"Bemerkungen zum Prinzip des kleinsten Zwanges," Sitz. Heidelberger Akad. Wiss. Abt. A, math.-naturw. Klasse, Carl Winters Universitätsbuchhandlung, Heidelberg, 1919.

# Remarks on the principle of least constraint

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PART ONE

## Mechanical systems with equality constraints

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### § 1. – Generalities on point-systems with holonomic and non-holonomic constraint equations.

One deals with a system of $n$ mass-points. In order to simplify the calculations, the mass of the $\nu^{\text{th}}$ point will be denoted by $m_{3\nu-2} = m_{3\nu-1} = m_{3\nu}$. Let its rectangular Cartesian coordinates at time $t$ be $x_{3\nu-2}$, $x_{3\nu-1}$, $x_{3\nu}$. The *position* of the system at time $t$ will be determined by the $3n$ coordinates $(x_\rho)$. Differentiation with respect to $t$ yields the velocity components $(\dot{x}_\rho)$; the quantities $(x_\rho, \dot{x}_\rho)$ characterize the *state of motion* of the system at time $t$. By repeated differentiation, one will obtain the acceleration components $(\ddot{x}_\rho)$. Finally, let the components of the force that acts upon the $\nu^{\text{th}}$ point be $X_{3\nu-2}$, $X_{3\nu-1}$, $X_{3\nu}$; they are assumed to be single-valued functions of the quantities $(x_\rho, \dot{x}_\rho)$ and time $t$.

The system might be subjected to $k$ mutually independent, consistent, holonomic equations:

$$f_\kappa(x_1, x_2, \ldots, x_{3n}; t) = 0 \qquad (\kappa = 1, 2, \ldots, k), \tag{1}$$

which shall be written more briefly as $f_\kappa(x_\rho; t) = 0$. If will be assumed of the functions $f_\kappa(x_\rho; t)$ (as for all of the functions that occur in what follow) that they admit the appropriate differentiations. In general, the $k$ mutually-independent, consistent equations:

(2) $$\sum_{\rho=1}^{3n} \frac{\partial f_k}{\partial x_\rho} \dot{x}_\rho + \frac{\partial f_k}{\partial t} = 0$$

will then exist between the $3n$ velocity components. In addition, it should be stated that *in general* an equation of higher degree will enter in place of at least one of the linear equations (2) for special systems of values ($x_\rho$ ; $t$), namely, for the systems of values for which all of the $k^{\text{th}}$-order determinants of the matrix $\partial f_\kappa / \partial x_\rho$ vanish.

At the level of velocity, $l$ non-holonomic equations:

(3) $$\sum_{\rho=1}^{3n} \varphi_{\lambda\rho}(x_\rho;t)\, \dot{x}_\rho + \varphi_{\lambda 0}(x_\rho;t) = 0 \qquad (\lambda = 1, 2, \ldots, l)$$

can be combined with equations (2). It will be assumed that equations (2) and (3) collectively make up a system of $m = k + l$ linear equations for the velocity components:

(4) $$\sum_{\rho=1}^{3n} F_{\mu\rho}(x_\rho;t)\, \dot{x}_\rho + F_{\mu 0}(x_\rho;t) = 0 \qquad (\mu = 1, 2, \ldots, m) ,$$

for which at least one $m^{\text{th}}$-order determinant of the matrix $\| F_{\mu\rho} \|$ does not vanish; otherwise, they are *singular*.

With those preparations, the basic problem of analytical mechanics for the point-system in question reads:

*Given a state of the system that satisfies the constraints at any time, find the accelerations that pertain to that time by means of the conditions and forces at that time.*

For a regular position, the equations:

(5) $$\sum_{\rho=1}^{3n} F_{\mu\rho}(x_\rho;t)\, \ddot{x}_\rho + H_\mu(x_\rho;\dot{x}_\rho;t) = 0$$

will follow from equations (4) by differentiation. The expressions $H_\mu(x_\rho;\dot{x}_\rho;t)$ are functions of degree two in the quantities $(\dot{x}_\rho)$. (5) will yield $m$ suitably-chosen acceleration components as linear functions of the remaining $3n - m$ .

With that, everything that can be inferred from the prescribed conditions for the velocities and accelerations has been exhausted. In order to determine the accelerations by means of the constraints and the forces completely, one must add a *principle* of analytical mechanics.

## § 2. – D'Alembert's principle for systems with holonomic and non-holonomic constraint equations.

**D'Alembert**'s principle demands that *for the virtual displacements that satisfy the equations:*

$$\sum_{\rho=1}^{3n} F_{\mu\rho}(x_\rho; t)\,\delta x_\rho = 0, \tag{6}$$

*the work done by the system reactions will vanish:*

$$\sum_{\rho=1}^{3n} (m_\rho \ddot{x}_\rho - X_\rho)\,\delta x_\rho = 0. \tag{7}$$

It is equivalent to the equations:

$$m_\rho \ddot{x}_\rho = X_\rho + \sum_{\rho=1}^{3n} F_{\mu\rho} L_\mu , \tag{8}$$

in which $L_1, L_2, \ldots, L_m$ mean undetermined multipliers. If the expressions for the quantities $(\ddot{x}_\rho)$ in (5) are substituted in (8) then one will get the $m$ linear equations:

$$\sum_{\mu=1}^{m}\left(\sum_{\rho=1}^{3n} \frac{1}{m_\rho} F_{\mu\rho} F_{\nu\rho}\right) L_\mu + J_\nu(x_\rho; \dot{x}_\rho; t) = 0 \qquad (\nu = 1, 2, \ldots, m) \tag{9}$$

for the multipliers, whose determinant is equal to the sum of the squares of the $m^{\text{th}}$-order determinants that belong to the matrix $\left\| \frac{1}{\sqrt{m_\rho}} F_{\mu\rho} \right\|$, from a known theorem ([1]). It follows from this that *for regular positions of the system*, the accelerations will be determined uniquely from the state of motion ([2]).

