, m_6, respectively, are equal, for better clarity, then we will have:

$$2L = m_1\,\dot{x}_1^2 + m_2\,\dot{x}_2^2 + m_3\,\dot{x}_3^2 + m_4\,\dot{x}_4^2 + m_5\,\dot{x}_5^2 + m_6\,\dot{x}_6^2 = a_{11}\dot{p}_1^2 + 2a_{12}\,\dot{p}_1\dot{p}_2 + a_{22}\dot{p}_2^2,$$

such that:

$$\left.\begin{aligned}
a_{11} &= m_1\,f_{11}^2 + m_2\,f_{21}^2 + m_3\,f_{31}^2 + m_4\,f_{41}^2 + m_5\,f_{51}^2 + m_6\,f_{61}^2,\\
a_{22} &= m_1\,f_{12}^2 + m_2\,f_{22}^2 + m_3\,f_{32}^2 + m_4\,f_{42}^2 + m_5\,f_{52}^2 + m_6\,f_{62}^2,\\
a_{12} &= m_1\,f_{11}\,f_{12} + m_2\,f_{21}\,f_{22} + m_3\,f_{31}\,f_{32} + m_4\,f_{41}\,f_{42} + m_5\,f_{51}\,f_{52} + m_6\,f_{61}\,f_{62}\,.
\end{aligned}\right\} \tag{2}$$

It follows from this that for:

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

one will have:

$$D = \begin{vmatrix} m_1\,f_{11}^2 + m_2\,f_{21}^2 + \cdots + m_6\,f_{61}^2 & m_1\,f_{11}\,f_{12} + m_2\,f_{21}\,f_{22} + \cdots + m_6\,f_{61}\,f_{62} \\ m_1\,f_{11}\,f_{12} + \cdots + m_6\,f_{61}\,f_{62} & m_1\,f_{12}^2 + m_2\,f_{22}^2 + \cdots + m_6\,f_{62}^2 \end{vmatrix},$$

and by step-wise decomposition:

$$D = m_1^2 \begin{vmatrix} f_{11}^2 & f_{11}\,f_{12} \\ f_{11}\,f_{12} & f_{12}^2 \end{vmatrix} + \cdots + m_1\,m_6 \begin{vmatrix} f_{11}^2 & f_{61}\,f_{62} \\ f_{11}\,f_{12} & f_{62}^2 \end{vmatrix} + m_2\,m_1 \begin{vmatrix} f_{21}^2 & f_{11}\,f_{12} \\ f_{21}\,f_{22} & f_{12}^2 \end{vmatrix} + \cdots + m_2\,m_6 \begin{vmatrix} f_{21}^2 & f_{61}\,f_{62} \\ f_{21}\,f_{22} & f_{62}^2 \end{vmatrix}$$

$$+ \cdots + \cdots + m_5\,m_6 \begin{vmatrix} f_{51}^2 & f_{61}\,f_{62} \\ f_{51}\,f_{52} & f_{62}^2 \end{vmatrix}$$

$$\begin{aligned}
= \quad & m_1^2\,f_{11}\,f_{12}\,F_{11} + m_1\,m_2\,f_{11}\,f_{22}\,F_{12} + \ldots + m_1\,m_6\,f_{11}\,f_{62}\,F_{16} \\
& + m_2\,m_1\,f_{21}\,f_{12}\,F_{21} + m_2^2\,f_{21}\,f_{22}\,F_{22} + \ldots + m_2\,m_6\,f_{21}\,f_{62}\,F_{26} \\
& \qquad + \ldots\ldots\ldots\ldots\ldots\ldots \\
& \qquad + m_5\,m_6\,f_{51}\,f_{62}\,F_{56}
\end{aligned}$$

$$= m_1\,m_2\,F_{12}^2 + m_1\,m_3\,F_{13}^2 + \ldots + m_1\,m_6\,F_{16}^2 + m_2\,m_3\,F_{23}^2 + \ldots + m_5\,m_6\,F_{56}^2\,. \tag{3}$$

That formula can also be obtained directly from the equation:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{vmatrix} \cdot \begin{vmatrix} x_1 & y_1 & z_1 & u_1 \\ x_2 & y_2 & z_2 & u_2 \end{vmatrix} = \begin{vmatrix} a_1\,x_1 + \cdots + d_1\,u_1 & a_1\,x_2 + \cdots + d_1\,u_2 \\ a_2\,x_1 + \cdots + d_2\,u_1 & a_2\,x_2 + \cdots + d_2\,u_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \cdot \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \cdot \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} + \cdots + \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix} \cdot \begin{vmatrix} z_1 & u_1 \\ z_2 & u_2 \end{vmatrix}$$

for:

$$x_1 = m_1 f_{11}, \quad y_1 = m_2 f_{21}, \ldots, \quad x_2 = m_1 f_{12}, \quad y_2 = m_2 f_{22}, \ldots$$
$$a_1 = f_{11}, \quad b_1 = f_{21}, \ldots, \quad a_2 = f_{12}, \quad b_2 = f_{22}, \ldots, \text{etc.}$$

Moreover, one has:

$$DZ' = a_{22}\, Q_1^2 - 2\, a_{12}\, Q_1\, Q_2 + a_{11}\, Q_2^2,$$
$$Q_1 = y_1 f_{11} + y_2 f_{21} + y_3 f_{31} + y_4 f_{41} + y_5 f_{51} + y_6 f_{61},$$
$$Q_2 = y_1 f_{12} + y_2 f_{22} + y_3 f_{32} + y_4 f_{42} + y_5 f_{52} + y_6 f_{62},$$

and

$$D \cdot \Phi = D\left[\frac{y_1^2}{m_1} + \frac{y_2^2}{m_2} + \cdots + \frac{y_6^2}{m_6}\right] - a_{22}\, Q_1^2 + 2a_{12}\, Q_1\, Q_2 + a_{11}\, Q_2^2, \tag{4}$$

such that Φ will take the form of a homogeneous, quadratic function of the $y_1, \ldots, y_6$. One would expect, in turn, that Φ includes the quantities $u_{\lambda\mu\nu}$ in quadratic form, such that we can also set:

$$D \cdot \Phi = C_{123} \cdot u_{123}^2 + C_{124} \cdot u_{124}^2 + \ldots + C_{456} \cdot u_{456}^2, \tag{5}$$

and Φ will take the form of a sum of squares. Namely, it can be shown that terms of the form $u_{123} \cdot u_{124}$, etc., *cannot* occur. The constants C are then determined when one differentiates the expressions (4) and (5) twice with respect to the y. For example, (4) yields:

$$\frac{1}{2} D \frac{\partial \Phi}{\partial y_1} = \frac{D}{m_1} y_1 - a_{22}\, f_{11}\, Q_1 + a_{12}\, f_{11}\, Q_2 + a_{12}\, f_{12}\, Q_1 - a_{11}\, f_{12}\, Q_2$$

and

$$\frac{1}{2} D \frac{\partial^2 \Phi}{\partial y_1^2} = \frac{D}{m_1} - a_{22}\, f_{11}^2 + 2a_{12}\, f_{11}\, f_{12} - a_{11}\, f_{12}^2,$$

in which a_{22}, a_{12}, a_{11} are taken to have the values in (2) and (3).

When one is careful to collect terms with *equal* masses, one will step-wise obtain:

$$\frac{1}{2} D \frac{\partial^2 \Phi}{\partial y_1^2} = \frac{D}{m_1} - f_{11}^2 [m_1\, f_{12}^2 + m_2\, f_{22}^2 + m_3\, f_{32}^2 + \cdots + m_6\, f_{62}^2]$$
$$+ 2 f_{11}\, f_{22} [m_1\, f_{11}\, f_{12} + m_2\, f_{21}\, f_{22} + \cdots + m_6\, f_{61}\, f_{62}] - f_{12}^2 [m_1\, f_{11}^2 + m_2\, f_{21}^2 + \cdots + m_6\, f_{61}^2]$$

$$= \frac{D}{m_1} - m_1 (f_{11}\, f_{12} - f_{11}\, f_{12})^2 - m_2 (f_{11}\, f_{22} - f_{12}\, f_{21})^2 \cdots - m_6 (f_{11}\, f_{62} - f_{12}\, f_{61})^2$$

$$= \frac{D}{m_1} - m_2 F_{12}^2 - m_3 F_{13}^2 - m_4 F_{14}^2 - m_5 F_{15}^2 - m_6 F_{16}^2 .$$

It follows from this, with the use of the value of D in (3), that:

$$m_1 \frac{1}{2} D \frac{\partial^2 \Phi}{\partial y_1^2} = D - m_1 m_2 F_{12}^2 - m_1 m_3 F_{13}^2 - \cdots - m_1 m_6 F_{16}^2$$
$$= m_2 m_3 F_{23}^2 + m_2 m_4 F_{24}^2 + \ldots + m_5 m_6 F_{56}^2 . \tag{4.a}$$

On the other hand, differentiating the expression (5) twice with respect to y_1 will yield:

$$\frac{1}{2} D \frac{\partial^2 \Phi}{\partial y_1^2} = C_{123} m_2^2 m_3^2 F_{23}^2 + C_{124} m_2^2 m_4^2 F_{24}^2 + \cdots + C_{156} m_5^2 m_6^2 F_{56}^2 .$$

One sees that one merely has to take:

$$C_{123} = \frac{1}{m_1 m_2 m_3}, \quad C_{124} = \frac{1}{m_1 m_2 m_4}, \quad \cdots, \quad C_{156} = \frac{1}{m_1 m_5 m_6}$$

and take the C to be constants that depend upon only the masses in order to achieve complete agreement between both forms.

One will arrive at the same result when one seeks the value of $\frac{\partial^2 \Phi}{\partial y_1 \partial y_2}$ instead of $\frac{\partial^2 \Phi}{\partial y_1^2}$.

One then finds from (4) that:

$$\frac{1}{2} D \frac{\partial^2 \Phi}{\partial y_1 \partial y_2} = - m_3 F_{13} F_{23} - m_4 F_{14} F_{24} - \cdots - m_6 F_{16} F_{26}$$
$$= + m_3 F_{31} F_{23} + m_4 F_{41} F_{24} + \ldots + m_6 F_{61} F_{26} , \tag{4.b}$$

and from (5) that:

$$\frac{1}{2} D \frac{\partial^2 \Phi}{\partial y_1 \partial y_2} = C_{123} \cdot m_3 m_1 \cdot m_2 m_3 F_{31} F_{23} + C_{124} \cdot m_4 m_1 \cdot m_2 m_4 F_{41} F_{24} + \cdots,$$

such that the values of the C above will then exhibit equality; analogous statements are true for the remaining quantities y. If one had included terms of the form $\psi = G \cdot u_{123} u_{124}$ in Φ then an additional one $\frac{\partial^2 \psi}{\partial y_1^2} = 2G m_2 m_4 \cdot m_2 m_3 F_{24} F_{23}$ would result. However, since a term with the factor $F_{24} F_{23}$ does *not* appear in (4.a), one must have $G = 0$. The same thing is true for the form $\psi = H \cdot u_{123} u_{234}$. In that case, one probably has $\frac{\partial^2 \psi}{\partial y_1^2} = 0$, but for but for $\frac{\partial^2 \Phi}{\partial y_1 \partial y_2}$, one will find the value $H m_3 m_4 \cdot m_2 m_3 F_{34} F_{23}$ – i.e., $H = 0$ – since such a term does not appear in (4.b). Terms that include products of two of the quantities u cannot occur in Φ then. We see that:

$$D\cdot\Phi=\frac{1}{m_1\,m_2\,m_3}u_{123}^2+\frac{1}{m_1\,m_2\,m_4}u_{124}^2+\cdots+\frac{1}{m_4\,m_5\,m_6}u_{456}^2\,, \tag{6}$$

where twenty such quantities u of the form:

$$u_{234} = y_2\, m_3\, m_4\, F_{34} - y_3\, m_2\, m_4\, F_{24} + y_4\, m_2\, m_4\, F_{23}$$

appear on the right.

The twenty quantities u_{123}, … are not mutually independent. In complete analogy to the first case, one finds that *two* relations exist between any *four* of the quantities u whose indices come from the same four numbers. Since the determinant for the combination 1234 is:

$$\begin{vmatrix} y_1 & m_1\,f_{11} & m_1\,f_{12} & m_1\,f_{11} \\ y_2 & m_2\,f_{21} & m_2\,f_{22} & m_2\,f_{21} \\ y_3 & m_3\,f_{31} & m_3\,f_{32} & m_3\,f_{31} \\ y_4 & m_4\,f_{41} & m_4\,f_{42} & m_4\,f_{41} \end{vmatrix}=0,$$

one will obtain the relation:

$$-\,m_1\,f_{11}\,u_{234} + m_2\,f_{21}\,u_{134} - m_3\,f_{31}\,u_{124} + m_4\,f_{41}\,u_{123} = 0\,,$$

and likewise:

$$-\,m_1\,f_{12}\,u_{234} + m_2\,f_{22}\,u_{134} - m_3\,f_{32}\,u_{124} + m_4\,f_{42}\,u_{123} = 0\,.$$

Hence, u_{134} and u_{234} can be expressed in terms of u_{123} and u_{124}. The combination (1235) likewise shows that u_{135} and u_{235} are expressible in terms of u_{123} and u_{125}. One can then consider all combinations of four elements and convince oneself that only *four* mutually-independent ones will appear in that case. In that case, one will also have two equations:

$$\frac{\partial\Phi}{\partial\ddot{p}_1}=0=m_1\;f_{11}\,\frac{\partial\Phi}{\partial y_1}+\cdots+m_6\;f_{61}\,\frac{\partial\Phi}{\partial y_6}$$

and

$$\frac{\partial\Phi}{\partial\ddot{p}_2}=0=m_1\;f_{12}\,\frac{\partial\Phi}{\partial y_1}+\cdots+m_6\;f_{62}\,\frac{\partial\Phi}{\partial y_6}\,,$$

such that Φ must be expressed in terms of 6 – 2 = 4 new mutually-independent variables, from the aforementioned theorem.

It is *not* superfluous to show that one can also arrive at that form for Φ by a *direct* path. With no loss of generality, one can set $m_5 = m_6 = 0$, $y_5 = y_6 = 0$, for clarity, and thus obtain:

$$\begin{aligned} D\cdot\Phi &= \frac{y_1^2}{m_1}[D-a_{22}\,m_1\;f_{11}^2+2a_{12}\,m_1\;f_{11}\;f_{22}-a_{11}\,m_1\;f_{12}^2] \\ &\quad+\frac{y_2^2}{m_2}[D-a_{22}\,m_2\;f_{21}^2+2a_{12}\,m_2\;f_{21}\;f_{22}-a_{11}\,m_2\;f_{22}^2] \end{aligned}$$

$$+\frac{y_3^2}{m_3}[D - a_{22}\, m_3\, f_{31}^2 + 2a_{12}\, m_3\, f_{31}\, f_{32} - a_{11}\, m_3\, f_{32}^2]$$
$$+\frac{y_4^2}{m_4}[D - a_{22}\, m_4\, f_{41}^2 + 2a_{12}\, m_4\, f_{41}\, f_{42} - a_{11}\, m_4\, f_{42}^2]$$
$$+ 2\, y_1\, y_2\, [- a_{22}\, f_{11}\, f_{21} + a_{12}\, f_{11}\, f_{22} + a_{12}\, f_{12}\, f_{21} - a_{11}\, f_{12}\, f_{22}]$$
$$+ 2\, y_1\, y_3\, [- a_{22}\, f_{11}\, f_{31} + a_{12}\, f_{11}\, f_{32} + a_{12}\, f_{12}\, f_{31} - a_{11}\, f_{12}\, f_{32}]$$
$$+ 2\, y_1\, y_4\, [- a_{22}\, f_{11}\, f_{41} + a_{12}\, f_{11}\, f_{42} + a_{12}\, f_{12}\, f_{41} - a_{11}\, f_{12}\, f_{42}]$$
$$+ 2\, y_2\, y_3\, [- a_{22}\, f_{21}\, f_{31} + a_{12}\, f_{21}\, f_{32} + a_{12}\, f_{22}\, f_{31} - a_{11}\, f_{22}\, f_{32}]$$
$$+ 2\, y_2\, y_4\, [- a_{22}\, f_{21}\, f_{41} + a_{12}\, f_{21}\, f_{42} + a_{12}\, f_{22}\, f_{41} - a_{11}\, f_{22}\, f_{42}]$$
$$+ 2\, y_3\, y_4\, [- a_{22}\, f_{31}\, f_{41} + a_{12}\, f_{31}\, f_{42} + a_{12}\, f_{32}\, f_{41} - a_{11}\, f_{32}\, f_{42}]\,.$$

If one sets a_{22}, a_{12}, a_{11}, and therefore also D, equal to the values above then the coefficient of y_1^2 / m_1 will be:

$$\begin{aligned}
&= D - m_1^2\, f_{11}^2\, f_{12}^2 - m_1\, m_2\, f_{11}^2\, f_{22}^2 - m_1\, m_3\, f_{11}^2\, f_{32}^2 - m_1\, m_4\, f_{11}^2\, f_{42}^2 \\
&\quad + 2m_1^2\, f_{11}^2\, f_{12}^2 + 2m_1\, m_2\, f_{11}\, f_{12}\, f_{21}\, f_{22} + 2m_1\, m_3\, f_{11}\, f_{12}\, f_{31}\, f_{32} \\
&\quad + 2m_1\, m_4\, f_{11}\, f_{12}\, f_{41}\, f_{42} - 2m_1^2\, f_{11}^2\, f_{12}^2 - m_1\, m_2\, f_{12}^2\, f_{21}^2 \\
&\quad - 2m_1\, m_3\, f_{12}^2\, f_{31}^2 - m_1\, m_4\, f_{12}^2\, f_{41}^2
\end{aligned}$$

$$\begin{aligned}
&= m_1\, m_2\, F_{12}^2 + m_1\, m_3\, F_{13}^2 + m_1\, m_4\, F_{14}^2 + m_2\, m_3\, F_{23}^2 + m_2\, m_4\, F_{24}^2 + m_3\, m_4\, F_{34}^2 \\
&\quad - m_1\, m_2\, F_{12}^2 - m_1\, m_3\, F_{13}^2 - m_1\, m_4\, F_{14}^2 \\
&= m_2\, m_3\, F_{23}^2 + m_2\, m_4\, F_{24}^2 + m_3\, m_4\, F_{34}^2\,.
\end{aligned}$$

One likewise finds that the coefficient of y_2^2 / m_2 is:

$$m_3\, m_4\, F_{34}^2 + m_3\, m_1\, F_{31}^2 + m_4\, m_1\, F_{41}^2\,, \text{ etc.}$$

One gets the following value for the coefficient of $2\, y_1\, y_2$ in a similar way:

$$- m_1\, f_{11}\, f_{21}\, f_{12}^2 - m_2\, f_{11}\, f_{21}\, f_{22}^2 - m_3\, f_{11}\, f_{21}\, f_{32}^2 - m_4\, f_{11}\, f_{21}\, f_{42}^2$$

$$+ m_1\, f_{11}\, f_{21}\, f_{11}\, f_{22} + m_2\, f_{21}\, f_{22}\, f_{11}\, f_{22} + m_3\, f_{31}\, f_{32}\, f_{11}\, f_{22} + m_4\, f_{41}\, f_{42}\, f_{11}\, f_{22}$$

$$+ m_1\, f_{11}\, f_{12}\, f_{21}\, f_{12} + m_2\, f_{21}\, f_{22}\, f_{21}\, f_{12} + m_3\, f_{31}\, f_{32}\, f_{21}\, f_{12} + m_4\, f_{41}\, f_{42}\, f_{21}\, f_{12}$$

$$- m_1\, f_{11}^2\, f_{12}\, f_{22} - m_2\, f_{21}^2\, f_{12}\, f_{22} - m_3\, f_{31}^2\, f_{12}\, f_{22} - m_4\, f_{41}^2\, f_{12}\, f_{22}$$

$$= m_3\, F_{31}\, F_{23} + m_4\, F_{41}\, F_{24}\,.$$

The coefficient of $2y_3\, y_1$ would likewise be:

$$m_2\, F_{23}\, F_{12} + m_4\, F_{43}\, F_{14}\ ,$$

and that of $2y_3\, y_4$ would be:

$$m_1\, F_{13}\, F_{41} + m_2\, F_{23}\, F_{42}\ , \text{ etc.}$$

Thus, one will have:

$$D\cdot\Phi = y_1^2\left[\frac{1}{m_1\,m_2\,m_3}m_2^2\,m_3^2\,F_{23}^2+\frac{1}{m_1\,m_2\,m_4}m_2^2\,m_4^2\,F_{24}^2+\frac{1}{m_1\,m_3\,m_4}m_3^2\,m_4^2\,F_{34}^2\right]$$

$$+y_2^2\left[\frac{1}{m_2\,m_3\,m_4}m_3^2\,m_4^2\,F_{34}^2+\frac{1}{m_1\,m_2\,m_3}m_3^2\,m_1^2\,F_{31}^2+\frac{1}{m_1\,m_2\,m_4}m_4^2\,m_1^2\,F_{41}^2\right]$$

$$+y_3^2\left[\frac{1}{m_1\,m_3\,m_4}m_4^2\,m_1^2\,F_{41}^2+\frac{1}{m_2\,m_3\,m_4}m_4^2\,m_2^2\,F_{42}^2+\frac{1}{m_1\,m_2\,m_3}m_1^2\,m_2^2\,F_{12}^2\right]$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+2y_1\,y_2\left[\frac{1}{m_1\,m_2\,m_3}m_1\,m_2\,m_3^2\,F_{31}\,F_{23}+\frac{1}{m_1\,m_2\,m_4}m_1\,m_2\,m_4^2\,F_{41}\,F_{24}\right]$$

$$+2y_3\,y_1\left[\frac{1}{m_1\,m_2\,m_3}m_1\,m_2^2\,m_3\,F_{23}\,F_{12}+\frac{1}{m_1\,m_3\,m_4}m_1\,m_3\,m_4^2\,F_{43}\,F_{14}\right]$$

$$+2y_3\,y_4\left[\frac{1}{m_1\,m_3\,m_4}m_1^2\,m_3\,m_4\,F_{13}\,F_{41}+\frac{1}{m_2\,m_3\,m_4}m_2^2\,m_3\,m_4\,F_{23}\,F_{42}\right]$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$=\frac{1}{m_1\,m_2\,m_3}\left[y_1\cdot m_2\,m_3\,F_{23}-y_2\cdot m_1\,m_3\,F_{13}+y_3\cdot m_1\,m_2\,F_{12}\right]^2$$

$$+\frac{1}{m_1\,m_2\,m_4}\left[y_1\cdot m_2\,m_4\,F_{24}-y_2\cdot m_1\,m_4\,F_{14}+y_4\cdot m_1\,m_2\,F_{12}\right]^2+\cdots,$$

and in general:

$$D\cdot\Phi=\frac{1}{m_1\,m_2\,m_3}u_{123}^2+\frac{1}{m_1\,m_2\,m_4}u_{124}^2+\cdots+\frac{1}{m_4\,m_5\,m_6}u_{456}^2\ .$$

The formula will, in turn, simplify greatly when only *one* point $m_1 = m_2 = m_3 = m$ is present; it will become:

$$m\cdot[F_{12}^2+F_{13}^2+F_{23}^2]\cdot\Phi=[y_1\,F_{23}-y_2\,F_{13}+y_3\,F_{12}]^2.$$

When the expressions above for $\dot{x}_1$, $\dot{x}_2$, and $\dot{x}_3$ are multiplied by m, it will follow that:

$$y_1\,F_{23}+y_2\,F_{31}+y_3\,F_{12}=m[D\cdot\dot{p}_1^2+2D'\cdot\dot{p}_1\,\dot{p}_2+D''\cdot\dot{p}_2^2]-[X_1\,F_{23}+X_2\,F_{13}+X_3\,F_{12}]\,,$$

in which Gauss's notations (from the theory of surfaces) now find an application:

$$
\begin{aligned}
D\ \ &= f_{111} \cdot F_{23} + f_{211} \cdot F_{31} + f_{311} \cdot F_{12}\,, \\
D' &= f_{112} \cdot F_{23} + f_{212} \cdot F_{31} + f_{312} \cdot F_{12}\,, \\
D'' &= f_{122} \cdot F_{23} + f_{222} \cdot F_{31} + f_{322} \cdot F_{12}\,.
\end{aligned}
$$

If p_1 and p_2 are two systems of curves on a surface then we shall focus our attention upon a point m on a surface with the coordinates p_1 and p_2, so the reciprocal value of the radius of curvature ρ of a normal section will be:

$$\frac{1}{\rho} = \frac{1}{\sqrt{F_{23}^2 + F_{31}^2 + F_{12}^2}} \, \frac{D \cdot \dot{p}_1^2 + 2D' \cdot \dot{p}_1\, \dot{p}_2 + D'' \cdot \dot{p}_2^2}{(a_{11}\, \dot{p}_1^2 + 2a_{12}\, \dot{p}_1\, \dot{p}_2 + a_{22} \cdot \dot{p}_2^2) \cdot \dfrac{1}{m}}\,,$$

which will make:

$$v^2 = \left(\frac{ds}{dt}\right)^2 = \frac{1}{m}[a_{11}\, \dot{p}_1^2 + 2a_{12}\, \dot{p}_1\, \dot{p}_2 + a_{22} \cdot \dot{p}_2^2]\,.$$

One will then have:

$$D \cdot \dot{p}_1^2 + 2D' \cdot \dot{p}_1\, \dot{p}_2 + D'' \cdot \dot{p}_2^2 = \frac{v^2}{\rho} \sqrt{F_{23}^2 + F_{31}^2 + F_{12}^2}\,,$$

and therefore:

$$m \cdot \Phi = \left[\frac{mv^2}{\rho} - \frac{X_1\, F_{23} + X_3\, F_{31} + X_3\, F_{12}}{\sqrt{F_{23}^2 + F_{31}^2 + F_{12}^2}} \right]^2,$$

or when the projection of the resultant R of X_1, X_2, and X_3 onto the normal is denoted by N, one will finally have:

$$m \cdot \Phi = \left[\frac{mv^2}{\rho} - N \right]^2,$$

which is a simple and uncommonly intuitive form that will reproduce the effect of the constraint under the motion that is *actually* present. From that equation, as with the most general one $\Phi = Z' - Z$, it will follow that the dimension of Φ is:

$$[\Phi] = m\, l^2\, t^{-4} = \frac{\text{work}}{\text{time-squared}}\,,$$

as it should be.

Increasing the number of points and parameters

The process up to now – i.e., the use of two types of methods – also allows one to see how one can proceed in the *most general* cases. For example, if we have three points with masses $m_1 = m_2 = m_3$, $m_4 = m_5 = m_6$, $m_7 = m_8 = m_9$, and only the two parameters p_1 and p_2, as before, then the specification of the components of accelerations $\dot{x}_1, \ldots, \dot{x}_9$ would yield nine equations, of which, $\frac{9\cdot8\cdot7}{1\cdot2\cdot3} = 84$ expressions that include each three of the quantities y can be derived from the forms $u_{149}, \ldots$, which are likewise free of $\ddot{p}_1$ and $\ddot{p}_2$. Supported by the foregoing, we conclude that $D \cdot \Phi$ will have the form:

$$D\cdot\Phi = \frac{1}{m_1\,m_2\,m_3}u_{123}^2 + \frac{1}{m_1\,m_2\,m_4}u_{124}^2 + \cdots + \frac{1}{m_7\,m_8\,m_9}u_{789}^2\;;$$

differentiating twice with respect to y will confirm that. One can rise to even greater numbers of points in that way.

On the other hand, if one increases the number of parameters (e.g., one assumes that three parameters p_1, p_2, and p_3 for two points are given) then it will now be necessary to eliminate the accelerations $\ddot{p}_1$, $\ddot{p}_2$, and $\ddot{p}_3$ from every four equations, or in other words, expression of the form:

$$u_{1245} = \begin{vmatrix} y_1 & m_1\,f_{11} & m_1\,f_{12} & m_1\,f_{13} \\ y_2 & m_2\,f_{21} & m_2\,f_{22} & m_2\,f_{23} \\ y_4 & m_4\,f_{41} & m_4\,f_{42} & m_4\,f_{43} \\ y_5 & m_5\,f_{51} & m_5\,f_{52} & m_5\,f_{53} \end{vmatrix}$$

will be free of the accelerations $\ddot{p}_1$, $\ddot{p}_2$, and $\ddot{p}_3$. If one again develops that into subdeterminants, and indeed using the elements of the first column, and sets:

$$F_{\kappa\lambda\mu} = \begin{vmatrix} f_{\kappa1} & f_{\kappa2} & f_{\kappa3} \\ f_{\lambda1} & f_{\lambda2} & f_{\lambda3} \\ f_{\mu1} & f_{\mu2} & f_{\mu3} \end{vmatrix},$$

to abbreviate then one will have:

$$u_{1245} = y_1 \cdot m_2\,m_4\,m_5\,F_{245} - y_2 \cdot m_1\,m_4\,m_5\,F_{145} + y_4 \cdot m_1\,m_2\,m_5\,F_{125} - y_5 \cdot m_1\,m_2\,m_4\,F_{124}$$

or

$$\frac{1}{m_1\,m_2\,m_4\,m_5}u_{1245} = \frac{y_1}{m_1}F_{245} - \frac{y_2}{m_2}F_{145} + \frac{y_4}{m_4}F_{125} - \frac{y_5}{m_5}F_{124}\;;$$

for two points (i.e., six coordinates), a total of $\binom{6}{4} = 15$ such quantities u will appear. From the foregoing, we would suspect that:

$$D \cdot \Phi = \frac{1}{m_1\, m_2\, m_3\, m_4} u_{1234}^2 + \frac{1}{m_1\, m_2\, m_3\, m_5} u_{1235}^2 + \cdots + \frac{1}{m_3\, m_4\, m_5\, m_6} u_{3456}^2 \,. \tag{7}$$

In this case, the six coordinates $x_1, \ldots, x_6$ of the two points are functions of the three parameters p_1, p_2, p_3, such that:

$$\dot{x}_1 = f_{11}\, \dot{p}_1 + f_{12}\, \dot{p}_2 + f_{13}\, \dot{p}_3 \,, \quad \ldots, \quad \dot{x}_6 = f_{61}\, \dot{p}_1 + f_{62}\, \dot{p}_2 + f_{63}\, \dot{p}_3 \,,$$

and

$$2L = m_1\, \dot{x}_1^2 + \cdots + m_6\, \dot{x}_6^2 = a_{11}\, \dot{p}_1^2 + 2a_{12}\, \dot{p}_1\, \dot{p}_2 + \cdots + a_{33}\, \dot{p}_3^2 \,,$$

so one will have:

$$a_{\mu\nu} = \sum_{i=1}^{6} m_i\, f_{i\mu}\, f_{i\nu} = m_1\, f_{1\mu}\, f_{1\nu} + m_2\, f_{2\mu}\, f_{2\nu} + \cdots m_6\, f_{6\mu}\, f_{6\nu} \,.$$

If the subdeterminants of:

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

are denoted by A_{11}, A_{12}, etc., then one will have:

$$D \cdot \Phi = D \cdot [Z - Z'] = D \left[\frac{y_1^2}{m_1} + \cdots + \frac{y_6^2}{m_6} \right] - [A_{11}\, Q_1^2 + 2A_{12}\, Q_1\, Q_2 + \cdots + A_{33}\, Q_3^2], \tag{8}$$

in which:

$$Q_1 = y_1\, f_{11} + y_2\, f_{21} + \ldots + y_6\, f_{61} \,,$$
$$Q_2 = y_1\, f_{12} + y_2\, f_{22} + \ldots + y_6\, f_{62} \,,$$
$$Q_3 = y_1\, f_{13} + y_2\, f_{23} + \ldots + y_6\, f_{63} \,.$$

Similarly, as before, it can also be shown here that the form (8) can go to the expression (7). Differential quotients of Φ with respect to any of the quantities y will drop out when calculated from (8) or (7). One will then find from (8) that:

$$\begin{aligned} \frac{1}{2} D \cdot \frac{\partial^2 \Phi}{\partial y_1^2} = \frac{D}{m_1} - [\, & f_{11} (A_{11}\, f_{11} + A_{12}\, f_{12} + A_{11}\, f_{13}) \\ & + f_{12}\, (A_{12}\, f_{11} + A_{22}\, f_{12} + A_{32}\, f_{13}) \\ & + f_{13}\, (A_{13}\, f_{11} + A_{23}\, f_{12} + A_{33}\, f_{13})] \end{aligned} \tag{8.a}$$

and

$$\begin{aligned} \frac{1}{2} D \cdot \frac{\partial^2 \Phi}{\partial y_1\, \partial y_2} = -[\, & f_{21} (A_{11}\, f_{11} + A_{12}\, f_{12} + A_{11}\, f_{13}) \\ & + f_{22}\, (A_{12}\, f_{11} + A_{22}\, f_{12} + A_{32}\, f_{13}) \end{aligned}$$

$$+ f_{23} (A_{13} f_{11} + A_{23} f_{12} + A_{33} f_{13})] \,. \tag{8.b}$$

If one substitutes the corresponding value for D, decomposes the determinant D into sums in such a way that quantities that include products of *equal* masses are always grouped together then one will find with no difficulty that:

$$D = \sum_{\lambda=1}^{6}\sum_{\mu=1}^{6}\sum_{\nu=1}^{6} m_\lambda\, m_\mu\, m_\nu\, F_{\lambda\mu\nu}^2 = m_1\, m_2\, m_3\, F_{123}^2 + m_1\, m_2\, m_4\, F_{124}^2 + \cdots + m_4\, m_5\, m_6\, F_{456}^2 \,. \tag{9}$$

If one introduces the subdeterminants of $F_{\kappa\lambda\mu}$, which one then calls:

$$F_{\kappa\lambda} = \begin{vmatrix} f_{\kappa 1} & f_{\lambda 1} \\ f_{\kappa 2} & f_{\lambda 2} \end{vmatrix}, \quad G_{\kappa\lambda} = \begin{vmatrix} f_{\kappa 2} & f_{\lambda 2} \\ f_{\kappa 3} & f_{\lambda 3} \end{vmatrix}, \quad H_{\kappa\lambda} = \begin{vmatrix} f_{\kappa 3} & f_{\lambda 3} \\ f_{\kappa 1} & f_{\lambda 1} \end{vmatrix}, \tag{10}$$

such that:

$$F_{\kappa\lambda\mu} = f_{\kappa 1}\, G_{\lambda\mu} + f_{\kappa 2}\, H_{\lambda\mu} + f_{\kappa 3}\, F_{\lambda\mu} \,, \tag{11}$$

then one will find that:

$$A_{11} = m_1\, m_2\, G_{12}^2 + m_1\, m_3\, G_{13}^2 + \ldots + m_5\, m_6\, G_{56}^2 \,,$$
$$A_{22} = m_1\, m_2\, H_{12}^2 + m_1\, m_3\, H_{13}^2 + \ldots + m_5\, m_6\, G_{56}^2 \,,$$
$$A_{33} = m_1\, m_2\, F_{12}^2 + m_1\, m_3\, F_{13}^2 + \ldots + m_5\, m_6\, F_{56}^2 \,,$$

$$A_{21} = A_{12} = m_1\, m_2\, G_{12}\, H_{12} + m_1\, m_3\, G_{13}\, H_{13} + \ldots + m_5\, m_6\, G_{56}\, H_{56} \,,$$
$$A_{31} = A_{13} = m_1\, m_2\, G_{12}\, F_{12} + m_1\, m_3\, G_{13}\, F_{13} + \ldots + m_5\, m_6\, G_{56}\, F_{56} \,,$$
$$A_{32} = A_{23} = m_1\, m_2\, H_{12}\, F_{12} + m_1\, m_3\, H_{13}\, F_{13} + \ldots + m_5\, m_6\, H_{56}\, F_{56} \,.$$

While continually considering the rule (11), one will find from this that:

$$f_{11}\, [A_{11}\, f_{11} + A_{12}\, f_{12} + A_{13}\, f_{13}] = f_{11}\, [m_1\, m_2\, G_{12}\, F_{112} + m_1\, m_3\, G_{13}\, F_{113} + \ldots + m_5\, m_6\, G_{56}\, F_{156}] \,,$$
$$f_{12}\, [A_{12}\, f_{11} + A_{22}\, f_{12} + A_{23}\, f_{13}] = f_{12}\, [m_1\, m_2\, H_{12}\, F_{112} + m_1\, m_3\, H_{13}\, F_{113} + \ldots + m_5\, m_6\, H_{56}\, F_{156}] \,,$$
$$f_{13}\, [A_{13}\, f_{11} + A_{23}\, f_{12} + A_{33}\, f_{13}] = f_{13}\, [m_1\, m_2\, F_{12}\, F_{112} + m_1\, m_3\, F_{13}\, F_{113} + \ldots + m_5\, m_6\, F_{56}\, F_{156}] \,.$$

If one defines the sum of those three expressions then one will get:

$$\frac{1}{2} D \frac{\partial^2 \Phi}{\partial y_1^2} = \frac{D}{m_1} - [m_2\, m_3\, F_{123}^2 + m_2\, m_4\, F_{124}^2 + \cdots + m_5\, m_6\, F_{156}^2],$$

or, from (9):

$$= \frac{1}{m_1} [m_2\, m_3\, m_4\, F_{234}^2 + m_2\, m_4\, m_5\, F_{235}^2 + \cdots + m_4\, m_5\, m_6\, F_{456}^2] \,.$$

One will arrive at precisely the same form by differentiating (7) twice. That will give:

$$\frac{1}{2} D \cdot \frac{\partial \Phi}{\partial y_1} = \frac{1}{m_1} [F_{234}\, u_{1234} + F_{235}\, u_{1235} + F_{236}\, u_{1236} + \cdots + F_{245}\, u_{1245} + F_{456}\, u_{1456}] \,,$$

$$\frac{1}{2}D\cdot\frac{\partial^2\Phi}{\partial y_1^2}=\frac{1}{m_1}[m_2\,m_3\,m_4\,F_{234}^2+\cdots+m_4\,m_5\,m_6\,F_{445}^2]\,.$$

In a completely analogous way, one will get from (8.b) that:

$$\begin{aligned}\frac{1}{2}D\cdot\frac{\partial^2\Phi}{\partial y_1\,\partial y_2}=-[\,&f_{21}(m_2\,m_3\,G_{23}\,F_{123}+\cdots+m_5\,m_6\,G_{56}\,F_{156})\\&+f_{22}\,(m_2\,m_3\,H_{23}\,F_{123}+\ldots+m_5\,m_5\,H_{23}\,F_{123})\\&+f_{23}\,(m_2\,m_3\,F_{23}\,F_{123}+\ldots+m_5\,m_5\,F_{23}\,F_{123})]\end{aligned}$$

$$\begin{aligned}=-\,[&m_3\,m_4\,F_{134}\,F_{234}+m_3\,m_5\,F_{135}\,F_{235}+m_3\,m_6\,F_{136}\,F_{236}\\&+m_4\,m_5\,F_{145}\,F_{245}+m_4\,m_6\,F_{146}\,F_{246}+m_5\,m_6\,F_{156}\,F_{256}]\,.\end{aligned}$$

By contrast, upon differentiating (7) with respect to y_1, one will get:

$$\frac{1}{2}D\cdot\frac{\partial\Phi}{\partial y_1}=\frac{1}{m_1}[F_{234}\cdot u_{1234}+\cdots+F_{456}\cdot u_{1456}],$$

as above, from which, a further differentiation with respect to y_2 will yield the foregoing six-term expression for $\frac{1}{2}D\,\frac{\partial^2\Phi}{\partial y_1\,\partial y_2}$.

Generalities

The process up to now admits the transition to general cases with no difficulty. Namely, we find that for κ points, when we consider only those combinations of the *abc*… that have no repetition of the indices, we will have for *one* parameter:

$$D_1\cdot\Phi_1=\sum_{a,b=1}^{3n}\frac{1}{m_a\,m_b}u_{ab}^2\,,\quad u_{ab}=\begin{vmatrix}y_a & m_a\,f_{a1}\\ y_b & m_b\,f_{b1}\end{vmatrix}=y_a\cdot m_b\,f_{b1}-y_b\cdot m_a\,f_{a1}\,,$$

$$D_1=\sum_{a=1}^{3n}m_a\,f_{a1}^2\,;$$

for *two* parameters:

$$D_2\cdot\Phi_2=\sum_{a,b,c=1}^{3n}\frac{1}{m_a\,m_b\,m_c}u_{abc}^2\,,\quad u_{abc}=\begin{vmatrix}y_a & m_a\,f_{a1} & m_a\,f_{a2}\\ y_b & m_b\,f_{b1} & m_b\,f_{b2}\\ y_c & m_c\,f_{c1} & m_c\,f_{c2}\end{vmatrix},$$

$$u_{abc}=y_a\cdot m_b\,m_c\,F_{bc}-y_b\cdot m_a\,m_c\,F_{ac}+y_c\cdot m_a\,m_b\,F_{ab}\,,$$

$$F_{ab} = \begin{vmatrix} f_{a1} & f_{a2} \\ f_{b1} & f_{b2} \end{vmatrix}, \quad D_2 = \sum_{a,b=1}^{3n} m_a\, m_b\, F_{ab}^2\,;$$

for *three* parameters:

$$D_3 \cdot \Phi_3 = \sum_{a,b,c,d=1}^{3n} \frac{1}{m_a\, m_b\, m_c\, m_d} u_{abcd}^2\,, \quad u_{abcd} = \begin{vmatrix} y_a & m_a\, f_{a1} & m_a\, f_{a2} & m_a\, f_{a3} \\ y_b & m_b\, f_{b1} & m_b\, f_{b2} & m_b\, f_{b3} \\ y_c & m_c\, f_{c1} & m_c\, f_{c2} & m_c\, f_{c3} \\ y_d & m_d\, f_{d1} & m_d\, f_{d2} & m_d\, f_{d3} \end{vmatrix},$$

$$u_{abcd} = y_a \cdot m_b\, m_c\, m_d\, F_{bcd} - y_b \cdot m_a\, m_c\, m_d\, F_{acd} + y_c \cdot m_a\, m_b\, m_d\, F_{abd} - y_d \cdot m_a\, m_b\, m_c\, F_{abc}\,,$$

$$D_3 = \sum_{a,b,c=1}^{3n} m_a\, m_b\, m_c\, F_{abc}^2\,,$$

$$F_{abc} = \begin{vmatrix} f_{a1} & f_{a2} & f_{a3} \\ f_{b1} & f_{b2} & f_{b3} \\ f_{c1} & f_{c2} & f_{c3} \end{vmatrix} = f_{a1} \cdot G_{bc} + f_{a2} \cdot H_{bc} + f_{a3} \cdot F_{bc}\,,$$

where

$$G_{bc} = \begin{vmatrix} f_{b2} & f_{c2} \\ f_{b3} & f_{c3} \end{vmatrix}, \quad H_{bc} = \begin{vmatrix} f_{b3} & f_{c3} \\ f_{b1} & f_{c1} \end{vmatrix}, \quad \text{and} \quad F_{bc} = \begin{vmatrix} f_{b1} & f_{c1} \\ f_{b2} & f_{c2} \end{vmatrix}$$

represent subdeterminants. We conclude that for *four* parameters, we will have:

$$D_4 \cdot \Phi_4 = \sum_{a,b,c,d,e=1}^{3n} \frac{1}{m_a\, m_b\, m_c\, m_d\, m_e} u_{abcde}^2\,, \quad u_{abcde} = \begin{vmatrix} y_a & m_a\, f_{a1} & m_a\, f_{a2} & m_a\, f_{a3} & m_a\, f_{a4} \\ \vdots & \vdots & \vdots & \vdots & \\ y_e & m_e\, f_{e1} & m_e\, f_{e2} & m_e\, f_{e3} & m_e\, f_{e4} \end{vmatrix},$$

$$D_4 = \sum_{a,b,c,d=1}^{3n} m_a\, m_b\, m_c\, m_d\, F_{abcd}^2\,,$$

$$F_{abcd} = \begin{vmatrix} f_{a1} & f_{a2} & f_{a3} & f_{a4} \\ \vdots & \vdots & \vdots & \vdots \\ f_{d1} & f_{d2} & f_{d3} & f_{d4} \end{vmatrix},$$

and we can test the validity of the formulas by differentiating with respect to the y. We can proceed further, and we will then convince ourselves that the accelerations $\ddot{p}_i$ will no longer occur in the quantities y when we set the y equal to their values.

"Bemerkungen über die Principien der Mechanik," Sitz. Kon. Bay. Akad. Wiss (1901), 167-182.

Remarks on the principles of mechanics

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I. – On the energetic foundation of mechanics.

From the extraordinary importance of the energy principle in all questions of physical mechanics, it is no wonder that one might seek to also derive it from the foundations of theoretical mechanics themselves. In all of those attempts, one deals with the problem of arriving at **d'Alembert**'s principle, or any form of the equations of motion that is equivalent to it, from the energy principle.

For the conception of mechanics that knows of only conservative forces that depend upon the coordinates of points, but are completely devoid of *conditions*, as, e.g., **Boussinesq** ([1]) developed in his lectures, that poses no difficulty. By differentiating the equation:

$$E = T + V = C$$

with respect to time t, in which V is the potential energy, which depends upon only the coordinates x, y, z, and T is the kinetic energy, one will get:

$$\sum m(x' x'' + y' y'' + z' z'') + \sum \left(\frac{\partial V}{\partial x} x' + \frac{\partial V}{\partial y} y' + \frac{\partial V}{\partial z} z' \right) = 0. \tag{1}$$

If it were now assumed that the accelerations, multiplied by the masses, are completely independent of the velocities and the constant C then it would follow from (1) that:

$$m x_i'' + \frac{\partial V}{\partial x_i} = 0, \qquad m y_i'' + \frac{\partial V}{\partial y_i} = 0, \qquad m z_i'' + \frac{\partial V}{\partial z_i} = 0.$$

([1]) **J. Boussinesq**, "Recherches sur les principes de la mécanique," J. de Math. (2) **18** (1873), pp. 315; *Leçons synthétiques de mécanique générale*, Paris, 1889, pp. 23.

However, the conclusion can no longer be applied when conditions between the coordinates are assumed, since in that case, the $m\,x''$, etc., actually depend upon the velocities ([1]). **Helm** ([2]) then sought the assistance of variational procedures and gave the basic principle of energetics the form: *The variation of the energy $E = T + V$ in any possible direction is equal to zero*. However, at the same time, one must demand that the concept of variation is introduced into *both* types of energy *in a consistent way*. Now, if E were varied in some direction then one would have to replace x, y, z with the quantities $x + \varepsilon\,\xi$, $y + \varepsilon\,\eta$, $z + \varepsilon\,\zeta$, in which ξ, η, ζ are arbitrary functions of t and ε is a constant that converges to zero. One understands the variation δA of an expression A to mean the coefficient of ε in the development of A in powers of ε.

In fact, one then has:

$$\delta V = \sum \frac{\partial V}{\partial x}\xi + \frac{\partial V}{\partial y}\eta + \frac{\partial V}{\partial z}\zeta\,,$$

but one finds the following value for δT :

$$\delta T = \frac{d}{dt}\sum m\,(x'\,\xi + y'\,\eta + z'\,\zeta) - \sum m\,(x''\,\xi + y''\,\eta + z''\,\zeta)\,,$$

and that expression is in no way equal to:

$$\sum m\,(x''\,\xi + y''\,\eta + z''\,\zeta)\,,$$

which would be necessary if one were to assert the identity of this principle with that of **d'Alembert**. Since the discussion between **Boltzmann** and **Helm** on the derivation of the equations of motion has not led to any entirely-conclusive result ([3]), it would nonetheless not be superfluous to summarize those simple relationships, and all the more so since **Helm** emphasized his viewpoint with particular vigor in his *Energetik*, and it has also been assumed by others since then ([4]).

I do not believe that I should go into the principle of the *superposition* of energy that was expressed by **Planck** ([5]) and **Boltzmann** with a similar purpose. In fact, it is nothing but an arbitrarily-chosen representation that forces the identity with **d'Alembert**'s principle. By contrast, **Schütz** ([6]) presented a principle of *absolute conservation of energy* in order to avoid **Helm**'s variational process. It will generally achieve the desired purpose for *one* material point, but it does

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([1]) See the remark of **R. Lipschitz** on **Helmholtz**'s conservation of force, **Ostwald**'s Klassiker-Bibliothek, no. 1, pp. 55, and likewise, **L. Boltzmann**, "Ein Wort der Mathematik an die Energetik," Wiedem. Ann. **57** (1896), pp. 39.

([2]) Namely, cf., **G. Helm**, *Die Energetik in ihrer geschichtlichen Entwicklung*, Leipzig, 1898, pp. 220, *et seq.*

([3]) Cf., **G. Helm**, "Zur Energetik," Wiedemann's Ann. **57**, pp. 646; **L. Boltzmann**, *ibid*. **58** (1896), pp. 595.

([4]) Cf., **P. Gruner**, "Die neueren Ansichten über Materie und Energie," Mitt. d. naturforsch. Ges. zu Bern, 1897.

([5]) **M. Planck**, "Das Prinzip der Erhaltung der Energie," Leipzig, 1887, pp. 148; **L. Boltzmann**, Wied. Ann. **57**, pp. 39 *et seq*. Cf., also the note by **C. Neumann** in **Helm**'s *Energetik*, pp. 229.

([6]) **J. Schütz**, "Das Princip der absoluten Erhaltung der Energie," Gött. Nachr. (1897), pp. 110. I do not quite understand the derivation of the equations of motion from the law of energy that **E. Padova** carried out ["Sulle equazioni della dinamica," Atti Ist. Veneto (7) **5** (1893), pp. 1641], due to the assumptions that were made in it.

not admit an extension to a system of them and might not be compatible, in and of itself, with the representation of the relativity of all states of motion, either.

However, one can avoid the incorrectness that was just pointed out by a *more general* variational process. Namely, if one also varies *time*, along with the coordinates x, y, z such that x, y, z, t go to $x + \varepsilon\,\xi$, $y + \varepsilon\,\eta$, $z + + \varepsilon\,\zeta$, $t + + \varepsilon\,\tau$, in which ξ, η, ζ, τ are arbitrary functions of t then x' will go to:

$$\frac{x' + \varepsilon\,\xi'}{1 + \varepsilon\,\tau'} = x' + \varepsilon\,(x' - \tau' x') + \ldots,$$

and that will imply that:

$$\delta(V + T) = \sum\left(\frac{\partial V}{\partial x} + m\,x''\right)\xi + \left(\frac{\partial V}{\partial y} + m\,y''\right)\eta + \left(\frac{\partial V}{\partial z} + m\,z''\right)\zeta$$

$$+ \frac{d}{dt}\sum m(x'\,\xi + y'\,\eta + z'\,\zeta) - 2\sum m(\xi\,x'' + \eta\,y'' + \zeta\,z'') - 2\tau' T\,.$$

One is now free to choose τ' in such a way that the right-hand side reduces to **d'Alembert**'s formula, and that is *always possible*, since T does not vanish. The desired result will be achieved *in that way*. However, one can hardly see anything but an abstract formalism in such an arbitrary representation. Since one also has that **Ostwald**'s principle of the maximum of energy exchange can be used only for the case of relative rest, but in general it must be replaced with an entirely different consideration ([1]), it would seem that the attempts that have been made *up to now* do not suggest the possibility of an unforced derivation of **d'Alembert**'s principle, or that of **Gauss**, from the law of energy.

II. – On Hamilton's principle.

It was proved in no. **1** that one can give rise to *any arbitrary relation* for the varied quantities by a suitably-generalized variational process. **Hölder** ([2]) employed such general variations in order to prove that the principles of **Hamilton** and **Maupertuis** are completely equivalent to **d'Alembert**'s principle. However, that viewpoint can be expressed in a much more general form by the following theorem:

Under the assumption of a suitable variational process, the variation of the integral:

$$J = \int_{t_0}^{t_1} (\alpha T + \beta U)\,dt\,,$$

([1]) Cf., **A. Voss**, "Ueber ein energetisches Grundsetz der Mechanik," these Situngsber. (1901), pp. 53.
([2]) **O. Hölder**, "Ueber die Principien von **Hamilton** und **Maupertuis**," Gött. Nachrichten (1896), issue 2.

in which α, β are two generally completely-arbitrary constants, will be equal to zero because of the differential equations of motion, and conversely, the requirement that δJ should vanish for all allowable displacements will lead to the differential equations of motion ([1]).

Ordinarily, one adds the condition that the variations of the coordinates x, y, z should *vanish* at the limits of the integral. That can generally be in the best interests of a mechanical interpretation, but in itself that further condition is generally superfluous and *inessential*.

Therefore, one might next understand δU to mean the virtual work done by the forces x, y, z under the displacement that corresponds to ξ, η, ζ, so one sets:

$$\delta U = \sum (X\,\xi + Y\,\eta + Z\,\zeta) .$$

Now, in order to vary the integral ([2]):

$$I' = \int_{t_0}^{t_1} F(x, x', t)\,dt ,$$

one can, by the substitution ([3]):

$$t = k\,u + k_0 ,$$

where

$$k = \frac{t_1 - t_0}{1 - t_0}, \qquad k_0 = \frac{1 - t_1}{1 - t_0} ,$$

reduce that to the integral between *constant* limits 0 and 1:

$$I' = \int_0^1 F\left(x, \frac{1}{k}\frac{dx}{du}, ku + k_0\right) k\,dt .$$

If one then lets x, y, z, u go to $x + \varepsilon\,\xi$, $y + \varepsilon\,\eta$, $z + \varepsilon\,\zeta$, $u + \varepsilon\,v$, then $k\,v$ will be the arbitrary function that was denoted by τ in no. **1**. At the same time, $\frac{1}{k}\frac{dx}{du}$ will go to ([4]):

([1]) Obviously, one can also substitute *any arbitrary function* of x, y, z; x', y', z' for the function under the integral sign. However, the linear function of U and T will lead to the forms that are essential from the mechanical standpoint.

([2]) For the sake of brevity, all differential quotients with respect to t are denoted with a prime, such that $x' = \frac{dx}{dt}$, $x'' = \frac{d^2 x}{dt^2}$.

([3]) If $t_0 = 1$ then one switches t_1 with t_0 or sets:

$$t = u\,(1 - t_1) + t_1 .$$

([4]) The symbols x', ξ', v' in brackets mean the differential quotients with respect to u here.

$$x' + \varepsilon \frac{[(\xi') - (x')(v')]}{k} + \dots$$

One will then get:

$$\delta I' = \int_0^1 \left\{ \frac{\partial F}{\partial x} \xi + \frac{\partial F}{\partial x'} \left[\frac{(\xi') - (x')(v')}{k} \right] + \frac{\partial F}{\partial t} kv + F(v') \right\} k\, du\,,$$

which, by means of the identities:

$$\frac{(\xi')}{k} = \frac{1}{k} \frac{d\xi}{du} = \frac{d\xi}{dt} = \xi',$$

$$\frac{(x')}{k} = \frac{1}{k} \frac{dx}{du} = \frac{d\xi}{dt} = x',$$

$$(v') = \frac{dv}{du} = \frac{k\,dv}{k\,du} = \frac{d\tau}{dt} = \tau',$$

will once more go to:

$$\text{(A)} \qquad \delta I' = \int_0^1 \left[\frac{\partial F}{\partial x} \xi + \frac{\partial F}{\partial x'} (\xi' - x'\tau') + \frac{\partial F}{\partial t} \tau + F\tau' \right] du\,.$$

Obviously, one can also deduce this formula *immediately* from the *concept of a variation* ([1]). In view of the misunderstanding that arises in presenting the variation by the use of the δ sign, it seems to me that the above consideration, which is also cumbersome, does not seem preferable for entirely elementary purposes. If formula (A) were addressed by the method of partial integration in the well-known way then that would produce the useful formula ([2]):

$$\delta I' = \left| \frac{\partial F}{\partial x'} (\xi' - x'\tau') + F\tau \right|_{t_0}^{t_1} + \int_{t_0}^{t_1} \left(\frac{\partial F}{\partial x} - \frac{d}{dt} \frac{\partial F}{\partial x'} \right) (\xi' - \tau x')\, du\,.$$

I shall now consider the integral:

$$J = \int_{t_0}^{t_1} (\alpha T + \beta U)\, dt$$

and set:

$$V = \sum (X\,\xi + Y\,\eta + Z\,\zeta)\,,$$

to abbreviate, which it is equal to the *virtual work done by the given forces*, and:

([1]) See **Hölder**, *loc. cit.*, § **2**, remark.
([2]) It was assumed in that form in, e.g., **Routh**, *Dynamik starrer Körper*, transl. by **A. Schepp**, v. 2, pp. 327.

$$S = \sum m(x'\xi + y'\eta + z'\zeta),$$

which is equal to the *virtual moment of the quantities of motion*, and:

$$W = \sum m(x''\xi + y''\eta + z''\zeta),$$

which is equal to the *virtual moment of the accelerations times the masses.* That will then yield:

$$\delta J = \int_{t_0}^{t_1} [(\beta U - \alpha T)\tau' + \alpha S' - \alpha W + \beta V]\,dt$$

or

$$\text{(I)} \qquad \delta J = \beta \int_{t_0}^{t_1} (V - W)\,dt + \int_{t_0}^{t_1} [(\beta U - \alpha T)\tau' + (\beta - \alpha)W + \alpha S']\,dt\,,$$

$$\text{(II)} \qquad \delta J = \alpha \int_{t_0}^{t_1} (V - W)\,dt + \int_{t_0}^{t_1} [(\beta U - \alpha T)\tau' + (\beta - \alpha)V + \alpha S']\,dt\,.$$

If one then chooses the arbitrary function *t* in such a way that the second partial integral in formulas (I), (II) vanishes then one will have:

$$\delta J = \beta \int_{t_0}^{t_1} (V - W)\,dt\,,$$

$$\delta J = \alpha \int_{t_0}^{t_1} (V - W)\,dt\,;$$

i.e., *the demand that* $\delta J = 0$ *will be completely equivalent to d'Alembert's principle.* Depending upon the choice of constants α, β, there can be various *special forms for the general variational principle.*

First: If one sets $\alpha = \beta$ then, from (I), that will demand the condition:

$$(U - T)\,\tau' + S' = 0\,;$$

i.e., when the part S' is dropped by integration, as usual, and the variations of the *x*, *y*, *z* are equal to zero at the limits then τ = const. or 0, resp. [1] In particular, if $U - T$ = const. = h then one can also set $\tau h + S = 0$. *That is Hamilton's principle.*

[1] When one adds the variations at the limits, one will find that the principle is *true without exception;* it is applicable even when $U - T$ vanishes between the limits of integration.

Secondly: If one takes $\beta = 0$ and one now sets, from (II):

$$T\,\tau' + V + S' = 0$$

then one will have the *extended form of the principle of least action* ([1]). Since T is not zero, *that way of determining* τ *is always possible*, which should be emphasized here especially.

Third: By contrast, if one takes $\alpha = 0$ then, from (I), one sets:

$$U\,\tau' + W = 0,$$

which means that a possible addition to the variations at the limits would be entirely superfluous to further simplification. However, it must be assumed here that *U does not vanish between the limits of the integral* ([2]). Under those circumstances, the expression:

$$\delta \int_{t_0}^{t_1} U\,dt = 0$$

will also lead to the differential equation of motion.

Fourth: Finally, one will get:

$$\delta \int_{t_0}^{t_1} E\,dt = 0$$

for $\beta = -\alpha$, with the condition $(T + U)\,\tau' + 2V - S' = 0$.

A generally useful form for the principle will arise only in the first two cases. In the last two, as well as in the general case, the appearance of the symbolic expression U will already be a hindrance, even when one overlooks the fact that $\alpha\,T - \beta\,U$ cannot vanish insider the limits on the integral, which is generally not possible for arbitrary values of α, β. One can, however, avoid the symbolic expression U completely in a variational concept that is this general.

Namely, if one varies the expression:

$$A = \int_{t_0}^{t_1} \sum (X\,x' + Y\,y' + Z\,z')\,dt\,,$$

which represents the *total* work that is done by the effective forces from t_0 to the variable time t, so from formula (A), that will yield:

([1]) Cf., **Hölder**, *loc. cit.*, § **2**.

([2]) Naturally, a similar assumption must always be made when one takes an arbitrary function under the integral sign (cf., remark 1 on pp. 4). It will be fulfilled by itself from the principle of least action and **Hamilton**'s principle.

(B) $$\delta A = V - V_0 + \int_{t_0}^{t_1} \sum (Z)\, dt\,,$$

in which:

$$Z = \left[y'\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) + z'\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) - \frac{\partial X}{\partial t}\right](\xi - \tau\, x')$$

$$+ \left[z'\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right) + x'\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) - \frac{\partial Y}{\partial t}\right](\eta - \tau\, y')$$

$$+ \left[x'\left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right) + y'\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) - \frac{\partial Z}{\partial t}\right](\zeta - \tau\, z')\,,$$

and one only has to show that the arbitrary function t is subjected to the conditions that arise when one employs the non-symbolic equation (B) in the variation of the integral:

$$\int_{t_0}^{t_1} (\alpha T + \beta A)\, dt\,,$$

in place of the previous equation $\delta U = V$.

If one considers that **d'Alembert**'s principle can be expressed in the forms:

$$\delta \int (T + A)\, dt = 0, \qquad \delta \int T\, dt = 0, \qquad \delta \int U\, dt = 0, \qquad \delta \int E\, dt = 0,$$

$$\delta \int (\alpha\, T + \beta A)\, dt = 0$$

then this variational principle, in its general form, will prove to be a *completely conventional rule* that no longer has anything to do with special representations that belong to the actual realm of mechanical intuitions, but are solely conceived for the sake of expressing the differential equations of motion in the most condensed form possible. I do not consider it trivial to once more repeat that remark (which is obvious by itself), which I already made on a previous occasion ([1]), since very differing opinions seem to be circulating at present in the conception of **Hamilton**'s principle, in principle. From the abstract standpoint, one can even see how the special form of the principle that employs the *energy integral* $\int E\, dt$ can have an advantage. However, there seems to be no doubt that the *actual **Hamilton** integral* is likewise recommended for its simplicity and general validity. Therefore, it was also used by **v. Helmholtz** in all of his investigations (under the name of the principle of least action).

([1]) **A. Voss**, "Ueber die Differentialgleichungen der Mechanik," Math. Ann. **25** (1884), pp. 267.

III. – On the principle of least constraint.

If one denotes the coordinates of the points of a material system indifferently by x_i ([1]) then the *vis viva* will be:

$$T = \tfrac{1}{2}\sum m_i\, x_i^2 \;.$$

Now, if one introduces just as many new variables y_i , which are mutually-independent functions of the x_i that can include time t, as well, in place of the x_i , then, by assumption, the functional determinant:

$$\Delta = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \cdots & \cdots & \dfrac{\partial x_1}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial x_n}{\partial y_1} & \cdots & \cdots & \dfrac{\partial x_n}{\partial y_n} \end{vmatrix}$$

will be non-zero, so:

$$m_1 \ldots m_n\, \Delta^2 = A$$

will also vanish, in which A is the determinant of the elements ([2]):

$$a_{s\sigma} = \sum m_i \frac{\partial x_i}{\partial y_s}\frac{\partial x_i}{\partial y_\sigma} \tag{1}$$

of the positive-definite quadratic form:

$$\sum a_{s\sigma}\, u_s\, u_\sigma = \sum m_i \left(\frac{\partial x_i}{\partial y_s} u_s\right)^2 .$$

If one denotes the sub-determinant of the elements (1), divided by A, by $A_{s\sigma}$ then:

$$\sum A_{s\tau}\, a_{s\sigma} = (\sigma\, \tau),$$

in which $(\sigma\, \tau)$ means the known sign. However, since one also has:

$$\sum \frac{\partial y_\tau}{\partial x_i}\frac{\partial x_i}{\partial y_\sigma} = (\sigma\, \tau),$$

it will follow that:

([1]) For the notation, see **H. Hertz**, Ges. Werke, III, pp. 62.

