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On the integration of the equation $dx^2 + dy^2 + dz^2 = ds^2$

By J.-A. SERRET

Translated by D. H. Delphenich

I.

The question that I propose to answer in the following one:

If x, y, z are four functions of one independent variable θ that are subject to verify the equation:

$$dx^2 + dy^2 + dz^2 = ds^2 \tag{1}$$

then express the general values of those function in finite form and without any integration sign.

It is obvious that one can satisfy the preceding equation by taking x, y, z, and s to be the rectangular coordinates and arc length of an arbitrary curve, resp., so it will follow that the general values of x, y, z, and s must contain two arbitrary functions of the independent variable in their expressions.

Consider an arbitrary curve. The developable surface that is the geometric locus of its tangents can be represented by the set of two equations:

$$z = p\,x + q\,y - u,$$

$$0 = x\,dp + y\,dq - du,$$

in which p, q, and u are functions of one parameter θ whose differentials are dp, dq, du, respectively, and the curve itself, which is the edge of regression of the surface, will be represented by the set of three equations:

$$\left\{\begin{aligned} z &= px + qy - u, \\ 0 &= x\,dp + y\,dq - du, \\ 0 &= x\,d^2p + y\,d^2q - d^2u. \end{aligned}\right. \tag{2}$$

One deduces the values of x, y, z as functions of the parameter θ, which we take to be the independent variable, from this. Namely:

$$(3)\qquad \begin{cases} x = \dfrac{dq\,d^2u - du\,d^2q}{dq\,d^2p - dp\,d^2q}, \\[2mm] y = \dfrac{du\,d^2p - dp\,d^2u}{dq\,d^2p - dp\,d^2q}, \\[2mm] z = px + qy - u. \end{cases}$$

Equations (2), when combined with the ones that one deduces from them by differentiation, will give:

$$dz = p\,dx + q\,dy,$$

$$dp\,dx + dq\,dy = 0,$$

$$d^2p + d^2q\,dy = d^3u - x\,d^3p - q\,d^3q,$$

from which one infers that:

$$(4)\qquad \begin{cases} dx = \dfrac{d^3u - x\,d^3p - y\,d^3q}{dq\,d^2p - dp\,d^2q}\,dq, \\[2mm] dy = \dfrac{d^3u - x\,d^3p - y\,d^3q}{dq\,d^2p - dp\,d^2q}\,dp, \\[2mm] dz = \dfrac{d^3u - x\,d^3p - y\,d^3q}{dq\,d^2p - dp\,d^2q}\,(p\,dq - q\,dp). \end{cases}$$

If one substitutes those values of dx, dy, dz in the proposed equation and then takes the square roots of the two sides then one will have:

$$(5)\qquad ds = \frac{d^3u - x\,d^3p - y\,d^3q}{dq\,d^2p - dp\,d^2q}\sqrt{dq^2 + dp^2 + (p\,dq - q\,dp)^2}\,.$$

Finally, if one replaces x and y with the values that are provided by equations (3) and sets:

$$(6)\qquad \begin{cases} A = \dfrac{d\theta^2\sqrt{dq^2 + dp^2 + (p\,dq - q\,dp)^2}}{dq\,d^2p - dp\,d^2q}, \\[2mm] B = \dfrac{d\theta\,(dq\,d^3p - dp\,d^3q)\sqrt{dq^2 + dp^2 + (p\,dq - q\,dp)^2}}{(dq\,d^2p - dp\,d^2q)^2}, \\[2mm] C = \dfrac{(d^2q\,d^3p - d^2p\,d^3q)\sqrt{dq^2 + dp^2 + (p\,dq - q\,dp)^2}}{(dq\,d^2p - dp\,d^2q)^2}, \end{cases}$$

to abbreviate, then the value of *ds* will be:

$$ds = A\frac{d^3u}{d\theta^3}d\theta + B\frac{d^2u}{d\theta^2}d\theta + C\frac{du}{d\theta}d\theta,$$

and upon integrating each term in *ds* by parts, one will find that:

$$\int A\frac{d^3u}{d\theta^3}d\theta = A\frac{d^2u}{d\theta^2} - \frac{dA}{d\theta}\frac{du}{d\theta} + \frac{d^3A}{d\theta^3}u - \int\frac{d^3A}{d\theta^3}u\,d\theta,$$

$$\int B\frac{d^2u}{d\theta^2}d\theta = B\frac{du}{d\theta} - \frac{dB}{d\theta}u + \int\frac{d^2B}{d\theta^2}u\,d\theta,$$

$$\int C\frac{du}{d\theta}d\theta = C\,u - \int\frac{dC}{d\theta}u\,d\theta,$$

and one will have:

$$(7)\qquad s = A\frac{d^2u}{d\theta^2}d\theta - \left(\frac{dA}{d\theta} - B\right)\frac{du}{d\theta} + \left(\frac{d^2A}{d\theta^2} - \frac{dB}{d\theta} + C\right)u - \int\left(\frac{d^3A}{d\theta^3} - \frac{d^2B}{d\theta^2} + \frac{dC}{d\theta}\right)u\,d\theta$$

for the value of *s*. Now the quantities *A*, *B*, *C* do not contain *u*, which must be an arbitrary function of θ. One can then express *s* in finite form by setting:

$$(8)\qquad u = \frac{\psi'(\theta)}{\dfrac{d^3A}{d\theta^3} - \dfrac{d^2B}{d\theta^2} + \dfrac{dC}{d\theta}},$$

in which $\psi'(\theta)$ denotes the derivative of an arbitrary function $\psi(\theta)$, because one will have:

$$(9)\qquad s = A\frac{d^2u}{d\theta^2}d\theta - \left(\frac{dA}{d\theta} - B\right)\frac{du}{d\theta} + \left(\frac{d^2A}{d\theta^2} - \frac{dB}{d\theta} + C\right)u - \psi(\theta),$$

and upon replacing *u* with its value that is given by equation (8) in equations (3) and (9), one will have the values of *x*, *y*, *z*, and *s* expressed in finite form, and with no integration sign. Those expressions contain three functions of θ, namely, $\psi(\theta)$, *p*, and *q*, only two of which must be considered to be arbitrary, because one can obviously take θ to be one of the two quantities *p* and *q* without altering the generality of the results, or if one prefers, one can equate one of those two quantities to any function of θ that one likes. The independent variable that has been left undetermined must be chosen in such a manner that one gets the simplest-possible formulas. That is a detail that we shall address in the following paragraph, but in conclusion, we point out that the question that we have posed is answered completely by the preceding, and also that the same method can be extended with no modification to the more general equation:

$$dx^2 + \ldots + dy^2 + dz^2 = ds^2,$$

which contains an arbitrary number m of variables $x, \ldots, y, z$.

Indeed, one can replace those m variables with m other quantities that are determined by the m equations:

$$z = p\,x + \ldots + q\,y - u,$$

$$0 = x\,dp + \ldots + y\,dq - du,$$

$$0 = x\,d^2 p + \ldots + y\,d^2 q - d^2 u,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$0 = x\,d^{m-1} p + \ldots + y\,d^{m-1} q - d^{m-1} u,$$

and upon following the same path as before, one will express the $m + 1$ quantities $x, \ldots, y, z$, and s in finite form with the aid of the m functions of the independent variable, $m - 1$ of which can be considered to be arbitrary.

The same transformation further applies to the more general equation:

$$dx^n + \ldots + dy^n + dz^n = ds^n,$$

in which n is an arbitrary number.

II.

Let us return to the question that was posed and address the choice of independent variable.

Since q and p are two functions of the same independent variable, they can be considered to be the rectilinear coordinates of an arbitrary plane curve. The general equation of the tangents to that curve will be:

$$q \cos \theta - p \sin \theta + \varphi(\theta) = 0,$$

in which θ is a variable parameter and φ is an arbitrary function. In addition, since any envelope is the envelope of its tangents, one can set:

$$\left\{\begin{aligned} q \cos\theta - p \sin\theta + \varphi &= 0, \\ q \sin\theta + p \cos\theta - \varphi' &= 0, \end{aligned}\right. \tag{10}$$

and consider those equations to belong to the curve. The second of those equations is the derivative of the first one with respect to the parameter θ that we have taken to be the independent variable. Finally, we put simply φ in place of φ (θ) and denote the derivatives in the manner of Lagrange.