The fact that the assumption of a regular position is essential is shown by the following example:

Let a point of unit mass be constrained to move on the surface of a cone:

$$x_1^2 + x_2^2 - x_3^2 = 0. \tag{1$'$}$$

That condition will imply the equations:

$$x_1\,\dot{x}_1 + x_2\,\dot{x}_2 - x_3\,\dot{x}_3 = 0, \tag{2$'$}$$

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([1]) C. G. J. JACOBI, "De formatione et proprietatibus determinatium," J. reine angew. Math. **22** (1841), pp. 312; *Werke*, Bd. III, Berlin 1884, pp. 386; Ostwald's *Klassiker der exakten Wissenschaften*, Heft 77, Leipzig 1896, pp. 40.

([2]) C. G. JACOBI, *Vorlesungen über Dynamik*, published 1842/43, 2nd edition, Berlin 1884, Lecture 17, especially pp. 140. The fact that JACOBI assumed holonomic constraints does not obstruct the generality of his process, since it involves only the behavior of the matrix $\| F_{\mu\rho} \|$.

(5′) $$x_1\,\ddot{x}_1 + x_2\,\ddot{x}_2 - x_3\,\ddot{x}_3 + \dot{x}_1^2 + \dot{x}_2^2 - \dot{x}_3^2 = 0,$$

and the virtual displacements will be specified by the equations:

(6′) $$x_1\ \delta x_1 + x_2\ \delta x_2 + x_3\ \delta x_3 = 0.$$

At time $t$, the mass-point is found at the vertex of the cone, and in fact at rest. The original specification of the virtual displacements breaks down at that singular point. However, one will also have to consider displacements that move the mass-point from the given position to a position that is compatible with the constraints, and equations (6′) will then have to be replaced with the equations:

(6″) $$\delta x_1^2 + \delta x_2^2 - \delta x_3^2 = 0.$$

One gets the equation:

(5″) $$\ddot{x}_1^2 + \ddot{x}_2^2 - \ddot{x}_3^2 = 0,$$

and most simply by geometric arguments. By means of **d'Alembert**'s principle, that will lead to the equations:

(8″) $$\ddot{x}_1 = X_1, \quad \ddot{x}_2 = X_2, \quad \ddot{x}_3 = X_3\,.$$

*By themselves, they generally contradict equation* (5″).

One can seek to explain that result by the fact that the vertex of the cone is not able to produce a reaction. However, it will be shown that reactions will appear when one applies the principle of least constraint.

## § 3. – The principle of least constraint for systems with holonomic and non-holonomic constraint equations.

According to **GAUSS**, *for a given state of motion, the constraint:*

(10) $$Z(\ddot{x}_\rho) = \sum_{\rho=1}^{3n} \frac{1}{m_\rho}(m_\rho \ddot{x}_\rho - X_\rho)^2$$

*will be a minimum for all quantities* $(\ddot{x}_\rho)$ *that are compatible with the constraints.*

For regular positions, the admissible quantities $(\ddot{x}_\rho)$ will be specified by the linear equations (5). For singular positions, at least one of those equations will be replaced with an equation of order two or higher. In both cases, the constraint will have at least one minimum. Namely, it is initially a continuous function of the independent variables $(\ddot{x}_\rho)$ and will preserve that property when its variability is restricted by algebraic equations. There will then be at least one system of values $(\ddot{\xi}_\rho)$ that makes the constraint a minimum.

It will now be shown that the constraint possesses *only one* minimum *for regular positions*.

Let $(\ddot{\xi}_\rho)$ be a location of the minimum, so $Z(\ddot{\xi}_\rho + u_\rho)$ will be greater than $Z(\ddot{\xi}_\rho)$ for all sufficiently small, admissible systems of values $(u_\rho)$, and indeed a system of values $(u_\rho)$ will be admissible when equations (5) are fulfilled for the quantities $(\ddot{\xi}_\rho + u_\rho)$, so when the $m$ equations exist:

$$\sum_{\rho=1}^{3n} F_{\mu\rho} u_\rho = 0. \tag{11}$$

It follows from this that for a system of values $(u_\rho)$, the system of values $(g\, u_\rho)$ will be admissible for arbitrary positive or negative $g$. Now, one has:

$$Z(\ddot{\xi}_\rho + u_\rho) = Z(\ddot{\xi}_\rho) + \sum_{\rho=1}^{3n} m_\rho u_\rho^2 + 2\sum_{\rho=1}^{3n} (m_\rho \ddot{\xi}_\rho - X_\rho) u_\rho, \tag{12}$$

and as a result, one must have the expression:

$$\sum_{\rho=1}^{3n} (m_\rho \ddot{\xi}_\rho - X_\rho) u_\rho = 0 \tag{13}$$

for a minimum, initially for sufficiently small systems of values $(u_\rho)$, but then for all admissible ones. If there is a second location for a minimum $(\ddot{\eta}_\rho)$ then one can put $\ddot{\eta}_\rho$ in place of $\ddot{\xi}_\rho + u_\rho$ in equation (12), so $Z(\ddot{\eta}_\rho)$ would be greater than $Z(\ddot{\xi}_\rho)$, with the exclusion of equality. One can show in the same way that $Z(\ddot{\xi}_\rho)$ is greater than $Z(\ddot{\eta}_\rho)$, with the exclusion of equality. As a result, the assumption that there is a second location for a minimum must be rejected.