([2]) In all cases in which nothing further is said about a summation, it will be extended over all indices s, σ, τ, … that appear *more than once* from 1 to 3.

$$\sum \left(A_{s\tau} \frac{\partial x_i}{\partial y_s} m_i - \frac{\partial y_\tau}{\partial x_i} \right) \frac{\partial x_i}{\partial y_\sigma} = 0,$$

or, since $\Delta \neq 0$ ([1]):

(2) $$\sum A_{s\tau} \frac{\partial x_i}{\partial y_s} m_i = \frac{\partial y_\tau}{\partial x_i}.$$

If one now introduces the equations:

$$x_i' = \sum \frac{\partial x_i}{\partial y_s} y_s' + \frac{\partial x_i}{\partial t}$$

into the expression T then it will follow that:

$$T = \tfrac{1}{2} \sum a_{s\sigma} \, y_s' \, y_\sigma' + \sum a_s \, y_s' + a,$$

if one sets:

(3) $$\begin{cases} a_s = \sum m_i \dfrac{\partial x_i}{\partial y_s} \dfrac{\partial x_i}{\partial t}, \\ a = \tfrac{1}{2} \sum m_i \left(\dfrac{\partial x_i}{\partial t} \right)^2. \end{cases}$$

T is then a function of second order in the y_s' that is not generally homogeneous. At the same time, one will have:

(4) $$x_i'' = \sum \frac{\partial^2 x_i}{\partial y_s \, \partial y_\sigma} y_s' \, y_\sigma' + 2 \sum \frac{\partial^2 x_i}{\partial y_s \, \partial t} y_s' + \sum \frac{\partial x_i}{\partial y_s} y_s' + \frac{\partial^2 x_i}{\partial t^2}.$$

We employ the value (4) in order to calculate the *constraint Z:*

(5) $$Z = \sum m_i \left(x_i'' - \frac{X_i}{m_i} \right)^2,$$

in which we understand the X_i to mean the components of the effective forces. From a very simple calculation, we find from (4) that:

$$Z = \sum m_i \, \Xi_i^2 + \sum A_{s\sigma} \left[\frac{d}{dt} \left(\frac{\partial T}{\partial y_s'} \right) - \frac{\partial T}{\partial y_s} - Y_s \right] \left[\frac{d}{dt} \left(\frac{\partial T}{\partial y_\sigma'} \right) - \frac{\partial T}{\partial y_\sigma} - Y_\sigma \right] - \sum A_{s\sigma} Q_s \, Q_\sigma,$$

([1]) In formula (2), the summation over i is obviously *not* performed.

in which we have set:

$$Y_s = \sum X_i \frac{\partial x_i}{\partial y_s},$$

$$\Xi_s = \sum\left(\frac{\partial^2 x_i}{\partial y_s\,\partial y_\sigma} y_s'\, y_\sigma' + 2\frac{\partial^2 x_i}{\partial y_s\,\partial t} y_s' + \frac{\partial^2 x_i}{\partial t^2} - \frac{X_i}{m_i}\right),$$

$$Q_s = \sum m_i \frac{\partial x_i}{\partial y_s}\frac{\partial^2 x_i}{\partial y_r\,\partial y_\sigma} y_r'\, y_\sigma' + 2\sum \frac{\partial x_i}{\partial y_s}\frac{\partial^2 x_i}{\partial t\,\partial y_\sigma} m_i\, y_\sigma' + \sum \frac{\partial x_i}{\partial y_s}\frac{\partial^2 x_i}{\partial t^2} m_i - Y_s,$$

to abbreviate, while we have:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial y_s'}\right) - \frac{\partial T}{\partial y_s} = \sum a_{sr} y_r'' + \sum m_i \frac{\partial x_i}{\partial y_s}\frac{\partial^2 x_i}{\partial y_r\,\partial y_\sigma} y_r'\, y_\sigma' + 2\sum m_i \frac{\partial x_i}{\partial y_s}\frac{\partial^2 x_i}{\partial t\,\partial y_\sigma} y_\sigma' + \sum m_i \frac{\partial x_i}{\partial y_s}\frac{\partial^2 x_i}{\partial t^2}.$$

One now sees immediately that the first sum in Z will cancel the last one. In order to do that, one needs only to replace the Q_s with their values again in:

$$W = \sum A_{s\sigma}\, Q_s\, Q_\sigma\,.$$

If one also expresses Y_s in terms of the X_i again then that will imply that:

$$W = \sum \frac{\partial x_i}{\partial y_s}\frac{\partial x_j}{\partial y_{s'}} m_i\, m_j A_{ss'}\, \Xi_i\, \Xi_j\,,$$

which will go to:

$$W = \sum m_j (i\ j)\Xi_i\, \Xi_j = \sum m_j\, \Xi_i^2\,,$$

from (2).

It follows further by differentiation that:

$$2\sum m_i \frac{\partial x_i}{\partial y_s}\frac{\partial^2 x_j}{\partial y_r \partial y_\sigma} = 2\begin{bmatrix} r\ \sigma \\ s \end{bmatrix} = \frac{\partial a_{s\sigma}}{\partial y_r} + \frac{\partial a_{sr}}{\partial y_\sigma} - \frac{\partial a_{r\sigma}}{\partial y_s},$$

$$2\sum m_i \frac{\partial x_i}{\partial y_s}\frac{\partial^2 x_i}{\partial t\,\partial y_\sigma} = [s\ \sigma] = \frac{\partial a_{s\sigma}}{\partial t} + \frac{\partial a_s}{\partial y_\sigma} - \frac{\partial a_\sigma}{\partial y_s},$$

$$\sum m_i \frac{\partial x_i}{\partial y_s}\frac{\partial^2 x_i}{\partial t^2} = [s] = \frac{\partial a_s}{\partial t} - \frac{\partial a}{\partial y_s},$$

such that:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial y'_s}\right)-\frac{\partial T}{\partial y_s}=\sum a_{s\sigma}y''_\sigma+\sum\begin{bmatrix} r\ \sigma \\ s\end{bmatrix}y'_r\ y'_\sigma+[s\ \sigma]\,y'_\sigma+[s]\,.$$

With that, the following theorem is proved:

If one replaces the variables x with just as many new variables y by means of the equations:

(6) $$x_i = f_i\,(y_1, y_2, \ldots, y_{3n}, t), \qquad y_i = \varphi_i\,(x_1, x_2, \ldots, x_{3n}, t),$$

*which are mutually-independent relative to the y, then the **Gauss**ian constraint Z will be expressed by the function:*

$$Z = \sum A_{s\sigma}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial y'_s}\right)-\frac{\partial T}{\partial y_s}-Y_s\right]\left[\frac{d}{dt}\left(\frac{\partial T}{\partial y'_\sigma}\right)-\frac{\partial T}{\partial y_\sigma}-Y_\sigma\right],$$

which is covariant in the vis viva T.

That is a generalization of result that **Lipschitz** ([1]) derived in the case where the functions f do not include time as a result of his general investigations into the transformation of *homogeneous* differential expressions. However, it is in the nature of things that it cannot be restricted to the case of a homogeneous form T. In that case, it would probably be simpler to derive the transformation result directly.

An *essential* condition for that is, however, that the number of variables y must be just as large as that of the x, because it is only under that assumption that the identity (2), upon which the entire calculation is based, can be applied ([2]).

Now one can choose the variables y in such a way ([3]) that for a mechanical problem with k condition equations:

$$\varphi_l\,(x_1, \ldots, x_n, t) = 0, \qquad l = 1, 2, \ldots, k,$$

when the first k functions y are set equal to zero, they will represent just those conditions, i.e.:

$$y_l = \varphi_l\,,$$

([1]) **R. Lipschitz**, "Bemerkungen zu dem Prinzip des kleinsten Zwanges," J. f. Math. **82** (1877), pp. 328.

([2]) **A. Wassmuth** has ["Ueber die Anwendung des Princips des kleinsten Zwanges auf die Elektrodynamik," these Sitzungsber. (1894), pp. 219] taken advantage of **Lipschitz**'s formula for Z for the case in which the number of variables y is also smaller than that of the x. The fact that this is not permissible could already be seen from the fact that under those circumstances, the constraint that he also denoted by Z would be equal to zero, which only happens for free motions of a system, while condition equations were nonetheless assumed on pp. 220. The formulas that are developed in the further course of the paper must also be replaced with the ones that are derived later in the text, insofar as they do not refer to free motions.

Incidentally, in **Lipschitz**, the fact that the number of variables cannot change is made an assumption expressly (*loc. cit.*, pp. 316 and 328).

([3]) Obviously, one can also drop some of the conditions just as simply by introducing general coordinates.

while the last $h = 3n - k$ of them can be regarded as general coordinates q_m , $m = 1, \ldots, h$. Under that assumption, one will then have:

$$y'_l = 0, \qquad y''_l = 0 \qquad \text{for } l = 1, 2, \ldots, k.$$

Now, should the constraint Z be a minimum, one would get the equations:

$$\sum A_{s\sigma} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial y'_s}\right) - \frac{\partial T}{\partial y_s} - Y_s \right] a_{sl} = \lambda_s \,,$$

$$\sum A_{s\sigma} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial y'_s}\right) - \frac{\partial T}{\partial y_s} - Y_s \right] a_{sm+k} = 0$$

in the known way by means of the method of **Lagrange** multipliers, or:

(a) $$\frac{d}{dt}\left(\frac{\partial T}{\partial y'_s}\right) - \frac{\partial T}{\partial y_s} - Y_s = \lambda_l \,, \qquad l = 1, \ldots, k,$$

(b) $$\frac{d}{dt}\left(\frac{\partial T}{\partial y'_{m+k}}\right) - \frac{\partial T}{\partial y_{m+k}} - Y_{m+k} = 0, \qquad m = 1, \ldots, h \,.$$

One can drop equations (a) entirely since they only serve to determine the multipliers λ. Equations (b) imply the equations of motion, as long as one sets:

$$y_l = 0 \qquad \text{for} \qquad l = 1, \ldots, k,$$

$$y_{m+k} = q_m \qquad \text{for} \qquad m = 1, \ldots, h$$

in them, and that will imply the value:

$$Z = \sum A_{ij} \, \lambda_i \, \lambda_j \,, \qquad i, j = 1, \ldots, k$$

for the constraint Z .

"Über die Ableitung des Gauß'schen Prinzips des kleinsten Zwanges aus den allgemeinsten Lagrange'schen Gleichungen zweiter Art," Sitz. Kais. Akad. Wiss. **116** (1907), 1321-1336.

On the derivation of Gauss's principle of least constraint from Lagrange's equations of the second kind

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It is known that the principle of least constraint that **Gauss** presented (1829) can be derived from Lagrange's equations of the first kind for rectangular coordinates in a simple way, in which the expression for the constraint Z – i.e., the function to be minimized – was first formulated by **Scheffler** (1858):

$$Z = \sum_{\nu=1}^{n} \frac{1}{m_\nu} [(m_\nu \ddot{x}_\nu - X_\nu)^2 + (m_\nu \ddot{y}_\nu - Y_\nu)^2 + (m_\nu \ddot{z}_\nu - Z_\nu)^2] .$$

Now, **Lipschitz** (1877) tried to introduce general coordinates into the condition equations that the variable must fulfill identically in place of the rectangular ones that are coupled to each other by the condition equations, and **Wassmuth** (1895) succeeded in performing the transformation of constraint into general coordinates in a very simple way, at least under the assumption that the conditions do not include time explicitly ([1]). In addition, **Radakovich** and others have addressed that problem in detail.

That raises the question of whether it might be possible to derive **Gauss**'s principle for mutually-independent generalized coordinates, as well, and in particular, the general expression for the constraint Z, from the most general Lagrange equations of the *second* kind *directly* in a way that is similar to what one does with rectangular coordinates.

Now, such a direct derivation shall be attempted in what follows in which in total four different cases shall initially be brought under consideration, since the form of the condition equations that exist between the rectangular coordinates, as well as the transformation equations by which the generalized coordinates p_1 , p_2 , …, p_s will be introduced in place of the rectangular ones, can be

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([1]) **Waßmuth**, "Über die Transformation des Zwanges," These Sitzungsberichte, CIV, Part II.

1. *holonomic* or 2. *non-holonomic*, and the transformation equations can themselves once more include time t or not in both cases, and thus be *rheonomic* or *scleronomic*, resp.

Meanwhile, it will be shown that the non-holonomity of the generalized coordinates generally exerts no essential influence upon the method of derivation, such that actually only *two* main cases will come under consideration in the present article, namely, according to whether the generalized coordinates are scleronomic or rheonomic.

The simpler, and at the same, more common of those two cases shall be attacked first.

A. – Scleronomic generalized coordinates.

At first, the transformation equations shall not include time t explicitly, so they will either read like [1]:

$$x_\nu = f_\nu(p_1, p_2, \ldots, p_s) \qquad \text{for } \nu = 1, 2, \ldots, 3n,$$

in case they are holonomic, or:

$$dx_\nu = \sum_{h=1}^{s} \pi_h^\nu \, dp_h \qquad \text{for } \nu = 1, 2, \ldots, 3n,$$

in case they are non-holonomic, in which the quantities π_h^ν are any functions of the p_h that do not, however, include time explicitly. The s general or generalized coordinates $p_1, p_2, \ldots, p_s$ that are introduced by these $3n$ equations shall fulfill the τ condition equations that exist between the rectangular coordinates:

$$\varphi_1(x_1, x_2, \ldots, x_{3n}) = 0, \quad \varphi_2(x_1, x_2, \ldots, x_{3n}) = 0, \quad \ldots, \quad \varphi_\tau(x_1, x_2, \ldots, x_{3n}) = 0$$

identically, but are completely independent of each other, which is naturally possible only when their number s is equal to the number of degrees of freedom in the system $3n - \tau$. The Lagrange equations of the second kind will then read:

$$Q_h = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_h}\right) - \frac{\partial L}{\partial p_h} - P_h = 0 \qquad \text{for} \qquad h = 1, 2, \ldots, s, \tag{I}$$

for holonomic coordinates, and for non-holonomic coordinates, they read:

$$Q_h = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_h}\right) - \frac{\partial L}{\partial p_h} - P_h + \sum_{\nu=1}^{3n} m_\nu \, \dot{x}_\nu \left(\zeta_h^\nu + \sum_{k=1}^{s} \zeta_{hk}^\nu\right) = 0, \qquad \text{for } h = 1, 2, \ldots, s, \tag{II}$$

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(1) Following **Boltzmann**, all rectangular coordinates for the system will be denoted with the same symbol x, and likewise, for the sake of simplicity in notation, each mass of each point will be expressed by three symbols; e.g., $m_1 = m_2 = m_3$.

as **Boltzmann** first found ([1]), in which L is the *vis viva*, and P_h means a generalized force component.

One can now derive Gauss's principle and the general expression for the constraint directly from these generalized Lagrange equation in the present case as follows:

It follows from the equations:

$$Q_1 = 0, \qquad Q_2 = 0, \ldots, \qquad Q_s = 0 \tag{1}$$

that:

$$Q_1\,\delta\ddot{p}_1 + Q_2\,\delta\ddot{p}_2 + Q_3\,\delta\ddot{p}_3 + \cdots + Q_s\,\delta\ddot{p}_s = 0. \tag{2}$$

Furthermore, differentiating the transformation equations will give either:

$$\dot{x}_\nu = \sum_{h=1}^{s} \frac{\partial x_\nu}{\partial p_h}\dot{p}_h \qquad \text{or} \qquad \dot{x}_\nu = \sum_{h=1}^{s} \pi_h^\nu\,\dot{p}_h\,,$$

such that the *vis viva* of the point system will take the form of a quadratic form in the quantities $\dot{p}_h$:

$$L = \sum_{\nu=1}^{3n} \frac{m_\nu}{2}\dot{x}_\nu^2 = \tfrac{1}{2}\sum_{h,k=1}^{s} a_{hk}\,\dot{p}_h\,\dot{p}_k\,. \tag{3}$$

The coefficients $a_{hk} = a_{kh}$ in that form are composed of the quantities $\partial x_\nu / \partial p_k$ (π_h^ν, respectively), so they are functions of the p_h (which do not include time explicitly), and it can be shown that the determinant of the quadratic form L is non-zero ([2]):

$$D \equiv \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ a_{s1} & a_{s2} & \cdots & a_{ss} \end{vmatrix} \neq 0$$

for all values of time t.

One then multiplies equation (2) by $2D$ and gets:

$$\sum_{h=1}^{s} 2Q_h \cdot D \cdot \delta\ddot{p}_h = 0. \tag{4}$$

Now, as is known:

$$D = a_{1h}\,A_{1h} + a_{2h}\,A_{2h} + \ldots + a_{sh}\,A_{sh}\,, \qquad \text{for } h = 1, 2, 3, \ldots, s, \tag{5}$$

([1]) **Boltzmann**, *Prinzipe der Mechanik*, Part II, pp. 109. See also pp. 8 of this article.
([2]) **Boltzmann**, *Prinzipe der Mechanik*, Part II, pp. 35.

in which the A_{rh} shall be the adjoints of D. Equation (4) can then also be written in form:

$$\sum_{h=1}^{s} 2Q_h\,[a_{1h}\,A_{1h} + a_{2h}\,A_{2h} + \ldots + a_{sh}\,A_{sh}]\,\delta\ddot{p}_h = 0. \tag{6}$$

Now, there is another theorem from the theory of determinants that when r is different from h :

$$a_{1h}\,A_{1h} + a_{2h}\,A_{2h} + \ldots + a_{sh}\,A_{sh} = 0. \tag{7}$$

If one then adds those terms (which vanish identically) to the left-hand sides of equation (4) or (6) then one will also have:

$$\begin{aligned}\sum_{h=1}^{s} 2Q_h\,\{&[a_{11}\,A_{1h} + a_{21}\,A_{2h} + \ldots + a_{s1}\,A_{sh}]\,\delta\ddot{p}_1\\ &+ [a_{12}\,A_{1h} + a_{22}\,A_{2h} + \ldots + a_{s2}\,A_{sh}]\,\delta\ddot{p}_2\\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &+ [a_{1s}\,A_{1h} + a_{2s}\,A_{2h} + \ldots + a_{ss}\,A_{sh}]\;\delta\ddot{p}_s\,\} = 0.\end{aligned} \tag{8}$$

For every well-defined h, from (7), all terms that appear in equation (6), except for one, will vanish then. If one adds all of the terms that appear in a column in equation (8) then it will follow that:

$$\begin{aligned}\sum_{h=1}^{s} 2Q_h\,\{&A_{1h}\,(a_{11}\;\delta\ddot{p}_1 + a_{12}\;\delta\ddot{p}_2\, + \ldots + a_{1s}\;\delta\ddot{p}_2\,)\\ &\qquad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &+ A_{sh}\,(a_{s1}\;\delta\ddot{p}_1 + a_{s2}\;\delta\ddot{p}_2\, + \ldots + a_{ss}\;\delta\ddot{p}_2\,)\,\} = 0.\end{aligned} \tag{9}$$

Now equation (3) implies that:

$$\frac{\partial L}{\partial \dot{p}_h} = \sum_{k=1}^{s} a_{hk}\;\dot{p}_k\;, \tag{10}$$

so

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_h}\right) = \sum_{k=1}^{s} a_{hk}\;\ddot{p}_k\;,$$

and therefore $Q_h = a_{h1}\;\ddot{p}_1 + a_{h2}\;\ddot{p}_2 + \cdots + a_{hs}\;\ddot{p}_s\; + \ldots = 0$.

One will then have:

$$\frac{\partial Q_h}{\partial \ddot{p}_1} = a_{h1}\,, \quad \frac{\partial Q_h}{\partial \ddot{p}_2} = a_{h2}\,, \ldots, \frac{\partial Q_h}{\partial \ddot{p}_s} = a_{hs}\,. \tag{11}$$

For that reason, one can also write equation (9) as follows:

$$\begin{aligned}\sum_{h=1}^{s} 2Q_h \Bigg\{ & A_{1h}\left(\frac{\partial Q_1}{\partial \ddot{p}_1}\delta\ddot{p}_1 + \frac{\partial Q_1}{\partial \ddot{p}_2}\delta\ddot{p}_2 + \cdots + \frac{\partial Q_1}{\partial \ddot{p}_s}\delta\ddot{p}_s\right) \\ & + A_{2h}\left(\frac{\partial Q_2}{\partial \ddot{p}_1}\delta\ddot{p}_1 + \frac{\partial Q_2}{\partial \ddot{p}_2}\delta\ddot{p}_2 + \cdots + \frac{\partial Q_2}{\partial \ddot{p}_s}\delta\ddot{p}_s\right) \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + A_{sh}\left(\frac{\partial Q_s}{\partial \ddot{p}_1}\delta\ddot{p}_1 + \frac{\partial Q_s}{\partial \ddot{p}_2}\delta\ddot{p}_2 + \cdots + \frac{\partial Q_s}{\partial \ddot{p}_s}\delta\ddot{p}_s\right)\Bigg\} = 0,\end{aligned} \tag{12}$$

or in summation form:

$$\sum_{h=1}^{s} 2Q_h \sum_{r=1}^{s} A_{rh}\left(\frac{\partial Q_r}{\partial \ddot{p}_1}\delta\ddot{p}_1 + \frac{\partial Q_r}{\partial \ddot{p}_2}\delta\ddot{p}_2 + \cdots + \frac{\partial Q_r}{\partial \ddot{p}_s}\delta\ddot{p}_s\right) = 0. \tag{12′}$$

If one now expands the first sum over h then that will yield the equation:

$$\begin{aligned} & + 2Q_1 \sum_{r=1}^{s} A_{r1}\left(\frac{\partial Q_r}{\partial \ddot{p}_1}\delta\ddot{p}_1 + \frac{\partial Q_r}{\partial \ddot{p}_2}\delta\ddot{p}_2 + \cdots + \frac{\partial Q_r}{\partial \ddot{p}_s}\delta\ddot{p}_s\right) \\ & + 2Q_2 \sum_{r=1}^{s} A_{r2}\left(\frac{\partial Q_r}{\partial \ddot{p}_1}\delta\ddot{p}_1 + \frac{\partial Q_r}{\partial \ddot{p}_2}\delta\ddot{p}_2 + \cdots + \frac{\partial Q_r}{\partial \ddot{p}_s}\delta\ddot{p}_s\right) \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + 2Q_s \sum_{r=1}^{s} A_{rs}\left(\frac{\partial Q_r}{\partial \ddot{p}_1}\delta\ddot{p}_1 + \frac{\partial Q_r}{\partial \ddot{p}_2}\delta\ddot{p}_2 + \cdots + \frac{\partial Q_r}{\partial \ddot{p}_s}\delta\ddot{p}_s\right) = 0.\end{aligned} \tag{13}$$

Now one can once more sum over all terms in the same column of this equation, which are all multiplied by the same factor $\frac{\partial Q_r}{\partial \ddot{p}_h}\delta\ddot{p}_h$. Equation (13) will then go to:

$$\begin{aligned}\sum_{r=1}^{s}\Bigg\{&(2A_{r1}\,Q_1+2A_{r2}\,Q_2+\cdots+2A_{rs}\,Q_s)\frac{\partial Q_r}{\partial\ddot{p}_1}\,\delta\ddot{p}_1\\&+(2A_{r1}\,Q_1+2A_{r2}\,Q_2+\cdots+2A_{rs}\,Q_s)\frac{\partial Q_r}{\partial\ddot{p}_2}\,\delta\ddot{p}_2\\&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\&+(2A_{r1}\,Q_1+2A_{r2}\,Q_2+\cdots+2A_{rs}\,Q_s)\frac{\partial Q_r}{\partial\ddot{p}_s}\,\delta\ddot{p}_s\Bigg\}=0.\end{aligned}\tag{14}$$

If one now divides equation (14) by D and then sets:

$$\frac{1}{D}(2A_{r1}\,Q_1+2A_{r2}\,Q_2+\cdots+2A_{rs}\,Q_s)=\frac{\partial Z}{\partial Q_r}\quad\text{for } r=1,2,3,\ldots,s\tag{15}$$

then Z will likewise be defined to be a quadratic function of the quantities $Q_1, Q_2, \ldots, Q_s$ by those s equations that will take the form:

$$Z=\frac{1}{D}\sum_{\mu,\nu=1}^{s}A_{\mu,\nu}\,Q_\mu\,Q_\nu+\varphi(p_1,\ldots,p_s,\dot{p}_1,\ldots,\dot{p}_s)\tag{16}$$

after one integrates, as one easily convinces oneself by differentiating the latter equation. The function φ enters into it because only the quantities $\ddot{p}_h$ were regarded as variable in equations (11).

Equation (14) will now go to:

$$\sum_{r=1}^{s}\frac{\partial Z}{\partial Q_r}\left(\frac{\partial Q_r}{\partial\ddot{p}_1}\,\delta\ddot{p}_1+\frac{\partial Q_r}{\partial\ddot{p}_2}\,\delta\ddot{p}_2+\cdots+\frac{\partial Q_r}{\partial\ddot{p}_s}\,\delta\ddot{p}_s\right)=0.\tag{17}$$

However, the last expression is nothing but the first variation of Z when only the quantities $\ddot{p}_h$ are varied (so in the Gaussian sense), and everything else is considered to be constant. Therefore, from equation (17):

$$\delta Z=0.\tag{18}$$

The second variation is obviously positive, such that the last equation says that when one varies the motion in question in the Gaussian sense, the constraint:

$$Z = \frac{1}{D} \sum_{\mu,\nu=1}^{s} A_{\mu,\nu}\, Q_\mu\, Q_\nu + \varphi(p_1, \ldots, p_s, \dot{p}_1, \ldots, \dot{p}_s)$$

must be a *minimum* for the actual motion, which results from just the general Lagrangian equations (1).

It is probably immediately obvious that this derivation will be valid for holonomic, as well as non-holonomic, *scleronomic* coordinates, when one now understands the Q_h to mean the left-hand sides of the Lagrangian equations in the form (I) in the former case, but in the form (II) in the latter case, because the argument will remain the same in either case.

B. – Rheonomic, generalized coordinates.

The second, and significantly more difficult, of the two main cases that were cited above shall now be brought under consideration, namely, the case in which *time also enters explicitly* into the transformation equations. Those transformation equations will then read simply:

$$x_\nu = f_\nu(t, p_1, p_2, \ldots, p_s) \qquad \text{for } \nu = 1, 2, 3, \ldots, 3n$$

when they are holonomic, or:

$$dx_\nu = \vartheta_\nu\, dt + \sum_{h=1}^{s} \pi_h^\nu\, dp_h \qquad \text{for } \nu = 1, 2, 3, \ldots, 3n$$

when they are non-holonomic ([1]), but in which the functions ϑ_ν and π_h^ν also include time explicitly. The rheonomic generalized coordinates $p_1, p_2, \ldots, p_s$ shall once more fulfill the conditions on the system:

$$\varphi_1(x_1, x_2, \ldots, x_{3n}) = 0, \quad \varphi_2(x_1, x_2, \ldots, x_{3n}) = 0, \quad \ldots, \quad \varphi_\tau(x_1, x_2, \ldots, x_{3n}) = 0$$

identically, and just as before, they must be *completely-independent* of each other.

The Lagrangian equations of the second kind do not change under that assumption, so they will also appear in the form (I) and (II) here according to whether the coordinates p_h are holonomic or non-holonomic, because a change in those equations would occur only when certain relations exist between the coordinates p_h themselves.

However, in order to be able to go deeper into that second case at all, from the mathematical standpoint, it will be necessary to make some assumption about the nature of the functions that appear here, and for that reason, one will perhaps assume that all of the functions that enter are *single-valued*, *analytic* functions, which is an assumption that can probably be regarded as too broad, rather than too narrow, for the physical problem, in general.

([1]) Cf., **Boltzmann**, *Prinzipe der Mechanik*, Part II.

Now, as differentiating the transformation equations will show, in the present case, one will have either:

$$\dot{x}_\nu = \frac{\partial x_\nu}{\partial t} + \sum_{h=1}^{s} \frac{\partial x_\nu}{\partial p_h} \dot{p}_h \tag{1}$$

or

$$\dot{x}_\nu = \vartheta_\nu + \sum_{h=1}^{s} \pi_h^\nu \dot{p}_h ,$$

so the *vis viva* of the point-system L will no longer take the form of a quadratic function of the quantities $\dot{p}_h$ now, but of a function of degree two in those quantities of the type:

$$L = \sum_{\nu=1}^{3n} \frac{m_\nu}{2} \dot{x}_\nu^2 = \tfrac{1}{2} \sum_{h,k=1}^{s} a_{hk} \dot{p}_h \dot{p}_k + \sum_{h,k=1}^{s} b_h \dot{p}_h + c, \tag{2}$$

in which the coefficients a_{hk}, b_h, and c will themselves include time and the coordinates p_h in any way that depends upon the form of the functions f_ν (ϑ_ν and π_h^ν, resp.).

If one now attempts to actually combine the expressions Q_h here again then one will get from equation (2) that:

a) $$\frac{\partial L}{\partial \dot{p}_h} = \sum_{k=1}^{s} a_{hk} \dot{p}_k + b_h ,$$

b) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_h}\right) = \sum_{k=1}^{s}\left[\frac{\partial a_{hk}}{\partial t} \dot{p}_k + \sum_{l=1}^{s} \frac{\partial a_{hk}}{\partial p_l} \dot{p}_k \dot{p}_l + a_{hk} \ddot{p}_k\right] + \frac{\partial b_h}{\partial t} + \sum_{l=1}^{s} \frac{\partial b_h}{\partial p_l} \dot{p}_l , \tag{3}$$

c) $$\frac{\partial L}{\partial p_h} = \tfrac{1}{2} \sum_{\rho,k=1}^{s} \frac{\partial a_{\rho k}}{\partial p_h} \dot{p}_k \dot{p}_\rho + \sum_{\rho=1}^{s} \frac{\partial b_\rho}{\partial p_h} \dot{p}_\rho + \frac{\partial c}{\partial p_h} .$$

Moreover, it is known that:

d) $$P_h = \sum_{\nu=1}^{3n} X_\nu \frac{\partial x_\nu}{\partial p_h} ,$$

or for non-holonomic coordinates:

$$P_h = \sum_{\nu=1}^{3n} X_\nu \pi_h^\nu ,$$

and finally, from **Boltzmann** ([1]):

([1]) **Boltzmann**, *Prinzipe der Mechanik*, Part II, pp. 106.

e)
$$\zeta_h^\nu + \sum_{k=1}^{s} \zeta_{hk}^\nu = \frac{\partial \dot{x}_\nu}{\partial p_h} - \frac{d}{dt}\left(\frac{\partial x_\nu}{\partial p_h}\right).$$

If one then forms the expression Q_h from these relations 3*b*, *c*, *d*, and possibly *e*, according to the forms (I) or (II) that were presented to begin with, then one will see immediately that for the rheonomic coordinates, as well as for the scleronomic ones, one will have in full generality:

$$\frac{\partial Q_h}{\partial \ddot{p}_r} = a_{hr}, \qquad \text{for} \qquad \begin{cases} h = 1, 2, \ldots, s, \\ r = 1, 2, \ldots, s. \end{cases} \tag{4}$$

For that reason, one should also consider the determinant D of the quadratic parts in the expression for L [equation (2)] here; i.e., the determinant of degree s :

$$D \equiv \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ a_{s1} & a_{s2} & \cdots & a_{ss} \end{vmatrix}.$$

However, one can no longer assert now, as perhaps one would in the first case, that this determinant D must be non-zero for all values of time t, because the proof that was given before was essentially based upon the assumption that L was a homogeneous form, which is indeed no longer the case here. In order to examine the determinant D more closely, it is therefore necessary to actual construct it, and indeed initially under the assumption that the transformation equation are *holonomic* in form. One will then have:

$$\dot{x}_\nu = \frac{\partial x_\nu}{\partial t} + \sum_{h=1}^{s} \frac{\partial x_\nu}{\partial p_h} \dot{p}_h,$$

and the *vis viva* will be:

$$L = \sum_{\nu=1}^{3n} \frac{m_\nu}{2} \dot{x}_\nu^2 = \sum_{\nu=1}^{3n} \frac{m_\nu}{2} \left[\left(\frac{\partial x_\nu}{\partial t}\right)^2 + 2\frac{\partial x_\nu}{\partial t} \sum_{h=1}^{s} \frac{\partial x_\nu}{\partial p_h} \dot{p}_h + \left(\sum_{h=1}^{s} \frac{\partial x_\nu}{\partial p_h} \dot{p}_h\right)^2\right]$$

or also:

$$L = \tfrac{1}{2} \sum_{\nu=1}^{3n} m_\nu \left(\frac{\partial x_\nu}{\partial p_1} \dot{p}_1 + \cdots + \frac{\partial x_\nu}{\partial p_s} \dot{p}_s\right)^2 + \sum_{\nu=1}^{3n} m_\nu \frac{\partial x_\nu}{\partial t} \sum_{h=1}^{s} \frac{\partial x_\nu}{\partial p_h} \dot{p}_h + \tfrac{1}{2} \sum_{h=1}^{3n} m_\nu \left(\frac{\partial x_\nu}{\partial t}\right)^2.$$

If one then compares that to the general form [equation (2)]:

$$L = \tfrac{1}{2} a_{11}\, \dot{p}_1^2 + \tfrac{1}{2} a_{22}\, \dot{p}_2^2 + \cdots + \tfrac{1}{2} a_{ss}\, \dot{p}_s^2 + a_{12}\, \dot{p}_1\, \dot{p}_2 + \cdots + a_{s-1,s}\, \dot{p}_{s-1}\, \dot{p}_s + b_1\, \dot{p}_1 + \ldots$$

then one will see immediately that the determinant of the coefficients a_{hk} will take the following form in the case of rheonomic, holonomic coordinates:

$$D = \begin{vmatrix} \sum_{\nu=1}^{3n} m_\nu \left(\frac{\partial x_\nu}{\partial p_1}\right)^2 & \sum_{\nu=1}^{3n} m_\nu \frac{\partial x_\nu}{\partial p_1}\frac{\partial x_\nu}{\partial p_2} & \cdots & \sum_{\nu=1}^{3n} m_\nu \frac{\partial x_\nu}{\partial p_1}\frac{\partial x_\nu}{\partial p_s} \\ \sum_{\nu=1}^{3n} m_\nu \frac{\partial x_\nu}{\partial p_1}\frac{\partial x_\nu}{\partial p_2} & \sum_{\nu=1}^{3n} m_\nu \left(\frac{\partial x_\nu}{\partial p_2}\right)^2 & \cdots & \sum_{\nu=1}^{3n} m_\nu \frac{\partial x_\nu}{\partial p_2}\frac{\partial x_\nu}{\partial p_s} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{\nu=1}^{3n} m_\nu \frac{\partial x_\nu}{\partial p_1}\frac{\partial x_\nu}{\partial p_s} & \cdots & \cdots & \sum_{\nu=1}^{3n} m_\nu \left(\frac{\partial x_\nu}{\partial p_s}\right)^2 \end{vmatrix}.$$

However, there is nothing more that one can say about the vanishing of that determinant from its form. It therefore important to point out that this symmetric determinant D can also be regarded as the product of two rectangular matrices (†):

$$\begin{bmatrix} m_1 \frac{\partial x_1}{\partial p_1} & m_2 \frac{\partial x_2}{\partial p_1} & \cdots & m_{3n} \frac{\partial x_{3n}}{\partial p_1} \\ m_1 \frac{\partial x_1}{\partial p_2} & m_2 \frac{\partial x_2}{\partial p_2} & \cdots & m_{3n} \frac{\partial x_{3n}}{\partial p_2} \\ \cdots & \cdots & \cdots & \cdots \\ m_1 \frac{\partial x_1}{\partial p_s} & m_2 \frac{\partial x_2}{\partial p_1} & \cdots & m_{3n} \frac{\partial x_{3n}}{\partial p_s} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial p_1} & \frac{\partial x_1}{\partial p_2} & \cdots & \frac{\partial x_1}{\partial p_s} \\ \frac{\partial x_2}{\partial p_1} & \frac{\partial x_2}{\partial p_2} & \cdots & \frac{\partial x_2}{\partial p_1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_{3n}}{\partial p_1} & \frac{\partial x_{3n}}{\partial p_2} & \cdots & \frac{\partial x_{3n}}{\partial p_s} \end{bmatrix}$$

However, since the number of columns s is smaller than the number of rows $3n$ in any event, from a theorem in the theory of determinants ([1]), this symbolic product can also be represented as a sum of $\binom{3n}{s}$ squares, each of which has the general form:

(†) [Translator: I have taken the liberty of modernizing the matrix notation in this product, so not all of the discussion that follows in regard to the theory of determinants is applicable to it. I have included the remarks only for the sake of completeness.]

([1]) **Balzer**, *Determinanten*, § **6**, pp. 48, *et seq.* – **E. Pascal**, *Determinanten*, I, § **7**.

$$m_{r_1}\cdot m_{r_2}\cdots m_{r_s}\cdot\begin{vmatrix}\dfrac{\partial x_{r_1}}{\partial p_1} & \dfrac{\partial x_{r_1}}{\partial p_2} & \cdots & \dfrac{\partial x_{r_1}}{\partial p_s}\\ \dfrac{\partial x_{r_1}}{\partial p_1} & \dfrac{\partial x_{r_2}}{\partial p_2} & \cdots & \dfrac{\partial x_{r_2}}{\partial p_1}\\ \cdots & \cdots & \cdots & \cdots\\ \dfrac{\partial x_{r_1}}{\partial p_1} & \dfrac{\partial x_{r_2}}{\partial p_2} & \cdots & \dfrac{\partial x_{r_s}}{\partial p_s}\end{vmatrix}^2,$$

in which $r_1, r_2, r_3, \ldots, r_s$ mean any combination of class s of the elements $1, 2, 3, \ldots, 3n$ without repetition. However, the coordinates $x_1, x_2, \ldots, x_{3n}$ are not independent of each other, but are coupled by the τ condition equations:

$$\varphi_1(x_1, x_2, \ldots, x_{3n}) = 0, \qquad \varphi_2(x_1, x_2, \ldots, x_{3n}) = 0, \qquad \ldots, \qquad \varphi_\tau(x_1, x_2, \ldots, x_{3n}) = 0,$$

which is why one can think of representing τ of the quantities x in these equations as functions of the remaining $3n - \tau = s$ quantities x, which will then be independent of each other – say, in the form:

$$x_{s+1} = \psi_1(x_1, x_2, \ldots, x_s), \qquad x_{s+2} = \psi_1(x_1, x_2, \ldots, x_s), \quad \ldots, \quad x_{3n} = \psi_1(x_1, x_2, \ldots, x_s)\,.$$

However, one will then have:

$$\frac{\partial x_{s+\lambda}}{\partial p_h} = \frac{\partial\psi_\lambda}{\partial x_1}\frac{\partial x_1}{\partial p_h} + \frac{\partial\psi_\lambda}{\partial x_2}\frac{\partial x_2}{\partial p_h} + \cdots + \frac{\partial\psi_\lambda}{\partial x_s}\frac{\partial x_s}{\partial p_h} \tag{5}$$

for all values of $\lambda = 1, 2, \ldots, \tau$, and $h = 1, 2, \ldots, s$.

If one now introduces this representation (5) into the individual determinant-squares in the development above then, as is clear immediately, a factor that is combined with $\left(\dfrac{\partial\psi_\lambda}{\partial x_h}\right)^2$ will emerge from each of them, with exception of the first one, and the remaining determinant-squares will then coincide with the first one. For that reason, one can also represent the entire original determinant D as the square of those individual first sub-determinants in the form:

$$D = \left[m_1\, m_2 \cdots m_s + \Phi\left(m_1, \cdots, m_s, \left(\frac{\partial \psi_1}{\partial x_1}\right)^2, \ldots, \left(\frac{\partial \psi_\tau}{\partial x_s}\right)^2 \right)\right] \begin{vmatrix} \dfrac{\partial x_1}{\partial p_1} & \cdots & \dfrac{\partial x_s}{\partial p_1} \\ \dfrac{\partial x_1}{\partial p_2} & \cdots & \dfrac{\partial x_s}{\partial p_2} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial x_1}{\partial p_s} & \cdots & \dfrac{\partial x_s}{\partial p_s} \end{vmatrix}^2 .$$

As would emerge immediately from its form, the expression in square brackets cannot vanish, and can only be positive, because the m_ν are positive quantities, since they are material masses, and the function Φ can also be only positive or zero, since it is a sum of quadratic terms. Therefore, the vanishing of the determinant D is possible only when the determinant of degree s :

$$\Delta \equiv \begin{vmatrix} \dfrac{\partial x_1}{\partial p_1} & \dfrac{\partial x_2}{\partial p_1} & \cdots & \dfrac{\partial x_s}{\partial p_1} \\ \dfrac{\partial x_1}{\partial p_2} & \dfrac{\partial x_2}{\partial p_2} & \cdots & \dfrac{\partial x_s}{\partial p_2} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial x_1}{\partial p_s} & \dfrac{\partial x_2}{\partial p_s} & \cdots & \dfrac{\partial x_s}{\partial p_s} \end{vmatrix}$$

vanishes.

However, the determinant Δ, whose vanishing then represents a necessary and sufficient condition for the vanishing of the original determinant D, is now nothing but the functional determinant of the single-valued, analytic functions:

$$x_1 = f_1(t, p_1, p_2, \ldots, p_s), \quad x_2 = f_2(t, p_1, p_2, \ldots, p_s), \quad \ldots, \quad x_s = f_s(t, p_1, p_2, \ldots, p_s) .$$

For that reason, Δ itself is a single-valued, analytic function of the quantities $p_1, p_2, \ldots, p_s$, and as such, must vanish identically when it also vanishes identically in any arbitrarily-small time-interval t' to t''. In that way, however, one would obviously define a relation between the quantities $p_1, p_2, \ldots, p_s$, which would contradict the complete independent of the generalized coordinates from each other that was assumed. One will then have the *identical* vanishing of the original determinant D in any time interval that is also not too small is *excluded*, and all that will remain is that the determinant vanishes for only isolated special moments t_0, t_1, t_2, …, while it will otherwise *be non-zero, in general*.

However, as long as that determinant D is non-zero, from of the results that were obtained in equation (4), one can, with no further analysis, also repeat precisely the same conclusions for the case of rheonomic coordinates that one reached in the case of scleronomic coordinates, such that one will get the same expression for the constraint Z (which must be a minimum for the actual motion) as in the first case, namely:

$$Z = \frac{1}{D}\sum_{\mu,\nu=1}^{s} A_{\mu\nu}\, Q_\mu\, Q_\nu + \varphi(p_1, \ldots, p_s, \dot{p}_1, \ldots, \dot{p}_s),$$

for all intervals in which the assumption $D \neq 0$ is fulfilled.

Naturally, time will also appear *explicitly* in that expression for the constraint Z now, just as it does in the quantities $A_{\mu\nu}$, and D.

However, as far as each individual moment is concerned in which the determinant D actually vanishes (and for which the derivation that was given above would no longer be meaningful then), from a purely-mathematical standpoint, one can probably find the modified general expression for the constraint at those moments by a passing to the limit under special assumptions on the nature of the functions that appear. However, such detailed mathematical investigations must be skipped over here, since more detailed specializations of the assumptions are not permissible with no further discussion for the present general physical problem, and since on the other hand the mechanical principles will no longer be actually applicable when one selects individual moments at which the time and coordinates do not vary, but are regarded as fixed. In general, one also cannot decide when that exceptional case can actually occur, or how often, without going into special cases.

Finally, in regard to the *non-holonomity* of the rheonomic coordinates, which has been excluded up to now, it should be pointed out that it will exert no essential influence on the course of the investigation here either, just as in the first case, except that the single-valued analytic functions π_k^ν (which are generally not better known) will enter in place of the differential quotients $\partial x_\nu / \partial p_k$, and the Lagrangian equations will again be employed in the form (II). However, all conclusions will remain correct, with the exception of the decomposition of the determinant D, which can no longer be performed here, in general. Nonetheless, that fact, which will become clear later, has no further influence on the entire argument, because just like the functional determinant Δ, D is obviously also itself a single-valued analytic function of the quantities p_1, p_2, …, p_s that cannot vanish identically without disturbing the assumed mutual-independence of the generalized coordinates. Under the assumption of single-valued, analytic functions, *the general expression for the constraint Z is precisely the same for rheonomic coordinates, and indeed for holonomic, as well as non-holonomic, as it is for sceleronomic coordinates* (except for individual moments that might possibly arise when $D = 0$).

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"Über eine dem Gauß'schen Prinzipe des kleinsten Zwanges entsprechende Integralform," Sitz. Kais. Akad. Wiss. **122** (1913), 721-738.

On an integral form that corresponds to Gauss's principle of least constraint

By

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Contents

I. – Statement of the problem.

It is known that the principles of mechanics that **A. Voss** ([1]) called principles of the "third kind," can be reduced to ones of a different kind that **Voss** (*loc. cit.*) called the "second kind," which are the ones that involve the equations of motion of mechanics directly. Principles of the "third kind" include, e.g., Hamilton's principle and the principle of least action, while ones of the "second kind" include the principle of virtual velocities, d'Alembert's principle, and Gauss's principle of least constraint. Lagrange's equations, in their first form – that is, the one in which the Lagrange multipliers appear – can be counted amongst the "second kind," while the ones of the "third kind" include the ones in which knowledge of the *vis viva* must be assumed (viz., Lagrange's equations in their second form).

For example, Hamilton's principle can be reduced to that of **d'Alembert**, and can be shown to be completely equivalent to it; i.e., if the one were true then the other would be true, as well, and conversely. Now, as is known, the cited principles of the "second kind" are completely equivalent to each other. On the other hand, the principles of the "third kind" are likewise completely equivalent to each other, since they can always be reduced to ones of the "second kind."

([1]) **A. Voß**, "Die Prinzipien der rationellen Mechanik," Encykl. der math. Wiss. IV, 1, pp. 10.

Such considerations prompted Herrn Prof. Dr. **A. Wassmuth** to remark in a seminar on mathematical physics in Graz that there must be a principle that has the form of a time integral between fixed time limits and can be reduced to Gauss's principle of least constraint in a manner that is analogous to the way that **Hamilton**'s principle or the principle of least action reduces to d'Alembert's principle. With the terminology that was given above, a principle of the "third kind" should be given that relates formally to Gauss's principle of least constraint, which belongs to the "second kind," in perhaps the way that Hamilton's principle relates to **d'Alembert**'s. Elaborating upon that formal analogy is the goal of the present article.

II. – Generalities concerning the variational principle and the associated variational conditions.

The proof of the equivalence of a principle of the "third kind" with one of the "second kind" will be accomplished with the help of an identity that has the form:

$$\int_{t_1}^{t_2} (\delta L - A)\, dt = \int_{t_1}^{t_2} \sum_{i=1}^{3n} (X_i - m_i\, \ddot{x}_i)\, \delta \ddot{x}_i \cdot dt\,, \tag{1}$$

in the special case of the equivalence of Hamilton's and d'Alembert's principles.

[In this, L means the *vis viva*, and A means the virtual work done on the system, which is thought of as consisting of n mass-points. (There are therefore $3n$ rectilinear coordinates for a point in it.) $\ddot{x}$ means an acceleration, X means an explicit force, t_1 and t_2 are fixed time limits (viz., the starting point and end point of the motion), and δ is the symbol for a variation. (For more details, see below.)]

In the case in question, in order to prove the equivalence of an integral that is presented with Gauss's principle of least constraint from this, one must start by finding an identity that is expressed analogously to equation (1) and whose right-hand side is obviously composed of the expression:

$$\int_{t_1}^{t_2} \sum_{i=1}^{3n} (X_i - m_i\, \ddot{x}_i)\, \delta \ddot{x}_i \cdot dt\,, \tag{2}$$

since that is the form of Gauss's principle that is analogous to d'Alembert's principle.

If the identity, thus-found, can lead to that conclusion in a manner that is analogous to the way that one concludes the equivalence of Hamilton's principle and d'Alembert's ([1]) from the identity (1) then the left-hand side of that new identity will yield the desired integral form.

The considerations to be made shall be referred to a system of n discrete mass-points, so one will be dealing with $3n$ independent variables in the case of rectilinear coordinates. However, the number of degrees of freedom in the system can be diminished by the condition equations that are imposed on the system. The same assumptions shall be made about those condition equations that

([1]) **Boltzmann**, *Mechanik*, pp. 7, *et seq.*

Boltzmann ([1]) drew upon in order to represent the equivalence of Hamilton's principle with d'Alembert's by analogy. The $3n$ rectilinear coordinates will all be denoted by the symbol x with indices added [likewise following **Boltzmann** ([2])]. Each of the n masses will then enter into only three of the notations in different forms, and indeed m_i , m_{i+1} , m_{i+2} ($i = 1, 4, 7, \ldots, 3n - 2$) will always be regarded as identical. The dots over the symbols x_i will mean derivatives of the coordinate values with respect to time, and indeed, one dot will mean the first derivative, or the velocity in the direction of the coordinate in question, two dots will mean the second derivative (i.e., the acceleration in that direction), and so on. The X_i refer to the explicit forces that correspond to the coordinates with the same indices, that is, the forces that do not arise form constraints that are expressed by the condition equations. The so-called variation of a quantity will be suggested by the operation symbol δ; i.e., a fully-general arbitrary increase in the quantity in question that is "infinitely small," as one says in the customary terminology of mechanics. The demand that is expressed in that way can be posed more precisely by saying that the absolute values of the variations must all lie below a positive quantity that is regarded as arbitrarily-small, but well-defined. Now, those variations are *not* by any means *identical* to the increases in the coordinates or other variables that *actually occur* in the course of motion. The latter increases will be referred to as *differentials* of the quantities in question, in the sense that they should be the changes in the quantities in question that *actually* take place in a time interval dt. One will further require that the variations must be *compatible* with the *conditions* on the system. Should the value of a variable – e.g., – always remain smaller than a well-defined quantity, then the "varied" value of that variable could not be greater than or equal to that quantity, either.

If the three coordinates of a mass-point were to take on values that are "varied," with the meaning that was just given to that term, then one would have to refer to the position of the mass-point that is established in that way as a *varied position*. One must once more note in that regard that the varied values of the coordinates must also satisfy the condition equations. For example, if a mass-point is to remain on a given surface then the varied coordinates must also fulfill the equation of the surface, i.e., the varied position must belong to the surface. Corresponding statements will be true for the "varied position" of several mass-points. Any motion that consists of a series of positions that are compatible with the condition equations will be called a *possible* motion of the system. Any of those possible motions will be distinguished as *unvaried*, and the values of the variables at each time point that belong to that special motion will be called *unvaried* values of the variables for that time point. Those values are only to be regarded as *arbitrary zero-points* of the variations ([3]). The remaining possible motions are called *varied motions*, and correspondingly, the term *varied value* of a variable at a time point will now mean each possible value that is different from the *unvaried* value.

It is known that the principles of the same kind (in the terminology that was used in the introduction) differ from each other by the individual variational conditions that are associated with each of them, that is, by well-defined instructions for isolating certain manifolds of narrower

([1]) *Mech.*, II, pp. 2, *et seq.*

([2]) *Mech.*, II, pp. 228, *et seq.*

([3]) The unvaried motion is often referred to as "actual," which is however premature and unclear. Indeed, it is only by means of the principle in question that one can first show which of all possible motions are actual, since one can choose any possible motion to be unvaried at the outset.

scope from the infinite manifold of all motions of the system that are compatible with the conditions from which the variations are to arise, and which must still be infinite in order to justify the necessary arbitrariness. For example, only those variations of the coordinates should be employed in Hamilton's principle that *do not include a variation of the time*; i.e., each state of a varied motion will be assigned to a state of the unvaried motion that are each passed through simultaneously, while the general case is the one in which a state of unvaried motion is assigned an entirely arbitrary state varied motion. Another case of variational conditions is this one: For Gauss's principle of least constraint, the so-called "Gaussian variation" will be employed, whose variational conditions read: For the state of motion in question, the variations of all coordinates and velocities will vanish, and only the variations of the acceleration should be non-zero. Now, the latter variational conditions are the ones that must be set down in order to succeed in exhibiting the desired identity for the case in question, because the right-hand side of the identity to be exhibited includes only variations of the accelerations [(2)].

III. – The variational conditions in special cases.

In what follows, Gauss's method of variation shall be considered more closely by using **Boltzmann**'s representation ([1]).

A *motion that is varied in the Gaussian manner* shall be characterized by the fact that, just as the coordinates are certain functions of time:

$$x_i = f_i(t), \qquad i = 1, 2, 3, \ldots, 3n \tag{3}$$

for the unvaried motion, here, they will be somewhat different functions $\varphi_i(t)$ of time that likewise fulfill the condition equations of the system, and one further requires of them that at a well-defined time (which can be chosen arbitrarily), the values of the functions themselves and their first derivatives of the coordinates with respect to time (so, the velocities) are all equal to the corresponding values for the unvaried motion. Only their second derivatives with respect to time (i.e., the accelerations) at that time point shall have somewhat different values from the ones in the unvaried motion, and indeed the increases that the accelerations experience under the transition from the unvaried motion to the varied one shall be denoted $\delta\ddot{x}_i$, which is consistent with the previous conventions. All δx_i and $\delta\dot{x}_i$ shall be equal to zero for any arbitrarily-chosen, but well-defined, time point, while the $\delta\ddot{x}_i$ should be non-zero. Naturally, that is true for only the precise moment in time considered, but for, say, a finite time interval, since otherwise all also $\delta\ddot{x}_i$ would have be zero during that interval. That type of variation is therefore *point-wise*; i.e., it initially makes no sense to speak of a global variation of the motion. In that regard, however, there exists no fundamental distinction from the variational method that appears for Hamilton's (d'Alembert's, respectively) principle, because in the latter, an arbitrary, but well-defined, point in the unvaried motion is initially associated with a point of the varied motion for which the variations of the

([1]) *Mech. I*, pp. 209, *et seq.*

coordinates have certain non-zero values. Now, in both cases, one therefore treats the transition from the variation of the motion at one time point to the *global* variation of the motion ([1]). In the case of Hamilton's (d'Alembert's, respectively) principle, that will be effected as follows ([2]): One advances from one moment of time to the next and applies the aforementioned point-wise variation everywhere. One will then obtain values of the variations δx_i that are arranged with no connection to time, such that the integral $\int \delta x_i \cdot dt$ would make no sense, since the quantity δx_i under the integral sign is an absolutely-discontinuous function of time. However, one can always select infinitely many arrangements of variations from the infinite manifold of those variations that are subject to the restriction that all δx_i should be continuous, but otherwise arbitrary, functions of time, so ones that might be represented in the form:

$$\delta x_i = \varepsilon f_i(t), \qquad i = 1, 2, 3, \ldots, 3n, \tag{4}$$

in which the $f_i(t)$ represent arbitrary, but continuous and finite functions of time, and ε is a positive quantity that is common to all coordinates and all times, and which can be made arbitrarily small in order to correspond to the requirement that was imposed above that the variations should be "infinitely small." Now, one can call the temporal sequence of those varied positions a "varied motion" and then the integral that was defined above will make sense as an integral of a continuous function.

However, if one would like to take advantage of the same notions for the variations $\delta \ddot{x}_i$ in the Gaussian method of variation then one would find that this process would not be possible, because when the variations of the accelerations are continuous functions of time, the variations of the velocities, and therefore those of the coordinates, as well, would *not be continually zero* or "infinitely-small" of higher order than the variations of the accelerations, as one must have for the corresponding advance from one moment in time to the next in the case of the Gaussian method of variation. Namely, if one $\delta \ddot{x}_i$ is a continuous function then it must be always positive or always negative in a time interval that is chosen to be correspondingly-small (as long as it is not, say always zero, which must be excluded here, however). As a result of the relation:

$$\delta \ddot{x}_i = \frac{d\delta \dot{x}_i}{dt}, \tag{5}$$

the $\delta \dot{x}_i$ that belongs to that time interval must increase or decrease, so it must be distinct from the value zero. However, as was mentioned expressly here already, the tacit assumption is made in that ([3]) that *time is not varied*, which was just expressed by equation (5).

([1]) The variational method for the principle of least action also behaves in precisely that way, since the variation of the time will first become essential when one considers (exhibits, respectively) the globally-varied motion.

([2]) **Boltzmann**, *Mech.*, II, pp. 3, *et seq.*

([3]) **Boltzmann**, *Mech.*, II, pp. 4, remark.

One must then look for another method in order to make it possible to construct integrals of the type $\int \delta \ddot{x}_i \, dt$ in order to go from the point-wise variations of the motion to global ones, respectively.

To that end, the aforementioned advance from one moment in time to the next in the motion shall be subjected to a more detailed consideration for the case in question. A mass-point will be considered that traverses a given path with a given velocity at each point under the unvaried motion. The motions that are varied in the Gaussian manner in this case can be exhibited as follows: One imagines a second path that has a point in common with the original one and whose tangent at that point coincides with the tangent to the original path at that point. Hence, if the mass-point at the point common to both paths advances simultaneously under the two motions (namely, the unvaried motion and the one that arises by traversing the varied path) with velocities that have the same absolute value and the same direction then one will, in fact, be dealing with motion that is varied in the Gaussian manner, which is generally the case, so the acceleration of the mass-point at the common point to the paths will not be the same for the varied and unvaried motions. That can be ensured, e.g., in such a way that the accelerations in the common direction of the path are equal, and the varied path possesses a different curvature from the original path at the common point. Now, Gauss' principle of least constraint says that for every individual point that is considered in that way, of the infinite manifold of possible motions that are varied in that way, the motion that actually takes place will be the one for which the relation:

$$\sum_{i=1}^{3n} (m \, \ddot{x}_i - X_i) \, \delta \ddot{x}_i = 0 \tag{6}$$

exists.

The advance from one moment in time to the next in the motion shall now be carried out. As was suggested, one should not regard the path as being globally varied, but rather, one should regard it as composed of very many pieces that always deviate from the paths that arise from the point-wise variation that was just described, and which one represents at each point of the unvaried path. Geometrically, that is expressed by saying: One thinks of each point of the original path as being endowed with a family of paths that are *varied in the Gaussian manner* (those paths shall also be referred to in that way now). In that way, one will get a doubly-infinite manifold of curves that possesses an *envelope*, however (say, when they are all represented in a plane), that is just the original path. Now, the motion of the mass-point at any point of that path is to be varied in such a way that the state of motion is taken, not from the unvaried motions, but from the simply-infinite manifold of motions that are varied in the Gaussian manner whose motions at that point are appropriate. However, the mass-point should continually remain on the original path and possess the prescribed velocity at each point. One can see from this that the choice of varied path from which the state of motion shall be taken will no longer be entirely arbitrary, since the acceleration of the varied motion (i.e., now that would be the motion that is assembled, so to speak, from the individual motions that are varied in the Gaussian manner) must not be always greater than or always less than the unvaried motion in any small enough time interval, since otherwise the mass-point would reach a velocity that is different from the prescribed one. If one now lets the pieces that are associated with the individual point-wise-varied paths decrease without limit then the variations of the acceleration when one traverses the path that one thinks of as composite, which now represents the globally-varied path, cannot be always positive or always negative in any sufficiently-small time interval. If they are not to be constantly zero then they can only be represented by a function of time that *changes its sign arbitrarily often in any sufficient small time interval*. However, such a function is *not integrable*, and that is, in turn, the same result that Boltzmann considered in his remark.

As was mentioned before, this argument is true only for the case in which time is not varied. If one were to introduce a *variation of time* [i.e., one would no longer compare the acceleration of the unvaried motion at any well-defined moment in time t with the acceleration of the (globally) varied motion that takes place at the same moment in time, but with an acceleration that belongs to the moment in time $t + \delta t$ of the varied motion] then one could no longer justify the statement that was just made about the variation of the acceleration as a function of time, and that suggests the idea of seeking to exhibit the varied motion with the help of the variation of time.

IV. – Exhibiting the variational conditions that are suitable to the case in question.

It shall now be shown that one can always arrive at a varied motion for which the variations of the accelerations are *integrable* and which likewise correspond to the other requirements that were just imposed by introducing a variation of time.

The argument will proceed along the following train of thought:

1. It will be shown that there are always infinitely many different system of $\delta \ddot{x}_i$ that are defined with no variation of time, yield a Gaussian variation, and therefore, as was shown, are not integrable.

2. It will be shown that for each such arrangement of $\delta \ddot{x}_i$, there is a variation of time (which will be denoted by δt) by whose introduction, the distribution of variations of the accelerations in time will become integrable, which shall be suggested by the notation $\bar{\delta} \ddot{x}_i$.

3. It will be shown that the time variation δt can be chosen in such a way that the variations of the velocities and coordinates that it produces will become infinitely small of higher order than the variations of the accelerations when the latter decrease without limit.

4. It will be shown that, in addition, the variation δt can be chosen in such a way that the integral of the form $\int F \, \bar{\delta} \ddot{x}_i \, dt$ will not always be zero, regardless of the function F, which will be important for our later conclusions.

It will then follow that:

1. It was already recognized to be necessary that the variations of the accelerations should be functions of time that change their signs arbitrarily often in any sufficiently small time interval. (From now on, in this argument, a single function will be considered by which the variations of a single acceleration can be represented. The results will then be valid for all accelerations directly.) Such a function can be given in the following way: One divides the interval t_a to t_b in which that function is to be regarded into an even number n of equal pieces τ_n and then determines that: The values of the functions $\delta\ddot{x}_i$ at a point of the subdivision that is separated from the starting point t_a of the interval by an even number of sub-intervals τ_n, and thus, at those points that are determined by:

$$t_+ = t_a + \mu\,\tau_n\,, \qquad \mu = 0, 2, 4, \ldots, n,$$

have finite, positive values that are taken from an arbitrarily-given function $f(t)$ that is to be continuous. [$f(t)$ must then have values that are always positive.] In that, one can ignore a constant that can be made arbitrarily small and which multiplies the all values of the function. Furthermore (with the same addendum), the values of the function shall be negative at the remaining "odd" points of the subdivision, namely:

$$t_- = t_a + \nu\,\tau_n\,, \qquad \mu = 1, 3, 5, \ldots, n-1,$$

which will be (up to absolute value) the arithmetic mean of the values of the function at the two neighboring "even" points of the subdivision:

$$\left|\,(\delta\ddot{x}_i)_\nu\right| = \frac{(\delta\ddot{x}_i)_{\nu-1} + (\delta\ddot{x}_i)_{\nu+1}}{2}.$$

One now lets n get bigger and bigger, which will make τ_n become smaller and smaller. The arrangement of values that arises in that way when lim $n = \infty$ or lim $\tau_n = 0$ will then represent a function with the required properties. However, due to the arbitrary choice of the positive values of the function, there will obviously be infinitely many such functions. Such a function can be represented graphically as two quasi-curves that are reflected in the axis $\delta\ddot{x}_i = 0$; i.e., the t-axis.

2. The variation δt, which is to be regarded as a function of time, shall be chosen in such a way that it has the value zero for all $t_+ = t_a + \mu\,\tau_n$, for which $\delta\ddot{x}_i$ is positive, by assumption, and has the value τ_n for all $t_- = t_a + \nu\,\tau_n$, for which $\delta\ddot{x}_i$ is negative. In that way, one will find that *all* points of the subdivision will be assigned a *positive* value of the function now. In the lim $\tau_n = 0$, the new assignment of values (which is not expressed by $\bar{\delta}\ddot{x}_i$) will go to the positive quasi-branch of the function $\delta\ddot{x}_i$, which is, however, an *actual curve* now. As is immediately clear, the function $\bar{\delta}\ddot{x}_i$ is continuous, and therefore *integrable*. It should be remarked here that this time variation δt has no connection whatsoever to the accelerations $\delta\ddot{x}_i$, and indeed should not be put on the same level as them in that regard.

