THE BASIC FORMULAS

FOR

THE GENERAL THEORY OF SURFACES

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FOREWORD

Since it was founded by **Monge** and **Gauss**, the general theory surfaces has piqued the interest of mathematicians to an appreciable degree. That interest led up to a series of survey presentations ([1]) that were mostly connected with the purely geometric treatment of **Gauss**, while the more recent especially rich and distinctive work of **Darboux** is essentially built upon kinematical foundations. Although it is not entirely easy to survey the wealth of formulas, theorems, and problems in the theory of surfaces, that fact does seem to point to the fact that the requirement of a general, as well as simple and unified, treatment of the matter has still not been sufficiently addressed. Thus, that points directly to the goal of the present brief discussion of the fundamental formulas and their most important applications. This little book might be suitable as a foundation for the study of more thorough works (such as the stimulating book by **Bianchi** and the works of **Darboux**) in which the analytical, as well as geometric, questions are treated in detail or of lectures in which the analytical development is extended and enlivened by developing the models ([2]) for special classes of surfaces.

The basic impetus to publish this book came from my lectures on the theory of surfaces over four years. The actual writing of it came about in collaboration with **Kommerell**, in whose dissertation (Tübingen, 1890) individual sections of it were treated already in the same spirit. For the sake of gaining a better overview, the subject is divided into three parts. The first section (§ **1-6**) gives the formulas that are needed for the study of a given surface, the second one (§ **7-12**) gives the formulas for the derivation of a surface with given properties, and the third one (§ **13-18**) gives the formulas for the study of curves on surfaces. The applications can be expanded quite easily. However, for the sake of a simple overview, a restriction to the most important groups of general problems also seems preferable here. At the same time, its connection with the original treatises is provided by numerous references, except that at a few places, it was not possible to refer to the literature. **Darboux**'s works deserve special attention.

Tübingen, August, 1892

H. Stahl

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([1]) Cf., pp. *iv*.

([2]) *Mathematische Modelle* in the publication of **L. Brill** in Darmstadt.

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ORIGINAL WORKS AND SURVEY PRESENTATIONS

Monge, *Application de l'analyse à la géometrie*, 1st ed. 1795, and the title: *Feuilles d'analyse appliquées à la géometrie*, 5th ed., 1850, with notes by J. Liouville.

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CHAPTER I

STUDY OF A GIVEN SURFACE

§ 1. – Gauss's equations. First-order fundamental quantities.

We shall first present **Gauss**'s most important formulas ([1]) that will facilitate the study of a surface, with some deviation in notation.

Let *the coordinates x, y, z of a point on the surface* be given as *functions of two variable parameters* (u, v) of the form:

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v). \tag{1}$$

To abbreviate, from time to time, we shall set:

$$\frac{\partial x}{\partial u} = x_1, \qquad \frac{\partial x}{\partial u} = x_2, \qquad \frac{\partial^2 x}{\partial u^2} = x_{11}, \qquad \frac{\partial^2 x}{\partial u\,\partial v} = x_{12}, \quad \frac{\partial^2 x}{\partial v^2} = x_{22}. \tag{2}$$

The expression for the *line element ds* is:

$$ds^2 = dx^2 + dy^2 + dz^2 = e\,du^2 + 2f\,du\,dv + g\,dv^2, \tag{3}$$

where

$$\begin{aligned} e &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 &&= \sum\left(\frac{\partial x}{\partial u}\right)^2, \\ f &= \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} &&= \sum\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}, \\ g &= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 &&= \sum\left(\frac{\partial x}{\partial v}\right)^2, \end{aligned} \tag{4}$$

in which only the term in the sum that refers to x is written out, to abbreviate, since all terms are symmetric in x, y, z. The three quantities e, f, g, which include only first derivatives of x, y, z, are called *first-order fundamental quantities.* We set:

$$\delta^2 = eg - f^2, \tag{5}$$

which is an expression that is always assumed to be non-zero. Furthermore, let:

([1]) **Gauss**, *Disq. gen.*, art. 3, *et seq.*

$$
(6)\qquad
\begin{aligned}
m &= \sum \frac{\partial x}{\partial u}\frac{\partial^2 x}{\partial u^2} = \frac{1}{2}\frac{\partial e}{\partial u}, &\qquad n &= \sum \frac{\partial x}{\partial v}\frac{\partial^2 x}{\partial u^2} = -\frac{1}{2}\frac{\partial e}{\partial v} + \frac{\partial f}{\partial u},\\
m' &= \sum \frac{\partial x}{\partial u}\frac{\partial^2 x}{\partial u\,\partial v} = \frac{1}{2}\frac{\partial e}{\partial v}, & n' &= \sum \frac{\partial x}{\partial v}\frac{\partial^2 x}{\partial u\,\partial v} = -\frac{1}{2}\frac{\partial g}{\partial u},\\
m'' &= \sum \frac{\partial x}{\partial u}\frac{\partial^2 x}{\partial v^2} = -\frac{1}{2}\frac{\partial g}{\partial u} + \frac{\partial f}{\partial v}, & n'' &= \sum \frac{\partial x}{\partial v}\frac{\partial^2 x}{\partial v^2} = \frac{1}{2}\frac{\partial g}{\partial v}.
\end{aligned}
$$

Thus, the derivatives of e, f, g with respect to u, v can be expressed in terms of the six quantities m, m', m'', n, n', n''.