One has repeatedly stated that the principle of least constraint yields an absolute minimum, and one would like to conclude from this that the constraint possesses *only one* minimum location. However, an absolute minimum can appear at several places at once; for instance, take the function $y = \sin x$. However, in addition, equation (11) will be true only under the assumption of a regular position of the system. How things work at the singular positions will be explained at the conclusion of this paragraph in an example.

Under the assumption of a regular position, there is always one and only one system of accelerations that satisfy the principle of least constraint for a given state of motion. It is easy to see that one will obtain the same accelerations that are given by **d'Alembert**'s principle, because equations (11) will be converted into equations (6) when one sets:

$$\delta x_\rho = u_\rho\, \delta t, \tag{14}$$

and in that way, equations (13) will, at the same time, go to equations (7).

It would seem that the fact that the admissible changes in the accelerations are coupled with the virtual displacements by equations (14) was first pointed out by **GIBBS** ([3]), but generally without expressly stating the essential assumption that the position of the system must be regular.

With that, we have arrived at the lemma that ***d'Alembert****'s principle and the principle of least constraint are equivalent for point-systems that are subject to holonomic and non-holonomic constraint equations, assuming that the position is regular.* Due to equations (14), the demand (7) that the virtual work done by reactions should vanish is identical to the necessary and sufficient condition (13) for the minimum of the constraint. In order to prove that the accelerations are determined uniquely by **d'Alembert**'s principle for a regular position, one can then depart from **JACOBI**'s path and appeal to the principle of least work and then arrive at that goal by simple conceptual arguments without using the theory of determinants.

In conclusion, we shall once more take up the example that was treated in the foregoing paragraphs of the motion of a mass-point on a cone. The principle of least constraint demands that the expression:

$$Z(\ddot{x}_1, \ddot{x}_2, \ddot{x}_3) = (\ddot{x}_1 - X_1)^2 + (\ddot{x}_2 - X_2)^2 + (\ddot{x}_3 - X_3)^2 \tag{10'}$$

must be a minimum, with the constraint (5″). Geometrically, that can be interpreted by saying that it is the shortest distance from the point $X_1$, $X_2$, $X_3$ to the surface of the cone.

One sees immediately that two points of the cone can yield a minimum in some situations. The line segments from the vertex of the cone to the two points will give the directions and magnitudes of the desired accelerations, while the two shortest distances will represent the associated reactions. When there is also a means of preferring one of the accelerations that are found, it must however, break down when the direction of the force lies along the axis of the cone.

In the textbooks on mechanics and physics, one often finds it maintained that the motion of a mechanical system should be established completely by the initial state of motion (i.e., determinism). It is then "self-explanatory" that the accelerations will be determined *uniquely* by the principles of mechanics. Here, however, the objection is raised that, first of all, it might be just as conceivable, in and of itself, that one must also know the initial accelerations. However, secondly, one must know the meaning of the principles. Those are starting points for the calculations, and their implementation belongs to the domain of mathematics. It is up to the physicists to verify the physical admissibility of the results of calculation. However, it would be wrong to reject a principle of mechanics just because the accelerations would not be determined uniquely in some situations. The fault might, in fact, lie in the way that the problem was posed. Thus, the motion in the vicinity of the vertex of the cone that was required in the example will not be realized by mechanics; an inadmissible idealization was made here.

In the present context, the example shows that ***d'Alembert****'s principle and the principle of least constraint do not need to be equivalent for singular positions*, and indeed, it shows that **Gauss**'s principle achieves more than **d'Alembert**'s. It follows from this that *it is not possible to derive the principle of least constraint from* ***d'Alembert****'s principle for singular positions.* Rather, one will have to pose **Gauss**'s principle *axiomatically* for singular positions.

([3]) J. W. GIBBS, "On the fundamental formulae of dynamics," Am. J. Math. **2** (1879), pp. 49.

## § 4. – Geometric interpretation.

It is often useful to regard the systems of quantities that appear in the mechanics of point-systems as the coordinates of a point in a Euclidian space of several extensions. In particular, that is true of the $3n$ acceleration components $(\ddot{x}_\rho)$ .

The geometric interpretation will become more transparent when one performs an affine transformation and introduces the $3n$ new coordinates:

$$\frac{1}{\sqrt{m_\rho}}(m_\rho \ddot{x}_\rho - X_\rho). \tag{15}$$

In that way, one will get the square of the distance from the point $(y_\rho)$ in a $3n$-fold extended Euclidian space $R_{3n}$ to the origin $O$ of the coordinates. The constraint equations (5) now read:

$$\sum_{\rho=1}^{3n} \frac{1}{\sqrt{m_\rho}} F_{\mu\rho} y_\rho + \bar{H}_\mu = 0, \tag{16}$$

and the principle of least constraint says that for a regular position of the system, the shortest distance from the point $O$ to the $(3n - m)$-fold extended Euclidian space $R_{3n-m}$ shall be singled out by the one that is suggested for $R_{3n}$ by equations (16). It is known from the study of multiply-extended Euclidian spaces that the desired shortest distance is the perpendicular that is dropped from $O$ to $R_{3n-m}$ and that there is always one and only one such perpendicular ([4]).