3. Naturally, the velocity $\dot{x}_i$ of the unvaried motion will be associated with another velocity of the varied motion, which will be denoted by $\dot{x}_i + \bar{\delta}\dot{x}_i$, point-by-point, by the introduction of a variation δt of time, as long as δt is not just equal to zero. Whereas $\delta\dot{x}_i$ would be equal to zero at any time (which is an assumption of the Gaussian variation), that is not the case for $\bar{\delta}\dot{x}_i$. One has the relation:

$$\bar{\delta}\dot{x}_i = \ddot{x}_i \cdot \delta t . \tag{7}$$

One now remarks that: The variations $\delta\ddot{x}_i$ (and naturally, the variations $\bar{\delta}\ddot{x}_i$, as well) shall be *infinitely small*; i.e., they shall be represented by:

$$\delta\ddot{x}_i = \varepsilon\,\varphi_i\,(t),$$

in which ε is a constant that can be made infinitely small. Likewise, one has:

$$\bar{\delta}\ddot{x}_i = \varepsilon\,\bar{\varphi}_i(t) .$$

Now, the arguments that were presented in 1. and 2. are also true for finite values of $\delta\ddot{x}_i$, since the constant ε is virtually absent from it. In contrast to those finite values of the function, the quantity τ_n, which converges to zero, is infinitely small. There is therefore nothing that prevents one from assuming that in the case where the $\delta\ddot{x}_i$ themselves become infinitely small upon multiplying by the constant ε, the $\tau_n = \delta t$ will become *infinitely small of higher order* than the $\delta\ddot{x}_i$ (say, finite functions that are multiplied by ε^2), since no special assumptions at all were make about the type of passage to the limit lim $\tau_n = 0$. As a result of the relation (7) and the further one:

$$\bar{\delta}x_i = \dot{x}_i \cdot \delta t, \tag{8}$$

the variations $\bar{\delta}\dot{x}_i$ and $\bar{\delta}x_i$ will now also be *infinitely small of higher order* than the variations $\delta\ddot{x}_i$ (the variations $\bar{\delta}\ddot{x}_i$, respectively), since $\ddot{x}_i$ and $\dot{x}_i$ are finite.

4. One can see immediately that an integral of the form $\int F\,\bar{\delta}\ddot{x}_i\,dt$ will certainly not be zero, in general, since the function $\bar{\delta}\ddot{x}_i$ is positive over the entire interval.

It should be added that, as is immediately clear, one can also impose the condition on the functions $\bar{\delta}\ddot{x}_i$ that their values should be zero at two well-defined fixed time-points, which might be called t_1 and t_2.

With that, it is shown that, as predicted, there are always infinitely many different systems of motions that will yield a global variation of the motion when they are varied (point-wise) in the

Gaussian manner and that correspond to the following conditions on the variations: The variations of all coordinates and velocities are equal to zero at all times, and the variations of the accelerations are non-zero, integrable functions of time.

One can add yet another condition on the variations arbitrarily that is expressed by:

$$\bar{\delta}\dddot{x}_i = \frac{d\bar{\delta}\ddot{x}_i}{dx}. \tag{9}$$

The justification for it will become clear immediately, since that convention implies no consequences for the variations of the derivatives of the coordinates with respect to time of order less than three.

It is now possible to address the formal calculation, and indeed on the grounds of the following four variational conditions:

$$\delta x_i = 0 \text{ for all times,} \tag{I}$$

$$\bar{\delta}\dot{x}_i = 0 \text{ for all times,} \tag{II}$$

$$\bar{\delta}\ddot{x}_i \text{ will be non-zero and integrable,} \tag{III}$$

$$\bar{\delta}\dddot{x}_i = \frac{d\bar{\delta}\ddot{x}_i}{dx}. \tag{IV}$$

V. – Formal implementation.

We shall now address the problem of establishing the identity that was mentioned in section II, whose right-hand side is given already.

To that end, we shall first establish the assumptions of a mechanical nature. The *vis viva* of the system of n mass-points will be considered to be a quadratic form in the velocities $\dot{x}_i$:

$$L \equiv \tfrac{1}{2}\sum_{i=1}^{3n} m_i\,\dot{x}_i^2\ . \tag{10}$$

The virtual work of the explicit forces X_i shall be denoted by δA :

$$\delta A \equiv \sum_{i=1}^{3n} X_i\,\delta x_i\ . \tag{11}$$

(From now on, for the sake of brevity, only the indices will be written in the summations, without specifying the range over which they vary, since it will always be 1 to $3n$.)

One now forms the following expressions:

$$\frac{dL}{dt} = \sum_i m_i \, \dot{x}_i \, \ddot{x}_i \,,$$

$$\frac{d^2 L}{dt^2} = \sum_i m_i \, [\ddot{x}_i^2 + \dot{x}_i \, \dddot{x}_i] \,,$$

$$\delta \frac{d^2 L}{dt^2} = \sum_i m_i \, [2 \ddot{x}_i \, \delta \ddot{x}_i + \dot{x}_i \, \delta \dddot{x}_i + \dddot{x}_i \, \delta \dot{x}_i \,] \,, \tag{12}$$

and furthermore:

$$\frac{d\delta A}{dt} = \sum_i \left(\frac{dX_i}{dt} \delta x_i + X_i \frac{d\delta x_i}{dt} \right),$$

$$\frac{d^2 \delta A}{dt^2} = \sum_i \left(\frac{d^2 X_i}{dt^2} \delta x_i + 2 \frac{dX_i}{dt} \frac{d\delta x_i}{dt} + X_i \frac{d^2 \delta x_i}{dt^2} \right). \tag{13}$$

Those relations will be valid in full generality when the operation on a function of several variables that is suggested by the symbol δ is performed according to the rule:

$$\delta F \,(x_1, x_2, \ldots) = \frac{\partial F}{\partial x_1} \delta x_1 + \frac{\partial F}{\partial x_2} \delta x_2 + \ldots \tag{14}$$

When one brings the variational conditions (I) to (IV) into play, one can see that one can calculate formally as if the time were not varied at all. Namely, one must first perform the operations of differentiation and variations according to the rule in equation (14), which implies the results in equations (12) and (13) in a completely general way. One then replaces the differential quotients $\frac{d\delta x_i}{dt}$ ($\frac{d^2 \delta x_i}{dt^2}$, respectively) with $\delta \dot{x}_i$ ($\delta \ddot{x}_i$, respectively) in the right-hand side of equation (13), since they are considered to have no time variation, corresponding to equation (5). One now carries out the time variation that was described thoroughly above by setting the variations δx_i and $\delta \dot{x}_i$ equal to zero, which means that the $\delta \ddot{x}_i$ must be replaced with the $\bar{\delta} \ddot{x}_i$, and from 3. in section IV, that must produce no variations in the values of the velocities and coordinates. To abbreviate, the variations $\bar{\delta} \ddot{x}_i$ will once more be denoted by $\delta \ddot{x}_i$ now. That will legitimize the aforementioned formal method of calculation.

One also gets equations (12) and (13) by performing the aforementioned operations:

$$\delta \frac{d^2 L}{dt^2} = \sum_i m_i \, [2 \ddot{x}_i \, \delta \ddot{x}_i + \dot{x}_i \, \delta \dddot{x}_i \,] \,, \tag{15}$$

$$\frac{d^2\delta A}{dt^2} = \sum_i X_i\,\delta\ddot{x}_i\,. \tag{16}$$

If one subtracts equation (16) from equation (15) and integrates the difference between fixed, but arbitrary, time limits t_1 and t_2, for which, as was suggested above, the convention is made that:

$$(\delta\ddot{x}_i)_{t_1} = (\delta\ddot{x}_i)_{t_2} = 0, \qquad i = 1, 2, 3, \ldots, 3n, \tag{V}$$

then one will get:

$$\int_{t_1}^{t_2}\left(\delta\frac{d^2L}{dt^2} - \frac{d^2\delta A}{dt^2}\right)dt = \int_{t_1}^{t_2}\sum_i\left(2m_i\,\ddot{x}_i\,\delta\ddot{x}_i + m_i\,\dot{x}_i\,\frac{d\delta\ddot{x}_i}{dx} - X_i\,\delta\ddot{x}_i\right)dt\,. \tag{17}$$

If one integrates the second term with the help of partial integration then it will give:

$$\left|\sum_i m_i\,\dot{x}_i\,\delta\ddot{x}_i\right|_{t_1}^{t_2} - \int_{t_1}^{t_2}\sum_i m_i\,\ddot{x}_i\,\delta\ddot{x}_i\cdot dt\,.$$

The first sum is equal to zero, as a result of the condition (V). One will then get:

$$\int_{t_1}^{t_2}\left(\delta\frac{d^2L}{dt^2} - \frac{d^2\delta A}{dt^2}\right)dt = \int_{t_1}^{t_2}\sum_i(m_i\,\ddot{x}_i - X_i)\,\delta\ddot{x}_i\cdot dt\,. \tag{VI}$$

That is the desired identity. It shows the *complete* equivalence of the principle that corresponds to setting the left-hand side equal to zero:

$$\int_{t_1}^{t_2}\left(\delta\frac{d^2L}{dt^2} - \frac{d^2\delta A}{dt^2}\right)dt = 0 \tag{VII}$$

and *Gauss's principle of least constraint* under the assumptions that were made here. That requires that one must prove that the principle in equation (VII) follows from Gauss's principle, and conversely. The first part of that proof is obvious. If one starts, conversely, from the validity of the form (VII) then one must reason as follows: The integral is a definite integral between fixed, but arbitrary, limits, so it must assume the value zero for all pairs t_1 and t_2 that correspond to the condition (V), which can happen only when the integrand itself is zero; i.e., Gauss's principle of least constraint will be true.

That proves the equivalence of the form (VII) and Gauss's principle of least constraint, and the form (VII) now represents the *desired integral form of that principle.*

One can also arrive at an extension of that result to the case of generalized coordinates, as well as the inclusion of non-mechanical processes, on the basis of lengthy and cumbersome developments by introducing the kinetic potential in a manner that is analogous to what **A. Wassmuth** ([1]) did for the principle of least action.

([1]) "Die Bewegungsgleichungen des Elektrons und das Prinzip der kleinsten Aktion," Wien. Ber. CXX, Abt. IIa (1911).

"Bemerkungen zum Prinzip des kleinsten Zwanges," Sitz. Heidelberger Akad. Wiss. Abt. A, math.-naturw. Klasse, Carl Winters Universitätsbuchhandlung, Heidelberg, 1919.

Remarks on the principle of least constraint

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PART ONE

Mechanical systems with equality constraints

§ 1. – Generalities on point-systems with holonomic and non-holonomic constraint equations.

One deals with a system of n mass-points. In order to simplify the calculations, the mass of the ν^{th} point will be denoted by $m_{3\nu-2} = m_{3\nu-1} = m_{3\nu}$. Let its rectangular Cartesian coordinates at time t be $x_{3\nu-2}$, $x_{3\nu-1}$, $x_{3\nu}$. The *position* of the system at time t will be determined by the $3n$ coordinates (x_ρ). Differentiation with respect to t yields the velocity components $(\dot{x}_\rho)$; the quantities $(x_\rho, \dot{x}_\rho)$ characterize the *state of motion* of the system at time t. By repeated differentiation, one will obtain the acceleration components $(\ddot{x}_\rho)$. Finally, let the components of the force that acts upon the ν^{th} point be $X_{3\nu-2}$, $X_{3\nu-1}$, $X_{3\nu}$; they are assumed to be single-valued functions of the quantities $(x_\rho, \dot{x}_\rho)$ and time t.

The system might be subjected to k mutually independent, consistent, holonomic equations:

$$f_\kappa(x_1, x_2, \ldots, x_{3n}; t) = 0 \qquad (\kappa = 1, 2, \ldots, k), \tag{1}$$

which shall be written more briefly as $f_\kappa(x_\rho; t) = 0$. If will be assumed of the functions $f_\kappa(x_\rho; t)$ (as for all of the functions that occur in what follow) that they admit the appropriate differentiations. In general, the k mutually-independent, consistent equations:

(2)
$$\sum_{\rho=1}^{3n} \frac{\partial f_k}{\partial x_\rho} \dot{x}_\rho + \frac{\partial f_k}{\partial t} = 0$$

will then exist between the $3n$ velocity components. In addition, it should be stated that *in general* an equation of higher degree will enter in place of at least one of the linear equations (2) for special systems of values (x_ρ ; t), namely, for the systems of values for which all of the k^{th}-order determinants of the matrix $\partial f_\kappa / \partial x_\rho$ vanish.

At the level of velocity, l non-holonomic equations:

(3)
$$\sum_{\rho=1}^{3n} \varphi_{\lambda\rho}(x_\rho ; t)\, \dot{x}_\rho + \varphi_{\lambda 0}(x_\rho ; t) = 0 \qquad (\lambda = 1, 2, \ldots, l)$$

can be combined with equations (2). It will be assumed that equations (2) and (3) collectively make up a system of $m = k + l$ linear equations for the velocity components:

(4)
$$\sum_{\rho=1}^{3n} F_{\mu\rho}(x_\rho ; t)\, \dot{x}_\rho + F_{\mu 0}(x_\rho ; t) = 0 \qquad (\mu = 1, 2, \ldots, m) ,$$

for which at least one m^{th}-order determinant of the matrix $\| F_{\mu\rho} \|$ does not vanish; otherwise, they are *singular*.

With those preparations, the basic problem of analytical mechanics for the point-system in question reads:

Given a state of the system that satisfies the constraints at any time, find the accelerations that pertain to that time by means of the conditions and forces at that time.

For a regular position, the equations:

(5)
$$\sum_{\rho=1}^{3n} F_{\mu\rho}(x_\rho ; t)\, \ddot{x}_\rho + H_\mu(x_\rho ; \dot{x}_\rho ; t) = 0$$

will follow from equations (4) by differentiation. The expressions $H_\mu(x_\rho ; \dot{x}_\rho ; t)$ are functions of degree two in the quantities $(\dot{x}_\rho)$. (5) will yield m suitably-chosen acceleration components as linear functions of the remaining $3n - m$.

With that, everything that can be inferred from the prescribed conditions for the velocities and accelerations has been exhausted. In order to determine the accelerations by means of the constraints and the forces completely, one must add a *principle* of analytical mechanics.

§ 2. – D'Alembert's principle for systems with holonomic and non-holonomic constraint equations.

D'Alembert's principle demands that *for the virtual displacements that satisfy the equations:*

(6) $$\sum_{\rho=1}^{3n} F_{\mu\rho}(x_\rho; t)\,\delta x_\rho = 0,$$

the work done by the system reactions will vanish:

(7) $$\sum_{\rho=1}^{3n} (m_\rho \ddot{x}_\rho - X_\rho)\,\delta x_\rho = 0.$$

It is equivalent to the equations:

(8) $$m_\rho \ddot{x}_\rho = X_\rho + \sum_{\rho=1}^{3n} F_{\mu\rho} L_\mu ,$$

in which $L_1, L_2, \ldots, L_m$ mean undetermined multipliers. If the expressions for the quantities $(\ddot{x}_\rho)$ in (5) are substituted in (8) then one will get the m linear equations:

(9) $$\sum_{\mu=1}^{m} \left(\sum_{\rho=1}^{3n} \frac{1}{m_\rho} F_{\mu\rho} F_{\nu\rho} \right) L_\mu + J_\nu(x_\rho; \dot{x}_\rho; t) = 0 \qquad (\nu = 1, 2, \ldots, m)$$

for the multipliers, whose determinant is equal to the sum of the squares of the m^{th}-order determinants that belong to the matrix $\left\| \frac{1}{\sqrt{m_\rho}} F_{\mu\rho} \right\|$, from a known theorem ([1]). It follows from this that *for regular positions of the system*, the accelerations will be determined uniquely from the state of motion ([2]).

The fact that the assumption of a regular position is essential is shown by the following example:

Let a point of unit mass be constrained to move on the surface of a cone:

(1′) $$x_1^2 + x_2^2 - x_3^2 = 0.$$

That condition will imply the equations:

(2′) $$x_1\,\dot{x}_1 + x_2\,\dot{x}_2 - x_3\,\dot{x}_3 = 0,$$

([1]) C. G. J. JACOBI, "De formatione et proprietatibus determinatium," J. reine angew. Math. **22** (1841), pp. 312; *Werke*, Bd. III, Berlin 1884, pp. 386; Ostwald's *Klassiker der exakten Wissenschaften*, Heft 77, Leipzig 1896, pp. 40.

([2]) C. G. JACOBI, *Vorlesungen über Dynamik*, published 1842/43, 2nd edition, Berlin 1884, Lecture 17, especially pp. 140. The fact that JACOBI assumed holonomic constraints does not obstruct the generality of his process, since it involves only the behavior of the matrix $\| F_{\mu\rho} \|$.

(5′) $$x_1\,\ddot{x}_1 + x_2\,\ddot{x}_2 - x_3\,\ddot{x}_3 + \dot{x}_1^2 + \dot{x}_2^2 - \dot{x}_3^2 = 0,$$

and the virtual displacements will be specified by the equations:

(6′) $$x_1\,\delta x_1 + x_2\,\delta x_2 + x_3\,\delta x_3 = 0.$$

At time t, the mass-point is found at the vertex of the cone, and in fact at rest. The original specification of the virtual displacements breaks down at that singular point. However, one will also have to consider displacements that move the mass-point from the given position to a position that is compatible with the constraints, and equations (6′) will then have to be replaced with the equations:

(6″) $$\delta x_1^2 + \delta x_2^2 - \delta x_3^2 = 0.$$

One gets the equation:

(5″) $$\ddot{x}_1^2 + \ddot{x}_2^2 - \ddot{x}_3^2 = 0,$$

and most simply by geometric arguments. By means of **d'Alembert**'s principle, that will lead to the equations:

(8″) $$\ddot{x}_1 = X_1, \quad \ddot{x}_2 = X_2, \quad \ddot{x}_3 = X_3.$$

By themselves, they generally contradict equation (5″).

One can seek to explain that result by the fact that the vertex of the cone is not able to produce a reaction. However, it will be shown that reactions will appear when one applies the principle of least constraint.

§ 3. – The principle of least constraint for systems with holonomic and non-holonomic constraint equations.

According to **GAUSS**, *for a given state of motion, the constraint:*

(10) $$Z(\ddot{x}_\rho) = \sum_{\rho=1}^{3n} \frac{1}{m_\rho}(m_\rho \ddot{x}_\rho - X_\rho)^2$$

will be a minimum for all quantities $(\ddot{x}_\rho)$ *that are compatible with the constraints.*

For regular positions, the admissible quantities $(\ddot{x}_\rho)$ will be specified by the linear equations (5). For singular positions, at least one of those equations will be replaced with an equation of order two or higher. In both cases, the constraint will have at least one minimum. Namely, it is initially a continuous function of the independent variables $(\ddot{x}_\rho)$ and will preserve that property when its variability is restricted by algebraic equations. There will then be at least one system of values $(\ddot{\xi}_\rho)$ that makes the constraint a minimum.

It will now be shown that the constraint possesses *only one* minimum *for regular positions*.

Let $(\ddot{\xi}_\rho)$ be a location of the minimum, so $Z(\ddot{\xi}_\rho + u_\rho)$ will be greater than $Z(\ddot{\xi}_\rho)$ for all sufficiently small, admissible systems of values (u_ρ), and indeed a system of values (u_ρ) will be admissible when equations (5) are fulfilled for the quantities $(\ddot{\xi}_\rho + u_\rho)$, so when the m equations exist:

(11)
$$\sum_{\rho=1}^{3n} F_{\mu\rho} u_\rho = 0.$$

It follows from this that for a system of values (u_ρ), the system of values ($g\, u_\rho$) will be admissible for arbitrary positive or negative g. Now, one has:

(12)
$$Z(\ddot{\xi}_\rho + u_\rho) = Z(\ddot{\xi}_\rho) + \sum_{\rho=1}^{3n} m_\rho u_\rho^2 + 2\sum_{\rho=1}^{3n} (m_\rho \ddot{\xi}_\rho - X_\rho) u_\rho,$$

and as a result, one must have the expression:

(13)
$$\sum_{\rho=1}^{3n} (m_\rho \ddot{\xi}_\rho - X_\rho) u_\rho = 0$$

for a minimum, initially for sufficiently small systems of values (u_ρ), but then for all admissible ones. If there is a second location for a minimum $(\ddot{\eta}_\rho)$ then one can put $\ddot{\eta}_\rho$ in place of $\ddot{\xi}_\rho + u_\rho$ in equation (12), so $Z(\ddot{\eta}_\rho)$ would be greater than $Z(\ddot{\xi}_\rho)$, with the exclusion of equality. One can show in the same way that $Z(\ddot{\xi}_\rho)$ is greater than $Z(\ddot{\eta}_\rho)$, with the exclusion of equality. As a result, the assumption that there is a second location for a minimum must be rejected.

One has repeatedly stated that the principle of least constraint yields an absolute minimum, and one would like to conclude from this that the constraint possesses *only one* minimum location. However, an absolute minimum can appear at several places at once; for instance, take the function $y = \sin x$. However, in addition, equation (11) will be true only under the assumption of a regular position of the system. How things work at the singular positions will be explained at the conclusion of this paragraph in an example.

Under the assumption of a regular position, there is always one and only one system of accelerations that satisfy the principle of least constraint for a given state of motion. It is easy to see that one will obtain the same accelerations that are given by **d'Alembert**'s principle, because equations (11) will be converted into equations (6) when one sets:

(14)
$$\delta x_\rho = u_\rho\, \delta t,$$

and in that way, equations (13) will, at the same time, go to equations (7).

It would seem that the fact that the admissible changes in the accelerations are coupled with the virtual displacements by equations (14) was first pointed out by **GIBBS** ([1]), but generally without expressly stating the essential assumption that the position of the system must be regular.

With that, we have arrived at the lemma that ***d'Alembert****'s principle and the principle of least constraint are equivalent for point-systems that are subject to holonomic and non-holonomic constraint equations, assuming that the position is regular.* Due to equations (14), the demand (7) that the virtual work done by reactions should vanish is identical to the necessary and sufficient condition (13) for the minimum of the constraint. In order to prove that the accelerations are determined uniquely by **d'Alembert**'s principle for a regular position, one can then depart from **JACOBI**'s path and appeal to the principle of least work and then arrive at that goal by simple conceptual arguments without using the theory of determinants.

In conclusion, we shall once more take up the example that was treated in the foregoing paragraphs of the motion of a mass-point on a cone. The principle of least constraint demands that the expression:

$$Z(\ddot{x}_1, \ddot{x}_2, \ddot{x}_3) = (\ddot{x}_1 - X_1)^2 + (\ddot{x}_2 - X_2)^2 + (\ddot{x}_3 - X_3)^2 \tag{10'}$$

must be a minimum, with the constraint (5″). Geometrically, that can be interpreted by saying that it is the shortest distance from the point X_1, X_2, X_3 to the surface of the cone.

One sees immediately that two points of the cone can yield a minimum in some situations. The line segments from the vertex of the cone to the two points will give the directions and magnitudes of the desired accelerations, while the two shortest distances will represent the associated reactions. When there is also a means of preferring one of the accelerations that are found, it must however, break down when the direction of the force lies along the axis of the cone.

In the textbooks on mechanics and physics, one often finds it maintained that the motion of a mechanical system should be established completely by the initial state of motion (i.e., determinism). It is then "self-explanatory" that the accelerations will be determined *uniquely* by the principles of mechanics. Here, however, the objection is raised that, first of all, it might be just as conceivable, in and of itself, that one must also know the initial accelerations. However, secondly, one must know the meaning of the principles. Those are the starting points for the calculations, and their implementation belongs to the domain of mathematics. It is up to the physicist to verify the physical admissibility of the results of calculation. However, it would be wrong to reject a principle of mechanics just because the accelerations would not be determined uniquely in some situations. The fault might, in fact, lie in the way that the problem was posed. Thus, the motion in the vicinity of the vertex of the cone that was required in the example will not be realized by mechanics; an inadmissible idealization was made here.

In the present context, the example shows that ***d'Alembert****'s principle and the principle of least constraint do not need to be equivalent for singular positions*, and indeed, it shows that **Gauss**'s principle achieves more than **d'Alembert**'s. It follows from this that *it is not possible to derive the principle of least constraint from* ***d'Alembert****'s principle for singular positions.* Rather, one will have to pose **Gauss**'s principle *axiomatically* for singular positions.

([1]) J. W. GIBBS, "On the fundamental formulae of dynamics," Am. J. Math. **2** (1879), pp. 49.

§ 4. – Geometric interpretation.

It is often useful to regard the systems of quantities that appear in the mechanics of point-systems as the coordinates of a point in a Euclidian space of several extensions. In particular, that is true of the 3*n* acceleration components $(\ddot{x}_\rho)$.

The geometric interpretation will become more transparent when one performs an affine transformation and introduces the 3*n* new coordinates:

$$\frac{1}{\sqrt{m_\rho}}(m_\rho \ddot{x}_\rho - X_\rho). \tag{15}$$

In that way, one will get the square of the distance from the point (y_ρ) in a 3*n*-fold extended Euclidian space R_{3n} to the origin *O* of the coordinates. The constraint equations (5) now read:

$$\sum_{\rho=1}^{3n} \frac{1}{\sqrt{m_\rho}} F_{\mu\rho} y_\rho + \bar{H}_\mu = 0, \tag{16}$$

and the principle of least constraint says that for a regular position of the system, the shortest distance from the point *O* to the $(3n - m)$-fold extended Euclidian space R_{3n-m} shall be singled out by the one that is suggested for R_{3n} by equations (16). It is known from the study of multiply-extended Euclidian spaces that the desired shortest distance is the perpendicular that is dropped from *O* to R_{3n-m} and that there is always one and only one such perpendicular ([1]).

D'Alembert's principle also takes on a simple geometric meaning. The base *F* of the perpendicular has the coordinates (η_ρ) . The point $(\eta_\rho + v_\rho)$ then belongs to the space when the quantities (v_ρ) satisfy the equations:

$$\sum_{\rho=1}^{3n} \frac{1}{\sqrt{m_\rho}} F_{\mu\rho} y_\rho = 0. \tag{17}$$

However, that will be converted into equations (11) when one sets:

$$v_\rho = \sqrt{m_\rho} \cdot u_\rho . \tag{18}$$

The requirement (7) of **d'Alembert**'s principle is then identical to the orthogonality condition:

$$\sum_{\rho=1}^{3n} \eta_\rho v_\rho = 0. \tag{19}$$

([1]) See, for instance, P. H. SCHOUTE, *Mehrdimensionale Geometrie*, Erster Teil: Lineare Räume, Sammlung Schubert, Band XXV, Leipzig 1902.

In fact, the shortest distance OF from the point to O to the space R_{3n-m} is always perpendicular to all of the directions in R_{3n} that belong to that space. The fact that the two principles are equivalent in the regular case will then become obvious.

PART TWO

Mechanical systems with inequality constraints ([1])

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§ 5. – Generalities on point-systems with holonomic and non-holonomic constraint inequalities.

The holonomic and non-holonomic constraint equations shall be combined with k' inequalities:

$$g_{\kappa'}(x_\rho \, ; t) > 0 \qquad (\kappa' = 1, 2, \ldots, k') . \tag{20}$$

In them, it will only be assumed that there are positions of the system that are compatible with all constraints at time t .

When one of the functions $g_{\kappa'}(x_\rho \, ; t)$ is positive for an admissible system of values (x_ρ) at time t, the condition that $g_{\kappa'} > 0$ will be called *passive to the change in position of the system,* because *all* systems of values will then be admissible in a sufficiently-small neighborhood of the system of values $(x_\rho \, ; t)$. However, when one of the functions $g_{\kappa'}(x_\rho \, ; t)$ vanishes at time t and assumes positive, as well as negative, values in the neighborhood of $(x_\rho \, ; t)$, the velocity components $(\dot{x}_\rho)$ must satisfy the condition:

$$\sum_{\rho=1}^{3n} \frac{\partial g_{\kappa'}}{\partial x_\rho} \dot{x}_\rho + \frac{\partial g_{\kappa'}}{\partial t} > 0. \tag{21}$$

If the value on the left-hand side is positive for one system of values $(x_\rho \, ; t)$ then all states of motion in a sufficiently-small neighborhood of the state of motion $(x_\rho ; \dot{x}_\rho ; t)$ will also be admissible. Such an inequality will then be called *passive to the change in the state of motion.*

l' non-holonomic inequalities:

([1]) For the history and literature on this topic, cf., the articles in the *Encyklopädie der mathematischen Wissenschaften*, Bd. IV 1, by A. VOSS, "Die Prinzipien der rationellen Mechanik," especially pp. 73 and 85, and by P. STÄCKEL, "Elementare Dynamik der Punktsysteme und starren Körper," especially pp. 460. The statements that were made there will be extended in various directions here.

(22) $$\sum_{\rho=1}^{3n} \psi_{\lambda'\rho}(x_\rho; t)\dot{x}_\rho + \psi_{\lambda'0}(x_\rho; t) \geq 0 \qquad (\lambda' = 1, 2, \ldots, l')$$

can be added to the k' holonomic inequalities (20). They will be active or passive to the changes of the state of motion according to whether equality or inequality exists, respectively.

In total, s equations that the velocity components must satisfy:

(23) $$\sum_{\rho=1}^{3n} G_{\sigma\rho}(x_\rho; t)\,\dot{x}_\rho + G_{\sigma 0}(x_\rho; t) = 0 \qquad (\sigma = 1, 2, \ldots, s)$$

will be obtained for the given state of motion in the manner that was described. The position (x_ρ) of the system at time t will be called *regular* when the $m + s$ equations (4) and (23) allow one to represent just as many suitably-chosen velocity components as linear functions of the remaining $3n - m$.

An inequality for the acceleration components will follow from equations (23):

(24) $$\sum_{\rho=1}^{3n} G_{\sigma\rho}(x_\rho; t)\ddot{x}_\rho + K_\sigma(x_\rho; \dot{x}_\rho; t) > 0.$$

However, whereas the knowledge of the state of motion will make it possible to decide whether the equality sign is or is not valid for one of the conditions (21) and (22), that will not be true of the conditions (24). It will be shown that for regular positions of the system, the acceleration components $(\ddot{\xi}_\rho)$ that actually exist are determined uniquely by the principle of least constraint, and indeed when the $(\ddot{\xi}_\rho)$ are substituted for the $(\ddot{x}_\rho)$, the equality sign will be valid for some of the conditions (24), the *greater than sign* will be valid for the rest of them. Those inequalities can be dropped from the outset; for that reason, they shall be called *passive for the accelerations*.

§ 6. – The D'ALEMBERT-FOURIER principle for systems with holonomic and non-holonomic constraint inequalities.

On the basis of arguments that go back to **Fourier** ([1]), one must replace **d'Alembert**'s principle with the requirement that *the virtual work done by reactions can have no negative values* for systems with inequalities. An example was given in the cited treatise by **Gibbs** for which the **d'Alembert-Fourier** principle was not sufficient to determine the acceleration components completely. The following simpler example will also accomplish the same objective.

A point of unit mass moves in space; let it be subject to the inequality $x_3 \geq 0$. In order for the condition to be active for the change in position at the time t, let it be found in the $x_1\,x_2$-plane. The velocity components must then satisfy the condition that $\dot{x}_3 \geq 0$, and that condition will be active

([1]) J. FOURIER, "Mémoire sur la statique," J. de l'École polyt. **5** (1798), pp. 30; *Œuvres*, t. II, 1890, pp. 488.

for the change in the state of motion when $\dot{x}_3 = 0$. That implies that one must have $\ddot{x}_3 \geq 0$ for the acceleration components.

When $x_3 = 0$ at time t, the virtual displacements must fulfill the inequality $\delta x_3 \geq 0$. Furthermore, the **d'Alembert-Fourier** principle demands that one must have:

$$(7') \qquad (\ddot{x}_1 - X_1)\,\delta x_1 + (\ddot{x}_2 - X_2)\,\delta x_2 + (\ddot{x}_3 - X_3)\,\delta x_3 > 0.$$

Due to the arbitrariness in δx_1 and δx_2, one must have $\ddot{x}_1 = X_1$, $\ddot{x}_2 = X_2$, such that (7) will go to the condition that X_3.

The result is that $\ddot{x}_3$ cannot be less than the greater of the two values 0 and X_3. *Therefore, $\ddot{x}_3$ will not be determined completely by the **d'Alembert-Fourier** principle.*

Just as **Gibbs** remarked for his own example, it is also possible to ascertain the value of $\ddot{x}_3$ here by means of simple arguments on the progress of the motion. Namely, if the component X_3 is negative then it will be annihilated by the reaction of the boundary surface $x_3 = 0$, and one will have $\ddot{x}_3 = 0$. However, if X_3 were zero or positive then the mass-point would move as if it were free, and one would have $\ddot{x}_3 = X_3$. That is correct. Merely performing the calculations from the Ansatz that the **d'Alembert-Fourier** principle prescribes will yield only the previously-posed inequality condition for $\ddot{x}_3$, and the example then shows that *this principle will not lead to the determination of the accelerations in general.*

§ 7. – The principle of least constraint for systems with holonomic and non-holonomic constraint inequalities.

It seems that **Jacobi** was the first to examine the application of the principle of least constraint to systems with inequality constraints in detail, in his lectures on dynamics during the Winter semester in 1848/49 ([1]). He was followed by RITTER (1853) in a dissertation that was supervised by **Gauss** ([2]), **Gibbs**, in the cited 1879 treatise, and **Boltzmann** ([3]).

Without knowing of the aforementioned publications, **Mayer** ([4]), prompted by some older work by **Ostragradsky** (1834 and 1836), addressed inequality constraints, and on the basis of some remarks by **Study** in the year 1899, he showed how one could determine the accelerations by means of the principle of least constraint; a regular position was tacitly assume there. It still remained doubtful whether several systems of acceleration might be obtained in some situations. Generally, **Mayer** believed that "For that reason, one can probably regard it as obvious that two different systems of accelerations of the given character cannot exist, because if they did exist then

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([1]) According to A. VOSS, *loc. cit.*, pp. 87. A copy of detailed calculations by SCHEIBNER that were cited in it can be found in the Bibliothek der Berliner Akademie der Wissenschaften.

([2]) A. RITTER, "Über das Princip des kleinsten Zwanges," Dissertation, Göttingen, 1853.

([3]) L. BOLTZMANN, *Vorlesungen über die Principe der Mechanik*, Part I, Section VI, Leipzig, 1897.

([4]) A. MAYER, "Über die Aufstellung der Differentialglichungen der Bewegung für reibungslose Punktsystemem die Bedingungsgleichungen unterworfen sind," Leipziger Berichte, math-phys. Klasse **51** (1899), pp. 224.

there would absolutely no means of finding out which of the two systems were the correct one." However, as was proved in § **3**, it is very probable that two systems of accelerations might make the constraint a minimum at singular positions. The fact that uniqueness prevails under **Mayers**'s assumption then requires proof.

While tacitly assuming a regular position, **Jacobi** already remarked in the cited lecture that "the nature of the minimum at hand excludes several minima," and **Boltzmann** asserted that "the constraint must be an absolute minimum for the actual motion and be capable of having several minima" (*loc. cit.*, pp. 240). Soon after, **Zermelo** ([1]) proved that the constraint possessed *only one* minimum in full rigor for regular positions of the system, but generally under certain restricting assumptions, and in that way demonstrated the uniqueness of the accelerations.

The process that was used in § **3** can be adapted to the case in which any holonomic or non-holonomic inequalities are added to the constraint equations, and that would allow one to derive the theorem on the uniqueness of the accelerations in its most general form.

In the R_{3n} of the components $(\ddot{x}_\rho)$, the constraint is a continuous function of position for the part of space that contains all of the points $(\ddot{x}_\rho)$ that are compatible with the constraints, and it will therefore attain a smallest value for at least one location; one does not need to assume that this position is regular in order to reach that conclusion.

Let the point $(\ddot{\xi}_\rho)$ be one location of the minimum, such that $Z(\ddot{\xi}_\rho + u_\rho)$ will be greater than $Z(\ddot{\xi}_\rho)$ for all sufficiently-small changes (u_ρ) in its coordinate that are compatible with the constraints. As a result of equation (12) the necessary and sufficient condition for that is that:

$$\sum_{\rho=1}^{3n}(m_\rho \ddot{\xi}_\rho - X_\rho)u_\rho > 0. \tag{25}$$

For a regular position, it can, in turn, be shown that the validity of the condition (25) for all sufficiently-small admissible systems of values (u_ρ) will imply its validity for all systems of values (u_ρ) that that exist in general.

In order for a system of values (u_ρ) to be admissible, it must first satisfy equations (11). Secondly, the conditions (24), which were fulfilled for $\ddot{x}_\rho = \ddot{\xi}_\rho$, must remain fulfilled when $\ddot{x}_\rho$ is replaced with the value $\ddot{\xi}_\rho + u_\rho$.

Now let ϑ be a quantity that lies between 0 and 1. The system of values (ϑU_ρ) will always be admissible when the system of values (U_ρ) is. The fact that equations (11) are valid for that system of values is indeed obvious. However, if the two inequalities are true:

$$\sum_{\rho=1}^{3n} G_{\sigma\rho}(x_\rho;t)\,\ddot{\xi}_\rho + K_\sigma(x_\rho;\dot{x}_\rho;t) \geq 0 \tag{26}$$

and

([1]) E. ZERMELO, "Über die Bewegung eines Punktsystems bei Bedingungsungleichungen," Göttinger Nachrichten, math-phys. Klasse, (1899), pp. 306; that note was presented on 3 February 1900.

(27)
$$\sum_{\rho=1}^{3n} G_{\sigma\rho}(x_\rho;t)\,\ddot{\xi}_\rho + \sum_{\rho=1}^{3n} G_{\sigma\rho}(x_\rho;t)U_\rho + K_\sigma(x_\rho;\dot{x}_\rho;t) \geq 0$$

for one value of the index σ then the inequality:

(28)
$$\sum_{\rho=1}^{3n} G_{\sigma\rho}(x_\rho;t)\,\ddot{\xi}_\rho + \vartheta\sum_{\rho=1}^{3n} G_{\sigma\rho}(x_\rho;t)U_\rho + K_\sigma(x_\rho;\dot{x}_\rho;t) \geq 0$$

will also be true, as one easily convinces oneself. The result can be interpreted geometrically by saying that for a regular position of the system, the part of space that contains all of the points $(\ddot{x}_\rho)$ that are compatible with the constraints is simply connected and everywhere convex.

Moreover, if one chooses the quantity ϑ to be sufficiently small that the requirement (25) is fulfilled for $u_\rho = \vartheta\, U_\rho$ then it will also be fulfilled for $u_\rho = U_\rho$. One might infer the conclusion of the proof word-for-word from § **3**.

§ 8. – Determining the accelerations by means of the principle of least constraint.

The determination of the accelerations from the principle of least constraint will be eased when one likewise employs the geometric interpretation that was developed in § **4**. Under the assumption that the position of the system is regular, equations (5) between the coordinates (y_ρ) will then correspond to m linear equations

(29)
$$\sum_{\rho=1}^{3n} A_{\mu\rho} y_\rho + A_{\mu 0} = 0$$

that are mutually independent and mutually consistent, and correspond to the constraints (24), and there will be s inequalities:

(30)
$$\sum_{\rho=1}^{3n} B_{\mu\rho}\, y_\rho + B_{\mu 0} \geq 0$$

that are compatible with each other and with (29). A Euclidian R_{3n-m} will be specified in R_{3n} by equations (29), and the inequalities (30) will have the effect that only the points of a certain N-fold-extended, simply-connected, everywhere-convex region of space S_N in it that is bounded by Euclidian spaces with $N-1$, $N-2$, …, 2, 1 extensions will come under consideration; therefore $N \leq 3n - m$. The principle of least constraint will then follow from the fact that the following problem has then be solved:

A region of space in a Euclidian space is specified by linear equations and inequalities. Find its shortest distance from a given point in space.

The fact that there is only one such shortest distance was proved in the foregoing paragraphs. Two cases are to be distinguished in its solution: First of all, the given point O belongs to the region of space S_N, including the boundary. The shortest distance to the point O itself will then be obtained. Secondly O can lie outside the region of space S_N. If the perpendicular OF were then dropped from O to the N-fold-extended Euclidian space R_N that includes S_N then OF would be the minimum of the distances from all points of R_N to O. Thus, if the point F belongs to the spatial region S_N then OF will be the desired shortest distance. It is easy to see that in this case the point F lies on the boundary of the spatial region S_N with the space R_{3n} . Finally, if the point F does not belong to the spatial region S_N then the minimum of the distance will reached at a point A that is different from F, and which necessarily lies on the boundary of the spatial region S_N with R_N . Namely, since the perpendicular OF is perpendicular to all of the directions of R_{3n} that lie in R_N , one will have the equation:

$$OA^2 = OF^2 + AF^2, \tag{31}$$

and as a result, AF will be the minimum of the distance from the point F to the points of the spatial region S_N .

The original problem with the auxiliary conditions (29) is then reduced to the same problem without the auxiliary conditions by the argument that was just presented. **Gauss** addressed precisely the problem of *finding the minimum of the expression:*

$$z_1^2 + z_2^2 + \cdots + z_N^2 \tag{32}$$

when the linear inequalities:

$$C_{\sigma 1}\, z_1 + C_{\sigma 1}\, z_1 + \ldots + C_{\sigma N}\, z_N \geq 0 \tag{33}$$

are prescribed for the quantities $z_1, z_2, \ldots, z_N$ in a lecture that he gave during the Winter semester of 1850/51 on the method of least squares ([1]). I have thoroughly presented the process that he suggested for finding the location of the minimum, and at the same time, I have proposed another way of treating it that is likewise based upon geometric considerations ([2]). Finally, one can also employ the method of multipliers to solve it. One might refer to the 1917 article for the details ([3]).

([1]) C. F. GAUSS, *Werke*, Bd. X 1, Göttingen 1917, pp. 473; reprint of part of RITTER's calculations.

([2]) P. STÄCKEL, "Eine von GAUSS gestellte Aufgabe des Minimums," these Sitzungsberichte (1917), 11th treatise. The note by Zermelo, which seemed to consider an entirely different subject, was not available to me at the time. Once I later recognized the connection, I did not neglect to point out that part of what had been done in the 1917 article had already been done by ZERMELO.

([3]) My colleague PERRON was kind enough to write to me that there are two places in § **7** that can be improved. When one of the possible points for the minimum of the function $f(x_1, x_2, \ldots, x_n)$ that is to be found lies in a spatial region S_n inside of an $(n - 1)$-extended boundary manifold, it does not need to yield a minimum. Rather, the same test that is required for less than $n - 1$ extensions can be carried out for it. For example, consider the case in which the shortest distance from a plane ($n = 2$) to a point on the surface of a square is to be found. That is connected with the fact that the sign convention for the multipliers is not given expressly. Moreover, one finds that OSTRAGRADSKY and MAYER already went into that in detail; cf., also L. HENNEBERG, "Über den Fall der Statik, in dem das virtuelle Moment einen negative Wert besitzt," J. f. reine u. angew. Math. **113** (1894), pp. 179.

§ 9. – Connection between GAUSS's minimum problem and the principle of least constraint.

Judging from **Ritter**'s report, **Gauss** did not say in the lecture during the Winter semester of 1850/51 what had led him to pose his problem of the minimum with inequality conditions, and the question then arose of whether he knew of its connection with the principle of least constraint. Now, it certainly remains puzzling how he would otherwise arrive at the problem. However, one can also arrive at an affirmative answer by other considerations.

Statements that were made in letters and articles show that **Gauss** repeatedly dealt with multiply-extended manifolds. Here, it will suffice to mention a statement that he made to **Sartorius v. Walterhausen** at roughly the time of that lecture: "We can (he said) perhaps empathize with beings that are aware of only two dimensions. A being that is above us would perhaps look down on us similarly, and he would (he continued jokingly) have to overlook certain problems that he thought had been treated geometrically in a higher state." ([1])

In his 1829 note "Über ein neues allgemeines Grundgesetz der Mechanik," **Gauss** explained that the restriction to condition equations was "unnecessary and not always reasonable in nature" and demanded that one should likewise express the law of virtual velocities in such a way that it would subsume all cases from the beginning. At the conclusion, he said that the analogy with the method of least squares could not be pursued any further, which does not, however, seem to be his present opinion on that subject ([2]). He also referred to the importance of the condition inequalities in the 28 September 1829 treatise "Principia generalia theoriae figurae fluidorum in statu aequilibrii" ([3]), and he returned to that topic in a letter to **Möbius** on 29 September 1837 ([4]).

In **Ritter**'s 1853 dissertation, which goes back to **Gauss**, the principle of least constraint was applied to systems with holonomic inequality constraints, and indeed **Ritter** appealed to the language of multidimensional geometry in it. He dealt in depth with the general problem of finding the minimum of a function of position in a multiply-extended Euclidian space for a spatial region in that is defined by inequalities. **Ritter** remained at that level of generality. He did not follow through on the Ansatz of **Gauss**'s principle with computations, nor did he present the linear inequalities ([5]). One can be certain that **Gauss** took that latter step.

§ 10. – Virtual displacements and admissible variations of the accelerations.

The fact that **d'Alembert**'s principle and the principle of least constraint can replace each other for equality constraints, assuming a regular position of the system, is based upon the equations:

$$\delta x_\rho = u_\rho \, \delta t \,, \tag{14}$$

([1]) Cf., my essay: "GAUSS *als Geometer*," Materielen für eine wissenschaftliche Biographie von GAUSS, Heft V, Leipzig, 1918, pp. 136.

([2]) C. F. GAUSS, *Werke*, Bd. V, pp. 25.

([3]) C. F. GAUSS, *Werke*, Bd. V, pp. 35.

([4]) First published by C. NEUMANN, "Über das Princip der vituellen oder fakultativen Verrückungen," Leipziger Berichte, math.-phys. Klasse **31** (1879), pp. 61; reprinted in C. F. GAUSS, *Werke*, Bd. XI 1, pp. 17.

([5]) The relevant section of RITTER's dissertation is printed in C. F. GAUSS, *Werke*, Bd. XI 1, pp. 469.

which exhibit the invertible, one-to-one correspondence between the virtual displacements and the admissible changes to the acceleration components. By contrast, the fact that **d'Alembert**'s principle and **Gauss**'s principle are not equivalent for inequality constraints, even when the position is regular, was shown in the example that was treated in § **6**.

One asks, "What relationships exist between the quantities (δx_ρ) and the quantities (u_ρ), and first of all in the example?" The only case that comes under consideration is the one in which the condition $x_3 \geq 0$ is active for the change in the state of motion, so x_3 and $\dot{x}_3$ vanish at time t. The virtual displacements δx_1 and δx_2 can then be chosen arbitrarily, and one must have $\delta x_3 \geq 0$. The condition that $\ddot{x}_3 \geq 0$ is valid for the acceleration components. Therefore, u_1 and u_2 are small quantities that can be chosen arbitrarily, and one can set $\delta x_1 = u_1\ \delta t$, $\delta x_2 = u_2\ \delta t$. For u_3, there are two possibilities to be distinguished: If the condition for $\ddot{x}_3$ is active then $\ddot{\xi}_3 = 0$, so one must have $u_3 \geq 0$, and one can set $\delta x_3 = u_3\ \delta t$. However, if that condition is passive then $\ddot{\xi}_3 > 0$, and u_3 is a small quantity that can be chosen arbitrarily. Therefore, the condition of the minimum requires that one must now have $\ddot{\xi}_3 = X_3$. The equation $\delta x_3 = u_3\ \delta t$ loses its validity for negative values of u_3, so *the domain of the admissible changes* (u_ρ) *is more extensive than the domain of the virtual displacements* (δx_ρ). The fact that the condition for the minimum is fulfilled for those changes in acceleration that are produced by means of the equation $\delta x_\rho = u_\rho\ \delta t$ is, in fact, necessary, but not sufficient, because the constraint must be a minimum when it is regarded as a function of acceleration. However, the demand that the virtual work done by reactions cannot be negative $(\ddot{\xi}_3 - X_3)\ \delta x_3 \geq 0$ says less than the demand that $(\ddot{\xi}_3 - X_3)\ u_3 \geq 0$, which is necessary and sufficient for a minimum of the constraint. That explains the fact that the first demand on $\ddot{\xi}_3$ yields only an inequality, while the second one implies the unique determination of the acceleration.

One further sees that **Gauss**'s principle cannot be a consequence of the **d'Alembert-Fourier** principle, because if it could be derived from the latter principle then the acceleration could be determined uniquely from the **d'Alembert-Fourier** principle. However, that principle probably follows from **Gauss**'s, namely, when u_3 is subject to the restriction that it is not negative.

What is true for the example proves to be correct in general. When one restricts oneself to sufficiently-small systems of values (u_ρ), as is allowed by the proof of the minimum, the quantities (u_ρ) must initially satisfy equations (11). As the conditions (27) show, for those values σ' of the index σ for which the equality sign is valid in the constraints (26), those equations must be combined with the inequalities:

$$\sum_{\rho=1}^{3n} G_{\sigma'\rho}\, u_\rho \geq 0. \tag{34}$$

By contrast, when the greater than sign is valid in the conditions (26), they will be passive to the changes in the accelerations.

One initially has equations (6), which correspond to equations (11), for the virtual displacements. However, when the inequalities (20) are active for the change in positions, they will be combined with the conditions:

(35) $$\sum_{\rho=1}^{3n} G_{\sigma\rho}\,\delta x_\rho > 0,$$

and indeed for all values $\sigma = 1, 2, 3, \ldots, s$.

A glimpse at the formulas shows that any system of virtual displacements (δx_ρ) will produce a system of admissible changes (u_ρ) in the acceleration components by means of equations (14). In general, however, the converse is not true, since the displacements (δx_ρ) must satisfy all s inequalities (35), while the conditions (34) on the changes (u_ρ) are fulfilled for only some of the values of σ, in general. Therefore, in general, the domain of the admissible changes (u_ρ) will be more extensive than the domain of the virtual displacements (δx_ρ), and the demand of the **d'Alembert-Fourier** principle that:

(36) $$\sum_{\rho=1}^{3n} (m_\rho \ddot{\xi}_\rho - X_\rho)\,\delta x_\rho > 0$$

will say less than the demand of **Gauss**'s principle that:

(24) $$\sum_{\rho=1}^{3n} (m_\rho \ddot{\xi}_\rho - X_\rho)\,u_\rho > 0.$$

Boltzmann's geometric proof of the principle of least constraint ([1]) might lead to the suspicion that the principle is also a consequence of the **d'Alembert-Fourier** principle for inequality constraints, as well. The fact that this is not the case was shown already by **Gibbs**'s example, which **Boltzmann** communicated in detail. It is generally true that a system of admissible changes (u_ρ) in the acceleration components $\ddot{x}_\rho$ will emerge from any system of virtual displacements (δx_ρ), and it cannot be denied that $Z(\ddot{\xi}_3 + u_\rho)$ is greater than $Z(\ddot{\xi}_3)$ for those changes. However, that only means that a necessary condition for the minimum has been fulfilled, but one lacks a proof that the value of the constraint proves to be greater than it is for $(\ddot{\xi}_3)$ for *all* admissible, sufficiently-small changes in the acceleration components.

Gauss underestimated the profundity of his "new fundamental law" when he explained that it was already included in the combination of **d'Alembert**'s principle with the extended principle of virtual displacements as far as the matter was concerned. The fact that the constraint on a mechanical system that is subjected to arbitrary equality and inequality constraints is a minimum for the actual accelerations cannot be proved. Rather, it is an *axiom* that first becomes accessible to mathematical investigation in the case of inequality constraints. *Such a concept only enhances the significance of* ***Gauss****'s principle. In that way, it attains the status of a fundamental law for analytical mechanics.*

([1]) L. BOLTZMANN, *loc. cit.*, pp. 216-220.

“Sur une forme nouvelle des équations de la Dynamique,” C. R. Acad. Sci. Paris **129** (1899), 459-460.

On a new form of the equations of dynamics

Note by **P. APPELL**

Translated by D. H. Delphenich

The new form of the equations of dynamics that was indicated in the *Comptes rendus* in 7 and 28 August 1899 can be summarized by the following theorem, which is attached to GAUSS's principle of least constraint:

Let a system with constraints be given that is subject to forces that can depend upon position, velocity, and time. Let **J** *denote the acceleration of an arbitrary point of the system, let m denote its mass, and let* **F** *denote the given force that is applied to it. Then form the function:*

$$R = \tfrac{1}{2}\sum m\,J^2 - \sum F\,J\cos F, J\ .$$

At an arbitrary instant, if the position of the system and the state of the velocities are regarded as determined then the accelerations will have values that make the function R a minimum.

In that statement, one can obviously replace the function R with the function:

$$\tfrac{1}{2}\sum \frac{1}{m}((m\,\mathbf{J} - \mathbf{F}))^2,$$

in which $((m\,\mathbf{J} - \mathbf{F}))$ is the geometric difference between the vectors $m\,\mathbf{J}$ and $\mathbf{F}$, or with any other function that differs from R by only terms that are independent of the accelerations.

The preceding equations are then obtained by equating the partial derivatives of the function R with the respect to $q''_1, q''_2, \ldots, q''_n$ to zero.

"Sur une forme générale des équations de la dynamique," J. reine angew. Math. 121 (1900), 310-319.

On a general form of the equations of dynamics

(By **Paul Appell** at St. Germain-en-Laye)

Translated by D. H. Delphenich

1.

The **Lagrange** equations are not applicable when certain constraints are expressed by non-integrable differential relations or when one introduces parameters that are coupled with the coordinates by non-integrable differential relations. That difficulty has been the subject of various studies, and one will find a detailed bibliography in an article that I just published in the collection *Scientia* (Carré and Naud, editors) that was entitled "Les mouvements de roulement en dynamique."

We propose to indicate a general form for the equation of motion here that is not subject to the exceptions that we just stated. In order to write the equations in that new form, it will suffice to calculate the function:

$$S = \tfrac{1}{2}\sum m J^2 ,$$

in which m denotes the mass of any of the points of the system, and J denotes that absolute acceleration of that point: One sees that this function S is composed of the accelerations in the same way that one-half the *vis viva* is composed of the velocities.

We have indicated the principle of the method that follows here in a note that was published in the Comptes Rendus des Séances de l'Académie des Sciences de Paris on 7 August 1899.

2.

Imagine a system that is subject to constraints such that in order to obtain the most general virtual displacement that is compatible with the constraints at the instant t, it will suffice to subject the n parameters $q_1, q_2, \ldots, q_n$ to arbitrary variations $\delta q_1, \delta q_2, \ldots, \delta q_n$. If we then call the coordinates of any of the points of the system with respect to the fixed axes x, y, z then the virtual displacement of that point will have projections onto those axes that are:

(1)
$$\begin{cases} \delta x = a_1\,\delta q_1 + a_2\,\delta q_2 + \cdots + a_n\,\delta q_n, \\ \delta y = b_1\,\delta q_1 + b_2\,\delta q_2 + \cdots + b_n\,\delta q_n, \\ \delta z = c_1\,\delta q_1 + c_2\,\delta q_2 + \cdots + c_n\,\delta q_n, \end{cases}$$

in which δq_1, δq_2, …, δq_n are arbitrary. In those formulas, the coefficients a_1, a_2, …, c_n can depend upon time t, the parameters q_1, q_2, …, q_n, and *some other parameters* q_{n+1}, q_{n+2}, …, q_{n+p} whose variations are coupled with those of the q_1, q_2, …, q_n by relations of the form:

(2)
$$\begin{cases} \delta q_{n+1} = \alpha_1\,\delta q_1 + \alpha_2\,\delta q_2 + \cdots + \alpha_n\,\delta q_n, \\ \delta q_{n+2} = \beta_1\,\delta q_1 + \beta_2\,\delta q_2 + \cdots + \beta_n\,\delta q_n, \\ \dotfill \\ \delta q_{n+p} = \lambda_1\,\delta q_1 + \lambda_2\,\delta q_2 + \cdots + \lambda_n\,\delta q_n, \end{cases}$$

in which the coefficients α_1, α_2, …, λ_n likewise depend upon t and the set of parameters q_1, q_2, …, q_n, q_{n+1}, q_{n+2}, …, q_{n+p}. Under those conditions, the real displacement of a system during the time dt will be defined by relations of the form:

(3)
$$\begin{cases} dx = a_1\,dq_1 + a_2\,dq_2 + \cdots + a_n\,dq_n + a\,dt, \\ dy = b_1\,dq_1 + b_2\,dq_2 + \cdots + b_n\,dq_n + b\,dt, \\ dz = c_1\,dq_1 + c_2\,dq_2 + \cdots + c_n\,dq_n + c\,dt, \end{cases}$$

with

(4)
$$\begin{cases} dq_{n+1} = \alpha_1\,dq_1 + \alpha_2\,dq_2 + \cdots + \alpha_n\,dq_n + \alpha\,dt, \\ dq_{n+2} = \beta_1\,dq_1 + \beta_2\,dq_2 + \cdots + \beta_n\,dq_n + \beta\,dt, \\ \dotfill \\ dq_{n+p} = \lambda_1\,dq_1 + \lambda_2\,dq_2 + \cdots + \lambda_n\,dq_n + \gamma\,dt, \end{cases}$$

in which the coefficients a, b, c, α, β, …, λ can depend upon t, q_1, q_2, …, q_{n+p}.

One can then obtain the equations of motion as follows:

The general equation of dynamics, which is deduced from **d'Alembert**'s principle and the principle of virtual work, is:

(5)
$$\sum m\,(x''\,\delta x + y''\,\delta y + z''\,\delta z) = \sum\,(X\,\delta x + Y\,\delta y + Z\,\delta z),$$

in which x'', y'', z'' are the second derivatives of the coordinates with respect to time, and X, Y, Z are the projections of any of the forces.

That equation can be true for all displacements (1) that are compatible with the constraints: They will then decompose into the following n equations:

$$(6)\quad \begin{cases} \sum m(x''a_1 + y''b_1 + z''c_1) = \sum (X\,a_1 + Y\,b_1 + Z\,c_1), \\ \sum m(x''a_2 + y''b_2 + z''c_2) = \sum (X\,a_2 + Y\,b_2 + Z\,c_2), \\ \qquad\qquad\qquad\qquad\qquad\qquad \cdots\cdots\cdots\cdots \\ \sum m(x''a_n + y''b_n + z''c_n) = \sum (X\,a_n + Y\,b_n + Z\,c_n). \end{cases}$$

The right-hand sides of those equations are calculated as they are for the **Lagrange** equations. Upon replacing δx, δy, δz with their values (1), one will have:

$$\sum (X\ \delta x + Y\ \delta y + Z\ \delta z) = Q_1\ \delta q_1 + Q_2\ \delta q_2 + \ldots + Q_n\ \delta q_n$$

for the sum of the virtual works done by applied forces. The quantities $Q_1, Q_2, \ldots, Q_n$ are the right-hand sides of equations (6):

$$Q_1 = \sum (X\ a_1 + Y\ b_1 + Z\ c_1),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

In order to calculate the left-hand sides, divide the relations (3) that define the real displacement by dt and let $x', y', z', q_1', q_2', \ldots, q_n'$ denote the total derivatives $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}, \frac{dq_1}{dt}, \frac{dq_2}{dt}, \ldots, \frac{dq_n}{dt}$. We have:

$$\begin{aligned} x' &= a_1\,q_1' + a_2\,q_2' + \cdots + a_n\,q_n' + a\,, \\ y' &= b_1\,q_1' + b_2\,q_2' + \cdots + b_n\,q_n' + b\,, \\ z' &= c_1\,q_1' + c_2\,q_2' + \cdots + c_n\,q_n' + c\,, \end{aligned}$$

in which the unwritten terms do not contain $q_1'', q_2'', \ldots, q_n''$. However, one will then obviously have:

$$a_1 = \frac{\partial x''}{\partial q_1''}, \qquad b_1 = \frac{\partial y''}{\partial q_1''}, \qquad c_1 = \frac{\partial z''}{\partial q_1''},$$

$$a_2 = \frac{\partial x''}{\partial q_2''}, \qquad b_2 = \frac{\partial y''}{\partial q_2''}, \qquad c_2 = \frac{\partial z''}{\partial q_2''},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

The equations of motion are then written:

$$(8)\quad \begin{cases} \sum m\left(x''\frac{\partial x''}{\partial q_1''} + y''\frac{\partial x''}{\partial q_1''} + z''\frac{\partial x''}{\partial q_1''}\right) = Q_1, \\ \sum m\left(x''\frac{\partial x''}{\partial q_2''} + y''\frac{\partial x''}{\partial q_2''} + z''\frac{\partial x''}{\partial q_2''}\right) = Q_2, \end{cases}$$

……………………………………

Now consider the function:

$$S = \tfrac{1}{2}\sum m(x''^2 + y''^2 + z''^2) = \tfrac{1}{2}\sum m J^2,$$

in which J is the absolute acceleration of the point m. The equations of motion (8) take the form:

$$\frac{\partial S}{\partial q_1''} = Q_1, \quad \frac{\partial S}{\partial q_2''} = Q_2, \ldots, \quad \frac{\partial S}{\partial q_n''} = Q_n. \tag{9}$$

One sees that in order to write them out, it will suffice to calculate just the function S and to express it in such a manner that it no longer contains any other second derivatives than those of the parameters $q_1, q_2, \ldots, q_n$, whose variations are regarded as arbitrary. It can happen that when this function S is calculated as a function of the $q_1, q_2, \ldots, q_{n+p}$, it will contain their first derivatives q_1', $q_2', \ldots, q_{n+p}'$ and the second derivatives $q_1'', q_2'', \ldots, q_{n+p}''$. When the relations (4) are divided by dt, that will give $q_{n+1}', q_{n+2}', \ldots, q_{n+p}'$ as linear functions of $q_1', q_2', \ldots, q_n'$, and when one differentiates them with respect to time, one will likewise obtain $q_{n+1}'', q_{n+2}'', \ldots, q_{n+p}''$ as linear functions of $q_1'', q_2'', \ldots, q_n''$. One can always do that in such a way that the function S will no longer contain any other second derivatives than the $q_1'', q_2'', \ldots, q_n''$. Furthermore, it will contain those quantities in the second degree. Once the function S has been prepared in that way, one can write out equations (9). Those equations, when combined with the conditions (4), form a system of $n + p$ equations that define $q_1, q_2, \ldots, q_n$ as functions of time.

3.

