"Ueber die Entstehung von Wirbelbewegungen in einer reibunglosen Flüssigkeit," Bull. Int. de l'Acad. des sci. de Cracovie (1896), 280-290.

On the creation of vortex motions in a frictionless fluid.

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In volume 56 of Wiedemann's Annalen (1895, pp. 144-147), **Schütz** referred to the fact that the famous theorem by **von Helmholtz** on the impossibility of creating or destroying vortex motions in a frictionless fluid by conservative forces is valid only under certain restricting conditions.

In the present article, in connection with the aforementioned work of **Schütz**, the following problem will now be treated: How must the pressure and density be distributed in a frictionless fluid that is exposed to the action of conservative forces exclusively, in order for vorticial motions to be created in it at a given moment in time? Moreover: With what velocity and what direction will the vorticial motions that were created begin to take the form of vortex filaments?

In that way, we shall not in the least deal with a proof that the conditions for the creation or destruction of vortices in a frictionless fluid by means of conservative force can actually be represented or even possible, but only with a precise analytical investigation of those conditions and the derivation of the corresponding lemmas that might possibly serve to prove the physical impossibility of the realization of those conditions as a result of its clarity.

According to **Schütz** (*loc. cit.*), under the assumption of conservative forces, one will get the following equations from the general equations of hydrodynamics for a fluid particle that possesses no vorticial motion at a given moment:

$$
(1) \qquad \left\{
\begin{aligned}
\xi' &= \frac{d\xi}{dt} = \frac{\partial}{\partial y}\left(\frac{1}{2\rho}\frac{\partial p}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{1}{2\rho}\frac{\partial p}{\partial y}\right), \\
\eta' &= \frac{d\eta}{dt} = \frac{\partial}{\partial z}\left(\frac{1}{2\rho}\frac{\partial p}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{1}{2\rho}\frac{\partial p}{\partial z}\right), \\
\zeta' &= \frac{d\zeta}{dt} = \frac{\partial}{\partial x}\left(\frac{1}{2\rho}\frac{\partial p}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{1}{2\rho}\frac{\partial p}{\partial x}\right),
\end{aligned}
\right.
$$

in which ξ, η, ζ mean the components of the (just-created) vorticity at the point x, y, z along the coordinate axes x, y, z, resp., t means the time, p is the pressure, and ρ is the density. The author started from those equations in order to arrive at a solution to the questions that were posed, and indeed in an intuitive, geometric form.

If one performs the operations that are suggested in (1) then it will follow that:

$$(2)\qquad \left\{\begin{aligned}
\xi' &= \frac{1}{2\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial z} - \frac{\partial p}{\partial z}\frac{\partial \rho}{\partial y}\right) = \frac{1}{2\rho^2}\left(\frac{p,\rho}{y,z}\right),\\
\eta' &= \frac{1}{2\rho^2}\left(\frac{\partial p}{\partial z}\frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial \rho}{\partial z}\right) = \frac{1}{2\rho^2}\left(\frac{p,\rho}{z,x}\right),\\
\zeta' &= \frac{1}{2\rho^2}\left(\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}\right) = \frac{1}{2\rho^2}\left(\frac{p,\rho}{x,y}\right),
\end{aligned}\right.$$

in which $\left(\frac{p,\rho}{y,z}\right)$, etc., are introduced as notations for the corresponding determinants for the sake of brevity. One now considers the surfaces:

1. constant pressure:

$$(3)\qquad p\,(x, y, z) = \text{const.},$$

and

2. constant density:

$$(4)\qquad \rho\,(x, y, z) = \text{const.},$$

which go through the given point at the given moment. If the two surfaces intersect, and therefore the given point lies along their line of intersection, then the projections *dx*, *dy*, *dz* of the arc-length element *ds* along the line of intersection, as measured from that point (*x*, *y*, *z*), will satisfy the following two equations:

$$(5)\qquad \left\{\begin{aligned}
\frac{\partial p}{\partial x}\frac{dx}{dz} + \frac{\partial p}{\partial y}\frac{dy}{dz} &= -\frac{\partial p}{\partial z},\\
\frac{\partial \rho}{\partial x}\frac{dx}{dz} + \frac{\partial \rho}{\partial y}\frac{dy}{dz} &= -\frac{\partial \rho}{\partial z}.
\end{aligned}\right.$$