**D'Alembert**'s principle also takes on a simple geometric meaning. The base $F$ of the perpendicular has the coordinates $(\eta_\rho)$ . The point $(\eta_\rho + v_\rho)$ then belongs to the space when the quantities $(v_\rho)$ satisfy the equations:

$$\sum_{\rho=1}^{3n} \frac{1}{\sqrt{m_\rho}} F_{\mu\rho} y_\rho = 0. \tag{17}$$

However, that will be converted into equations (11) when one sets:

$$v_\rho = \sqrt{m_\rho} \cdot u_\rho . \tag{18}$$

The requirement (7) of **d'Alembert**'s principle is then identical to the orthogonality condition:

$$\sum_{\rho=1}^{3n} \eta_\rho v_\rho = 0. \tag{19}$$

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([4]) See, for instance, P. H. SCHOUTE, *Mehrdimensionale Geometrie*, Erster Teil: Lineare Räume, Sammlung Schubert, Band XXV, Leipzig 1902.

In fact, the shortest distance $OF$ from the point to $O$ to the space $R_{3n-m}$ is always perpendicular to all of the directions in $R_{3n}$ that belong to that space. The fact that the two principles are equivalent in the regular case will then become obvious.

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PART TWO

**Mechanical systems with inequality constraints** ([5])

## § 5. – Generalities on point-systems with holonomic and non-holonomic constraint inequalities.

The holonomic and non-holonomic constraint equations shall be combined with $k'$ inequalities:

$$g_{\kappa'}(x_\rho\,;\,t) > 0 \qquad (\kappa' = 1, 2, \ldots, k')\,. \tag{20}$$

In them, it will only be assumed that there are positions of the system that are compatible with all constraints at time $t$ .

When one of the functions $g_{\kappa'}(x_\rho\,;\,t)$ is positive for an admissible system of values $(x_\rho)$ at time $t$, the condition that $g_{\kappa'} > 0$ will be called *passive to the change in position of the system*, because *all* systems of values will then be admissible in a sufficiently-small neighborhood of the system of values $(x_\rho\,;\,t)$ . However, when one of the functions $g_{\kappa'}(x_\rho\,;\,t)$ vanishes at time $t$ and assumes positive, as well as negative, values in the neighborhood of $(x_\rho\,;\,t)$, the velocity components $(\dot{x}_\rho)$ must satisfy the condition:

$$\sum_{\rho=1}^{3n} \frac{\partial g_{\kappa'}}{\partial x_\rho}\,\dot{x}_\rho + \frac{\partial g_{\kappa'}}{\partial t} > 0\,. \tag{21}$$

If the value on the left-hand side is positive for one system of values $(x_\rho\,;\,t)$ then all states of motion in a sufficiently-small neighborhood of the state of motion $(x_\rho; \dot{x}_\rho; t)$ will also be admissible. Such an inequality will then be called *passive to the change in the state of motion.*

$l'$ non-holonomic inequalities:

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([5]) For the history and literature on this topic, cf., the articles in the *Encyklopädie der mathematischen Wissenschaften*, Bd. IV 1, by A. VOSS, "Die Prinzipien der rationellen Mechanik," especially pp. 73 and 85, and by P. STÄCKEL, "Elementare Dynamik der Punktsysteme und starren Körper," especially pp. 460. The statements that were made there will be extended in various directions here.

(22) $$\sum_{\rho=1}^{3n} \psi_{\lambda'\rho}(x_\rho; t)\dot{x}_\rho + \psi_{\lambda'0}(x_\rho; t) \ge 0 \qquad (\lambda' = 1, 2, \ldots, l')$$

can be added to the $k'$ holonomic inequalities (20). They will be active or passive to the changes of the state of motion according to whether equality or inequality exists, respectively.

In total, $s$ equations that the velocity components must satisfy:

(23) $$\sum_{\rho=1}^{3n} G_{\sigma\rho}(x_\rho; t)\,\dot{x}_\rho + G_{\sigma 0}(x_\rho; t) = 0 \qquad (\sigma = 1, 2, \ldots, s)$$

will be obtained for the given state of motion in the manner that was described. The position ($x_\rho$) of the system at time $t$ will be called *regular* when the $m + s$ equations (4) and (23) allow one to represent just as many suitably-chosen velocity components as linear functions of the remaining $3n - m$ .

An inequality for the acceleration components will follow from equations (23):

(24) $$\sum_{\rho=1}^{3n} G_{\sigma\rho}(x_\rho; t)\ddot{x}_\rho + K_\sigma(x_\rho; \dot{x}_\rho; t) > 0.$$

However, whereas the knowledge of the state of motion will make it possible to decide whether the equality sign is or is not valid for one of the conditions (21) and (22), that will not be true of the conditions (24). It will be shown that for regular positions of the system, the acceleration components $(\ddot{\xi}_\rho)$ that actually exist are determined uniquely by the principle of least constraint, and indeed when the $(\ddot{\xi}_\rho)$ are substituted for the $(\ddot{x}_\rho)$, the equality sign will be valid for some of the conditions (24), the *greater than sign* will be valid for the rest of them. Those inequalities can be dropped from the outset; for that reason, they shall be called *passive for the accelerations*.

## § 6. – The D'ALEMBERT-FOURIER principle for systems with holonomic and non-holonomic constraint inequalities.

On the basis of arguments that go back to **Fourier** ([6]), one must replace **d'Alembert**'s principle with the requirement that *the virtual work done by reactions can have no negative values* for systems with inequalities. An example was given in the cited treatise by **Gibbs** for which the **d'Alembert-Fourier** principle was not sufficient to determine the acceleration components completely. The following simpler example will also accomplish the same objective.

