"Zur Krümmungstheorie der Integralkurven der Pfaffschen Gleichung," Math. Ann. **101** (1928), 261-272.

On the curvature of integral curves of the Pfaff equation

By

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1. – The properties of systems of integral curves of the Pfaff equation:

$$P\,dx + Q\,dy + R\,dz = 0 \tag{1}$$

have attracted the attention of distinguished researchers on several occasions. The ground-breaking papers of **A. Voss** (Math. Ann. **16**, **23**, **25**), S. Lie (*Geometrie der Berührungstransformationen*, Bd. I), and R. v. Lilienthal (*Grundlagen der Krümmunstheorie der Kurvenscharen*, 1896 and Math. Ann. **32** and **38**) were followed by Rogers [Proc. Roy. Irish Acad. (A) **29**, no. 6, (1912)] and G. Darboux [Bull. Soc. Math. (1912) and *Théorie des Surfaces*, 2nd ed., v. II, 1915]. The analogy with the theory of surfaces prompted Abbot Issaly to choose the term "pseudo-surfaces" in numerous publications.

However, I would not like to stress the analogies, but the deviations from them. It seems to me that such an investigation will give a glimpse into the intrinsic structure of surface elements, since the integral curves of (1) that go through the point (x, y, z) just like curves in a surface, have all of their tangents in a plane:

$$P\,(X - x) + Q\,(Y - y) + R\,(Z - z) = 0, \tag{2}$$

but when the integrability condition:

$$G = P\,(R'_y - Q'_z) + Q\,(P'_z - R'_z) + R\,(Q'_x - P'_y) = 0 \tag{3}$$

is not fulfilled, they will not define a surface element at (x, y, z).

For that reason, even in the case of $G = 0$, it is convenient to not consider the individual surface $F = 0$, but all of the integral surfaces $F = \text{const.}$ of (1).

2. Principle tangents. – The point $(x + dx, y + dy, z + dz)$ corresponds to the plane:

$$(P + dP)\,(X - x - dx) + (Q + dQ)\,(Y - y - dy) + (R + dR)\,(Z - z - dz) = 0. \tag{2′}$$

The perpendicular that is dropped at the point (x, y, z) has the length:

$$\delta = \frac{dP\,dx + dQ\,dy + dR\,dz}{\sqrt{P^2 + Q^2 + R^2}}, \tag{4}$$

when one keeps only the second-order terms, except that δ will have order three in the directions (dx, dy, dz) that satisfy the equation:

$$dP \cdot dx + dQ \cdot dy + dR \cdot dz = 0. \tag{5}$$

Those are the *principle tangent directions*. One will come to the same equation (5) when one looks for the integral curves of (1) whose osculating planes coincide with the corresponding planes (2). One can regard those two explanations as basically equivalent.

3. Singular points and minimal lines. – If one has:

$$P = 0, \quad Q = 0, \quad R = 0 \tag{6}$$

simultaneously at the point (x, y, z) then the plane (2) will be indeterminate. The tangents will lie on the surface of the cone:

$$\begin{aligned} &P'_x(X-x)^2 + Q'_y(Y-y)^2 + R'_z(Z-z)^2 \\ &+ (P'_y + Q'_z)(X-x)(Y-y) + (P'_z + R'_x)(X-x)(Z-z) + (Q'_z + R'_y)(Y-y)(Z-z) = 0, \end{aligned} \tag{7}$$

which can degenerate into a line that is the axis of a pencil of planes, two planes (biplanar point), or even into a double plane (uniplanar point) in certain cases. Such points will be ignored in what follows. Minimal lines will also be ignored, which are lines whose tangents possess imaginary directions:

$$(P^2 + R^2)\,dx^2 + 2PQ\,dx\,dy + (Q^2 + R^2)\,dy^2 = 0. \tag{8}$$

The properties of such lines are entirely analogous to the properties of minimal lines of surfaces.

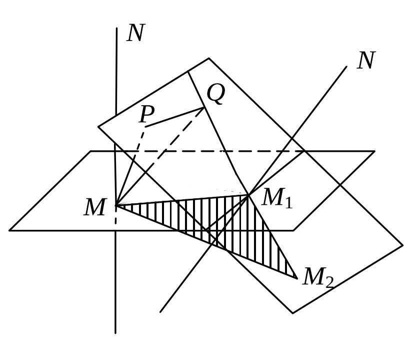

Figure 1.

