"Ueber die Differentialgleichungen der Dynamik und den Begriff der analytischer Aequivalenz dynamischer Probleme," J. reine angew. Math. **107** (1891), 319-348.

On the differential equations of dynamics and concept of the analytical equivalence of dynamical problems.

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I.

Concept of a dynamical problem. Normal form for the differential equations of dynamics.

In his first lecture on *dynamics* (edited by **A. Clebsch** in 1886 and by **E. Lottner** in 1884), **Jacobi** said that he would restrict himself to those problems in mechanics for which one dealt with the motion of a system of a *finite* number of material *points* with the property that the constraints, as well as the forces that act upon the system depend upon only the configuration of points, but not on their velocities. In what follows, it will prove convenient to also allow certain systems with an unbounded number of material points, namely, all of the ones whose position at time t is given by the values of a *finite* number of *determining data*, as would be the case for a rigid body. By contrast, **Jacobi**'s restriction in regard to the constraints on the system as well as the applied forces, will be fixed throughout. The problems that are defined in that way shall be called, briefly, *dynamical problems*.

In order for a dynamical problem to be exhibited in the form of the differential equations of motion that **Lagrange** gave in Part Two of his mechanics (*Mécanique analytique*, Paris 1788, pp. 226), the determining data $p_1, p_2, \ldots, p_n$ of the system will be introduced in the following way: Let the coordinates of the i^{th} material point at time t relative to an arbitrary fixed rectangular coordinate system be x_{3i-2}, x_{3i-1}, x_{3i}. Certain equations will then exist between the quantities $x_1, x_2, \ldots, x_n$ due to the constraints on the system, and in dynamical problems, it is their nature that they will be fulfilled identically when one expresses the x_h as functions of n independent variables $p_1, p_2, \ldots, p_n$. However, if one regards the $p_1, p_2, \ldots, p_n$ as functions of $n + r$ new independent variables $p_1, p_2, \ldots, p_n$ then one will get expressions for the x_h in terms of those new variables by which the constraint equations will likewise be fulfilled. For that reason, **Lagrange** (*loc. cit.*, pp. 217) added that n shall be the *smallest* number of independent variables by means of which x_h can be represented. **Jacobi** (*Dynamik*, pp. 62) expressed that requirement (and likewise **Kirchhoff**, *Mechanik*, pp. 29) in such a way the number n is equal to the difference between the number of all coordinates x_h and the number of mutually-independent constraint equations between those quantities. That will be permissible as long as one has only a finite number of quantities x_h.

However, it is possible to give a form to the criterion for n to actually be the smallest number of independent variables that is free of that restriction. Namely, **Kronecker** has proved the following theorem in his lectures:

If one has a system of at least n functions $x_1, x_2, \ldots, x_n$ of the independent variables $p_1, p_2, \ldots, p_n$ then the smallest number of independent variables in terms of which those functions can be expressed will be precisely equal to the number that gives the ***rank*** (*) *of the system:*

$$\frac{\partial x_h}{\partial p_\kappa} \qquad \begin{pmatrix} h = 1, 2, 3, \ldots \\ \kappa = 1, 2, \ldots, n \end{pmatrix}.$$

It is equal to n if and only if that rank is equal to precisely n.

If the x_h are now represented by the n determining data $p_1, p_2, \ldots, p_n$ of the required type then according to **Lagrange**, one must represent the components along the coordinate axes of the force $X_{3i-2}, X_{3i-1}, X_{3i}$ that acts upon the i^{th} material point, which depend upon only the x_h, by assumption, in terms of the p_κ. The expressions for x_h and X_h must then be replaced:

First of all, in the expression for the *virtual work* done on the system during the time interval $(t, t + dt)$:

$$\sum_h X_h\, \delta x_h\,,$$

which might then go to:

$$U' = \sum_\kappa P_\kappa\, \delta p_\kappa\,. \tag{1}$$

Secondly, in the expression for the *vis viva* of the system at time t, which will read:

$$\tfrac{1}{2}\sum_h m_h \left(\frac{dx_h}{dt}\right)^2$$

when the mass of the i^{th} point is denoted by $m_{3i-2} = m_{3i-1} = m_{3i}$, and that might go to:

$$T = \tfrac{1}{2}\sum_{\kappa,\lambda} a_{\kappa\lambda}\,\frac{dp_\kappa}{dt}\,\frac{dp_\lambda}{dt} \qquad (a_{\kappa\lambda} = a_{\lambda\kappa}). \tag{2}$$

In that way, the P_κ and $a_{\kappa\lambda}$ are functions of only $p_1, p_2, \ldots, p_n$. Here, as in what follows, the small Greek symbols refer to the sequence of numbers 1, 2, …, n.

If that is the case then that will finally give the desired equations in the form:

(*) On the concept of the rank of a system of quantities, cf., Sitzungsberichte der Berliner Akademie (1884), issue II, pp. 1192.

(*a*) $$\frac{d}{dt}\frac{\partial T}{\partial \frac{dp_\mu}{dt}} - \frac{\partial T}{\partial p_\mu} - P_\mu = 0 .$$

However, another completely-equivalent system can be derived from that one in which the second derivatives of the p_κ with respect to time are expressed in terms of the quantities p_κ themselves and their first derivatives with respect to time. In order to do that, we appeal to the following argument:

The *vis viva* of the system T can (for real values of the x_h) vanish only when all of the dx_h / dt vanish. Therefore:

$$T = \tfrac{1}{2}\sum_{\kappa,\lambda} a_{\kappa\lambda}\frac{dp_\kappa}{dt}\frac{dp_\lambda}{dt}$$

can also vanish only when the equations:

(3) $$\sum_\mu \frac{\partial x_h}{\partial p_\mu}\frac{dp_\mu}{dt} \qquad (h = 1, 2, \ldots)$$

are all fulfilled. Now, the minimum number n of independent variables will be characterized by the fact that *at least one* determinant:

$$\left|\frac{\partial x_h}{\partial p_\mu}\right| \qquad (\lambda, \mu = 1, 2, \ldots, n)$$

will not vanish identically, from which it will follow that equations (3) can be satisfied only by the vanishing of all dp_μ / dt . Therefore, T is a quadratic form in the dp_μ / dt that will vanish only when all of the dp_μ / dt vanish and will otherwise have a positive value. In the theory of quadratic forms, it is proved that this property is inseparably linked with the other one that the principal subdeterminants of the symmetric system of coefficients $a_{\kappa\lambda}$ does not vanish identically but are positive quantities (except for the singular systems of values of the p_μ). In particular, that will imply that for every dynamical problem, the determinant:

(4) $$a = | a_{\kappa\mu} | \qquad (\lambda, \mu = 1, 2, \ldots, n)$$

will not vanish identically. The same thing is true for the quantities $a_{11}, a_{22}, \ldots, a_{nn}$.

Since the determinant of the quadratic differential form:

$$2T\,dt^2 = \sum_{\kappa,\lambda} a_{\kappa\lambda}\,dp_\kappa\,dp_\lambda$$

does not vanish identically, one finds an application to it of the theorem about such forms that **Christoffel** and **Lipschitz** developed in the simultaneously-appearing treatises ["Ueber die Transformation der homogenen Differentialausdrücke zweiten Grades," this Journal, **70** (1869), pp. 46 and 241; "Untersuchungen in Betreff der ganzen homogenen Functionen von *n* Differentialen," *ibidem*, pp. 71]. In those investigations, a special role was played by the combination:

$$\frac{\partial a_{\kappa\mu}}{\partial p_\lambda} + \frac{\partial a_{\lambda\mu}}{\partial p_\kappa} - \frac{\partial a_{\kappa\lambda}}{\partial p_\mu} ,$$

which **Lipschitz** denoted by $f_{\mu\kappa\lambda}$, and **Christoffel** denoted by $2\begin{bmatrix}\kappa\,\lambda\\ \mu\end{bmatrix}$. Since different differential forms had to be considered along with them later on, the procedure of **Weingarten** ("Ueber die Theorie der auf einander abwickelbaren Oberflächen," Festschrift der technischen Hochschule. Berlin, 1884) shall be discussed here:

(5) $$\tfrac{1}{2}\left(\frac{\partial a_{\kappa\mu}}{\partial p_\lambda} + \frac{\partial a_{\lambda\mu}}{\partial p_\kappa} - \frac{\partial a_{\kappa\lambda}}{\partial p_\mu}\right) = \begin{bmatrix}\kappa\,\lambda\\ \mu\end{bmatrix}_a .$$

With the use of that symbol, equations (*a*) read explicitly:

(a') $$\sum_\kappa a_{\kappa\mu}\frac{d^2 p_\kappa}{dt^2} + \sum_{\kappa,\lambda}\begin{bmatrix}\kappa\,\lambda\\ \mu\end{bmatrix}\frac{dp_\kappa}{dt}\frac{dp_\lambda}{dt} - P_\mu = 0 .$$

However, since the determinant:

$$a = |\, a_{\kappa\mu}\,| \qquad (\lambda, \mu = 1, 2, \ldots, n)$$

does not vanish identically, those equations can be solved for the $\frac{d^2 p_\kappa}{dt^2}$. To that end, one introduces (likewise following the procedure of **Christoffel**, **Lipschitz**, and **Weingarten**):

(6) $$\sum_\mu \begin{bmatrix}\kappa\,\lambda\\ \mu\end{bmatrix}_a a'_{\mu\nu} = \begin{Bmatrix}\kappa\,\lambda\\ \nu\end{Bmatrix}_a ,$$

in which $a'_{\mu\nu}$ denotes the system that is reciprocal to the system of $a_{\mu\nu}$. One will then get the system that is completely equivalent to (a'):

(a^*) $$\frac{d^2 p_\nu}{dt^2} = -\sum_{\kappa,\lambda}\begin{Bmatrix}\kappa\,\lambda\\ \nu\end{Bmatrix}_a \frac{dp_\kappa}{dt}\frac{dp_\lambda}{dt} + \sum_\mu P_\mu\, a'_{\mu\nu} .$$

Equations (a^*) shall be referred to as the *normal form* for the system of differential equations of motion for the dynamical problem.