A point of unit mass moves in space; let it be subject to the inequality $x_3 \ge 0$ . In order for the condition to be active for the change in position at the time $t$, let it be found in the $x_1\,x_2$-plane. The velocity components must then satisfy the condition that $\dot{x}_3 \ge 0$, and that condition will be active

([6]) J. FOURIER, "Mémoire sur la statique," J. de l'École polyt. **5** (1798), pp. 30; *Œuvres*, t. II, 1890, pp. 488.

for the change in the state of motion when $\dot{x}_3 = 0$. That implies that one must have $\ddot{x}_3 \geq 0$ for the acceleration components.

When $x_3 = 0$ at time $t$, the virtual displacements must fulfill the inequality $\delta x_3 \geq 0$. Furthermore, the **d'Alembert-Fourier** principle demands that one must have:

$$(7') \qquad (\ddot{x}_1 - X_1)\,\delta x_1 + (\ddot{x}_2 - X_2)\,\delta x_2 + (\ddot{x}_3 - X_3)\,\delta x_3 > 0.$$

Due to the arbitrariness in $\delta x_1$ and $\delta x_2$, one must have $\ddot{x}_1 = X_1$, $\ddot{x}_2 = X_2$, such that (7) will go to the condition that $X_3$.

The result is that $\ddot{x}_3$ cannot be less than the greater of the two values 0 and $X_3$. *Therefore, $\ddot{x}_3$ will not be determined completely by the **d'Alembert-Fourier** principle.*

Just as **Gibbs** remarked for his own example, it is also possible to ascertain the value of $\ddot{x}_3$ here by means of simple arguments on the progress of the motion. Namely, if the component $X_3$ is negative then it will be annihilated by the reaction of the boundary surface $x_3 = 0$, and one will have $\ddot{x}_3 = 0$. However, if $X_3$ were zero or positive then the mass-point would move as if it were free, and one would have $\ddot{x}_3 = X_3$. That is correct. Merely performing the calculations from the Ansatz that the **d'Alembert-Fourier** principle prescribes will yield only the previously-posed inequality condition for $\ddot{x}_3$, and the example then shows that *this principle will not lead to the determination of the accelerations in general.*

## § 7. – The principle of least constraint for systems with holonomic and non-holonomic constraint inequalities.

It seems that **Jacobi** was the first to examine the application of the principle of least constraint to systems with inequality constraints in detail, in his lectures on dynamics during the Winter semester in 1848/49 ([7]). He was followed by RITTER (1853) in a dissertation that was supervised by **Gauss** ([8]), **Gibbs**, in the cited 1879 treatise, and **Boltzmann** ([9]).

Without knowing of the aforementioned publications, **Mayer** ([10]), prompted by some older work by **Ostragradsky** (1834 and 1836), addressed inequality constraints, and on the basis of some remarks by **Study** in the year 1899, he showed how one could determine the accelerations by means of the principle of least constraint; a regular position was tacitly assume there. It still remained doubtful whether several systems of acceleration might be obtained in some situations. Generally, **Mayer** believed that "For that reason, one can probably regard it as obvious that two different systems of accelerations of the given character cannot exist, because if they did exist then

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([7]) According to A. VOSS, *loc. cit.*, pp. 87. A copy of detailed calculations by SCHEIBNER that were cited in it can be found in the Bibliothek der Berliner Akademie der Wissenschaften.

([8]) A. RITTER, "Über das Princip des kleinsten Zwanges," Dissertation, Göttingen, 1853.

([9]) L. BOLTZMANN, *Vorlesungen über die Principe der Mechanik*, Part I, Section VI, Leipzig, 1897.

([10]) A. MAYER, "Über die Aufstellung der Differentialglichungen der Bewegung für reibungslose Punktsystemem die Bedingungsgleichungen unterworfen sind," Leipziger Berichte, math-phys. Klasse **51** (1899), pp. 224.

there would absolutely no means of finding out which of the two systems were the correct one." However, as was proved in § **3**, it is very probable that two systems of accelerations might make the constraint a minimum at singular positions. The fact that uniqueness prevails under **Mayers**'s assumption then requires proof.

While tacitly assuming a regular position, **Jacobi** already remarked in the cited lecture that "the nature of the minimum at hand excludes several minima," and **Boltzmann** asserted that "the constraint must be an absolute minimum for the actual motion, and be capable of having several minima" (*loc. cit.*, pp. 240). Soon after, **Zermelo** ([11]) proved that the constraint possessed *only one* minimum in full rigor for regular positions of the system, but generally under certain restricting assumptions, and in that way demonstrated the uniqueness of the accelerations.

The process that was used in § **3** can be adapted to the case in which any holonomic or non-holonomic inequalities are added to the constraint equations, and that would allow one to derive the theorem on the uniqueness of the accelerations in its most general form.

In the $R_{3n}$ of the components $(\ddot{x}_\rho)$, the constraint is a continuous function of position for the part of space that contains all of the points $(\ddot{x}_\rho)$ that are compatible with the constraints, and it will therefore attain a smallest value for at least one location; one does not need to assume that this position is regular in order to reach that conclusion.

Let the point $(\ddot{\xi}_\rho)$ be one location of the minimum, such that $Z(\ddot{\xi}_\rho + u_\rho)$ will be greater than $Z(\ddot{\xi}_\rho)$ for all sufficiently-small changes $(u_\rho)$ in its coordinate that are compatible with the constraints. As a result of equation (12) the necessary and sufficient condition for that is that:

$$\sum_{\rho=1}^{3n} (m_\rho \ddot{\xi}_\rho - X_\rho) u_\rho > 0. \tag{25}$$

*For a regular position*, it can, in turn, be shown that the validity of the condition (25) for all sufficiently-small admissible systems of values $(u_\rho)$ will imply its validity for all systems of values $(u_\rho)$ that that exist in general.