4. Radius of curvature. The Meusnier circle. – We take a second point M_2 in the plane (2′) in (Fig. 1) and draw the perpendicular MP to the plane (2′) and the perpendicular MQ to the line $M_1\,M_2$. Let φ be the angle between the planes $N_1\,M_1\,M_2$ and $M\,M_1\,M_2$ [$N_1\,M_1$ is the normal to (2′)]. Since $MP \parallel M_1\,N_1$, we will also have:

(9) $$\sphericalangle PMQ = \varphi,$$

$$\overline{MP} = \overline{MQ}\cdot\cos\varphi\,.$$

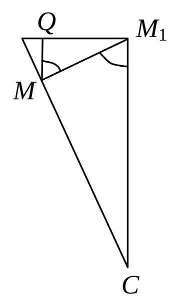

Figure 2.

Let lines $M\,C \parallel M\,M_1$ and $M_1\,C \parallel M_1\,Q$ be drawn in the plane $M\,Q\,M_1$, and let C be their point of intersection (Fig. 2). One will then have $\dfrac{M\,Q}{M\,M_1} = \dfrac{M\,M_1}{M_1\,C}$, so:

$$\frac{1}{r_\varphi} = \frac{dP\,dx + dQ\,dy + dR\,dz}{\cos\varphi\cdot ds^2\cdot\sqrt{P^2+Q^2+R^2}}\,.$$

If $\varphi = 0$ then one will have:

(10) $$\frac{1}{r} = \frac{dP\,dx + dQ\,dy + dR\,dz}{ds^2\sqrt{P^2+Q^2+R^2}}\,.$$

That will yield Meusnier's theorem:

$$\boxed{r_\varphi = r\cdot\cos\varphi.}$$

When φ changes from 0 to 2π, the center of curvature will trace out a circle, namely, the *Meusnier circle.*

5. Umbilic points. – If we eliminate dz with the help of (1) and write:

$$(1) = R\,P'_x - P\,P'_z\,,\qquad (2) = R\,P'_y - Q\,P'_z\,,\qquad (3) = R\,Q'_x - P\,Q'_z\,,$$

$$(4) = R\,Q'_y - Q\,Q'_z\,,\qquad (5) = R\,R'_x - P\,R'_z\,,\qquad (6) = R\,R'_y - Q\,R'_z\,,$$

to abbreviate, then (10) will assume the form:

$$(10') \qquad \frac{1}{r} = \frac{[R(1)-P(5)]\,dx^2+[R(2+3)-P(6)-Q(5)]\,dx\,dy+[R(4)-Q(6)]\,dy^2}{[(R^2+P^2)\,dx^2+2PQ\,dx\,dy+(R^2+Q^2)\,dy^2]\sqrt{P^2+Q^2+R^2}}.$$

r will then be independent of the direction dy / dx only when:

$$(11) \qquad \frac{R(1)-P(5)}{R^2+P^2} = \frac{R(2+3)-P(6)-Q(5)}{2PQ} = \frac{R(4)-Q(6)}{R^2+Q^2},$$

Such points will be called umbilic points (*umbilica*). In general, they trace out a curve, namely, the *line of circle points* (*Kreispunktlinie*).

6. Indicatrix. – We lay line segments equal to $\sqrt{|r|}$ in every direction in the plane (2) from M. If M is the coordinate origin and (2) is the XY-plane then we will have:

$$(12) \qquad \frac{1}{r} = P'_x \cos^2\alpha + (P'_y + Q'_x)\sin\alpha\cos\alpha + Q'_y \sin^2\alpha,$$

and the endpoint of the aforementioned segment will have the coordinates:

$$(13) \qquad \xi = \sqrt{|r|}\cdot\frac{dx}{ds}, \qquad \eta = \sqrt{|r|}\cdot\frac{dy}{ds}.$$