If one solves those equations for *dx* / *dz* and *dy* / *dz* then that will give:

$$(6)\qquad dx : dy : dz = \left(\frac{p,\rho}{y,z}\right) : \left(\frac{p,\rho}{z,x}\right) : \left(\frac{p,\rho}{x,y}\right),$$

i.e., from (2):

$$(7)\qquad dx : dy : dz = \xi' : \eta' : \zeta' .$$

One will then get the following:

Theorem I:

If a vorticial motion is created at a given particle in a frictionless fluid that is acted upon by conservative forces exclusively then the initial vortex axis of the particle will coincide with the element of the curve of intersection of the surfaces of constant pressure and constant density to which the particle momentarily belongs. The vortex lines that are defined in that way will then coincide with those lines of intersection completely.

However, it obviously does not follow from this that a vorticial motion must always arise as soon as the surfaces in question merely intersect. Nonetheless, we would like to prove that this is, in fact, the case:

From (2), the resulting vorticial *acceleration* ω' of the vortex that is created will be:

$$\omega' = (\xi'^2 + \eta'^2 + \zeta'^2) = \frac{1}{2\rho^2}\left\{\left(\frac{p,\rho}{y,x}\right)^2 + \left(\frac{p,\rho}{z,x}\right)^2 + \left(\frac{p,\rho}{x,y}\right)^2\right\}^{1/2} . \tag{8}$$

The direction cosines of the normals n, ν of the surfaces p = const., ρ = cont. at the point x, y, z are:

$$a = \frac{1}{\sqrt{P}}\frac{\partial p}{\partial x}, \quad b = \frac{1}{\sqrt{P}}\frac{\partial p}{\partial y}, \quad c = \frac{1}{\sqrt{P}}\frac{\partial p}{\partial z}, \tag{9}$$

or

$$\alpha = \frac{1}{\sqrt{P}}\frac{\partial \rho}{\partial x}, \quad \beta = \frac{1}{\sqrt{P}}\frac{\partial \rho}{\partial y}, \quad \gamma = \frac{1}{\sqrt{P}}\frac{\partial \rho}{\partial z}, \tag{10}$$

respectively, in which one sets:

$$P = \left(\frac{\partial p}{\partial x}\right)^2 + \left(\frac{\partial p}{\partial y}\right)^2 + \left(\frac{\partial p}{\partial z}\right)^2, \qquad R = \left(\frac{\partial \rho}{\partial x}\right)^2 + \left(\frac{\partial \rho}{\partial y}\right)^2 + \left(\frac{\partial \rho}{\partial z}\right)^2, \tag{11}$$

to abbreviate. If one assumes that the directions of n, ν that point towards *increasing* pressure (density, resp.) are *positive* then one will have to take the square roots in (9) and (10) with the plus signs. The angle θ that the two normals n, ν define between themselves is determined from:

$$\cos\theta = \frac{1}{\sqrt{PR}}\left[\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial x} + \frac{\partial p}{\partial y}\frac{\partial \rho}{\partial y} + \frac{\partial p}{\partial z}\frac{\partial \rho}{\partial z}\right] . \tag{12}$$

On the other hand, the bracketed sum on the right-hand side of equation (8) is equal to:

$$(13)\quad \begin{cases} \omega'^2 \cdot 4\rho^4 = \left(\dfrac{\partial p}{\partial y}\right)^2 \cdot \left(\dfrac{\partial \rho}{\partial z}\right)^2 + \left(\dfrac{\partial p}{\partial z}\right)^2 \cdot \left(\dfrac{\partial \rho}{\partial y}\right)^2 + \cdots + \cdots - 2\dfrac{\partial p}{\partial y}\dfrac{\partial \rho}{\partial y}\dfrac{\partial p}{\partial z}\dfrac{\partial \rho}{\partial z} - \cdots - \cdots \\ = \left[\left(\dfrac{\partial p}{\partial x}\right)^2 + \left(\dfrac{\partial p}{\partial y}\right)^2 + \left(\dfrac{\partial p}{\partial z}\right)^2\right] \cdot \left[\left(\dfrac{\partial \rho}{\partial x}\right)^2 + \left(\dfrac{\partial \rho}{\partial y}\right)^2 + \left(\dfrac{\partial \rho}{\partial z}\right)^2\right] - \left[\dfrac{\partial p}{\partial x}\dfrac{\partial \rho}{\partial x} + \dfrac{\partial p}{\partial y}\dfrac{\partial \rho}{\partial y} + \dfrac{\partial p}{\partial z}\dfrac{\partial \rho}{\partial z}\right]^2. \end{cases}$$

