A. Sommerfeld: Review of R. von Lilienthal, *Grundlagen einer Krümmungslehre der Curvenschaaren*, Leipzig, B. G. Teubner, 1896. 114 pages, octavo. Published in Göttingische gelehrte Anzeigen **160** (1898), 901-912.

Review of the book:

Grundlagen einer Krümmungslehre der Curvenschaaren

By R. von Lilienthal

Translated by D. H. Delphenich

In the spirit of his previous papers (Math. Ann. Bd. **32**, **38**, and **42**: "Ueber die Krümmung von Curvenschaaren," "Zur Krümmungtheorie der Curvenschaaren," "Ueber geodätische Krümmung"), the author of this book gave a summary presentation of the theory that was mentioned in the title that was rich in content, extremely concise in form, and restricted to the necessary superstructure.

The first part of the book treats *simply-infinite families of curves* in the plane and on an arbitrary surface. The second part, in which the center of gravity of the books lies, considers *doubly-infinite families of curves* in space. There, they were thought of as being given by *finite equations* of the form:

$$x = f(p, q, r), \qquad y = f_1(p, q, r), \qquad z = f_2(p, q, r),$$

in which x, y, z mean the rectangular coordinates of a point along any curve of the family, for which only the parameter r will change along each individual curve, while a change in p and q means the transition from one individual curve of the family to another. Finally, in Part Three, *doubly-infinite families of curves* are examined once more, but this time they are established by the *differential equation:*

$$dx : dy : dz = \xi : \eta : \zeta$$

(by means of three arbitrarily given functions ξ, η, ζ of x, y, z).

Let us next say a few words about the general problem of doubly-infinite families of curves, or as one usually says, *congruences* of curves, and about what makes that this situation inherently interesting.

As is known, under certain circumstances, a doubly-infinite family of curves possesses a system of orthogonal surfaces, in which case, the author refers to that family as a *normal family*. In general, that property is not satisfied. In the former case, one can borrow from a terminology that Hertz introduced into mechanics and aptly call the family *holonomic*, while in general it is *non-holonomic*. (The mechanical problem, which we would like to recall with the terminology that is proposed here, is this: Investigate the motion of a material point that is constrained to move continually perpendicular to the curve of the family that goes through it at each point in time. For a normal family, such a point always remains on a well-defined orthogonal surface to the family. By contrast, for a general, non-surface-normal family, that can be achieved by the addition of suitable jumps at each point of the region of space that the family of curves goes through.)

The simplest example of a normal family is defined by the normal system of an arbitrary surface, for which the curves of the family will be straight lines. In this simplest case, the study of the orthogonal trajectories to the family of curves and its curvature behavior is identical with the study of the curves that lie on the surface and the curvature theory of the surface. Above all, the *theory of holonomic families of curves* proves to be essentially identical with the *general theory of surfaces*; i.e., with the theory of the families of curves that are orthogonal to surfaces. Analogously, the *theory of non-holonomic families of curves* represents an interesting *extension of the theory of surfaces*. If one, like the author, imagines that orthogonal trajectories to the family of curves have been constructed at each point then one will obtain a distribution of surface elements in space that one might, in turn, be able to refer to as a non-holonomic system of surface elements, in connection with Hertz's terminology. Indeed, the elements of such a system do not unite into surfaces, as in the holonomic case. However, their arrangement and positions are accessible to investigations that are similar to what one does with holonomic systems of surface elements; i.e., they can represent the system of tangent planes to an ordinary surface that one might in study in the theory of curvature. Thus, we can perhaps refer to the actual objective of the book as this one: *To adapt the methods of surface theory from holonomic systems of surface elements to non-holonomic ones.*

The basic principle that will lead to that adaptation is clear from the outset. One chooses the notations and concepts (as is also customary in the cited literature) in such a way that they will go to the notations and concepts of the ordinary theory of surfaces in the case of a holonomic family of curves. In that sense, the surface-theoretic concept of a surface normal for a family of curves will take the form of the tangents to the curve of the family that goes through the point under consideration, and furthermore, the tangent plane to the surface will go to the normal plane to the curve, a normal section of the surface will be a plane that contains the tangent to the curve, the curves on the surface will go to the orthogonal trajectories to the family of curves, etc. One can define the *normal curvature* to the orthogonal trajectories to the family of curves, which is the analogue of the curvature of a normal section in surface theory, as well as the *geodetic curvature*, which is the analogue of the geodetic curvature of the curves on the surface, as follows: At the point P under consideration, one constructs the curvature axis of the orthogonal trajectory that is spoken of; it will cut the tangent to the curve of the family that goes through P at Q and the normal plane to the family of curves that goes through P at R. Q is then the center of the normal curvature, and R is the geodetic curvature. At the same time, the reciprocal values of the lengths PQ and PR determine the magnitudes of the normal curvature and geodetic curvature, up to sign. Beyond that, one can determine a series of further concepts that are requisite for the curvatures of the orthogonal trajectories, which we will not, however, go into here.