Hence, the position will be determined by the equation:

$$(14) \qquad P'_x\xi^2 + (P'_y + Q'_x)\,\xi\eta + Q'_y\eta^2 = 1.$$

It will be an ellipse, hyperbola, or parabola (≡ pair of parallel lines) according to whether:

$$(P'_y + Q'_x)^2 - 4P'_x Q'_y \equiv \tfrac{1}{4}G^2 + \Delta = \tfrac{1}{4}\Delta'$$

is negative, positive, or zero, resp. The corresponding points in space are then called *elliptic*, *hyperbolic*, or *parabolic*, resp. In the first case, the principal tangents are imaginary, in the second case, they are real, but distinct, and in the third case, they coincide. The projective behavior of the directions $MM_1 = (dx, dy, dz)$ and $M_1 M' \equiv (d'x, d'y, d'z)$ [i.e., the intersection of the tangent planes (2) and (2′) at the points M and M_1] (Fig.1) ([1]):

$$\frac{d'x}{Q\,dR - R\,dQ} = \frac{d'y}{R\,dP - P\,dR} = \frac{d'y}{P\,dQ - Q\,dP}$$

([1]) The notation M' is missing from Fig. 1.

will also be elliptic in the first case, hyperbolic, in the second, and parabolic, in the third, but involutory only when $G = 0$.

The axes of the indicatrix are determined by the angle α_0 :

$$\tan 2\alpha_0 = \frac{P'_y + Q'_x}{P'_x - Q'_y}.$$

If one chooses them to be the coordinate axes then equation (14) will become:

$$(14') \qquad (P'_x)\cdot\xi^2 + (Q'_y)\cdot\eta^2 = 1.$$

The directions of the principal tangents will then coincide with the asymptotes of the indicatrix.

7. Directions of principal curvature. – When α_1 changes, the radius r in (12) will also change, and it will assume its extreme values when:

$$(Q'_y - P'_x)\cos\alpha\sin\alpha + (P'_y + Q'_x)(\cos^2\alpha - \sin^2\alpha) = 0,$$

and thus in the directions of the axes of the indicatrix.

If α were replaced with $\alpha + \pi / 2$ in (12) then one would have:

$$(12') \qquad \frac{1}{r'} = P'_x\cos\alpha\sin\alpha - (P'_y + Q'_x)\sin\alpha\cos\alpha + Q'_y\cos^2\alpha.$$

The sum would then give:

$$(15) \qquad \frac{1}{r} + \frac{1}{r'} = P'_x + Q'_y = (P'_x) + (Q'_y),$$

and would be independent of direction.

However, since (12) will assume the form:

$$(P'_x)\cos^2\alpha + (Q'_y)\sin^2\alpha = \frac{1}{r}$$

when the axes of the indicatrix are coordinate axes, the extremal radii r_1 and r_2 will be:

$$(P'_x) = r_1, \qquad (Q'_y) = r_2,$$

and formulas (12) and (14) will then take on the form:

$$(12'') \qquad \frac{1}{r} = \frac{1}{r_1}\cos^2\alpha + \frac{1}{r_2}\sin^2\alpha$$

$$(14'') \qquad \frac{1}{r_1}\xi^2 + \frac{1}{r_2}\eta^2 = \pm 1.$$

However, equation (15) will become:

$$(15') \qquad \frac{1}{r} + \frac{1}{r'} = \frac{1}{r_1} + \frac{1}{r_2}.$$

One can also call one-half the sum of the principle curvatures the *mean curvature* here.

The locus of the curvature circles will now be:

$$(X^2 + Y^2 + Z^2) + 2\,Z\,(X^2 + Y^2) = 0,$$

which will have the Z-axis [viz., the normal to (2)] for a double line and will contain the two isotropic lines $X^2 + Y^2 = 0$, $Z = 0$.