One deduces the following values of p and q from equations (10), which contain an arbitrary function φ, and from the preceding, they are the most general ones that one can imagine:

$$\left\{\begin{aligned} q &= \varphi' \sin\theta - \varphi\cos\theta, \\ p &= \varphi' \cos\theta + \varphi\sin\theta. \end{aligned}\right. \tag{11}$$

One deduces from them that:

$$\left\{\begin{aligned} dq &= (\varphi'' + \varphi)\sin\theta\, d\theta, \\ dp &= (\varphi'' + \varphi)\cos\theta\, d\theta, \\ p\,dq - q\,dp &= (\varphi'' + \varphi)\varphi\, d\theta. \end{aligned}\right. \tag{12}$$

Hence:

$$\sqrt{dq^2 + dp^2 + (p\,dq - q\,dp)^2} = (\varphi'' + \varphi)\sqrt{1+\varphi^2}\, d\theta\,. \tag{13}$$

One further has:

$$\left\{\begin{aligned} d^2q &= [(\varphi''' + \varphi)\sin\theta + (\varphi'' + \varphi)\cos\theta]\, d\theta^2, \\ d^2p &= [(\varphi''' + \varphi)\cos\theta - (\varphi'' + \varphi)\sin\theta]\, d\theta^2, \end{aligned}\right. \tag{14}$$

so

$$dq\, d^2p - dp\, d^2q = -(\varphi'' + \varphi)^2\, d\theta^3. \tag{15}$$

We further have need for third-order differentials; we find that:

$$\left\{\begin{aligned} d^3q &= [(\varphi^{\mathrm{IV}} - \varphi)\sin\theta + 2(\varphi'' + \varphi)\cos\theta]\, d\theta^3, \\ d^3p &= [(\varphi^{\mathrm{IV}} + \varphi)\cos\theta - 2(\varphi'' + \varphi)\sin\theta]\, d\theta^3, \end{aligned}\right. \tag{16}$$

and we can deduce from (15) by differentiation that:

$$dq\, d^3p - dp\, d^3q = -2\,(\varphi'' + \varphi)(\varphi''' + \varphi')\, d\theta^4. \tag{17}$$

Finally, one infers from equations (14) and (16) that:

$$d^2q\, d^3p - d^2p\, d^3q = [(\varphi'' + \varphi)(\varphi^{\mathrm{IV}} + \varphi'') - 2\,(\varphi''' + \varphi') - (\varphi'' + \varphi)^2]\, d\theta^3. \tag{18}$$

Having said that, by virtue of equations (13), (15), (17), and (18), the values of A, B, C become:

$$(19)\qquad \begin{cases} A = -\dfrac{\sqrt{1+\varphi^2}}{\varphi''+\varphi}, \\[2mm] B = \dfrac{2(\varphi'''+\varphi')\sqrt{1+\varphi^2}}{(\varphi''+\varphi)^2}. \\[2mm] C = \left[\dfrac{(\varphi^{\mathrm{IV}}+\varphi'')}{(\varphi''+\varphi)^2} - \dfrac{(\varphi'''+\varphi')^2}{(\varphi''+\varphi)^3} - \dfrac{1}{\varphi''+\varphi}\right]\sqrt{1+\varphi^2}. \end{cases}$$

If one differentiates the first of equations (19) and then subtracts the result from the second one then one will have:

$$(20)\qquad \frac{dA}{d\theta} - B = -\frac{(\varphi'''+\varphi')\sqrt{1+\varphi^2}}{(\varphi''+\varphi)^2} - \frac{\varphi\varphi'}{(\varphi''+\varphi)\sqrt{1+\varphi^2}}.$$