II.

Concept of and conditions on analytically-equivalent dynamical problems.

One can introduce new variables $q_1, q_2, \ldots, q_n$ in place of the independent variables $p_1, p_2, \ldots, p_n$ when one sets:

$$(S) \qquad p_\kappa = p_\kappa(q_1, q_2, \ldots, q_n) .$$

In so doing, the substitution (S) is subject to only the one condition that the functional determinant:

$$\left| \frac{\partial p_\kappa}{\partial q_\lambda} \right| \qquad (\kappa, \lambda = 1, 2, \ldots, n)$$

must not vanish identically. Since the integration of the differential equations of motion is based upon the possibility of introducing the n determining data in various ways, the transformation of those equations has fundamental importance, and it will be eased precisely by taking the viewpoint that **Lagrange** assumed when he exhibited the differential equations of motion in the form (a) (cf., *loc. cit.*, pp. 217). Namely, if one introduces the new variables into the expressions for U' and T, from which one might get:

$$(1') \qquad U' = \sum_\kappa Q_\kappa \, \delta q_\kappa$$

$$(2') \qquad T = \tfrac{1}{2} \sum_{\kappa,\lambda} b_{\kappa\lambda} \frac{dq_\kappa}{dt} \frac{dq_\lambda}{dt} \qquad (b_{\kappa\lambda} = b_{\lambda\kappa}),$$

then the new differential equations of motion will read:

$$(b) \qquad \frac{d}{dt} \frac{\partial T}{\partial \frac{dq_\kappa}{dt}} - \frac{\partial T}{\partial q_\kappa} - Q_\kappa = 0 .$$

It will then suffice to convert the expressions for the virtual work and the *vis viva*, because one can derive the differential equations of motion from them immediately. In that way, one will also get the normal form of the equations of motion directly. To that end, one must only form:

$$(5') \qquad \tfrac{1}{2}\left[\frac{\partial b_{\kappa\mu}}{\partial q_\lambda} + \frac{\partial b_{\lambda\mu}}{\partial q_\kappa} - \frac{\partial b_{\kappa\lambda}}{\partial q_\mu}\right] = \begin{bmatrix}\kappa\ \lambda \\ \mu\end{bmatrix}_b ,$$

$$(6') \qquad \sum_\mu \begin{bmatrix}\kappa\ \lambda \\ \mu\end{bmatrix}_b b'_{\mu\nu} = \begin{Bmatrix}\kappa\ \lambda \\ \nu\end{Bmatrix}_b$$

from the coefficients of (2′), in which $b'_{\mu\nu}$ means the system that is reciprocal to the system of $b_{\mu\nu}$. One will then have:

$$(b^*) \qquad \frac{d^2 q_\nu}{dt^2} = -\sum_{\kappa,\lambda} \begin{Bmatrix}\kappa\ \lambda \\ \nu\end{Bmatrix}_b \frac{dq_\kappa}{dt}\frac{dq_\lambda}{dt} + \sum_\mu Q_\mu\, b'_{\mu\nu} .$$

The fact that one needs to know only the expressions for the virtual work and the *vis viva* in terms of the smallest number of determining data from the system in question in order to exhibit the differential equations of motion for a dynamical problem seems significant to me from another angle. It explains the phenomenon that various problems in mechanics that are totally disparate in their formulation have led to the same system of differential equations of motion, so to the same analytical problem. One must then distinguish between the *mechanical* trappings of the dynamical problem and its *analytical* kernel. The solution of a single analytical problem frequently produces the means to resolve entirely-different types of dynamical problems, since they only come down to the problem of interpreting and discussing the functions of time that one obtains from the standpoint of mechanics. Such phenomena are common in other branches of mathematical physics, and potential theory is a beautiful example of one. However, it is precisely in dynamics that the aforementioned distinction does not seem to be recognized with sufficient clarity and followed through on. The consequences of that will be shown in some illustrative examples below.

When two problems in dynamics relate to each other in such a way that they lead to the same system of differential equations of motion, they shall be called *analytically equivalent*.

If one would like to make the concept of the analytical equivalence of dynamical problems useful for the investigation of such problems then one will immediately encounter a serious difficulty insofar as that equivalence in terms of the identity of the systems of differential equations of motion will become obvious only when the determining data for the two problems being considered are chosen in a special way. In general, only the following is true: If Problem I leads to equations (a^*) and Problem II leads to equations (b^*) then both problems will be analytically equivalent when a substitution (S) exists that takes the system (a^*) to the system (b^*).

One can immediately give a *sufficient* condition for that to be the case. Since the time of **Lagrange**, one has known that when the related expressions for virtual work and *vis viva* for two dynamical problems:

$$\text{(I)}\quad \begin{cases} U' = \sum_\kappa P_\kappa \, \delta p_\kappa, \\ T = \frac{1}{2} \sum a_{\kappa\lambda} \frac{dp_\kappa}{dt} \frac{dp_\lambda}{dt}, \end{cases} \qquad \text{(II)}\quad \begin{cases} \mathfrak{U}' = \sum_\kappa \mathfrak{Q}_\kappa \, \delta q_\kappa, \\ \mathfrak{T} = \frac{1}{2} \sum \mathfrak{b}_{\kappa\lambda} \frac{dq_\kappa}{dt} \frac{dq_\lambda}{dt}, \end{cases}$$

go to each other under a substitution (*S*), the same substitution will take the differential equations of motion to each other, as well.

In order to decide whether such a substitution (*S*) exist, one proceeds as follows: One examines whether the quadratic differential form $2T\,dt^2$ can be transformed into $2\mathfrak{T}\,dt^2$. The question of whether two given quadratic differential forms can be transformed into each other was treated by **Christoffel** (*loc. cit.*, pps. 65 and 244). One next excludes the case in which the domain of the variables p_κ can be displaced into itself without changing $2T\,dt^2$, since "the substitution (*S*) would necessarily include elements that are capable of continuous variation." Therefore, if that transformation is possible at all then it will be well-defined, and one will decide whether it is possible when one investigates the possibility of the simultaneous linear transformation of a series of *algebraic* forms.

If one has investigated the transformation (*A*) that takes $2T\,dt^2$ to $2\mathfrak{T}\,dt^2$ in that way then one needs only to ascertain whether U' also goes to $\mathfrak{U}'$ under it. However, the identical existence of the equations:

$$\mathfrak{Q}_\kappa = \sum_\mu P_\mu \frac{\partial p_\mu}{\partial q_\kappa} \qquad (\kappa = 1, 2, \ldots, n)$$

is necessary and sufficient for that. Thus, if those equations are true then the problems will be analytically equivalent.

Now, since analytical equivalence requires only that the differential equations of motion can be made identical, that now raises the question of what sort of relationship the expressions for the virtual work and *vis viva* might have for two analytically-equivalent dynamical problems.

Two such problems necessarily belong to the same smallest number *n* of determining data. It would then be natural to combine all dynamical problems for which that number has the same value into *orders* that are characterized by the value of *n*.

Since the two problems considered are analytically equivalent, one can then imagine that the determining data were introduced from the outset in such a way that the equivalence will become obvious. Therefore, let:

$$U' = \sum_\kappa P_\kappa \, \delta p_\kappa, \qquad \mathfrak{U}' = \sum_\kappa V_\kappa \, \delta p_\kappa,$$

$$T = \frac{1}{2} \sum a_{\kappa\lambda} \frac{dp_\kappa}{dt} \frac{dp_\lambda}{dt}, \qquad \mathfrak{T} = \frac{1}{2} \sum w_{\kappa\lambda} \frac{dp_\kappa}{dt} \frac{dp_\lambda}{dt}.$$

The relevant differential equations of motion will then read:

$$(a^*) \qquad \frac{d^2 p_\nu}{dt^2} = -\sum_{\kappa,\lambda} \begin{Bmatrix} \kappa\,\lambda \\ \nu \end{Bmatrix}_a \frac{dp_\kappa}{dt}\frac{dp_\lambda}{dt} + \sum_\mu P_\mu\, a'_{\mu\nu} ,$$

$$(w^*) \qquad \frac{d^2 p_\nu}{dt^2} = -\sum_{\kappa,\lambda} \begin{Bmatrix} \kappa\,\lambda \\ \nu \end{Bmatrix}_w \frac{dp_\kappa}{dt}\frac{dp_\lambda}{dt} + \sum_\mu V_\mu\, w'_{\mu\nu} .$$

By assumption, the two systems are identical, i.e., they define the p_κ as the same functions of time. However, in order for that to be true, it is necessary and sufficient that both of them imply the same values of the $\frac{d^2 p_\kappa}{dt^2}$ for the same arbitrary initial values of the p_κ and $\frac{dp_\kappa}{dt}$, and that means nothing but the idea that the equations:

$$(A) \qquad \begin{Bmatrix} \kappa\,\lambda \\ \nu \end{Bmatrix}_w = \begin{Bmatrix} \kappa\,\lambda \\ \nu \end{Bmatrix}_a \qquad (\kappa, \lambda, \nu = 1, 2, \ldots, n),$$

$$(B) \qquad \sum_\kappa V_\kappa\, \delta p_\kappa = \sum_\kappa P_\kappa\, \delta p_\kappa \qquad (\nu = 1, 2, \ldots, n)$$

must be true identically in the $p_1, p_2, \ldots, p_n$.