8. Total curvature. – In order to arrive at general formulas, we determine the extremal values of:

$$(16) \qquad \frac{\sqrt{P^2+Q^2+R^2}}{r} = P'_x a^2 + Q'_y b^2 + R'_z c^2 + (P'_z + Q'_x)\,ab + (P'_z + R'_x)\,a\,c + (Q'_z + R'_y)\,b\,c$$

with the conditions that:

$$P\,a + Q\,b + R\,c = 0, \qquad a^2 + b^2 + c^2 = 1.$$

We will come to the equations:

$$(17) \qquad \left\{\begin{aligned} 2P'_x a + (P'_y + Q'_x)\,b + (P'_z + R'_x)\,c + \lambda P - Sa &= 0, \\ (P'_y + Q'_x)\,a + 2Q'_y\,b + (Q'_z + R'_y)\,c + \lambda Q - Sb &= 0, \\ (P'_z + R'_x)\,2\,a + (Q'_z + R'_y)\,b + 2R'_z\,c + \lambda R - Sc &= 0, \end{aligned}\right.$$

and

$$P \cdot a + Q \cdot b + R \cdot c = 0.$$

One obtains the quadratic equation for S from:

$$\text{(18)} \qquad \begin{vmatrix} 2P'_x - S & P'_y + Q'_x & P'_z + R'_x & P \\ P'_y + Q'_x & 2Q'_y - S & Q'_z + R'_y & Q \\ P'_z + R'_x & Q'_z + R'_y & 2R'_z - S & R \\ P & Q & R & 0 \end{vmatrix} = 0.$$

The quantity S is equal to $2\dfrac{\sqrt{P^2+Q^2+R^2}}{r}$, which one will find when one multiplies (17) by a, b, c, respectively, and sums.

If we then denote:

$$\text{(19)} \qquad 4H = \begin{vmatrix} 2Q'_y & Q'_z + R'_y & Q \\ Q'_z + R'_y & 2R'_z & R \\ Q & R & 0 \end{vmatrix} + \begin{vmatrix} 2P'_x & P'_z + R'_x & P \\ P'_z + R'_x & 2R'_z & R \\ P & R & 0 \end{vmatrix} + \begin{vmatrix} 2P'_x & P'_y + Q'_x & P \\ P'_y + Q'_x & 2Q'_y & Q \\ P & Q & 0 \end{vmatrix}$$

then (18) will assume the form:

$$\text{(18}'\text{)} \qquad S^2 (P^2 + Q^2 + R^2) - 4H \cdot S - \Delta' = 0.$$

Hence:

$$\text{(20)} \qquad \begin{cases} \dfrac{1}{r_1} + \dfrac{1}{r_2} = \dfrac{2H}{(P^2+Q^2+R^2)^{3/2}}, \\[2ex] \dfrac{1}{r_1 r_2} = -\dfrac{\Delta'}{(P^2+Q^2+R^2)^{3/2}}. \end{cases}$$

The two radii of principal curvature have the same signs at an elliptic point, so $\dfrac{1}{r_1 r_2} > 0$, while they will have different signs at a hyperbolic point, so $\dfrac{1}{r_1 r_2} < 0$, and at a parabolic point, one will have $\dfrac{1}{r_1 r_2} = 0$. The quantity $K = \dfrac{1}{r_1 r_2}$ characterizes the point in space in the same way that the total curvature does in the theory of surfaces. One can then also call that quantity by that name here, as well.

9. Lines of curvature (of the first kind) as envelopes of directions of principal curvature. – If one eliminates λ and S from (17), instead of a, b, c, then one will get the quadratic form in dx, dy, dz :

(21)
$$\begin{vmatrix} 2P'_x\,dx+(P'_y+Q'_x)\,dy+(P'_z+R'_x)\,dz & P & dx \\ (P'_y+Q'_x)\,dx+2Q'_y\,dy+(Q'_z+R'_y)\,dz & Q & dy \\ (P'_z+R'_x)\,dx+(Q'_z+R'_y)\,dy+2R'_z\,dz & R & dz \end{vmatrix} = 0.$$

For the special coordinate system with M as its coordinate origin and (2) as the XY-plane, it will take the simpler form:

(21′)
$$0 = (P'_y+Q'_x)(dy^2-dx^2)+2\,(P'_x-Q'_y)\,dx\,dy\,.$$

The two directions are mutually-perpendicular and coincide with the axes of the indicatrix in question. Since the asymptotic lines are determined by the equation:

(5′)
$$(P'_x\,dx^2+(P'_y+Q'_x)\,dx\,dy+Q'_y\,dy^2 = 0,$$

they will have the lines of curvature as angle bisectors. If the axes of the coordinates coincide with those of the indicatrix then (5′) will become:

$$\frac{1}{r_1}dx^2+\frac{1}{r_2}dy^2 = 0,$$

and (21′) will become:

$$2\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\cdot dx\,dy = 0.$$

10. Lines of curvature of the second kind. – We now ask about the directions in (2) along which infinitely-close normals to (2) will intersect. The condition that:

$$\frac{X-x}{P}=\frac{Y-y}{Q}=\frac{Z-z}{R} \qquad \text{and} \qquad \frac{X-x-dx}{P+dP}=\frac{Y-y-dy}{Q+dQ}=\frac{Z-z-dz}{R+dR}$$

should intersect is that:

(22)
$$0 = \begin{vmatrix} dx & dy & dz \\ P & Q & R \\ dP & dQ & dR \end{vmatrix}.$$

If M is the coordinate origin and (2) is the XY-plane then the equation will become:

$$dQ\;dx - dP\;dy = 0$$

or

(22′)
$$Q'_x\,dx^2+(Q'_y-P'_x)\,dx\,dy-P'_y\,dy^2 = 0.$$

Since:

$$P'_x = \frac{1}{r_1}, \qquad Q'_y = \frac{1}{r_2}, \qquad P'_y + Q'_x = 0, \quad \text{and} \quad P'_y - Q'_x = G,$$

in this case, (22) will assume the form:

$$(22'') \qquad -\tfrac{1}{2} G\,(dx^2 + dy^2) + \left(\frac{1}{r_2} - \frac{1}{r_1}\right) dx\,dy = 0,$$

and it is clear that both systems (22″) and (21′) will coincide for $G = 0$.

The angle between the directions (22″) for $G \neq 0$ is different form $\pi / 2$. Namely, if we divide by ds^2 and set $dx / ds = \cos \alpha$, $dy / ds = \sin \alpha$ then that will give:

$$-\tfrac{1}{2} G + \left(\frac{1}{r_2} - \frac{1}{r_1}\right) \sin \alpha \cos \alpha = 0,$$

so $\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \sin 2\alpha = G$.

However, if $G = 0$ then $\sin 2\alpha = 0$, so $\alpha = 0$ or $\pi / 2$.

11. Gaussian curvature. – We would now like to calculate the product of the radii of curvature r'_1 and r'_2 that would correspond to the directions that were just found. In order to shorten the calculations, I shall take $R = -1$ and introduce the notations:

$$P'_x + P\,P'_z = (\mathrm{I}), \qquad P'_y + Q\,P'_z = (\mathrm{II}),$$
$$Q'_x + P\,Q'_z = (\mathrm{III}), \qquad Q'_y + Q\,Q'_z = (\mathrm{IV}).$$

We will then have:

$$\frac{1}{r} = \frac{dP\,dx + dQ\,dy}{ds^2\sqrt{1 + P^2 + Q^2}} = \frac{(\mathrm{I})\,dx^2 + (\mathrm{II} + \mathrm{III})\,dx\,dy + (\mathrm{IV})\,dy^2}{ds^2\sqrt{1 + P^2 + Q^2}},$$

and equation (22) will become:

$$(24) \qquad 0 = \begin{vmatrix} dP & P & dx \\ dQ & Q & dy \\ 0 & -1 & P\,dx + Q\,dy \end{vmatrix}$$

$$\equiv dP\,[PQ\,dx + (1 + Q^2)\,dy] - dQ\,[dx\,(1 + P^2) + PQ\,dy].$$

Thus, we can also write:

(25)
$$\begin{cases} dP = \lambda[dx(1+P^2)+dy\cdot PQ], \\ dQ = \lambda[PQ\,dx+(1+P^2)\,dy], \end{cases}$$

or

(25′)
$$\begin{cases} (\mathrm{I})\,dx+(\mathrm{II})\,dy = \lambda[PQ\,dy+(1+P^2)\,dx], \\ (\mathrm{III})\,dx+(\mathrm{IV})\,dy = \lambda[PQ\,dx+(1+Q^2)\,dy]. \end{cases}$$