The concepts of normal curvature and geodetic curvature are linked with the definitions of *asymptotic lines*, *geodetic lines*, and *lines of curvature*. Among the orthogonal trajectories to a family of curves, the asymptotic lines are the ones whose normal curvature vanishes everywhere. One further says "geodetic lines" to mean the ones whose geodetic curvature is continually equal to zero. As far as the surface-theoretic concept of lines of curvature is concerned, they separate into two distinct concepts, as is known already from the special case of ray systems, namely, according to whether a line of curvature is defined to be an orthogonal trajectory along which the tangents to the family of curves define a developable surface or a line whose tangent belong to a principle normal plane everywhere. That is, in one of the two planes in which one finds a maximum or minimum of the normal curvature, one will obtain two generally different systems of curves that will be distinguished from each other as curvature lines of the second an first kind, respectively.

In regard to that, one can remark that even the surface-theoretic concept of geodetic lines is capable of two extensions to families of curves. The two different types of curves to which one will be led in that way are best distinguished as being the *straightest* and *shortest* lines, respectively. The definition of geodetic lines that von Lilienthal gave, which is reproduced above, corresponds to the one that we just called the straightest line. They have the properties that their principal normal coincides with the tangent to the family of curves everywhere and that they exhibit a minimum of the first curvature when compared to certain neighboring curves. For the mechanical problem that is closely connected with the theory of families of curves that was mentioned above, those straightest lines are likewise the actual paths of the material point in question. In contrast to that, the shortest lines – i.e., the lines which possess the smallest arc-length of all orthogonal trajectories that connect those two points – have no simple mechanical meaning. The differential equation that determines them is also more complicated than that of the straightest line. Moreover, Hertz simply referred to precisely those shortest lines as geodetic lines. Naturally, for a holonomic family of curves, the concepts of straightest and shortest lines coalesce into one. Perhaps a book like the present one might consider the straightest lines along with the shortest ones and, in fact, touch upon the mechanical meaning of the entire problem statement, which might raise the interest of many readers in the subject essentially. Unfortunately, as is known, detailed examples that would explain the interesting distinction between shortest and straightest paths are still lacking. Von Lilienthal could easily provide us with them, based upon his deep knowledge of the subject.

We shall not go into the many details of the contents of the book here. One can hardly present them more briefly than what is done in the book, anyway. By contrast, we would like to expand a bit on the curious analytical method that author applied to his deductions; we mean the *differentiation with respect to arc-lengths.*

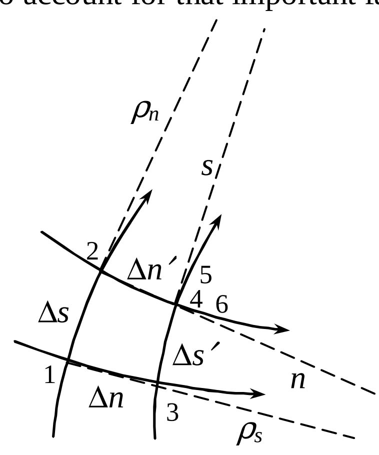
Without question, for geometric calculations, it would be preferable to employ only those operations that are independent of the provisionally-introduced coordinate system and possess an invariant geometric meaning. That procedure will reward one with the fact that all of the formulas that occur will have geometric interpretations and one's attention will not be misdirected by inessential auxiliary quantities from the true geometric invariants, which are the only ones to which one arrives.

Now, for a system of curves (for the sake of simplicity, we would like to speak of a simply-infinite system of curves in the plane, and we would like to think of a quantity F as a function of position in that plane), the differential quotients of F with respect to the arbitrarily-chosen coordinates are something inessential, while the differential quotients $\partial F / \partial s$ and $\partial F / \partial n$ with respect to the arc-length of a curve of the system or with respect to the arc-length of a curve in the orthogonal family, resp., mean something that is geometrically invariant. What one must understand by those differential quotients hardly requires explanation. One considers two points on a curve of the family (or the orthogonal family) that are separated from each other by an arc-length of Δs (or Δn), forms the difference between F at those points, divides it by Δs (or Δn), and passes to the limit.