Equations (A) and (B) can also be regarded as the Ansatz for the following problem: If a dynamical problem with the expressions U' and T is given then the P_κ and $a_{\kappa\lambda}$ are known. One seeks the expressions $\mathfrak{U}'$ and $\mathfrak{T}$, so the V_κ and $w_{\kappa\lambda}$, that will belong to dynamical problems that are analytically equivalent to the given one. By means of equations (A) and (B), that problem is reduced to a discussion of a system of simultaneous partial differential equations. In so doing, it should be noted that the only solutions $w_{\kappa\lambda}$ to that system that can be allowed are the ones for which $2\mathfrak{T}\, dt^2$ is a positive quadratic form in the dp_κ . That will immediately make it possible to simplify the discussion immensely, since the determinant:

$$(7) \qquad w = |\, w_{\kappa\lambda}\,| \qquad (\kappa, \lambda = 1, 2, \ldots, n)$$

cannot vanish identically, equations (B) can be solved for the V_λ, so they will be equivalent to:

$$(B') \qquad V_\lambda = \sum_{\kappa,\nu} P_\kappa\, a'_{\kappa\nu}\, w_{\lambda\nu} .$$

One then needs only to investigate what the most general solution of equations (A) for which $2\mathfrak{T}\, dt^2$ is a positive quadratic form. If one has found it then one will get the most general expression for $\mathfrak{U}'$ when one defines the V_λ by equations (B') .

Everything then comes down to a discussion of equations (A). One easily sees that those equations are fulfilled identically by:

$$w_{\kappa\lambda} = c\, a_{\kappa\lambda} ,$$

in which c is a positive constant. One will then have:

$$\mathfrak{T} = c\,T,$$

so $2\mathfrak{T}\,dt^2$ will be a positive quadratic form in the dp_κ. Furthermore, that will imply that:

$$\mathfrak{U}' = cU'.$$

In order for two dynamical problems to be analytical equivalent, it will then suffice that the related expressions for virtual work and *vis viva* should differ by only the same multiplicative constant.

It is easy to find a problem with expressions for U' and T that will lead to $\mathfrak{U}' = cU'$, $\mathfrak{T} = c\,T$. One needs only to replace every length l in the first problem with $\alpha\,l$, every mass m with $\beta\,m$, every force X with γX, and every time t with $\varepsilon\,t$, where α, β, γ, ε mean constants. Thus, U' will go to $\alpha\gamma U'$, and T will go to $(\alpha^2\beta/\varepsilon^2)T$. Hence, if one has only:

$$\varepsilon^2 = \frac{\alpha\beta}{\gamma}$$

then U' and T will change by the same multiplicative constant, and the differential equations of motion will remain unchanged.

Newton (*Philosophiae naturalis principia mathematica*, 1686, Liber. I, prop. 87) had already recognized that special type of analytical equivalence for some problems of a special nature. It is formulated in full generality in **Bertrand** [“Note sur la similitude en mécanique,” J. de l’École Polytechnique **19** (1848) Cah. 32, pp. 189] and can be referred to as *mechanical similarity*. **Bertrand** stressed that this principle, which he regarded as very fruitful, did not, in fact, give the solution to a mechanical problem, but probably just the connection between different problems, which would be problems “de difficulté analytique équivalente,” as he phrased it. It should be mentioned that **H. von Helmholtz** made repeated use of mechanical similarity in his work (cf., e.g., *Wissenschaftliche Abhandlungen*, Bd. I, pp. 158).

One now asks whether the system (A) possesses other solutions besides the obvious solution $w_{\kappa\lambda} = c\,a_{\kappa\lambda}$. The fact that this can be very likely be the case for certain special problems can be shown by the following example. Let the $a_{\kappa\lambda}$ all be constants. All $\begin{bmatrix}\kappa\,\lambda\\ \mu\end{bmatrix}_a$, and therefore all $\begin{Bmatrix}\kappa\,\lambda\\ \mu\end{Bmatrix}_a$, as well, will be equal to zero, and equations (A) will become $\begin{Bmatrix}\kappa\,\lambda\\ \mu\end{Bmatrix}_a = 0$.

However, those equations are clearly fulfilled identically when one assigns any constant values to the $w_{\kappa\lambda}$. In order for $2\mathfrak{T}\,dt^2$ to be a positive quadratic form in the dp_κ, only the principal subdeterminants of the symmetric system of $w_{\kappa\lambda}$ must be positive. Therefore, it is not at all unnecessary for the ratios $a_{\kappa\lambda} : w_{\kappa\lambda}$ to have the same constant values for all systems of values κ, λ.

However, it can probably be shown (and this will define the subject of the following section) that *in general* the system (*A*) has only the solution $w_{\kappa\lambda} = c\, a_{\kappa\lambda}$, i.e., that the demand that there should exist another solution in addition to that one, defines an actual constraint on the choice of coefficients $a_{\kappa\lambda}$. The following theorem will be proved in the following section:

Let a symmetric system of functions of the variables $p_1, p_2, \ldots, p_n$: $a_{\kappa\lambda}$ ($\kappa, \lambda = 1, 2, \ldots, n$) *whose principal subdeterminants do not vanish identically be given. One seeks a system* $w_{\kappa\lambda}$ *with the property that the* $w_{\kappa\lambda}$ *satisfy the partial differential equations:*

$$\left\{ \begin{matrix} \kappa\ \lambda \\ \mu \end{matrix} \right\}_w = \left\{ \begin{matrix} \kappa\ \lambda \\ \mu \end{matrix} \right\}_a \qquad (\kappa, \lambda, \mu = 1, 2, \ldots, n).$$

Assume that the $a_{\kappa\lambda}$ *satisfy certain partial differential equations that can be given precisely. The most general system* $w_{\kappa\lambda}$ *of the required type will then be given by:*

$$w_{\kappa\lambda} = c\, a_{\kappa\lambda},$$

in which c is an arbitrary constant.

The conclusion of the section will consist of the consequences that we can infer from that theorem in terms of our dynamical questions.

According to **Lagrange**, in order to exhibit the differential equations of motion for a dynamical problem of the system in question, one needs to know only the expressions for the virtual work and the *vis viva* in terms of the smallest number of determining data. It will follow from the theorem that was just given that, except for certain singular cases, one can, conversely, determine the expressions for the virtual work up to a multiplicative constant. The appearance of those constants explains how it already emerges from the remark on page 9 that as long as the choice of units of length, mass, force, and time remain arbitrary, those expressions will be determined only up to a proportionality factor. One might then say that for a suitable choice of units, the identity of the system of differential equations for two problems will generally also imply the identity of the expressions for virtual work and the *vis viva*. Under that condition, one will then have the theorem:

Except for some singular cases (that must be examined individually), in order for two dynamical problems to be analytically equivalent, it is necessary and sufficient that the relevant expressions for virtual work and vis viva in terms of the smallest number of determining data are identical or can be made identical by simultaneous transformation of the variables.

It follows from this that every general principle of mechanics with whose help the differential equations of motion for a dynamical problem can be exhibited must be such that one can determine the expressions for the virtual work and *vis viva* of the system with its help. The knowledge those expressions can then be characterized as the minimum amount of information that would be required in order to characterize a dynamical problem analytically.

However, the result of the foregoing investigation of the analytical equivalence of dynamical problems first came into the proper light when one linked it up with certain considerations that **Lipschitz** presented in his treatises "Untersuchung eines Problem der Variationsrechnung" [this journal, **74** (1871), pp. 746] and "Bemerkungen zu dem Princip des kleinsten Zwanges" [this journal, **82** (1877), pp. 316] (*). It is especially important in that to work through what one finds in *loc. cit.*, Bd. 82, pp. 331, and whose essential content is reproduced here in the notation that we have applied.

A problem in dynamics has led to the expressions:

$$U' = \sum_{\kappa} P_{\kappa}\, \delta p_{\kappa}\,, \qquad T = \tfrac{1}{2}\sum_{\kappa,\lambda} a_{\kappa\lambda}\, \frac{dp_{\kappa}}{dt}\frac{dp_{\lambda}}{dt}\,.$$

Now, one can extend the problem in dynamics in such a way that the square of the line element ds in space is assumed to be equal to an arbitrary essentially-positive quadratic form in the differentials of the coordinates. One can choose that differential form to have precisely the expression:

$$\sum_{\kappa,\lambda} a_{\kappa\lambda}\, dp_{\kappa}\, dp_{\lambda} = ds^2\,. \tag{8}$$

In agreement with that, the *vis viva* of a point of mass 1 is then equal to:

$$T = \tfrac{1}{2}\left(\frac{ds}{dt}\right)^2 = \tfrac{1}{2}\sum_{\kappa,\lambda} a_{\kappa\lambda}\, \frac{dp_{\kappa}}{dt}\frac{dp_{\lambda}}{dt}\,.$$

However, one can also give the corresponding meaning of the virtual work for the motion that is now being considered to the expression:

$$U' = \sum_{\kappa} P_{\kappa}\, \delta p_{\kappa}\,.$$

If one does that then, by means of the theorems that **Lipschitz** developed in the first-cited treatise, one will obtain a system of differential equations of motion for a point of mass 1 in the n-fold manifold whose line element ds is given by eq. (8) that has precisely the same form as the system of differential equations (a) that was exhibited for that dynamical problem.