If one combines that relation with (12) then one will get:

$$\cos^2\theta = 1 - \frac{4\rho^4\omega'^2}{P\,R}, \qquad 4\rho^4\omega'^2 = P\,R\sin^2\theta,$$

so

$$(14)\qquad \omega' = \frac{1}{2\rho^2}\sqrt{P}\sqrt{R}\sin\theta,$$

or since n, ν are merely the directions in which the pressure (density, resp.) increases fastest:

$$(15)\qquad \omega' = \frac{1}{2\rho^2}\frac{\partial p}{\partial n}\frac{\partial \rho}{\partial \nu}\sin\theta.$$

With that, we have the proof of the following:

Theorem II:

The necessary and sufficient condition for the creation of a vorticial motion at a particle in a frictionless fluid that is acted upon by conservative forces exclusively is that a surface of constant pressure and a surface of constant density that did not simultaneously include the given particle at all up to a point in time or contact each other there ($\theta = 0$ or $\theta = \pi$) must begin to intersect in such a way that the particle in question will move along their line of intersection at the given moment in time. The axis of the vorticial motion that is created will coincide with the corresponding arc-length element of the line of intersection, and indeed the particle will begin to exhibit a vorticial motion around that element in the sense from ν to n (along the shortest path) with an acceleration of its vorticity that equal:

$$(15)\qquad \omega' = \frac{1}{2\rho^2}\frac{\partial p}{\partial n}\frac{\partial \rho}{\partial \nu}\sin(\nu, n) = \frac{1}{2\rho^2}V\,\frac{\partial \rho}{\partial \nu}\,\frac{\partial p}{\partial n}.$$

We can also express the result that was just obtained as follows: *No vorticial motion will be created* at a vortex-free particle *when and only when p is a function of only ρ* such that the corresponding p (and ρ) surfaces overlap, or when ρ or p are both quantities that are independent of position.

However, if the surfaces intersect from a certain moment onwards then they will simultaneously define a vortex line along the intersection curve, and indeed such that after a time element *dt* has elapsed, the individual particle on the vortex line will achieve the corresponding vorticial velocity:

$$d\omega = \frac{1}{2\rho^2}\sin\theta\,\frac{\partial p}{\partial n}\cdot\frac{\partial \rho}{\partial \nu}\,dt\,. \tag{16}$$

If a family of p-surfaces intersects a family of ρ-surfaces then at the same time, that will define a complete vortex filament whose momentum over the time interval *dt* can be given directly by (16). One considers an infinitely-thin vortex filament that fills up the channel between two neighboring surfaces of constant pressure p and $p + \frac{\partial p}{\partial n}dn$ and two neighboring surfaces of constant density ρ and $\rho + \frac{\partial \rho}{\partial \nu}d\nu$ that begin to intersect each other from a certain moment onward. Without reverting to the original expressions for the components ξ, η, ζ of the vorticity, one can see directly from (16) that the momentum of the vortex filament that is created will have one and the same value for its entire *length*. Namely, the cross-section q of the vortex filament in question is a parallelogram everywhere whose sides are:

$$a = dn : \sin\theta, \quad b = d\nu : \sin\theta, \tag{17}$$

which subtend and angle of $\theta = (\nu, n)$. One will then have:

$$q = a\,b\,\sin\theta = \frac{dn\cdot d\nu}{\sin\theta}, \tag{18}$$

so the moment over the time interval *dt* :

$$q\,d\omega = \frac{dt}{2\rho^2}\frac{\partial p}{\partial n}\frac{\partial \rho}{\partial \nu}\,dn\,d\nu = -\frac{dt}{2}\cdot\frac{\partial p}{\partial n}dn\cdot\frac{\partial}{\partial \nu}\left(\frac{1}{\rho}\right)d\nu\,. \tag{19}$$

However, since the pressure difference between the two p-surfaces, and likewise the density difference (so the difference between the values of $1 / \rho$) between the two ρ-surfaces, will remain constant along the entire channel, *the momentum will also have one and the same value along the entire vortex filament*. It will follow immediately from this in a known way that *already at the instant of its creation* the vortex filament will extend between two locations on the boundary surfaces of the fluid or define closed rings.