In order for a system of values $(u_\rho)$ to be admissible, it must first satisfy equations (11). Secondly, the conditions (24), which were fulfilled for $\ddot{x}_\rho = \ddot{\xi}_\rho$, must remain fulfilled when $\ddot{x}_\rho$ is replaced with the value $\ddot{\xi}_\rho + u_\rho$.

Now let $\vartheta$ be a quantity that lies between 0 and 1. The system of values $(\vartheta U_\rho)$ will always be admissible when the system of values $(U_\rho)$ is. The fact that equations (11) are valid for that system of values is indeed obvious. However, if the two inequalities are true:

$$\sum_{\rho=1}^{3n} G_{\sigma\rho}(x_\rho; t)\, \ddot{\xi}_\rho + K_\sigma(x_\rho; \dot{x}_\rho; t) \geq 0 \tag{26}$$

and

([11]) E. ZERMELO, "Über die Bewegung eines Punktsystems bei Bedingungsungleichungen," Göttinger Nachrichten, math-phys. Klasse, (1899), pp. 306; that note was presented on 3 February 1900.

(27) $$\sum_{\rho=1}^{3n} G_{\sigma\rho}(x_\rho;t)\,\ddot{\xi}_\rho + \sum_{\rho=1}^{3n} G_{\sigma\rho}(x_\rho;t)U_\rho + K_\sigma(x_\rho;\dot{x}_\rho;t) \geq 0$$

for one value of the index $\sigma$ then the inequality:

(28) $$\sum_{\rho=1}^{3n} G_{\sigma\rho}(x_\rho;t)\,\ddot{\xi}_\rho + \vartheta\sum_{\rho=1}^{3n} G_{\sigma\rho}(x_\rho;t)U_\rho + K_\sigma(x_\rho;\dot{x}_\rho;t) \geq 0$$

will also be true, as one easily convinces oneself. The result can be interpreted geometrically by saying that for a regular position of the system, the part of space that contains all of the points (…) that are compatible with the constraints is simply connected and everywhere convex.

Moreover, if one chooses the quantity $J$ to be sufficiently small that the requirement (25) is fulfilled for $u_\rho = \vartheta\, U_\rho$ then it will also be fulfilled for $u_\rho = U_\rho$ . One might infer the conclusion of the proof word-for-word from § **3**.

### § 8. – Determining the accelerations by means of the principle of least constraint.

The determination of the accelerations from the principle of least constraint will be eased when one likewise employs the geometric interpretation that was developed in § **4**. Under the assumption that the position of the system is regular, equations (5) between the coordinates ($y_\rho$) will then correspond to $m$ linear equations

(29) $$\sum_{\rho=1}^{3n} A_{\mu\rho}\,y_\rho + A_{\mu0} = 0$$

that are mutually independent and mutually consistent, and correspond to the constraints (24), and there will be $s$ inequalities:

(30) $$\sum_{\rho=1}^{3n} B_{\mu\rho}\,y_\rho + B_{\mu0}$$

that are compatible with each other and with (29). A Euclidian $R_{3n-m}$ will be specified in $R_{3n}$ by equations (29), and the inequalities (30) will have the effect that only the points of a certain $N$-fold-extended, simply-connected, everywhere-convex region of space $S_N$ in it that is bounded by Euclidian spaces with $N-1$, $N-2$, …, 2, 1 extensions will come under consideration; therefore $N \leq 3n-m$ . The principle of least constraint will then follow from the fact that the following problem has then be solved:

*A region of space in a Euclidian space is specified by linear equations and inequalities. Find its shortest distance from a given point in space.*

The fact that there is only one such shortest distance was proved in the foregoing paragraphs. Two cases are to be distinguished in its solution: First of all, the given point $O$ belongs to the region of space $S_N$ , including the boundary. The shortest distance to the point $O$ itself will then be obtained. Secondly $O$ can lie outside the region of space $S_N$ . If the perpendicular $OF$ were then dropped from $O$ to the $N$-fold-extended Euclidian space $R_N$ that includes $S_N$ then $OF$ would be the minimum of the distances from all points of $R_N$ to $O$ . Thus, if the point $F$ belongs to the spatial region $S_N$ then $OF$ will be the desired shortest distance. It is easy to see that in this case the point $F$ lies on the boundary of the spatial region $S_N$ with the space $R_{3n}$ . Finally, if the point $F$ does not belong to the spatial region $S_N$ then the minimum of the distance will reached at a point $A$ that is different from $F$, and which necessarily lies on the boundary of the spatial region $S_N$ with $R_N$ . Namely, since the perpendicular $OF$ is perpendicular to all of the directions of $R_{3n}$ that lie in $R_N$ , one will have the equation:

$$OA^2 = OF^2 + AF^2, \tag{31}$$

and as a result, $AF$ will be the minimum of the distance from the point $F$ to the points of the spatial region $S_N$ .

The original problem with the auxiliary conditions (29) is then reduced to the same problem without the auxiliary conditions by the argument that was just presented. **Gauss** addressed precisely the problem of *finding the minimum of the expression:*

$$z_1^2 + z_2^2 + \cdots + z_N^2 \tag{32}$$

*when the linear inequalities:*

$$C_{\sigma 1}\, z_1 + C_{\sigma 1}\, z_1 + \ldots + C_{\sigma N}\, z_N \geq 0 \tag{33}$$

*are prescribed for the quantities* $z_1, z_2, \ldots, z_N$ in a lecture that he gave during the Winter semester of 1850/51 on the method of least squares ([12]). I have thoroughly presented the process that he suggested for finding the location of the minimum, and at the same time, I have proposed another way of treating it that is likewise based upon geometric considerations ([13]). Finally, one can also employ the method of multipliers to solve it. One might refer to the 1917 article for the details ([14]).