If one then considers dynamical problems that correspond to that extended concept of mechanics then one will regard all of the ones that lead to the same system of differential equations of motion as analytically equivalent. In order for two such problems to be analytically equivalent, it is generally (except for those singular cases) necessary and sufficient that they should lead to the same differential forms:

(*) In regard to that, I shall remark that **Lipschitz** was also kind enough to communicate some letters to me on the subject that were of great use to me in the present investigation and for which I feel that I owe him my deepest gratitude.

$$U' = \sum_{\kappa} P_\kappa \, \delta p_\kappa \,, \qquad 2T\,dt^2 = \sum_{\kappa,\lambda} a_{\kappa\lambda}\, dp_\kappa\, dp_\lambda \,.$$

Among the infinitude of analytically-equivalent dynamical problems, there is always one of them whose mechanical formulation is simplest, namely, the problem that says that one must determine the motion of *one* point of mass 1 that always remain in an n-fold manifold whose line element ds is given by eq. (8), while the expression for the virtual work is represented by U'. However, such a reduction is possible even in those singular cases, although it can happen that more than one *normal problem* of the type that was just characterized might exist.

With that conception of things, the equations:

$$\left\{ \begin{matrix} \kappa\ \lambda \\ \mu \end{matrix} \right\}_w = \left\{ \begin{matrix} \kappa\ \lambda \\ \mu \end{matrix} \right\}_a \qquad (A)$$

(whose discussion everything comes back to, moreover) will take on a simple dynamical meaning. They can be regarded as the necessary and sufficient condition for the system of differential equations:

$$\frac{d^2 p_\mu}{dt^2} = - \sum_{\kappa,\lambda} \left\{ \begin{matrix} \kappa\ \lambda \\ \mu \end{matrix} \right\}_a \frac{dp_\kappa}{dt} \frac{dp_\lambda}{dt} \qquad (a^0)$$

to be identical to the system:

$$\frac{d^2 p_\mu}{dt^2} = - \sum_{\kappa,\lambda} \left\{ \begin{matrix} \kappa\ \lambda \\ \mu \end{matrix} \right\}_w \frac{dp_\kappa}{dt} \frac{dp_\lambda}{dt} \,. \qquad (w^0)$$

However, the first system can be regarded as the system of differential equations of the *geodetic lines* of the n-fold manifold whose line element ds will be given by equation (8), and the question of the most general solution to equation (A) for a given $a_{\kappa\lambda}$ will then be identical to the question of the most general form for the line element of an n-fold manifold whose geodetic lines satisfy the differential equations (a^0). Since that solution will generally be $w_{\kappa\lambda} = c\, a_{\kappa\lambda}$, as will be shown in the following section, that will likewise imply that (except for singular cases in which the question must remain open) the line element of the n-fold manifold in question is determined uniquely up to an arbitrary constant multiplicative factor by the differential equations of the geodetic lines, so by general equations of those lines, as well. The fact that the converse is true, namely, that n-fold manifolds whose line elements will agree also lead to the same general equations of geodetic lines, is immediately obvious.

III.

Investigating the system of partial differential equations:

$$\left\{\begin{matrix}\kappa\,\lambda\\ \nu\end{matrix}\right\}_w = \left\{\begin{matrix}\kappa\,\lambda\\ \nu\end{matrix}\right\}_a \qquad (\kappa, \lambda, \nu = 1, 2, \ldots, n).$$

Let a symmetric system of functions of the n independent variables $p_1, p_2, \ldots, p_n$ be given:

$$a_{\kappa\lambda} \qquad (\kappa, \lambda = 1, 2, \ldots, n),$$

of which one assumes only that the principal subdeterminants do not vanish identically. One seeks the most general system of them with the same property:

$$w_{\kappa\lambda} \qquad (\kappa, \lambda = 1, 2, \ldots, n)$$

that relates to the given one in such a way that:

$$(A) \qquad \left\{\begin{matrix}\kappa\,\lambda\\ \nu\end{matrix}\right\}_w = \left\{\begin{matrix}\kappa\,\lambda\\ \nu\end{matrix}\right\}_a \qquad (\kappa, \lambda, \nu = 1, 2, \ldots, n).$$

The question of whether such a system $w_{\kappa\lambda}$ exists is meaningful, since $w_{\kappa\lambda} = c\, a_{\kappa\lambda}$, where c means a constant, is one solution of (A).

For the investigation of equations (A), it is of the greatest importance that they can be replaced by a completely-equivalent system of homogeneous linear first-order partial differential equations for the $w_{\kappa\lambda}$. That conversion is based upon the identity that **Christoffel** gave (*loc. cit.*, pp. 50):

$$(9) \qquad \left[\begin{matrix}\mu\,\kappa\\ \lambda\end{matrix}\right]_w + \left[\begin{matrix}\mu\,\lambda\\ \kappa\end{matrix}\right]_w = \frac{\partial w_{\kappa\lambda}}{\partial p_\mu}\,.$$

Moreover, that identity is only a special case of a more general one that **Lipschitz** discovered [*loc. cit.*, Bd. 70, formula (11)]. With the help of the definition of the $\left\{\begin{matrix}\kappa\,\lambda\\ \mu\end{matrix}\right\}_w$, one concludes from equation (9) that along with equations (A), one also has the equations:

$$(A^*) \qquad \frac{\partial w_{\kappa\lambda}}{\partial p_\mu} = \sum_\sigma \left\{\begin{matrix}\mu\,\kappa\\ \sigma\end{matrix}\right\}_a w_{\sigma\lambda} + \sum_\sigma \left\{\begin{matrix}\mu\,\lambda\\ \sigma\end{matrix}\right\}_a w_{\sigma\kappa}\,.$$

However, a simple calculation will show that conversely equations (A) will again emerge from equations (A^*), so they can be replaced by equations (A^*) completely.

Equations (A^*) answer the question of the most general solution to equations (A) quite directly in the aforementioned special case on page 9 in which all $a_{\kappa\lambda}$ were constants. Namely, one will then have:

$$\frac{\partial w_{\kappa\lambda}}{\partial p_\mu} = 0 \qquad (\kappa, \lambda, \mu = 1, 2, \ldots, n),$$

such that the solution $w_{\kappa\lambda}$ = const. that was given above is even the most general solution of equations (A) for this special case, which can then be regarded as solved from now on.

Should equations (A^*) be compatible with each other, then the partial derivative of $\frac{\partial w_{\kappa\lambda}}{\partial p_\mu}$ with respect to p_ν would have to agree with the partial derivative of $\frac{\partial w_{\kappa\lambda}}{\partial p_\nu}$ with respect to p_μ. One easily sees that these *integrability conditions* appear in the form of homogeneous linear relations between the $w_{\kappa\lambda}$. The coefficients in those relations are closely related to the coefficients of the quadrilinear form Ψ that **Lipschitz** exhibited [this journal, Bd. 79, pp. 84, formula (34)], which is covariant of the quadratic differential form:

$$\sum_{\kappa,\lambda} a_{\kappa\lambda}\, dp_\kappa\, dp_\lambda\,,$$

and whose identical vanishing is the necessary and sufficient condition for that form to be transformable into a form with constant coefficients. However, the same expressions, up to a constant factor, also occur in a different context for **Christoffel** [*ibidem*, pp. 54, formula (13)], where it was denoted symbolically by ($\kappa\,\lambda\,\mu\,\nu$). Since that symbol was also used by **Lipschitz**, but with a completely different meaning, I shall regard a different notation to be preferable and set:

$$(10) \qquad \begin{bmatrix} \kappa & \lambda \\ \mu & \nu \end{bmatrix}_a = \frac{\partial}{\partial p_\nu}\begin{bmatrix} \mu\,\kappa \\ \lambda \end{bmatrix}_a - \frac{\partial}{\partial p_\mu}\begin{bmatrix} \nu\,\kappa \\ \lambda \end{bmatrix}_a + \sum_{\alpha,\beta} a'_{\alpha\beta}\left(\begin{bmatrix} \mu\,\nu \\ \alpha \end{bmatrix}_a \begin{bmatrix} \mu\,\lambda \\ \beta \end{bmatrix}_a - \begin{bmatrix} \kappa\,\mu \\ \alpha \end{bmatrix}_a \begin{bmatrix} \nu\,\lambda \\ \beta \end{bmatrix}_a\right).$$

One will then have:

$$(11) \qquad \Psi = -\,2 \sum_{\kappa,\lambda,\mu,\nu} \begin{bmatrix} \kappa & \lambda \\ \mu & \nu \end{bmatrix}_a du_\kappa\, \delta u_\lambda\, dv_\mu\, \delta v_\nu\,,$$

in which the u_ρ and v_σ mean new variables. Just as the expressions $\begin{Bmatrix} \kappa\,\lambda \\ \mu \end{Bmatrix}_a$ can be derived from the expressions $\begin{bmatrix} \kappa\,\lambda \\ \mu \end{bmatrix}_a$ using equation (6), new expressions can be derived from the $\begin{bmatrix} \kappa & \lambda \\ \mu & \nu \end{bmatrix}_a$ by way of:

(12) $$\sum_\sigma \begin{bmatrix} \kappa & \sigma \\ \mu & \nu \end{bmatrix}_a a'_{\sigma\lambda} = \begin{Bmatrix} \kappa & \lambda \\ \mu & \nu \end{Bmatrix}_a .$$

Conversely, one then has:

(13) $$\sum_\sigma \begin{Bmatrix} \kappa & \sigma \\ \mu & \nu \end{Bmatrix}_a a_{\sigma\lambda} = \begin{bmatrix} \kappa & \lambda \\ \mu & \nu \end{bmatrix}_a ,$$

such that the vanishing of the $\begin{bmatrix} \kappa & \lambda \\ \mu & \nu \end{bmatrix}_a$ will also imply the vanishing of the $\begin{Bmatrix} \kappa & \lambda \\ \mu & \nu \end{Bmatrix}_a$, and conversely.