If one differentiates equation (20) and adds the result to the third of equations (19) then one will have:

$$(21)\qquad \frac{d^2A}{d\theta^2} - \frac{dB}{d\theta} + C = -\frac{\varphi}{\sqrt{1+\varphi^2}} - \frac{1+\varphi^2+\varphi'^2}{(\varphi''+\varphi)(1+\varphi^2)^{3/2}},$$

and upon setting:

$$(22)\qquad -\frac{\varphi}{\sqrt{1+\varphi^2}} - \frac{1+\varphi^2+\varphi'^2}{(\varphi''+\varphi)(1+\varphi^2)^{3/2}} = P,$$

to abbreviate, one will then have:

$$(23)\qquad u = \frac{\psi'(\theta)}{P'},$$

and as a result:

$$(24)\qquad \begin{cases} \dfrac{du}{d\theta} = \dfrac{\psi''(\theta)}{P'} - \dfrac{P''\psi'(\theta)}{P'^2}, \\[2mm] \dfrac{d^2u}{d\theta^2} = \dfrac{\psi'''(\theta)}{P'} - \dfrac{2P''\psi''(\theta)}{P'^2} + \dfrac{(2P''^2 - P'P''')\psi'(\theta)}{P'^2}. \end{cases}$$

One will get the values of the quantities x, y, z, and s from that with the aid of formulas (3) and (9) upon eliminating the quantities p, q, u from their expressions. That can be done with no difficulty by appealing to some formulas that we gave, but we shall dispense with writing those values out here, due to their extreme complexity.

III.

We just found the general solution to the equation:

$$dx^2 + dy^2 + dz^2 = ds^2,$$

which includes two arbitrary functions φ and ψ, as we just saw. However, it is very remarkable that this same equation admits yet another solution that includes only one arbitrary function, and which will not be contained in the general solution that we just found. As a consequence, I will call that second solution the *singular* solution of the proposed equation. That singular solution relates to the case in which the quantity that we have called P reduces to a constant. Indeed, equations (23) and (24) will become illusory. The function $\varphi(\theta)$ will then be determined by the differential equation:

$$P = \text{constant},$$

i.e.:

$$\frac{\varphi}{\sqrt{1+\varphi^2}} + \frac{1+\varphi^2+\varphi'^2}{(\varphi''+\varphi)(1+\varphi^2)^{3/2}} = \text{constant} = m, \tag{25}$$

and the function u will be absolutely arbitrary. Equations (3) continue to give the values of x, y, and z, and the value of s will be given by equation (7), which will reduce to:

$$s = A\frac{d^2u}{d\theta^2} - \left(\frac{dA}{d\theta} - B\right)\frac{du}{d\theta} + m\,u.$$

One arrives at some simple formulas in the case of $m = 0$. Equation (25) will reduce to:

$$\varphi + \frac{1+\varphi^2+\varphi'^2}{(\varphi''+\varphi)(1+\varphi^2)} = 0,$$

and one will have:

$$\varphi = \sqrt{n^2\cos^2(\theta-\theta_0)-1}$$

for its integral, in which n and θ_0 are two arbitrary constants. Nonetheless, I do not believe that I must insist upon that particular case.

NOTE

ON A PARTIAL DIFFERENTIAL EQUATION

By J.-A. SERRET.

One knows from the beautiful theorem of Gauss that if one performs an arbitrary deformation of a surface then the product of the radii of principal curvature will preserve its value at every point. Hence, it will follow, in particular, that the surfaces that can be developed to a sphere will enjoy the property that the product of their radii of principal curvature will have the same value at every point.

The recent work of Liouville and Bertrand on Gauss's theorem has attracted my attention to that subject, and I shall undertake the study of the partial differential equation of surfaces whose radii of principal curvature have a constant product. Up to now, my research has not led me to any result that is satisfying from a geometric viewpoint, but I have found a solution to the partial differential equation that I just spoke of, which is a solution that contains one arbitrary function and represents only imaginary surfaces. That solution is quite remarkable in the sense that it presents itself as a sort of *singular solution* to the partial differential equation. I think that I will be doing something agreeable to the geometers by publishing that result here.