For example, take a solid body that moves around a fixed point O and calculate the function S by referring the motion to a system of axes $Oxyz$ that move along with the body in space. Let Ω denote the instantaneous rotation of the trihedron $Oxyz$ and let P, Q, R, resp., be its components along the axes. Let ω be the rotation of the body, and let p, q, r, resp., be its components. A molecule m of the body with coordinates x, y, z possesses an absolute velocity v whose projections are:

$$v_x = q\,z - r\,y, \quad \ldots$$

That molecule possesses an absolute acceleration J whose projections are:

$$J_x = \frac{d}{dt} v_x + Q\,v_x - R\,v_x, \ldots, \tag{10}$$

which would result from the fact that J is the absolute velocity of the point whose coordinates are v_x, v_y, v_z. Now, upon calling the derivatives of p, q, r with respect to time p', q', r', one will have:

$$\frac{dv_x}{dt} = q\frac{dz}{dt} - r\frac{dy}{dt} + z\,q' - y\,r', \;\ldots,$$

in which $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, which are projections of the relative velocity of the molecule with respect to the axes $Oxyz$ are:

$$\text{(11)} \qquad \frac{dx}{dt} = qz - ry - (Q\,z - R\,y), \;\ldots$$

Indeed, the relative velocity is the geometric difference between the absolute velocity and the velocity of the frame. From that, one will have the following expression for J, which we arrange with respect to x, y, z :

$$\text{(12)} \qquad J_x = -\,x\,(q^2 + r^2) + y\,[q\,(p - P) + pQ - r'] + z\,[r\,(p - P) + pR + q']\,.$$

One gets J_y and J_z similarly, and finally:

$$2S = \sum m\,(J_x^2 + J_y^2 + J_z^2)\,.$$

In order to simplify this, we write out that sum by supposing that the axes $Oxyz$ are the principal axes of inertia at the point O and calling the moments of inertia with respect to those axes A, B, C, resp. Upon confining ourselves to the terms in p', q', r', we will have:

$$\text{(13)} \qquad \left\{ \begin{array}{l} 2S = A\,p'^2 + B\,q'^2 + C\,r'^2 + 2[(C-B)\,q\,r + A(r\,Q - q\,R)]\,p' \\ \quad + 2\,[(A-C)\,r\,p + B(\,p\,R - r\,P)]\,q' + 2\,[(B-A)\,p\,q + C\,(q\,P - p\,Q)]\,r' + \cdots \end{array} \right.$$

Euler equations: Take the moving axes to be three axes that are *invariably linked* to the body and coincide with the three principal axes of inertia. We will then have:

$$P = p, \;\; Q = q, \;\; R = r,$$

$$2S = A\,p'^2 + B\,q'^2 + C\,r'^2 + 2\,(C - B)\,q\,r\,p' + 2\,(A - C)\,r\,p\,q' + 2\,(B - A)\,p\,q\,r' + \ldots$$

Call the sums of the moments of the applied forces with respect to the axes L, M, N, and let:

$$\delta\lambda, \qquad \delta\mu, \qquad \delta\nu$$

be the elementary angles through which the body must turn around the axes in order to go from one position to an infinitely-close one. We shall make λ, μ, ν play the role of the parameters q_1, q_2, …, q_n . One has, on the one hand:

$$\sum (X\,\delta x + Y\,\delta y + Z\,\delta z) = L\;\delta\lambda + M\;\delta\mu + N\;\delta\nu\,,$$

and on the other hand, the components p, q, r of the instantaneous rotation of the body are:

$$p = \frac{d\lambda}{dt} = \lambda', \qquad q = \frac{d\mu}{dt} = \mu', \qquad r = \frac{d\nu}{dt} = \nu'.$$

The function S is then:

$$S = \tfrac{1}{2}(A\lambda''^2 + B\mu''^2 + C\nu''^2) + (C - B)\;\mu'\,\nu'\,\lambda'' + (A - C)\;\nu'\,\lambda'\,\mu'' + (B - A)\;\lambda'\,\mu'\,\nu'' + \ldots,$$

in which the unwritten terms do not contain λ'', μ'', ν''. The equations of motion are then:

$$\frac{\partial S}{\partial \lambda''} = L, \qquad \frac{\partial S}{\partial \mu''} = M, \qquad \frac{\partial S}{\partial \nu''} = N.$$

For example, the first of them is written:

$$A\;\lambda'' + (C - B)\;\mu''\,\nu'' = L\;.$$

From the values of p, q, r, that is precisely one of **Euler**'s equations.

5.

Body of revolution suspended by a point *O* on its axis. – Draw a fixed axis $O\alpha$ through O and take the axis Oz to be the axis of revolution, the axis Oy to be the perpendicular to the plane αOz, and the axis Ox to be the perpendicular to the plane yOz. When the position of the trihedron $Oxyz$ is known, in order to get the position of the body, it will suffice to know the angle φ that Oy makes with a ray that issues from O and is invariably coupled with the body in the xy-plane. The derivative φ' of that angle with respect to time represents the proper rotation of the body around Oz. The rotation ω of the body is then the resultant of the rotation Ω of the trihedron and the rotation φ'. One will then have:

$$p = P, \qquad q = Q, \qquad r = R + \varphi'.$$

Since $A = B$, the function S that is defined by the expression (13) will then become:

(14) $$2S = A\,(p'^2 + q'^2) + C\,r'^2 + 2\,(AR - Cr)\,(p\,q' - q\,p') + \dots$$

Once more, let $\delta\lambda$, $\delta\mu$, $\delta\nu$ be the elementary angles through which one must turn the body around the axes *Ox*, *Oy*, *Oz* in order to take it from one position to a neighboring one, and let *L*, *M*, *N* be the moments of the forces with respect to the axes, so one will have:

$$p = \lambda'', \qquad q = \mu'', \qquad r = \nu'',$$

as above, and the equations of motion will be:

$$\frac{\partial S}{\partial \lambda''} = L, \qquad \frac{\partial S}{\partial \mu''} = M, \qquad \frac{\partial S}{\partial \nu''} = N\,;$$

i.e., since the component *R* of the rotation Ω does not depend upon λ'', μ'', ν'':

$$\begin{aligned} A\,p' - (AR - Cr)\,q &= L, \\ A\,q' + (AR - Cr)\,p &= M, \\ C\,r' &= N. \end{aligned}$$

6.

Hoop. – In order to calculate the function *S* relative to an arbitrary system, one can employ a theorem that is analogous to one by **Koenig** for the calculation of *vis viva*. For example, take a hoop or a homogeneous disc of negligible thickness that is subject to roll in a horizontal plane. Call the radius of the hoop *a* and call its center *G*. Let $G\alpha$ be the ascending vertical that is drawn through *G*, and let *Gz* be the normal to the plane of the hoop; i.e., the axis of revolution of the body. We let Θ denote the angle $\alpha\,G\,z$.

As in the preceding example, take the axis *Gy* to be the perpendicular to the plane αGz and the *Gx* to be the perpendicular to the plane *yGz*. In that way, *Gy* is a horizontal in the plane of the hoop, and *Gx* is the line of greatest slope in that plane that starts from the point *H* where the hoop touches the fixed plane.

Take the mass of the hoop to be unity. Let J_0 denote the acceleration of the point *G*, and let J' denote the relative acceleration of a point *m* on the hoop with respect to some axes with fixed directions that pass through *G*. Upon applying a theorem that is analogous to **Koenig**'s theorem, one will have:

$$\tfrac{1}{2}\sum m\,J^2 = \tfrac{1}{2}J_0^2 + \tfrac{1}{2}\sum m\,J'^2,$$

which is a formula that we write:

$$S = \tfrac{1}{2}J_0^2 + S'.$$

The relative motion of the hoop around the point G is the motion of a body of revolution that is suspended by a point on its axis. Upon applying the notations of the preceding number to that motion, from (14), one will have:

$$2S' = A\,(p'^2 + q'^2) + C\,r'^2 + 2\,(AR - Cr)\,(p\,q' - q\,p') + \dots$$

It then remains to calculate J_0^2. In order to do that, let u, v, w denote the projections of the absolute velocity of the point G onto the axes Gx, Gy, Gz. In order to express the idea that the hoop rolls, one must write out that the material point on the hoop that is found to be in contact with the base has zero velocity at the point H. Since the velocity of that point is the resultant of its relative velocity around G and the velocity of the frame of G, one will then have:

$$u = 0, \qquad v + r\,a = 0, \qquad w - qa = 0. \tag{15}$$

The coordinates of the point H with respect to the axes $Gxyz$ are indeed a, 0, 0.

Since the instantaneous rotation of the trihedron $Gxyz$ is Ω, the absolute acceleration of the point G will have the following projections onto the axes Gx, Gy, Gz :

$$\frac{du}{dt} + Qw - Rv,$$
$$\frac{dv}{dt} + Ru - Pw,$$
$$\frac{dw}{dt} + Pv - Qu\,;$$

i.e., form (15):

$$q\,(Qq + Rr), \quad -ar' - aPq, \quad aq' - aPr,$$

and upon forming the sum of the squares and remarking that $P = p$, $Q = q$, one will have:

$$J_0^2 = a\,(q'^2 + r'^2) + 2a^2\,p\,(p\,r' - r\,q') + \dots,$$

in which the terms that do not contain p', q', r' are not written out. Finally, one will then have:

$$2S = A\,p'^2 + (A + a^2)\,q'^2 + (C + a^2)\,r'^2 + 2\,(AR - Cr)\,(pq' - qp') + 2a^2\,p\,(qr' - rq') + \dots$$

Once more, let:

$$\delta\lambda, \qquad \delta\mu, \qquad \delta\nu$$

denote the infinitely-small angles through which one must turn the hoop around the axes Gx, Gy, Gz, resp., in order to move it from one position to an infinitely-close one. Those quantities are arbitrary and are determined completely by the displacement of the hoop. We take λ, μ, ν to be the parameters $q_1, q_2, \ldots, q_n$, and we will once more have:

$$p' = \lambda'', \qquad q' = \mu'', \qquad r' = \nu''.$$

We can then write the left-hand sides of the equations of motion such as (9). It remains for us to calculate the right-hand sides. In order to do that, we can calculate the sum of the works done by the applied forces:

$$\sum (X\,\delta x + Y\,\delta y + Z\,\delta z)$$

and put it into the form:

$$L_1\,\delta\lambda + M_1\,\delta\mu + N_1\,\delta\nu\,.$$

L_1, M_1, N_1 will be the right-hand sides of the equations. Those quantities have a simple meaning: Draw three axes Hx_1, Hx_2, Hx_3 through the point of contact H with the base that are parallel to the axes Gx, Gy, Gz, resp. L_1, M_1, N_1 will be the sums of the moments of the applied forces with respect to those new axes, respectively. Indeed, the velocity of the molecule that is placed at H is zero for a displacement that is compatible with the constraints, so the infinitely-small displacement of the hoop is the resultant displacement of the three elementary rotations $\delta\lambda$, $\delta\mu$, $\delta\nu$ around the axes Hx_1, Hx_2, Hx_3, resp., without displacing H. That proves the proposition.

If the only applied force is the weight g applied to G then one will obviously have:

$$L_1 = 0, \qquad N_1 = 0,$$

$$M_1 = -\,g\,a\cos\Theta.$$

The equations of motion are then:

$$\frac{\partial S}{\partial \lambda''} = 0, \qquad \frac{\partial S}{\partial \mu''} = -\,g\,a\cos\Theta, \qquad \frac{\partial S}{\partial \nu''} = 0\,;$$

i.e., from the value of S :

$$Ap' - (AR - Cr)\,q = 0,$$

$$(A + a^2)\,q' + (AR - Cr)\,p - a^2\,p\,r = -\,g\,a\cos\Theta,$$

$$(C + a^2)\,r' + a^2\,p\,q = 0.$$

Korteweg and myself have pointed out (almost at the same time) that integrating those equations comes down to integrating **Gauss**'s hypergeometric equation, which follows from one quadrature. (See an article in the Rendiconti del Circolo Matematico di Palermo, which is followed by a letter by **Korteweg**, in the first fascicle of 1900.)

7.

In the preceding, we have deduced equations (9) from **d'Alembert**'s principle, along with the principle of virtual work. One can also attach it to **Gauss**'s principle of least constraint (Crelle's Journal, t. IV). Furthermore, things could not be otherwise, since, as **Gauss** pointed out, all principles of equilibrium and motion that are equivalent to the principle of virtual velocities and **d'Alembert**'s principle must necessarily be equivalent to each other.

If one forms the function:

$$R = S - (Q_1\, q_1'' + Q_2\, q_2'' + \cdots + Q_n\, q_n''),$$

which contains the symbols q'' in degree two, then one will see that the equations of motion (9) can be written:

$$\frac{\partial R}{\partial q_1''} = 0, \qquad \frac{\partial R}{\partial q_2''} = 0, \qquad \ldots, \qquad \frac{\partial R}{\partial q_n''} = 0. \tag{16}$$

Those are the equations that one has to write down in order to find the values of q_1'', q_2'', …, q_n'' that make R a *minimum*. Conversely, the values of q'' that one infers from those equations will make R a minimum, because the terms in R that are homogeneous of degree two will come from S and constitute a positive-definite quadratic form. Since the values of q'' determine the acceleration, one can interpret that result by saying that *the values of the acceleration at each instant will make R a minimum.*

One can replace the function R in that statement with any other function that differs from it only by terms that are independent of the accelerations: for example, by the following two functions:

$$\tfrac{1}{2}\sum m\,(x''^2 + y''^2 + z''^2) - \sum (X\,x'' + Y\,y'' + Z\,z''),$$
$$\tfrac{1}{2}\sum \frac{1}{m}[(m\,x'' - X)^2 + (m\,y'' - Y)^2 + (m\,z'' - Z)^2].$$

The fact that the accelerations make the latter function a minimum is an immediate consequence of the **Gauss**'s principle of least constraint, as **A. Mayer** showed in an interesting article that was entitled "Ueber die Aufstellung der Differentialgleichungen der Bewegung für reibungslose Punktsysteme, die Bedingungsgleichungen unterworfen sind," and was followed by "Zur Regulierung der Stösse in reibungslosen Punktsystemen, die dem Zwange von Bedingungsgleichungen unterliegen," which was printed in the Berichten der mathematisch-physikalischen Klasse der Königl. Sächs. Gesellschaft der Wissenschaft zu Leipzig, session on 2 July 1899. The statement of **Gauss**'s principle that I gave, for my own part, following **Mayer**, in the Comptes Rendus in 11 September 1899, is already found in volume III of the works of **Hertz**, page 224 (Leipzig, 1894).

"Sur la function *S* introduite par M. Appell dans les équations de la Dynamique," C. R. Acad. Sci. Paris **130** (1900), 1174.

On the function *S* that Appell introduced into the equations of dynamics

Note by **A. DE SAINT-GERMAN**

Presented by Appell

Translated by D. H. Delphenich

One knows that Appell [Journal de Mathématiques (1900)] recently obtained the general equations of dynamics in a form that is convenient to several problems in which the Lagrange equations do not apply. The Appell equations are established by a very direct analysis that is free from any calculational gimmicks. It would be convenient to give a name to the function S that appears in them: I think that the name *energy of acceleration* is perfectly appropriate, by analogy with the term *kinetic energy* or *energy of velocity* that is given to one-half the *vis viva* T ([1]).

That energy of acceleration possesses a property that is analogous to the property of the *vis viva* that is expressed by Koenig's theorem. When the system considered is solid, one can demand, as Ph. Gilbert did for the *vis viva*, that any point A will share the same property that the center of gravity exhibits, i.e., that the energy of acceleration of the solid is equal to the energy of acceleration of a mass that is equal to the mass of the solid and concentrated at the point A, that energy being augmented by the energy of acceleration that corresponds to the relative motion with respect to some axes that issue from the point A and have invariable directions.

Draw three rectangular axes Gx, Gy, Gz through the center of gravity, the last of which is parallel to the instantaneous axis of rotation ω. Let u, v, w be the components of the acceleration of the point G, while p', q', r' are those of the angular acceleration. The locus of the point A is defined by the following equation, into which neither mass nor moments of inertia enter:

$$\left(\omega^2 x + r'y - q'z - \frac{u}{2}\right)^2 + \left(\omega^2 y + p'z - r'x - \frac{v}{2}\right)^2 + \left(q'x - p'y - \frac{w}{2}\right)^2 = \frac{u^2 + v^2 + w^2}{4}.$$

It is an ellipsoid that is homothetic to the ellipsoid on which the acceleration has a given magnitude. The center of gravity and the center of acceleration are at the extremities of one of its diameters. When p' and q' are zero, the locus will become a cylinder of revolution.

([1]) Compare that with HERTZ, *Gesammelte Abhandlungen*, Bd. III, pp. 84-85.

"Remarques d'ordre analytique sur une nouvelle forme des équations de la Dynamique," J. Math. pures appl. (5) **7** (1901), 5-12.

Remarks of an analytical order about a new form of the equations of dynamics

By **PAUL APPELL**

Translated by D. H. Delphenich

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1. – As we showed in a paper that was included in the first fascicle of this collection in the year 1900, a material system is characterized by the function:

$$S = \tfrac{1}{2}\sum m\,J^2 ,$$

in which J denotes the acceleration of a point of mass m. Upon calling the parameters $q_1, q_2, \ldots, q_n$, whose virtual velocities are arbitrary, the function S will become a function of degree two in q_1'', q_2'', …, q_n'' that one can suppose to be reduced to only the terms in q_1'', q_2'', …, q_n''. The coefficients of that function can depend upon $q_1, q_2, \ldots, q_n$ and some other parameters whose virtual variations are given linear, homogeneous functions of the variations of $q_1, q_2, \ldots, q_n$. For an arbitrary virtual displacement that is imposed upon the system, the sum of the works done by the applied force will be:

$$Q_1\,\delta q_1 + Q_2\,\delta q_2 + \ldots + Q_n\,\delta q_n .$$

Moreover, the equations of motion are written:

$$\frac{\partial S}{\partial q_\alpha''} = Q_\alpha \qquad (\alpha = 1, 2, \ldots, n) .$$

De Saint-Germain proposed [Comptes rendus **130** (1900), pp. 1174] to call the function S the *energy of acceleration*, by analogy with the terms *kinetic energy* or *energy of velocity* that are given to one-half the *vis viva T*.

We now propose to show that the function S can be chosen arbitrarily as a function of the parameters under only certain conditions on the degrees of q_1'', q_2'', …, q_n'' and q_1', q_2', …, q_n'. If the function S is supposed to be known then we will show how we can deduce the correction terms in the Lagrange equations. Finally, we will give some indications of how we can apply transformation methods to the problems of dynamics to which the Lagrange equations do not apply.

We suppose, to simplify, that the constraints do not depend upon time and that the coefficients of S contain only $q_1, q_2, \ldots, q_n$.

2. – From the expression for S that was given in the preceding paper:

$$S = \tfrac{1}{2}\sum m(x''^2 + y''^2 + z''^2)\,,$$

that function will have the following form:

$$S = \varphi(q_1'', q_2'', \ldots, q_n'') + \psi_1\, q_1' + \psi_2\, q_2' + \cdots + \psi_n\, q_n'\,, \tag{1}$$

in which φ is a quadratic form in the q'':

$$\varphi(q_1'', q_2'', \ldots, q_n'') = \sum a_{ij}\, q_i'' q_j'' \qquad (a_{ij} = a_{ji}) \tag{2}$$

whose coefficients a_{ij} are supposed to depend upon on $q_1, q_2, \ldots, q_n$, and in which $\psi_1, \psi_2, \ldots, \psi_n$ are quadratic forms in $q_1', q_2', \ldots, q_n'$ whose coefficients also depend upon $q_1, q_2, \ldots, q_n$.

One-half the *vis viva* of the system:

$$T = \tfrac{1}{2}\sum m(x'^2 + y'^2 + z'^2)$$

is a quadratic form in the $q_1', q_2', \ldots, q_n'$ whose coefficients are the same as the ones in the form φ in such a way that:

$$T = \varphi(q_1', q_2', \ldots, q_n') = \sum a_{ij}\, q_i' q_j' \,. \tag{3}$$

That results from calculating with the two functions S and T. In order to simplify the notation, we set:

$$\varphi(q_1'', q_2'', \ldots, q_n'') = \varphi_2\,,$$
$$\varphi(q_1', q_2', \ldots, q_n') = \varphi_1\,.$$

We will then have:

$$\begin{cases} S = \varphi_2 + \psi_1\, q_1'' + \psi_2\, q_2'' + \cdots + \psi_n\, q_n'', \\ T = \varphi_1. \end{cases} \tag{4}$$

3. Necessary conditions that *S* must fulfill. – As is easy to show, and as we showed at the end of the preceding paper, we will have:

(5) $$\frac{dT}{dt} = \frac{\partial S}{\partial q_1''}\, q_1'' + \frac{\partial S}{\partial q_2''}\, q_2'' + \cdots + \frac{\partial S}{\partial q_n''}\, q_n'' .$$

Let us see what the identity gives with the forms (4) for S and T. It will become:

(6) $$\left\{ \begin{aligned} & q_1' \frac{\partial \varphi_2}{\partial q_1''} + q_2' \frac{\partial \varphi_2}{\partial q_2''} + \cdots + q_n' \frac{\partial \varphi_2}{\partial q_n''} + \psi_1\, q_1' + \psi_2\, q_2' + \cdots + \psi_n\, q_n' , \\ & \quad = q_1'' \frac{\partial \varphi_1}{\partial q_1'} + q_2'' \frac{\partial \varphi_1}{\partial q_2'} + \cdots + q_n'' \frac{\partial \varphi_1}{\partial q_n'} + q_1' \frac{\partial \varphi_1}{\partial q_1} + q_2' \frac{\partial \varphi_1}{\partial q_2} + \cdots + q_n' \frac{\partial \varphi_1}{\partial q_n} \end{aligned} \right.$$

identically. The right-hand side of this is the developed expression for dT / dt that would result from T by the intermediary of q_1', q_2', …, q_n', q_1, q_2, …, q_n. Now the first part of the left-hand side of (6) is identical to the first part of the right-hand side, from an elementary property of quadratic forms. The identity (6) will then reduce to:

(7) $$\psi_1\, q_1' + \psi_2\, q_2' + \cdots + \psi_n\, q_n' = q_1' \frac{\partial \varphi_1}{\partial q_1} + q_2' \frac{\partial \varphi_1}{\partial q_2} + \cdots + q_n' \frac{\partial \varphi_1}{\partial q_n} .$$

That relation must be true for any q_1, q_2, …, q_n, q_1', q_2', …, q_n'. It then establishes necessary conditions between the coefficients of the forms ψ_1, ψ_2, …, ψ_n and the coefficients a_{ij} of φ_1. To abbreviate the notation, we shall denote the two sides of the identity (7) by the same symbol. When we set:

(8) $$E = \frac{\partial \varphi_1}{\partial q_1}\, q_1' + \frac{\partial \varphi_1}{\partial q_2}\, q_2' + \cdots + \frac{\partial \varphi_1}{\partial q_n}\, q_n' \equiv \psi_1\, q_1' + \psi_2\, q_2' + \cdots + \psi_n\, q_n' ,$$

the function E will be a cubic form in the q_1', q_2', …, q_n'.

4. Correction terms in the Lagrange equations. – Suppose that the identity (7) is fulfilled and look for an expression for the difference:

(9) $$\Delta_1 = \frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) - \frac{\partial T}{\partial q_1} - \frac{\partial S}{\partial q_1''} .$$

Since we have set $T = \varphi_1$, we will have:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) = \frac{\partial^2 \varphi_1}{\partial q_1'^2}\, q_1'' + \frac{\partial^2 \varphi_1}{\partial q_1'\, \partial q_2'}\, q_2'' + \cdots + \frac{\partial^2 \varphi_1}{\partial q_1'\, \partial q_n'}\, q_n'' + \frac{\partial^2 \varphi_1}{\partial q_1'\, \partial q_1}\, q_1 + \frac{\partial^2 \varphi_1}{\partial q_1'\, \partial q_2}\, q_2' + \cdots + \frac{\partial^2 \varphi_1}{\partial q_1'\, \partial q_n}\, q_n' ,$$

because $\frac{dT}{dq_1'}$ or $\frac{\partial \varphi_1}{\partial q_1}$ depend upon t only by the intermediary of q_1', q_2', …, q_n', q_1, q_2, …, q_n.

Upon specifying the first row and taking the expression for E into account, one can write:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) = 2(a_{11}\,q_1'' + a_{12}\,q_2'' + \cdots + a_{1n}\,q_n'') + \frac{\partial E}{\partial q_1'} - \frac{\partial \varphi_1}{\partial q_1}\,.$$

On the other hand:

$$\frac{\partial T}{\partial q_1'} = \frac{\partial \varphi_1}{\partial q_1},$$

$$\frac{\partial S}{\partial q_1''} = 2(a_{11}\,q_1'' + a_{12}\,q_2'' + \cdots + a_{1n}\,q_n'') + \psi_1\,.$$

After reduction, the difference (9) that was called Δ_1 will then become:

$$\Delta_1 = \frac{\partial E}{\partial q_1'} - 2\frac{\partial \varphi_1}{\partial q_1} - \psi_1\,.$$

Upon setting:

$$\Delta_\alpha = \frac{d}{dt}\left(\frac{\partial T}{\partial q_\alpha'}\right) - \frac{\partial T}{\partial q_\alpha} - \frac{\partial S}{\partial q_\alpha''},$$

one will have:

(10) $$\Delta_\alpha = \frac{\partial E}{\partial q_\alpha'} - 2\frac{\partial \varphi_1}{\partial q_\alpha} - \psi_\alpha\,.$$

Having said that, the equations of motion will be:

$$\frac{\partial S}{\partial q_\alpha''} = Q_\alpha\,.$$

One can then write:

(11) $$\frac{d}{dt}\left(\frac{\partial T}{\partial q_\alpha'}\right) - \frac{\partial T}{\partial q_\alpha} - \frac{\partial S}{\partial q_\alpha''} = Q_\alpha + \Delta_\alpha \qquad (\alpha = 1, 2, \ldots, n),$$

in which the term Δ_α is expressed by the quantity in (10). Those quantities Δ_α constitute what one can call the *correction terms* in the Lagrange equations. One sees that the Lagrange equations can apply to the system only when those terms Δ_α are identically zero. That situation will come about when the system considered is subject to constraints that can all be expressed in finite form and the parameters are true coordinates. Following Hertz, one would then call the system *holonomic*.

If the system is not holonomic then the motion of the system will be the same as that of a holonomic system that admits the same *vis viva* $2T$ as the first one and is subjected to forces:

$$Q_1 + \Delta_1 , \qquad Q_2 + \Delta_2 , \qquad , \ldots, \quad Q_n + \Delta_n .$$

The fact that a non-holonomic system and a holonomic system can have the same T identically can be proved by a simple example that we gave in the Journal für die reine und angewandte Mathematik, which was founded by Crelle, v. 122, pp. 205.

3. *Vis viva* equation: verification. – If the constraints are independent of time then the *vis viva* equation will be:

$$\frac{dT}{dt} = Q_1 \, q_1' + Q_2 \, q_2' + \cdots + Q_n \, q_n' . \tag{12}$$

In order to deduce that equation from (11), one must multiply the first of those equations by q_1', the second by q_2', …, and the last one by q_n', and then add them together.

One will then get equation (12), because one will have:

$$\Delta_1 \, q_1' + \Delta_2 \, q_2' + \cdots + \Delta_n \, q_n' = 0 \tag{13}$$

identically.

Indeed, from the expressions (10) for the quantities Δ_α and the definition of E [viz., equation (8)], one will have:

$$\Delta_1 \, q_1' + \Delta_2 \, q_2' + \cdots + \Delta_n \, q_n' = q_1' \frac{\partial E}{\partial q_1} + q_2' \frac{\partial E}{\partial q_2} + \cdots + q_n' \frac{\partial E}{\partial q_n} - 3E .$$

However, since E is homogeneous of degree three in q_1', q_2', …, q_n', the right-hand side of that will be zero identically, from the theorem on homogeneous functions.

6. Application of transformation methods. – We conclude by indicating a problem that presents itself naturally. If the components of the forces Q_1, Q_2, …, Q_n depend upon only q_1, q_2, …, q_n, and not upon the velocities then the right-hand sides of the equations of motion (11) can nonetheless contain the velocities in the terms Δ_α when the system is not holonomic: Those terms have degree two in q_1', q_2', …, q_n'.

Can one make the terms of that nature disappear by performing a change of variables that involves the parameters of time?

In particular, one can try a transformation of the form:

$$
(14) \qquad \begin{cases} p_\alpha = f_\alpha(q_1, q_2, \ldots, q_n) \quad (\alpha = 1, 2, \ldots, n), \\ dt = \lambda(q_1, q_2, \ldots, q_n)\, dt_1, \\ p'_\alpha = \dfrac{dp_\alpha}{dt_1}, \end{cases}
$$

in which the f_α and λ are functions of $q_1, q_2, \ldots, q_n$, the p_α are the new parameters, and t_1 is the new time. From a calculation that we made in an article "Sur des transformations de movement" (Crelle's Journal, Bd. 110, pp. 37), the equations of motion (11) will take the form:

$$
(15) \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial p''_2}\right) - \frac{\partial T}{\partial p_2} = \Phi_\alpha + \sum_{i=1}^{n} R^i_\alpha (Q_i + \Delta_i) \ ,
$$

in which Φ_α is a quadratic form in $q'_1, q'_2, \ldots, q'_n$, and in which the R^i_α depend upon only the $q_1, q_2, \ldots, q_n$. One can make the derivatives on the right-hand side disappear, moreover, if one can specialize the transformation in such a way that one has:

$$
(16) \qquad \Phi_\alpha + \sum_{i=1}^{n} R^i_\alpha (Q_i + \Delta_i) = 0 \qquad (\alpha = 1, 2, \ldots, n)
$$

identically.

Those conditions, whose left-hand sides are quadratic forms in $q'_1, q'_2, \ldots, q'_n$, must be true no matter what those derivatives are, so upon equating the coefficients of the various powers of $q'_1, q'_2, \ldots, q'_n$ to zero, one will get a larger number of defining equations for the functions $f_1, f_2, \ldots, f_n$, and λ. Those equations will generally be quite numerous, and the problem can be solved for only some special systems.

"Di alcune nouve forme delle equazioni della dinamica, applicabili ai sistemi anolonomi," Atti della Reale Accademia Naz. dei Lincei (5) **10** (1910), 287-292.

On some new forms of the equations of dynamics that are applicable to anholonomic systems

Note by correspondent GIAN ANTONIO MAGGI

Translated by D. H. Delphenich

The idea of forming the differential equations of motion for a constrained system of points that makes use of expressions for velocity as linear functions of the independent parameters can be found in Kirchhoff's *Mechanik*. There, that principle was used to deduce the final equations of **§ 4** of Lesson 3.a from Hamilton's theorem, which were then applied, in **§ 2** of Lesson 4, to the formation of the more general differential equations of motion of a free solid body or one that has a fixed point.

Volterra has applied the same principle in his article "Sopra una classe di equazioni dinamiche," which was published in volume XXXIII (1898) of the Atti della R. Accademia delle Scienze di Torino in order to deduce a form of the equations of motion of a system of points for which the constraints are independent of time from the equations of d'Alembert and Lagrange, when reduced to an expression by Beltrami ([1]), and expressions for the total differential equations between the coordinates that are just as valid whether or not they form an integrable system; that is to say, whether the moving system is holonomic or anholonomic. The differential equations (C) that are established between time, the coordinates, and the *characteristics of motion* are equal in number to the latter: There are thus as many parameters as there are degrees of freedom in the system, by means of which, by virtue of the constraints, one expresses the components of the velocity of any point as homogeneous linear functions. Moreover, Volterra proposed, in particular, to indicate the case in which those equations are sufficient to determine the characteristics as functions of time.

Finally, Appell used the same principle of the characteristics to deduce an elegant form for the differential equations of motion from the d'Alembert and Lagrange equations that was, like the preceding ones, applicable to holonomic or non-holonomic systems, as well as the case of constraints that depend upon time in his article "Sur les mouvements de roulement – Équations analogues à celles de Lagrange" that was included in volume CXXIX of the Comptes Rendus des Séances de l'Académie des Sciences in Paris (1899), as well as in "Sur une forme Générale des équations de la dynamique," which published in volume CXXI of Crelle's Journal (1900) ([2]).

([1]) Beltrami, "Sulle equazioni dinamiche di Lagrange," Rend. R. Ist. Lombardo **28** (1895).
([2]) See also, Appell, "Les mouvements de roulement en Dynamique," § **24** (Scientia 4º), Paris, 1899.

In this brief note, permit me to show how Appell's equations and Volterra's can be deduced from a form of the equations of dynamics that is found in § **493** of my *Meccanica* ([1]), which is, in turn, deduced quite directly from Hamilton's theorem. It would seem that it has remained unnoticed, although in the following §, it will be applied to the construction of the equations of motion for a solid by a method that seems to present the advantage of greater expediency when compared to that of Kirchhoff.

We begin by recalling the very simple deduction of the equations in question. If we start from Hamilton's theorem:

(1) $$\int_{t'}^{t''} (\delta T + \Pi)\, dt = 0,$$

in which if $q_1, q_2, \ldots, q_n$ denote any type of coordinates – free or not – for the moving system then we will have:

(2) $$\delta T = \sum_{i=1}^{n} \left(\frac{\partial T}{\partial q_i} \delta q_i + \frac{\partial T}{\partial \dot{q}_i} \delta \dot{q}_i \right), \qquad \dot{q}_i = \frac{dq_i}{dt}, \qquad \Pi = \sum_{i=1}^{n} Q_i\, \delta q_i .$$

Let the constraints be represented by:

(3) $$\sum_{i=1}^{n} \Phi_{i,j}\, dq_i = T_j\, dt \qquad (j = 1, 2, \ldots, m),$$

by means of which the $\delta q_1, \delta q_2, \ldots, \delta q_n$ are defined by:

(4) $$\sum_{i=1}^{n} \Phi_{ij}\, \delta q_i = 0 \qquad (j = 1, 2, \ldots, m) .$$

These last equations can always be supposed to be solved for $n - m$ of the $\delta q_1, \delta q_2, \ldots, \delta q_n$, which are, if needed, opportunely chosen to make:

(5) $$\delta q_i = \sum_{i=1}^{n} E_{ir}\, \varepsilon_r \qquad (i = 1, 2, \ldots, n) ,$$

in which the $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-m}$ are other arbitrary parameters.

Now (1), conforming with (2), can be written in the form:

$$\int_{t'}^{t''} dt \sum_{i=1}^{n} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} - Q_i \right) \delta q_i = 0,$$

which immediately yields:

([1]) *Principii della teoria matematica del movimento dei corpi*, Milan, 1896.

$$\sum_{i=1}^{n}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot q_i} - \frac{\partial T}{\partial q_i} - Q_i\right)\delta q_i = 0.$$

One substitutes (5) in that relation, selects one of the ε_r, equates its coefficient to 0, and sets:

$$\sum_{i=1}^{n} Q_i E_{ir} = E_r ,$$

so from (2):

$$\Pi = \sum_{r=1}^{n-m} E_r \varepsilon_r ,$$

and one will get:

$$\sum_{i=1}^{n}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot q_i} - \frac{\partial T}{\partial q_i}\right)E_{ir} = E_r \qquad (r = 1, 2, \ldots, n - m) . \tag{7}$$

Those are the desired equations, and along with (3), they define a system of n differential equations, where t serves as the independent variable and the $q_1, q_2, \ldots, q_n$ are unknowns. They are valid for both holonomic and non-holonomic systems, and for constraints that are or are not independent of time.

In order to put those equations into Appell form, it is enough to observe that (3), conforming to (5), will imply that:

$$\dot q_i = E_i + \sum_{r=1}^{n-m} E_{ir} e_r \qquad (i = 1, 2, \ldots, n),$$

in which $e_1, e_2, \ldots, e_{n-m}$ represent the characteristics of the motion of the system considered, in such a way that:

$$E_{ir} = \frac{\partial \dot q_i}{\partial e_r} = \frac{\partial \ddot q_i}{\partial \dot e_r} . \tag{8}$$

Meanwhile, on the other hand, with Appell, set:

$$S = \tfrac{1}{2}\sum m(\ddot x^2 + \ddot y^2 + \ddot z^2),$$

in which m and x, y, z represent the mass and coordinates of the generic point of the system, and the sum extends over all points, so one will have:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q_i} - \frac{\partial T}{\partial q_i} = \frac{\partial S}{\partial \ddot q_i} . \tag{9}$$

In fact:

$$\frac{\partial S}{\partial \ddot q_i} = \sum m\left(\ddot x\frac{\partial \ddot x}{\partial \ddot q_i} + \ddot y\frac{\partial \ddot y}{\partial \ddot q_i} + \ddot z\frac{\partial \ddot z}{\partial \ddot q_i}\right).$$

However, one has:

$$\dot{x} = \frac{\partial x}{\partial t} + \sum_{i=1}^{n} \frac{\partial x}{\partial q_i} \dot{q}_i ,$$

so:

$$\frac{\partial x}{\partial q_i} = \frac{\partial \dot{x}}{\partial \dot{q}_i} = \frac{\partial \ddot{x}}{\partial \ddot{q}_i},$$

and analogous expressions. Hence:

$$\frac{\partial S}{\partial \ddot{q}_i} = \sum m \left(\ddot{x} \frac{\partial \dot{x}}{\partial \dot{q}_i} + \ddot{y} \frac{\partial \dot{y}}{\partial \dot{q}_i} + \ddot{z} \frac{\partial \dot{z}}{\partial \dot{q}_i} \right),$$

and what will remain is:

$$T = \tfrac{1}{2} \sum m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2), \tag{10}$$

as well as:

$$\frac{\partial S}{\partial \ddot{q}_i} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \sum m \left(\dot{x} \frac{d}{dt} \frac{\partial x}{\partial \dot{q}_i} + \dot{y} \frac{\partial y}{\partial \dot{q}_i} + \dot{z} \frac{\partial z}{\partial \dot{q}_i} \right) = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i};$$

therefore:

$$\frac{d}{dt} \frac{\partial x}{\partial q_i} = \frac{\partial^2 x}{\partial q_i \, \partial t} + \sum_{j=1}^{n} \frac{\partial^2 x}{\partial q_i \, \partial q_j} \dot{q}_j = \frac{\partial^2 x}{\partial t \, \partial q_i} + \sum_{j=1}^{n} \frac{\partial^2 x}{\partial q_j \, \partial q_i} \dot{q}_j = \frac{\partial}{\partial q_i} \frac{dx}{dt} = \frac{\partial \dot{x}}{\partial q_i}.$$

Having done that, from (8) and (9), (7) will become:

$$\sum_{i=1}^{n} \frac{\partial S}{\partial \ddot{q}_i} \frac{\partial \ddot{q}_i}{\partial \dot{e}_r} = E_r \qquad (r = 1, 2, \ldots, n - m)$$

or

$$\frac{\partial S}{\partial \dot{e}_i} = E_r \qquad (r = 1, 2, \ldots, n - m), \tag{7′}$$

which are Appell's equations, when they are free of any special hypothesis about the choice of the characteristics.

One will get Volterra's equations when one puts (7) into the form:

$$\sum_{i=1}^{n} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} \right) \frac{\partial \dot{q}_i}{\partial \dot{e}_r} = E_r ,$$

conforming to (8), or:

$$\frac{d}{dt} \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial e_r} = \sum_{i=1}^{n} \frac{dE_{ir}}{dt} \frac{\partial T}{\partial \dot{q}_i} + \sum_{i=1}^{n} E_{ir} \frac{\partial T}{\partial q_i} + E_r ,$$

or finally:

$$\frac{d}{dt}\frac{\partial T}{\partial e_r} = \sum_{i=1}^{n} \frac{dE_{ir}}{dt}\frac{\partial T}{\partial \dot{q}_i} + \sum_{i=1}^{n} E_{ir}\frac{\partial T}{\partial q_i} + E_r \qquad (r = 1, 2, \ldots, n - m) . \tag{7''}$$

Those equations properly apply to any type of coordinates $q_1, q_2, \ldots, q_n$ and to constraints that also depend upon time. Suppose that the $q_1, q_2, \ldots, q_n$, with $n = 3\nu$, represent the orthogonal Cartesian coordinates x, y, z of a system of ν points, and that the constraints are independent of time. Conforming to (10), one will have:

$$\frac{\partial T}{\partial q_i} = 0, \qquad \frac{\partial T}{\partial \dot{q}_i} = m_i\,\dot{q}_i = m_i \sum_{u=1}^{n-m} E_{iu} e_u \qquad (i = 1, 2, \ldots, 3\nu;\ m_i = m_{i+1} = m_{i+2}) .$$

Furthermore:

$$\frac{dE_{ir}}{dt} = \sum_{j=1}^{n} \frac{\partial E_{ir}}{\partial q_j}\dot{q}_j = \sum_{\nu=1}^{n-m} e_\nu \sum_{j=1}^{n} \frac{\partial E_{ir}}{\partial q_j} E_{j\nu} .$$

Therefore, under those hypotheses, the preceding equations will become:

$$\frac{d}{dt}\frac{\partial T}{\partial e_i} = \sum_{u=1}^{n-m}\sum_{\nu=1}^{n-m} b_{r\nu}^{(u)}\, u_u\, e_\nu , \tag{7'''}$$

in which

$$b_{r\nu}^{(u)} = \sum_{j=1}^{3\nu} E_{j\nu} \sum_{j=1}^{3\nu} m_i \frac{\partial E_{ir}}{\partial q_j} E_{iu} .$$

The simple form to which Volterra's equations (C) reduce (except for different symbols) makes use of the assumed relations and follows from the indicated operations.

We conclude with some brief observations on the possibility and legitimacy of using Hamilton's theorem.

It seems to me that as far as the possibility is concerned, the equation that translates that theorem can always be considered to be a reduction of a more concise form of d'Alembert's and Lagrange's equations that is almost spontaneous from the way that the equations of motion in general coordinates are deduced.

As for their legitimacy, with the exception of Appell, who addressed that topic in his article "Sur les équations de Lagrange et le principe d'Hamilton" in volume XXVI of the Bulletin de la Société mathématique de France (1898), one can object that the proof of the incompatibility of:

$$d\,\delta x = \delta\, dx, \qquad d\,\delta q_i = \delta\, dq_i \tag{11}$$

in the anholonomic case is based upon the deduction of:

$$\delta\,[dx - (A_1\, dq_1 + A_2\, dq_2)] = 0$$

from the equations that translate the constraint:

$$dx - (A_1\, dq_1 + A_2\, dq_2) = 0\ .$$

Now, that signifies that the virtual motion is forced to satisfy the same constraint as the effective motion. Conforming to the usual canon, it is the variation that relates to the passage from a virtual motion that is defined by (11) to:

$$\delta x - (A_1\, \delta q_1 + A_2\, \delta q_2) = 0$$

in the case at hand. In that case, Appell's argument shows how holonomity is the necessary and sufficient condition for the virtual motion to coincide with the motion that satisfies the same constraints as the effective motion ([1]).

([1]) Cf., Hölder, "Ueber die Principien von Hamilton und Maupertuis," § **6**, Nachrichten der Gesellschaft der Wissenschaften in Göttingen (1896).

"Aperçu sur l'emploi possible de l'énergie d'accélération dans les équations de l'électrodynamique," C. R. Acad. Sci. Paris **154** (1912), 1037-1040.

Outline of the possible use of the energy of acceleration in the equations of electrodynamics

Note by **PAUL APPELL**

Translated by D. H. Delphenich

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1. – In a volume of the collection *Scientia* that was published in 1902 under the title of "L'électricité déduite de l'expérience et ramenée au principe des travaux virtuels," Carvallo, following Maxwell's theory, studied the application of the Lagrange to electrodynamical phenomena ([1]), and notably to the cases of two or three-dimensional conductors. He explained (pp. 81) the inadequacy of the Lagrange equations by remarking that the three parameters θ, q_1, q_2 whose arbitrary infinitely small variations define the most general displacement of the system in Barlow's experiment with the wheel are not true coordinates, and that the system can be composed of a hoop in an analogous way, to which the Lagrange equations do not apply, as is known from a remark by Ferrers (Quarterly Journal of Mathematics 1871-1873). In other words, in Hertz's terminology, the system is not *holonomic*.

2. – Under those conditions, if one can hope to attach the equations of electrodynamics to those of analytical mechanics then one must seek to attach them to a general form of equations that is applicable to all systems, whether holonomic or not.

In order to form such equations, as I showed in the *Comptes rendus* (session on 7 August 1899), one can proceed as follows ([2]): Imagine a material system whose virtual displacement (which is compatible with the constraints at the time t) is defined by the arbitrary variations δq_1, δq_2, …, δq_k of the parameters q_1, q_2, …, q_k. For that displacement, the sum of the elementary forces done by the given forces Q is:

$$Q_1\,\delta q_1 + Q_2\,\delta q_2 + \ldots + Q_k\,\delta q_k\,.$$

On the other hand, let the energy of acceleration be:

(1) $$S = \tfrac{1}{2}\sum m\,J^2\,,$$

([1]) Likewise, see a book by Carvallo that is entitied *Leçons d'Électricité*, Béranger, editor, 1904.

([2]) See also my *Traité de Mécanique rationelle*, t. II, Chap. XXIV, § **6**.

which is equal to one-half the sum of the products that are obtained by multiplying the mass *m* of each point by the square of its acceleration *J*. That expression *S* is a function of degree two in the second derivatives q_1'', q_2'', …, q_k'' of the parameters $q_1, q_2, \ldots, q_k$, resp., with respect to time. The equations of motion are then:

$$\frac{\partial S}{\partial q_\nu''} = Q_\nu \qquad (n = 1, 2, \ldots, k) ; \tag{2}$$

they express the idea that the accelerations at each instant *t* minimize the function:

$$R = S - \sum Q\,J \cos Q\,J . \tag{3}$$

That is the form of the equations that one would like to be able to extend to electrodynamic phenomena that depend upon a finite number of parameters. The difficulty will obviously be in calculating the energy of acceleration *S*. From a purely formal standpoint, one can calculate it for all phenomena for which the Lagrange equations apply, because when one calls the energy of velocity or kinetic energy *T*, one will then have:

$$\frac{\partial S}{\partial q_\nu''} = \frac{d}{dt}\left(\frac{\partial T}{\partial q_\nu'}\right) - \frac{\partial T}{\partial q_\nu} \qquad (\nu = 1, 2, \ldots, k),$$

which are equations that give *S*, up to a term that is independent of q''. However, when the Lagrange equations do not apply, the initial condition for equations (2) to be capable of accounting for the phenomenon is that when the equations of motion have been put into the form:

$$f_\nu(q_1'', q_2'', \ldots, q_k'') = Q_\nu \qquad (\nu = 1, 2, \ldots, k),$$

the f_ν must be partial derivatives of the same function *S* with respect to the q_ν''.

3. – For the example of the Barlow wheel, when one uses the notations of Carvallo (*loc. cit.*, pp 78-80), the parameters will be θ, q_1, q_2, while the right-hand sides Q_ν of equations will be Q, $E_1 - r_1 q_1'$, $E_2 - r_2 q_2'$; the equations themselves will be:

$$\begin{aligned} I\,\theta'' - K\,q_1'\,q_2' &= Q, \\ L_1\,q_1'' + K\,\theta''\,q_2' &= E_1 - r_1\,q_1', \\ L_2\,q_2'' &= E_2 - r_2\,q_2'. \end{aligned}$$

Now, the left-hand sides are the partial derivatives with respect to θ, q_1'', and q_2'' of the function:

$$S = \tfrac{1}{2}[I\,\theta''^2 + L_1\,q_1''^2 + L_2\,q_2''^2 + 2K\,q_2'(\theta'\,q_1'' - \theta''\,q_1') + \cdots], \tag{4}$$

in which the unwritten terms do not contain second derivatives. The equations of motion are then indeed of the form (2). They express the idea that the accelerations at each instant minimize the function:

$$R = S - Q\,\theta'' - (E_1 - r_1\,q_1')\,q_1'' - (E_2 - r_2\,q_2')\,q_2''.$$

By analogy, the function S that is given by equation (4) must then be regarded as the energy of acceleration of the system. However, the truly important point will be to know if that function S, which is thus-formed analytically, can be obtained directly by physical considerations that are attached to the defining formula (1).

4. – As an example of a case with an infinite number of parameters, in the next volume, which will be published in Italy, in honor of Lagrange, I will show how one can deduce the general equations of hydrodynamics from the principle of the minimum of the function R that is defined by equation (3).

I recently indicated (C. R. Acad. Sci. Paris, session on 8 May 1911; Rendiconti del Circolo matematico di Palermo, t. XXXII, session on 14 May 1911, and t. XXXIII, session on 25 February 1912) how that same principle can be applied to the motion of systems that are subject to constraints that are nonlinear with respect to the velocities. That question was examined more deeply in a very general and complete way by Delassus in various notes that were included in *Comptes rendus* ([1]).

([1]) E. Delassus, "Sur la réalisation matérielle des liaisons," C. R. Acad. Sci. Paris **152**, session on 19 June 1911, pp. 1739-1743; "Sur les liaisons non linéaires," *ibid.*, **153**, session on 2 October 1911, pp. 626-628; "Sur les liaisons non linéaires et les mouvements étudiés par M. Appell," *ibid.*, **153**, session on 16 October 1911, pp. 707-710; "Sur les liaisons d'ordre quelconque des systèmes matériels," *ibid.*, **154**, session on 15 April 1912, pp. 964-967.

"Les équations du mouvement d'un fluide parfait déduites de la considération de l'énergie d'accélération," Ann. mat. pura appl. (3) **20** (1913), 37-42.

The equations of motion for a perfect fluid, deduced from the consideration of the energy of acceleration.

By PAUL APPELL in Paris

Translated by D. H. Delphenich

I. – J. L. Lagrange deduced the equations of hydrostatics from the principle of virtual work in his *Mécanique analytique* (3rd edition, I, pp. 173-206; *Oeuvres* **11**, pp. 197-236). As for the equations of hydrodynamics, BASSET, in the work entitled *A Treatise on hydrodynamics*, Cambridge, 1888, said this on page 32: "As Larmor showed, the equations of motion can be deduced by the use of the principle of least action, combined with Lagrange's method."

I propose to deduce those equations from the following principle:

In a material system with arbitrary constraints without friction (whether holonomic or not) that is subject to forces X, Y, Z that depend upon time, the positions, and the velocities, the components x, y, z of the accelerations of the various points have values at an arbitrary instant that render the function:

$$R = \sum\left[\frac{m}{2}(x''^2 + y''^2 + z''^2) - (X\,x'' + Y\,y'' + Z\,z'')\right]$$

a minimum.

One will find the proof of that principle in a note that was entitled "Sur les mouvements de roulement; équations du mouvement analogues à celles de Lagrange" [Comptes Rendus de l'Académie des Sciences de Paris **129** (7 August 1899), pp. 317-320].

The quantity:

$$S = \sum\frac{m}{2}(x''^2 + y''^2 + z''^2),$$

which is analogous to the LAGRANGE function:

$$T = \sum\frac{m}{2}(x'^2 + y'^2 + z'^2),$$

has received the name of *energy of acceleration.*

II. – Having said that, imagine a perfect fluid in motion. Let ρ denote the density of a particle whose coordinates are x, y, z at the instant t, and let X, Y, Z denote the components of the force per unit mass. Use the variables that one calls the *Lagrange variables*, and let a, b, c denote the initial coordinate of the particle x, y, z at the instant $t = t_0$, and let ρ_0 denote the initial density of that particle. The coordinates x, y, z, and the density ρ are functions of a, b, c, t.

The continuity equation is:

$$\rho D = \rho_0 , \tag{1}$$

in which D denotes the functional determinant:

$$D = \frac{d(x,y,z)}{d(a,b,c)},$$

and we shall denote minors such as $\frac{\partial y}{\partial b}\frac{\partial z}{\partial c} - \frac{\partial z}{\partial b}\frac{\partial y}{\partial c}$ by $\frac{d(y,z)}{d(b,c)}$. Finally, the derivatives of x, y, z with respect t will be denoted by primes, following LAGRANGE's notation. The function that is analogous to R is presently expressed by a triple integral that is extended over the volume V of the fluid at the instant t :

$$R = \iiint \left[\tfrac{1}{2}\rho (x''^2 + y''^2 + z''^2) - \rho (X\,x'' + Y\,y'' + Z\,z'') \right] dx\, dy\, dz ,$$

or, upon taking the integration variables to be a, b, c :

$$R = \iiint \left[\tfrac{1}{2}\rho_0 (x''^2 + y''^2 + z''^2) - \rho_0 (X\,x'' + Y\,y'' + Z\,z'') \right] da\, db\, dc, \tag{2}$$

because one must replace $dx\, dy\, dz$ by $D\, da\, db\, dc$ or $(\rho_0 / \rho)\, da\, db\, dc$. In that integral, x'', y'', z'' are, at the instant t, functions of the a, b, c that are subject to the following relation, which is deduced from the continuity equation (1). Differentiate that equation twice with respect to t ; it will become:

$$D'' + 2\frac{\rho'}{\rho} D' + \frac{\rho''}{\rho} D = 0. \tag{3}$$

In that equation, the terms in x'', y'', z'' are provided by only D''. Now, one has:

$$\left.\begin{aligned} D'' = {} & \frac{\partial x''}{\partial a}\frac{d(y,z)}{d(b,c)} + \frac{\partial y''}{\partial a}\frac{d(z,x)}{d(b,c)} + \frac{\partial z''}{\partial a}\frac{d(x,y)}{d(b,c)} \\ & + \frac{\partial x''}{\partial b}\frac{d(y,z)}{d(c,a)} + \frac{\partial y''}{\partial b}\frac{d(z,x)}{d(c,a)} + \frac{\partial z''}{\partial b}\frac{d(x,y)}{d(c,a)} \\ & + \frac{\partial x''}{\partial c}\frac{d(y,z)}{d(a,b)} + \frac{\partial y''}{\partial c}\frac{d(z,x)}{d(a,b)} + \frac{\partial z''}{\partial c}\frac{d(x,y)}{d(a,b)} \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}\right\} \tag{4}$$

in which the unwritten terms do not contain x'', y'', z''.

Having said that, from LAGRANGE's method for the calculus of variations, upon denoting an arbitrary function of a, b, c, t by λ, one must determine x'', y'', z'' in such a fashion that the variation of the integral:

$$I = \iiint \left[\tfrac{1}{2}\rho_0 \,(x''^2 + y''^2 + z''^2) - \rho_0 \,(X\,x'' + Y\,y'' + Z\,z'') + \lambda\,D'' \right] da\; db\; dc$$

will be zero when x'', y'', z'' are subjected to arbitrary infinitely-small variations $\delta x''$, $\delta y''$, $\delta z''$. Now:

$$\delta I = \iiint \left[\rho_0 \,(x'' - X)\,\delta x'' + \cdots + \lambda\,\delta D'' \right] da\; db\; dc\,,$$

in which, from (4):

$$\delta D'' = \frac{\partial \delta x''}{\partial a}\frac{d(y,z)}{d(b,c)} + \frac{\partial \delta y''}{\partial a}\frac{d(z,x)}{d(b,c)} + \frac{\partial \delta z''}{\partial a}\frac{d(x,y)}{d(b,c)} + \ldots + \ldots,$$

in which we have written only the first line. We now apply the formula for integration by parts, which would result from GREEN's theorem (see, for example, my *Traité de mécanique*, t. III, Chap. XXVIII) to terms such as:

$$\iiint \lambda \,\frac{d(y,z)}{d(b,c)}\frac{\partial \delta x''}{\partial a}\, da\; db\; dc. \tag{5}$$

Upon letting $d\sigma_0$ denote an element of the bounding surface S_0 of the fluid, and letting α_0, β_0, γ_0 denote the direction cosines of the exterior normal to it, one will have

$$\iiint \frac{\partial (PQ)}{\partial a}\, da\; db\; dc = \iint_{S_0} P\,Q\,\alpha_0\, d\sigma_0$$

for two any functions P and Q, so:

$$\iiint P\,\frac{\partial Q}{\partial a}\, da\; db\; dc = \iint_{S_0} P\,Q\,\alpha_0\, d\sigma_0 - \iiint Q\,\frac{\partial P}{\partial a}\, da\; db\; dc.$$

The term (5) can then be replaced with:

$$\iint_{S_0} \lambda \frac{d(y,z)}{d(b,c)}\, \delta x''\, \alpha_0\, d\sigma_0 - \iiint \frac{\partial}{\partial a}\left(\lambda \frac{d(y,z)}{d(b,c)}\right) \delta x''\, da\, db\, dc.$$

We treat each of the nine terms that are provided by $\delta D''$ similarly, and remark that:

$$\frac{\partial}{\partial a}\left(\lambda \frac{d(y,z)}{d(b,c)}\right) + \frac{\partial}{\partial b}\left(\lambda \frac{d(y,z)}{d(c,a)}\right) + \frac{\partial}{\partial c}\left(\lambda \frac{d(y,z)}{d(a,b)}\right)$$

$$= \frac{\partial \lambda}{\partial a}\frac{d(y,z)}{d(b,c)} + \frac{\partial \lambda}{\partial b}\frac{d(y,z)}{d(c,a)} + \frac{\partial \lambda}{\partial c}\frac{d(y,z)}{d(a,b)}$$

$$= D\frac{\partial \lambda}{\partial x} = \frac{\rho_0}{\rho}\frac{\partial \lambda}{\partial x},$$

as one will see upon writing:

$$\frac{\partial \lambda}{\partial a} = \frac{\partial \lambda}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial \lambda}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial \lambda}{\partial z}\frac{\partial z}{\partial a},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

We finally have:

$$\left.\begin{aligned} \delta I = \iint_{S_0} \left(\alpha_0 \frac{d(y,z)}{d(b,c)} + \beta_0 \frac{d(y,z)}{d(c,a)} + \gamma_0 \frac{d(y,z)}{d(a,b)}\right) \lambda\, \delta x''\, d\sigma_0 \\ + \cdots + \cdots \\ + \iiint \left(x'' - X - \frac{1}{\rho}\frac{\partial \lambda}{\partial x}\right) \delta x''\, \rho_0\, da\, db\, dc + \cdots + \cdots, \end{aligned}\right\} \tag{6}$$

in which we have written only the terms in $\delta x''$ in both the partial integral (viz., the double integral) and the triple integral. Later, we shall see that the partial integral is *zero*. We equate the coefficients of $\delta x''$, $\delta y''$, $\delta z''$ in the triple integral to zero. We will then get the classical equations:

$$x'' - X - \frac{1}{\rho}\frac{\partial \lambda}{\partial x} = 0, \qquad y'' - Y - \frac{1}{\rho}\frac{\partial \lambda}{\partial y} = 0, \qquad z'' - Z - \frac{1}{\rho}\frac{\partial \lambda}{\partial z} = 0,$$

in which the pressure is equal to $-\lambda$.

III. – It remains for us to see that the partial integral (viz., the double integral) in formula (6) is *zero*, as a result of the values of $\delta x''$, $\delta y''$, $\delta z''$ on the boundary surface.

One knows that the fluid particles that are on the boundary surface S_0 at the initial instant will remain on the boundary surface S at the instant t. The element $d\sigma_0$ of S_0 will become an element $d\sigma$ on S. One lets α, β, γ denote the direction cosines of the normal to $d\sigma$. One will then have:

$$\left(\alpha_0 \frac{d(y,z)}{d(b,c)} + \beta_0 \frac{d(y,z)}{d(c,a)} + \gamma_0 \frac{d(y,z)}{d(a,b)}\right) d\sigma_0 = \alpha \, d\sigma . \tag{7}$$

Indeed, a, b, c are functions of two parameters p and q on S_0, and when one sets:

$$d\sigma_0 = k \, dp \, dq,$$

one will have:

$$\alpha_0 = \frac{1}{k}\frac{d(b,c)}{d(p,q)}, \qquad \beta_0 = \frac{1}{k}\frac{d(c,a)}{d(p,q)}, \qquad \gamma_0 = \frac{1}{k}\frac{d(a,b)}{d(p,q)} .$$

Since x, y, z are functions of p and q by the intermediary of a, b, c, one will have:

$$d\sigma = h \, dp \, dq,$$

$$\alpha = \frac{1}{h}\frac{d(y,z)}{d(p,q)} = \frac{1}{h}\left[\frac{d(y,z)}{d(b,c)}\frac{d(b,c)}{d(p,q)} + \frac{d(y,z)}{d(c,a)}\frac{d(c,a)}{d(p,q)} + \frac{d(y,z)}{d(a,b)}\frac{d(a,b)}{d(p,q)}\right],$$

which is precisely formula (7). The double integral that figures in δI can then be written:

$$\iint_S (\alpha \, \delta x'' + \beta \, \delta y'' + \gamma \, \delta z'') \lambda \, d\sigma ,$$

in which the integration is extended over the surface S of the fluid at the instant t. However, the differential elements will then be *zero*. Indeed, let:

$$f(x, y, z, t) = 0$$

be the equation of the bounding surface S. Upon differentiating with respect to t and following the particle x, y, z in its motion, one will have:

$$\frac{\partial f}{\partial x} x' + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial z} z' + \frac{\partial f}{\partial t} = 0,$$

and then:

$$\frac{\partial f}{\partial x} x'' + \frac{\partial f}{\partial y} y'' + \frac{\partial f}{\partial z} z'' + \ldots = 0, \tag{8}$$

in which the unwritten terms do not contain x'', y'', z''. The relation (8) shows that the variations $\delta x''$, $\delta y''$, $\delta z''$ will verify the condition:

$$\frac{\partial f}{\partial x}\delta x'' + \frac{\partial f}{\partial y}\delta y'' + \frac{\partial f}{\partial z}\delta z'' = 0$$

on the surface, or when one calls the direction cosines of the normal α, β, γ:

$$\alpha\,\delta x'' + \beta\,\delta y'' + \gamma\,\delta z'' = 0.$$

Paris, 14 February 1912.

“Sur les liaisons cachées et les forces gyroscopiques apparentes dans les systèmes non holonomes,” C. R. Acad. Sci. Paris **162** (1916), 27-29.

On hidden constraints and apparent gyroscopic forces in non-holonomic systems

Note by PAUL APPELL

Translated by D. H. Delphenich

I. – In the search for a mechanical representation of a phenomenon, one can assume *a priori* only that the hidden constraints are of a special nature: In order to embrace the most general case, one must then suppose that those constraints are not holonomic and employ neither the Lagrange equations nor the canonical equations, but the general equations that would result from considering the energy of acceleration.

That viewpoint, which seems to have interested the physicists, was pointed out in a note “Sur l’emploi possible de l’énergie d’accelération dans les équations de l’Électrodynamique” that I presented to the Academy during the session on 22 April 1922 (C. R. Acad. Sci. Paris **154**, pp. 1037). It was developed in a note by Édouard Guillaume, “Sur l’extension des équations mécanique de M. Appell à la physique des milieu continus; application à la théorie des électrons” (C. R. Acad. Sci. Paris **156**, 10 March 1913, pp. 875).

If one mistakenly employs the Lagrange equations then one will be led to introduce, along with the forces that are actually applied, some *apparent* forces, which, in the terminology of Sir William Thomson (*Treatise on Natural Philosophy*, Vol. I, Part I, new edition, Cambridge, 1879, pp. 391-415), are *gyroscopic forces*, like the ones that present themselves in certain electromagnetic phenomena.

It is that fact that I propose to illuminate in a general manner.

II. – Although the consideration of the isolated system under study would suffice, it seems preferable to me to make the comparison that I would like to employ.

Imagine two systems (A) and (B), with hidden constraints that are independent of time and frictionless, the one (A) being holonomic, while the other (B) is non-holonomic. Suppose that those two systems have the same number k of degrees of freedom and the same expressions for their kinetic energies:

$$2T = \sum a_{ij}\, q_i'\, q_j' ,$$

in which the coefficients a_{ij} are functions of the parameters $q_1, q_2, \ldots, q_k$. Finally, suppose that the forces that are actually applied to the two systems are derived from the same force function:

$$U(q_1, q_2, \ldots, q_k),$$

or, more generally, that the sum of the works done by those forces for an arbitrary displacement $\delta q_1, \delta q_2, \ldots, \delta q_k$ have the same expression:

$$Q_1\,\delta q_1 + Q_2\,\delta q_2 + \ldots + Q_k\,\delta q_k$$

in the two systems.

The fact that systems of that type exist results from an example that I gave in my *Traité de Mécanique* (2nd ed., t. 2, pp. 385, no. 469).

Under those conditions, the equations of motion of system (A) are:

$$\text{(A)} \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right) - \frac{\partial T}{\partial q_i} = Q_i \quad (i = 1, 2, \ldots, k)\,.$$

Those of the system (B) can be written:

$$\frac{\partial S}{\partial q_i''} = Q_i\,,$$

in which S denotes the energy of acceleration of that system. One can also put them into the form:

$$\text{(B)} \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right) - \frac{\partial T}{\partial q_i} = Q_i + \Delta_i\,,$$

in which the Δ_i are correction terms that are homogeneous of order two in the components q_1', q_2', …, q_k' of the velocities. The analytical composition of the terms Δ_i was indicated in an article that was entitled "Remarques d'ordre analytique sur une nouvelle forme des équations de la Dynamique" and that I published in Jordan's *Journal de Mathématiques* [(5) **7** (1901), 5-12]. One can also obtain those terms by using Hertz's calculations (*Gesammelte Abhandlungen*, Bd. 3).