If one compares formula (13) with the one that **Christoffel** derived (*loc. cit.*, pp. 53):

$$\begin{bmatrix} \kappa & \lambda \\ \mu & \nu \end{bmatrix}_a = \sum_\sigma a_{\sigma\lambda} \left\{ \frac{\partial}{\partial p_\nu} \begin{Bmatrix} \mu\,\kappa \\ \sigma \end{Bmatrix}_a - \frac{\partial}{\partial p_\mu} \begin{Bmatrix} \nu\,\kappa \\ \alpha \end{Bmatrix}_a + \sum_\rho \left(\begin{Bmatrix} \kappa\,\mu \\ \rho \end{Bmatrix}_a \begin{Bmatrix} \rho\,\nu \\ \sigma \end{Bmatrix}_a - \begin{Bmatrix} \kappa\,\nu \\ \rho \end{Bmatrix}_a \begin{Bmatrix} \rho\,\mu \\ \sigma \end{Bmatrix}_a \right) \right\}$$

then one will easily convince oneself that the following equations are true:

(14) $$\begin{Bmatrix} \kappa & \lambda \\ \mu & \nu \end{Bmatrix}_a = \frac{\partial}{\partial p_\nu} \begin{Bmatrix} \mu\,\kappa \\ \lambda \end{Bmatrix}_a - \frac{\partial}{\partial p_\mu} \begin{Bmatrix} \nu\,\kappa \\ \lambda \end{Bmatrix}_a + \sum_\rho \left(\begin{Bmatrix} \kappa\,\mu \\ \rho \end{Bmatrix}_a \begin{Bmatrix} \rho\,\nu \\ \lambda \end{Bmatrix}_a - \begin{Bmatrix} \kappa\,\nu \\ \rho \end{Bmatrix}_a \begin{Bmatrix} \rho\,\mu \\ \lambda \end{Bmatrix}_a \right).$$

Moreover, if one defines the integrability conditions:

$$\frac{\partial w_{\kappa\lambda}}{\partial p_\mu\,\partial p_\nu} - \frac{\partial w_{\kappa\lambda}}{\partial p_\nu\,\partial p_\mu} = 0$$

then one will get (*):

(*J*) $$\sum_\sigma \begin{Bmatrix} \kappa & \sigma \\ \mu & \nu \end{Bmatrix}_a w_{\sigma\lambda} + \sum_\sigma \begin{Bmatrix} \lambda & \sigma \\ \mu & \nu \end{Bmatrix}_a w_{\sigma\kappa} = 0$$

after some reductions.

If one set $\lambda = \kappa$ and gives κ a fixed value from the sequence 1, 2, 3, …, *n*, while μ and ν remain arbitrary, so they can be set equal to the $\frac{1}{2}n(n-1)$ different combinations of two of the *numbers* 1, 2, …, *n*, then one will get the group $\frac{1}{2}n(n-1)$ equations that are included among (*J*):

(*) Equations (*J*) will exist identically when all $\begin{Bmatrix} \kappa & \sigma \\ \mu & \nu \end{Bmatrix}_a$ vanish identically. That is the only time when all $\begin{Bmatrix} \lambda & \sigma \\ \mu & \nu \end{Bmatrix}_a$ will also vanish identically, so $\Psi = 0$, and one can assume from the outset that the $a_{\kappa\lambda}$ are constants. However, it is precisely that case that was resolved already.

$$(J_\kappa) \qquad \sum_\sigma \begin{Bmatrix} \kappa & \sigma \\ \mu & \nu \end{Bmatrix}_a w_{\sigma\kappa} = 0 \qquad (\mu,\ \nu = 1, 2, \ldots, n).$$

Those equations can be regarded as homogeneous linear equations for the unknowns $w_{1\kappa}$, $w_{2\kappa}$, …, $w_{n\kappa}$. The number of them amounts to 1, 3, 6 for $n = 2, 3, 4$, respectively. Equations (J_κ) are satisfied identically by $w_{\sigma\kappa} = a_{\sigma\kappa}$. Now, since $a_{\kappa\kappa}$ is non-zero, one can satisfy equations (J_κ) without having to set all n quantities $w_{\sigma\kappa}$ equal to zero. As a result, the rank of the system:

$$(S_\kappa) \qquad \begin{Bmatrix} \kappa & \sigma \\ \mu & \nu \end{Bmatrix} \qquad \begin{pmatrix} \sigma = 1, 2, \ldots, n, \\ \mu, \nu = 1, 2, \ldots, n \end{pmatrix}$$

will necessarily be smaller than n.

If the rank of that system is equal to precisely $n - 1$ then the $w_{\sigma\kappa}$ will be determined by equations (J_κ), up to an arbitrary multiplicative factor. However, one can satisfy those equations by $w_{\sigma\kappa} = a_{\sigma\kappa}$, so their most general solution is:

$$w_{\sigma\kappa} = z_\kappa\, a_{\sigma\kappa},$$

in which z_κ denotes a still-undetermined function of the $p_1, p_2, \ldots, p_n$. In order to determine that functions, I shall substitute those expressions for the $w_{\sigma\kappa}$ in the equations of the system (A^*):

$$(A_\kappa^*) \qquad \frac{\partial w_{\kappa\kappa}}{\partial p_\mu} = 2\sum_\sigma \begin{Bmatrix} \mu\ \kappa \\ \sigma \end{Bmatrix}_a w_{\sigma\kappa} \qquad (\mu = 1, 2, \ldots, n).$$

That will then give:

$$a_{\kappa\kappa}\frac{\partial z_\kappa}{\partial p_\mu} = -\, z_\kappa\left(\frac{\partial a_{\kappa\kappa}}{\partial p_\mu} - 2\sum_\sigma \begin{Bmatrix} \mu\ \kappa \\ \sigma \end{Bmatrix}_a a_{\sigma\kappa}\right).$$

However, the factor of z_κ will vanish, since one can satisfy equations (A_κ^*) by $w_{\sigma\kappa} = a_{\sigma\kappa}$. As a result:

$$a_{\kappa\kappa}\frac{\partial z_\kappa}{\partial p_\mu} = 0 \qquad (\mu = 1, 2, \ldots, n),$$

and since $a_{\kappa\kappa}$ does not vanish identically, $\frac{\partial z_\kappa}{\partial p_\mu} = 0$, so z_κ will be equal to a constant C_κ. If the rank of the system (S_κ) is equal to $n - 1$ then the $w_{1\kappa}$, $w_{2\kappa}$, …, $w_{n\kappa}$ will be determined by equations (J_κ), in conjunction with equations (A_κ^*), up to the multiplicative constant C_κ.

That will be true for all n groups (J_1), …, (J_n), and if the rank of the system (S_1), …, (S_n) is equal to $n - 1$ then all $w_{\mu\nu}$ will be determined up to multiplicative constants C_κ. However, it can now be shown that under that assumption, the n constants C_κ must necessarily all have the same values c. Namely, one has:

$$w_{\kappa\lambda} = w_{\lambda\kappa} = C_\kappa\, a_{\kappa\lambda} = C_\lambda\, a_{\lambda\kappa} = C_\lambda\, a_{\kappa\lambda},$$

so

$$(C_\kappa - C_\lambda)\, a_{\kappa\lambda} = 0 .$$

It follows from this that $C_\kappa = C_\lambda$, except when $a_{\kappa\lambda} = 0$. The assertion is then proved, unless one has precisely:

$$a_{\kappa\lambda} = 0 \qquad (\lambda = 1, 2, \ldots, \kappa - 1, \kappa + 1, \ldots, n)$$

for a fixed value of κ, which is a case that requires a special examination. Moreover, it will occur, and even for every κ, when the differential form in question includes only the squares of the differentials $dp_1, dp_2, \ldots, dp_n$ as is often the case.

Equations (J_κ) are satisfied in any case by $w_{\sigma\kappa} = a_{\sigma\kappa}$, so by:

$$w_{1\kappa} = 0 , \quad \ldots, \quad w_{\kappa-1,\kappa} = 0 , \quad w_{\kappa\kappa} = a_{\kappa\kappa}, \quad w_{\kappa+1,\kappa} = 0 , \quad \ldots, \quad w_{n\kappa} = 0$$

in the case that is considered here. If one substitutes those values for the $w_{\sigma\kappa}$ in (J_κ) then that will make:

$$\begin{Bmatrix} \kappa & \sigma \\ \mu & \nu \end{Bmatrix}_a a_{\kappa\kappa} = 0 ,$$

such that one must have $\begin{Bmatrix} \kappa & \sigma \\ \mu & \nu \end{Bmatrix}_a = 0$ in this case, and indeed for all $\frac{1}{2} n (n-1)$ pairs of values μ, ν. Equations (J_κ) can then be fulfilled by only:

$$w_{\sigma\kappa} = 0 \qquad (\sigma = 1, 2, \ldots, \kappa - 1, \kappa + 1, \ldots, n),$$

while $w_{\kappa\kappa}$ does not appear in it at all, so it remains completely undetermined. However, equations (A_κ^*) will imply that $w_{\kappa\kappa} = C_\kappa\, a_{\kappa\kappa}$, as before.