Conforming to the adopted usage, we shall let x, y, z denote the rectangular coordinates of the surface, while p and q denote the first-order derivatives $\frac{dz}{dx}$ and $\frac{dz}{dy}$, and r, s, t denote the second-order ones $\frac{d^2z}{dx^2}, \frac{d^2z}{dx\,dy}, \frac{d^2z}{dy^2}$. As everyone knows, the equation that we shall consider will then be:

$$a^2 (r\,t - s^2) = -(1 + p^2 + q^2)^2, \tag{1}$$

in which a denotes a real or imaginary constant.

We employ the known Legendre transformation and set:

$$u = p\,x + q\,y - z\,. \tag{2}$$

If we then take p and q to be independent variables then we will have:

$$x = \frac{du}{dp}, \qquad y = \frac{du}{dq}, \tag{3}$$

and

(4) $$\frac{d^2u}{dp^2} = \frac{t}{rt - s^2}, \qquad \frac{d^2u}{dp\,dq} = \frac{-s}{rt - s^2}, \qquad \frac{d^2u}{dq^2} = \frac{r}{rt - s^2},$$

so:

(5) $$\frac{d^2u}{dp^2} \cdot \frac{d^2u}{dq^2} - \left(\frac{d^2u}{dp\,dq}\right)^2 = \frac{1}{rt - s^2}.$$

Equations (2) and (3) will give z, y, and x as functions of p and q as soon as one knows the value of u . Finally, due to equation (5), the proposed equation (1) will become:

$$\frac{d^2u}{dp^2} \cdot \frac{d^2u}{dq^2} - \left(\frac{d^2u}{dp\,dq}\right)^2 = -\frac{a^2}{(1 + p^2 + q^2)^2},$$

or

$$\frac{d^2u}{dp^2} \cdot \frac{d^2u}{dq^2} = \left(\frac{d^2u}{dp\,dq} - \frac{a}{1 + p^2 + q^2}\right)\left(\frac{d^2u}{dp\,dq} + \frac{a}{1 + p^2 + q^2}\right),$$

and will result from eliminating the quantity λ from the following two:

(6) $$\begin{cases} \lambda \dfrac{d^2u}{dp^2} - \dfrac{d^2u}{dp\,dq} + \dfrac{a}{1 + p^2 + q^2} = 0, \\[2ex] \lambda \dfrac{d^2u}{dp\,dq} - \dfrac{d^2u}{dq^2} + \dfrac{a\,\lambda}{1 + p^2 + q^2} = 0. \end{cases}$$

If one differentiates the first of those equations by q and the second one by p then one will have:

$$\lambda \frac{d^3u}{dp^2dq} - \frac{d^3u}{dp\,dq^3} + \frac{d\lambda}{dq}\frac{d^2u}{dp^2} - \frac{2aq}{(1 + p^2 + q^2)^2} = 0,$$

$$\lambda \frac{d^3u}{dp^2\,dq} - \frac{d^3u}{dp\,dq^2} + \frac{d\lambda}{dp}\frac{d^2u}{dp\,dq} + \frac{a\dfrac{d\lambda}{dp}}{1 + p^2 + q^2} - \frac{2a\,p\,\lambda}{(1 + p^2 + q^2)^2} = 0,$$

and subtracting them will give:

(7) $$\frac{d\lambda}{dq}\frac{d^2u}{dp^2} - \frac{d\lambda}{dp}\frac{d^2u}{dp\,dq} + \frac{a\dfrac{d\lambda}{dp}}{1 + p^2 + q^2} - \frac{2a\,(q - p\,\lambda)}{(1 + p^2 + q^2)^2} = 0.$$

One can infer the values of the three derivatives $\dfrac{d^2u}{dp^2}$, $\dfrac{d^2u}{dp\,dq}$, $\dfrac{d^2u}{dq^2}$ from equations (6) and (7); namely:

(8)
$$\begin{cases} \left(\frac{d\lambda}{dq} - \lambda\frac{d\lambda}{dp}\right)\frac{d^2u}{dp^2} = \frac{a\frac{d\lambda}{dp}}{1+p^2+q^2} + \frac{2a(q-p\lambda)}{(1+p^2+q^2)^2}, \\ \left(\frac{d\lambda}{dq} - \lambda\frac{d\lambda}{dp}\right)\frac{d^2u}{dp\,dq} = \frac{a\left(\frac{d\lambda}{dq} + \lambda\frac{d\lambda}{dp}\right)}{1+p^2+q^2} + \frac{2a(q-p\lambda)}{(1+p^2+q^2)^2}, \\ \left(\frac{d\lambda}{dq} - \lambda\frac{d\lambda}{dp}\right)\frac{d^2u}{dq^2} = \frac{2a\lambda\frac{d\lambda}{dq}}{1+p^2+q^2} + \frac{2a\lambda^2(q-p\lambda)}{(1+p^2+q^2)^2}. \end{cases}$$

One sees that if λ is known then u will also be, since one knows its second differential d^2u. The quantity λ depends upon a second-order equation that we shall dispense with writing out, and which can be obtained easily with the aid of equations (8). For example, it will suffice to infer the values of $\frac{d^2u}{dp^2}$ and $\frac{d^2u}{dp\,dq}$ from the first two and set the values of their derivatives $\frac{d}{dq}\frac{d^2u}{dp^2}$ and $\frac{d}{dp}\frac{d^2u}{dp\,dq}$ equal to each other.

We remark that equations (8) will become illusory for any value of λ that satisfies the equation:

(9)
$$\frac{d\lambda}{dq} - \lambda\frac{d\lambda}{dp} = 0.$$

Now, I say that such a value for λ can correspond to a solution of our partial differential equation. That cannot happen unless the right-hand sides of equations (8) are zero; i.e., unless one has:

(10)
$$\frac{d\lambda}{dp} + \frac{q-\lambda p}{1+p^2+q^2} = 0.$$

It is very remarkable that one can satisfy equations (9) and (10) by the same value of λ, so it will follow that equations (8) will then be verified in their own right. Indeed, the general integral of equation (9) is:

$$p + \lambda q = \varphi(\lambda),$$

in which $\varphi(\lambda)$ denotes an arbitrary function. One then deduces that:

$$\frac{d\lambda}{dp} = \frac{1}{\frac{d\varphi}{d\lambda} - q},$$

and as a result, upon replacing p and $d\lambda / dp$ with their values, equation (10) will become:

$$\left(1+\varphi^2-\lambda\varphi\frac{d\varphi}{d\lambda}\right)+q\left[(1+\lambda^2)\frac{d\varphi}{d\lambda}-\lambda\varphi\right]=0.$$

In order for that equation to be true for any q, it is necessary that one must have both:

(11) $$1-\varphi^2-\lambda\varphi\frac{d\varphi}{d\lambda}=0 \quad\text{and}\quad (1+\lambda^2)\frac{d\varphi}{d\lambda}-\lambda\varphi=0.$$

Upon eliminating $d\varphi / d\lambda$ from equations (11), one will have:

$$1+\lambda^2+\varphi^2=0, \qquad \text{hence,} \qquad \varphi(\lambda)=\sqrt{-1-\lambda^2}\,,$$

and one easily assures oneself that this value of $\varphi(\lambda)$ satisfies each of equations (11).

If one then sets:

(12) $$p+\lambda q=\sqrt{-1-\lambda^2}$$

then equations (8) will be found to be verified in their own right, and one can, moreover, integrate equations (6), which are each linear and first-order, if one considers the du / dp in the first one and the du / dq in the second one to be the principal variable.