However, the value of the vortex momentum will *vary in time* for one and the same vortex filament, and indeed will continue to do so as long as the p-surfaces and ρ-surfaces continue to intersect each other in such a way that their angle of inclination θ remains unvarying. Nonetheless, if that angle becomes 0 or π (i.e., the p-surfaces coincide with the ρ-surfaces at a certain instant) then the vortex filament that is created will keep the value of the momentum that was reached up to that point in time unvarying in the further course of time as long as the surfaces do not begin to define a line of intersection with each other. One can mathematically apply those theorems to a two-dimensional motion of the fluid for the sake of simplicity. The general equations that couple the components of the vorticity with its variation in time (**Schütz**, *loc. cit.*) will go to:

$$\eta = 0, \quad \zeta = 0, \quad \eta' = 0, \quad \zeta' = 0,$$

(20) $$\omega' = \xi' = \frac{d\xi}{dt} = \frac{1}{2\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial z} - \frac{\partial p}{\partial z}\frac{\partial \rho}{\partial y}\right) - \xi\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right),$$

in this case when we make the "plane of motion" be, e.g., the *yz*-plane, in which *v*, *w* mean the components of the velocity in the directions of the *y*-axis (*z*-axis, resp.). In this case, the *p*-surfaces and the ρ-surfaces are merely cylindrical surfaces that are perpendicular to the *yz*-plane and whose normals n, ν (so normals to the *yz*-plane) are everywhere parallel. Upon introducing the normals n, ν and the angle θ, equation (20) will go to:

(21) $$\frac{d\xi}{dt} = \frac{1}{2\rho^2}\frac{\partial p}{\partial n}\frac{\partial \rho}{\partial \nu}\sin\theta - \xi\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right).$$

The continuity equation in our case reads:

(22) $$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial y}(\rho v) - \frac{\partial}{\partial y}(\rho w),$$

in which $\partial\rho / \partial t$ mean the time variation of the density in a fixed element of space. It follows from this that:

(23) $$-\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \frac{1}{\rho}\left[\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}\right] = \frac{1}{\rho}\frac{d\rho}{dt},$$

in which $d\rho / dt$ refers to an individual moving fluid particle. If one substitutes that value of $\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$ in (21) then one will get:

(24) $$\frac{d\xi}{dt} = \frac{1}{2\rho^2}\frac{\partial p}{\partial n}\frac{\partial \rho}{\partial \nu}\sin\theta + \xi\frac{1}{\rho}\frac{d\rho}{dt},$$

or

(25) $$\frac{d}{dt}\left(\frac{\xi}{\rho}\right) = \frac{1}{2\rho^2}\frac{\partial p}{\partial n}\frac{\partial \rho}{\partial \nu}\sin\theta,$$

which is an equation that is valid for every value of *t*.

Now, since the density ρ is inversely proportional to the volume (so the two-dimensional motion of the cross-section of the vortex filament, in our case), it will follow that ξ / ρ will differ from the vortex momentum by only a multiplicative constant, so (25) will, in fact, imply that *the momentum of a vortex filament will be independent of time if and only if the p-surfaces and ρ-surfaces intersect at all or at least neither in the interior of the vortex filament nor on the surface. However, the law of time variation of the vortex momentum will otherwise be given by the general formula* (25) (in the case of two-dimensional motion).