If we multiply (25) by *dx* and *dy*, respectively, and add them then that will give:

$$dP\,dx + dQ\,dy = \lambda\,[dx^2\,(1 + P^2) + 2PQ\,dx\,dy + (1 + Q^2)\,dy^2] = \lambda\,ds^2,$$

so

$$\lambda = \frac{dP\,dx+dQ\,dy}{ds^2} = \frac{\sqrt{1+P^2+Q^2}}{r}.$$

We must now calculate λ. Eliminating *dx* and *dy* from (25′) will give the quadratic equation in λ:

(26)
$$\begin{vmatrix} (\mathrm{I})-\lambda(1+P^2) & (\mathrm{II})-\lambda\,PQ \\ (\mathrm{III})-\lambda\,PQ & (\mathrm{IV})-\lambda(1+Q^2) \end{vmatrix} = 0,$$

or, when calculated out:

(26′)
$$\lambda^2\,(1 + P^2 + Q^2) - \lambda\,[(1 + Q^2)\,(\mathrm{I}) - (\mathrm{II} + \mathrm{III})\,PQ + (1 + P^2)(\mathrm{IV})] + (\mathrm{I})(\mathrm{IV}) - (\mathrm{II})(\mathrm{III}) = 0.$$

Hence:

$$\lambda_1\,\lambda_2 = \frac{(\mathrm{I})(\mathrm{IV})-(\mathrm{II})(\mathrm{III})}{(1+P^2+Q^2)^2},$$

so:

$$\frac{1}{r_1'r_2'} = \frac{(\mathrm{I})(\mathrm{IV})-(\mathrm{II})(\mathrm{III})}{(1+P^2+Q^2)^2} = \begin{vmatrix} P_x' & P_y' & P_z' & P \\ Q_x' & Q_y' & Q_z' & Q \\ 0 & 0 & 0 & -1 \\ P & Q & -1 & 0 \end{vmatrix} \frac{1}{(1+P^2+Q^2)^2}.$$

If we return to the general case (i.e., we introduce $-P/R$ and $-Q/R$, in place of *P* and *Q*) then after an easy calculation, we will have:

$$\frac{1}{r_1'r_2'} = -\frac{\Delta}{(1+P^2+Q^2)^2}.$$

We will give that quantity the name of *Gaussian curvature*. The sum of the roots of (26) is $\lambda_1 + \lambda_2 = \sqrt{1+P^2+Q^2}\left(\frac{1}{r_1'}+\frac{1}{r_2'}\right)$, so:

$$\frac{1}{r_1'}+\frac{1}{r_2'}=\frac{(1+Q^2)(\mathrm{I})-(\mathrm{II}+\mathrm{III})P\,Q+(1+P^2)(\mathrm{IV})}{(1+P^2+Q^2)^{3/2}}.$$

The right-hand side is equal to the sum of the principal curvatures, so:

$$\frac{1}{r_1'}+\frac{1}{r_2'}=\frac{1}{r_1}+\frac{1}{r_2}.$$

However, from (15′), one has:

$$\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{r_1'}+\frac{1}{r_1''},$$

in which r_1'' is the radius of curvature in the direction that is perpendicular to the r_1'-direction. Hence:

$$r_2''=r_1'.$$

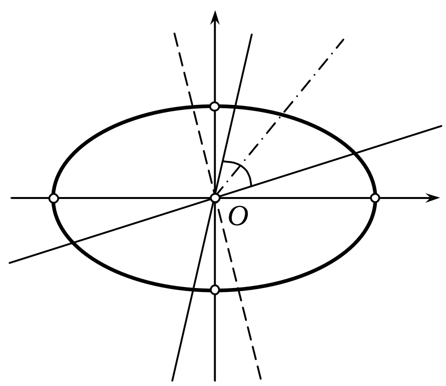

Figure 3.

As a result of the symmetry of the indicatrix relative to its axes, the angle that a line of curvature of the second kind makes with one of the first kind and the one that a line of curvature of the second first kind makes with one of the second second kind will be equal to each other. The lines of curvature of the first and second kind will then have the same angle bisectors (Fig. 3).