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([12]) C. F. GAUSS, *Werke*, Bd. X 1, Göttingen 1917, pp. 473; reprint of part of RITTER's calculations.

([13]) P. STÄCKEL, "Eine von GAUSS gestellte Aufgabe des Minimums," these Sitzungsberichte (1917), 11th treatise. The note by Zermelo, which seemed to consider an entirely different subject, was not available to me at the time. Once I later recognized the connection, I did not neglect to point out that part of what had been done in the 1917 article had already been done by ZERMELO.

([14]) My colleague PERRON was kind enough to write to me that there are two places in § **7** that can be improved. When one of the possible points for the minimum of the function $f(x_1, x_2, \ldots, x_n)$ that is to be found lies in a spatial region $S_n$ inside of an $(n - 1)$-extended boundary manifold, it does not need to yield a minimum. Rather, the same test that is required for less than $n - 1$ extensions can be carried out for it. For example, consider the case in which the shortest distance from a plane ($n = 2$) to a point on the surface of a square is to be found. That is connected with the fact that the sign convention for the multipliers is not given expressly. Moreover, one finds that OSTRAGRADSKY and MAYER already went into that in detail; cf., also L. HENNEBERG, "Über den Fall der Statik, in dem das virtuelle Moment einen negative Wert besitzt," J. f. reine u. angew. Math. **113** (1894), pp. 179.

## § 9. – Connection between GAUSS's minimum problem and the principle of least constraint.

Judging from R**itter**'s report, **Gauss** did not say in the lecture during the Winter semester of 1850/51 what had led him to pose his problem of the minimum with inequality conditions, and the question then arose of whether he knew of its connection with the principle of least constraint. Now, it certainly remains puzzling how he would otherwise arrive at the problem. However, one can also arrive at an affirmative answer by other considerations.

Statements that were made in letters and articles show that **Gauss** repeatedly dealt with multiply-extended manifolds. Here, it will suffice to mention a statement that he made to **Sartorius v. Walterhausen** at roughly the time of that lecture: "We can (he said) perhaps empathize with beings that are aware of only two dimensions. A being that is above us would perhaps look down on us similarly, and he would (he continued jokingly) have to overlook certain problems that he thought had been treated geometrically in a higher state." ([15])

In his 1829 note "Über ein neues allgemeines Grundgesetz der Mechanik," **Gauss** explained that the restriction to condition equations was "unnecessary and not always reasonable in nature" and demanded that one should likewise express the law of virtual velocities in such a way that it would subsume all cases from the beginning. At the conclusion, he said that the analogy with the method of least squares could not be pursued any further, which does not, however, seem to be his present opinion on that subject ([16]). He also referred to the importance of the condition inequalities in the 28 September 1829 treatise "Principia generalia theoriae figurae fluidorum in statu aequilibrii" ([17]), and he returned to that topic in a letter to **Möbius** on 29 September 1837 ([18]).

In **Ritter**'s 1853 dissertation, which goes back to **Gauss**, the principle of least constraint was applied to systems with holonomic inequality constraints, and indeed **Ritter** appealed to the language of multidimensional geometry in it. He dealt in depth with the general problem of finding the minimum of a function of position in a multiply-extended Euclidian space for a spatial region in that is defined by inequalities. **Ritter** remained at that level of generality. He did not follow through on the Ansatz of **Gauss**'s principle with computations, nor did he present the linear inequalities ([19]). One can be certain that **Gauss** took that latter step.

## § 10. – Virtual displacements and admissible variations of the accelerations.

The fact that **d'Alembert**'s principle and the principle of least constraint can replace each other for equality constraints, assuming a regular position of the system, is based upon the equations:

$$\delta x_\rho = u_\rho \, \delta t \,, \tag{14}$$

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([15]) Cf., my essay: "GAUSS *als Geometer*," Materielen für eine wissenschaftliche Biographie von GAUSS, Heft V, Leipzig, 1918, pp. 136.

([16]) C. F. GAUSS, *Werke*, Bd. V, pp. 25.

([17]) C. F. GAUSS, *Werke*, Bd. V, pp. 35.

([18]) First published by C. NEUMANN, "Über das Princip der vituellen oder fakultativen Verrückungen," Leipziger Berichte, math.-phys. Klasse **31** (1879), pp. 61; reprinted in C. F. GAUSS, *Werke*, Bd. XI 1, pp. 17.

([19]) The relevant section of RITTER's dissertation is printed in C. F. GAUSS, *Werke*, Bd. XI 1, pp. 469.

which exhibit the invertible, one-to-one correspondence between the virtual displacements and the admissible changes to the acceleration components. By contrast, the fact that **d'Alembert**'s principle and **Gauss**'s principle are not equivalent for inequality constraints, even when the position is regular, was shown in the example that was treated in § **6**.