One then sees that for the observer who believes that the constraints on the system (B) are holonomic, it would seem that the system is subject to not only the real forces that give rise to the terms $Q_1, Q_2, \ldots, Q_k$, but also to apparent forces that give rise to the correction terms $\Delta_1, \Delta_2, \ldots, \Delta_k$.

Furthermore, those apparent forces are *gyroscopic*. Indeed, an application of the *vis viva* theorem shows that the systems of equation (A) and (B) will imply the same *vis viva* equation:

$$\frac{dT}{dt} = Q_1\,q_1' + Q_2\,q_2' + \cdots + Q_k\,q_k'\,,$$

which is obtained by adding them, after they have been multiplied by q_1', q_2', …, q_k', respectively.

One then obtains the relation:

(C) $$\Delta_1 q_1' + \Delta_2 q_2' + \cdots + \Delta_k q_k' = 0,$$

which will be true for any velocity components q_1', q_2', …, q_k' and parameters $q_1, q_2, \ldots, q_k$, since all of those quantities can be taken arbitrarily at the initial instant. The sum of the works done by apparent forces Δ_i is therefore zero under the real displacement: Those forces are *gyroscopic*.

If one makes a change of variables *in finite form:*

$$q_i = f_i(p_1, p_2, \ldots, p_k) \qquad (i = 1, 2, \ldots, k)$$

then equations (A) and (B) will keep the same forms, in which the q and q' are replaced by p and p', resp., the Q are replaced with the P that are deduced from the identity:

$$Q_1\,\delta q_1 + Q_2\,\delta q_2 + \ldots + Q_k\,\delta q_k \equiv P_1\,\delta p_1 + P_2\,\delta p_2 + \ldots + P_k\,\delta p_k,$$

and in which the Δ that define the apparent forces are replaced by the Γ that are deduced from the analogous identity:

$$\Delta_1\,\delta q_1 + \Delta_2\,\delta q_2 + \ldots + \Delta_k\,\delta q_k \equiv \Gamma_1\,\delta p_1 + \Gamma_2\,\delta p_2 + \ldots + \Gamma_k\,\delta p_k.$$

The apparent forces then behave like true forces under changes of variables.

The simplest example of the considerations that were just developed is provided by the theory of the hoop and the bicycle, such as was presented in Carvallo's memoir [Journal de l'École Polytechnique (11) (1900-1901), 5th and 6th Cahiers].

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FASCICLE 1.

On a general form of the equations of dynamics

By Paul APPELL

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TABLE OF CONTENTS

INTRODUCTION

Here, one must take the word *dynamics* in its old sense, namely, the sense of Galilei, Newton, Lagrange, d'Alembert, Carnot, Lavoisier, and Mayer.

As H. Poincaré said in his book *La valeur de la Science* (pp. 231):

> "Perhaps we must all construct a new mechanics that we can only glimpse in which inertia increases with velocity so the velocity of light will become an impassable obstacle. The simpler ordinary mechanics will remain a first approximation, since it will be true for velocities that are not very large, in such a way that we will again recover the old dynamics from the new one. We should not regret that we believed in those principles, and since very large velocities will never be anything but exceptions to the old formulas, we can even be most certain in practice that we can continue to work as if we still believed them. They are useful enough that they will still have their place. To wish to exclude them completely would be to deprive ourselves of a valuable weapon. In conclusion, I hasten to say that we have not reached that point and that nothing suggests that we will not leave it victorious and intact."

The equations that we have in mind then refer to the classical mechanics of today. As we will see, they apply regardless of the nature of the constraints, provided that the constraints are realized in such a fashion that the general equation of dynamics is exact.

One will see that in order to obtain those equations, one will be obliged to calculate the energy of acceleration of the system $S = \frac{1}{2}\sum m J^2$; i.e., to go to the second order of derivation with respect to time. If one would like to take the first order of derivation, like Lagrange, then one would be led to some very complicated equations that generalize those of Lagrange [**37**], and that one can call the Euler-Lagrange equations. That method was first studied by Volterra in 1898 ([**38**] and [**39**]). One can also consult the papers of Tzenoff [**46**] and Hamel [**47**]. We shall give some applications of it to some questions of rational mechanics. However, we hope that those equations can also be useful to physicists in the cases where the Lagrange equation and Hamilton's canonical equations, which are deduced from them, are no longer applicable.

For H. Poincaré: "The mathematician must not be simply a provider of formulas to the physicist. There must be a closer collaboration between them."

Along those lines, it is important to recall that Édouard Guillaume in Bern has applied the general equations that we shall develop to various physical theories ([**23**] and [**24**]).

I agree with Mach (Paris, Librarie Hermann, 1904, translation by Émile Betrand, with a preface by Émile Picard) when he said (pp. 465) that there exist no purely-mechanical phenomena and that all phenomena belong to all branches of physics:

> "The opinion that puts mechanics at the fundamental basis for all other branches of physics today, and according to which physical phenomena must have a *mechanical* explanation is, to me, a prejudice."

However, one must seek to explain the most possible physical phenomena *mechanically,* and then, as one has done up to now, abandon those phenomena in order to return to rational mechanics, and in that regard, to the general form that one gives to the equations that embraces more cases than the form that is due to Lagrange, which supposes that the constraints can be expressed in finite terms; i.e., in Hertz's terminology, that the constraints considered are *holonomic*. Now, one knows nothing about the constraints that are realized in the universe. H. Poincaré said: "It is a machine that is more complicated than all of those of industry, and almost all of its parts are hidden deeply from us." From the English mathematician Larmor, it is the principle of *least action* that seems to have persisted for the longest time. On the contrary, the general form that I shall present is attached to Gauss's principle of *least constraint* ([**1**], [**2**], [**3**], [**4**], [**5**], [**45**]), which Mach discussed on pages 343, *et seq.*, in the cited work. Notably, he said:

> "The examples that we just treated show that this theorem *does not represent an essentially-new concept*… The equations of motion will be the same (as they are from a direct application of the general equation of dynamics that results from the combination of d'Alembert's principle with the equation of virtual work), as one will see, moreover, by treating the same problems by d'Alembert's theorem and then by that of Gauss."

I think that the value of Gauss's principle is found in precisely that identity.

Mach's opinion is, moreover, that of Gauss himself, who said in presenting his theorem in volume IV of Crelle's Journal:

> "As one knows, the principle of virtual velocities transforms any problem in statics into a question of pure mathematics, and dynamics is, in turn, reduced to statics by d'Alembert's principle. It results from this that no fundamental principle of equilibrium and motion can be essentially distinct from the ones that we just cited, and that, be that as it may, one can always regard that principle as a more or less immediate consequence of the former ones.
>
> One must not conclude from this that any new theorem will be without merit. On the contrary, it will always be interesting and instructive to study the laws of nature from a new viewpoint that might then allow us to treat this or that particular question more simply or only obtain a much greater precision to the statements.
>
> The great geometer, who has so brilliantly made the science of motion rest upon the principle of virtual velocities did not despair to perfect and generalize Maupertuis's principle, which relates to *least action,* and one

knows that this principle is often employed by geometers in a very advantageous manner."

The great geometer that Gauss spoke of is Lagrange. One will find the works of Lagrange *on the principle of least action* on page 281 of the first volume of the third edition of *Mécanique analytique*, which was edited, corrected, and annotated by J. Bertrand (Mallet-Bachelier, 1853).

Among the applications of the general equations, I must cite the ones that Henri Beghin just made to the Anschütz and Sperry gyrostatic compass in a thesis that he presented to the science faculty in Paris [**29**] in November 1922.

I. – NATURE OF THE CONSTRAINTS.

1. Essentially holonomic or essentially non-holonomic systems. Order of a non-holonomic system. – Imagine a material system with k degrees of freedom that is composed of n points with masses m_μ ($\mu = 1, 2, \ldots, n$) that have rectangular coordinates x_μ, y_μ, z_μ in an oriented trihedron of axes, and are animated with a motion of uniform, rectilinear translations with respect to axes that are considered to be fixed in classical mechanics. The displacements, velocities, and accelerations that we consider are displacements, velocities, and accelerations with respect to that trihedron.

In order to obtain the most general displacement of the system that is compatible with the constraints that exist at the instant t, it will suffice to vary k conveniently-chosen parameters $q_1, q_2, \ldots, q_k$ by arbitrary infinitely-small quantities $\delta q_1, \delta q_2, \ldots, \delta q_k$. One will then have that the virtual displacement of the point m_μ is:

$$(1) \qquad \left\{ \begin{aligned} \delta x_\mu &= a_{\mu,1}\,\delta q_1 + a_{\mu,2}\,\delta q_2 + \cdots + a_{\mu,k}\,\delta q_k, \\ \delta y_\mu &= b_{\mu,1}\,\delta q_1 + b_{\mu,2}\,\delta q_2 + \cdots + b_{\mu,k}\,\delta q_k, \\ \delta z_\mu &= c_{\mu,1}\,\delta q_1 + c_{\mu,2}\,\delta q_2 + \cdots + c_{\mu,k}\,\delta q_k, \end{aligned} \right.$$

and for the actual displacement of the same point during the time interval dt, one has:

$$(2) \qquad \left\{ \begin{aligned} dx_\mu &= a_{\mu,1}\,dq_1 + a_{\mu,2}\,dq_2 + \cdots + a_{\mu,k}\,dq_k + a_\mu\,dt, \\ dy_\mu &= b_{\mu,1}\,dq_1 + b_{\mu,2}\,dq_2 + \cdots + b_{\mu,k}\,dq_k + b_\mu\,dt, \\ dz_\mu &= c_{\mu,1}\,dq_1 + c_{\mu,2}\,dq_2 + \cdots + c_{\mu,k}\,dq_k + c_\mu\,dt. \end{aligned} \right.$$

In those equations, the coefficients $a_{\mu,\nu}$, $b_{\mu,\nu}$, $c_{\mu,\nu}$, a_μ, b_μ, c_μ ($\mu = 1, 2, \ldots, n$; $\nu = 1, 2, \ldots, k$) are arbitrary. They depend upon only the position of the system at the instant t and upon time t itself. The nature of the coefficients plays no role in the general case.

In Hertz's terminology, a system is called *holonomic* when the constraints that are imposed upon it are expressed by relations with finite terms between the coordinates that determine the positions of the various bodies that it is composed of. One can choose q_1, q_2, …, q_k to be variables whose numerical values at the instant t determine the position of the system. The quantities q_1, q_2, …, q_k are then the coordinates of the holonomic system whose position is determined by the figurative point whose rectangular coordinates in k-dimensional space are q_1, q_2, …, q_k. The coordinates x_μ , y_μ , z_μ are functions of the q_1, q_2, …, q_k and time t that are expressible in finite terms, and the right-hand sides of equations (2) are total differentials of functions of a_μ , b_μ , c_μ and time t. The equations of motion will then take the form that was given by Lagrange. On the contrary, it can happen that the constraints between certain bodies of the system are expressed by *non-integrable* differential relations between the coordinates that the positions of those bodies depend upon. For example, that is what happens when a solid in the system is bounded by a surface or line that is subject to rolling without slipping on a fixed surface or on the surface of another solid of the system. Indeed, that constraint is expressed, in the former case, by writing that the velocity of the material point is *zero* at the point of contact, and in the latter, by writing that the velocities of two material points are the same at the point of contact. According to Hertz, one says that the system is not holonomic in those cases. Even if one supposes that the $a_{\mu,\nu}$, $b_{\mu,\nu}$, $c_{\mu,\nu}$ can be expressed with the aid of only the variables q_1, q_2, …, q_k, t, the right-hand sides of formulas (2) are not supposed to be exact differentials.

In the preceding, we, with Hertz, have considered only the systems themselves. In order to distinguish them, we say that they are *essentially holonomic* or *essentially non-holonomic*. One can also define the nature of a system for a certain choice of parameters. In that regard, one can define the *order of a non-holonomic system for a choice of parameters.* There will then be two elements that one must address, namely, the material system and the choice of parameters. One says that a system is holonomic for a certain choice q_1, q_2, …, q_k if the Lagrange equations apply to all the parameters. One says the *order* of a non-holonomic system for a certain choice of parameters q_1, q_2, …, q_k to mean the number of parameters to which the Lagrange equations do not apply [**33**]. In nos. **15** and **16**, we shall see how that order can be determined when one defines the energy of the velocities $T = \frac{1}{2}\sum m\, V^2$ and the energy of the accelerations $S = \frac{1}{2}\sum m\, J^2$ for a system.

From that, a system that is *non-holonomic of order zero* for a certain choice of parameters will be *holonomic*.

The order can remain the same or change when one replaces the system of parameters q_1, q_2, …, q_k with another one.

Example. – Here is an example in which the order passes from 0 to 2. Take a system that is composed of just one point in the xOy plane with coordinates x, y, 0. It is a system with two degrees of freedom, so it is essentially holonomic. That system will be holonomic when one chooses the parameters for the coordinates of the point in an arbitrary system. For example, if one takes polar coordinates r, θ in the plane then one will have:

$$x = r \cos \theta, \qquad y = r \sin \theta, \qquad z = 0,$$

$$T = \frac{m}{2}(x'^2 + y'^2 + z'^2) = \frac{m}{2}(r'^2 + r^2\ \theta'^2)\ .$$

Upon calling the component of the force $(X, Y, 0)$ along the perpendicular to the radius vector P and its component along the prolongation of the radius vector Q, one will have:

$$X\ \delta x + Y\ \delta y = Pr\ \delta\theta + Q\ \delta r\ .$$

The Lagrange equations apply to the parameters r and θ, but in place of θ, they take the area σ that is described by the radius vector as their parameter:

$$\delta\sigma = \tfrac{1}{2} r^2\ \delta\theta, \qquad d\sigma = \tfrac{1}{2} r^2\ d\theta,$$

$$(5) \qquad \left\{ \begin{aligned} T &= \frac{m}{2}\left(r'^2 + \frac{4\sigma'^2}{r^2}\right), \\ X\ \delta x + Y\ \delta y &= \frac{2P}{r}\delta\sigma + Q\,\delta r. \end{aligned} \right.$$

Neither of the two Lagrange equations apply, as one verifies immediately.

For the new choice of variables r and σ, the system is then non-holonomic of order 2.

One sees that the order of a non-holonomic system is defined with respect to a certain choice of parameters and that when one varies that choice, one can vary the *order*. However, there nonetheless exists *an essential order* that is attached to that system, which is the *minimum* ω of the orders that are obtained by varying the choice of parameters in an arbitrary way. For example, an essentially-holonomic system is a non-holonomic system of essential order zero.

2. Examples: Top and hoop. – The two favorite toys of children – viz., the top and the hoop – provide examples of systems that are essentially holonomic or essentially non-holonomic. In order to show that, we first define the six coordinates of an entirely-free solid body (which is an essentially-holonomic system). Let $O\xi\eta\zeta$ be three rectangular fixed axes. Call the coordinates of the center of gravity G of the solid body with respect to those axes ξ, η, ζ. Let θ, φ, ψ be the Euler angles that a system of rectangular axes $Gxyz$ that are coupled with the body makes with the axes with fixed directions $Gx_1\,y_1\,z_1$ that are parallel to the fixed axes. Those six coordinates $\xi, \eta, \zeta, \theta, \varphi, \psi$ define the position of a free solid body. The coordinates of an arbitrary point of the body are functions of those six coordinates. If one imposes constraints on the solid then, depending upon the case, that will amount to establishing certain relations in finite terms between the six coordinates or also establishing certain non-integrable first-order differential relations. The number of degrees of freedom will then be diminished.

1. *Top. Essentially-holonomic system with five degrees of freedom*. – In the absence of friction or slipping, the top is a ponderous body of revolution whose axis is terminated by a point *P* that slides on a perfectly-polished fixed plane Π. If one takes the axis *Gz* to be the axis of revolution (when counted to be positive in the sense that goes from *P* to *G*), and one lets *a* denote the distance *PG* then one will have:

$$\zeta = a \cos \theta,$$

which is a constraint equation in finite terms. The position of the top is then defined by the five coordinates:

$$\xi, \eta, \theta, \varphi, \psi .$$

The coordinates of an arbitrary point on the top with respect to the fixed axes are expressed as functions of those five coordinates. The top is then an *essentially-holonomic* system; that system is holonomic for the choice of parameters $\xi, \eta, \theta, \varphi, \psi$.

2. *Hoop. Non-holonomic system with three degrees of freedom and essential order two.* – A hoop is a solid body of revolution that is bounded by a circular edge *C* that is subject to *rolling without slipping* on a fixed horizontal plane Π (one neglects friction while it rolls). The center of gravity *G* of the hoop is supposed to be in the plane of the edge *C*. The axes *Gxyz* that are coupled to the body here will be the axis of the circle *Gz*, which is perpendicular to the plane of the edge, and two rectangular axes *Gx* and *Gy* that are situated in the plane of the edge; the radius of the edge *C* is *a*.

As one will see in *Traité de Mécanique* by P. Appell (t. II, no. 462), one will have:

$$\left\{\begin{array}{l} d\xi - a\sin\psi\sin\theta\,d\theta + a\cos\psi\cos\theta\,d\psi + a\cos\psi\,d\varphi = 0, \\ d\eta + a\cos\psi\sin\theta\,d\theta + a\sin\psi\cos\theta\,d\psi + a\sin\psi\,d\varphi = 0, \\ d\zeta - a\cos\theta\,d\theta = 0 \end{array}\right.$$

for the actual displacement and:

$$\left\{\begin{array}{l} \delta\xi - a\sin\psi\sin\theta\,\delta\theta + a\cos\psi\cos\theta\,\delta\psi + a\cos\psi\,\delta\varphi = 0, \\ \delta\eta + a\cos\psi\sin\theta\,\delta\theta + a\sin\psi\cos\theta\,\delta\psi + a\sin\psi\,\delta\varphi = 0, \\ \delta\zeta - a\cos\theta\,\delta\theta = 0 \end{array}\right. \tag{8}$$

for the virtual displacement that is compatible with the constraints.

The last of the preceding relations is equivalent to the relation in finite terms:

$$\zeta = a \sin \theta, \tag{9}$$

which is obvious geometrically. However, neither the first two relations in (8) nor any linear combination of the relations (8) in which at least one of the first two appears, can be integrated and written in a finite form. The system considered will then be non-holonomic. It has *three degrees of freedom* ($k = 3$), because the most general virtual displacement that is compatible with the constraints is obtained by giving arbitrary values to $\delta\theta$, $\delta\varphi$, $\delta\psi$; $\delta\xi$, $\delta\eta$, $\delta\zeta$ are then determined by the relations (8). It remains to see that the system is *holonomic of order two*. Indeed, since the position of the hoop around its center of gravity is defined by the Euler angles θ, φ, ψ, Ferrer already showed [**6**] that the Lagrange equation can apply to the inclination θ, but it does not apply to φ and ψ. The *order* of the non-holonomic system will then be $\omega = 2$.

II. – REALIZING THE CONSTRAINTS. SUBORDINATION.

3. Realizing constraints. – In the foregoing, the constraints were considered from a purely-analytical viewpoint that was independent of the particular manner by which they were realized. (BEGHIN [**29**], *Thesis*, pp. 8). Can one now abstract the manner by which a constraint is realized? That question has been the object of numerous studies. Here are some general considerations that are borrowed from Beghin (*loc. cit.*) and Delassus ([**26**] and [**27**]). A constraint L on a system Σ can be realized with or without the help of an auxiliary system Σ_1 . In the former case, the realization of the constraint is called *perfect*; in the latter case, the realization of the constraint is still called *perfect* if the introduction of the auxiliary system Σ_1 does not imply any restriction on the virtual displacements of the system Σ, which will all remain compatible with the constraint L then. However, it is *imperfect* if the introduction of the system Σ_1 does imply restrictions on the virtual displacements of the system Σ.

Delassus [**27**] then gave the following example of the *imperfect* constraint $z = a$ that is imposed upon a material point with coordinates x, y, z. Imagine a hoop of radius a that rolls without slipping on the plane xOy. Suppose that the plane of the hoop (i.e., the plane of the circular edge C) is kept vertical by means of a tripod that carries the axis of the hoop and slides without friction on the horizontal plane xOy. The material point x, y, z is attached to the center G of the hoop C. That constitutes the system Σ ; the hoop with the tripod and the accessories constitutes the system Σ_1 . The apparatus obviously realizes the constraint $z = a$. It permits the material point to occupy all of the possible positions in the plane $z = a$. However, if one imposes a virtual displacement on the system that is compatible with the constraints then the displacement of the material point in the plane of the edge of the hoop and not an arbitrary direction in the plane $z = a$. The constraint is then realized *imperfectly*.

If, on the contrary, the material point is attached to the center of a sphere of radius a that is subject to rolling without slipping on the plane xOy then that point will be subject to the same constraint $z = a$, but it will then be realized perfectly.

4. Work done by constraint forces. – When one proves the theorem of virtual work for a system, one appeals to the hypothesis that for any virtual displacement of the system that is compatible with the constraints, *the sum of the works done by the constraint forces is zero.* Here, we take that hypothesis to be something that defines the constraints that we consider. It is that hypothesis that one then utilizes in order to apply d'Alembert's principle by writing that by virtue of the constraints that exist at the instant t there will be equilibrium between the forces of inertia and the applied forces.

5. Case of subordination. – However, one must point out that even if one confines oneself to perfect constraints, there will exist an important category of mechanisms in which the constraints are found to be realized by methods that are different from the ones that permit the pure and simple application of the general equation of dynamics. For those special constraints, one cannot abstract from the mode of realization, and one must be content with their analytical expression. Those constraints are the ones that one obtains by *subordination.* We say that there is *subordination* when the corresponding constraints, rather than being realized in a fashion that is, in some way, passive (such as the contact between two solids that slide or roll on each other, by way of example), they are realized by the appropriate use of arbitrary forces (e.g., electromagnetic forces, fluid, pressure, forces produced by a living entity, etc.). Those subordinate forces imply the constraint forces that Beghin [**29**] called ones of the *second type* and whose virtual work is generally non-zero, even when the displacement is compatible with the constraint. That means that we shall pass over that type of constraint and refer the study of that case to Beghin's thesis, which used the general form of the equations that we shall indicate. We shall confine ourselves to the classical constraints that were defined above (no. **4**).

III. – EQUATIONS.

6. General equations of motion. – We write the general equation of dynamics in such a way that it will result from d'Alembert's principle, combined with the theory of virtual work. In all of what follows, we shall employ Lagrange's notation of primes to denote the derivatives with respect to time. The general equation of dynamics is then:

$$\sum_\mu m_\mu (x''_\mu\,\delta x_\mu + y''_\mu\,\delta y_\mu + z''_\mu\,\delta z_\mu) - \sum_\mu (X_\mu\,\delta x_\mu + Y_\mu\,\delta y_\mu + Z_\mu\,\delta z_\mu) = 0, \tag{10}$$

in which the first summation is extended over all material points of the system, but the second one comprises only the material points to which the forces are applied. Upon replacing δx_μ , δy_μ , δz_μ with their values in (1), one will have an equation of the form:

$$P_1\,\delta q_1 + P_2\,\delta q_2 + \ldots + P_k\,\delta q_k - (Q_1\,\delta q_1 + Q_2\,\delta q_2 + \ldots + Q_k\,\delta q_k) = 0. \tag{11}$$

Since $\delta q_1, \delta q_2, \ldots, \delta q_k$ are arbitrary, equation (11) will reduce to k equations:

$$P_1 = Q_1\,, \qquad P_2 = Q_2\,, \qquad \ldots, \qquad P_k = Q_k\,, \tag{12}$$

which define the k parameters $q_1, q_2, \ldots, q_k$ as functions of t.

In order to write those equations, we remark that:

$$P_\nu = \sum_\mu m_\mu (x''_\mu\, a_{\mu,\nu} + y''_\mu\, b_{\mu,\nu} + z''_\mu\, c_{\mu,\nu})\,.$$

Now, from the relations (2), one will have:

$$\left\{\begin{aligned} x'_\mu &= a_{\mu,1}\, q'_1 + a_{\mu,2}\, q'_2 + \cdots + a_{\mu,\nu}\, q'_\nu + \cdots + a_{\mu,k}\, q'_k + a_\mu\,, \\ y'_\mu &= b_{\mu,1}\, q'_1 + b_{\mu,2}\, q'_2 + \cdots + b_{\mu,\nu}\, q'_\nu + \cdots + b_{\mu,k}\, q'_k + b_\mu\,, \\ z'_\mu &= c_{\mu,1}\, q'_1 + c_{\mu,2}\, q'_2 + \cdots + c_{\mu,\nu}\, q'_\nu + \cdots + c_{\mu,k}\, q'_k + c_\mu\,, \end{aligned}\right. \tag{13}$$

so, upon differentiating once with respect to time, one will get:

$$\begin{aligned} x''_\mu &= \sum_{\nu=1}^{k} \left[a_{\mu,\nu}\, q''_\nu + \frac{da_{\mu,\nu}}{dt}\, q'_\nu \right] + \frac{da_\mu}{dt}\,, \\ y''_\mu &= \sum_{\nu=1}^{k} \left[b_{\mu,\nu}\, q''_\nu + \frac{db_{\mu,\nu}}{dt}\, q'_\nu \right] + \frac{db_\mu}{dt}\,, \\ z''_\mu &= \sum_{\nu=1}^{k} \left[c_{\mu,\nu}\, q''_\nu + \frac{dc_{\mu,\nu}}{dt}\, q'_\nu \right] + \frac{dc_\mu}{dt}\,. \end{aligned}$$

One concludes that the only term on the right-hand side that contains q''_ν is $a_{\mu,\nu}\, q''_\nu$ in the first expression, $b_{\mu,\nu}\, q''_\nu$ in the second, and $c_{\mu,\nu}\, q''_\nu$ in the third. One will then have:

$$a_{\mu,\,\nu} = \frac{\partial x''_\mu}{\partial q''_\nu}\,, \qquad b_{\mu,\,\nu} = \frac{\partial y''_\mu}{\partial q''_\nu}\,, \qquad c_{\mu,\,\nu} = \frac{\partial z''_\mu}{\partial q''_\nu}\,,$$

and the expression for P_ν will be written:

$$P_\nu = \sum_\mu m_\mu \left(x''_\mu\, \frac{\partial x''_\mu}{\partial q''_\nu} + y''_\mu\, \frac{\partial y''_\mu}{\partial q''_\nu} + z''_\mu\, \frac{\partial z''_\mu}{\partial q''_\nu} \right).$$

If one finally sets:

$$S = \tfrac{1}{2} \sum_{\mu=1}^{n} m_\mu \left(x''^2_\mu + y''^2_\mu + z''^2_\mu \right)$$

then one will have:

$$P_\nu = \frac{\partial S}{\partial q''_\nu}.$$

On the other hand, the term Q_ν has a known value. If one imposes upon the system the special virtual displacement in which all of the δq are *zero*, except δq_ν, then the sum $\mathcal{T}_\nu$ of the virtual works of the applied forces will be precisely:

$$\mathcal{T}_\nu = Q_\nu \, \delta q_\nu ,$$

which gives a simple meaning to the Q_ν. One will then have the k equations of motion:

$$\frac{\partial S}{\partial q''_1} = Q_1, \qquad \frac{\partial S}{\partial q''_2} = Q_2, \qquad \ldots, \qquad \frac{\partial S}{\partial q''_k} = Q_k, \tag{14}$$

which are the desired general equations, which are applicable to all systems – holonomic or not and for all choices of parameters – under the indicated restrictions that relate to subordination. In order to write those equations, one must form the function S [**19**].

7. Energy of acceleration of a system. – The semi-*vis viva*, or kinetic energy:

$$T = \tfrac{1}{2}\sum_{\mu=1}^{n} m_\mu \left(x'^2_\mu + y'^2_\mu + z'^2_\mu\right) = \tfrac{1}{2}\sum m\, V^2, \tag{15}$$

in which V denotes the velocity of the point with mass m, can be called the *energy of velocity* of the system. The function S :

$$S = \tfrac{1}{2}\sum_{\mu=1}^{n} m_\mu \left(x''^2_\mu + y''^2_\mu + z''^2_\mu\right) = \tfrac{1}{2}\sum m\, J^2, \tag{16}$$

in which J denotes the acceleration of the point with mass m, will be called the *energy of acceleration* of the system. That terminology was introduced by A. de Saint-Germain [**20**]. In order to write the equations of an arbitrary system with k degrees of freedom, with an arbitrary choice of the k parameters $q_1, q_2, \ldots, q_k$, it will then suffice to form the energy of the accelerations S of that system. In each case, the quantity S will be a function of second degree in $q''_1, q''_2, \ldots, q''_k$. One can then, in turn, write the equations of motion by a simple differentiation.

One knows that if a system is essentially holonomic and if its position at the instant t depends upon k geometrically-independent coordinates then the equations of motion of the system can be written in the form that was given by Lagrange:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q'_\nu}\right)-\frac{\partial T}{\partial q_\nu}=Q_\nu \qquad (\nu = 1, 2, \ldots, k). \tag{17}$$

However, that form is not applicable to non-holonomic systems. It is not even adapted to an arbitrary choice of parameters for holonomic systems. In order to obtain an absolutely-general form, one agrees to calculate S as was said, i.e., to go to the second order of differentiation with respect to t.

8. Case in which the Lagrange equations apply to certain parameters. – The coefficient P_ν of δq_ν in the general equation of dynamics (10) is:

$$P_\nu = \sum_\mu m_\mu\,(x''_\mu\, a_{\mu,\nu} + y''_\mu\, b_{\mu,\nu} + z''_\mu\, c_{\mu,\nu}) .$$

In the case of a holonomic system (which is the only one that Lagrange considered), that coefficient can be written:

$$P_\nu = \frac{d}{dt}\left(\frac{\partial T}{\partial q'_\nu}\right)-\frac{\partial T}{\partial q_\nu} .$$

In any case, one can obviously write:

$$P_\nu = \frac{d}{dt}\sum_\mu m_\mu\,(x'_\mu\, a_{\mu,\nu} + y'_\mu\, b_{\mu,\nu} + z'_\mu\, c_{\mu,\nu}) - \sum_\mu m_\mu\left(x'_\mu\,\frac{da_{\mu,\nu}}{dt}+y'_\mu\,\frac{db_{\mu,\nu}}{dt}+z'_\mu\,\frac{dc_{\mu,\nu}}{dt}\right).$$

Now since, from (13), $a_{\mu,\nu}$, $b_{\mu,\nu}$, $c_{\mu,\nu}$ are equal to $\frac{\partial x'_\mu}{\partial q'_\nu}$, $\frac{\partial y'_\mu}{\partial q'_\nu}$, $\frac{\partial z'_\mu}{\partial q'_\nu}$, one will have:

$$P_\nu = \frac{d}{dt}\left(\frac{\partial T}{\partial q'_\nu}\right) - \sum_\mu m_\mu\left(x'_\mu\,\frac{da_{\mu,\nu}}{dt}+y'_\mu\,\frac{db_{\mu,\nu}}{dt}+z'_\mu\,\frac{dc_{\mu,\nu}}{dt}\right).$$

If one sets:

$$R_\nu = \sum_\mu m_\mu\left(x'_\mu\,\frac{da_{\mu,\nu}}{dt}+y'_\mu\,\frac{db_{\mu,\nu}}{dt}+z'_\mu\,\frac{dc_{\mu,\nu}}{dt}\right)$$

then one will see that:

$$P_\nu = \frac{d}{dt}\left(\frac{\partial T}{\partial q'_\nu}\right)-\frac{\partial T}{\partial q_\nu}-\left(R_\nu-\frac{\partial T}{\partial q_\nu}\right).$$

The Lagrange equation will then be applicable to the parameter q_ν if one has:

$$\Delta_\nu \equiv R_\nu - \frac{\partial T}{\partial q_\nu} = 0.$$

Now, one has:

$$(18) \qquad \Delta_\nu \equiv R_\nu - \frac{\partial T}{\partial q_\nu} = \sum_\mu m_\mu \left[x'_\mu \left(\frac{da_{\mu,\nu}}{dt} - \frac{\partial x'_\mu}{\partial q_\nu} \right) + y'_\mu \left(\frac{db_{\mu,\nu}}{dt} - \frac{\partial y'_\mu}{\partial q_\nu} \right) + z'_\mu \left(\frac{dc_{\mu,\nu}}{dt} - \frac{\partial z'_\mu}{\partial q_\nu} \right) \right].$$

If one replaces x'_μ, y'_μ, z'_μ with their expressions in terms of q'_1, q'_1, …, q'_k [eq. (13)] then one will see that $R_\nu - \frac{\partial T}{\partial q_\nu}$ is a function of degree two in q'_1, q'_1, …, q'_k. In order to apply the Lagrange equation to the parameter q_ν, it is necessary and sufficient that that function must be zero for any positions and velocities of the points of the system that are compatible with the constraints, since at each instant (which is considered to be initial), those quantities can be taken arbitrarily.

Particular case. – Suppose that the coefficients $a_{\mu,\nu}$ are functions of the q_1, q_2, …, q_k, and t, so:

$$\frac{da_{\mu,\nu}}{dt} = \frac{\partial a_{\mu,\nu}}{\partial q_1} q'_1 + \frac{\partial a_{\mu,\nu}}{\partial q_2} q'_2 + \cdots + \frac{\partial a_{\mu,\nu}}{\partial q_\nu} q'_\nu + \cdots + \frac{\partial a_{\mu,\nu}}{\partial q_k} q'_k + \frac{\partial a_{\mu,\nu}}{\partial t},$$

$$\frac{\partial x'_\mu}{\partial q_\nu} = \frac{\partial a_{\mu,1}}{\partial q_\nu} q'_1 + \frac{\partial a_{\mu,2}}{\partial q_\nu} q'_2 + \cdots + \frac{\partial a_{\mu,\nu}}{\partial q_\nu} q'_\nu + \cdots + \frac{\partial a_{\mu,k}}{\partial q_k} q'_k + \frac{\partial a_\mu}{\partial q_\nu},$$

…………………………………………………………………

The coefficient of x_μ in the difference (18) is:

$$\left(\frac{\partial a_{\mu,\nu}}{\partial q_1} - \frac{\partial a_{\mu,1}}{\partial q_\nu} \right) q'_1 + \left(\frac{\partial a_{\mu,\nu}}{\partial q_2} - \frac{\partial a_{\mu,2}}{\partial q_\nu} \right) q'_2 + \cdots + \left(\frac{\partial a_{\mu,\nu}}{\partial q_k} - \frac{\partial a_{\mu,k}}{\partial q_\nu} \right) q'_k + \frac{\partial a_{\mu,\nu}}{\partial t} - \frac{\partial a_\mu}{\partial q_\nu}.$$

If that coefficient is zero for any μ, as well as the analogous coefficients of y'_μ, z'_μ, then the quantity R_ν will be zero. The Lagrange equation will then apply to the parameter q_ν if one has:

$$(19)\qquad \left\{\begin{array}{l}
\dfrac{\partial a_{\mu,\nu}}{\partial q_1}=\dfrac{\partial a_{\mu,1}}{\partial q_\nu},\quad \dfrac{\partial a_{\mu,\nu}}{\partial q_2}=\dfrac{\partial a_{\mu,2}}{\partial q_\nu},\quad \cdots\quad \dfrac{\partial a_{\mu,\nu}}{\partial q_k}=\dfrac{\partial a_{\mu,k}}{\partial q_\nu},\quad \dfrac{\partial a_{\mu,\nu}}{\partial t}=\dfrac{\partial a_{\mu}}{\partial q_\nu},\\
\dfrac{\partial b_{\mu,\nu}}{\partial q_1}=\dfrac{\partial b_{\mu,1}}{\partial q_\nu},\quad \dfrac{\partial b_{\mu,\nu}}{\partial q_2}=\dfrac{\partial b_{\mu,2}}{\partial q_\nu},\quad \cdots\quad \dfrac{\partial b_{\mu,\nu}}{\partial q_k}=\dfrac{\partial b_{\mu,k}}{\partial q_\nu},\quad \dfrac{\partial b_{\mu,\nu}}{\partial t}=\dfrac{\partial b_{\mu}}{\partial q_\nu},\\
\dfrac{\partial c_{\mu,\nu}}{\partial q_1}=\dfrac{\partial c_{\mu,1}}{\partial q_\nu},\quad \dfrac{\partial c_{\mu,\nu}}{\partial q_2}=\dfrac{\partial c_{\mu,2}}{\partial q_\nu},\quad \cdots\quad \dfrac{\partial c_{\mu,\nu}}{\partial q_k}=\dfrac{\partial c_{\mu,k}}{\partial q_\nu},\quad \dfrac{\partial c_{\mu,\nu}}{\partial t}=\dfrac{\partial c_{\mu}}{\partial q_\nu}
\end{array}\right.$$

for any μ.

One can characterize this case in a different way. If the conditions (19) are assumed to have been fulfilled then determine the functions U_μ, V_μ, W_μ of and t by the formulas:

$$U_\mu = \int_{q_\nu^0}^{q_\nu} a_{\mu,\nu}\, dq_\nu\,, \qquad V_\mu = \int_{q_\nu^0}^{q_\nu} b_{\mu,\nu}\, dq_\nu\,, \qquad W_\mu = \int_{q_\nu^0}^{q_\nu} c_{\mu,\nu}\, dq_\nu\,,$$

in which q_ν^0 denotes a *constant*. From (19), one has immediately:

$$\frac{\partial U_\mu}{\partial q_1} = \int_{q_\nu^0}^{q_\nu} \frac{\partial a_{\mu,\nu}}{\partial q_1} dq_\nu = \int_{q_\nu^0}^{q_\nu} \frac{\partial a_{\mu,1}}{\partial q_\nu} dq_\nu = a_{\mu,1} - a_{\mu,1}^0,$$

in which $a_{\mu,1}^0$ is what $a_{\mu,1}$ will become when one replaces q_ν with the constant q_ν^0. Similarly:

$$\frac{\partial U_\mu}{\partial q_2} = a_{\mu,2} - a_{\mu,2}^0, \qquad \ldots, \qquad \frac{\partial U_\mu}{\partial q_k} = a_{\mu,k} - a_{\mu,k}^0\,,$$

$$\frac{\partial U_\mu}{\partial t} = \int_{q_\nu^0}^{q_\nu} \frac{\partial a_{\mu,\nu}}{\partial t} dq_\nu = \int_{q_\nu^0}^{q_\nu} \frac{\partial a_{\mu}}{\partial q_\nu} dq_\nu = a_\mu - a_\mu^0,$$

$$\frac{\partial V_\mu}{\partial q_\rho} = b_{\mu,\rho} - b_{\mu,\rho}^0\,, \qquad \frac{\partial V_\mu}{\partial t} = b_{\mu,} - b_\mu^0\,,$$

$$\frac{\partial W_\mu}{\partial q_\rho} = c_{\mu,\rho} - c_{\mu,\rho}^0\,, \qquad \frac{\partial W_\mu}{\partial t} = c_{\mu,} - c_\mu^0\,.$$

If one replaces $a_{\mu,\rho}$, $b_{\mu,\rho}$, $c_{\mu,\rho}$, a_μ, b_μ, c_μ, with their expressions that one infers from the preceding formulas then formulas (1) will become:

(20) $$\begin{cases} \delta x_\mu = \delta U_\mu + a^0_{\mu,1}\,\delta q_1 + a^0_{\mu,2}\,\delta q_2 + \cdots + a^0_{\mu,k}\,\delta q_k , \\ \delta y_\mu = \delta V_\mu + b^0_{\mu,1}\,\delta q_1 + b^0_{\mu,2}\,\delta q_2 + \cdots + b^0_{\mu,k}\,\delta q_k , \\ \delta z_\mu = \delta W_\mu + c^0_{\mu,1}\,\delta q_1 + c^0_{\mu,2}\,\delta q_2 + \cdots + c^0_{\mu,k}\,\delta q_k , \end{cases}$$

in which δU_μ, δV_μ, δW_μ are total differentials that are taken while regarding t as constant, and in which $a^0_{\mu,\nu}$, $b^0_{\mu,\nu}$, $c^0_{\mu,\nu}$ (which are the coefficients of the δq_ν) are zero.

Formulas (2) likewise become:

(20′) $$\begin{cases} dx_\mu = dU_\mu + a^0_{\mu,1}\,dq_1 + a^0_{\mu,2}\,dq_2 + \cdots + a^0_{\mu,k}\,dq_k , \\ dy_\mu = dV_\mu + b^0_{\mu,1}\,dq_1 + b^0_{\mu,2}\,dq_2 + \cdots + b^0_{\mu,k}\,dq_k , \\ dz_\mu = dW_\mu + c^0_{\mu,1}\,dq_1 + c^0_{\mu,2}\,dq_2 + \cdots + c^0_{\mu,k}\,dq_k . \end{cases}$$

One sees that the Lagrange equation will apply to the q_ν when δx_μ, δy_μ, δz_μ and dx_μ, dy_μ, dz_μ can be put *into the form of a total differential, followed by an expression that contains neither q_ν not δq_ν nor dq_ν* for any point of the system. One can also say that the Lagrange equation will apply to the parameter q_ν when the other parameters q_1, q_2, …, $q_{\nu-1}$, $q_{\nu+1}$, …, q_k are known as functions of t, so q_ν will become a true coordinate, in such a fashion that x_μ, y_μ, z_μ can be expressed in finite form as functions of q_ν and t.

In order for the Lagrange equations to be applicable to the parameters q_1, q_2, …, q_k, it is sufficient for that condition to be true for $\nu = 1, 2, \ldots, s$; i.e., that the virtual displacements δx_μ, δy_μ, δz_μ and the actual displacements dx_μ, dy_μ, dz_μ can be put into the form:

$$\delta x_\mu = \delta U_\mu + \alpha_{\mu,\,s+1}\,\delta q_{s+1} + \ldots + \alpha_{\mu,\,k}\,\delta q_k ,$$

$$\delta y_\mu = \delta V_\mu + \beta_{\mu,\,s+1}\,\delta q_{s+1} + \ldots + \beta_{\mu,\,k}\,\delta q_k ,$$

$$\delta z_\mu = \delta W_\mu + \gamma_{\mu,\,s+1}\,\delta q_{s+1} + \ldots + \gamma_{\mu,\,k}\,\delta q_k ,$$

$$dx_\mu = dU_\mu + \alpha_{\mu,\,s+1}\,dq_{s+1} + \ldots + \alpha_{\mu,\,k}\,dq_k + \alpha_\mu\,dt ,$$

$$dy_\mu = dV_\mu + \beta_{\mu,\,s+1}\,dq_{s+1} + \ldots + \beta_{\mu,\,k}\,dq_k + \beta_\mu\,dt ,$$

$$dz_\mu = dW_\mu + \gamma_{\mu,\,s+1}\,dq_{s+1} + \ldots + \gamma_{\mu,\,k}\,dq_k + \gamma_\mu\,dt ,$$

in which the coefficients $\alpha_{\mu,\,s+1}$, …, $\alpha_{\mu,\,k}$, α_μ, $\beta_{\mu,\,s+1}$, …, $\beta_{\mu,\,k}$, β_μ, $\gamma_{\mu,\,s+1}$, …, $\gamma_{\mu,\,k}$, γ_μ no longer contain the q_1, q_2, …, q_s. The system is then non-holonomic of order $k - s$ for the choice of parameters q_1, q_2, …, q_k.

IV. – APPLICATIONS.

9. Motion of a point in polar coordinates for the plane. – The equations:

$$x = r \cos \theta, \qquad y = r \sin \theta, \qquad z = 0$$

give

$$S = \frac{m}{2}(x''^2 + y''^2 + z''^2) = \frac{m}{2}[(r'' - r\,\theta'')^2 + (r\,\theta'' + 2r'\,\theta')^2] .$$

Upon adopting the notations of the example at the end of no. **1**, one will see that the equations of motion are:

$$\frac{\partial S}{\partial r''} = Q, \qquad \frac{\partial S}{\partial \theta''} = P\,r,$$

or

$$m\,(r'' - r\,\theta'') = Q, \qquad m\,(r\,\theta'' + 2r'\,\theta') = P.$$

Those equations are identical to those of Lagrange. With those parameters r and θ, the system is holonomic. However, if one takes the area σ that is swept out by the radius vector to be the parameter then one will have:

$$\delta\sigma = \tfrac{1}{2}\,r^2\,\delta\theta, \qquad d\sigma = \tfrac{1}{2}\,r^2\,d\theta,$$

$$\delta x = \cos\,\theta\,\delta r - \frac{2\sin\theta}{r}\delta\sigma,$$

$$\delta y = \sin\,\theta\,\delta r + \frac{2\cos\theta}{r}\delta\sigma,$$

$$x' = \cos\,\theta\, r' - \frac{2\sin\theta}{r}\sigma', \qquad y' = \sin\,\theta\, r' + \frac{2\cos\theta}{r}\sigma',$$

$$T \equiv \frac{m}{2}\left(r'^2 + \frac{4\sigma'^2}{r}\right),$$

$$x' = \cos\,\theta\left(r'' - \frac{4}{r^3}\sigma'^2\right) + 2\sin\theta\,\frac{\sigma''}{r},$$

$$y' = \sin\,\theta\left(r'' - \frac{4}{r^3}\sigma'^2\right) + 2\cos\theta\,\frac{\sigma''}{r},$$

$$S = \frac{m}{2}\left[\left(r'' - \frac{4}{r^3}\sigma'^2\right)^2 + \frac{4}{r^2}\sigma''^2\right],$$

$$X\,\delta x + Y\,\delta y = \frac{2P}{r}\delta\sigma + Q\,\delta r.$$

The equations are then:

$$\frac{\partial S}{\partial r''} = Q, \qquad \frac{\partial S}{\partial \sigma''} = \frac{2P}{r},$$

or upon making things explicit:

$$m\left(r'' - \frac{4}{r^3}\sigma'^2\right) = Q, \qquad m\,\sigma'' = \frac{2P}{r}.$$

If the force is central then $P = 0$, and the second equation will give:

$$\sigma'' = 0, \qquad \sigma' = C,$$

which expresses the theorem of areas.

Neither of the quantities:

$$\Delta_1 = \left[\frac{d}{dt}\left(\frac{\partial T}{\partial r'}\right) - \frac{\partial T}{\partial r}\right] - \frac{\partial S}{\partial r''},$$

$$\Delta_2 = \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \sigma'}\right) - \frac{\partial T}{\partial \sigma}\right] - \frac{\partial S}{\partial \sigma''}$$

is zero. With the choice of parameters r and σ, the system has become non-holonomic of order 2.

10. Motion of a solid body around a fixed point. – We calculate the energy of acceleration S of a solid body that moves around a point O while placing ourselves in the most general case. For each particular example, it will then suffice to employ that function S when it has been calculated once and for all. Refer the motion of the body to a tri-rectangular trihedron $Oxyz$ with its origin at O and which is animated with a known motion. Let Ω be the instantaneous rotation of that trihedron, let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ be the components of that rotation along the edges Ox, Oy, Oz. Similarly, let ω be the absolute instantaneous rotation of the solid body, and let p, q, r be its components along the axes $Oxyz$. A molecule m of the body with coordinates x, y, z possesses an absolute velocity v with projections:

$$(21)\qquad \begin{cases} v_x = qz - ry, \\ v_y = rx - pz, \\ v_z = py - qx. \end{cases}$$

That molecule possesses an absolute acceleration J that has projections:

$$(22)\qquad \begin{cases} J_x = \dfrac{dv_x}{dt} + \mathcal{Q}\, v_z - \mathcal{R}\, v_y, \\ J_y = \dfrac{dv_y}{dt} + \mathcal{R}\, v_x - \mathcal{P}\, v_z, \\ J_z = \dfrac{dv_z}{dt} + \mathcal{R}\, v_y - \mathcal{Q}\, v_z. \end{cases}$$

Those formulas can be written down immediately when one remarks that the acceleration J is the absolute velocity of the geometric point that has v_x, v_y, v_z with respect to the moving axes $Oxyz$.

Having said that, one will have:

$$\frac{dv_x}{dt} = q\,\frac{dz}{dt} - r\,\frac{dy}{dt} + z\,q' - y\,r', \ldots,$$

in which p', q', r' denote the derivatives of p, q, r with respect to time. The quantities $\dfrac{dx}{dt}$, $\dfrac{dy}{dt}$, $\dfrac{dz}{dt}$ are the projections onto Ox, Oy, Oz of the relative velocity v_r of the molecule m with respect to those axes. If one calls the guiding velocity of that same molecule v_e then one will have:

$$(v_r)_x = v_x - (v_e)_x\,;$$

i.e.:

$$\frac{dx}{dt} = q\,z - r\,y - (\mathcal{Q}\,z - \mathcal{R}\,y).$$

One will get $\dfrac{dy}{dt}$ and $\dfrac{dz}{dt}$ similarly, by permutation. From that, the expressions (22) for J_x , J_y , J_z will take the following form, in which we write only J_x :

$$J_x = q\,[(p - \mathcal{P})\,y - (q - \mathcal{Q})\,x] - r\,[(r - \mathcal{R})\,x - (p - \mathcal{P})\,z]$$
$$+ z\,q' - y\,r' + \mathcal{Q}\,(p\,y - q\,x) - \mathcal{R}\,(r\,x - p\,z),$$

or, upon rearranging:

$$J_x = -x\,(q^2 + r^2) + y\,[q\,(p - \mathcal{P})\,y + p\,\mathcal{Q} - r'] - z\,[r\,(p - \mathcal{P}) + p\,\mathcal{R} + q'] .$$

One will get J_y and J_z upon permuting. When one takes the sum of the squares, one will get J^2, and then the function:

$$S = \tfrac{1}{2}\sum m\,(J_x^2 + J_y^2 + J_z^2).$$

The coefficients of the moment of inertia:

$$A = \sum m(y^2 + z^2), \quad B = \sum m(z^2 + x^2), \quad C = \sum m(x^2 + y^2)$$

enter into that sum, along with the products of inertia:

$$D = \sum m\,y\,z, \quad E = \sum m\,z\,x, \quad F = \sum m\,x\,y$$

with respect to the axes *Oxyz*. In general, those six quantities will vary in time since the axes *Oxyz* displace in the body.

At present, the parameters are the angles that fix the orientation of the body around the point O. The quantities p, q, r contain the *first derivatives* of those parameters with respect to time. If the trihedron *Oxyz* is animated with a known motion then $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ must be regarded as known functions of time. If the motion of the trihedron is coupled to that of the body in some fashion then $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ will depend upon only the first derivatives of the parameters. The second derivatives of the parameters then enter into only the p', q', r'. From a preceding remark, it will then suffice to calculate the terms in S that depend upon the accelerations (i.e., the p', q', r'), because only those terms depend upon second derivatives of the parameters.

Set:

$$q\,\mathcal{R} - r\,\mathcal{Q} = \mathcal{P}_1, \qquad r\,\mathcal{P} - p\,\mathcal{R} = \mathcal{Q}_1, \qquad p\,\mathcal{Q} - q\,\mathcal{P} = \mathcal{R}_1, \tag{23}$$

to abbreviate, and let a, b, c denote the sums $\sum m\,x^2$, $\sum m\,y^2$, $\sum m\,z^2$, for the moment. One can write:

$$\begin{aligned}
2S &= a\,[(q' - \mathcal{Q}_1 - p\,r)^2 + (r' - \mathcal{R}_1 + p\,q)^2] \\
&\quad + b\,[(r' - \mathcal{R}_1 - q\,p)^2 + (p' - \mathcal{P}_1 + q\,r)^2] \\
&\quad + c\,[(p' - \mathcal{P}_1 - r\,q)^2 + (q' - \mathcal{Q}_1 + r\,p)^2] \\
&\quad - 2D\,[(q^2 - r^2)\,p' + (q' - \mathcal{Q}_1 + p\,r)\,(r' - \mathcal{R}_1 - p\,q)] \\
&\quad - 2E\,[(r^2 - p^2)\,q' + (r' - \mathcal{R}_1 + q\,p)\,(p' - \mathcal{P}_1 - q\,r)]
\end{aligned} \tag{24}$$

$$- 2F\,[(p^2 - q^2)\,r' + (p' - \mathcal{P}_1 + r\,q)\,(q' - \mathcal{Q}_1 - r\,p)] + \ldots$$

We develop this and rearrange the result with respect to $p' - \mathcal{P}_1$, $q' - \mathcal{Q}_1$, $r' - \mathcal{R}_1$, while noting that:

$$\begin{array}{lll} b + c = A, & c + a = B, & a + b = C, \\ b - c = C - B, & c - a = A - C, & a - b = B - A\,. \end{array}$$

Upon dropping the terms that are independent of p', q', r', we can write:

$$\begin{aligned} (25) \qquad 2S &= A\,(p' - \mathcal{P}_1)^2 + B\,(q' - \mathcal{Q}_1)^2 + C\,(r' - \mathcal{R}_1)^2 \\ &\quad - 2D\,(q' - \mathcal{Q}_1)\,(r' - \mathcal{R}_1) - 2E\,(r' - \mathcal{R}_1)\,(p' - \mathcal{P}_1) - 2F\,(p' - \mathcal{P}_1)\,(q' - \mathcal{Q}_1) \\ &\quad + 2\,[(C - B)\,q\,r - D\,(q^2 - r^2) - E\,p\,q + F\,p\,r]\,(p' - \mathcal{P}_1) \\ &\quad + 2\,[(A - C)\,r\,p - E\,(r^2 - p^2) - F\,q\,r + D\,q\,p]\,(q' - \mathcal{Q}_1) \\ &\quad + 2\,[(B - A)\,p\,q - F\,(p^2 - q^2) - D\,r\,p + E\,r\,q\,]\,(r' - \mathcal{R}_1) + \ldots \end{aligned}$$

Remark. – If the axes *Oxyz* are *fixed in space* then $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ will be zero, and one will have:

$$(26) \qquad \mathcal{P}_1 = \mathcal{Q}_1 = \mathcal{R}_1 = 0.$$

The same fact will be true if the axes are *fixed in the body*, because in that case:

$$(27) \qquad \mathcal{P} = p, \qquad \mathcal{Q} = q, \qquad \mathcal{R} = r.$$

Upon making this explicit, one will get the Euler equations, which one establishes easily.

Similarly, upon suitably specializing the formulas, one will get the equations of motion for the classical case in which the ellipsoid of inertia relative to the fixed point O is one of revolution. One takes the axis Oz to be the axis of revolution and the axes Ox and Oy to be two axes that move in both the body and space that are defined as follows: Let Ox_1, Oy_1, Oz_1 be three fixed axes: The axis Oy is perpendicular to the plane zOz_1 and the axis Ox is perpendicular to the plane yOz. The angle θ is then the angle z_1Oz, and ψ is the angle x_1Oy. The instantaneous rotation Ω of the trihedron $Oxyz$ is the resultant of two rotations, one of which $d\theta / dt = \theta'$ is around Oy, while the other $d\psi / dt = \psi'$ is around Oz_1 . The components of that rotation around Ox, Oy, Oz are then:

$$(28) \qquad \mathcal{P} = -\,\psi' \sin\,\theta, \qquad \mathcal{Q} = \theta', \qquad \mathcal{R} = \psi' \sin\,\theta.$$

Once the trihedron *Oxyz* has been located, one must define the position of the solid with respect to that trihedron. In order to do that, it will suffice to know the angle φ that a line that is coupled to the body in the *xOy* plane makes with the axis *Oy*. The derivative $d\varphi / dt = \varphi'$ of that angle measures the proper rotation of the body around *Oz*. The instantaneous rotation ω of the body is then the resultant of the rotation Ω of the trihedron and the proper rotation φ around *Oz*. One will then have that the projections *p*, *q*, *r* of ω onto the axes *Oxyz* are:

$$p = \mathcal{P} = -\psi'\sin\theta, \qquad q = \mathcal{Q} = \theta', \qquad r = \mathcal{R} + \varphi' = \psi'\cos\theta + \varphi'. \tag{29}$$

Upon differentiating with respect to *t*, one will conclude that:

$$p' = -\psi''\sin\theta + \ldots, \quad q' = \theta'', \qquad r' = \psi''\cos\theta + \varphi'' + \ldots$$

In addition, since the ellipsoid of inertia is one of revolution around *Oz* :

$$B = A.$$

When one replaces *P* and *Q* with *p* and *q*, resp., and remarks that $D = E = F = 0$, since the moving axes are the principal axes of inertia, the general expression (25) for *S* will be:

$$2S = A\,(p'^2 + q'^2) + C\,r'^2 + 2\,(A\,\mathcal{R} - C\,r)\,(p\,q' - q\,p') + \ldots \tag{30}$$

For a variation $\delta\theta$, $\delta\varphi$, $\delta\psi$ of the three angles, the sum of the works done by the applied forces will take the form:

$$\Theta\,\delta\theta + \Phi\,\delta\varphi + \Psi\,\delta\psi\,.$$

Since the virtual displacement that is obtained by setting $\delta\varphi = \delta\psi = 0$ is a rotation around *Oy*, Θ is the sum $\mathcal{M}_y$ of the moments of the forces with respect to *Oy*. Similarly, Φ is the sum $\mathcal{M}_z$ of the moments of the forces with respect to *Oz*, and Ψ is the sum $\mathcal{M}_{z_1}$ of the moments of the forces with respect to Oz_1. The equations are then easy to write out.

One will get them in the definitive form more quickly by introducing (as one can do in the general case) the three quantities λ, μ, ν that are defined by the relations:

$$\delta\lambda = -\sin\theta\,\delta\psi, \qquad \delta\mu = \delta\theta, \qquad \delta\nu = \cos\theta\,\delta\psi + \delta\varphi \tag{31}$$

as the parameters, so the actual displacement will be:

(32) $$\begin{cases} p = \lambda' = -\sin\theta\,\psi', \quad q = \mu' = \theta', \quad r = \nu' = \cos\theta\,\psi' + \varphi', \\ p' = \lambda'', \quad q' = \mu'', \quad r' = \nu''. \end{cases}$$

The quantities $\delta\lambda$, $\delta\mu$, $\delta\nu$ are then the elementary rotations around Ox, Oy, Oz, and one will have:

$$\sum (X\,\delta x + Y\,\delta y + Z\,\delta z) = \mathcal{M}_x\,\delta\lambda + \mathcal{M}_y\,\delta\mu + \mathcal{M}_z\,\delta\nu\,.$$

The function $2S$ that is given by the expression (30) is expressed immediately as a function of λ'', μ'', ν'', and the equations of motion are:

$$\frac{\partial S}{\partial \lambda''} = \mathcal{M}_x\,, \qquad \frac{\partial S}{\partial \mu''} = \mathcal{M}_y\,, \qquad \frac{\partial S}{\partial \nu''} = \mathcal{M}_z\,,$$

or, since $\lambda'' = p'$, $\mu'' = q'$, $\nu'' = r'$:

(33) $$\frac{\partial S}{\partial p'} = \mathcal{M}_x\,, \qquad \frac{\partial S}{\partial q'} = \mathcal{M}_y\,, \qquad \frac{\partial S}{\partial r'} = \mathcal{M}_z\,.$$

These are the three equations in their simplest form. With the parameters λ, μ, ν, the system will be non-holonomic of order 3 [**31**-2].

11. Theorem analogous to that of Koenig. – With the applications that follow in mind, it will be useful to establish a theorem that is analogous to that of Koenig, in order to abbreviate the calculations. Let x, y, z be the absolute coordinates of a point in a certain system with respect to some fixed axes. Let m be the mass of that point, let ξ, η, ζ be the coordinates of the center of gravity G of the system, let $M = \sum m$ be the total mass, and let x_1, y_1, z_1 be the coordinates of the point m with respect to the axes $Gx_1\,y_1\,z_1$, which are drawn through G parallel to the fixed axes. Let J_0 denote the absolute value of the acceleration of the point G :

$$J_0^2 = \xi''^2 + \eta''^2 + \zeta''^2,$$

let J denote the absolute value of the acceleration of the point m :

$$J^2 = x''^2 + y''^2 + z''^2,$$

and let J_1 be its acceleration relative to the axes $Gx_1\,y_1\,z_1$:

$$J_1^2 = x_1''^2 + y_1''^2 + z_1''^2\,.$$

One has:

$$x = \xi + x_1\,, \qquad y = \eta + y_1\,, \qquad z = \zeta + z_1\,.$$

The expression:

$$S = \tfrac{1}{2}\sum m\,(x''^2 + y''^2 + z''^2) = \tfrac{1}{2}\sum m\,[(\xi'' + x_1'')^2 + (\eta'' + y_1'')^2 + (\zeta'' + z_1'')^2]\,,$$

when one takes into account the fact that:

$$\sum m\,x_1 = 0, \qquad \sum m\,x_1'' = 0, \qquad \ldots,$$

will become:

$$S = \tfrac{1}{2} M\,J_0^2 + \tfrac{1}{2}\sum m\,J_1^2\,,$$

which one can write:

$$S = \tfrac{1}{2} M\,J_0^2 + S_1\,,$$

in which S_1 is the energy of acceleration that is calculated for the relative motion around G; one will then have the theorem:

The energy of acceleration S of a system is equal to the energy of acceleration that one would have if the total mass were concentrated at its center of gravity, plus the energy of acceleration of the system that is calculated for the relative motion of the system around its center of gravity.

12. Totally-free solid body. – One can obtain the function S for a free solid body by applying the theorem in no. **11** that is analogous to Koenig's theorem. The term $S_1 = \tfrac{1}{2}\sum m\,J_1^2$, which relates to the motion of the body around its center of gravity, will be given by formula (25) , which relates to the motion of a solid around a fixed point. One will then have:

$$2S' = M\,J_0^2 + 2S_1\,.$$

That formula is easily applied to the motion of a ponderous homogeneous body of revolution that is subject to sliding without friction on a fixed plane [**22**].

It will likewise permit one to write out the equations of motion of a ponderous homogeneous body of revolution that is subject to rolling without slipping on a fixed horizontal plane.

13. Application to a solid body that moves parallel to a fixed plane. – In the study of the motion of a solid around a fixed point, one essentially supposes that the point is at a finite distance. If it is at infinity then the solid can move parallel to a fixed plane. Take the plane of the figure to be the plane of the curve that is described by the center of gravity. Let two axes Ox and Oy be fixed in the plane, and let ξ and η be the coordinates of G. It will obviously suffice to know the motion of the plane figure (P), which is a section of the body by the plane xOy. Let θ denote the angle that Ox makes with a radius GA that is invariably coupled to that planar figure (P), while $M\,k^2$ is the moment of inertia of the body with respect to the axis that is drawn through G perpendicular to the plane xOy.

The motion of the body around the center of gravity G is a rotation around an axis that is fixed in the body, while the angular velocity of rotation is θ'. One will then have:

$$S_1 = \frac{M\,k^2}{2}(\theta''^2 + \theta'^4)$$

for the function S_1 that is calculated for the motion of the body around G.

Therefore:

$$S = \frac{M}{2}[\xi''^2 + \eta''^2 + k^2\theta''^2 + \ldots],$$

in which it is pointless to write out the terms that do not contain the second derivatives.

On the other hand, if one calls the projections of the general resultant of the applied forces X_0, Y_0, and lets N_0 denote the sum of the moments of those forces with respect to the axis that is drawn through G perpendicular to the plane xOy then one will have:

$$\sum (X\,\delta x + Y\,\delta y + Z\,\delta z) = X_0\;\delta\xi + Y_0\;\delta\eta + N_0\;\delta\theta\,.$$

If the body is not supposed to be subject to any constraint then the parameters ξ, η, ζ will be independent, and the equations of motion will be:

$$\frac{\partial S}{\partial \xi''} = X_0\,, \qquad \frac{\partial S}{\partial \eta''} = Y_0\,, \qquad \frac{\partial S}{\partial \theta''} = Z_0\,,$$

$$M\;\xi'' = X_0\,, \qquad M\;\eta'' = Y_0\,, \qquad M\,k^2\;\theta'' = N_0\,.$$

One will then recover the equations that give the general theorems immediately.