Since equations (J_κ) and (A_κ^*) have been exhausted, one must take recourse to the other equations in the system (J), in which κ has the fixed value that comes under consideration here. Since $w_{\sigma\kappa} = 0$ for $\sigma = 1, 2, \ldots, \kappa - 1, \kappa + 1, \ldots, n$, those equations will be:

$$\sum_\sigma \begin{Bmatrix} \kappa & \sigma \\ \mu & \nu \end{Bmatrix}_a w_{\sigma\lambda} + \begin{Bmatrix} \lambda & \kappa \\ \mu & \nu \end{Bmatrix}_a w_{\kappa\kappa} = 0 \qquad \begin{pmatrix} \lambda = 1, 2, \ldots, \kappa - 1, \kappa + 1, \ldots, n \\ \mu, \nu = 1, 2, \ldots, n \end{pmatrix}.$$

Therefore, if:

$$C_\lambda \sum_\sigma \begin{Bmatrix} \kappa & \sigma \\ \mu & \nu \end{Bmatrix}_a a_{\sigma\lambda} + C_\kappa \begin{Bmatrix} \lambda & \kappa \\ \mu & \nu \end{Bmatrix}_a a_{\kappa\kappa} = 0 .$$

Now, one has:

$$\sum_\sigma \begin{Bmatrix} \kappa & \sigma \\ \mu & \nu \end{Bmatrix}_a a_{\sigma\lambda} + \begin{Bmatrix} \lambda & \kappa \\ \mu & \nu \end{Bmatrix}_a a_{\kappa\kappa} = 0$$

identically, so one must have:

$$(C_\lambda - C_\kappa) \sum_\sigma \begin{Bmatrix} \kappa & \sigma \\ \mu & \nu \end{Bmatrix}_a a_{\sigma\lambda} = 0 .$$

However, one infers from this that $C_\lambda = C_\kappa$ for $\lambda = 1, 2, \ldots, \kappa - 1, \kappa + 1, \ldots, n$ since it is impossible that *all* expressions:

(15) $$\sum_\sigma \begin{Bmatrix} \kappa & \sigma \\ \mu & \nu \end{Bmatrix}_a a_{\sigma\lambda} = 0 \qquad (\mu, \nu = 1, 2, \ldots, n).$$

Namely, that system of equations will go to (J_κ) when one replaces $a_{\sigma\lambda}$ with $w_{\sigma\lambda}$, and as a result, its most general solution will be:

$$a_{1\lambda} = 0 , \qquad \ldots, \qquad a_{\kappa-1,\lambda} = 0 , \qquad a_{\kappa+1,\lambda} = 0 , \qquad \ldots, \qquad a_{n\lambda} = 0 .$$

The existence of all equations (15) would then have $a_{\lambda\lambda} = 0$ as a consequence, since λ is different from κ, which would contradict the assumption.

One has then arrived at the following result:

If each of the n systems:

(S_κ) $$\begin{Bmatrix} \kappa & \sigma \\ \mu & \nu \end{Bmatrix}_a \qquad \begin{pmatrix} \sigma = 1, 2, \ldots, n \\ \mu, \nu = 1, 2, \ldots, n \end{pmatrix}$$

has rank precisely n – 1 *then the most general solution of:*

(A) $$\begin{Bmatrix} \kappa\ \lambda \\ \nu \end{Bmatrix}_w = \begin{Bmatrix} \kappa\ \lambda \\ \nu \end{Bmatrix}_a \qquad (\kappa, \lambda, \nu = 1, 2, \ldots, n)$$

will be given by:

$$w_{\kappa\lambda} = c\, a_{\kappa\lambda} \qquad (\kappa, \lambda = 1, 2, \ldots, n),$$

in which c means a constant.

Everything will then come down to showing that the requirement that the rank of one of those n systems is less than $n - 1$ means an actual restriction on the choice of functions $a_{\kappa\lambda}$, so it is not true for, say, *each* system of $a_{\kappa\lambda}$ that at least one of the systems (S_κ) has a rank less than $n - 1$ in its own right.

To that end, it would suffice to prove that when the $a_{\rho\sigma}$ are regarded as arbitrary functions, at least one of the subdeterminants of $(n - 1)^2$ elements of each of the systems (S_κ) is non-zero. It is only with the same expenditure of calculation that one can show then that when one selects an arbitrary equation from the $\frac{1}{2} n (n-1)$ equations (J_κ), one can always associate it with $n - 2$ of the other equations in such a way that under that assumption, the ratios of the $w_{1\kappa}, w_{2\kappa}, \ldots, w_{n\kappa}$ will be determined completely by those $n - 1$ equations.

If the $a_{\rho\sigma}$ are regarded as arbitrary functions of the p_τ then their derivatives with respect to those variables can be regarded as new arbitrary functions. Therefore, the determinant:

$$\left| \begin{Bmatrix} \kappa & \sigma_\beta \\ \mu_\alpha & \nu_\alpha \end{Bmatrix} \right| \qquad (\alpha, \beta = 1, 2, \ldots, n-1)$$

can vanish identically only when the set of all terms in that determinant that include the second derivatives of the $a_{\rho\sigma}$ vanish by themselves. Along with that determinant, the determinant:

$$\left| \sum_\lambda \left(\frac{\partial^2 a_{\lambda\mu_a}}{\partial p_\kappa\, \partial p_{\nu_\alpha}} + \frac{\partial^2 a_{\kappa\nu_a}}{\partial p_\lambda\, \partial p_{\mu_\alpha}} - \frac{\partial^2 a_{\lambda\nu_a}}{\partial p_\kappa\, \partial p_{\mu_\alpha}} - \frac{\partial^2 a_{\kappa\mu_a}}{\partial p_\lambda\, \partial p_{\nu_\alpha}} \right) a'_{\lambda\sigma_\beta} \right| \qquad (\alpha, \beta = 1, 2, \ldots, n-1)$$

must vanish identically then. Since the term with $\lambda = \kappa$ vanishes identically in each of those $(n - 1)^2$ sums of n terms, that determinant will be equal to the product:

$$\left| \frac{\partial^2 a_{\lambda'\mu_a}}{\partial p_\kappa\, \partial p_{\nu_\alpha}} + \frac{\partial^2 a_{\kappa\nu_a}}{\partial p_{\lambda'}\, \partial p_{\mu_\alpha}} - \frac{\partial^2 a_{\lambda'\nu_a}}{\partial p_\kappa\, \partial p_{\mu_\alpha}} - \frac{\partial^2 a_{\kappa\mu_a}}{\partial p_{\lambda'}\, \partial p_{\nu_\alpha}} \right| \cdot \left| a'_{\lambda'\sigma_\beta} \right| \qquad \begin{pmatrix} \lambda' = 1, 2, \ldots, \kappa-1, \kappa+1, \ldots, n \\ \alpha, \beta = 1, 2, \ldots, n-1 \end{pmatrix}.$$

As long as $a_{\rho\sigma} = a_{\sigma\rho}$ can be regarded as arbitrary quantities, the second factor cannot vanish identically. However, the second factor is also non-zero. One can see that as follows:

If one first considers the quantities that occur in an element of the determinant (in which one must observe that the indices of the $a_{\rho\sigma}$ can be switched without changing the value, like the sequence of differentiations) then one will see that the first term can only be equal to the second one and the third one can only be equal to the fourth, and conversely, and that this will occur for $\mu_a = \kappa$, $\nu_a = \lambda$ and $\nu_a = \kappa$, $\mu_a = \lambda$, respectively. One now chooses the $n - 1$ different pairs of values μ, ν as follows: Let all μ have the same value μ_0, and let $\nu = 1, 2, \ldots, \mu_0 - 1, \mu_0 + 1, \ldots, n$. Since one can have $\mu_0 = 1, 2, \ldots, n$, one will get n times $n - 1$ pairs μ, ν in that way, and one will easily see that these $n\,(n - 1)$ pairs μ, ν will include all possible pairs μ, ν, and indeed each of them twice. Therefore, if an arbitrary equation of the system (J_κ) is given then one can always associate it with $n - 2$ other equations in such a way that μ will remain constant. Now, it can be shown that of those $n - 1$ equations, the associated n subdeterminants of $(n - 1)^2$ elements that determine the ratios of the $w_{\sigma\kappa}$ when the $a_{\rho\sigma}$ are arbitrary functions will not vanish identically.

To that end, I will select from them the quantity:

$$\frac{\partial^2 a_{\lambda'\mu_0}}{\partial p_\kappa\, \partial p_\nu},$$

in which λ' and n have any fixed values (except that $\lambda' \neq \kappa$, $\nu \neq \mu_0$) and ask where that quantity will occur in the determinant:

$$\left|\frac{\partial^2 a_{\lambda'\mu_0}}{\partial p_\kappa\,\partial p_\nu}+\frac{\partial^2 a_{\kappa\nu}}{\partial p_{\lambda'}\,\partial p_{\mu_0}}-\frac{\partial^2 a_{\lambda'\nu}}{\partial p_\kappa\,\partial p_{\mu_0}}-\frac{\partial^2 a_{\kappa\mu_0}}{\partial p_{\lambda'}\,\partial p_\nu}\right| \qquad \begin{pmatrix}\lambda'=1,2,\dots,\kappa-1,\kappa+1,\dots,n\\ \nu=1,2,\dots,\mu_0-1,\mu_0+1,\dots,n-1\end{pmatrix}.$$

One easily sees that it can occur in *only one* element of the determinant and generally once in that element. It is only for $\mu_0 = \kappa$, $\lambda' = n$ that it can occur twice.