Finally, it will be shown how a vorticial motion arises *mechanically* under the conditions of formula (15) that were cited in Theorem II. To that end, imagine an infinitely-small parallelepiped in the interior of the fluid that is determined by the edges dn, $d\nu$, and one of the edges ds that is perpendicular to them. The latter is, at the same time, an element of the line of intersection of a p-surface and a ρ-surface: For the sake of simplicity, let the angle $\theta = (n, \nu)$ be a right angle, such that the parallelepiped will be a rectangular one. If one divides the volume of the parallelepiped into two equal parts by a plane that is parallel to the ds, dn-wall and is ρ means the mean density in the interior of the entire volume then one can assume that the two parts have the homogeneous densities $\rho - \frac{\partial \rho}{\partial \nu}\frac{d\nu}{4}$ and $\rho + \frac{\partial \rho}{\partial \nu}\frac{d\nu}{4}$, resp. (which are assigned to exactly their midpoints 1 and 2, resp.), so they will possess masses:

(26) $$m_1 = \left(\rho - \frac{1}{4}\frac{\partial \rho}{\partial \nu} d\nu\right) ds\, dn\, \frac{d\nu}{2},$$

(27) $$m_2 = \left(\rho + \frac{1}{4}\frac{\partial \rho}{\partial \nu} d\nu\right) ds\, dn\, \frac{d\nu}{2}.$$

(The direction $\overline{12}$ coincides with the positive direction of ν.) Now, each of the two parts will be acted upon in the $-n$ direction by the force:

(28) $$N = \frac{\partial p}{\partial n} dn \cdot ds\, \frac{d\nu}{2}.$$

If one imagines rigidifying each of the two fluid parts and concentrating their masses m_1, m_2 at the midpoints 1, 2 then that force will impart the accelerations:

(29) $$w_1 = N : m_1 = \frac{\partial p}{\partial n} : \left(\rho - \frac{1}{4}\frac{\partial \rho}{\partial \nu} d\nu\right),$$

(30) $$w_2 = N : m_2 = \frac{\partial p}{\partial n} : \left(\rho + \frac{1}{4}\frac{\partial \rho}{\partial \nu} d\nu\right),$$

upon those points 1, 2, resp., in the $-n$ direction, such that $|w_1| > w_2$. If one sets:

$$w_1 = w_0 + \tfrac{1}{2}(w_1 - w_2), \tag{31}$$

$$w_2 = w_0 - \tfrac{1}{2}(w_1 - w_2) \tag{32}$$

then $w_0 = \tfrac{1}{2}(w_1 + w_2)$ will mean the *translatory* acceleration of the midpoint O of the entire parallelepiped (in the direction of $-n$), which bisects the line segment $\overline{12}$, and:

$$\tfrac{1}{2}(w_1 - w_2) : \tfrac{1}{4}d\nu = 2(w_1 - w_2) : d\nu, \tag{33}$$

i.e., from (20) and (30):

$$2(w_1 - w_2) : d\nu = 2\frac{\partial p}{\partial n}\cdot\left(\frac{1}{\rho - \frac{1}{4}\frac{\partial \rho}{\partial \nu}d\nu} - \frac{1}{\rho + \frac{1}{4}\frac{\partial \rho}{\partial \nu}d\nu}\right) : d\nu = \frac{1}{\rho^2}\frac{\partial p}{\partial n}\frac{\partial p}{\partial \nu} \tag{34}$$

is the *angular acceleration* of the system of points 1, 2 in the sense of $\nu - n$ around the axis that goes through the midpoint O and is parallel to ds, or in fact, twice the acceleration of the vorticity ω' at O, which was to be proved. In a completely-similar way, one gets the expression (34), multiplied by $\sin\theta$, in the general case in which the p-surfaces define an arbitrary angle θ with the ρ-surfaces, which agrees with Theorem II.

The author presented the lemmas above as merely mathematical consequences of the mechanical conditions in a frictionless fluid that is acted upon by conservative forces. Of course, there is an entirely-different problem that will not be touched upon here, namely, the question of the physical conditions for the existence of actual lines of intersection of the p-surfaces and the ρ-surfaces. The only thing that seems clear from the outset is that when one imposes pressure and density distributions upon a fluid mass with the aforementioned character in any way such that the p-surfaces intersect the ρ-surfaces and give rise to new vorticity, and one then leaves the fluid to itself, the required distributions in it will change *in a very short time* in such a way that all of the p-surfaces will overlap with the corresponding ρ-surfaces. However, the momenta of the vortex filaments that are created in that short time interval will no longer vary in time then, and no further new vortices will be created.