Suppose that the body is subject to a new constraint, which can be expressed by a relation in finite terms:

$$f(\xi,\,\eta,\,\theta,\,t) = 0,$$

or by a differential relation:

$$A\;d\xi + B\;d\eta + C\;d\theta + D\;dt = 0,$$
$$A\;\delta\xi + B\;\delta\eta + C\;\delta\theta + \qquad = 0,$$

in which A, B, C, D are functions of ξ, η, θ, t . One can then express η'' as a function of ξ'' and θ'', for example, and $\delta\eta$ as a function of $\delta\xi$ and $\delta\theta$. As a result, one can calculate S as a function of ξ'' and θ'' and make $\sum (X\,\delta x + Y\,\delta y + Z\,\delta z)$ linear and homogeneous in $\delta\xi$ and $\delta\theta$, and then equate $\dfrac{\partial S}{\partial \xi''}$ to the coefficients of $\delta\xi$ and $\dfrac{\partial S}{\partial \theta''}$ to that of $\delta\theta$.

V. – REMARKS OF AN ANALYTICAL ORDER.

14. Some properties of the function *S*. – In this number, we suppose that the constraints do not depend upon time:

$$a_\mu = b_\mu = c_\mu = 0,$$

and that the coefficients $a_{\mu,\nu}$, $b_{\mu,\nu}$, $c_{\mu,\nu}$ depend solely upon the $q_1, q_2, \ldots, q_k$, and not on t. The same thing will then be true of the coefficients of S.

From the expression for S that was given above:

$$S = \tfrac{1}{2}\sum m\,(x''^2 + y''^2 + z''^2),$$

when that function is confined to only the useful terms, it will have the following form:

$$S = \varphi(q_1'', q_2'', \ldots, q_n'') + \psi_1\, q_1'' + \psi_2\, q_2'' + \cdots + \psi_n\, q_n'', \tag{32}$$

in which φ is a quadratic form in the q'':

$$\varphi(q_1'', q_2'', \ldots, q_n'') = \sum \alpha_{ij}\, q_i'' q_j'' \qquad (\alpha_{ij} = \alpha_{ji}),$$

whose coefficients α_{ij} are supposed to depend upon solely the $q_1, q_2, \ldots, q_k$, and in which the $\psi_1, \psi_2, \ldots, \psi_k$ are quadratic forms in the $q_1', q_2', \ldots, q_k'$ whose coefficients also depend upon the $q_1, q_2, \ldots, q_k$.

The semi-*vis viva* of the system:

$$T = \tfrac{1}{2}\sum m\,(x'^2 + y'^2 + z'^2)$$

is a quadratic form in the $q_1', q_2', \ldots, q_k'$ whose coefficients are the same as those of the form φ, in such a way that:

$$T = \varphi(q_1', q_2', \ldots, q_k') = \sum \alpha_{ij}\, q_i' q_j' ; \tag{34}$$

that fact results from calculating the two functions S and T. In order to simplify the writing, we make:

$$\varphi(q_1'', q_2'', \ldots, q_n'') = \varphi_2,$$
$$\varphi(q_1', q_2', \ldots, q_k') = \varphi_1,$$

so

$$\left\{\begin{aligned} &S = \varphi_2 + \psi_1\,q_1'' + \psi_2\,q_2'' + \cdots + \psi_k\,q_k'', \\ &T = \varphi_1. \end{aligned}\right. \tag{35}$$

It is easy to verify that one has:

$$\frac{dT}{dt} = \frac{\partial S}{\partial q_1''}\,q_1' + \frac{\partial S}{\partial q_2''}\,q_2' + \cdots + \frac{\partial S}{\partial q_k''}\,q_k' \tag{36}$$

identically.

Let us see what that identity gives from the forms (35) of S and T. It becomes:

$$\begin{aligned} &q_1'\frac{\partial\varphi_2}{\partial q_1''} + q_2'\frac{\partial\varphi_2}{\partial q_1''} + \cdots + q_k'\frac{\partial\varphi_2}{\partial q_1''} + \psi_1\,q_1' + \psi_2\,q_2' + \cdots + \psi_k\,q_k' \\ &\quad = q_1''\frac{\partial\varphi_1}{\partial q_1'} + q_2''\frac{\partial\varphi_1}{\partial q_2'} + \cdots + q_k''\frac{\partial\varphi_1}{\partial q_k'} + q_1'\frac{\partial\varphi_1}{\partial q_1} + q_2'\frac{\partial\varphi_1}{\partial q_2} + \cdots + q_k'\frac{\partial\varphi_1}{\partial q_k}. \end{aligned} \tag{37}$$

The right-hand side of this is the developed expression for dT / dt that would result from the fact that T depends upon t by the intermediary of q_1', q_2', …, q_k', q_1, q_2, …, q_k. Now, the first part of the left-hand side of (37) is identical to the first part on the right-hand side, from an elementary property of quadratic forms. The identity (37) will then reduce to:

$$\psi_1\,q_1' + \psi_2\,q_2' + \ldots + \psi_k\,q_k' = \frac{\partial\varphi_1}{\partial q_1}q_1' + \frac{\partial\varphi_1}{\partial q_2}q_2' + \cdots + \frac{\partial\varphi_1}{\partial q_k}q_k'. \tag{38}$$

That relation must be true for any q_1, q_2, …, q_k, q_1', q_2', …, q_k'. It will then establish the necessary relations between the coefficients of the forms ψ_1, ψ_2, …, ψ_k and the coefficients α_{ij} of φ_1. To abbreviate the writing, we denote both sides of the identity (38) by a single letter and set:

$$E = \frac{\partial\varphi_1}{\partial q_1}q_1' + \frac{\partial\varphi_1}{\partial q_2}q_2' + \cdots + \frac{\partial\varphi_1}{\partial q_k}q_k' = \psi_1\,q_1' + \psi_2\,q_2' + \ldots + \psi_k\,q_k', \tag{39}$$

so the function E will be a cubic form in the q_1', q_2', …, q_k'.

15. Correction terms in the Lagrange equations. – If the identity (38) is supposed to be fulfilled then look for an expression for the difference:

$$\Delta_1 = \frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) - \frac{\partial T}{\partial q_1} - \frac{\partial S}{\partial q_1''}. \tag{40}$$

With the notations of no. **8**, one will have:

$$\Delta_1 = R_1 - \frac{\partial T}{\partial q_1}.$$

Since we have set $T = \varphi_1$, we will have:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) &= \frac{\partial^2 \varphi_1}{\partial q_1'^2}\,q_1'' + \frac{\partial^2 \varphi_1}{\partial q_1'\,\partial q_2'}\,q_2'' + \cdots + \frac{\partial^2 \varphi_1}{\partial q_1'\,\partial q_k'}\,q_k'' \\ &\quad + \frac{\partial^2 \varphi_1}{\partial q_1'\,\partial q_1}\,q_1' + \frac{\partial^2 \varphi_1}{\partial q_1'\,\partial q_2}\,q_2' + \cdots + \frac{\partial^2 \varphi_1}{\partial q_1'\,\partial q_k}\,q_k',\end{aligned}$$

because $\frac{\partial T}{\partial q_1'}$ or $\frac{\partial \varphi_1}{\partial q'}$ depend upon t by the intermediary of $q_1', q_2', \ldots, q_k', q_1, q_2, \ldots, q_k$.

Upon specifying the first row and taking into account the expression for E, one can write:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) = 2(\alpha_{11}\,q_1'' + \alpha_{11}\,q_2'' + \cdots + \alpha_{11}\,q_k'') + \frac{\partial E}{\partial q_1'} - \frac{\partial \varphi_1}{\partial q_1}.$$

On the other hand:

$$\frac{\partial T}{\partial q_1} = \frac{\partial \varphi_1}{\partial q_1},$$

$$\frac{\partial S}{\partial q_1''} = 2(\alpha_{11}\,q_1'' + \alpha_{11}\,q_2'' + \cdots + \alpha_{11}\,q_k'') + \psi_1.$$

After reduction, the difference (40), which is called Δ_1, will then become:

$$\Delta_1 = \frac{\partial E}{\partial q_1'} - 2\frac{\partial \varphi_1}{\partial q_1} - \psi_1.$$

Upon setting:

$$\Delta_\nu = \frac{d}{dt}\left(\frac{\partial T}{\partial q_\nu'}\right) - \frac{\partial T}{\partial q_\nu} - \frac{\partial S}{\partial q_\nu''} = R_\nu - \frac{\partial T}{\partial q_\nu},$$

one will likewise have:

(41) $$\Delta_\nu = \frac{\partial E}{\partial q'_\nu} - 2\frac{\partial \varphi_1}{\partial q_\nu} - \psi_\nu .$$

Having said that, the equations of motion can be written:

(42) $$\frac{d}{dt}\left(\frac{\partial T}{\partial q'_\nu}\right) - \frac{\partial T}{\partial q_\nu} = Q_\nu + \Delta_\nu \qquad (\nu = 1, 2, \ldots, k),$$

in which the term Δ_ν is expressed by the quantity (41). Those quantities Δ_ν form what one can call the *correction terms* in the Lagrange equations. One sees that the Lagrange equations can apply to the system if those terms Δ_ν are all identically zero. That will be the case when the system considered is holonomic and the parameters are true coordinates.

If the system is not holonomic then the motion of the system is the same as that of a holonomic system that admits the same *vis viva* $2T$ as the first one and is acted upon by "generalized forces":

$$Q_1 + \Delta_1, \qquad Q_2 + \Delta_2, \qquad \ldots, \qquad Q_k + \Delta_k .$$

The proof of the fact that a non-holonomic system and a holonomic system can have the same T identically can be found in a simple example that we gave in the Journal für die reine und angewandte Mathematik (Crelle's Journal), v. **122**, pp. 205.

The order of a non-holonomic system for the choice of parameters $q_1, q_2, \ldots, q_k$ is the number of Δ_ν that are non-zero.

Equation of *vis viva*. Verification. – If the constraints are independent of time then the equation of the *vis viva* will be:

(43) $$\frac{dT}{dt} = Q_1\, q'_1 + Q_2\, q'_2 + \cdots + Q_k\, q'_k .$$

In order to deduce that equation from equations (42), one must multiply the first of those equations by q'_1, the second one by q'_2, etc., the last one by q'_k, and add them.

One will then get equation (43), because one has:

(44) $$\Delta_1\, q'_1 + \Delta_2\, q'_2 + \cdots + \Delta_k\, q'_k = 0$$

identically; in other words, the apparent forces that are characterized by $\Delta_1, \Delta_2, \ldots, \Delta_k$ are *gyroscopic*, in the terminology of Sir. W. Thomson [**44**].

Indeed, from the expressions (41) for the quantities Δ_ν and the definition of E, one will have:

$$\Delta_1\, q_1' + \Delta_2\, q_2' + \cdots + \Delta_k\, q_k' \;=\; q_1' \frac{\partial E}{\partial q_1} + \cdots + q_k' \frac{\partial E}{\partial q_k} \;-\; 3E\ .$$

However, since E is homogeneous of degree three in q_1', q_2', …, q_k', the right-hand side will be zero identically, from the theorem on homogeneous functions.

16. General case. – If the constraints depend upon time, one can once more set:

$$\Delta_\nu = \frac{d}{dt}\left(\frac{\partial T}{\partial q_\nu'}\right) - \frac{\partial T}{\partial q_\nu} - \frac{\partial S}{\partial q_\nu''}\ .$$

The order of the non-holonomic system for the choice of parameters q_1, q_2, …, q_k will again be the number of Δ_ν (n = 1, 2, …, k) that are not zero [**33**].

VI. – FORMULATING THE EQUATIONS OF A PROBLEM IN DYNAMICS REDUCES TO THE SEARCH FOR THE MINIMUM OF A SECOND-DEGREE FUNCTION. GAUSS'S PRINCIPLE OF LEAST CONSTRAINT.

17. Problem of the minimum of a second-degree function. – If one considers the function R :

$$R = S - Q_1\, q_1'' - Q_2\, q_2'' - \cdots - Q_k\, q_k''\ ,$$

which one can call the analytical expression for the constraint, then R will be a function of degree two in the q_1'', q_2'', …, q_k''. The equations of motion are written:

$$\frac{\partial R}{\partial q_1''} = 0, \qquad \frac{\partial R}{\partial q_2''} = 0, \qquad \ldots, \qquad \frac{\partial R}{\partial q_k''} = 0\ .$$

The values of q_1'', q_2'', …, q_k'' that are inferred from those equations will then make R a maximum or minimum. Since R is a function of degree two in the q_1'', q_2'', …, q_k'' whose second-degree terms constitute a positive-definite form, the function R will be a *minimum* for the values of q_ν'' that correspond to the motion. It is obvious that one can make any function that differs from R by terms that are independent of the q_ν'' play the same role as R. From the expressions for the x_μ'', y_ν'', z_μ'', δx_μ, δy_μ, δz_μ, that function R will have the same terms in q_1'', q_2'', …, q_k'' as:

$$S - \sum_{\mu} [X\, x''_\mu + Y\, y''_\mu + Z\, z''_\mu]$$

or

$$\tfrac{1}{2}\sum m\, J^2 - \sum F\, J \cos FJ$$

or

$$R_0 = \sum \frac{1}{m}[(m\,x'' - X)^2 + (m\,y'' - Y)^2 + (m\,z'' - Z)^2].$$

One can then say that the accelerations that take the system from each instant, which are characterized by the values of q''_1, q''_2, …, q''_k, will make R_0 a *minimum*. If the system is free then that minimum will obviously be *zero*. If there are no external forces then R_0 will reduce to S.

18. Gauss's principle of least constraint. – From the translation of Gauss's paper, the principle of least constraint can be stated as follows:

> "The new principle is the following one:
>
> The motion of a system of material points that are coupled to each other in an arbitrary manner and subject to arbitrary influences will, at each instant, happen with the most perfect agreement possible with the motion that it would have if its were entirely free; i.e., with the smallest constraint possible, by taking the measure of the constraint that it experiences during an infinitely-small time interval to be the sum of the products of the mass of each point with the square of the quantity by which it deviates from the position that it would have taken had it been free.
>
> Let m, m', m'' be the masses of the points, let a, a', a'' be their respective positions, and let b, b', b'', resp., be the positions that they will occupy after an infinitely-small time dt by virtue of the forces that act upon them and the velocities that had had acquired at the beginning of that instant. The preceding statement amounts to saying that the positions c, c', c'', resp., that they will take will be, among all of the ones that are allowed by the constraints, the ones for which the sum:
>
> $$m\,\overline{bc}^2 + m'\,\overline{b'c'}^2 + m''\,\overline{b''c''}^2 + \ldots$$
>
> will be a minimum.
>
> Equilibrium is a special case of the general law that will be true when the sum:
>
> $$m\,\overline{ab}^2 + m'\,\overline{a'b'}^2 + \ldots$$

is a minimum (since the points have no velocity), or in other words, when the conservation of the system of points in the rest state is *closer* to the free motion that each would tend to take than any possible displacement that one can imagine."

Here is the proof of that principle:

The preceding equation proves that principle. One can say that the equation is its analytical expression. On page 343 of his book [**11**], Mach, speaking of Gauss's principle, considered the expression:

$$N = \sum m \left[\left(\frac{X}{m} - \xi \right)^2 + \left(\frac{Y}{m} - \eta \right)^2 + \left(\frac{Z}{m} - \zeta \right)^2 \right],$$

(in which ξ, η, ζ denote the projections of the acceleration of the point m), and sought the conditions that ξ, η, ζ must fulfill in order for N to be a *minimum*; he then came back to the general equation of dynamics.

In the German edition of the *Enzyklopädie der mathematischen Wissenschaften* and on page 84 of his article "Die Prinzipien der rationelle Mechanik," A. Voss [**28**] proceeded as follows in order to establish Gauss's principle: The position c of the point m has the following abscissa at the instant $t + dt$:

$$x + x'dt + \frac{x''}{1\cdot 2}dt^2 .$$

The sum that Gauss considered to be the measure of the constraint, namely:

$$m\overline{bc}^{\,2} + m'\overline{b'c'}^{\,2} + \dots$$

will then be:

$$\tfrac{1}{4}dt^4 \sum m \left[\left(x'' - \frac{X}{m} \right)^2 + \left(y'' - \frac{Y}{m} \right)^2 + \left(z'' - \frac{Z}{m} \right)^2 \right].$$

Now, that sum is precisely:

$$\tfrac{1}{4}dt^4 R_0 .$$

It will be a minimum among all of the possible motions because the accelerations will make R_0 a minimum.

VII. – APPLICATIONS TO MATHEMATICAL PHYSICS.

19. Electrodynamics. – In a volume in the *Collection Scientia*, "L'électricité déduite de l'expérience et ramenée au principe des travaux virtuels," Carvallo studied the application of the Lagrange equations to electrodynamical phenomena according to Maxwell's theory. In regard to the Barlow wheel, he pointed out that those equations are not always applicable to electrodynamical phenomena, notably in the case of two or three-dimensional conductors. He observed that the phenomenon of the Barlow wheel depends upon three parameters θ, q_1, q_2 whose arbitrary variations define the most general displacement of the system. He indicated that those parameters are not true coordinates and that the system behaves in regard to them in the same way that a hoop behaves in regard to the three parameters θ, φ, and ψ (no. **2**). Under those conditions, the Lagrange equations will not be applicable, and if one can hope to attach the equations of electrodynamics to those of analytical mechanics then one must choose a form for the equations that will be applicable to all systems, whether holonomic or not [**19**].

For the Barlow wheel, when one employs Carvallo's notations (*loc. cit.*, pp. 76 and 80), the equations of motion will be:

$$\begin{aligned} I\,\theta'' - K\,q_1'\,q_2' &= Q, \\ L_1\,q_1'' + K\theta'\,q_2' &= E_1 - r_1\,q_1', \\ L_2\,q_2'' &= E_\nu - r_2\,q_2', \end{aligned}$$

in which the right-hand sides are the generalized forces that we have previously denoted by Q_1, Q_2, Q_3. Now, the left-hand sides of those equations are written:

$$\frac{\partial S}{\partial \theta''}, \quad \frac{\partial S}{\partial q_1''}, \quad \frac{\partial S}{\partial q_2''}$$

if one sets:

$$S = \tfrac{1}{2}[I\,\theta''^2 + L_1\,q_1''^2 + L_2\,q_2''^2 + 2K\,q_2''(\theta' q_1'' - q_1'\,\theta'') + \cdots],$$

in which the unwritten terms no longer contain the second derivatives of the parameters. The equations of motion are indeed of the general form that was studied in that volume, but it would be important to know whether the function S, thus-formed analytically, can be shown directly to be the energy of acceleration $S = \frac{1}{2}\sum m\,J^2$ by physical considerations.

20. Extension to the physics of continuous media. Application to the theory of electrons. – In this number, we shall reproduce almost *verbatim* a note by Guillame in Bern [**24**].

"One can remark that if a system possesses a potential energy W then one will have:

$$\sum_{i=1}^{k} Q_i\, q_i'' = -W'' + U,$$

in which the sum is extended over the parameters that the potential energy depends upon, U denotes a term that is independent of the q_i'', and Q_i are the forces that are derived from the potential. The remaining forces will be called forces *external* to the system, and we set:

$$E = \sum_{i=1}^{k} Q_i\, q_i'',$$

in which the sum extends over those forces. If there are constraint equations $L_j = 0$ then one can introduce functions λ_j by a generalization of the method of Lagrange multipliers, as Poincaré showed in his *Leçons sur la théorie de l'élasticité*, in such a fashion that $\sum_j \lambda_i L_j$ can be considered to be a supplementary potential energy.

In the particular case where the kinetic energy:

$$T = \tfrac{1}{2}\sum m\,(x'^2 + y'^2 + z'^2)$$

is expressed in Cartesian coordinates, one will have:

$$T' = \sum m\,(x'\,x'' + y'\,y'' + z'\,z''),$$

$$T'' = \sum m\,(x''^2 + y''^2 + z''^2) + \sum m\,(x'\,x''' + y'\,y''' + z'\,z'''),$$

from which, one will deduce that:

$$\frac{\partial S}{\partial x''} = \frac{1}{2}\frac{\partial T''}{\partial x''}.$$

The expression:

$$R = S - \sum_{\nu=1}^{k} Q_\nu\, q_\nu''$$

wiil then be replaced by:

$$R = \tfrac{1}{2}T'' + W'' + \sum_j \lambda_j\, L_j - E. \tag{45}$$

If the coordinates are arbitrary then one must put S in place of $\frac{1}{2}T''$. From that, it will be easy to write R for continuous media. In that case, instead of the motion of a point m, one considers the motion of an element $d\tau$ of a certain volume V that is bounded by a surface Σ. The functions S, T, or W become integrals that are extended over the volume V. The term that relates to the constraint equations will be obtained upon multiplying the left-hand sides of those equations by $\lambda_j\, d\tau$, adding them together, and integrating over the volume V. The term E can give both a volume integral and a surface integral. By definition, R has the form:

$$R = \iiint \varphi_0\, d\tau + \iint \psi_0\, d\sigma,$$

in which φ_0 and ψ_0 can contain the accelerations and their partial derivatives. One then specifies the accelerations in such a fashion as to put R into the form:

$$R = \iiint \varphi_1\, d\tau + \iint \psi_1\, d\sigma,$$

in which φ_1 and ψ_1 are polynomials of degree two or three in the accelerations. That transformation will be possible if the system is mechanical. Upon varying the accelerations, one will form the variation δR, which must be zero for any variation of the accelerations. Upon annulling the coefficients of those variations, one will obtain the desired equations.

Application to the theory of electrons. – In order to establish a mathematical link between mechanics and electrical phenomena, Maxwell appealed to Lagrange's equations; he then supposed that the corresponding systems were holonomic. H. A. Lorentz [**40**] reprised and generalized Maxwell's ideas. In particular, he showed the following: Consider the energy of the magnetic field:

$$T = \tfrac{1}{2}\iiint \mathfrak{h}^2\, d\tau \tag{46}$$

to be a kinetic energy, and the energy of the electric field:

$$W = \tfrac{1}{2}\iiint \mathfrak{d}^2\, d\tau \tag{47}$$

to be a potential energy, where the vectors $\mathfrak{h}$ and $\mathfrak{d}$ satisfy the two constraint equations:

$$c \text{ rot } \mathfrak{h} - \mathfrak{v} \text{ div } \mathfrak{d} - \mathfrak{d}' = 0, \tag{48}$$

$$\text{div } \mathfrak{h} = 0, \tag{49}$$

in which $\mathfrak{v}$ denotes the velocity of matter, and c denotes the velocity of light. One can then establish the fundamental equation:

$$\text{rot } \mathfrak{d} = -\frac{1}{c}\mathfrak{h}' \tag{50}$$

by means of d'Alembert's principle.

The proof demands certain restrictions that are due to the use of quantities of electricity as coordinates and the introduction of all the virtual displacements. Lorentz was then led to define a new class of constraints that he called *quasi-holonomic*: He supposed that a system of electrons belongs to that class. Upon starting from the expression (45), and being given equations (46), (47), (48), (49), one can then establish equation (50), by supposing that the system is non-holonomic, in a general fashion.

Indeed, conforming to the meanings of T and W, the magnetic field $\mathfrak{h}$ is analogous to a velocity, so its derivative $\mathfrak{h}'$ will be analogous to an acceleration, and the electric field $\mathfrak{d}$ measures the deformation that produces the potential energy, so its first derivative $\mathfrak{d}'$ will be the rate of the variation of that deformation, and $\mathfrak{d}''$ will be its acceleration. Equation (48) permits one to immediately express $\mathfrak{d}''$ as a function of $\mathfrak{h}'$, in such a way that one will no longer have an equation of constraint (49) to consider. Let $\mathcal{F}\,d\sigma$ denote the force that acts on the element $d\sigma$, so one has:

$$R = \iiint \left[\tfrac{1}{2}\mathfrak{h}'^2 + c\,\mathfrak{d}\,\text{rot}\,\mathfrak{h}' - 2\lambda'\,\text{div}\,\mathfrak{h}'\right] d\tau - \iint \mathcal{F}\,\mathfrak{h}'\,d\sigma + \ldots$$

$$= \iiint \left[\tfrac{1}{2}\mathfrak{h}'^2 + c\;\mathfrak{h}'\,\text{rot}\,\mathfrak{d} + 2\mathfrak{h}'\,\text{grad}\,\lambda'\right] d\tau - \iint [c\,(\mathfrak{d}\,\mathfrak{h}')_n + \lambda'\,\mathfrak{h}' + \mathcal{F}\,\mathfrak{h}']\,d\sigma + \ldots$$

One infers from the volume integral that:

$$\mathfrak{h}' = -c \text{ rot } \mathfrak{d} - 2 \text{ grad } \lambda'. \tag{51}$$

In order to determine λ', it suffices to form div $\mathfrak{h}'$, while taking equations (49) into account. One will then find that λ' must be constant. Its gradient will then be zero, and equation (51) will reduce to the desired equation (50).

The surface integral permits one to determine the force $\mathcal{F}$. In order to find its significance, it will suffice to look for the work done per unit time. One finds that by taking the constant λ' to be equal to zero:

$$\mathcal{F}\,\mathfrak{h} = -\,c\,[\mathfrak{d}\ \mathfrak{h}]_n\ ;$$

i.e., the Poynting energy flux.

If one starts with the same equations, while remaining in the ether, then the expression (45) will permit one to determine equation (48) without the term that relates to matter. One can then exhibit the duality that often observed in electricity in a striking way.

The fecundity of the method that was proposed here comes from the fact that one substitutes virtual accelerations for virtual displacements. The quantities of electricity do not come into play. There is no need to go deeper into the mechanism for the phenomenon. The possibility of establishing the expressions φ_1 and ψ_1 for the theory of electrons will imply the possibility of a mechanical interpretation for that theory. In addition to d'Alembert's principle, one has tried (above all since Helmholtz) to extended Hamilton's principle to all of physics. Now, those principles apply poorly to the theory of electrons. One has the right to think that Appell's principle, thus-generalized, can be substituted for them advantageously; at least, in a number of cases.

One can see that the considerations above extend to the mechanics of Einstein. In it, one introduces the function (41):

$$H = -\,m_0 c\sqrt{1-\frac{\mathfrak{v}^2}{c^2}}$$

in order to form the equations of Lagrange and Hamilton in his mechanics. It is easy to see that H is the analogue of T in ordinary mechanics. Indeed, one has:

$$\frac{1}{2}\frac{\partial H''}{\partial \mathfrak{v}'} = \mathcal{F},$$

in which $\mathcal{F}$ denotes a force. That is the fundamental equation of motion in the new mechanics. The function R is obtained by replacing T'' with H'' in the expression (45)."

VIII. – CONSTRAINTS THAT ARE NONLINEAR IN THE VELOCITIES.

21. Possibility of non-linear constraints. – In his book on mechanics [**10**], Hertz showed that constraints are expressed by linear relations. However, it is possible that when certain masses or certain geometric quantities tend to zero, a set of linear constraints will produce a non-linear constraint that is imposed upon a point of a system in the limit. One can then apply the preceding general equations to the corresponding motions. That is what I did in 1911 in a note in Comptes rendus [**25**] and then in two articles in the Rendiconti del Circolo Matematico di Palermo [**25**-2].

Delassus, a professor on the Science Faculty at Bordeaux, dedicated some important notes that were included in Comptes rendus de l'Académie des Sciences de Paris [**26**] in 1911 to a general study of the question and several papers "Sur les liaisons et les mouvements des systèmes matériels" that were printed in the Annales de l'École Normale supérieure [**27**]. In a letter that he wrote to me in 1911, Professor Hamel in Brünn (without knowing of the research of Delassus), likewise pointed out the difficulties that present themselves when one passes to the limit.

Delassus called "the motions studied by Appell" or "abstract motions" the motions that are obtained by the extending the principle of the minimum of the function:

$$R = \sum \left[\frac{m}{2}(x''^2 + y''^2 + z''^2) - (X\,x'' + Y\,y'' + Z\,z'') \right]$$

to nonlinear constraints. In his note to the Comptes rendus on 16 October 1911 [**26**], he realized those motions as motions that were limited by means of *realizations with perfect tendency*. For example, if L is the constraint:

$$(L) \qquad x'^2 + y'^2 = z'^2$$

then Delassus considered a linear constraint L' that contained arbitrary constants that gave the single relation:

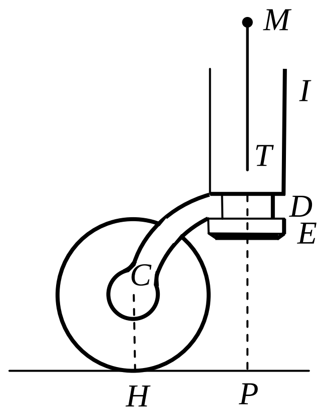
$$x'^2 + y'^2 = z'^2$$

between x', y', z', but produced some supplementary relations between x'', y'', z'' that disappeared in the limit.

My viewpoint will be different in what follows: In order to arrive at the limiting realization of the constraint L, I will consider a linear constraint L that contains an arbitrary constant ρ *that does not give any relation between* x', y', z', *but which produces the relation* (L) *in the limit* $\rho = 0$.

From the mechanical viewpoint, those two concepts are quite distinct.

I shall present that passage to the limit in an example. One will find some examples of the other viewpoint in the publications [**26** and **27**].



22. Example. – Imagine a caster that rolls without slipping on the horizontal plane *xOy*. The caster swivels about *EI*. Its wheel turns around a horizontal axis *C* that is carried by a fork *CD* that surrounds the swivel axis with a collar *D*. That collar can turn freely about the swivel axis in such a fashion that when one wishes to push the caster in a certain direction, the wheel will turn around the axle and will be located in a vertical plane through the direction of the swivel axis. The system poses no resistance to displacement in any direction.

In order to now complete our mechanism, we must suppose that there is just one wheel and just one swivel *EI* that is constrained to remain vertical by lateral shafts that rest on the floor without friction or sliding. A vertical shaft *TM* slides without friction inside the swivel. It is activated by the wheel with the aid of a mechanism that is easy to imagine, in such a way that it will be raised or lowered by a distance that is proportional to the angle φ through which the wheel has turned, in one sense or the other. The extremity of that shaft carries a point *M* of mass *m* and rectangular coordinates *x*, *y*, *z* on which an arbitrary force *F* acts. That system will give a quadratic constraint of the form:

$$\delta z^2 = k^2 (\delta x^2 + \delta y^2)$$

in the limit, in which *k* denotes a constant when one supposes that:

1. All of the masses, except for *M*, first become zero.

2. The distance *HP* from the center *C* of the wheel to the shaft *TM* then tends to zero.

Indeed, in that limit, if the wheel turns through $\delta\varphi$ then its center *C* will experience a displacement whose projection onto the horizontal plane *xOy* will have components δx, δy such that:

$$\sqrt{\delta x^2 + \delta y^2} = a\ \delta\varphi ,$$

in which a denotes the radius of the wheel. On the other hand, the point M will experience a vertical displacement that is proportional to $\delta\varphi$:

$$\delta z = b\ \delta\varphi\,;$$

one will then have:

$$\delta z^2 = k^2\,(\delta x^2 + \delta y^2), \qquad k^2 = \frac{b^2}{a^2}\,.$$

Before passing to the limit, one will have a system with linear constraints with two parameters x and y, to which one can apply the general equations that consist of writing out that under the motion, the values of the acceleration are the ones that make the function:

$$R = \sum\left[\frac{m}{2}(x''^2 + y''^2 + z''^2) - (X\,x'' + Y\,y'' + Z\,z'')\right]$$

a minimum. If one writes out those equations and passes to the aforementioned limit on the indicated order then one will find equations for the motion of M that express the idea that the function:

$$\tfrac{1}{2}m(x''^2 + y''^2 + z''^2) - (X\,x'' + Y\,y'' + Z\,z'')$$

is a minimum when x, y, z are coupled by the relation:

$$k^2\,(x'x'' + y'y'') - z'z'' = 0$$

that is obtained by differentiating the constraint equation:

$$k^2\,(x'^2 + y'^2) - z'^2 = 0.$$

When one employs the method of Lagrange multipliers in order find the minimum, those equations will be:

$$m\,x'' = X + \lambda\,k^2\,x',$$
$$m\,y'' = Y + \lambda\,k^2\,y',$$
$$m\,z'' = Z - \lambda\,z'.$$

The force of constraint, whose projections are $\lambda\,k^2\,x'$, $\lambda\,k^2\,y'$, $-\lambda\,z'$, is perpendicular at M to the tangent plane to the cone whose summit is M and is defined by the set of all virtual displacements such that:

$$\delta z^2 = k^2\,(\delta x^2 + \delta y^2),$$

while the plane is tangent along the actual displacement dx, dy, dz.

The work done by that constraint force is zero for the actual displacement: *It will not be zero for a virtual displacement that is compatible with the constraint.*

We perform the calculation that we just indicated. In the system of the figure: Let x, y, z denote the coordinates of M, let ξ, η denote those of the center C of the wheel, and z will remain constant for that point. ρ is the distance HP, and θ is the angle between HP and Ox. One will then have:

$$x = \xi + \rho \cos \theta, \qquad y = \eta + \rho \sin \theta.$$

The virtual displacements are defined by the relations:

$$\delta\xi = a \cos \theta \, \delta\varphi, \qquad \delta\eta = a \sin \theta \, \delta\varphi,$$

$$\begin{aligned} \delta x &= a \cos \theta \, \delta\varphi - \rho \sin \theta \, \delta\theta, \\ \delta y &= a \sin \theta \, \delta\varphi + \rho \cos \theta \, \delta\theta, \\ \delta z &= b \, \delta\varphi . \end{aligned}$$

The actual displacement is subject to the following conditions:

$$\xi' = a \cos \theta \, \varphi', \qquad \eta' = a \sin \theta \, \varphi',$$

$$\begin{aligned} x' &= a \cos \theta \, \varphi' - \rho \sin \theta \, \theta', \\ y' &= a \sin \theta \, \varphi' + \rho \cos \theta \, \theta', \\ z' &= b \, \varphi', \end{aligned}$$

so

$$\text{(52)} \qquad \left\{ \begin{aligned} \xi'' &= a \cos \theta \, \varphi'' - a \sin \theta \, \varphi' \, \theta', \\ \eta'' &= a \sin \theta \, \varphi'' + a \cos \theta \, \varphi' \, \theta', \end{aligned} \right.$$

$$\begin{aligned} x'' &= (a \, \varphi'' - \rho \, \theta'^2) \cos \theta - (\rho \, \theta'' + a \, \varphi' \, \theta') \sin \theta, \\ y'' &= (a \, \varphi'' - \rho \, \theta'^2) \sin \theta + (\rho \, \theta'' + a \, \varphi' \, \theta') \cos \theta, \\ z'' &= b \, \varphi'', \end{aligned}$$

from which, one infers that:

$$\text{(53)} \qquad \left\{ \begin{aligned} a \, \varphi'' - \rho \, \theta'^2 &= x'' \cos \theta + y'' \sin \theta, \\ \rho \, \theta'' + a \, \varphi' \, \theta' &= -x'' \sin \theta + y'' \cos \theta. \end{aligned} \right.$$

The energy of acceleration S of the system is composed of the energy S_1 of the wheel and the energy S_2 of the point M upon neglecting the mass of the shaft and that of the piece CD :

$$S = S_1 + S_2 .$$

Now, from the preceding, one will have:

$$2S_1 = m\,(\xi''^2 + \eta''^2) + A\,\theta''^2 + B\,\varphi''^2 + \ldots,$$

when one calls the total mass of the wheel μ, while A and B are its principle moments of inertia relative to its center; hence:

$$2S_2 = m\,(x''^2 + y''^2 + z''^2)\,.$$

Since $z'' = b\,\varphi''$, from (52), one will have:

$$2S = (\mu\,a^2 + B + m\,b^2)\,\varphi''^2 + A\,\theta''^2 + m\,(x''^2 + y''^2) + \ldots,$$

or, upon replacing φ'' and θ'' with their expressions that are inferred from (53):

$$\begin{aligned} 2S = {} & \frac{\mu\,a^2 + B + mb^2}{a^2}\,(x''\cos\theta + y''\sin\theta + \rho\,\theta'^2)^2 \\ & + \frac{A}{\rho^2}\,(x''\sin\theta - y''\cos\theta + a\,\varphi'\,\theta')^2 + m\,(x''^2 + y''^2) + \ldots, \end{aligned} \tag{54}$$

in which the unwritten terms no longer contain second derivatives.

Now, make a force act upon the point M whose projections are X, Y, Z. The elementary work done by that force under a virtual displacement is:

$$X\,\delta x + Y\,\delta y + Z\,\delta z,$$

in which:

$$\delta z = b\,\delta\varphi = (\delta x\cos\theta + \delta y\sin\theta),$$

$$\delta z = k\,(\delta x\cos\theta + \delta y\sin\theta)\,.$$

The virtual work is then:

$$(X + k\,Z\cos\theta)\,\delta x + (Y + k\,Z\sin\theta)\,\delta y\,.$$

The equations of motion are then:

$$\left\{\begin{aligned} \frac{\partial S}{\partial x''} &= X + k\,Z\cos\theta, \\ \frac{\partial S}{\partial y''} &= Y + k\,Z\sin\theta. \end{aligned}\right. \tag{55}$$

Now, pass to the limit, while making the mass μ of the wheel and ρ go to zero. The coefficients B and A will also tend to zero. However, here we see the indeterminacy that Delassus pointed out in the general case. If A and ρ tend to zero at the same time then the limiting value of S will depend upon the behavior of K / ρ^2. We shall arrange that in such a way that A / ρ^2 tends to *zero*. S will then tend to the limit:

$$S = \tfrac{1}{2} m\, k^2 (x'' \cos\theta + y'' \sin\theta)^2 + m\,(x''^2 + y''^2) + \ldots$$

The equations of motion keep the form (55). They are then:

$$m\, k^2 (x'' \cos\theta + y'' \sin\theta) \cos\theta + m\, x'' = X + k\, Z \cos\theta,$$
$$m\, k^2 (x'' \cos\theta + y'' \sin\theta) \sin\theta + m\, y'' = Y + k\, Z \sin\theta.$$

On the other hand, one will then have:

$$x' = a \cos\theta\, \varphi', \qquad y' = a \sin\theta\, \varphi', \qquad z' = b\, \varphi',$$

$$\cos\theta = k\,\frac{x'}{z'}, \quad \sin\theta = k\,\frac{y'}{z'},$$

when one sets:

$$\frac{Z - m\, z''}{z'} = \lambda .$$

IX. – REMARKS ON NON-HOLONOMIC SYSTEMS THAT ARE SUBJECT TO PERCUSSIONS OR ANIMATED WITH VERY SLOW MOTIONS.

23. Application of Lagrange's equations in the case of percussions. – Beghin and Rousseau showed in a paper in the Journal de Mathématiques [**30**] that the form of the equations of the theory of percussions, which I have deduced from the Lagrange equations for the holonomic systems, further applies to non-holonomic systems, even though the Lagrange equations will then break down. One can establish that result by a method that is analogous to the one that I pointed out in no. **15**. Take the equations of an arbitrary system in the form of no. **15**:

$$\text{(56)} \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial q'_\nu}\right) - \frac{\partial T}{\partial q_\nu} = Q_\nu + \Delta_\nu \qquad (\nu = 1, 2, \ldots, k) .$$

The Δ_ν are correction terms that depend upon only the $q_1, q_2, \ldots, q_k, q'_1, q'_2, \ldots, q'_k$ and time. Those correction terms are *zero* if the system is holonomic. However, if the percussions take place during the very short interval $t_1 - t_0$ then multiply the two terms in equation (56) by dt and integrate from t_0 to t_1. The integrals of $\frac{\partial T}{\partial q_\nu} dt$ and $\Delta_\nu\, dt$ will be negligible, because the q_ν and q'_ν will remain finite, and the equations will give:

$$\left(\frac{\partial T}{\partial q'_\nu}\right)_1 - \left(\frac{\partial T}{\partial q'_\nu}\right)_0 = \int_{t_0}^{t_1} Q_\nu \, dt .$$

Up to the difference in notations, these are precisely the equations that one can deduce from those of Beghin and Rousseau [**48**].

24. Case of very slow motions. – One can make a remark of the same type for the application of the Lagrange equations to the very slow motions of a non-holonomic system with constraints that are independent of time.

If the motion is very slow then the velocities will be very small. Consequently, the quantities $q'_1, q'_2, \ldots, q'_k$ will remain very small. Suppose that one neglects the squares and products of those quantities then. The terms that enter into equations (56) are quadratic forms in the $q'_1, q'_2, \ldots, q'_k$, so they are *negligible*, and the approximate equations take the form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q'_\nu}\right) - \frac{\partial T}{\partial q_\nu} = Q_\nu ,$$

in which it will remain for one to suppress the terms of degree two in $q'_1, q'_2, \ldots, q'_k$.

In that case, the Lagrange equations will then provide some approximate equations of motion, although that form of the equations is not rigorously applicable.

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On the principles of Hamilton and Maupertuis

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In the introduction to his *Mechanik*, **Heinrich Hertz** ([1]) said that **Hamilton**'s principle often yields results that are physically false. In order to document that, he cited a case in which one could, as he himself remarked, assess the motions that can be performed, as well as the ones that would correspond to **Hamilton**'s principle, through a mere consideration without calculation. **Hertz** added that the result would not change if one employed the **Maupertuisian** principle of least action, instead of **Hamilton**'s principle. Let us consider an example. It consists of a ball whose entire inertia rolls without slipping on a fixed horizontal plane ([2]). According to **Hertz**, the motions that correspond to **Hamilton**'s principle here are the ones that arrive at the given goal in shortest time for a given constant *vis viva*, which would imply that passing from any initial point to any final point would have to be possible without the action of a force. That conclusion, which is more closely connected with the principle of least action than it is with that of **Hamilton**, was reached approximately. If one chooses the initial and final positions of the ball arbitrarily then there will always be a pure rolling passage ([3]) from the one to the other. Among those transitions, each of which should come about at constant *vis viva* and all of which should have the same constant *vis viva*, there will be one of them that takes the least time ([4]). In **Hertz**'s opinion, that will correspond to **Hamilton**'s principle and the principle of least action. **Hertz** contrasted that result with the fact that in reality, despite the arbitrariness that the initial

([1]) *Gesammelte Werke*, 1894, Bd. III, pp. 23

([2]) The ball does not need to be homogeneous. If it were homogeneous then a uniform motion of the center of the ball, combined with a uniform rotation of the ball around a fixed axis that goes through the center, would occur.

([3]) In regard to the existence of that passage, one should cf. the final remark of § **12**. The fact that the passage, which will be contrived here for the proof, is also one of rolling without slipping was not stated expressly by **Hertz** at that point. However, one could not arrive at the same conclusion without establishing that. The fact that I have echoed **Hertz**'s opinion correctly by adding the proof of the latter will emerge from numbers 347, 358, 112, 111.

([4]) That might be added here, although it could still be challenged from a rigorous mathematical standpoint.

velocity is stuck with, no natural transition from any position to any other one is possible without forces being involved.

The aforementioned truth, which was self-explanatory for **Hertz**, can, in fact, be inferred by mere observation. To that end, we need only to consider that the motion must be determined by the initial state of the ball. Other than the initial position, all that one needs to determine the initial state are the instantaneous axis of rotation (which we shall assume goes through the center of the ball), the associated angular velocity, and a displacement velocity. However, the magnitude and direction of the displacement velocity will then be determined because rolling with slipping must take place on the horizontal plane. Nothing can be said about the magnitude of the initial rotational velocity. However, since the initial rotational axis can be chosen in a double infinitude of ways, and each choice of axis will lead to a simply-infinite manifold of positions for the ball, one can arrive at only a triple infinitude of positions when one starts from a given position. By contrast, the totality of all positions of the ball define a five-fold manifold, because the center can be placed in a double infinitude of ways, while the ball can be positioned about its center in a triple infinitude of ways. That implies the impossibility of going from a given position to any other one without the action of forces.

The attempt to clarify the contradiction that comes about from the fact that, strictly speaking, no rolling occurs in nature that is not coupled with at least a small amount of slipping did not satisfy **Hertz**. It also emerges from the foregoing well enough that here one is not dealing with a contradiction in which ordinary mechanics should take the advice of experiments as much as with contradictory conclusions of the different arguments. The contradiction must then be removed from the theory.

The thorough developments in **Hertz**'s book contain such a solution. In order to understand that, one must focus on the condition equations by which the motion of a material system can be constrained. **Hertz** allowed only condition equations that did not contain time. However, the coordinates of the points of the system could also appear in the form of differentials. More precisely, the condition equations are assumed to have the form ([1]):

$$(1) \qquad \sum_{(\nu)} (\varphi_{i\nu}\, dx_\nu + \psi_{i\nu}\, dy_\nu + \chi_{i\nu}\, dz_\nu) = 0 \qquad (i = 1, 2, \ldots),$$

in which the symbols φ, ψ, χ denote dimensionless functions of the coordinates:

$$x_1, y_1, z_1, \qquad x_2, y_2, z_2, \qquad x_3, y_3, z_3, \qquad \ldots$$

of the material points. Now, there is a special case in which the totality of conditions (1) is equivalent to a complex of conditions of the form:

$$(2) \qquad d\Phi_1 = 0, \qquad d\Phi_1 = 0, \qquad \ldots;$$

([1]) Cf., no. **124**. **Voss** already treated such conditions before; cf., Math. Ann., v. **25** , pp. 258 *et seq.*

i.e., one that is "completely integrable." In that case, **Hertz** called the material system *holonomic* ([1]). His solution to the previous contradiction is this: The basic laws of mechanics that he presented were true in general for both holonomic systems and non-holonomic systems, but he arrived at **Hamilton**'s principle and the principle of least action only by adding the restriction to holonomic systems. The ball that rolls on the plane presents a non-holonomic system, which the tendency to slip would destroy.

If that solution were satisfied then that would not contradict the general belief that **Hamilton**'s principle is merely another form of **d'Alembert**'s, and that this would be true in general. The deviation from the usual picture of **Hertz**'s theory can also not explain the fact that it has placed a new law at the foundations, since his basic law is equivalent to **d'Alembert**'s principle in the cases that he considered ([2]). That raises the basically-mathematical question: Does the usual derivation of **Hamilton**'s principle from **d'Alembert**'s require a restricting condition? The present article will serve to answer that question. That answer that it will give is that when **d'Alembert**'s principle is true in general, **Hamilton**'s must also be generally true in its most complete formulation. However, if one chooses the formulation that **Hertz** assumed then the restriction that he pointed out will, in fact, enter into it. In this paper, I will explain yet another point, and in more detail that it has been given up to now: First of all, the concept of the variation of a motion itself will be discussed, and then the forms that the principle of least action can take, along with the relationship of that principle to **Hamilton**'s, which can encompass both principles with a more general integral principle. At the same time, it will be shown that the principle of least action can also be formulated in such a way that it will remain valid when time enters into the condition equations.

([1]) Cf., no. **123**, **132**, **133**.

([2]) Confer **Hertz**'s no. 394. As far as his basic law is concerned, which he unnecessarily restricted to *free* systems (nos. 309, 122, 117), it includes two statements: One of them determines the constancy of the differential quotients with respect to time ds / dt. The quantity s is the defined by the equation:

$$ds^2 \sum_{(\nu)} m_\nu = \sum_{(\nu)} m_\nu (dx_\nu^2 + dy_\nu^2 + dz_\nu^2),$$

in which $m_1, m_2, \ldots$ mean the masses of the system points. Obviously, that part of the basic law is nothing but the law of conservation of *vis viva*. The other part is derived from the fact that the quantity:

$$\sum_{(\nu)} m_\nu \left\{ \left(\frac{d^2 x_\nu}{ds^2}\right)^2 + \left(\frac{d^2 y_\nu}{ds^2}\right)^2 + \left(\frac{d^2 z_\nu}{ds^2}\right)^2 \right\}$$

is continually a minimum under the motion. If one sets $s = \text{const.} \times t$ in the last sum then one will obtain essentially the same expression that was supposed to be a minimum in **Gauss**'s principle of least constraint, in which all of the forces that **Hertz** excluded from the foundations are set to zero. One should compare this to my own presentation of his basic law in **Hertz** nos. 309, 266, 263, 55, 100, 106, 151, 152, 153. I have once more introduced the usual notation of rectangular coordinates in place of his notation for coordinates.

In my opinion, the significance of **Hertz**'s book does not take the form of a basic law, but the fact that the forces can be nonetheless constructed mathematically from a basic law that does not contain forces, as it is now formulated. I shall go no further into that construction, which first appears in the later parts of his book and lies beyond the scope of the present study.

Whenever both principles are at issue, above all, I would like to at least suggest that here by once more considering the motions of the ball. During its actual motion, which is one of pure rolling, the ball will assume a continuous succession of positions. The application of the aforementioned principles will demand only a small change in the motion. In order to accomplish that, we will first displace each of the original running positions of the ball slightly, such that a second continuous succession of positions will arise. At the same time, the positions of this new sequence can be related to the positions of the first sequence. The second motion is still not determined completely in that way, since it has not been stated that the corresponding positions of the two motions will be passed through at the same time. That is required by **Hamilton**'s principle, while the principle of least action requires something else. However, both principles will be applied here in such a way that aforementioned small displacement of the ball should result from a simple rolling motion, while **Hertz**, in contrast to that, employed the condition that the second motion – i.e., the varied one – should also itself exhibit rolling without slipping. If one performs the variations in the correct way then that will imply the rolling motion of the ball in precisely the way that **Hertz** said corresponds to the facts. It is not distinguished *from the motions of its kind*, even when it, in fact, requires less time. However, we have lived upon a different foundation for a long time that conceptualizes the principle of least action and **Hamilton**'s principle only to the extent that the variation of an integral or an integral that contains a variation should be set to zero. Of course, the name "principle of least action" will no longer be appropriate then.

§ 1. – Variation of a motion.

In order to make the concept of the variation of a motion clearer, we would first like to consider a free material point. Its motion should be varied in such a way that the initial position A and final position B will remain unchanged. The original motion is the one that actually takes place, while the new varied one is only an auxiliary mathematical notion. Now, one can choose the path of the new motion from A to B such that it differs slightly from the old path and runs approximately parallel to it ([1]), and arbitrarily, in general. After that, one let can the motion along the new path evolve over time in various ways. We imagine that both motions begin at A at the same time. They do not need to arrive at B at the same time, which will not be the case, e.g., when the actual motion takes less time than the varied one. Now, in order to have a precise picture of the variation in mind, one must refer each position that is assumed by the varied motion to a position that was assumed in the original motion ([2]). Without such a relation, e.g., the variation of the integral:

$$\int T\,dt\,,$$

([1]) Cf., the first remark in § **2**.

([2]) Naturally, this important state of affairs has already been observed in the geometric problems of the calculus of variations; **Weierstrass** always emphasized that in his own lectures.

in which T represents the *vis viva* and t represents the time, would probably be meaningful, but the equation:

$$\delta\int T\,dt = \int \delta(T\,dt)$$

would make no sense. One associates the (identical) initial positions with each other, and similarly, the final positions. In that way, it is clear that in the event that the motions do not arrive at B at the same time, the relation could not be presented in such a way that corresponding positions of both motions would be passed through simultaneously. One would then produce an arbitrary point-wise relationship between the two paths and observe that corresponding positions are separated from each other by very little distance ([1]). One might wonder if that point-wise association of the paths is physically meaningless here, and whether that association, like the variation of the motion itself, is only an auxiliary mathematical construction. We shall consider the simpler way of expressing things, so for the moment, the time-point at which both motions begin at A will be the origin of time. Now, if C and C' are associated positions of the motions then one could logically refer to the time that it takes for the original motion to flow from A to C by τ and the time that it takes for the varied motion to flow from A to C by $\tau + \delta\tau$. *The variation $\delta\tau$ of time is therefore nothing but the difference between the times at which corresponding positions will be passed through.* The variation of the time differential is the algebraic overshoot of the time that it takes a small part of the new motion to flow over the time that is used by the associated part of the old motion ([2]). If one compares the initial and final times for both motions along those small pieces then one will easily see that the variation of the time differential is equal to the differential of the time variation. That corresponds to the known general theorem on the commutability of the symbols d and δ.

Now, the variation of the motion of our points would be best carried out as follows: One first gives each point of the original path a displacement such that a new path will arise that relates to the old one point-wise. One then determines the velocity for each point of the new path. It must differ from the velocity at the corresponding location on the old path only slightly, but it can otherwise be taken to be arbitrary. However, we shall now distinguish between two special ways of doing that determination.

The *first* kind of variation arises from the condition that *corresponding locations on both paths are passed through simultaneously;* both motions would then have to arrive at B at the same time.

The *second* kind of variation relates to the forces under whose action the original motion proceeded. If we imagine the forces here in such a way that we can speak of a "potential energy" then we can define this kind of variation as follows: *The total energy of the corresponding states of the motions being compared must be the same.* That variational

([1]) More precisely, two corresponding, infinitely-small arcs of both paths must have a well-defined ratio for each location, and that ratio should differ from one only slightly. On that subject, cf., the first rem. in § **2**.

([2]) A more rigorous use of pure mathematics that would distinguish between differentials and changes and between variations and changes would be impractical here.

condition will be formulated somewhat differently later on such that it will also be suitable for the remaining cases. The total energy is composed of the *vis viva* and the potential energy. Now, since the original motion is thought of as being given, the *vis viva* and the potential energy will also be given for a location C on its path. For the corresponding location C' on the varied path, at first, only the potential, which depends upon just the position, will be known. One will then get the *vis viva* for the location C' from the variational condition that is required here, and thus, the velocity.

Hence, once the new path and its point-wise relationship to the old path has been established, the varied path will be determined completely by the first variational condition, as well as the second one, and in different ways each time. *The time is varied for the second kind of variation, but not for the first one.*

The relationships for the motion of a material system are analogous. If we, with **Hertz**, take the concept of the "position of a system" to include the totality of all positions of the points of the system then motion will consist of a continuous succession of system positions that follow in time in a certain way. In order to vary that original motion, one will first assign a small displacement to each system position such that a new continuous succession of system positions will arise. If the original sequence goes through a position twice then one will have two overlapping motions that can naturally be displaced in different ways. The new paths of system points and the association between the locations on those paths are now established. For that reason, one can choose the velocity at all locations of the new path for a system point for the most general kind of variation. However, if one establishes that either the new positions are passed through at the same times as the associated old positions or that the two associated states of both motions should have the same energy then one would determine how the new succession of system positions is to be passed through completely in that way.

Previously, we did not take any condition equations into account. If a motion is subject to some conditions then it will not be excluded in that way that we can compare it to a varied motion that does not satisfy those conditions.

§ 2. – Derivation of the integral principles.

I shall now consider a material system that moves under the influence of forces and the simultaneous constraint of condition equations in the sense of ordinary mechanics. Time can once more enter into the condition equations. It will suffice to assume that the coordinates are rectangular. Now, when I vary the motion, I shall temporarily not concern myself with the condition equations at all. If m_1, m_2, … are the masses of material points then that will imply the variation of the *vis viva*:

(3) $$\delta T = \sum_{(\nu)} m_\nu \left(\frac{dx_\nu}{dt}\,\delta\frac{dx_\nu}{dt} + \frac{dy_\nu}{dt}\,\delta\frac{dy_\nu}{dt} + \frac{dz_\nu}{dt}\,\delta\frac{dz_\nu}{dt} \right).$$

Now, one has, e.g. [1]:

$$\delta \frac{dx_\nu}{dt} = \frac{\delta dx_\nu \cdot dt - \delta dt \cdot dx_\nu}{dt^2} = \frac{d\delta x_\nu \cdot dt - d\delta t \cdot dx_\nu}{dt^2} .$$

If one now converts the right-hand side of (3) with the help of that equation and its analogous equations then one will find that:

$$\delta T = \sum_{(\nu)} m_\nu \left(\frac{dx_\nu}{dt}\frac{d\delta x_\nu}{dt} + \frac{dy_\nu}{dt}\frac{d\delta y_\nu}{dt} + \frac{dz_\nu}{dt}\frac{d\delta z_\nu}{dt} \right) - 2T \frac{d\delta t}{dt} .$$

This equation should be multiplied by dt and integrated over the time interval $t_0 \dots t_1$ in which the original motion takes place. After a partial integration, one will get [2]:

$$\text{(4)} \qquad \int_{t_0}^{t_1} \delta T \cdot dt = - \int_{t_0}^{t_1} \sum_{(\nu)} m_\nu \left(\frac{dx_\nu}{dt}\frac{d\delta x_\nu}{dt} + \frac{dy_\nu}{dt}\frac{d\delta y_\nu}{dt} + \frac{dz_\nu}{dt}\frac{d\delta z_\nu}{dt} \right) dt - 2\int_{t_0}^{t_1} T\, d\delta t .$$

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(1) This formula from the calculus of variations must be used here, since the quantities that are based upon differentiation will be varied. If one would like to avoid that then one would have to introduce yet another variable ϑ, as e.g., **v. Helmholtz** did. One then relates the positions of the original motion to the values of the parameter ϑ and associates the corresponding positions in the varied motion to the same values of ϑ. ϑ will not be varied in that way, but possibly the time t. The following picture is especially intuitive: Let τ be the time that it takes for the initial position A of the system to flow to the position C of the original motion, and let $\tau + \delta\tau$ be the time that elapses between the initial position and the corresponding position C' of the varied motion. All quantities, including $\delta\tau$, can be regarded as functions of τ. Now, $\delta\,(dx_\nu / dt)$ is the difference between the velocity components, taken along the x-axis, of the mass m_ν for the varied and unvaried motion. Obviously, one will then have:

$$\delta \frac{dx_\nu}{dt} = \frac{d(x_\nu + \delta x_\nu)}{d(\tau + \delta\tau)} - \frac{dx_\nu}{dt} = \frac{\dfrac{d}{d\tau}(x_\nu + \delta x_\nu)}{\dfrac{d}{d\tau}(\tau + \delta\tau)} - \frac{dx_\nu}{dt}$$

$$= \left(\frac{dx_\nu}{d\tau} + \frac{d\delta x_\nu}{d\tau} \right)\left(1 + \frac{d\delta\tau}{d\tau} \right) - \frac{dx_\nu}{dt} .$$

If one now develops this and neglects terms of higher order in the derivatives of the variations then one will get:

$$\frac{d\delta x_\nu}{d\tau} - \frac{dx_\nu}{dt}\frac{d\delta\tau}{d\tau} ;$$

viz., the formula in the text. At the same time, one will see that not only must the variations be assumed to be small, but also their derivatives.

(2) This formula is basically already in **Serret**, Comptes rendus de l'Acad. des Sciences **72** (1871), pp. 700, no. (7).

The position of the system is thought of as being unvaried for t_0 and t_1, which will make the terms that appear before the integral as a result of the partial integration vanish.

Now, if X_ν, Y_ν, Z_ν are the components of the force that acts upon the mass m_ν then the symbol $\delta' U$ shall be defined by the formula:

$$\delta' U = \sum_{(\nu)} (X_\nu \delta x_\nu + Y_\nu \delta y_\nu + Z_\nu \delta z_\nu) . \tag{5}$$

Equation (5) is once more multiplied by dt and integrated and then added to (4), which will yield:

$$\int_{t_0}^{t_1} [2T\, d\delta t + (\delta T + \delta' U)\, dt] \tag{6}$$

$$= \int_{t_0}^{t_1} dt \sum_{(\nu)} \left\{ \left(X_\nu - m_\nu \frac{d^2 x_\nu}{dt^2} \right) \delta x_\nu + \left(Y_\nu - m_\nu \frac{d^2 y_\nu}{dt^2} \right) \delta y_\nu + \left(Z_\nu - m_\nu \frac{d^2 z_\nu}{dt^2} \right) \delta z_\nu \right\} .$$

At the same time, if one performs that variation of the motion such that the quantities δx_ν, δy_ν, δz_ν, represent a *virtual* displacement of the system then the right-hand side of the last equation will be equal to zero, from **d'Alembert**'s principle. One then has the theorem:

If one compares the actual motion of a material system with a motion that deviates from it slightly and for which the starting and ending positions of the system remain unvaried, and the displacements of each position of the actual motion to the corresponding positions of the varied motion are virtual displacements then ([1]):

$$\int [2T\, d\delta t + (\delta T + \delta' U)\, dt] = 0. \tag{7}$$

In this equation, T means the vis viva and $\delta' U$ means the work that is done by the effective forces under the aforementioned displacement, which is merely imagined.

Here, one can specialize the variations by introducing the first or second of the special kinds of variation that were presented in § **1**.

1) We demand that the corresponding positions in the actual and varied motions must be passed through at the same time (i.e., we set $\delta t = 0$) and obtain:

$$\int (\delta T + \delta' U)\, dt = 0.$$

That is **Hamilton**'s principle.

([1]) Actually, the integral will only be infinitely-small of higher order when the quantities that were referred to as small up to now are made infinitely-small of order one.

2) We generalize the previous second kind of variation by setting:

$$\delta T = \delta' U. \tag{8}$$

We then require that the difference between the *vis viva* for corresponding states of the two motions should be equal to the work that the effective forces do a displacement that connects corresponding positions. That will determine how the continuous succession of varied positions should be traversed. One can then replace the quantity $\delta' U$ in (7) with δT and then get:

$$0 = \int (T\, d\delta t + \delta T\, dt) = \int (T\, \delta dt + \delta T\, dt) = \int \delta(T\, dt)$$

for those special variations, i.e. ([1]):

$$0 = \delta \int T\, dt\,.$$

That is *the principle of least action in its extended form* ([2]).

The other form of that principle will be discussed in § **4**.

§ 3. – Virtual displacements. Equivalence of the principles.

The concept of virtual displacement is to be understood here in precisely the same way as in the analytical formulation of **d'Alembert**'s principle. According to that principle, when one takes into account the couplings between the material points that exist at the moment, the external forces will be in equilibrium at each time. For example, if the material points are constrained to move in accordance with the conditions:

$$\omega_\iota (x_1, y_1, z_1, \ldots, x_r, y_r, z_r ; t) = 0 \qquad (i = 1, 2, \ldots) \tag{9}$$

([1]) The validity of the formula $\int \delta (T\, dt) = \delta \int T\, dt$ will not be impaired by the fact that the time interval changes under the variation. In order to see that, one subdivides $\int T\, dt$ into its elements and subtracts from each of those elements the quantity that corresponds to it in the varied motion.