Derivatives such as $\partial F / \partial s$ and $\partial F / \partial n$ have been useful for some time in mathematical physics, where one understandably places much value upon employing only those quantities in calculations that are physically meaningful and independent of coordinate systems. There, one can recall the "directional derivatives" that Maxwell successfully employed in his theory of electricity, as well as the methods of vector analysis, in general, in which one differentiates with respect to vectors in arbitrary directions and, above all, abhors the use of a coordinate system. We can say that the operations that von Lilienthal employs are special applications of differential vector analysis, in which the functions to be differentiated are scalars and the vectors with respect to which they are differentiated are chosen to have the directions of curves of the system or its orthogonal trajectories.

Moreover, the author rejected the notations $\partial F / \partial s, \partial F / \partial n$, since one cannot choose the arc-lengths s and n to be independent variables, and he introduced a less-intuitive new symbol in their place. By contrast, we will allow ourselves to preserve the symbols $\partial F / \partial s$, …, in this review, since they are entirely natural in various parts of mathematics and are understandable with no further discussion.

A fact that might perhaps seem amazing on first glance is that $\frac{\partial^2 F}{\partial s\, \partial n}$ *is not, in general, equal to* $\frac{\partial^2 F}{\partial n\, \partial s}$. Von Lilienthal showed that by calculation and determined the difference between the two aforementioned derivatives by a formula that will be given below. Here, we would like to account for that important fact geometrically.



We consider a curvilinear rectangle 1234 (cf., the figure) that is composed of two curves of our planar family and two of their orthogonal trajectories.

The arc-length 12 will be denoted by Δs and the arc-length 13, by Δn, such that the curves 12 and 34 will belong to the family of curves itself, while the curves 13 and 24 will belong to the system of its orthogonal trajectories. From 2 on the curve 24, we measure out the arc-length Δn, as well as the length Δs from 3 on the curve 34. The points to which we arrive in that way might be called 5 and 6. We let $F_1, \ldots, F_6$ denote the values of F at the points 1, …., 6, resp. The directions of increasing s and n are suggested in the figure by arrows. By definition, the symbols $\partial F / \partial s$ and $\partial F / \partial n$ then mean the limiting values that the quotients:

$$\frac{F_2 - F_1}{\Delta s}, \qquad \frac{F_2 - F_1}{\Delta n}, \text{ resp.,}$$

will approach. Moreover, one clearly understands:

$$\frac{\partial}{\partial s}\frac{\partial F}{\partial n} = \frac{\partial^2 F}{\partial s\,\partial n}$$

to mean the limiting value of the following quotient:

$$\frac{1}{\Delta s}\left\{\frac{F_5 - F_2}{\Delta n} - \frac{F_3 - F_1}{\Delta n}\right\} = \frac{F_1 - F_2 - F_3 + F_5}{\Delta s\,\Delta n}.$$

By contrast, $\dfrac{\partial^2 F}{\partial n\,\partial s}$ is the limiting value of the expression:

$$\frac{1}{\Delta n}\left\{\frac{F_6 - F_3}{\Delta s} - \frac{F_2 - F_1}{\Delta s}\right\} = \frac{F_1 - F_2 - F_3 + F_6}{\Delta s\,\Delta n}.$$

Thus the difference between our two derivatives will be:

$$\frac{\partial^2 F}{\partial s\,\partial n} - \frac{\partial^2 F}{\partial n\,\partial s} = \mathrm{Lim}\frac{F_5 - F_6}{\Delta s\,\Delta n} = \mathrm{Lim}\frac{F_5 - F_4}{\Delta s\,\Delta n} - \mathrm{Lim}\frac{F_6 - F_4}{\Delta s\,\Delta n}.$$