Finally, let:

$$
(7)\qquad
\begin{aligned}
m\,g - n\,f &= p\,\delta^2, &\qquad n\,e - m\,f &= q\,\delta^2,\\
m'\,g - n'\,f &= p'\,\delta^2, & n'\,e - m'\,f &= q'\,\delta^2,\\
m''\,g - n''\,f &= p''\,\delta^2, & n''\,e - m''\,f &= q''\,\delta^2,
\end{aligned}
$$

from which:

$$
(7a)\qquad p + q' = \frac{1}{\delta}\frac{\partial \delta}{\partial u}, \qquad p'' + q' = \frac{1}{\delta}\frac{\partial \delta}{\partial v}.
$$

The *angle* (C, C_1) *between the two curves* C *and* C_1 at the point (u, v), whose directions are given by the quotients $(du : dv)$ and $(d_1u : d_1v)$, is determined by:

$$
(8)\qquad
\begin{aligned}
\cos(C, C_1) &= \frac{e\,du\,d_1u + f\,(du\,d_1v + dv\,d_1u) + g\,dv\,d_1v}{ds\cdot d_1s},\\
\sin(C, C_1) &= \delta\cdot\frac{du\,d_1v - dv\,d_1u}{ds\cdot d_1s}.
\end{aligned}
$$

Hence, the angle ω that the two parameter curves v = const. and u = const. define with each other will be determined from:

$$
(9)\qquad \cos\omega = \frac{f}{\sqrt{e\,g}}, \qquad \sin\omega = \frac{\delta}{\sqrt{e\,g}}, \qquad \tan\omega = \frac{\delta}{f},
$$

and the equation:

$$
(10)\qquad f = 0
$$

is the condition for the *parameter curves to intersect at right angles.*

The *surface element do* has the value:

$$
(11)\qquad do = \sin\omega\,\sqrt{e\,g}\;du\,dv = \delta\,du\,dv\,.
$$

It follows from the last of equations (9) that:

$$d\omega = \frac{f\,d\delta - \delta\,df}{eg} = \frac{f\,(e\,dg + g\,de) - 2\,e\,g\,df}{2\,e\,g\,\delta}, \tag{12}$$

or

$$-\,d\omega = \left(\frac{\delta q}{e} + \frac{\delta p'}{g}\right) du + \left(\frac{\delta p''}{g} + \frac{\delta q'}{e}\right) dv, \tag{13}$$

and from that:

$$-\,\frac{\partial^2 \omega}{\partial u\,\partial v} = \frac{\partial}{\partial u}\left(\frac{\delta p''}{g} + \frac{\delta q'}{e}\right) = \frac{\partial}{\partial v}\left(\frac{\delta q}{e} + \frac{\delta p'}{g}\right). \tag{14}$$

We further denote the cosines of the inclination angles between the coordinate axes at the point (u, v) and the *surface normal* by (a, b, c) ([1]), while the ones defined by the *tangents* to the curves v = const. and u = const. will be denoted by (α_1, β_1, γ_1) and (α_2, β_2, γ_2), respectively.

Six relations exist between these nine cosines. In order to obtain them, one imagines two coordinate systems that emanate from the origin: the rectangular one (x, y, z) and a second one (x', y', z'), in which the z'-axis is parallel to the surface normal, while the x' and y' axes are parallel to the tangents to the curves v = const. and u = const. If one denotes the coordinates of an arbitrary point in those two systems by (x, y, z) and (x', y', z') then one will have the transformation formulas:

$$\begin{aligned} x &= \alpha_1\,x' + \alpha_2\,y' + \alpha\,z', \\ y &= \beta_1\,x' + \beta_2\,y' + \beta\,z', \\ z &= \gamma_1\,x' + \gamma_2\,y' + \gamma\,z'. \end{aligned} \tag{15}$$

The identity:

$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2 + 2\,x'y'\cos\omega \tag{16}$$

will yield:

$$\begin{aligned} x' + y'\cos\omega &= \alpha_1\,x + \beta_1\,y + \gamma_1\,z, \\ y' + x'\cos\omega &= \alpha_2\,x + \beta_2\,y + \gamma_2\,z, \\ z' &= a\,x + b\,y + c\,z \end{aligned} \tag{17}$$

as the solution to equations (15), and that will yield the desired relations in either the form:

$$a^2 + b^2 + c^2 = 1, \qquad \alpha_1\,\alpha_2 + \beta_1\,\beta_2 + \gamma_1\,\gamma_2 = \cos\omega,$$