([2]) **v. Helmholtz** discussed this form of the principle in detail in the Sitzungsberichten der Berliner Akademie for 1887 (**Helmholtz**'s *Ges. Abhandlungen*, 1895, Bd. III, pp. 249). It would probably be better to refer to the quantity F that he referred to as "potential energy" as the "negative force function." Namely, since F must include time, as well as the coordinates, the equation that motivates the term "potential energy" would break down. One can object to the presentation that **Helmholtz** gave in other ways. If one compares the equations on pp. 259 that were denoted by 1_f and 1_g then that will show that the term $(\partial F / \partial t)\, \delta t$ in the development of the variation δF will drop out and that it is only in that way that the equations of motion will be obtained correctly. **Helmholtz** based his procedure on the remark that F can also be regarded as a function of the coordinates and of ϑ, instead of t (cf., by first rem. in this paragraph). Now, ϑ will not be varied; however, it is expressly assumed that time will be varied. t will then be a different function of ϑ for the varied motion that it was for the original motion. For that reason, the process is not permissible. However, the entire presentation will become correct when one regards the quantity that **Helmholtz** referred to as δF as the work. Correspondingly, the variational condition must also be formulated differently then.

then one must introduce the momentary value for *t*. Those momentary couplings allow one to have displacements that satisfy the equations:

$$\frac{\partial\omega_\iota}{\partial x_1}\delta x_1+\frac{\partial\omega_\iota}{\partial y_1}\delta y_1+\frac{\partial\omega_\iota}{\partial z_1}\delta z_1+\cdots+\frac{\partial\omega_\iota}{\partial x_r}\delta x_r+\frac{\partial\omega_\iota}{\partial y_r}\delta y_r+\frac{\partial\omega_\iota}{\partial z_r}\delta z_r=0. \tag{10}$$

Those displacements are virtual, and they are introduced into the equilibrium condition for the external forces, i.e., into:

$$\sum_{(\iota)}\left\{\left(X_\nu-m_\iota\frac{d^2x_\nu}{dt^2}\right)\delta x_\nu+\left(Y_\nu-m_\iota\frac{d^2y_\nu}{dt^2}\right)\delta y_\nu+\left(Z_\nu-m_\iota\frac{d^2z_\nu}{dt^2}\right)\delta z_\nu\right\}=0.$$

The fact that there is no term of the form $(\partial\omega_\iota / \partial t)\ \delta t$ in equation (10) will also be paraphrased by the remark that time is not varied in the application of **d'Alembert**'s principle. We will satisfy that prescription when a variation of time takes place in a different way. Taking into account what we must logically denote by $\delta\omega_\iota$ here, the equations that determine the virtual displacements:

$$\delta\omega_\iota-\frac{\partial\omega_\iota}{\partial t}\,\delta t=0. \tag{11}$$

If the motion of the system is subject to condition equations of the form (1):

$$\sum_{(\nu)}(\varphi_{\iota\nu}\,dx_\nu+\psi_{\iota\nu}\,dy_\nu+\chi_{\iota\nu}\,dz_\nu)=0 \qquad (i=1,2,\ldots),$$

in which the functions φ, ψ, χ depend upon only the coordinates, then the virtual displacements will satisfy the relations ([1]):

$$\sum_{(\nu)}(\varphi_{\iota\nu}\,\delta x_\nu+\psi_{\iota\nu}\,\delta y_\nu+\chi_{\iota\nu}\,\delta z_\nu)=0 \qquad (i=1,2,\ldots).$$

It should be remarked that in all cases the displacement of the individual positions of the system are independent of each other. For that reason, the displacements can also be

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([1]) The analogy leads to the suggestion that **Voss** made (*loc. cit.*, pp. 286) that one should take the condition on the motion in the form:

$$\sum_{(\nu)}(\varphi_{\iota\nu}\,dx_\nu+\psi_{\iota\nu}\,dy_\nu+\chi_{\iota\nu}\,dz_\nu)=0 \qquad (i=1,2,\ldots),$$

in which the functions φ, ψ, χ, ω include the coordinates and time. That will yield the equations of the virtual displacements when one replaces dt by 0 and dx_ν, dy_ν, dz_ν with δx_ν, δy_ν, δz_ν. A suitable example of that would be that of a ball that rolls without slipping on a plane while the plane advances in a prescribed manner in time.

assumed to be non-zero only for an infinitely-small part of the motion. If one links that concept with equation (6) then a known argument from the calculus of variations will imply that the continual vanishing of the left-hand side of (6) will also bring with it the vanishing of each individual element in the integral that one finds on the right-hand side. The demand that the integral (7) should vanish for all variations will once more imply the fulfillment of **d'Alembert**'s principle. Let us consider the right-hand side of (6) in more detail. We think of the forces and actual motion of the material system as being given. Hence, the aforementioned right-hand side is determined merely by the displacements of the system positions. It does not depend upon how the new succession of positions in time that arises from the displacements is run through. For that reason, it does not matter whether we let the variation of the motion be general, except for the unallowable conditions, or we restrict ourselves to the first or second of the special kinds of variations.

That and the contents of the previous paragraphs will then imply that **Hamilton**'s principle, as well as the principle of least action in the form above, will be equivalent to **d'Alembert**'s principle ([1]).

§ 4. – Modifications of the principles.

If a force function U exists then equation (5) will assume the form:

$$\delta' U = \sum_{(\iota)} \left(\frac{\partial U}{\partial x_\nu} \delta x_\nu + \frac{\partial U}{\partial y_\nu} \delta y_\nu + \frac{\partial U}{\partial z_\nu} \delta z_\nu \right).$$

Now, when U includes the time t, along with the coordinate, in the event that time is not varied, one will then have:

$$\delta' U = \delta U, \tag{12}$$

and one can express **Hamilton**'s principle by means of the equation:

$$\delta \int (T + U)\, dt = 0.$$

([1]) The derivation of the differential equations of motion is also given by **Hamilton**'s principle and the principle of least action then, while new coordinates can likewise be introduced. It was given in symbolic form by **Voss** (*loc. cit.*, pp. 263), while our concept of the variation of a motion will always permit an actual application of the principle. The derivation of the equations from the least-action principle has led to various discussions (cf., **Rodrigues**, Correspondance sur l'école impériale polytechnique publ. par Hachette, vol. III, pp. 159, and **A. Mayer**, Ber. d. K. Sächs. Ges. d. W. math.-phys. Cl. 1886, pp. 343). It is simplest to follow our path above backwards. One develops $\delta \int T\, dt = \int (\delta T \cdot dt + T \cdot \delta dt)$ for arbitrary coordinates, and with the help of equation (8), one then replaces the quantity δdt that enters here explicitly and implicitly with an expression that is multiplied by dt and includes variations of position and their derivatives. One then removes those derivatives by partial integration. An integral will arise that is analogous to the right-hand side of (6). When that integral is set to zero, while observing that the virtual displacements are independent of the individual system positions, that will yield the differential equations of motion.

If one is dealing with the variations that the least-action principle demands then one must have a force function U that is free of time if equation (12) is to be true. The variational condition (8) can then be expressed in such a way that the quantity $T - U$ should have the same value for two corresponding states of the actual and varied motion. Now, if time does not enter into the condition equations, in addition, which might then be differential equations of the form (1) or ordinary equations, then $T - U$ would be constant for the actual motion ([1]). One then calls $-U$ the potential energy and $T - U$, the total energy, and it is clear that the total energy will not change at all during either the motion or the variation. In that way, one will get the following restricted form of the least-action principle:

That form of the principle assumes that the actual motion obeys the law of the constancy of energy, and that motion will determine more precisely the fact that when one compares it with a motion that deviates slightly from it and has the same constant energy motion, it will fulfill the condition:

$$\delta \int T\, dt = 0.$$

In that way, the variations of positions will be virtual displacements, and the initial and final positions will remain unvaried. That more restrictive form is applicable when a force-function that is independent of time exists, and time does not appear in the condition equations, either ([2]).

([1]) Cf., § **5**. Cf., also **Voss**, *loc. cit.*, pp. 266.

([2]) Insofar as one can already speak of a definite principle with **Maupertuis**, and which was shared by the ideas of others – in particular, **Euler** – I would not like to examine it here (cf., **A. Mayer**, *Geschichte des Princips der kleinsten Action*, 1877, and **Helmholtz**, *loc. cit.*). One finds a derivation of the least-action principle by **Lagrange** in his *Mécanique analytique* (2nd ed., 1811, t. I, pp. 296, *et seq.*). He assumed that the equation that expresses the law of conservation of energy will continue to be true under the variation, with no change in the constants that enter into it. In that way, he obtained a relation that corresponded to equation (8) in the present article. One must conclude from the stated assumption that **Lagrange** had the restricted form of the principle in mind at the cited place. However, if one introduces equation (8) directly as a variational condition then **Lagrange**'s method of proof will remain completely unchanged, and if one asks what the narrowest assumptions would be under which it is still true then one will be led to a broader form of the principle. One will also find that form suggested in his earlier work in the *Miscellanea Taurinensia*, t. II, 1760-1761 (*Oeuvres*, 1867, t. 1, pp. 365, *et seq.*). Namely, it was stated in no. XIII that the relation (U) that is presented in no. VIII, which is nothing but our equation (8), can be employed in the case of completely arbitrary forces. Most of **Lagrange**'s followers have taken only the more restricted form of the principle, such as, e.g., **Hamilton** in the Philosophical Transactions of 1834, pp. 253. **Jacobi** has granted that form of the principle with another expression, in which he expressed the quantity $R\, dt$ under the integral in terms of space-elements and a constant that is nothing but the value of $T - U$, which is constant and unvaried here. (*Vorlesungen über Dynamik*, 1866, 6th Lecture) **Helmholtz**, in the cited work, was the first to distill the broader form of the principle from **Lagrange**'s works. As far as the relationship between that form of the least-action principle and **Hamilton**'s principle (Philosophical Transactions, 1835, pp. 99) is concerned, in contrast to **Helmholtz**, I find that both of them can be obtained from each other rigorously. Since both of them are equivalent to **d'Alembert**'s principle, they will also be consequences of each other. Nevertheless, neither of the two principles is subordinate to the other one since they relate to different kinds of variations. However, both principles are implied when one specializes the integral principle that was contained in equation (7) in this article, in which, the variations of the motion were more general. Integral (7) has a close relationship to **Helmholtz**'s integral formula 2_b :

§ 5. – The broader form of the principle of least action and the law of conservation of energy.

The restricted form of the last-action principle assumes the law of the constancy of energy, but not the broader form. We can also derive the law of constancy of energy from the broader form of the principle when we assume that there is a force function that is free of time, along with the condition equations. We assume that the law is unknown, and we imagine that, e.g., the quantity $T - U$ continually increases, in the algebraic sense, during a time interval (t', t'') of the actual motion. Now, the positions that are taken when $t \leq t'$ and $t \geq t''$ might not be displaced. Every position C that is taken when $t' < t < t''$ will be varied into one C' that will itself be taken in the actual motion, but at a later time-point that lies between t' and t''. For that reason, since time does not appear in the condition equations, those displacements will be virtual displacements ([1]). From the variational conditions, in the form that they can now be expressed, where a force function that is free of time exists, $T - U$ must have the value for the position C of the varied motion that it had for the position C' of the actual motion. Now, we have imagined that $T - U$ increases from C to C' under the actual motion. $T - U$, and therefore the *vis viva* T, as well, must be smaller for the varied motion than it was for the actual one. We can perhaps assume that the ratio of these two *vis vivas* is like $\varepsilon^2 : 1$, where $\varepsilon < 1$. Now, under the transition to the system position C', all velocities of the varied motion will have the ratio $\varepsilon : 1$ with the ones that occur for the actual motion, since the system paths of both motions coincide. One now compares two small intervals of the varied and actual motion, and indeed intervals for which position are traversed that differ only slightly from C', and which are not regarded as associated under the variation then. The elapsed times would then relate to each other like $1 : \varepsilon$, while the partial integrals will relate like $\varepsilon : 1$. Hence, the partial integral $\int T\,dt$ that extends from t' to t'' and represents the "action" will be reduced by the chosen variation. A more precise consideration will show that the other part of the integral will not vary. That will imply a reduction in the total integral, and it can be shown that this reduction will generally have the same order as the variations of the coordinates and the quantity $1 - \varepsilon$. If the position of the system were displaced in the opposite sense then that would yield an increase in $\int T\,dt$ as a result of the variation. One would then think of the quantity $T - U$ as non-increasing for the actual motion, if one would not like to contradict the principle of least action; naturally, the same thing would be true if it were decreasing. $T - U$ is constant.

$$\delta \int \left[\lambda F + \left(1 + \frac{dt}{d\vartheta}\right) L\right] d\vartheta \,.$$

Namely, if one carried out the variation under the integral here, in which one leaves λ unvaried, varies dt, and (see the last rem. in § **2**) replaces δF with a form of work (viz., the $-\delta' U$ in this article) then an integral will come about that will coincide with one-half of our integral (7) for $L = T$, $\lambda = -1/2$, $\vartheta = t$. In order to be able to set $\vartheta = t$ *after the variation*, one must only regard ϑ as the time that is required to reach a certain position under the actual motion.

([1]) Cf., **G. Kirchhoff**'s *Mechanik*, pps. 25 and 34. The relationship between virtual and actual displacements that **Hertz** expressed in no. 111 is based upon the fact that he did not include time in his condition equations.

§ 6. – Inequivalence of the true and varied motions.

Everywhere, we have observed the condition that the variations of the positions must be virtual displacements. Something else would happen if we were to demand that the varied motion should satisfy the same condition equations as the actual one. For example, if the condition equations are given in the form (9) – i.e., as ordinary equations:

$$\omega_\iota (x_1, y_1, z_1, \ldots, x_r, y_r, z_r, t) = 0 \qquad (i = 1, 2, \ldots),$$

then the last demand would imply that:

$$\omega_\iota (x_1 + \delta x_1, , \ldots, z_r + \delta z_r, t + \delta t) = 0,$$

and thus, that one would also have $\delta\omega_\iota = 0$. However, an application of mechanical principles would call for equations (11):

$$\delta\omega_\iota - \frac{\partial \omega_\iota}{\partial t}\,\delta t = 0 \qquad (i = 1, 2, \ldots).$$

Indeed, those equations will agree with $\delta\omega_\iota = 0$ when:

$$\frac{\partial \omega_\iota}{\partial t} = 0,$$

i.e., when time does not enter into (9), and likewise when $\delta t = 0$, i.e., when **Hamilton**'s principle should be applied. By contrast, for the principle of least action, one should observe the aforementioned difference when time enters into the condition equations (9). The actual and varied motions are not equivalent in this case.

That inequivalence will also appear with **Hamilton**'s principle ([1]) when the condition equations are given as differential equations in the form (1) into which time does not enter. That will be illuminated by the example in the next paragraph. Here, it will only be stressed that the inequivalence of the motions will vanish again when one treats **Hertz**'s holonomic material systems. The conditions in that case can be assumed to also have the form (2):

$$d\Phi_\iota = 0 \qquad (i = 1, 2, \ldots).$$

That says that $\Phi_1, \Phi_2, \ldots$ should remain constant under the motion without those values having to be prescribed initially. Now, should the varied motion satisfy the same conditions, one could choose other constant values $\Phi_1, \Phi_2, \ldots$ for that motion, *per se.*

([1]) **C. Neumann** had already emphasized that fact for rolling motion, which I first noted during its publication. Compare Ber. d. Sächs. Ges. d. W. math.-phys. Cl., 1888, pp. 34, and especially the words: "By contrast, the fictitious motion will correspond to the character of the system in general."

However, that is excluded by the fact that one does not vary the initial and final positions. Now, one sees that the result would be the same if the variation were performed according to the equations:

$$\delta\Phi_\iota = 0 \qquad (i = 1, 2, \ldots).$$

However, the latter equations arise from the condition of motion when one replaces the coordinate differentials with coordinate variations. Those equations will then correspond here to the true demand that the variations of position must be virtual displacements. Now, that explains why the conception of the principles of **Maupertuis** and **Hamilton** that **Hertz** chose brought with it the restriction to holonomic systems. Namely, **Hertz** assumed that the varied was possible – i.e., as one that satisfied the same conditions as the actual path ([1]).

§ 7. – Special conditions of motion for a point.

It would be appropriate to explain the foregoing by way of an example. Since only the difficulties in the variation should be discussed here, it would seem permissible for me to choose a simple (although probably not realizable) motion. Furthermore, it belongs to the ones that **Hertz** allowed ([2]). The motion of a material point upon which no force acts shall be constrained by the condition equation:

$$\varphi(x, y, z)\, dx + \psi(x, y, z)\, dy + \chi(x, y, z)\, dz = 0. \tag{13}$$

The point will then be forced to move along a given surface element at each location. The direction cosines of the surface element that belongs to x, y, z will have the ratios:

$$\varphi(x, y, z) : \psi(x, y, z) : \chi(x, y, z).$$

Equation (13) can be integrated only in special cases in the form:

$$\omega(x, y, z) = \text{const.}$$

In those cases, we call equation (13) integrable. A function $\Omega(x, y, z)$ will then exist such that when it multiplies the left-hand side of the equation, it will go to a total differential. In order for that to happen, Ω must satisfy the conditions:

$$\frac{\partial(\Omega\cdot\varphi)}{\partial y} = \frac{\partial(\Omega\cdot\psi)}{\partial x}, \quad \frac{\partial(\Omega\cdot\psi)}{\partial z} = \frac{\partial(\Omega\cdot\chi)}{\partial y}, \quad \frac{\partial(\Omega\cdot\chi)}{\partial x} = \frac{\partial(\Omega\cdot\varphi)}{\partial z},$$

([1]) Cf., nos. 347, 358, 110, 112, 113.

([2]) It was treated already by **Voss** (*loc. cit.*, pp. 280).

which can be put into the forms:

$$\begin{aligned}
\Omega\,(\varphi_2 - \psi_1) &= \Omega_1\,\psi - \Omega_2\,\varphi, \\
\Omega\,(\psi_3 - \chi_2) &= \Omega_2\,\chi - \Omega_3\,\psi, \\
\Omega\,(\chi_1 - \varphi_3) &= \Omega_3\,\varphi - \Omega_1\,\chi,
\end{aligned}$$

in which the partial derivatives with respect to x, y, z are denoted by 1, 2, 3, resp. If one multiplies these equations by χ, φ, ψ and adds them then that will give:

$$\chi\,(\varphi_2 - \psi_1) + \varphi\,(\psi_3 - \chi_2) + \psi\,(\chi_1 - \varphi_3) = 0. \tag{14}$$

That is the integrability condition, which is not always fulfilled ([1]). It only when it is fulfilled that the material point subjected to the above condition will represent a holonomic system.

§ 8. – Varying the path.

The variation of a motion basically involves only the paths. Let us consider a path that obeys equation (13). The application of mechanical principles requires variations of position that are virtual displacements, i.e., they correspond to the equation:

$$\varphi\,\delta x + \psi\,\delta y + \chi\,\delta z = 0. \tag{15}$$

Since that is true for the displacements of all places along the original path, one will also have:

$$d\,(\varphi\,\delta x + \psi\,\delta y + \chi\,\delta z) = 0. \tag{16}$$

By contrast, if one would like to vary in such a way that the varied path satisfies the same condition as the original one then equation (13) must be true for two small corresponding pieces of both paths. Subtracting the two equations thus-obtained will yield:

$$\delta\,(\varphi\,dx + \psi\,dy + \chi\,dz) = 0. \tag{16}$$

The behavior of the two requirements that were posed here for the variation will become clearer when we look for the variations that fulfill both requirements. If equations (16) and (17) were developed and then subtracted from each other then the result would be the equation:

$$(\varphi_2 - \psi_1)(\delta x\,dy - \delta y\,dx) + (\psi_2 - \chi_1)(\delta y\,dz - \delta z\,dy) + (\chi_1 - \varphi_3)(\delta z\,dx - \delta x\,dz) = 0. \tag{18}$$

([1]) The integrability condition that we found is also sufficient (cf., **A. Mayer**, Math. Ann., Bd. 5, pp. 450 to 452 and *Theorie der Transformationsgruppen* by **Lie**, with the collaboration of **F. Engel**, 1888, first Sections, pp. 90 to 93).

Equation (13), together with the relation (13) that exists for the original path, gives the proportion:

$$(\delta x\, dy - \delta y\, dx) : (\delta y\, dz - \delta z\, dy) : (\delta z\, dx - \delta x\, dz) = \chi : \varphi : \psi.$$

However, that proportion is compatible with (18) only when either the integrability condition (14) is fulfilled or:

(19) $$\delta x : \delta y : \delta z = dx : dy : dz.$$

The latter case represents an entirely special variation, namely, a variation of the path; such a variation corresponds to the one that applied in § **5**. However, equation (15) admits a more general type of solution. Likewise, equation (17) – i.e., ([1]):

$$\left(\frac{\partial\varphi}{\partial x}\delta x + \frac{\partial\varphi}{\partial y}\delta y + \frac{\partial\varphi}{\partial z}\delta z\right)dx + \left(\frac{\partial\psi}{\partial x}\delta x + \frac{\partial\psi}{\partial y}\delta y + \frac{\partial\psi}{\partial z}\delta z\right)dy$$

$$+ \left(\frac{\partial\chi}{\partial x}\delta x + \frac{\partial\chi}{\partial y}\delta y + \frac{\partial\chi}{\partial z}\delta z\right)dx + \varphi\, d\,\delta x + \psi\, d\,\delta y + \chi\, d\,\delta z = 0$$

can be satisfied by variations that vanish at the end points of the path and do not satisfy the proportion (19) along the path.

If the integrability condition is fulfilled then the two requirements will lead to different type of variations. One can illustrate those variations roughly as follows: From (13), a surface element will belong to each point of the original path. Those planes will envelop a developable surface α. The varied path will always run parallel to the original one, so the two will collectively define a narrow ribbon. Now, under the variations that correspond to mechanical principles, the segment that consists of the components δx, δy, δz will lie in the surface element that belongs to the point x, y, z, and thus the surface α, as well; that is not the case for the other variations. For that reason, one can regard the ribbon

([1]) If one initially assumes that the variations are finite and that the varied path should satisfy the same condition (13) as the original one then that would actually imply the validity of the equation:

$$\varphi(x + \delta x, y + \delta y, z + \delta z)\,\frac{d}{d\sigma}\,(x + \delta x) + \psi(x + \delta x, y + \delta y, z + \delta z)\,\frac{d}{d\sigma}\,(y + \delta y)$$

$$+ \chi(x + \delta x, y + \delta y, z + \delta z)\,\frac{d}{d\sigma}\,(z + \delta z) = 0,$$

in which σ is any variable that one can make depend upon a variable point along the original path. The equation in the text will emerge from this equation after one subtracts:

$$\varphi(x, y, z)\frac{d}{d\sigma} + \psi(x, y, z)\frac{d}{d\sigma} + \chi(x, y, z)\frac{d}{d\sigma} = 0$$

and neglecting certain terms. One must then regard the variations and their derivatives as small quantities of first order and omit terms of higher order.

approximately as something that is cut from α in the former case, while in the latter case, it will generally make a finite angle with the developable surface α along the original path.

§ 9. – Equations of motion. True and geodetic paths.

We now define the differential equations for the motion of the material point. We would like to apply the principle of least work in its restricted form. If s means the arc length of the path then, from the conservation of energy, the velocity:

$$\frac{ds}{dt} = c \tag{20}$$

will be constant for the actual motion. One must think of the varied motion as having the same constant velocity. Now, that principle will imply that:

$$\frac{2}{mc}\delta\int T\,dt = \delta\int c\,dt = \delta\int ds = 0.$$

Upon developing that, one will find:

$$\begin{aligned}\delta\int ds = \int \delta\,ds &= \int \frac{dx\,\delta\,dx + dy\,\delta\,dy + dz\,\delta\,dz}{ds} \\ &= \int\left(\frac{dx}{ds}d\,\delta x + \frac{dy}{ds}d\,\delta y + \frac{dz}{ds}d\,\delta z\right) = 0\,.\end{aligned}$$

One separates the three terms in the last integral and partially integrates them. Afterward, due to the vanishing of the initial and final variations, one will get:

$$\int\left(-\frac{d^2x}{ds^2}\delta x + \frac{d^2y}{ds^2}\delta y + \frac{d^2z}{ds^2}\delta z\right)ds = 0\,. \tag{21}$$

The variations are determined in such a way that δx, δy, δz represent virtual displacements, i.e., they satisfy the equation:

$$\varphi\,\delta x + \psi\,\delta y + \chi\,\delta z = 0\,.$$

The left-hand side of that equation is multiplied by $\lambda \cdot ds$ and then added under the last integral. In that way, one first gets:

$$\int\left\{\left(\lambda\varphi - \frac{d^2x}{ds^2}\right)\delta x + \left(\lambda\psi - \frac{d^2y}{ds^2}\right)\delta y + \left(\lambda\chi - \frac{d^2z}{ds^2}\right)\delta z\right\}ds = 0\,,$$

and from that:

$$(22)\qquad \begin{cases} \dfrac{d^2x}{ds^2} = \lambda\varphi, \\ \dfrac{d^2y}{ds^2} = \lambda\psi, \\ \dfrac{d^2z}{ds^2} = \lambda\chi. \end{cases}$$

Since λ means an unknown variable here, the content of equations (22) consists of just the proportions:

$$(23)\qquad \frac{d^2x}{ds^2} : \frac{d^2y}{ds^2} : \frac{d^2z}{ds^2} = \varphi : \psi : \chi .$$

However, from a known theorem, the second differential quotients d^2x / ds^2, d^2y / ds^2, d^2z / ds^2 behave like the direction cosines of the normals to the path that lie in the osculating plane. That normal is then identical with the normal to the surface element that is assigned to the point x, y, z by way of (13). *Hence, the osculating plane at each point of the curve will be perpendicular to the surface element that is assigned to that point* ([1]).

Equations (20) and (22) correspond to the two statements in **Hertz**'s basic laws ([2]) that the differential equations (22), together with (13), determine what **Hertz** called the *straightest path* ([3]).

We have just now determined the actual path with the help of the equation:

$$(24)\qquad \delta\int ds = 0.$$

We now present the same equation, but with a different way of picturing the variation. We shall now no longer demand that variations of the positions should be virtual displacements, but we will demand that the varied path should satisfy the same differential equation (13) that we prescribed for the varied path. That will generally pose an entirely different problem in the calculus of variations from which the actual paths of material points will not generally emerge. In the problem, one must subject the variations to the condition (17); i.e., to the equation:

$$(25)\qquad \delta\varphi \cdot dx + \delta\psi \cdot dy + \delta\chi \cdot dz + \varphi \cdot d\delta x + \psi \cdot d\delta y + \chi \cdot d\delta z = 0.$$

([1]) Cf., **Voss**, pp. 280. If one treats the varied path as if it lived on the developable surface α of the previous paragraph then the condition $\delta\int ds = 0$ would yield the actual path in the form of the *geodetic line* on the surface α in the ordinary sense of the term. However, in that way, one would arrive directly at the geometric property that is expressed in the text.

([2]) No. **309**.

([3]) Cf., **Hertz**, no. **155**d. Since we have derived those paths from the least-action principle, it would not mean anything here to say that they are the "straightest" paths.

If one now develops (24) as before then that will yield (21) again. One must add λ times (25) under the integral in the latter equation ([1]). After one has partially integrated part of the terms under the integral, one will then get, in the known way:

$$\frac{d^2x}{ds^2} - \lambda\left(\frac{\partial\varphi}{\partial x}\frac{dx}{ds} + \frac{\partial\psi}{\partial x}\frac{dy}{ds} + \frac{\partial\chi}{\partial x}\frac{dz}{ds}\right) + \frac{d}{ds}(\lambda\varphi) = 0,$$

$$\frac{d^2y}{ds^2} - \lambda\left(\frac{\partial\varphi}{\partial y}\frac{dx}{ds} + \frac{\partial\psi}{\partial y}\frac{dy}{ds} + \frac{\partial\chi}{\partial y}\frac{dz}{ds}\right) + \frac{d}{ds}(\lambda\psi) = 0,$$

$$\frac{d^2z}{ds^2} - \lambda\left(\frac{\partial\varphi}{\partial z}\frac{dx}{ds} + \frac{\partial\psi}{\partial z}\frac{dy}{ds} + \frac{\partial\chi}{\partial z}\frac{dz}{ds}\right) + \frac{d}{ds}(\lambda\psi) = 0.$$

One can also give those equations the form:

$$(26)\qquad \left\{\begin{aligned}
&\frac{d^2x}{ds^2} + \varphi\frac{d\lambda}{ds} - \left(\frac{\partial\psi}{\partial x} - \frac{\partial\varphi}{\partial y}\right)\lambda\frac{dy}{ds} - \left(\frac{\partial\chi}{\partial x} - \frac{\partial\varphi}{\partial z}\right)\lambda\frac{dz}{ds} = 0,\\
&\frac{d^2y}{ds^2} + \psi\frac{d\lambda}{ds} - \left(\frac{\partial\chi}{\partial y} - \frac{\partial\psi}{\partial z}\right)\lambda\frac{dz}{ds} - \left(\frac{\partial\varphi}{\partial y} - \frac{\partial\psi}{\partial x}\right)\lambda\frac{dx}{ds} = 0,\\
&\frac{d^2z}{ds^2} + \chi\frac{d\lambda}{ds} - \left(\frac{\partial\varphi}{\partial z} - \frac{\partial\chi}{\partial x}\right)\lambda\frac{dx}{ds} - \left(\frac{\partial\psi}{\partial z} - \frac{\partial\chi}{\partial y}\right)\lambda\frac{dy}{ds} = 0.
\end{aligned}\right.$$

Together with (13), they determine the so-called *geodetic paths* ([2]).

Hertz showed that for his holonomic systems the geodetic paths coincided with the straightest ones – i.e., the actual ones ([3]). Namely, when the integrability condition is fulfilled, one can also think of equation (13) as being given in such a way that $\varphi\,dx + \psi\,dy + \chi\,dz$ will be a total differential. The following relations then will exist:

$$\frac{\partial\varphi}{\partial y} = \frac{\partial\psi}{\partial x}, \qquad \frac{\partial\varphi}{\partial z} = \frac{\partial\chi}{\partial x}, \qquad \frac{\partial\psi}{\partial z} = \frac{\partial\chi}{\partial y},$$

and if one recalls equations (26) then that means nothing but the validity of the proportion (23).

([1]) In regard to that rule from the calculus of variations, cf., **Scheeffer**, Math. Ann. **55** (1885), pp. 555 *et seq.* and **A Mayer**, Ber. d. Sächs. Ges. d. Wiss. (1885), (1895), and Math. Ann, **26** (1886).

([2]) **Hertz**, no. 181a. **Voss**, pp. 282.

([3]) No. 190.

§ 10. – Manifold of the true and geodetic paths.

The actual path of the material point is determined completely when the initial position and direction are given. That follows on mechanical grounds, but also allows one to verify the actual paths from its geometric properties. One can then still choose the initial direction in the surface element that is associated with the point *A* arbitrarily. A simple infinitude of actual paths will then emanate from a well-defined location.

The geodetic paths will behave differently when the integrability condition is not fulfilled. They will be determined by equations (26), to which one adds (13). When (13) is differentiated, that will give:

$$\varphi\frac{d^2x}{ds^2}+\psi\frac{d^2y}{ds^2}+\chi\frac{d^2z}{ds^2}+\left(\frac{\partial\varphi}{\partial x}\frac{dx}{ds}+\frac{\partial\varphi}{\partial y}\frac{dy}{ds}+\frac{\partial\varphi}{\partial z}\frac{dz}{ds}\right)$$

(27)

$$+\left(\frac{\partial\psi}{\partial x}\frac{dx}{ds}+\frac{\partial\psi}{\partial y}\frac{dy}{ds}+\frac{\partial\psi}{\partial z}\frac{dz}{ds}\right)\frac{dy}{ds}+\left(\frac{\partial\chi}{\partial x}\frac{dx}{ds}+\frac{\partial\chi}{\partial y}\frac{dy}{ds}+\frac{\partial\chi}{\partial z}\frac{dz}{ds}\right)\frac{dz}{ds}=0.$$

One can express $\frac{d^2x}{ds^2}$, $\frac{d^2y}{ds^2}$, $\frac{d^2z}{ds^2}$, $\frac{d\lambda}{ds}$ in terms of x, y, z, λ, $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$. The quantities x, y, z, λ can be determined as functions of s, with the help of the aforementioned equations when the initial values of x, y, z, $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$, λ are given for an initial value of s. On the other hand, if one performs the integration of (26) and (27) with any initial values whatsoever then due to equation (27) that will yield functions φ, ψ, χ that satisfy the condition:

$$\varphi\frac{dx}{ds}+\psi\frac{dy}{ds}+\chi\frac{dz}{ds}=C_1,$$

in which C_1 means a constant. Moreover, since equations (26), when multiplied by $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$, resp., and added, will have the consequence that:

$$\frac{d^2x}{ds^2}\frac{dx}{ds}+\frac{d^2y}{ds^2}\frac{dy}{ds}+\frac{d^2z}{ds^2}\frac{dz}{ds}+\left(\varphi\frac{dx}{ds}+\psi\frac{dy}{ds}+\chi\frac{dz}{ds}\right)=0,$$

the functions that one obtains must also fulfill the condition:

$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2+2C_1\,\lambda=C_2$$

then; C_2 is constant here. The initial values must now be chosen such that $C_1 = 0$ and $C_2 = 1$. In that way, it will be clear that as long as the initial position x, y, z is given, one can choose only the initial values of s, λ, and perhaps $\frac{dx}{ds} : \frac{dy}{ds}$ arbitrarily. Nothing can be said about the initial value of s, and one will thus introduce two constants into the geodetic path to be determined. However, that shows that the initial value of λ will influence the path only when the integrability condition (14) is not fulfilled ([1]). Therefore, if the quantity:

$$\varphi\,(\psi_3 - \chi_2) + \psi\,(\chi_1 - \varphi_3) + \chi\,(\varphi_2 - \psi_1) \tag{28}$$

at a location then a double infinitude of geodetic paths will emanate from the that place and only a simple infinitude of actual paths.

That result is analogous to the one that is found for the ball. For the ball, the motions that emanate from a given position and satisfy the minimum problem in the introduction will define a higher-dimensional manifold of those motions that the ball can perform starting from a given position without the action of forces.

§ 11. – Rolling motion of the ball. Condition equations.

We would now like to exhibit the differential equations for the rolling motion of the ball. Let ξ, η, ζ be the coordinates relative to a rectangular coordinate system that is fixed in space. The ball rolls without slipping on the fixed $\xi\eta$-plane. Let x, y, and z be coordinates that refer to a rectangular coordinate system that is invariably coupled with the ball. That system shall have its origin at the center of the ball. Now, if ξ, η, ζ and x, y, z are the coordinates of the same spatial point then the equations:

$$\begin{aligned}
\xi &= \alpha + \alpha_1\, x + \alpha_2\, y + \alpha_3\, z, \\
\eta &= \beta + \beta_1\, x + \beta_2\, y + \beta_3\, z, \\
\zeta &= \gamma + \gamma_1\, x + \gamma_2\, y + \gamma_3\, z,
\end{aligned}$$

and

$$x = \alpha_1\,(\xi - \alpha) + \beta_1\,(\eta - \beta) + \gamma_1\,(\zeta - \gamma),$$

([1]) When one takes (13) into account, along with the fact that s means the arc length, equations (26) will imply the relation:

$$\left(\psi\frac{d^2x}{ds^2} - \varphi\frac{d^2y}{ds^2}\right)\frac{dz}{ds} + \left(\chi\frac{d^2y}{ds^2} - \psi\frac{d^2z}{ds^2}\right)\frac{dx}{ds} + \left(\varphi\frac{d^2z}{ds^2} - \chi\frac{d^2x}{ds^2}\right)\frac{dy}{ds}$$
$$= [(\varphi_2 - \psi_1)\,\chi + (\psi_3 - \chi_2)\,\varphi + (\chi_1 - \varphi_3)\,\psi]\,\lambda.$$

If the expression (28) does not perhaps vanish for the entire path then λ will be a given function of position along a geodetic path λ. Two geodetic paths that emanate from the same location and belong to different initial values of λ will certainly be different then when the expression (28) is zero for the initial location, and therefore also for its neighborhood. However, if (28) vanishes for all values of x, y, z then the remark that was made at the conclusion of the last paragraph should be considered.

$$y = \alpha_2\,(\xi - \alpha) + \beta_2\,(\eta - \beta) + \gamma_2\,(\zeta - \gamma), \tag{29}$$
$$z = \alpha_3\,(\xi - \alpha) + \beta_3\,(\eta - \beta) + \gamma_3\,(\zeta - \gamma)\,.$$

In the first coordinate system, $\xi = \alpha$, $\eta = \beta$, $\zeta = \gamma$ are the coordinates of the center of the ball, and α, β, 0 are the coordinates of the point at which the ball contacts the $\xi\eta$-plane. γ is constant and equal to the radius a of the ball. The particle of the ball that is found at precisely the contact point must have the velocity 0 at that moment, since otherwise slipping would take place. Hence, the relations:

$$\frac{d\xi}{dt} = \frac{d\eta}{dt} = \frac{d\zeta}{dt} = 0$$

will be true for that particle at the moment of contact; i.e.:

$$\tag{30} \left\{ \begin{aligned} \frac{d\alpha}{dt} + \frac{d\alpha_1}{dt}x + \frac{d\alpha_2}{dt}y + \frac{d\alpha_3}{dt}z = 0, \\ \frac{d\beta}{dt} + \frac{d\beta_1}{dt}x + \frac{d\beta_2}{dt}y + \frac{d\beta_3}{dt}z = 0, \\ \frac{d\gamma}{dt} + \frac{d\gamma_1}{dt}x + \frac{d\gamma_2}{dt}y + \frac{d\gamma_3}{dt}z = 0. \end{aligned} \right.$$

Here, x, y, z mean those values that (29) will give when $\xi = \alpha$, $\eta = \beta$, $\zeta = 0$. One will then have to substitute:

$$x = -\gamma\gamma_1 = -a\gamma_1\,,$$
$$y = -\gamma\gamma_2 = -a\gamma_2\,,$$
$$z = -\gamma\gamma_3 = -a\gamma_3$$

in (30). In that way, one will get:

$$\tag{31} \left\{ \begin{aligned} \frac{d\alpha}{dt} = a\left(\gamma_1\frac{d\alpha_1}{dt} + \gamma_2\frac{d\alpha_2}{dt} + \gamma_3\frac{d\alpha_3}{dt}\right), \\ \frac{d\beta}{dt} = a\left(\gamma_1\frac{d\beta_1}{dt} + \gamma_2\frac{d\beta_2}{dt} + \gamma_3\frac{d\beta_3}{dt}\right), \\ \frac{d\gamma}{dt} = a\left(\gamma_1\frac{d\gamma_1}{dt} + \gamma_2\frac{d\gamma_2}{dt} + \gamma_3\frac{d\gamma_3}{dt}\right). \end{aligned} \right.$$

Of these equations, the last one is fulfilled by itself, since γ is constant, and the right-hand side will vanish by means of the relations of the orthogonal coordinate transformation. The first two of equations (31), together with $\gamma = a$, are the conditions for pure rolling then ([1]).

([1]) Cf., **Neumann**, Sächs. Ber. 1888, pp. 358.

§ 12. – Character of the condition equations.

In order to understand the character of the conditions, we express the coefficients of the coordinate transformation by the **Euler** formulas ([1]):

$$
(32)\qquad \begin{cases}
\alpha_1 = -\cos\varphi\cos f\cos\vartheta - \sin\varphi\sin f, \\
\beta_1 = -\sin\varphi\cos f\cos\vartheta + \cos\varphi\sin f, \\
\gamma_1 = \quad \cos f\sin\vartheta, \\
\alpha_2 = -\cos\varphi\sin f\cos\vartheta + \sin\varphi\cos f, \\
\beta_2 = -\sin\varphi\sin f\cos\vartheta - \cos\varphi\cos f, \\
\gamma_2 = \quad \sin f\sin\vartheta, \\
\alpha_3 = \quad \cos\varphi\sin\vartheta, \\
\beta_3 = \quad \sin\varphi\sin\vartheta, \\
\gamma_3 = \quad \cos\vartheta.
\end{cases}
$$

If those values were introduced into equations (31) then that would yield:

$$
(33)\qquad \begin{cases}
d\alpha = -\,a\sin\varphi\sin\vartheta\,df + a\cos\varphi\,d\vartheta, \\
d\beta = \quad a\cos\varphi\sin\vartheta\,df + a\sin\varphi\,d\vartheta.
\end{cases}
$$

Those equations are not completely integrable since they are not integrable to begin with ([2]). The ball that can roll on a plane but not slide on it will then represent a non-holonomic material system.

([1]) Novi Commentarii Acad. Petrop. **15** (1770), pp. 75. For the geometric meaning of the angles φ, f, ϑ, cf., e.g., **Kirchhoff**'s *Mechanik*, 1877, pp. 43 and 44.

([2]) That is, there is no function $\omega(\alpha, \beta, \varphi, f, \vartheta)$ whose differential vanishes because of equations (33) either. Namely, such a function would have to satisfy the partial differential equations [cf., **A. Mayer**, Math. Ann. **5** (1872), pp. 449 and **Lie** *Theorie der Transformationsgruppen*, Section I, pp. 91 and 92]:

$$
\begin{aligned}
&\frac{\partial\omega}{\partial\varphi} = 0, \\
&\frac{\partial\omega}{\partial f} - a\sin\varphi\sin\vartheta\frac{\partial\omega}{\partial\alpha} + a\cos\varphi\sin\vartheta\frac{\partial\omega}{\partial\beta} = 0, \\
&\frac{\partial\omega}{\partial\vartheta} + a\cos\varphi\frac{\partial\omega}{\partial\alpha} + a\sin\varphi\frac{\partial\omega}{\partial\beta} = 0.
\end{aligned}
$$

Those equations, which do not define a complete system (cf., **Clebsch**, Journal für reine und angewandte Math. **65**, pp. 258), can be extended to such a system and then show directly what one can also verify, namely, that they will be satisfied by only a constant.

The nonexistence of a function ω with the aforementioned property can be inferred from the fact that such a function would have to remain constant as a result of the differential equations (33). However, one can go from each system of values α_1, β_1, φ_1, f_1, ϑ_1 to another α_2, β_2, φ_2, f_2, ϑ_2 in which the ball can roll without slipping from any position to any other without harming equations (33) as a result of the transition. Since that fact was also employed by **Hertz**, it should be explained to some degree. If w means a constant angle

§ 13. – New form of the conditions.

The momentary state of motion of the ball will now be regarded as a combined rotation about an axis that goes through the center and a displacement. Let p, q, r be the components of the angular velocity, and let u, v, w be the displacement velocity, and both of them are replaced the x, y, z axes. Those components are given by the equations ([1]):

$$p = \alpha_3 \frac{d\alpha_2}{dt} + \beta_3 \frac{d\beta_2}{dt} + \gamma_3 \frac{d\gamma_2}{dt},$$

$$q = \alpha_1 \frac{d\alpha}{dt} + \beta_1 \frac{d\beta_3}{dt} + \gamma_1 \frac{d\gamma_3}{dt},$$

$$r = \alpha_2 \frac{d\alpha_1}{dt} + \beta_2 \frac{d\beta_1}{dt} + \gamma_2 \frac{d\gamma_1}{dt}$$

and

$$(34) \qquad \begin{cases} u = \alpha_1 \dfrac{d\alpha}{dt} + \beta_1 \dfrac{d\beta}{dt} + \gamma_1 \dfrac{d\gamma}{dt}, \\ v = \alpha_2 \dfrac{d\alpha}{dt} + \beta_2 \dfrac{d\beta}{dt} + \gamma_2 \dfrac{d\gamma}{dt}, \\ w = \alpha_3 \dfrac{d\alpha}{dt} + \beta_2 \dfrac{d\beta}{dt} + \gamma_3 \dfrac{d\gamma}{dt}. \end{cases}$$

Furthermore, the relations exist ([2]):

$$\frac{d\alpha_1}{dt} = \alpha_2\, r - \alpha_3\, q, \qquad \frac{d\alpha_2}{dt} = \alpha_3\, p - \alpha_1\, r, \qquad \frac{d\alpha_3}{dt} = \alpha_1\, q - \alpha_2\, p,$$

---

that is expressed in terms of arc length, and if ϑ is also assumed to be constant then equations (33) will be satisfied for:

$$\alpha = w \frac{a \sin \vartheta}{2\pi} \cos \varphi, \qquad \beta = w \frac{a \sin \vartheta}{2\pi} \sin \varphi, \qquad f = \frac{w}{2\pi} \varphi.$$

Here, φ can mean any function of time, and one will be dealing with a known motion. If φ runs through the interval from 0 to 2π then α, β and [cf., (32)] α_3, β_3, γ_3 will finally have the same values that they had to be begin with. One will obtain the same final position by a pure rolling that one would reach if one only rotated around the z-axis. Since f will then increase by w, w will be the magnitude of that rotation. The z-axis has no special direction. One can then replace every rotation around the center with just a rolling motion and produce the same final result. Now, in summary, it is clear that one can find a pure rolling transition from any initial position to any final position. Since that motion is composed of pieces, it would introduce discontinuities into the velocity that can, however, be eliminated by a small alteration.

([1]) Cf., e.g., **Kirchhoff**'s *Mechanik*, pp. 50.

([2]) *Ibidem*.

(35) $$\frac{d\beta_1}{dt} = \beta_2\, r - \beta_3\, q, \qquad \frac{d\beta_2}{dt} = \beta_3\, p - \beta_1\, r, \qquad \frac{d\beta_3}{dt} = \beta_1\, q - \beta_2\, p,$$

$$\frac{d\gamma_1}{dt} = \gamma_2\, r - \gamma_3\, q, \qquad \frac{d\gamma_2}{dt} = \gamma_3\, p - \gamma_1\, r, \qquad \frac{d\gamma_3}{dt} = \gamma_1\, q - \gamma_2\, p\, .$$

If one now replaces the quantities $\frac{d\gamma}{dt}$ in (34) with 0 and $\frac{d\alpha}{dt}, \frac{d\beta}{dt}$ with the right-hand sides of (31), but afterwards replaces the quantities:

$$\frac{d\alpha_1}{dt}, \frac{d\alpha_2}{dt}, \frac{d\alpha_3}{dt}, \qquad \frac{d\beta_1}{dt}, \frac{d\beta_2}{dt}, \frac{d\beta_3}{dt}$$

with the right-hand sides of (35) then, after one has employed the relations that pertain to orthogonal coordinate transformations, one will get:

(36) $$\begin{aligned} u &= a\,(\gamma_3\, q - \gamma_2\, r), \\ v &= a\,(\gamma_1\, r - \gamma_3\, p), \\ w &= a\,(\gamma_2\, p - \gamma_1\, q). \end{aligned}$$

When rolling without slipping takes place, those equations will represent the connection that exists between rotation and displacement ([1]). The components *p*, *q*, *r* of the angular velocity can be chosen arbitrarily.

§ 14. – Equations of motion.

With these preparations, the differential equations of motion ([2]) can be exhibited. I shall employ **Hamilton**'s principle. Since no forces are active, one must set:

$$\int \delta T \cdot dt = 0,$$

in which the type of variation must be observed. Since the system of coordinates *x, y, z* is thought of as fixed in the ball, the *vis viva T* will be a function of *p, q, r, u, v, w* that is given once and for all, and one will get:

(37) $$\int \left(\frac{\partial T}{\partial p}\delta p + \frac{\partial T}{\partial q}\delta q + \frac{\partial T}{\partial r}\delta r + \frac{\partial T}{\partial u}\delta u + \frac{\partial T}{\partial v}\delta v + \frac{\partial T}{\partial w}\delta w \right) dt = 0.$$

([1]) It is not difficult to derive those equations geometrically.

([2]) We could also employ **Neumann**'s general equations for rolling motion, Ber. d. Sächs. Ges. 1888, pps. 36 and 39.

The variations of the velocity components that enter here must be expressed. The variation of the motion will once more come about in such a way that initially each of the original positions that the ball can run through will take on a small displacement. The displacement will be decomposed into a rotation that takes place around the center and a parallel displacement. The rotation and displacement have the components p', q', r' and u', v', w' with respect to the x, y, z axes. Now, the variations of the velocity components are represented by the formulas ([1]):

$$\delta p = \frac{dp'}{dt} + qr' - q'r,$$

$$(38) \qquad \delta q = \frac{dq'}{dt} + rp' - r'p,$$

$$\delta r = \frac{dr'}{dt} + pq' - p'q,$$

and

$$\delta u = \frac{du'}{dt} + vr' - v'r + w'q - wq',$$

$$(39) \qquad \delta v = \frac{dv'}{dt} + wp' - w'p + u'r - ur',$$

$$\delta w = \frac{dw'}{dt} + uq' - u'q + v'p - vp'.$$

The derivation of these formulas is based upon a commutation of the symbols d / dt and δ ([2]). Such a commutation is allowed when time t is not varied; however, that is the type of variation that is required by **Hamilton**'s principle.

One now introduces the right-hand sides of (38) and (39) for δp, δq, δr, δu, δv, δw in (37). If one takes into account the fact that the variations should vanish for the beginning and end of the interval in question then one will get, after certain partial integrations:

$$(40) \qquad \int \left\{ \left(-\frac{d}{dt}\frac{\partial T}{\partial p} + r\frac{\partial T}{\partial q} - q\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial v} - v\frac{\partial T}{\partial w} \right) p' \right.$$

$$+ \left(-\frac{d}{dt}\frac{\partial T}{\partial q} + p\frac{\partial T}{\partial r} - r\frac{\partial T}{\partial p} + u\frac{\partial T}{\partial w} - w\frac{\partial T}{\partial u} \right) q'$$

([1]) **Kirchhoff**, pp. 58 and 59.
([2]) **Kirchhoff**, pp. 58.

$$+\left(-\frac{d}{dt}\frac{\partial T}{\partial r}+q\frac{\partial T}{\partial p}-p\frac{\partial T}{\partial q}+v\frac{\partial T}{\partial u}-u\frac{\partial T}{\partial v}\right)r'$$

$$+\left(-\frac{d}{dt}\frac{\partial T}{\partial u}+r\frac{\partial T}{\partial v}-q\frac{\partial T}{\partial w}\right)u'$$

$$+\left(-\frac{d}{dt}\frac{\partial T}{\partial v}+p\frac{\partial T}{\partial w}-r\frac{\partial T}{\partial u}\right)v'$$

$$\left.+\left(-\frac{d}{dt}\frac{\partial T}{\partial w}+q\frac{\partial T}{\partial u}-p\frac{\partial T}{\partial v}\right)w'\right\}dt=0.$$

Up to now, the condition that constrains the motion has not been used in this paragraph. Since the ball must roll without slipping, the displacements that correspond to the variations, which are virtual displacements, must also consist of a pure rolling. Each of those small displacements decomposes into a rotation and a displacement, and the components of such a rotation and displacement must be coupled by relations. Those relations are analogous to (36), and they are the following ones:

$$\begin{aligned} u' &= a\,(\gamma_3\,q' - \gamma_2\,r'), \\ v' &= a\,(\gamma_1\,r' - \gamma_3\,p'), \\ w' &= a\,(\gamma_2\,p' - \gamma_1\,q'). \end{aligned}$$

If one introduces those values of u', v', w' into (40) then, since the components p', q', r' of the rotation are arbitrary, one will get:

$$(41)\qquad \left\{\begin{aligned} &\frac{d}{dt}\frac{\partial T}{\partial p}-a\gamma_3\frac{d}{dt}\frac{\partial T}{\partial v}+a\gamma_2\frac{d}{dt}\frac{\partial T}{\partial w}-r\frac{\partial T}{\partial q}+q\frac{\partial T}{\partial r} \\ &-a(\gamma_3\,r+\gamma_2\,q)\frac{\partial T}{\partial r}+(-w+a\gamma_2\,p)\frac{\partial T}{\partial v}+(v+a\gamma_2\,q)\frac{\partial T}{\partial w}=0, \\[2ex] &\frac{d}{dt}\frac{\partial T}{\partial q}-a\gamma_1\frac{d}{dt}\frac{\partial T}{\partial w}+a\gamma_3\frac{d}{dt}\frac{\partial T}{\partial u}-p\frac{\partial T}{\partial r}+r\frac{\partial T}{\partial p} \\ &-a(\gamma_1\,p+\gamma_3\,r)\frac{\partial T}{\partial v}+(-u+a\gamma_3\,q)\frac{\partial T}{\partial w}+(w+a\gamma_1\,q)\frac{\partial T}{\partial u}=0, \\[2ex] &\frac{d}{dt}\frac{\partial T}{\partial r}-a\gamma_2\frac{d}{dt}\frac{\partial T}{\partial u}+a\gamma_1\frac{d}{dt}\frac{\partial T}{\partial v}-q\frac{\partial T}{\partial p}+p\frac{\partial T}{\partial q} \\ &-a(\gamma_2\,q+\gamma_1\,p)\frac{\partial T}{\partial w}+(-v+a\gamma_1\,r)\frac{\partial T}{\partial u}+(u+a\gamma_2\,r)\frac{\partial T}{\partial v}=0. \end{aligned}\right.$$

Those are the desired differential equations that will determine the motion, in combination with the condition (26) ([1]).

§ 15. – Special case.

We now assume that the ball has its center of gravity at its center, without actually being homogeneous. The coordinate system x, y, z shall be defined by the principal axes that are constructed at the center of gravity. The *vis viva* is then inferred from the equation:

$$2T = (u^2 + v^2 + w^2)\, M + Pp^2 + Qq^2 + Rr^2,$$

in which M means the mass, and P, Q, R mean the principal moments of inertia. Equations (41) then take on the form:

$$(42)\qquad \left\{\begin{aligned} &P\frac{dp}{dt} - aM\left(\gamma_3\frac{dv}{dt} - \gamma_2\frac{dw}{dt}\right) + (R-Q)\,qr + aM\{\gamma_2(pv-qu) + \gamma_3(pw-ru)\} = 0, \\ &Q\frac{dq}{dt} - aM\left(\gamma_1\frac{dw}{dt} - \gamma_3\frac{du}{dt}\right) + (P-R)\,rp + aM\{\gamma_3(qw-rv) + \gamma_1(qu-pv)\} = 0, \\ &R\frac{dr}{dt} - aM\left(\gamma_2\frac{du}{dt} - \gamma_1\frac{dv}{dt}\right) + (Q-P)\,p\,q + aM\{\gamma_1(ru-pw) + \gamma_2(rv-qw)\} = 0. \end{aligned}\right.$$

Differentiation of (36) will yield:

$$(43)\qquad \begin{aligned} \frac{du}{dt} &= a\left(\frac{d\gamma_3}{dt}q - \frac{d\gamma_2}{dt}r + \gamma_3\frac{dq}{dt} - \gamma_2\frac{dr}{dt}\right), \\ \frac{dv}{dt} &= a\left(\frac{d\gamma_1}{dt}r - \frac{d\gamma_3}{dt}p + \gamma_1\frac{dr}{dt} - \gamma_3\frac{dp}{dt}\right), \\ \frac{dw}{dt} &= a\left(\frac{d\gamma_2}{dt}p - \frac{d\gamma_1}{dt}q + \gamma_2\frac{dp}{dt} - \gamma_1\frac{dq}{dt}\right). \end{aligned}$$

([1]) The derivatives $\partial T / \partial p$, etc. in these equations are defined by partially-differentiating a function that represents the *vis viva* for a rolling and slipping motion of the ball. Namely, those derivatives emerge from the calculation of δT, and the *vis viva* $T + \delta T$ of the varied motion cannot be calculated from an expression that is valid for a pure rolling motion. That fact was overlooked in the development of the special **Neumann** formulas, which relate to rolling on a fixed plane. Those formulas (Ber. d. Sächs. Ges. math.-phys. Cl. 1888, pp. 42 and 1885, pp. 368) need to be corrected then.

Those equations serve to make the quantities $\frac{du}{dt}, \frac{dv}{dt}, \frac{dw}{dt}$ known by way of (42). Afterwards, one replaces $\frac{d\gamma_1}{dt}, \frac{d\gamma_2}{dt}, \frac{d\gamma_3}{dt}$ with the right-hand sides of (35), and furthermore, *u*, *v*, *w* with the right-hand side of (36), and one will ultimately get:

$$(44) \qquad \left\{ \begin{aligned} &[P + a^2 M(\gamma_2^2 + \gamma_3^2)]\frac{dp}{dt} - a^2 M\gamma_1\gamma_2\frac{dq}{dt} - a^2 M\gamma_1\gamma_3\frac{dr}{dt} = (Q - R)\,qr, \\ &[Q + a^2 M(\gamma_3^2 + \gamma_1^2)]\frac{dq}{dt} - a^2 M\gamma_2\gamma_3\frac{dr}{dt} - a^2 M\gamma_2\gamma_1\frac{dp}{dt} = (R - P)\,rq, \\ &[R + a^2 M(\gamma_1^2 + \gamma_2^2)]\frac{dr}{dt} - a^2 M\gamma_3\gamma_1\frac{dp}{dt} - a^2 M\gamma_3\gamma_1\frac{dq}{dt} = (P - Q)\;pq. \end{aligned} \right.$$

We now have equations that are linear in $\frac{dp}{dt}$, $\frac{dq}{dt}$, $\frac{dr}{dt}$ and have a positive determinant. That will yield expressions for $\frac{dp}{dt}$, $\frac{dq}{dt}$, $\frac{dr}{dt}$ in terms of the quantities γ_1, γ_2, γ_3, *p*, *q*, *r*, and except for the relation:

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1,$$

they are arbitrary for the initial state.

The simplest case is the one in which:

$$P = Q = R,$$

such as, e.g., the homogeneous ball. That will then imply equations (44):

$$\frac{dp}{dt} = \frac{dq}{dt} = \frac{dr}{dt} = 0;$$

i.e., one has a rotational axis that is fixed in the ball and a uniform rotation about it. With the help of the latter equations and the relations (43), (35), and (36), that will now yield:

$$\frac{d}{dt}(\alpha_1\, u + \alpha_2\, v + \alpha_3\, w) = 0,$$

$$\frac{d}{dt}(\beta_1\, u + \beta_2\, v + \beta_3\, w) = 0.$$

That means that the center moves with a uniform, rectilinear motion.

From now on, we shall once more assume that the moments of inertia *P*, *Q*, *R* are different, but that the initial state is such that $p = q = 0$. The rotational axis is one of the principal axes at the outset. Now, equations (44) imply that $p = q = 0$ and r remains constant, and the motion in this case will proceed as it does for the homogeneous ball. That can also be assumed. Namely, if one imagines that the initial state obeys the condition, but the ball is completely free and subject to no forces, then the motion will proceed as described. Hence, if the initial state corresponds to a pure rolling on the plane then the same thing will be true for all of the states that follow it. If one adds the constraint that slipping is prohibited then that will not change anything in regard to the motion.

Tübingen, May 1896.

"Les idées de Hertz sur la mécanique," Revue générale des sciences pures et appliquées, **8** (1897), 734-743.

Hertz's ideas on mechanics

By Henri Poincaré

Translated by D. H. Delphenich

In 1890, the great electrician Hertz had arrived at the apogee of his glory. All of the European academies had lavished all the awards upon him that they had to give. The entire world hoped that there would be many years ahead for him and that they would be as brilliant as the ones that he began with.

Unfortunately, the illness that was to take him from us so prematurely had already been contracted, and his experimental activity soon slackened and almost stopped completely. He barely had enough time to install his new laboratory in Bonn. Various maladies impeded him, as well as depriving us of the discoveries that he promised to make.

He further contributed to the physical sciences by the enormous influence that he exerted and the advice that he gave to his students. However, it is true that that period was distinguished by only a single personal discovery of surpassing importance, namely, the transparency of aluminum to cathode rays.

However, although he was thus cruelly diverted from the studies that had been so dear to him, nonetheless, he did not remain inactive. Perhaps his senses betrayed him, but his intellect remained, and he applied it to some profound reflections on the philosophy of mechanics. The results of those reflections were published in a posthumous book, and I would like to summarize them and discuss them briefly here.

Hertz first criticized the two principal systems that had been proposed up to now, and that I shall call the *classical system* and the *energetic system*, and he proposed a third one that I shall call the *Hertzian system*.

I. – **THE CLASSICAL SYSTEM**.

§ 1. – Definition of force.

The first attempt at coordinating the facts of mechanics is the one that we shall call the *classical system.* According to Hertz, it is:

> "…the royal route whose principal stations bear the names of Archimedes, Galileo, Newton, and Lagrange."
>
> "The fundamental notions that one finds at the point of departure are those of *space, time, force,* and *mass.* In that system, force is regarded as the cause of motion; it exists in advance of the motion and is independent of it."

I would like to explain the reasons why Hertz was not satisfied with that way of considering things.

First, one has the difficulties that one encounters when one wishes to define the fundamental notions. What is *mass*? Newton responded, "it is the product of volume with density." – Thomson and Tait responded, "it would be better to say that density is the quotient of mass by volume." – What is *force*? Lagrange responded, "it is a cause that produces the motion of body or tends to produce it." – Kirchhoff said, "it is the product of mass times *acceleration*." But then, why does one not say that mass is the quotient of force by acceleration?

Those difficulties are inextricable.

When one says that force is the cause of a motion, one is dealing with metaphysics, and if one accepts that definition then it would be absolutely sterile. In order for a definition to be useful, it must tell one how to *measure* the force. If it can do that, moreover, then it would not at all be necessary for it to tell one what force is *intrinsically* nor whether it is the cause or the effect of motion.

One must first define the equality of two forces then. When does one say that two forces are equal? It is, one answers, when they are applied to the same mass and they impart the same acceleration upon it or when they are directly opposed to each other and it is found to be in equilibrium. That definition is only an illusion. One cannot unhook a force that is applied to one body in order to hook it up with another, as one unhooks a locomotive in order to couple it with another train. It is therefore impossible to know what acceleration a force that is applied to one body would impart upon another body *if* it were applied to it. It is impossible to know how two forces that are not directly opposed to each other would behave *if* they were directly opposed.

That is the definition that one seeks to materialize, so-to-speak, when one measures a force with a dynamometer or by equilibrating it with a weight. Two forces F and F', which I shall suppose to be vertical and point from down to up, for simplicity, are applied to two bodies C and C'. I first suspend the same heavy body P from the body C and then from the body C'. If there is equilibrium in both cases then I conclude that the two forces F and F' are equal to each other, since they are both equal to the weight of the body P.

But am I sure that the body P has kept the same weight when I transport it from the first body to the second? Far from that being true, *I am certain of the contrary.* I know that the intensity of gravity varies from one point to another, and that it is stronger, for example, at the poles than at the equator. Without a doubt, the difference is quite weak, and in practice, I would not take it into account. However, a well-constructed definition must be mathematically rigorous: That rigor does not exist here. That is why I say that weight obviously applies to the force of the spring in a dynamometer, but that temperature and a host of circumstances can make that vary.

But that is not all. One cannot say that the weight of the body P is applied to the body C and equilibrates the force F directly. What is applied to the body C is the action A of the body P on the body C. The body P is, in its own right, subject to, on the one hand, its weight, and on the other, the reaction R of the body C on P. By definition, the force F is equal to the force A, because it equilibrates it. The force A is equal to R by virtue of the principle of the equality of action and reaction. Finally, the force R is equal to the weight of P because it equilibrates it. It is from those three equalities that we deduce the equality of F and the weight of P as a consequence.

We are then obliged to introduce the principle of the equality of action and reaction itself into the definition of the equality of two forces. *In that respect, that principle must not be regarded as an experimental law, but as a definition.*

We then come down to Kirchhoff's definition: *Force is equal to mass, multiplied by acceleration.* That "law of Newton," in turn, ceases to be regarded as an experimental law, since it is now just a definition. However, that definition is also insufficient, since we do not know what mass is. Without a doubt, it permits us to calculate the relationship between two forces that are applied to the same body at different instants, but it tells us nothing about the relationship between two forces that are applied to two different bodies.

In order to complete it, one must again resort to Newton's third law (viz., the equality of action and reaction), which is further regarded, not as an experimental law, but as a definition. Two bodies *A* and *B* act upon each other. The acceleration of *A*, multiplied by the mass of *A*, is equal to the action of *B* on *A*. Similarly, the product of the acceleration of *B* times its mass is equal to the reaction of *A* on *B*. Since the action is equal to the reaction, by definition, the masses of *A* and *B* are inversely proportional to the accelerations of those two bodies. The ratio of those two masses is then defined, and it is up to experiments to verify that the ratio is constant.

That will indeed be the case if the two bodies *A* and *B* are the only ones present and abstracted from the action of the rest of the world. Nothing of the sort is true: The acceleration of *A* is not due to solely the action of *B*, but to the action of a host of other bodies *C*, *D*… In order to apply the preceding rule, one must then decompose the acceleration of *A* into several components and discern which of those components is the one that is due to the action of *B*.