If one now imagines that the determinant is developed in the quantities:

$$\frac{\partial^2 a_{\lambda'\mu_0}}{\partial p_\kappa\,\partial p_\nu} \qquad \begin{pmatrix}\lambda'=1,2,\dots,\kappa-1,\kappa+1,\dots,n\\ \nu=1,2,\dots,\mu_0-1,\mu_0+1,\dots,n-1\end{pmatrix}$$

then they can vanish only when the sets of equal dimensions in those quantities vanish identically, so in particular, when the determinant:

$$\left|\frac{\partial^2 a_{\lambda'\mu_0}}{\partial p_\kappa\,\partial p_\nu}\right|$$

of those quantities vanishes identically. However, that will contain $(n-1)^2$ arbitrary functions when the $a_{\rho\sigma}$ are arbitrary functions.

With that, the proof is brought to completion, and investigation has gone as far as it should at this point. One might only remark that the discussion of the cases in which the n systems (S_κ) are *not all* of rank $n - 1$ seems to raise some appreciable difficulties.

IV.

Some applications of the concept of the analytical equivalence of dynamical problems to certain problems in mechanics.

It might be appropriate to explain the investigation in the foregoing section in the simplest case of dynamical problems, for which $n = 2$.

Then let:

$$U' = P_1\,\delta p_1 + P_2\,\delta p_2\,,$$

$$T = \tfrac{1}{2}\left[a_{11}\left(\frac{dp_1}{dt}\right)^2 + 2\,a_{12}\,\frac{dp_1}{dt}\,\frac{dp_2}{dt} + a_{22}\left(\frac{dp_2}{dt}\right)^2\right].$$

It will then follow that every second-order dynamical problem is analytically equivalent to the motion of *one* point on a surface whose line element ds is given by:

$$ds^2 = a_{11}\,dp_1^2 + 2\,a_{12}\,dp_1\,dp_2 + a_{22}\,dp_2^2 \;.$$

From the foregoing, in order for two problems of that type to be analytically equivalent, it is sufficient and also necessary, except for certain singular cases, that first of all the two associated surfaces should be developable to each other, and that secondly, U' must always have the same value for corresponding points of the two surfaces.

That condition is sufficient under all circumstances. One will then have the theorem:

If one bends a surface on which a material point moves, and in that way lets forces act upon it whose tangential components are always the same functions of p_1, p_2 *for each point* p_1, p_2 *of the surfaces, then with the same initial conditions, the moving point will traverse the same path on the bent surface that emerges from the original path by bending the surface, and indeed in such a way that corresponding points* p_1, p_2 *will always belong to the same value of time.*

Euler had already applied that *principle of bending* in his own mechanics ("Mechanica sive motus scientia analytice exposita," Petersburg 1736. T. II, §§ 869, 870), and in that way he reduced the problem of the motion of a massive point in a circular cylinder to the coordinate motion in the plane. Moreover, the same reduction is possible for any cylinder. For **Liouville** [**Liouville** Journal, **12** (1846), pp. 358], the principle of bending is expressed in words as (†): "que des formules d'analyse identiques entre ells serviront pour le movement d'un point sur deux surfaces susceptibles d'être appliquées l'une sur l'autre sans déchichure ni duplicature." **Betrand** remarked [**Liouville**'s Journal **17** (1851), pp. 121] that when the differential equations of motion for a point on a surface possess certain prescribed integrals, they will also lead to the same integral under the bending of the surfaces that arise from it. For the surfaces with curvature zero, **Wittenbauer** [Berichte der Wiener Akademie **71** (1880)] inferred the invariance of the paths under bending by purely-geometric considerations. Finally, in my Inaugural Dissertation ("Ueber die Bewegung eines Punktes auf einer Fläche," Berlin 1885), I expressed that theorem in the form that was given above, except with the unnecessary restriction that U' should be a total differential, and referred to the meaning that it would have for the solution of dynamical problems.

The principle of bending is closely related to the theory of the *geodetic curvature* of curves on a surface, and the invariance of that curvature under bending that **Minding** ["Bemerkung über die Abwicklung krummer Linien von Flächen," this journal, Bd. 6, (1830) pp. 159] first proved can be recognized immediately. A material point of mass 1 moves along the curve in question from the point p_1, p_2 or A and traverses the arc-length $AB = ds$ in the time interval dt. In the following time interval dt, it will then traverse an element of arc BC of the curve that differs from ds by only second-order quantities. Let BC' be the arc-length element of the geodetic continuation of AB that the point would cover in the time interval dt if it were left to itself, and which also deviates from ds by only second-order quantities. Let the angle CBC' be equal to $d\gamma$. When one neglects infinitely-small quantities of order one, the geodetic curvature ρ_g of the curve at the point B will then be equal to $d\gamma : ds$, and as a result, it will also be equal to $CC' : ds^2$. Therefore, ρ_g is nothing but the component of the driving force in the tangent plane to the surface that is perpendicular to

(†) Translator: "some formulas of analysis that are identical to each other can serve to describe the motion of a point on two surfaces that can be mapped to each other without tearing or duplication."

the tangent to the path, divided by the square of the velocity at that location on the path [cf., the article by **Resal** in **Liouville**'s Journal (3) **3** (1877), pp. 82 and the remark that **O. Bonnet** made on pp. 207 there]. If one now bends the surface and lets the point move with the same initial conditions and under the action of forces that are the same functions of p_1, p_2 as before then the curve that arises will be described by bending the given curve. Therefore, ρ_g will also have the old value for that curve, as well.

Conversely, I shall imagine that two surfaces are given. A material point moves on each of them under the influence of given forces. If one exhibits the differential equations of motion and finds precisely the same equations in both cases then one can infer from this that the surfaces will be developable onto each other, unless one of those singular cases occurs that requires a special examination. However, that examination is easy to carry out in the case of $n = 2$. In that way, it will be shown that the appearance of singular cases is in no way based upon a lack of methods of investigation, but rather upon the fact that the essential differences in the nature of quadratic forms demands a separate treatment of the individual cases.

Those singular cases are characterized by the fact that not all systems (S_1), (S_2), …, (S_n) have rank $n - 1$. For $n = 2$, a simple calculation will show that one has:

$$(S_1) \qquad -\frac{a_{12}}{a}k, \qquad \frac{a_{11}}{a}k,$$

$$(S_2) \qquad -\frac{a_{22}}{a}k, \qquad \frac{a_{12}}{a}k,$$

in which k is the curvature of the quadratic differential form:

$$a_{11}\,dp_1^2 + 2\,a_{12}\,dp_1\,dp_2 + a_{22}\,dp_2^2 \ .$$

The rank of those two systems is generally equal to 1. It can be equal to zero only when k vanishes. However, one can then assume from the outset that the coefficients a_{11}, a_{12}, a_{22} are constants. One will then get the most general solution of equations (A) when one likewise sets w_{11}, w_{12}, w_{22} equal to (arbitrary) constants. One will then have the second quadratic differential form:

$$w_{11}\,dp_1^2 + 2\,w_{12}\,dp_1\,dp_2 + w_{22}\,dp_2^2$$

likewise has zero curvature. Now, since all surfaces of zero curvature can be developed to each other, concluding the coincidence of the differential equations of motion from the developability of the surface will remain correct. In the regular cases, that conclusion is based upon only the fact that when the differential equations are identical, the line elements that produce those equations must also be identical (up to the constant of mechanical similarity), while in the singular cases, it will be possible only when the intrinsically-different line elements can go to each other under a transformation of the variables.

To conclude, an example of a system of infinitely-many material points will be given whose motion is analytically equivalent to the motion of a point on a surface, which is an example that I was induced to examine in the Winter of 1887/88 by a communication from my esteemed teacher **Kronecker**, with whom I had a lively conversation at the time.

The motion of a rigid line that is endowed with mass in space can be decomposed into the motion of the geometric lines that the rigid line contains, and their displacement in the latter is then a *fifth*-order problem. However, that order can be lowered when constraints are added. Since constraints that include time shall be excluded from this investigation from the outset, only two possibilities will remain. Either one allows the displacement of the rigid lines in the geometric lines, or one rejects it. If one would like to obtain a *second*-order problem then one must either assume that the rigid line always remains a member of a *system of rays*, and therefore its position in the ray system is determined by giving the ray, or that the rigid line is constrained to remain on a *rectilinear* surface. However, a more precise examination would show that the second case is already contained in the first one and can be regarded as a degenerate case of it. Assuming that the applied forces depend upon only the two determining data p_1, p_2 of the *rigid lines*, the two motions that were just given will be analytically equivalent to the motion of *one point* on certain surfaces. Therefore, above all, the problem will come down to the problem of exhibiting the line element of those surfaces.

If one treats the motion of a rigid line in a system of rays then one must base that on the study of the surfaces that include all possible positions of the center of mass of the line and associate each point p_1, p_2 of that surface with a point p_1, p_2 on the **Gauss**ian unit sphere by means of the ray that goes through it. Let the line element of the center of mass surface be ds, while that of the associated **Gauss**ian sphere is $d\sigma$.