Let us examine the first of equations (6) and consider $du / dp = x$ to be a function of the quantities q and λ. p will then be a function of q and λ that is defined by equation (12). One has:

$$\frac{d^2u}{dp^2}=\frac{dx}{d\lambda}\frac{d\lambda}{dp},$$

$$\frac{d^2u}{dp\,dq}=\frac{dx}{d\lambda}\frac{d\lambda}{dp}+\frac{dx}{dq}.$$

As a result, upon considering equation (9), the first of equations (6) will give:

$$\frac{dx}{dq}=\frac{a}{1+p^2+q^2}=\frac{-a}{\left(\lambda+q\sqrt{-1-\lambda^2}\right)^2},$$

so upon integrating and letting ψ denote an arbitrary function:

(13) $$x=\frac{a}{\sqrt{-1-\lambda^2}\left(\lambda+q\sqrt{-1-\lambda^2}\right)^2}+\psi(\lambda).$$

One can likewise consider $du / dq = y$ to be a function of the quantities q and λ in the second of equations (6). One will have:

$$\frac{d^2u}{dp\,dq} = \frac{dy}{d\lambda}\frac{d\lambda}{dp}.$$

$$\frac{d^2u}{dq^2} = \frac{dy}{d\lambda}\frac{d\lambda}{dq} + \frac{dy}{dq}.$$

The second of equations (6) will then become:

$$\frac{dy}{dq} = \frac{a\lambda}{1+p^2+q^2} = \frac{-a\lambda}{\left(\lambda+q\sqrt{-1-\lambda^2}\right)^2},$$

so upon letting φ denote an arbitrary function:

(14)
$$y = \frac{a\lambda}{\sqrt{-1-\lambda^2}\,(\lambda+q\sqrt{-1-\lambda^2})} + \varphi(\lambda).$$

The functions y and φ are not both arbitrary and depend upon each other, as one sees, due to the fact that x and y must be partial derivatives of the same function of p and q.

One has:

$$du = x\,dp + y\,dq,$$

and due to equation (12):

$$dp = -\lambda\,dq - \frac{\lambda+q\sqrt{-1-\lambda^2}}{\sqrt{-1-\lambda^2}}\,d\lambda,$$

so

$$du = (y - \lambda x)\,dq - \frac{\lambda+q\sqrt{-1-\lambda^2}}{\sqrt{-1-\lambda^2}}\,x\,d\lambda,$$

and upon replacing x and y with their values that are inferred from equations (13) and (14):

$$du = [\varphi(\lambda) - \lambda\,\psi(\lambda)]\,dq + \frac{a\,d\lambda}{1+\lambda^2} - \frac{\lambda+q\sqrt{-1-\lambda^2}}{\sqrt{-1-\lambda^2}}\,\psi(\lambda)\,d\lambda,$$

$$= \frac{a\,d\lambda}{1+\lambda^2} - \frac{\lambda\psi(\lambda)}{\sqrt{-1-\lambda^2}}\,d\lambda - \{q\,\psi(\lambda)\,d\lambda + [\lambda\,\psi(\lambda) - \varphi(\lambda)]\,dq\}.$$

The first two terms in that value of du are exact differentials, and in order for the third one to be that way too, it is necessary and sufficient that one must have:

$$\lambda\,\psi'(\lambda) = \varphi'(\lambda),$$

in which φ' and ψ' denote the derivatives of φ and ψ, respectively. Hence, if $F(\lambda)$ then denotes an arbitrary function then one can set:

$$\psi(\lambda) = F(\lambda), \qquad \varphi(\lambda) = \lambda F'(\lambda) - F(\lambda),$$

and one will have:

$$du = \frac{a\,d\lambda}{1+\lambda^2} - \frac{\lambda F'(\lambda)}{\sqrt{-1-\lambda^2}} - [q\,F'(\lambda)\,d\lambda + F(\lambda)\,dq],$$

and upon integrating:

$$u = -\,q\,F(\lambda) + a \arctan \lambda - \int \frac{\lambda F'(\lambda)}{\sqrt{-1-\lambda^2}}\,d\lambda\,.$$