Furthermore, that decomposition will be possible if we *assume* that the action of *C* on *A* is simply added to that of *B* on *A* without the presence of the body *C* modifying the action of *B* on *A* or the presence of *B* modifying the action of *C* on *A*. Consequently, if we assume that two arbitrary bodies attract each other than their mutual action will point along the line that connects them and depend upon only their separation distance. In a word, if we assume *the hypothesis of central forces.*

One knows that in order to determine the mass of celestial bodies, one must appeal to an entirely different principle. The law of gravitation tells us that the attraction of two bodies is proportional to their masses. If r is their distance, m and m' are their masses, and k is a constant then their attraction will be:

$$\frac{k\,m\,m'}{r^2}$$

What one measures then is not the mass – i.e., the ratio of force to acceleration – but the mass of attraction. That is not the inertia of the body, but its attracting force.

That is an indirect procedure whose use is not theoretically indispensable. It might very well be the case that the attraction is inversely proportional to the square of the distance without being proportional to the masses, which would make it equal to:

$$\frac{f}{r^2}$$

but without one having:

$$f = k\, m\, m'.$$

If that were true then one could nonetheless measure the masses of those bodies by observing the *relative* motions of the celestial bodies.

But do we have the right to assume the hypothesis of central forces? Is that hypothesis rigorously exact? Is it certain that it will never be contradicted by experiments? Who dares to assert that? Moreover, if we must abandon that hypothesis then the entire edifice that was raised so laboriously would collapse.

We no longer have the right to speak of the component of the acceleration of *A* that is due to the action of *B*. We have no means for discerning what is due to the action of *C* or some other body. The rule for measuring the masses will then become inapplicable.

What does the principle of the equality of action and reaction then rest upon? If one rejects the hypothesis of central forces then that principle must obviously be stated thus: The geometric resultant of all of the forces that are applied to the various bodies of a system that is isolated from any external action will be zero, or, in other words, *the motion of the center of gravity of that system will be uniform and rectilinear*.

It seems that one has a means for defining mass in that: The position of the center of gravity obviously depends upon the values that one attributes to the masses. One must arrange those values in such a fashion that the motion of that center of gravity is uniform and rectilinear. That will always be possible if Newton's third law is true, and that will be possible in only one way, in general.

However, systems that are isolated from all external action do not exist. All of the parts of the universe exert a more or less strong effect on all of the other parts. *The law of motion of the center of gravity is rigorously true only if one applies it to the entire universe.*

However, one would then have to observe the motion of the center of gravity of the universe in order to infer the values of the masses. The absurdity of that conclusion is obvious. We only know about its relative motions. The motion of the center of gravity of the universe will remain eternally unknown to us.

Nothing remains then, and our efforts have been fruitless. We are then forced to make the following definition, which is only a confession of our powerlessness: *Masses are convenient coefficients to introduce into the calculations.*

We can reconstruct all of mechanics by attributing different values to all masses. That new mechanics will not contradict either experiments or the general principles of dynamics (e.g., the principle of inertia, proportionality of masses and accelerations, equality of action and reaction, uniform, rectilinear motion of the center of gravity, principle of areas).

However, the equations of that new mechanics will be *less simple*. *Nota bene*: It will be only the first terms that will be less simple – i.e., the ones that experiment has already made known to us. Perhaps we can alter the masses of small quantities without the *complete* equations gaining or losing any simplicity.

I have insisted upon discussing that point much longer than Hertz himself. However, I wanted to show that Hertz did not seek to simply have a German quarrel with Galileo and Newton. We must conclude that with the classical system, *it is impossible to give a satisfactory conception to force and mass.*

§ 2. Various objections.

Hertz then demanded to know whether the principles of mechanics are rigorously true. He said:

> "In the opinion of many physicists, it would seem inconceivable that even the most extensive experiments could ever change anything about the unwavering principles of mechanics, and anyhow, that type of experiment can always be rectified by experiments."

From what we said, those fears seem superfluous. The principles of dynamics initially seem to be experimental truths to us. However, we have been obliged to appeal to appeal to them as definitions. It is *by definition* that force is equal to the product of mass times acceleration. That is a principle that is henceforth placed beyond the reach of any ultimate experiment. Similarly, it is by definition that action is equal to reaction.

But then, one says, those unverifiable principles will be absolutely devoid of any significance. Experiments cannot contradict them, but they can give us nothing useful. Why should one study dynamics then?

That condemnation very rapidly proves to be unjustified. There are no *perfectly* isolated systems in nature, namely, ones that are perfectly abstracted from any external action. However, there are *almost* isolated systems.

If one observes such a system then one can study not only the relative motion of its various parts with respect to each other, but the motion of its center of gravity with respect to the other parts of the universe. One then confirms that the motion of that center of gravity is *almost* uniform and rectilinear, which conforms to Newton's third law.

That is an experimental truth, but it could be invalidated by experiment. What would we learn from a more precise experiment? It would tell us that the law was only approximately true; however, we knew that already.

One now explains how experiments can serve as the basis for the principles of mechanics and still never contradict them.

But let us return to Hertz's argument. The classical system is incomplete, because not all of the motions that are compatible with the principles of dynamics are realized in nature, or even realizable. Indeed, is it not obvious that the principle of areas and the motion of the center of gravity are not the only laws that govern natural phenomena? Undoubtedly, it would be unreasonable to demand that dynamics should embrace all of the laws of physics that were discovered or could be discovered in the same formula. However, it is no less true that one must regard a system of mechanics in which the principle of the conservation of energy is passed over in silence as incomplete and insufficient.

Hertz concluded:

> "It is true that our system embraces *all* natural motions, but at the same time, it embraces many *other ones* that are not natural. A system that excludes some of those motions would better conform to the nature of things and would consequently constitute an advance."

Such a thing would be, for example, the energetic system that we shall speak of later on, in which the fundamental principle of the conservation of energy is introduced quite naturally.

Perhaps one can very well understand what prevents one from quite simply adding that fundamental principle to the other principles of the classical system.

However, Hertz posed another question:

The classical system gives us an image of the external world. Is that image *simple*? Is one spared the existence of parasitic traits that are introduced arbitrarily along with the essential traits? Are the forces that we are led to introduce not truly useless gears that turn in a vacuum?

A piece of iron rests upon a table. An observer is not prevented from believing that since there is no motion, there is no force. How wrong he would be! Physics teaches us that every atom of iron is attracted by all of the other atoms of the universe. Moreover, each atom of iron is magnetic, and consequently subject to the action of all the magnets in the universe. All of the electric currents in the world also act upon that atom. (I shall overlook the electrostatic forces, molecular forces, etc.)

If one of those forces were to act alone then their action would be enormous; the piece of iron would shatter. Fortunately, they act together, and they counterbalance in such a way that nothing of sort happens. Our observer who sees only a piece of iron at rest will obviously conclude that those forces exist only in our imagination.

Undoubtedly, there is nothing absurd about any of those suppositions, but a system that eliminates them would be better than ours, by that fact alone.

It is impossible to not be struck by the scope of that objection. Moreover, in order to show that is it not purely artificial, it will suffice for me to recall the memory of a polemic that has existed for some years between two entirely eminent scholars – namely, Helmholtz and Bertrand – in regard to the mutual actions of currents. Bertrand, who sought to translate Helmholtz's theory into classical language, collided with some insoluble contradictions. Each element of current must be subject to a couple. However, a couple is composed of two parallel forces that are equal and oppositely directed. Bertrand calculated that each of those two components must be considerable and large enough to lead to the destruction of the wire, so he concluded that one must reject the theory. On the contrary, Helmholtz, who was an advocate of the energetic system, did not see any difficulty.

Therefore, according to Hertz, the classical system must be abandoned, because:

1. A good definition of force is impossible.

2. It is incomplete.

3. It introduces parasitic hypotheses, and those hypotheses can often generate purely artificial difficulties that are meanwhile large enough to impede even the best minds.

II. – THE ENERGETIC SYSTEM.

§ 1. – Various objections.

The energetic system was born as a result of the discovery of the principle of the conservation of energy. It was Helmholtz who gave it its definitive form.

One begins by defining two quantities that play the fundamental roles in that theory. Those two quantities are: On the one hand, the *kinetic energy*, or *vis viva*, and on the other hand, the *potential energy*.

All of the changes that bodies in nature can submit to are governed by two experimental laws:

1. The sum of the kinetic energy and the potential energy is constant. That is the principle of the conservation of energy.

2. If a system of bodies is in the situation A at the instant t_0 and in the situation B at the instant t_1 then it will always go from the first situation to the second one by a path such that the *mean* value of the difference between those two types of energy over the time interval that separates the two instants t_0 and t_1 is as small as possible.

That is Hamilton's principle, which is one form of the principle of least action.

The energetic theory presents the following advantages over the classical theory:

1. It is less incomplete: i.e., the principles of the conservation of energy and Hamilton's principle tell us more than the fundamental principles of the classical theory and exclude certain motions that nature does not realize that would be compatible with the classical theory.

2. It allows us to dispense with the hypothesis of atoms, which was almost impossible to avoid with the classical theory.

However, it raises some new difficulties, in turn. Before speaking of Hertz's objections, I would like to point out two that come to my mind:

The definitions of the two types of energy raise difficulties that are almost as great as the ones that are raised by force and mass in the former system. Meanwhile, one can infer those definitions more easily, at least in the simplest cases.

Suppose that an isolated system is composed of a certain number of material points. Suppose that those points are subject to forces that depend upon only their relative positions and the mutual separation distances but are independent of their velocities. By virtue of the principle of the conservation of energy, one must have a mass function.

In that simple case, the statement of the principle of conservation of energy is one of extreme simplicity. A certain quantity that is accessible to experiment must remain constant. That quantity is the sum of two terms: The first one depends upon only the positions of the material points and is independent of their velocities. The second one is proportional to the square of those velocities. That decomposition can be accomplished in only one way.

The first of those terms, which I will call U, will be the potential energy. The second one, which I will call T, will be the kinetic energy.

It is true that if $T + U$ is constant then the same thing will be true for an arbitrary function of $T + U$:

$$q\,(T + U)\,.$$

However, that function $q\,(T + U)$ will not be the sum of two terms, one of which is independent of velocities, and the other of which is proportional to the square of those velocities. Among the functions that remain constant, there is only one that enjoys that property: namely, $T + U$ (or a linear function of $T + U$, which will change nothing, since that linear function can always be reduced to $T + U$ by a change of unit and origin). That is what we shall call the energy. It is the first term that we shall call the kinetic energy and the second one that we shall call the potential energy. The definitions of those two types of energy can then be pushed to their limits with no ambiguity.

The same thing is true for the definition of mass. The kinetic energy – or *vis viva* – is expressed very simply with the aid of the masses and relative velocities of all material points with respect to each other. Those relative velocities are accessible to observation, and when we have an expression for the kinetic energy as a function of those relative velocities, the coefficients of that expression will give us the masses.

Hence, in that simple cases, one can define the fundamental notions with no difficulty. However, the difficulties will reappear in the most complicated cases, if, for example, the forces depend upon the velocities, instead of upon only the distances. For example, Weber supposed that the mutual action of two electric molecules depends upon not only the distance between them, but their velocities and accelerations. If the material points attract each other according to an analogous law then U will depend upon the velocity, and it can contain a term that is proportional to the square of the velocity.

Among the terms that are proportional to the squares of the velocities, how can one discern the ones that are provided by T or U ? How does one consequently distinguish the two types of energy?

But there is more: How does one define energy itself? We no longer have any reason to take $T + U$ to be the definition, rather than any other function of $T + U$, when the property that characterizes $T + U$ disappears, namely, that it is the sum of two terms of a particular form.

But that is not all: One must take into account not just the mechanical energy, properly speaking, but the other forms of energy, such as heat, chemical energy, electric energy, etc. The principle of the conservation of energy is then written:

$$T + U + Q = \text{const.}$$

in which T represents the observable kinetic energy, U represents the potential energy of position, which depends upon only the positions of the bodies, and Q is the internal molecular energy, which can take the form of thermal, chemical, or electrical energy.

Everything would be fine if those three terms were absolutely distinct, namely, if T were proportional to the square of the velocities, U were independent of those velocities and the state of the body, and Q were independent of the velocities and positions of the bodies, but dependent upon only their internal states.

The expression for energy could be decomposed into three terms of that form in only one way.

However, that is not the case. Consider the charged bodies: The electrostatic energy that is due to their mutual action will obviously depend upon their charges – i.e., upon their states. If those bodies are in motion then they will act upon each other electrodynamically, and the electrodynamical energy will depend upon not only their states and positions, but upon their velocities.

We would no longer have any means of sorting out the terms that must belong to T, U, and Q then, and thus to separate the three types of energy.

If $(T + U + Q)$ is constant then the same thing will be true for an arbitrary function:

$$\varphi(T + U + Q) .$$

If $T + U + Q$ has the special form that I envisioned above then no ambiguity will result. Among the functions $\varphi(T + U + Q)$ that remain constant, there will be only one of them that has the special form, and that will be the one that I agree to call energy.

However, as I said, that is not rigorously true. Among the functions that remain constant, there are none that can be put into that special form rigorously. Moreover, how does one choose the one that must be called energy? We no longer have anything that can guide us in our choice.

It only remains for us to state the principle of the conservation of energy in the form: *There is something that remains constant.* In that form, it is, in turn, found to be beyond the scope of experiments and reduces to a sort of tautology. It is clear that if the world is governed by laws then there will have to be quantities that remain constant. Things like Newton's principles, and for an analogous reason, the principle of the conservation of energy, which are based upon experiments, can no longer be confirmed by them.

That discussion shows that one has made some progress upon passing from a classical system to an energetic system. However, at the same time, it shows that this progress is insufficient.

Another objection seems much more serious to me: The principle of least action applies to reversible phenomena. However, it is not remotely satisfied as far as irreversible phenomena are concerned. Helmholtz's attempt to extend to that class of phenomena did not succeed, nor can it succeed. Everything remains to be done in that respect.

There are other objections of an almost metaphysical order that Hertz developed at length.

If the energy is *materialized* – so to speak – then it must always be positive. Now, there are cases in which it difficult to avoid considering negative energy. For example, consider Jupiter orbiting around the Sun. The total energy will have the expression:

$$a\,v^2 - \frac{b}{r} + c,$$

in which a, b, c are three positive constants, v is Jupiter's velocity, and r is its distance to the Sun.

Since we can choose the constant c, we can suppose that it is large enough to make the energy positive. That is already something arbitrary that should come as a shock.

But there is more: Now, imagine that a celestial body of an enormous mass and an enormous velocity traverses the solar system. When it has passed through and once more gone out to an immense distance, the orbits of the planets will have been subjected to considerable perturbations. For example, we can imagine that the major axis of Jupiter has become very small, but that its orbit remains reasonably circular. No matter how big the constant c might be, if the new major axis is very small then the expression:

$$a\, v^2 - \frac{b}{r} + c$$

will become negative, and one will see the difficulty reappear that we believe to be obvious by giving a large value to c.

In summary, we cannot ensure that energy will always remain positive.

On the other hand, in order to *materialize* the energy, one must *localize* it. For the kinetic energy, that is easy, but the same thing is not true for the potential energy. Where does one localize the potential energy that is due to the attraction of two stars? Is it in one of the two stars? Is it in both of them? Is it in the intermediate medium?

The statement of the principle of least action itself includes something that should come as a shock the senses. In order to go from one point to another, a material molecule that is free from the action of any force, but subject to move on a surface, must take a geodesic line, i.e., the shortest path.

That molecule seems to know the point where it must go to, predict the time that it will take to reach it by following this or that path, and then choose the most convenient path. That statement makes it sound, so to speak, like a free and animate being. It is clear that one would do better to replace it with a statement that is less shocking, and in which, as the philosophers would say, the final causes do not seem to substitute for the effective causes.

§ 2. – Boule's objection.

The final objection, which seems to be the one that is most striking to Hertz, is of a slightly different nature.

One knows what one calls a system with constraints. First imagine two points that are connected by a rigid link in such a fashion that their separation distance is always kept invariable, or, more generally, suppose that an arbitrary mechanism maintains a relation between the coordinates of two or more points of the system. That is the first type of constraint, which one calls a "solid constraint."

Now suppose that a sphere is constrained to roll on a plane. The velocity of the point of contact must be zero. We then have a second kind of constraint that is expressed by a relation that is no longer just between the coordinates of the various points of the system, but between their coordinates and velocities.

The systems in which there are constraints of the second kind enjoy a curious property that I would like to explain in the simple example that I just cited, namely, that of a ball that rolls on a horizontal plane.

Let O be a point in the horizontal plane, and let C be the center of the sphere.

In order to define the location of the moving sphere, I will take three fixed coordinate axes Ox, Oy, Oz, the first two of which are located in the horizontal plane upon which the sphere rolls, and three coordinate axes that are invariably coupled with the sphere $C\xi$, $C\eta$, and $C\zeta$.

The location of the sphere will be defined completely when one is given the two coordinates of the contact point and the nine direction cosines of the moving axes with respect to the fixed axes. Let A be a position of the sphere where the contact point is at O, the origin, and the moving axes are parallel to the fixed axes.

The coordinates of the contact point are:

$$x = 0, \quad y = 0,$$

and the nine direction cosines are:

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

Give the sphere an infinitely small rotation ε around the axis $C\xi$. It will go to a position B where the contact point will become:

$$x = 0, \quad y = 0,$$

and the nine cosines will become:

$$\begin{matrix} 1 & 0 & 0 \\ 0 & \cos \varepsilon & \sin \varepsilon \\ 0 & -\sin \varepsilon & \cos \varepsilon \end{matrix}$$

However, that rotation is impossible, since it will make the sphere slide on the plane, not roll. It will then be impossible to pass from the position A to the infinitely close position B *directly*, i.e., by an infinitely small motion.

Nonetheless, we shall see that this passage can be made *indirectly*, i.e., by a finite motion.

Start from the position A. Roll the sphere on the plane in such a way that the instantaneous axis is situated in the horizontal plane and is parallel to the axis Oy at each instant, and stop when the axis $C\xi$ becomes vertical and parallel to Oz. One will arrive at a position D where the coordinates of the contact point have become:

$$x = \frac{\pi}{2} R, \qquad y = 0,$$

in which R is the radius of the sphere, and the nine cosines are:

$$\begin{matrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ +1 & 0 & 0 \end{matrix}$$

In the position D, the contact point is at the extremity of the axis $C\xi$, which is vertical.

Impart a rotation ε around the axis $C\xi$ to the sphere. That rotation is a pivoting around the vertical axis that passes through the contact point, so it will not include any sliding, and it will therefore be compatible with the constraints.

The sphere then goes to a position *E* where the coordinates of its contact point are:

$$x = \frac{\pi}{2} R, \qquad y = 0,$$

and the nine cosines are:

$$\begin{array}{ccc} 0 & 0 & -1 \\ \sin \varepsilon & \cos \varepsilon & 0 \\ \cos \varepsilon & -\sin \varepsilon & 0 \end{array}$$

Now, roll the sphere in such a fashion that the instantaneous axis of rotation remains constantly parallel to *Oy*, and consequently the contact will always be along the axis *Ox*. Stop when the contact point has returned to the origin *O*. It is easy to see that we have arrived at the position *B*.

One can then go from the position *A* to the position *B* by passing through the intermediate positions *D* and *E*.

Hertz called the systems such that if the constraints did not permit one to pass directly from a certain position to another infinitely close position then they would no longer permit one to pass from one to the other indirectly *holonomic*. Those are the systems where there are only solid constraints.

One sees that our sphere is not a holonomic system.

Now, it can happen that the principle of least action is not applicable to the non-holonomic systems.

Indeed, one can pass from the position *A* to the position *B* by the path that I just described, and without a doubt by many other paths. Among all of those paths there is obviously one of them that corresponds to an action that is smaller than all of the other ones. The sphere must then be able to follow it in order to go from *A* to *B*. Nothing of the sort is true. No matter what the initial conditions of motion are, the sphere will never go from *A* to *B*.

There is more: If the sphere effectively goes from the position *A* to another position *A′* then it does not always take the path that corresponds to the minimum action.

The principle of least action is no longer true.

As Hertz said:

> "In this case, a sphere that obeys that principle would seem to be a living being that consciously pursues a definite goal, while a sphere that follows the laws of nature will present the impression of being an inanimate mass that rolls uniformly… However, one says, such constraints do not exist in nature. The so-called rolling without slipping is only rolling with a small amount of slipping. That phenomenon enters the realm of irreversible phenomena such as friction, which are still poorly understood, and to which we further *do not know* how to apply the principles of mechanics."
>
> "I respond: Rolling without slipping is not contrary to either the principle of energy or any of the known laws of physics. That phenomenon

> can be realized in the observable world to such an approximation that one can appeal to it in order to construct the most delicate integration machines (e.g., planimeters, harmonic analyzers, etc.) We have no right to exclude it as impossible. However, it will be, and it can be realized only to a degree of approximation such that the difficulties still do not disappear. In order to adopt a principle, we must demand that when it is applied to a problem whose givens are approximately exact, it should also give results that are approximately exact. Furthermore, the other constraints – viz., the solid constraints – are also realized only approximately in nature; nonetheless, one does not exclude them…"

III. – THE HERTZIAN SYSTEM.

Now here is the system that Hertz proposed to substitute for the two theories that he criticized. That system is based upon the following hypotheses:

1. There are only systems with constraints in nature, which are free from the action of any external force.

2. If certain bodies seem to obey some forces then that is because they are *coupled* by other bodies that are invisible to us.

Meanwhile, a material point that seems free to us does not describe a rectilinear trajectory. The old mechanicians said that it deviated from such a trajectory because it was subject to a force. Hertz said that it deviated because it was not free but was coupled to other invisible points.

That hypothesis seems strange at first: Why introduce hypothetical invisible bodies, along with the visible ones? However, Hertz responded, the two old theories were likewise obliged to suppose who-knows-what sort of invisible entities, along with the visible ones. The classical theory introduced forces, and the energetic theory introduced energy, but those invisible entities of force and energy have an unknown and mysterious nature. On the contrary, the hypothetical entities that I imagine have entirely same nature as the visible bodies.

Is that not simpler and more natural?

We can argue that point and maintain that the entities in the old theories must be retained precisely because of their mysterious nature. To respect that mystery is to confess our ignorance, and since the fact of our ignorance is certain, would it not be better to acknowledge it than to deny it?

But let us move on, and see what Hertz inferred from his hypotheses.

The motions of systems with constraints in the absence of external forces are governed by a unique law.

Among the motions that are compatible with the constraints, the one that is realized will be the one that is such that the sum of the masses times the square of their accelerations is a minimum.

That principle is equivalent to that of least action when the system is holonomic, but it is more general, because it also applies to non-holonomic systems.

In order to better explain the scope of that principle, take a simple example: namely, that of a point that is constrained to move on a surface. Here, we have only one material point. The acceleration must then be a minimum. In order for that to be true, it is necessary for the tangential acceleration to be zero. Now, that acceleration is equal to dv / dt, where v is the velocity, and t is the time, so v is a constant, and the motion of the point is uniform. Moreover, it is necessary that the normal acceleration should be a minimum. Now, it is equal to v^2 / ρ, where ρ is the radius of curvature of the trajectory, or to $v^2 / (R \cos \varphi)$, where R is the radius of curvature of the normal section to the surface, and φ is the angle between the osculating plane to the trajectory and the normal to the surface.

Now, the magnitude and direction of the velocity is supposed to be known. Therefore, v and R will be known.

It will then be necessary to have $\cos \varphi = 1$; i.e., the osculating plane must be normal to the surface. That is, the moving point must describe a geodesic line.

In order to now understand how one might explain the motion of systems that *seem* to be subject to forces to us, I shall once more take a simple example, namely, that of the governor. That apparatus is known to consist of an articulated parallelogram *ABCD*. The opposite vertices *B* and *D* of that parallelogram carry balls whose mass is appreciable. The upper vertex *A* is fixed. The lower vertex *C* carries a ring that can slide along a fixed vertical rod *AX*. The entire apparatus is animated with a rapid rotational motion around the rod *AX*. A link *T* is suspended from the ring *C*.

The centrifugal force tends to move the balls apart, and consequently to raise the ring *C* and the link *T*. The link *T* is then subject to a traction that becomes greater as the rotation becomes more rapid.

Now suppose that an observer sees only that link and imagines that the balls, the rod *AX*, and the parallelogram are made of a material that is invisible to him. That observer will confirm that traction is exerted upon the link *T*. However, since he will not see the organs that produced it, he will attribute a mysterious cause – say, a "force" – to the attraction that is experienced by the point *A* on the link.

Indeed, according to Hertz, whenever we imagine a force, we are being duped by an analogous illusion.

That raises the question: Can one imagine an articulated system that imitates a system of forces that is defined by an arbitrary law or approaches one as close as one desires? The response must be in the affirmative. I shall be content to recall a theorem of Koenigs that can serve as the basis for a proof. This is the theorem: One can always imagine an articulated system such that a point of that system describes a curve or an arbitrary algebraic surface, or more generally, one can imagine an articulated system such that by virtue of its constraints, the coordinates of the various points of the system are subject to some given, but arbitrary, algebraic relations.

Nonetheless, the hypotheses to which one will be led can be very complicated.

That was not the first attempt that was made along those lines, moreover. It is impossible to not compare Hertz's hypotheses with Lord Kelvin's theory of gyrostatic elasticity.

As one knows, Lord Kelvin sought to explain the properties of the ether without introducing any forces. He even gave a definitive form to his hypothesis and represented the ether by a mechanical model that was like that of the English magnet. The English scholars, who were content to give a mechanism to their ideas in order to make it tangible, were not frightened by the complexity of models in which one has a multiplicity of links, rods, and slides, as in the mechanic's workshop.

To give some idea of that, let me describe the model that represented the gyrostatic ether. The ether is composed of a sort of mesh. Each intersection in that mesh is a tetrahedron. Each of the edges of the tetrahedron is composed of two rods, one of which is solid and the other of which is hollow, and the former slides inside the latter. That edge is then extensible, but not flexible.

At each intersection, one finds an apparatus that is composed of three rods that are coupled to each other invariably and form a tri-rectangular trihedron. Each of those three rods is supported by two opposite edges of the tetrahedron. Finally, each of them carries four gyroscopes.

In the system that I just described, there is no potential energy, but only kinetic energy, namely, that of the tetrahedra and the gyroscopes. Meanwhile, a medium thus constituted will behave like an elastic medium. It will transmit transverse undulations absolutely like the ether.

I shall add something more: One can not only imitate all forces that are found in nature with articulated systems of that type that contain gyroscopes, but also imitate some other ones that nature has not realized. That was precisely the goal that Lord Kelvin proposed to attain. He wished to explain certain properties of the ether that the usual hypotheses seemed incapable of accounting for.

One knows that the axis of the gyroscope tends to preserve a fixed direction in space. When it deviates from it, it tends to return as if it were acted upon by a guiding force. Unlike the real forces, the apparent force that tends to maintain the direction of the gyroscope is not counterbalanced by an equal and opposite reaction. It is thus liberated from the law of action and reaction and from its consequences, such as the law of areas, to which the natural forces are subject.

One then agrees that the gyrostatic hypothesis, in which one is freed from that restrictive rule, has accounted for some fact that could not be explained by the usual hypotheses upon which it rests.

What must one ultimately think of Hertz's theory? It is certainly interesting, but I do not find it entirely satisfying, because it places too much weight upon hypothesis.

Hertz is protected from some of the objections that have tormented us; he does not seem to have dismissed all of them.

The difficulties that we discussed at length at the beginning of this article can be summarized as follows:

One can present the principles of dynamics in two ways. However, one can never sufficiently distinguish what is a definition, what is an experimental truth, and what is a mathematical theorem. In the Hertzian system, the distinction is still not perfectly clear, and a fourth element is introduced, moreover: viz., hypothesis.

Nevertheless, that mode of exposition is useful due to the fact that it is new: It forces us to reflect and to free ourselves from the old ways of associating ideas. We still cannot see the monument in its entirety, but it is worth something to have a new perspective and to take a new viewpoint.

H. Poincaré

of the Academy of Sciences
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at the Sorbonne

"Eine Anfrage, betreffend ein Beispiel zu Hertz' Mechanik," Jber. d. Deutscher Math.-Verein. **7** (1899), 76-77.

A problem concerning an example in Hertz's mechanics

By **Ludwig Boltzmann** in Vienna.

Translated by D. H. Delphenich

Obviously, an application of **Hertz**'s mechanics to a special case would be of greatest importance for its understanding. I therefore ask my colleagues to pursue that application to a case in which I was not successful.

A point with a mass of m shall be coupled to a second one of much smaller mass in such a way that the distance between them must remain constantly equal to a. The second point is once more coupled to a third one of mass μ in such a way that the distance between them must once more be equal to a. In the sense of **Hertz**'s mechanics, that is an exhaustive picture of an elastic ball of mass μ and radius ρ that moves inside of a hollow elastic ball of mass m and inner radius r when $2a = r - \rho$.

Find a picture, in the sense of **Hertz**'s mechanics, for two completely-elastic solid balls that can collide according to the ordinary laws; i.e., a point system that is constrained by holonomic or non-holonomic conditions, so by equations (but not inequalities!) between the coordinates and their first derivatives with respect to time that are linear and homogeneous in them, in the event that they include the latter at all. The point-system shall therefore provide a picture of an ordinary, completely-elastic collision.

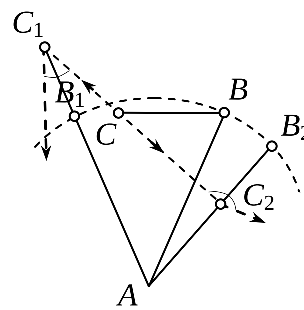
"Über ein Beispiel des Herrn Boltzmann zu der Mechanik von Hertz," Jahresb. Deutsch. Math.-Verein (1900), 200-204.

On Boltzmann's example in regard to Hertz's mechanics

By **A. Brill** in Tübingen

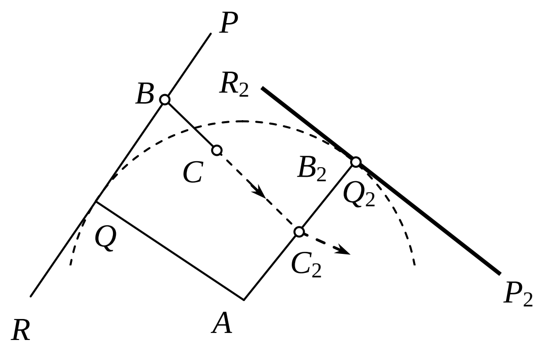
Translated by D. H. Delphenich

In the Jahresbericht der Deutschen Mathematiker-Vereinigung in 1898, **Boltzmann** cited the lack of suitable examples that might make **Hertz**'s book on mechanics more understandable, and then graciously proceeded to give an example of his own. He replaced the motion of a completely-elastic ball in the interior of a hollow sphere in the Hertzian sense that knew of neither forces at a distance nor elastic forces in the usual sense, but only rigid constraints on the masses, with the motion of a system of two massless rods *AB*, *BC* that are coupled to each other by a link *B* and at whose free ends *A*, *C*, one finds mass-points, while the location of the link *B* is a point that has vanishingly-small mass.



In fact, one imagines that the motion of the rod has been simplified in such a way that the one end point *A* is fixed, and writes out the principle of *vis viva* and the law of areas, referred to a planar coordinate that has the point *A* for its origin and its plane is determined by *A* and the initial velocity of *C*. One can then infer the conclusion from that combination of equations that the rectilinear path CC_1 that the end point *C* of the pair of rods sweeps out for a vanishingly-small mass *B* will be converted into another likewise-rectilinear path at the location where the rods define an elongated angle $AB_1C_1 = \pi$ that makes the same angle with the line AB_1C_1 as the one that *C* returns to. At the moment when the path of *C* changes its direction, the infinitely-small mass *B* will gain a *vis viva* whose carrier is otherwise *C* and which helps it to get over the "dead point" in the extended position at that moment by its infinitely-large velocity.

However, that is just the picture of the motion of an elastic ball of radius ρ that moves without the action of forces inside of a hollow sphere of inner radius $AB + BC + \rho$. **Boltzmann** let the rods *AB*, *BC* have equal length so that the center of the ball could go through that of the hollow sphere. However, if one makes $AB \neq BC$ and one chooses the initial direction CC_2 of C such that the angle ABC between the two rods is equal to zero at that time, then the mass-point C will behave at the location C_2 exactly as it does at the location where $\sphericalangle ABC = \pi$. Namely, it bounces off at the same angle with respect to AC_2 that it approached it with, and the system will behave like a solid elastic ball of radius ρ that appears inside of that hollow sphere on a solid ball of radius $AB - BC - \rho$ ($BC - AB - \rho$, resp.) that is concentric to it.



That modification thus implies the picture of the elastic collision of *two solid balls* that **Boltzmann** required (*loc. cit.*). If one would like to avoid the complications of the hollow sphere then one could let the two rods increase without bound while keeping the same difference in length. One might also require the end point B of the link BC, which carries an infinitely-small mass, instead of remaining on a ball of radius AB, to remain on a rectilinear "guide" (say, a tube with a slit in it) PQR that is kept at a constant distance AQ from the fixed point A by an arm $AQ \perp PQR$ such that it always remains tangent to the ball of radius AQ whose center is A. If the end point C of the rod BC carries a finite mass and $AQ > BC$ then one will again have the picture of the elastic collision of two solid balls. The idea behind the latter arrangement goes back to **Finsterwalder**.

All of those pictures can be reduced to the representation of the collision of smooth balls. They will break down when friction associated with the rotating motion of the balls changes the angle of reflection and the plane of reflection.

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Although **Hertz** always spoke only of "rigid constraints," the examples of hidden masses and motions that he might have had in mind when he imagined a replacement for the forces-at-a-distance that occur in nature are however hardly rods or otherwise-discrete mass-systems, but rather matter that fills up space uniformly and can displace in its own right. That is because it was in relation to just that notion that the introduction (*Mechanik*, pp. 31) referred to **Helmholtz**'s theory of hidden motions, which one will find to be developed in his treatises on monocyclic systems and the principle of least action [J. f. Math. **97** (100)], in which it is exemplified by fluids and gaseous bodies. He further cited the representations to which **Maxwell** arrived [in his articles "Über physikalische Kraftlinien" and "A dynamical theory of the electromagnetic field," Trans. Roy. Phil. Soc.

London (1864), etc], in which one must think of a "hidden" fluid medium that is switched on between two masses that seem to act upon each other at a distance and which is "coupled" to the visible masses, in **Hertz**'s terminology, and whose vorticial or otherwise cyclic "adiabatic" motions are the carrier of a certain total kinetic energy that ordinary mechanics cares to describe as the potential energy of the visible masses. **Hertz** also referred to **Lord Kelvin**'s vortex theory of atoms, etc.

One does not generally find representations of that sort developed in the text itself. Indeed, **Hertz**'s statements on the form of the permissible equations of motion that define the "rigid" constraints that enter in place of forces for him seem to contradict that assumption, since only either finite equations between the coordinates or homogeneous linear equations between the *differentials* of the coordinates of the system (129), along with a possible position of the system, are permissible, and according to pp. 43 in the introduction, that is the form in which all of the connections in nature must be clothed. If one now assumes that there is a space-filling intermediate medium that is assumed to be incompressible, in any event, then the condition of incompressibility will not, as **Hertz** concluded, already be expressed as a finite equation between the coordinates, but rather the partial differential quotients of the coordinates x, y, z of a system point with respect to its initial values a, b, c will enter into, e.g., **Lagrange**'s incompressibility condition. However, if one (as **Lagrange** himself did) regards that condition as the expression of the purely-geometric fact that the volume of the tetrahedron that is defined by the points (a, b, c), $(a + da, b, c)$, $(a, b + db, c)$, $(a, b, c + dc)$ does not change in time then, as one can easily show (Mitteil. d. math. naturw. Vereins in Württemberg 1900), that equation will also take the form of an equation between the coordinates of four (infinitely-close) system points. It will express only the "rigid constraint between the *smallest parts*" (Intro., pp. 49).

Moreover, one can also make the transition from discrete to continuous mass-points that fill up a line in the system of rods that was considered above without getting close to **Hertz**'s conception of things when one goes from two rods to n of them, and ultimately to an inextensible chain for which one will again have a condition equation in the partial differential quotients (cf., e.g., **Routh-Schepp**, *Dynamik*, II, § **602**).

Hertz might have also imagined that in the vicinity of two colliding elastic bodies a medium that is endowed with properties of the indicated kind would be capable of absorbing the *vis viva* that would be free at the moment of contact for a brief time in order to once more transmit it to the visible masses (no. **733**). He did not explain that any further, but rather referred expressly to "the individual consideration of that special relationship (for collisions) in the realm of general mechanics" (*ibid.*). Nonetheless, one must agree with **Boltzmann** when he placed precisely that special relationship at the center of the discussion of **Hertz**'s mechanics by means of the example that he treated.

Addendum: Later on, a speech was brought to my attention that **Boltzmann** had presented at one of the general sessions of the Munich Congress of Scientists (see these Jahresbericht, pp. 71, *et seq.*) in "Über die Entwicklung der Methoden der theoretischen Physik zu unser Zeit." In it, **Boltzmann** took an entirely different position in regard to the interpretation of hidden masses in **Hertz**'s mechanics when he explained that:

> "The structure of the formerly-useful medium [that is filled with fluid] and also **Maxwell**'s light ether do not need to be endowed with them [the hidden masses], since indeed forces are thought to act in all of those media of the kind that **Hertz** excluded expressly."

Only **Helmholtz** and **Maxwell** have derived the forces that appear in a vorticial incompressible fluid upon which no external forces act from the equations of hydrodynamics and the incompressibility condition. I believe that it needs to be proved that the latter also have the form that **Hertz** allowed. Now, **Lagrange** derived the system of hydrodynamical equations with the help of only **d'Alembert**'s principle on the basis of that one equation. However, **Hertz** also decreed the latter, since the formulation of it in no. **393** likewise implied the equation that **Lagrange** employed when one treated the coordinate increments δp_ρ , not as possible or virtual displacements, but as mutually-independent displacements, from the procedure in *Mécanique analytique*, t. II, sect. IV, no. 11, when one adds the condition equations that exist between them, suitably provided with undetermined multipliers, to the left-hand side. However, if one has put **d'Alembert**'s principle into that form then that would complete the transition from a finite number of mass-points to an infinite number, in the sense of the remark in no. **6** of **Hertz**'s *Mechanik*, and from there to the fluid media, with the help of that condition equation, precisely as **Lagrange** did (*Mécanique analytique*, t. II, sect. IV, no. 17; sect. XI, no. **2**, *et seq.*). It is therefore not clear why Hertz's hidden masses could not also be fluid masses.

Tübingen, 7 November 1899.

"Über die Mechanik von Hertz," Mitteilungen des Mathematisch-Naturwissenschaftlichen Verein in Württemberg **2** (1900), 1-16.

On Hertz's mechanics ([1])

By Professor **Brill** in Tübingen.

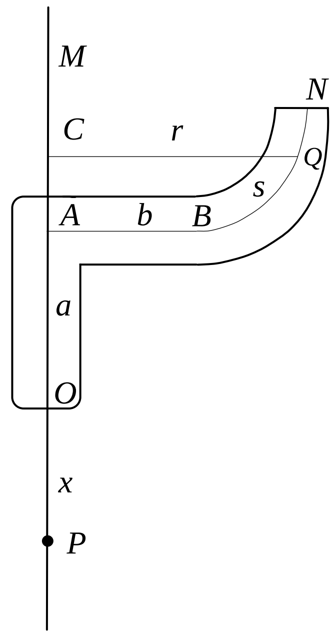
Translated by D. H. Delphenich

When in the year 1894, right after the death of the author, the book *Die Prinzipien der Mechanik* appeared by Heinrich Hertz, with a Foreword by H. von Helmholtz, physicists and mathematicians were gripped with the same enthusiasm for the book, which promised to be a brilliantly-written introduction to a completely new representation of mechanics that was envisioned and written in the spirit of mathematics. It was a mechanics that was built upon only one axiom and explained the controversy regarding the older concept of force in the spirit of the modern physics. However, many readers soon put the book down again, due to precisely the fact that the new conceptual structures that were contained in it were founded upon physical origins and were therefore hard to approach for the mathematicians, which is a fact that was made even more acute by the lack of examples.

Indeed, the difficulty that Helmholtz referred to more precisely in his Foreword still exists to this day that: "One must call upon a great degree of scientific imagination in order to explain even the simplest cases of physical forces in the sense of Hertz." In the meantime, however, the theory that Hertz might have originally had in mind when he presented the concept of "hidden mass" for the propagation of electrical and magnetic force effects through space has taken on more meaning and respectability in a broader context. In addition, many of the new concepts have already proved to be fruitful in other domains. That probably justifies the attempt to draw upon that theory (albeit one that is not free of contradictions) as an example for explaining Hertz's basic concept, and (as I have already tried to do in a lecture in the Winter of 1898-99) proceed from that example to the essence and concepts of the new mechanics without leaving the basis for the old. However, the book itself, whose study is indeed complicated by the peculiar form of the prose that often interrupts the train of thought, is most appealing for the rich, carefully-structured content and language that was chosen, so it might gain a few new friends from the following discussion.

([1]) The following is a transcription of an address that the author gave to the meeting of the Society in Plochingen on 14 May 1899.

Hertz (no. **469**) and Helmholtz (Foreword, pp. X, XX, etc.) used the terms *force-at-a-distance* and *action-at-a-distance* for the force effects between mass points that have *different* coordinates, and in particular for the force of attraction between gravitating or magnetic, electric, etc. masses. Hertz wished to eliminate such forces at a distance by introducing rigid (visible or hidden) constraints between the points. An example might show how one can understand that.



The apparatus that is drawn above, which is slightly altered from one that Boltzmann gave, allows the motion of a mass-point P, without the action of external forces, to be arranged such that it seems to move along the line AOP under a prescribed law of attraction (e.g., Newton's) with O as its center. The thin tube $OABQN$, a piece of which BQN is curved in a well-defined way, rotates about the axis OAM. An inextensible string that connects the material point Q with P runs through the tube, such that when $PO = x$, $OA = a$, $AB = b$, and BQ has arc-length s, the length of the string will be $l = x + a + b + s$. The tube is put into a rotational motion and left to itself. One must determine the rectilinear motion of P when one ignores friction and assumes that the apparatus is massless.

The *vis viva* principle and the law of areas imply that:

$$(P + Q)\left(\frac{ds}{dt}\right)^2 + Q\,r^2\,\omega^2 = h, \qquad r^2\,\omega = c,$$

in which r is the distance QC from the rotational axis, ω is the angular velocity, and h and c are constants.

Upon differentiating:

$$(P + Q)\left(\frac{ds}{dt}\right)^2 + \frac{Q\,c^2}{r^2} = h$$

with respect to time, one will get:

$$(P + Q)\frac{d^2 s}{dt^2} = \frac{Q c^2}{r^3}\frac{dr}{ds},$$

or:

$$(P + Q)\frac{d^2 x}{dt^2} = \frac{Q c^2}{r^3}\frac{dr}{dx}.$$

Now, if say $f(x)$ is the law of the force that seems to act upon P then one must set:

$$\frac{Q c^2}{r^3}\frac{dr}{dx} = -f(x)\frac{P+Q}{P},$$

so

$$\frac{1}{r^2} = -k\int f(\alpha - s)\,ds,$$

in which k, α are positive constants. r is expressed in terms of s in that way.

If z is the abscissa of the point Q, so:

$$ds^2 = dz^2 + dr^2,$$

then a second quadrature will also give z as a function of s, and with that the equation of the curve along which the tube BQN is bent in order to realize the prescribed motion $f(x)$ of P. The assumption that $f(x) = k / x^2$ will yield a hyperelliptic integral for z.

If one now imagines that the apparatus is invisible, except for the mass P, then the *rigidity constraint* that the string exhibits will effect that apparent force at a distance.

Certain curvilinear motions of a point that seem to result from the influence of forces, such as curves that roll on each other, can also be realized. I shall not go into that here.

The motion of a sphere that collides with another sphere and rebounds elastically can be represented by an apparatus that was described by Boltzmann and the author in the Jahresberichten der deutschen Mathematiker-Vereiningung for 1898 and 1899.

Moreover, Hertz's dynamical explanation for forces at a distance is not to be found in rigidity constraints of the type that was considered above, because the hidden motions that he assumed are not the motions of rods or other discrete masses, but they have a *cyclic* nature (no. **599**) and return to themselves in such a way that in place of each advancing mass-point, an equal one will enter immediately. One might do better to imagine the top that Helmholtz referred to when he spoke of forces that would be evoked by cyclic motion (J. f. Math. **100**, pp. 154; see also his *Vorles. über Dynamik*, pub. by Krigar-Menzel, pp. 321). However, the Introduction to Hertz's *Mechanik* (pp. 31) suggests that one must take yet another step. In order to make the following attempt at an explanation understandable, permit me to first recall some things that are known.

A system of gravitating masses that are distributed in space in any way exerts a force on a unit mass whose direction and magnitude are known. If one follows the direction of the force that issues from the point to a neighboring point and then makes the same construction there, and proceeds similarly then one will obtain a *force-line* whose behavior, namely, in the vicinity of the attracting mass will be determined by potential theory. One can think of the distribution of such force-lines, which run through all of space, as being defined such that the measure of the intensity of the force at each location will be determined by the density of the lines. Electrical force-lines belong to electrical masses in an analogous way, and magnetic ones belong to magnetic masses. The latter, which one can, as is known, make visible in the neighborhood of a magnetic pole by iron filings, run through the magnetic masses and close upon themselves ([1]). However, magnetic force-lines also fill up the neighborhood (i.e., the *field*) of *moving* electrical masses, and in particular, the field of an electric current that closes upon itself in the form of a ring, as one call also show with iron filings. Furthermore, the arrangement of the force-lines that are generated by a current exhibits an essential difference from the ones that are produced by magnetic masses. The latter can be contracted to a point, in the sense of *analysis situs*. However, for the former, the current (which one might think of as a tube that closes on itself) acts like a point of discontinuity for a function in the Gaussian plane when one takes its integral along a line that encloses the point: It makes space multiply-connected, and the potential becomes a multi-valued function of position.

Now, the new idea is that this system of force-lines, or rather their action along them, can be produced by the motion of a hidden mass *that fills all of space*, which one must imagine to be something that is intrinsic to the propagation of the outer surface of the mass (or a cavity in it).

The admissibility of that assumption might be shown by the example of the electromagnetic force-field that we would like to consider in what follows (but restricted to a dielectric that is filled with the invisible matter of "empty" space).

Since the time of Faraday ([2]), one of most distinguished problems in mathematical physics has been to replace the force-lines of the fields of gravitating, magnetic, etc., masses with identical *vector fields* ([3]) of a different type that define perhaps the displacements or velocities or stresses in a continuous mass (such as elastic or fluid bodies), whereby the potential energy of the force at a distance will go to the potential or kinetic energy of the medium.

As far as it concerns vectors of the sort that might come into question when they are applied to the outer surface of an *elastic* medium, one know that one can represent the pressures (i.e., stresses) that act upon a surface element that goes through an interior point by the radii of an ellipsoid, namely, the elasticity ellipsoid. Its *axes* at any point will yield

([1]) See the beautiful Tables in Herger, Leipzig, 1844.

([2]) Faraday's ground-breaking work appeared in German in Ostwald's Klassiker-Bibliothek der exakt. Wiss; *ibidem*, see also the relevant work of Maxwell, translated into German by Boltzmann.

([3]) As is known, a vector is a spatial magnitude that is endowed with a direction.

three distinguished directions, so not a vector that would point in *one* direction. By contrast, displacements are vectors. In fact, **W. Thomson** (Lord Kelvin) represented the *electric* force-field in that way in his treatise "On a mechanical representation, etc." (1847, *Papers* I, pp. 76), whereas the magnetic one found a less intuitive interpretation.

Any small change of position – namely, a small cube – in the interior of an elastic body can be composed of:

1. A parallel displacement.
2. A rotation around an axis through the center.

The position of that rotational axis, along with the magnitude of the angle of rotation, for each point that is present will, in turn, give a vector field that was used to advantage by Thomson for the magnetic (electromagnetic, resp.) force field, which is an assumption that bears upon the behavior of the calculations for the boundaries of conductors and non-conductors, in particular. Although one might find it hard to imagine that a medium would not resist translation, but probably rotation, such as the magnetic force would experience under the assumption above, Lord Kelvin made that apparent by an ingenious apparatus that exists in a context where numerous rapidly-rotating tops have been placed.

Of course, a medium of that kind is no longer elastic, in the usual sense of the word. It has hidden cyclic motions, and for that reason, in order to distinguish it from the latter, the discoverer called it a "jelly' that was given the name of "ether." In essence, that "ether" comes down to the fluid that Maxwell considered, which will be discussed shortly. In a more recent paper (*Papers* III, pp. 436), Lord Kelvin returned to such matters and showed that when one calculates the energy stored inside and outside of a solenoid, on the one hand, for the elastic forces in the "jelly" and on the other, for the rotational forces in the "ether" that the latter will have the advantage.

He was associated with other researchers, such as Boltzmann and Heaviside. By contrast, Sommerfeld and Reiff postulated that the magnetic force would be assigned to a displacement of the ether-particle, while the *electric* one would be assigned to a *rotation*, because it is best for one to perform the actual calculations in such a way that the energy of the electric current in conductors will be converted into heat by friction (which opposes the rotation), which is a theory that Boltzmann had even more misgivings about.

The experiments that were sketched out up to now, despite many gaps in the details, all refer to the fact that the electromagnetic force field can be represented by a vector field of *fluid* type. The fact that it must be incompressible was shown, *inter alia*, by Hertz's experiment on the propagation of the electric waves, which refer to absolutely transversal oscillations. Already in his 1858 treatise on vortex motions (Ostwald's Klassiker), Helmholtz had emphasized the analogy that exists between a line vortex in a fluid and an electric current, namely, the force that the vortex exerts on a particle in the surrounding mass of water is analogous to the force that a current exerts on a magnetic pole outside of it. Just as the latter moves perpendicular to the (rectilinearly-envisioned) conductor, so does a reaction on the water particle in the vicinity emanate from the outer surface of the line vortex, which one imagines to be closed, as one does for the current, and the one reaction is equal in magnitude and direction to the other.

However, whereas Helmholtz did not pursue that two-sided analogy any further, **Maxwell** arrived at a theory of the mutual dependency of electric and magnetic effects on the grounds of closely-related arguments that have been developed even further in a series of works, and today that theory defines the undisputed foundation for the entire study of electricity in the form of the "electromagnetic theory of light." Like Helmholtz, Maxwell (1861, 63, "Physikalische Kraftlinien," German trans. by Boltzmann, Ostwald's Klassiker) thought that the magneto-electric field was a fluid that was permeated by vortices, but magnetic force lines were arranged around the electric current, instead of line vortices, and the force itself prevailed along the vortex axis. Indeed, it is known that the pressure at a point in a fluid is the same in all directions, but on a surface element whose order of magnitude is that of a line vortex cross-section, the pressure in the direction of the axis will be smaller than it is on one that is perpendicular to it, such that there will be a suction in that direction. If one does not follow Maxwell in assuming that there are "particles of friction" between the vortices and if one overlooks the complication that arises for densely-packed rotating vortices with the same direction, or if one goes along with the hypotheses of the younger English physicists (cf., e.g., Lodge, *Electricity*) then one must once more represent the electric force by the displacement of particles (von Helmholtz, *Vorlesungen über elektromagn. Lichttheorie,* pub. by König and Runge, pp. 37).

Now, on the basis of those assumptions, one can exhibit a system of six differential equations (initially for the dielectric), by means of which one can conclude the spatial distribution of the magnetic force at a *given* moment from the change that the electric force at that location will experience at the *next* moment, and conversely, one can get the spatial change in the electric field from the temporal change in the magnetic field ([1]).

Maxwell's equations express a far-reaching duality between electric and magnetic forces. One derives an expression for the (combined electric and magnetic) energy in a spatial region in the dielectric from them. Moreover, they also imply (as Helmholtz has also derived from *his own* assumption) the known laws of action at a distance with no further assumptions, and in particular, the Biot-Savart law for the action of a current element on a magnetic pole, under which, an increase or decrease in the kinetic energy of the medium will enter in place of the equally-large change in potential of the force-at-a-distance.

Later on, Maxwell derived his equations from other foundations. However, the representation that was suggested first here is especially worthwhile as an example of Hertz's mechanics, because it illustrates the cyclic motion of the intervening medium by means of vortices.

([1]) One can find the details in the report that Planck presented to the Vers. des D. Math. Ver. in Düsseldorf in 1898.

If none of the attempts to explain the electromagnetic force-at-a-distance in a dynamical way by means of motion are also unimpeachable, nonetheless, they collectively give a picture of what Hertz meant when he posed the problem in his mechanics (no. **596**) of "determining the motions of the visible masses in a system in advance, despite the ignorance that prevails in regard to the positions of the hidden masses," and when he consigned the future of the problem (Intro., pp. 49) to "reducing the alleged effect of forces at a distance to processes of motion in a medium that fills up space and whose smallest parts are subject to rigidity constraints [see below]."

Furthermore, in the foregoing, one must accept the reasons that Hertz gave for endowing his hidden masses and motions with the following properties:

1. The motion of the hidden mass is cyclic, and indeed the velocity of that cyclic motion is considerably larger than the change in the acyclic coordinates (viz., the "parameters"), or more precisely, it is so large that the terms in the expression for the energy of the cyclic system that include the rate of change of the parameters can be dropped in comparison to the ones that include cyclic velocities (**549**). The cyclic coordinates themselves do not enter into the expression for the energy at all.

2. The cyclic motion of the hidden masses is "adiabatic," insofar as it deals with the representation of conservative forces (i.e., ones that possess a force function); i.e., "a free-willed direct influence [of forces] on the cyclic coordinates is excluded" (**600**, **562**), such as, e.g., the rotational speed of a gyroscope that has been turned on can no longer be influenced directly afterwards. If follows from the property that was given (as one sees perhaps by appealing to the second form of Lagrange's differential equations) that the cyclic momentum will always keep the same value for an adiabatic motion.

3. What ordinary mechanics calls potential energy is nothing but the kinetic energy of hidden masses (**605**).

Helmholtz had already introduced the concepts of "cyclic" and "hidden motion," and right from the beginning, he had not merely top-like, vorticial motions in mind, but also ones of the type that one assumes in the theory of heat in a gas; hence, densely-packed colliding elastic molecules that zip through space irregularly at a detailed level, and which first suggest the earlier definition in their totality. For that reason, the theory of heat also yields examples of Hertz's mechanics (cf., Helmholtz, J. f. Math. **97**, pp. 111; *ibid*., **100**, pp. 147).

The focal point of Hertz's mechanics is the introduction of hidden masses in place of forces at a distance; the other innovations first flow out of that. The removal of the concept of force from the elements then changes the axioms; their reformulation implies new conceptual structures. The problem of the first book is to prepare and introduce them, which is entirely independent of the new theory, and can be added as an autonomous appendix to any other mechanics.

The fact that this view excludes everything strange from the book shall now be shown.

Above all, the concept of a *free system* must be imagined in such a way that everywhere the old mechanics assumes potential energy, the hidden masses that produce it in kinetic form must appear as a necessary component of the free system. Thus, Hertz required of a free system that the "connections" between its points should be "legitimate"; i.e., independent of time (**119**, **122**). Those "connections" will be given (**124**) by a system of equations in the coordinates of the mass-points into which their first differentials need to enter only linearly and homogeneously. However, one cannot express (e.g., the connection that is established between two gravitating mass-points by perhaps Newton's law) by equations of *that* type. For that reason, two such points do not define a "free system" in the sense of numbers **122**, **309**, etc., by themselves, but only when they are coupled by hidden masses (that are left undetermined).

Now, as far as the *axioms* of the new mechanics are concerned, Newton's third axiom, which demands the equality of action and reaction, becomes unnecessary, in any case. Indeed, Hertz did not by any means relinquish the concept of "force," since he always sought the connection with the usual representation of mechanics. However, force appeared to him only as the reciprocal influence of two "coupled" (i.e., continually-contacting) systems that pertained to just the location where the contact took place. Both of them together define a free system, and if one drops – say – the second one then its effect on the motion of the (now not free) first one can be replaced with certain terms with multipliers that appear in the equations of motion for the latter, which will take on the meaning of a force in the sense above in that way. The converse will be likewise true when one considers the motion of only the second one. The fact that force and counter-force must be equal when they are regarded in that way can be easily proved by combining both systems (**468**), so it is no longer an axiom.

The content of Newton's second axiom, which states the proportionality of force and acceleration, is a necessary consequence of the aforementioned formal definition of the force as a multiplier (**459**).

However, the axiom of inertia remains intact, and ultimately one more integral principle or its equivalent will be necessary for it. Hertz chose Gauss's principle of least constraint, which he combined with the law of inertia into his *fundamental law* (**309**).

One must now give that fundamental law a conception that is concise, as well as easy to understand. In order to do that, in addition to the aforementioned extension of the concept of free system, one also needs a convenient formulation of the concept of *constraint*. To that end, Hertz introduced some new terminology. Namely, he adapted certain concepts that would be common to the individual mass-points, such as path element, velocity, acceleration, and the curvature of the path to a system of such things, in which he referred to the mean values of the magnitudes and directions in question by those words, as he did in numbers **55**, **265**, **275**. In particular, the *path element ds* of a system of points m_1, m_2, …, m_n is defined by:

$$ds^2 \cdot \sum m_i = \sum m_i \, ds_i^2 ,$$

and the expression for the *curvature* of the path, which is the reciprocal value of the radius of principal curvature for the individual point, is defined by a quantity c, which is defined by the equation (**106**):

$$c^2 \cdot \sum m_i = \sum m_i \, (x_i''^2 + y_i''^2 + z_i''^2) ,$$

if x'', y'', z'' are the second differential quotients of the rectangular coordinates of a point with respect to the path-element of the system.

If connections between the system-points exist – i.e., if they are linked by condition equations in the coordinates – then a free system that is left to itself under the influence of those connections and its initial velocities will adopt a certain natural path. The principle of least constraint (in the absence of external forces) then says that the curvature c will have a smaller value for the natural path than it will for any other conceivable one that is compatible with the conditions on the system, or that it is a *straightest path*, in Hertz's terminology. The straightest path is usually, but not always, the shortest one. Namely, they will differ in the case of *non-holonomic* systems, i.e. a system for which non-integrable differential equations will enter into the condition equations (no. **132**; moreover, cf. Voss, Math. Ann. **25**, pp. 258, where such condition equations were treated previously). A sphere that rolls on a surface serves as an example of a non-holonomic system (Hölder, "Über die Prinzipien, etc.," Gött. Nach. 1896, pp. 150), which obviously does not generally describe a shortest path when it rolls from one position to another.

From what was said, the idea behind Hertz's fundamental law will now become understandable" "Any free system will remain in its state of rest or a state of uniform motion along a straightest path."

The law of the *conservation of* (kinetic) *energy* follows immediately from that fundamental law for a free system. If one considers a "conservative" system (**602**), in particular – i.e., one that is composed of two coupled subsystems (450), one of which contains all visible masses, while the other contains all hidden masses (with adiabatic cycles) – then the "parameters" (i.e., non-cyclic coordinates) of the *hidden* subsystem of the system (which vary slowly compared to its cyclic coordinates) are likewise coordinates of the visible ones. Now, the energy of the system (605) splits into the energy of the visible masses and the energy of the hidden masses. Indeed, as was remarked above (pp. 7), the rates of change of the parameters will vanish in comparison to those of the cyclic coordinates in the expression for the energy of the hidden system. However, insofar as the parameters of the visible system are concerned, their velocities do not, in turn, vanish from the total energy. In order to resolve that contradiction, one might perhaps make the assumption that the hidden masses that come under consideration are very small compared to the visible masses.

It follows further from the fundamental law that the *time integral of the energy* is a minimum for the natural motion (**358**). If one defines it for a conservative system then if ($-U$) is the energy of the hidden masses and T is that of the visible ones then that condition can be written in the form (**626**):

$$\delta_{\mathrm{p}} \int (T - U)\, dt = 0,$$

in which the variation δ_{p} refers to all coordinates, namely, the visible ones (which also appear as parameters in U), as well as the hidden ones. If one introduces the momenta of the hidden coordinates here in place of their velocities, which can be arranged by a well-known process (cf., say, **Jacobi**, *Vorlesungen über Dynamik*, ed. by Clebsch, Lecture 9, pp. 69), then the equation will take on the form:

$$\delta_{\mathrm{p}} \int (T - U)\, dt = 0,$$

in which one must now perform the variation as if the cyclic momenta were not varied at all, because the motion is adiabatic, by assumption (see above, pp. 7). However, that is just the assumption under which ordinary mechanics (which does not know about hidden masses) varies the integral, such that equation above will now represent *Hamilton's principle* (**628**, **629**).

The connection to ordinary mechanics is achieved with that, and the trivial examples, such as perhaps a falling stone, that can be addressed directly by Hertz's method, and which will present the difficulty that one does not know the type of coupling between visible and invisible masses and the motion of the latter, will lead back to Lagrange's equations. The aforementioned formal conversion of the variation of the integral will necessitate some preparations that fill up numerous numbers (such as **593**, **555**, **493**, **292**, **68**).

In conclusion, let us say a few words about the *condition equations* that define the connection between the system points. When Hertz demanded that they should contain only the coordinates of a point and its first differentials, he seemed to exclude continuous masses from the treatment from the outset, and yet, one must think of the hidden masses as being the matter that fills up space, which is already due to their cyclic motion. In that way, if one recalls the analogy with fluid vortices, then one must admit the condition equation that expressed incompressibility, or "the rigidity constraint of the smallest part" (Intro., pp. 49). **Lagrange** represented it with the help of the partial differential quotients of the coordinates *x*, *y*, *z* of a point in the fluid with respect to their initial values *a*, *b*, *c*, namely, by the equation:

$$\frac{d(x,y,z)}{d(a,b,c)} = 1. \tag{1}$$

However, one can give that equation the form:

$$\frac{1}{da\cdot db\cdot dc}\begin{vmatrix} x & y & z & 1 \\ x+\dfrac{\partial x}{\partial a}da & y+\dfrac{\partial y}{\partial a}da & z+\dfrac{\partial z}{\partial a}da & 1 \\ x+\dfrac{\partial x}{\partial b}db & y+\dfrac{\partial y}{\partial b}db & z+\dfrac{\partial z}{\partial b}db & 1 \\ x+\dfrac{\partial x}{\partial c}dc & y+\dfrac{\partial y}{\partial c}dc & z+\dfrac{\partial z}{\partial c}dc & 1 \end{vmatrix} = 1, \tag{2}$$

in which the expression on the left includes only the coordinates of the four (infinitely-close) points that the mass-particles will assume, which will define the vertices (*a*, *b*, *c*); (*a* + *da*, *b*, *c*); (*a*, *b* + *db*, *c*), (*a*, *b*, *c* + *dc*) of a tetrahedron, as well as the three sides *da*, *db*, *dc* of that tetrahedron, such that equation will no longer have the form that Hertz admitted.

One can make the transition from equation (1) to the more useful condition for incompressibility:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0$$

in the way that was indicated in, say, Kirchhoff's *Mechanik*, Lect. 10, § 5, (pp. 107). That equation can also be just as well brought into the form of condition equation in the coordinates of neighboring points that is analogous to (2).

A continuation of Hertz's mechanics in the direction of introducing a space-filling mass instead of the discrete mass-points seems entirely possible, and for the sake of completeness, necessary, which is a step that even Hertz himself seemed to have had in mind (cf., no. **7**)([1]).

Tübingen, 13 October 1899.

([1]) As I saw later, the conception of "hidden masses" that was presented here differs essentially from the one that Boltzmann maintained in his speech to the Münchener Naturforscher-Versammlung in 1899. I have expounded in detail upon the basis for allowing me to persist in my opinion in a report to the Deutschen Mathematiker-Vereinigung in 1899.