One must now define the *vis viva* T of the rigid line at time t. Let the center of motion coordinates at time t, when referred to a rectangular coordinate system that is fixed in space, be a, b, c, and let the direction cosines of the rigid line at the same time be α, β, γ. If one then denotes the distance from a point x, y, z of the rigid line to its center of mass by λ then:

$$x = a + \lambda\,\alpha\,, \quad y = b + \lambda\,\beta\,, \quad z = c + \lambda\,\gamma\,.$$

The motion of the rigid line during the time interval $(t, t + dt)$ can be summarized as a parallel displacement of it under which α, β, γ remain unchanged and a, b, c change in such a way that:

$$da^2 + db^2 + dc^2 = ds^2\,,$$

and a rotation of the rigid line around the center of mass under which only α, β, γ will change in such a way that:

$$d\alpha^2 + d\beta^2 + d\gamma^2 = d\sigma^2\,.$$

If the mass at the point x, y, z is denoted by $\mu(\lambda)\,d\lambda$ then the mass of the massive line will be equal to:

$$\int \mu(\lambda)\,d\lambda\,,$$

so that integral can be set equal to 1. Since $\lambda = 0$ is the center of mass of the line, one must have:

$$\int \lambda\,\mu(\lambda)\,d\lambda = 0 .$$

Finally, the moment of inertia of the rigid line relative to its center of mass is given by:

$$\int \lambda^2\,\mu(\lambda)\,d\lambda = \kappa^2 .$$

With those preparations, one will have:

$$2\,T = \int \mu(\lambda)\,d\lambda \left\{\left(\frac{da}{dt} + \lambda\frac{d\alpha}{dt}\right)^2 + \left(\frac{db}{dt} + \lambda\frac{d\beta}{dt}\right)^2 + \left(\frac{dc}{dt} + \lambda\frac{d\gamma}{dt}\right)^2\right\}$$
$$= \left(\frac{ds}{dt}\right)^2 + \kappa^2\left(\frac{d\sigma}{dt}\right)^2 ,$$

from which, it will follow that:

The motion of a rigid line in the ray system is analytically equivalent to the motion of a point on a surface whose line element dS is given by:

$$dS^2 = ds^2 + \kappa^2\,d\sigma^2 .$$

If one now lets the ray system degenerate into a rectilinear surface, and at the same time, lets its center of mass surface go to a rectilinear surface then one will arrive at precisely the case of the motion of a rigid line on a rectilinear surface. However, the lines will produce only a *curve* on the **Gauss**ian sphere, and for a cylinder, it will even be a *point*. For the cylinder, one will then have $d\sigma = 0$, and therefore $dS = ds$, i.e.:

The motion of a rigid line on a cylinder is given by the motion of its center of mass on the cylinder when all masses of the rigid line are concentrated there.

In the general case, one can choose the arc-length of that curve on the **Gauss**ian sphere to be the variable p_2 . One will then get dS from the formula:

$$dS^2 = ds^2 + \kappa^2\,dp_2^2 .$$

The variable p_2 has the simple geometric meaning that the lines p_2 = const. are the generating lines of the surface in question. One can then choose the variable p_1 to be the length on such lines, as measured from a fixed curve on the surface. The rectilinear surface will then be represented by:

$$x = \varphi(p_2) + \alpha(p_2) \cdot p_1 , \qquad \alpha^2 + \beta^2 + \gamma^2 = 1,$$
$$y = \psi(p_2) + \beta(p_2) \cdot p_1 , \qquad d\alpha^2 + d\beta^2 + d\gamma^2 = dp_2^2 .$$
$$z = \chi(p_2) + \gamma(p_2) \cdot p_1 ,$$

If one sets:

$$\alpha\, d\varphi + \beta\, d\psi + \gamma\, d\chi = f\, dp_2 ,$$
$$d\varphi^2 + d\psi^2 + d\chi^2 = g\, dp_2^2 ,$$
$$d\alpha\, d\varphi + d\beta\, d\psi + d\gamma\, d\chi = h\, dp_2^2 ,$$

to abbreviate, then those equations will yield the surface:

$$ds^2 = dp_1^2 + 2f\, dp_1\, dp_2 + (g + 2h\, p_1 + p_1^2)\, dp_2^2 ,$$

and one will then have:

$$dS^2 = dp_1^2 + 2f\, dp_1\, dp_2 + (g + \kappa^2 + 2h\, p_1 + p_1^2)\, dp_2^2 .$$

Conversely, if one considers the rectilinear surface [cf., **Minding**, this journal, **18** (1836), pp. 297]:

$$X = \int (\cos p_2 \cdot f - \sin p_2 \cdot h)\, dp_2 + \cos p_2 \cdot p_1 ,$$
$$Y = \int (\sin p_2 \cdot f + \cos p_2 \cdot h)\, dp_2 + \sin p_2 \cdot p_1 ,$$
$$Z = \int \sqrt{g + \kappa^2 - f^2 - h^2}\, dp_2$$

then one will easily see that the square of the line element on that surface is equal to precisely dS^2, i.e.:

The motion of a rigid line on a rectilinear surface is analytically equivalent to the motion of a point on an associated surface that is likewise rectilinear.

However, rather than that second rectilinear surface, one can also choose any surface that arises from it by bending, although it by no means needs to be rectilinear.

If one sets:

$$f = 0 , \qquad g = \text{const.}, \qquad h = 0 ,$$

in particular, then one will have:

$$ds^2 = dp_1^2 + (g + p_1^2)\, dp_2^2 , \qquad dS^2 = dp_1^2 + (g + \kappa^2 + p_1^2)\, dp_2^2 ,$$

and *ds* and *dS* can be regarded as line elements of the helicoid (*hélicoïdes gauches*):

$$x = \cos p_2 \cdot p_1 , \qquad y = \sin p_2 \cdot p_1 , \qquad z = \sqrt{g} \cdot p_2 ,$$

and

$$X = \cos p_2 \cdot p_1 , \qquad Y = \sin p_2 \cdot p_1 , \qquad Z = \sqrt{g+\kappa^2} \cdot p_2 .$$

It is known (cf., **Minding**, *loc. cit.*, pp. 365) that any helicoid can be bent into the *surface of rotation of a catenary*, that takes the generating lines to the meridians. One can therefore also choose the catenoid to be the surface that is associated with the line element *ds* :

$$\mathfrak{X} = \sqrt{g+\kappa^2+p_1^2} \cdot \cos p_2 , \quad \mathfrak{Y} = \sqrt{g+\kappa^2+p_1^2} \cdot \sin p_2 , \quad \mathfrak{Z} = \int \frac{\sqrt{g+\kappa^2} \cdot dp_1}{\sqrt{g+\kappa^2+p_1^2}} .$$

Now, I have solved the **Jacobi** problem of motion of point on a surface of revolution when the force function is constant on the parallel circles in my aforementioned Inaugural Dissertation in full generality (*). However, the motion of a point on a helicoid is also resolved by that principle of bending when one has only that the force function is constant on the orthogonal trajectories to the generating lines, so it will be a function of only p_2 with the notation that is used here. As a result, the motion of a rigid line on a helicoid can be regarded as known when the force function depends upon p_2 alone.

Fernbach investigated the motion of a rigid line that is homogeneously endowed with mass in a *cone* whose vertex attracts the rigid line in his Inaugural Dissertation ("Ueber die Bewegung einer homogenen mit Masse belegten starren Geraden aud einer geradlinigen Fläche," Halle 1887). That motion is a special case of the aforementioned motion on a helicoid and can therefore lead to only a special case of the differential equation that I discussed in general. That explains the fact that **Fernbach** could adopt the fourth section of my own Inaugural Dissertation into his own almost word-for-word.

Even the remaining cases in which one investigates the motion of a rigid line on a rectilinear line can admit a similar treatment.

The motion of a massive rigid line on a *parabolic cylinder*, which **Lüttich** treated with a great expenditure of calculation (Inaugural Dissertation, Jena 1883), leads one back almost immediately to the elementary parabolic ballistic motion.

Tuphorn (Inaugural Dissertation, Halle 1883) examined the motion of a massive rigid line on a vertical *hyperboloid of revolution with one sheet*. If the equation of that surface is:

$$\frac{x^2+y^2}{a^2} - \frac{z^2}{a^2\,(\lambda^2-1)} = 1 \qquad (\lambda > 1)$$

then one can set:

$$x = -\,a \cos \lambda\, p_2 + \frac{1}{\lambda} \sin \lambda\, p_2 \cdot p_1 ,$$
$$y = +\,a \sin \lambda\, p_2 + \frac{1}{\lambda} \cos \lambda\, p_2 \cdot p_1 ,$$

(*) In Section Three, one will find a summary of the abundant literature on that subject.

$$z = \frac{1}{\lambda}\sqrt{\lambda^2 - 1}\cdot p_1 .$$

The square of the line element of that surface will then be:

$$ds^2 = dp_1^2 + 2a\,dp_1\,dp_2 + (a^2\lambda^2 + p_1^2)\,dp_2^2 ,$$

and the square of the line element of the associated surface will be:

$$dS^2 = dp_1^2 + 2a\,dp_1\,dp_2 + (a^2\lambda^2 + \kappa^2 + p_1^2)\,dp_2^2 .$$

One can therefore likewise choose the associated surface to be a hyperboloid of revolution with one sheet, namely, the one on which the λ^2 on the first surface is replaced with:

$$\lambda^2 + \frac{\kappa^2}{a^2} .$$

However, the motion of a massive point on such a vertical hyperboloid of revolution falls within the motions that I treated in my Inaugural Dissertation.

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