12. Extension of Gauss's theorem. – We take two infinitely-close points $M' \equiv (x + dx, y + dy, z + dz)$ and $M'' \equiv (x + d'x, y + d'y, z + d'z)$ in the plane (2) that corresponds to the point M (x, y, z). (Fig. 4). Three planes (2) belong to it with normals MN, $M'N'$, $M''N''$. We construct the spherical image of the triangle $M\,M'M''$: namely, M_1, with the coordinates $\xi = P / V$, $\eta = Q / V$, $\zeta = -1 / V$ ($V = \sqrt{1+P^2+Q^2}$), so $OM_1 \parallel MN$, etc.; $M_1' \equiv (\xi + d\xi, \eta + d\eta, \zeta + d\zeta)$, $M_1'' \equiv (\xi + d'\xi, \eta + d'\eta, \zeta + d'\zeta)$.

The surface area of the triangle $M\,M'M''$:

$$S = \tfrac{1}{2} V\,(dx\,d'y - dy\,d'y)$$

is equal to that of the triangle $M_1 M_1' M_1''$:

$$S = \frac{1}{2}\cdot\frac{1}{V^2}(dP\,d'Q - dQ\,d'P).$$

Hence:

$$\frac{S'}{S} = \frac{dP\,dQ - dQ\,dP}{V^4(dx\,d'y - dy\,d'x)}\,.$$

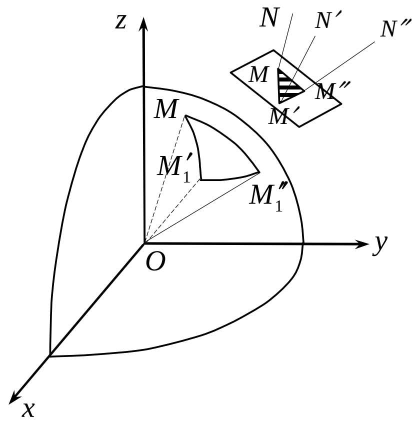

Figure 4.

However:

$$dP\,d'Q - dQ\,d'P = [(\mathrm{I})(\mathrm{IV}) - (\mathrm{II})(\mathrm{III})]\,(dx\,d'y - dy\,d'x)\,.$$

Hence:

$$\frac{S'}{S} = \frac{(\mathrm{I})(\mathrm{IV}) - (\mathrm{II})(\mathrm{III})}{(1 + P^2 + Q^2)^2} = \frac{1}{r_1' r_2'}\,.$$

We then get the theorem:

The quotient of the surface area of the spherical image of a triangle and that of the triangle itself has the product of the curvatures along the lines of curvature of the second kind as a limit, and thus, the Gaussian curvature,

as in the known theorem of Gauss.

We are then justified in calling the quantity:

$$\frac{S'}{S} = \frac{1}{r_1' r_2'} = -\frac{\Delta}{(P^2 + Q^2 + R^2)^2}\,.$$

the “Gaussian curvature.” Since:

$$\Delta' = G^2 + 4\Delta,$$

the quantities $-\dfrac{\Delta}{(P^2+Q^2+R^2)^2}$ and $-\dfrac{\Delta'}{(P^2+Q^2+R^2)^2}$ will coincide when $G = 0$.

The lines of curvature of the second kind are coupled with the Gaussian curvature in the same way that those of the first kind are coupled with the total curvature, and the properties of the lines of curvature of the surface are distributed over the two types of lines of curvature.

13. Analogue of the Enneper-Beltrami relation. – We considered two quadratic forms above:

I. The square of the line element: $ds^2 = dx^2 + dy^2 + dz^2$.

II. The left-hand side of the equation of the asymptotic lines:

$$dP\,dx + dQ\,dy + dR\,dz\,.$$

We then add:

III. The square of the line element of the spherical image:

$$d\left(\frac{P}{V}\right)^2 + d\left(\frac{Q}{V}\right)^2 + d\left(\frac{R}{V}\right)^2 .$$