Naturally, we tacitly assume that the function F, as well as the basic system of curves, possesses continuity properties that would be required for the existence of the limiting values that occur. All that remains is to determine the last two limiting values that were written down more closely. We would like to denote the arc-lengths between the points 4 and 5 (4 and 6, resp.) by $\Delta n'$ and $\Delta s'$, respectively. We will then have:

$$\mathrm{Lim}\frac{F_5 - F_4}{\Delta s\,\Delta n} = \mathrm{Lim}\frac{F_5 - F_4}{\Delta n'} \cdot \mathrm{Lim}\frac{\Delta n'}{\Delta s\,\Delta n} = \frac{\partial F}{\partial n} \cdot \mathrm{Lim}\frac{\Delta n'}{\Delta s\,\Delta n}$$

and

$$\operatorname{Lim}\frac{F_6 - F_4}{\Delta s\,\Delta n} = \operatorname{Lim}\frac{F_6 - F_4}{\Delta s'}\cdot\operatorname{Lim}\frac{\Delta n'}{\Delta s\,\Delta n} = \frac{\partial F}{\partial s}\cdot\operatorname{Lim}\frac{\Delta s'}{\Delta s\,\Delta n}.$$

We further remark that, in the limit where $\Delta s = 0$ and $\Delta n = 0$, the connecting lines 13 and 24 will go to two neighboring normals on the system curve *s*, and likewise, the lines 12 and 34 will go to neighboring normals to the trajectory *n*. In the limit, the intersection point of those two pairs of lines will then determine the centers of curvature of the two curves *s* and *n*; the associated radii of curvature might be called ρ_s and ρ_n, resp. We can now exhibit the proportions, which are exact in the limit:

$$\Delta s : \Delta s - \Delta s' = \rho_s : \rho_s - \Delta n$$

and

$$\Delta n : \Delta n - \Delta n' = \rho_n : \rho_n - \Delta s,$$

from which it will follow that:

$$\Delta s' : \Delta s = \Delta n : \rho_s \qquad \text{or} \qquad \frac{\Delta s'}{\Delta s\,\Delta n} = \frac{1}{\rho_s}$$

and

$$\Delta n' : \Delta n = \Delta s : \rho_n \qquad \text{or} \qquad \frac{\Delta n'}{\Delta s\,\Delta n} = \frac{1}{\rho_n}.$$

The limiting values in question are thus determined. When we substitute them, we will thereupon get the formula that von Lilienthal gave on page 4:

$$\frac{\partial^2 F}{\partial s\,\partial n} - \frac{\partial^2 F}{\partial n\,\partial s} = \frac{1}{\rho_n}\frac{\partial F}{\partial n} - \frac{1}{\rho_s}\frac{\partial F}{\partial s}.$$

Precisely the same argument can also be applied to a family of curves that lies on an arbitrarily-curved surface. It will then show that the foregoing formula also remains valid for this case, except that corresponding geodetic curvatures will appear in place of the curvatures ρ_s and ρ_n. On the other hand, for a twofold-infinite family of curves in space, one must consider a curved rectangular parallelepiped that is composed of curves of the family and eight of their orthogonal trajectories, in place of the curvilinear rectangle. If one denotes the lengths of three mutually-perpendicular edges of the parallelepiped by Δs, Δn, Δm then one will also have to measure out those lengths along the parallel edges, and in that way, arrive at points that are different from the vertices of the parallelepiped, just like the points 5 and 6 in the case above. That difference shows clearly the geometric basis for why the operations $\partial / \partial s$, $\partial / \partial n$, $\partial / \partial m$ do not commute with each other here, either. It is characteristic of calculations with arc-length that nothing but geometrically-important quantities will appear in the formulas that represent the difference between two differential quotients, namely, certain curvature measures that are characteristic of the family of curves, just like in the planar case. The formulas under consideration, which make up an essential component of the present theory, are developed by von Lilienthal on page 56.

In conclusion, we would like to express a divergent opinion on the most preferable definition of *Lamé's differential parameters*. Von Lilienthal described those quantities for the case of the plane, to which we would like to restrict ourselves here, and based upon rectangular coordinates by the formulas:

$$\Delta_1 F = \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2}\,, \qquad \Delta_2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\,.$$

That must then lead to a proof that they have a meaning that is independent of the coordinate system. In contrast, it seems to us to be more correct to define the differential parameters independently of coordinate systems from the outset, which would shed a brighter light on the concept. Furthermore, that is also often customary in the literature.