One asks, "What relationships exist between the quantities $(\delta x_\rho)$ and the quantities $(u_\rho)$, and first of all in the example?" The only case that comes under consideration is the one in which the condition $x_3 \geq 0$ is active for the change in the state of motion, so $x_3$ and $\dot{x}_3$ vanish at time $t$. The virtual displacements $\delta x_1$ and $\delta x_2$ can then be chosen arbitrarily, and one must have $\delta x_3 \geq 0$. The condition that $\ddot{x}_3 \geq 0$ is valid for the acceleration components. Therefore, $u_1$ and $u_2$ are small quantities that can be chosen arbitrarily, and one can set $\delta x_1 = u_1\ \delta t$, $\delta x_2 = u_2\ \delta t$. For $u_3$, there are two possibilities to be distinguished: If the condition for $\ddot{x}_3$ is active then $\ddot{\xi}_3 = 0$, so one must have $u_3 \geq 0$, and one can set $\delta x_3 = u_3\ \delta t$. However, if that condition is passive then $\ddot{\xi}_3 > 0$, and $u_3$ is a small quantity that can be chosen arbitrarily. Therefore, the condition of the minimum requires that one must now have $\ddot{\xi}_3 = X_3$. The equation $\delta x_3 = u_3\ \delta t$ loses its validity for negative values of $u_3$, so *the domain of the admissible changes* $(u_\rho)$ *is more extensive than the domain of the virtual displacements* $(\delta x_\rho)$. The fact that the condition for the minimum is fulfilled for those changes in acceleration that are produced by means of the equation $\delta x_\rho = u_\rho\ \delta t$ is, in fact, necessary, but not sufficient, because the constraint must be a minimum when it is regarded as a function of acceleration. However, the demand that the virtual work done by reactions cannot be negative $(\ddot{\xi}_3 - X_3)\ \delta x_3 \geq 0$ says less than the demand that $(\ddot{\xi}_3 - X_3)\ u_3 \geq 0$, which is necessary and sufficient for a minimum of the constraint. That explains the fact that the first demand on $\ddot{\xi}_3$ yields only an inequality, while the second one implies the unique determination of the acceleration.

One further sees that **Gauss**'s principle cannot be a consequence of the **d'Alembert-Fourier** principle, because if it could be derived from the latter principle then the acceleration could be determined uniquely from the **d'Alembert-Fourier** principle. However, that principle probably follows from **Gauss**'s, namely, when $u_3$ is subject to the restriction that it is not negative.

What is true for the example proves to be correct in general. When one restricts oneself to sufficiently-small systems of values $(u_\rho)$, as is allowed by the proof of the minimum, the quantities $(u_\rho)$ must initially satisfy equations (11). As the conditions (27) show, for those values $\sigma'$ of the index $\sigma$ for which the equality sign is valid in the constraints (26), that must be combined with the inequalities:

$$\sum_{\rho=1}^{3n} G_{\sigma'\rho}\, u_\rho \geq 0. \tag{34}$$

By contrast, when the greater than sign is valid in the conditions (26), they will be passive to the changes in the accelerations.

One initially has equations (6), which correspond to equations (11), for the virtual displacements. However, when the inequalities (20) are active for the change in positions, they will be combined with the conditions:

$$(35) \qquad \sum_{\rho=1}^{3n} G_{\sigma\rho}\,\delta x_\rho > 0,$$

*and indeed for all values* $\sigma = 1, 2, 3, \ldots, s$ .

A glimpse at the formulas shows that any system of virtual displacements ($\delta x_\rho$) will produce a system of admissible changes ($u_\rho$) in the acceleration components by means of equations (14). In general, however, the converse is not true, since the displacements ($\delta x_\rho$) must satisfy all $s$ inequalities (35), while the conditions (34) on the changes ($u_\rho$) are fulfilled for only some of the values of $\sigma$, in general. Therefore, in general, the domain of the admissible changes ($u_\rho$) will be more extensive than the domain of the virtual displacements ($\delta x_\rho$), and the demand of the **d'Alembert-Fourier** principle that:

$$(36) \qquad \sum_{\rho=1}^{3n} (m_\rho \ddot{\xi}_\rho - X_\rho)\,\delta x_\rho > 0$$

will say less than the demand of **Gauss**'s principle that:

$$(24) \qquad \sum_{\rho=1}^{3n} (m_\rho \ddot{\xi}_\rho - X_\rho)\,u_\rho > 0.$$

**Boltzmann**'s geometric proof of the principle of least constraint ([20]) might lead to the suspicion that the principle is also a consequence of the **d'Alembert-Fourier** principle for inequality constraints, as well. The fact that this is not the case was shown already by **Gibbs**'s example, which **Boltzmann** communicated in detail. It is generally true that a system of admissible changes ($u_\rho$) in the acceleration components $\ddot{x}_\rho$ will emerge from any system of virtual displacements ($\delta\, x_\rho$), and it cannot be denied that $Z(\ddot{\xi}_3 + u_\rho)$ is greater than $Z(\ddot{\xi}_3)$ for those changes. However, that only means that a necessary condition for the minimum has been fulfilled, but one lacks a proof that the value of the constraint proves to be greater than it is for $(\ddot{\xi}_3)$ for *all* admissible, sufficiently-small changes in the acceleration components.

**Gauss** underestimated the profundity of his "new fundamental law" when he explained that it was already included in the combination of **d'Alembert**'s principle with the extended principle of virtual displacements as far as the matter was concerned. The fact that the constraint on a mechanical system that is subjected to arbitrary equality and inequality constraints is a minimum for the actual accelerations cannot be proved. Rather, it is an *axiom* that first becomes accessible to mathematical investigation in the case of inequality constraints. *Such a concept only enhances the significance of* ***Gauss****'s principle. In that way, it attains the status of a fundamental law for analytical mechanics.*

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([20]) L. BOLTZMANN, *loc. cit.*, pp. 216-220.