If the coordinate axis is placed at the point itself and the axes of the indicatrix taken to be the axes OX, OY then one will have:

$$\mathrm{I} = dx^2 + dy^2,$$

$$\mathrm{II} = \frac{1}{r_1}dx^2 + \frac{1}{r_2}dy^2,$$

$$\mathrm{III} = dP^2 + dQ^2 = \left(\frac{1}{r_1^2}+\frac{1}{4}G^2\right)dx^2 + G\left(\frac{1}{r_1}-\frac{1}{r_2}\right)dx\,dy + \left(\frac{1}{r_2^2}+\frac{1}{4}G^2\right)dy^2 .$$

However, those three forms do not suffice for one to be able to exhibit a relation that would be analogous to the Enneper-Beltrami relation. One must employ a fourth form: namely, the left-hand side of the equation of lines of curvature of the first or second kind. It will now read:

$$\mathrm{IV} = \left(\frac{1}{r_1}-\frac{1}{r_2}\right)dx\,dy$$

and

$$\mathrm{V} = \tfrac{1}{2}G\left(dx^2+dy^2\right) + \left(\frac{1}{r_1}-\frac{1}{r_2}\right)dx\,dy \equiv \tfrac{1}{2}G\cdot\mathrm{I} + \mathrm{IV}\,.$$

One can also write the form III as:

$$\mathrm{III} = \left(\frac{1}{r_1^2}dx^2 + \frac{1}{r_2^2}dy^2\right) + \tfrac{1}{4}G^2\cdot\mathrm{I} + \mathrm{IV}$$

or

$$\mathrm{III} = \frac{1}{r_1^2}dx^2 + \frac{1}{r_2^2}dy^2 - \tfrac{1}{4}G^2\cdot\mathrm{I} + \mathrm{V}\,.$$

Eliminating dx^2 and dy^2 from the three equations will yield:

$$\begin{vmatrix} \mathrm{I} & 1 & 1 \\ \mathrm{II} & \dfrac{1}{r_1} & \dfrac{1}{r_2} \\ \mathrm{III} - \tfrac{1}{4}G^2\cdot\mathrm{I} - G\cdot\mathrm{IV} & \dfrac{1}{r_1^2} & \dfrac{1}{r_2^2} \end{vmatrix} = 0.$$

If we develop that and drop the common factor of $\dfrac{1}{r_2} - \dfrac{1}{r_1}$ then we will get the desired relation:

$$\left(K - \tfrac{1}{4}G^2\right)\cdot\mathrm{I} - 2\,H\cdot\mathrm{II} + \mathrm{III} - G\cdot\mathrm{IV} = 0$$

or

$$\left(K + \tfrac{1}{4}G^2\right)\cdot\mathrm{I} - 2\,H\cdot\mathrm{II} + \mathrm{III} - G\cdot\mathrm{V} = 0,$$

according to whether we employ the form IV or V. For $G = 0$, one will again get the Enneper-Beltrami formula. The same formula will also be true for an arbitrary coordinate choice when $P^2 + Q^2 + R^2 = 1$.

By contrast, when $P^2 + Q^2 + R^2 \neq 1$, one must replace G with $\dfrac{G}{P^2 + Q^2 + R^2}$, and that will give:

$$\left(K - \frac{1}{4}\frac{G^2}{(P^2 + Q^2 + R^2)^2}\right)\cdot\mathrm{I} - 2\,H\cdot\mathrm{II} + \mathrm{III} - \frac{G}{P^2 + Q^2 + R^2}\cdot\mathrm{IV} = 0$$

or

$$\left(K - \frac{1}{4}\frac{G^2}{(P^2 + Q^2 + R^2)^2}\right)\cdot\mathrm{I} - 2\,H\cdot\mathrm{II} + \mathrm{III} - \frac{G}{P^2 + Q^2 + R^2}\cdot\mathrm{V} = 0.$$

If one introduces the Gaussian curvature K', instead of the total curvature K, then the last formula will assume the form:

$$K' \cdot \mathrm{I} - 2\, H \cdot \mathrm{II} + \mathrm{III} - \frac{G}{P^2 + Q^2 + R^2} \cdot \mathrm{V} = 0.$$

I have treated this situation more thoroughly in vol. **3** of the Annals of the Scientific Lyceum in Ukraine and the Letters to the Kharkov Mathematical Society (4) **1**.

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