From that standpoint, one can say the following: The first differential parameter $\Delta_1 F$ at a point P is equal to the derivative of the function F in the direction of the normal to the curve F = const. that goes through P (†). Furthermore, one can define the second differential parameter $\Delta_2 F$ as follows: One surrounds the point P in question with an arbitrary closed curve σ and defines the line integral $\int \frac{\partial F}{\partial \nu} d\sigma$, which extends over the entire curve σ, in which $\partial F / \partial \nu$ means the derivative of F with respect to the outward-pointing normal to σ. One divides that line integral by the area of the surface that is bounded by σ and passes to the limit when one contracts the curve σ to the point P. *The limiting value of our quotient that arises in that way is precisely identical to the second Lamé differential parameter.*

Exactly the same definition can be adapted, word-for-word, to the case in which F is given as a function of position on an arbitrary curved surface. It therefore also extends to space in the same way when one replaces the words "curve" and "surface" with "surface" and "volume," respectively

The various properties and ways of representing the differential parameters are obtained from those definitions with the greatest ease. Thus, e.g., the expression for the second differential parameter in rectangular coordinates can be written down directly from Green's theorem. The same theorem will also shows that our limiting value is independent of the choice of curve σ. The fact that it cannot depend upon the coordinate system is self-explanatory, since nothing of the sort was mentioned in our definition. In particular, we would like to derive an expression that von Lilienthal gave for the second differential parameter in the plane in terms of differential quotients with respect to arc-length on the basis of the previous figure.

As before, we imagine a family of curves s in the plane and give the family its orthogonal trajectories n. Let the curve σ that we just spoke of be the infinitesimal curvilinear rectangle 1243. We then have:

(†) Translator: More precisely, $\Delta_1 F$ is the Euclidian norm $\| dF \|$ of that derivative; i.e., the length of the covector.

$$\int \frac{\partial F}{\partial \nu} d\sigma = \left(-\frac{\partial F}{\partial n}\right)_{12} \Delta s + \left(-\frac{\partial F}{\partial s}\right)_{13} \Delta n + \left(-\frac{\partial F}{\partial n}\right)_{34} (\Delta s - \Delta s') + \left(\frac{\partial F}{\partial s}\right)_{34} (\Delta n - \Delta n') .$$

In the products on the right-hand side, the second factor means the length of the side in question of our curvilinear rectangle, while the first term is a mean value of the respective differential quotient of F; i.e., the value of the differential quotient in question at a suitably-chosen point of the arc that is suggested by the index that is affixed to that quotient. The negative signs on the first two products arise from the fact that the outward-pointing normal ν is opposite to the positive directions of the arc-lengths s and u that is assumed in the figure. However, in order to develop those products, we can now write:

$$\left(\frac{\partial F}{\partial n}\right)_{34} = \left(\frac{\partial F}{\partial n}\right)_{12} + \Delta n \frac{\partial^2 F}{\partial n^2}$$

and

$$\left(\frac{\partial F}{\partial s}\right)_{34} = \left(\frac{\partial F}{\partial s}\right)_{12} + \Delta s \frac{\partial^2 F}{\partial s^2} .$$

If we neglect all terms that will be zero to order higher than two in the limit then when we drop the indices, that will give:

$$\int \frac{\partial F}{\partial \nu} d\sigma = \left(\frac{\partial^2 F}{\partial s^2} + \frac{\partial^2 F}{\partial n^2}\right) \Delta s \, \Delta n - \frac{\partial F}{\partial n} \Delta s' - \frac{\partial F}{\partial s} \Delta n' .$$

We must divide this by the area of our curvilinear rectangle – i.e., by Δs Δn, approximately – consider the expressions for $\Delta s'$ and $\Delta n'$ that were given above, and pass to the limit. The second differential parameter $\Delta_2 F$ will appear on the left-hand side, while the right-hand side will go to:

$$\frac{\partial^2 F}{\partial s^2} + \frac{\partial^2 F}{\partial n^2} - \frac{1}{\rho_s} \frac{\partial F}{\partial s} - \frac{1}{\rho_n} \frac{\partial F}{\partial n} .$$

That is the expression that we would like to derive, and which von Lilienthal ascertained on page 6 in a different and (we would like to confess) more rigorous way.

If we replace the plane with a curved surface then that expression will persist, except that ρ_s and ρ_n will then mean the geodetic curvatures of the family of curves under consideration. When one goes to space, in which surface integrals must be used in place of line integrals, that will yield an analogous manner of representation in which the coefficients of the first derivatives of F will once again mean certain quantities that are characteristic of the curvature of the family of curves considered.

Clausthal, July 1898.

A. Sommerfeld.