Finally, one determines z from the equation:

$$z = p\,x + q\,y - u,$$

and in summary, one will have the following values for x, y, z :

$$(15)\qquad \begin{cases} x = \dfrac{a}{\sqrt{-1-\lambda^2}\left(\lambda + q\sqrt{-1-\lambda^2}\right)} + F'(\lambda), \\[2ex] y = \dfrac{a\lambda}{\sqrt{-1-\lambda^2}\left(\lambda + q\sqrt{-1-\lambda^2}\right)} + \lambda F'(\lambda) - F(\lambda), \\[2ex] z = \dfrac{a}{\lambda + q\sqrt{-1-\lambda^2}} - a \arctan\lambda + \sqrt{-1-\lambda^2}\,F'(\lambda) + \displaystyle\int \frac{\lambda F'(\lambda)\,d\lambda}{\sqrt{-1-\lambda^2}}. \end{cases}$$

Those equations (15), which refer to an arbitrary function $F(\lambda)$, constitute a solution to the proposed partial differential equation. It belongs to an imaginary ruled surface, because one will obtain the following two equations when one eliminates q :

$$(16)\qquad \begin{cases} y = \lambda x - F(\lambda), \\[1ex] z = x\sqrt{-1-\lambda^2} - a \arctan\lambda + \displaystyle\int \frac{\lambda F'(\lambda)\,d\lambda}{\sqrt{-1-\lambda^2}}, \end{cases}$$

which are the equations of a straight line.

One can rid formulas (15) or (16) of the integration sign. If one sets:

$$F(\lambda) = (1+\lambda^2)\sqrt{-1-\lambda^2}\,f'(\lambda) - a\sqrt{-1-\lambda^2}\,,$$

in which $f(\lambda)$ denotes the derivative of an arbitrary function $f(\lambda)$, then equations (16) will become:

$$(17)\qquad \begin{cases} y = \lambda x + a\sqrt{-1-\lambda^2} - (1+\lambda^2)\sqrt{-1-\lambda^2}\, f'(\lambda), \\ z = x\sqrt{-1-\lambda^2} - a\lambda + \lambda(1+\lambda^2)\, f'(\lambda) - f(\lambda). \end{cases}$$

If one changes x and y into $x\sqrt{-1}$ and $y\sqrt{-1}$ then equations (17) will become:

$$(18)\qquad \begin{cases} y = \lambda x + a\sqrt{-1-\lambda^2} - (1+\lambda^2)^{3/2}\, f'(\lambda), \\ z = -x\sqrt{-1-\lambda^2} - a\lambda + \lambda(1+\lambda^2)\, f'(\lambda) - f(\lambda), \end{cases}$$

and equations (18) will constitute a real solution to the following partial differential equation:

$$a^2 (r\,t - s^2) = -(1 - p^2 - q^2)^2.$$

The imaginary surface that is represented by equations (15) or (16) or (17) then enjoys the property that the product of its two radii of principal curvature is constant at each point. However, what seems remarkable to me is that those two radii of curvature are themselves constant at each point and equal to each other; in other words, our surface, like the sphere, has the property that all of its points are umbilics.

Indeed, upon differentiating the equation:

$$p + \lambda\, q = \sqrt{-1-\lambda^2}$$

with respect to x and y and recalling equations (15) and (16), one will have:

$$r + \lambda\, s = \frac{\lambda\left(\lambda + q\sqrt{-1-\lambda^2}\right)^2}{a},$$

$$s + \lambda\, t = \frac{-\left(\lambda + q\sqrt{-1-\lambda^2}\right)^2}{a},$$

so

$$(19)\qquad t\,\lambda^2 + 2s\,\lambda + r = 0.$$

Moreover, the equation between p, q, and λ can be put into the form:

$$(20)\qquad (1 + q^2)\,\lambda^2 + 2\,p\,q\,\lambda + (1 + p^2) = 0.$$

If one eliminates λ from equations (19) and (20) then one will find that:

$$[(1 + q^2)\,r - 2\,p\,q\,s + (1 + p^2\, t]^2 - 4\,(1 + p^2 + q^2)\,(r\,t - s^2) = 0,$$

or, upon denoting the two radii of curvature by R and R':

$$(R - R')^2 = 0.$$

It is quite curious that this imaginary surface, which one can consider to be known, since Monge gave the complete integral of the equation:

$$R = R',$$

presents itself as a true singular solution of the equation:

$$RR' = \text{constant}.$$

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