([1]) Later on, (§ **11**), these quantities will also be denoted by (X, Y, Z).

$$(18)\qquad \alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1, \qquad a\,\alpha_1 + b\,\beta_1 + c\,\gamma_1 = 0,$$

$$\alpha_2^2 + \beta_2^2 + \gamma_2^2 = 1, \qquad a\,\alpha_2 + b\,\beta_2 + c\,\gamma_2 = 0,$$

or in the form:

$$(19)\qquad \begin{aligned} a^2 \sin^2 \omega + 2\,\alpha_1\,\alpha_2 \cos \omega &= \sin^2 \omega, \\ ab \sin^2 \omega + \alpha_1\,\beta_1 + \alpha_2\,\beta_2 - (\alpha_1\,\beta_2 + \beta_1\,\alpha_2) \cos \omega &= 0, \end{aligned}$$

along with the equations that follow from these by cyclic permutation.

From (18), one has:

$$(20)\qquad \begin{vmatrix} a & \alpha_1 & \alpha_2 \\ b & \beta_1 & \beta_2 \\ c & \gamma_1 & \gamma_2 \end{vmatrix} = |\, a\ \alpha_1\ \alpha_2 \,| = \pm \sin \omega,$$

when one writes only the row that contains a, α_1, α_2 in a determinant that is constructed symmetrically from a, b, c ; α_1, β_1, γ_1 , and α_2, β_2, γ_2 , to abbreviate. Under the assumption that one chooses the positive sign in (20), one will have:

$$(21)\qquad \begin{aligned} \beta_1\,\gamma_2 - \gamma_1\,\beta_2 &= a \sin \omega, \\ \sin \omega\,(b\,\gamma_1 - c\,\beta_1) &= \alpha_2 - \alpha_1 \cos \omega, \\ \sin \omega\,(c\,\beta_2 - b\,\gamma_2) &= \alpha_1 - \alpha_2 \sin \omega, \end{aligned}$$

along with the equations that follow from these by cyclic permutation.

The cosines (α_1, β_1, γ_1) and (α_2, β_2, γ_2) are expressed as functions of (u, v) by the equations:

$$(22)\qquad \begin{aligned} \alpha_1 &= \frac{1}{\sqrt{e}}\frac{\partial x}{\partial u}, & \beta_1 &= \frac{1}{\sqrt{e}}\frac{\partial y}{\partial u}, & \gamma_1 &= \frac{1}{\sqrt{e}}\frac{\partial z}{\partial u}, \\ \alpha_2 &= \frac{1}{\sqrt{g}}\frac{\partial x}{\partial v}, & \beta_2 &= \frac{1}{\sqrt{g}}\frac{\partial y}{\partial v}, & \gamma_2 &= \frac{1}{\sqrt{g}}\frac{\partial z}{\partial v}, \end{aligned}$$

and, from (21), the cosines (a, b, c) are expressed by the equations:

$$\delta \cdot a = \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v},$$

$$(23)\qquad \delta \cdot b = \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v},$$

$$\delta \cdot c = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v},$$

while (20) goes to:

(24) $$\delta = \begin{vmatrix} a & \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \end{vmatrix}.$$

For later use, we note the following facts: When one employs the abbreviations (2), (23) will imply:

(25) $$\delta \begin{vmatrix} b & c \\ y_1 & z_1 \end{vmatrix} = e\,x_2 - f\,x_1, \qquad \delta \begin{vmatrix} b & c \\ y_2 & z_2 \end{vmatrix} = f\,x_2 - g\,x_1.$$

It will further follow from the first equation (23), when one squares it and employs (4), that:

(26) $$\delta^2 a^2 = (e - x_1^2)\,(g - x_2^2) - (f - x_1\,x_2)^2,$$

or

(27) $$a = \sqrt{1 - \delta'(x)},$$

when one sets:

(28) $$\delta'(x) = \frac{1}{\delta^2}(e\,x_2^2 - 2\,f\,x_1\,x_2 + g\,x_1^2),$$

to abbreviate (cf., § **17**).

§ 2. – Second-order fundamental quantities.

In addition to the three functions e, f, g, which are defined by first derivatives of x, y, z with respect to u, v, there are three more functions [1] d, d', d'', which also include the second derivatives of x, y, z, and are therefore called *second-order fundamental quantities.* They are defined by:

$$d = a\frac{\partial^2 x}{\partial u^2} + b\frac{\partial^2 y}{\partial u^2} + c\frac{\partial^2 z}{\partial u^2} = \sum a\frac{\partial^2 x}{\partial u^2},$$

(1) $$d' = a\frac{\partial^2 x}{\partial u\,\partial v} + b\frac{\partial^2 y}{\partial u\,\partial v} + c\frac{\partial^2 z}{\partial u\,\partial v} = \sum a\frac{\partial^2 x}{\partial u\,\partial v},$$

[1] **Gauss**, *Disq. gen.*, art. 10.

$$d'' = a\frac{\partial^2 x}{\partial v^2} + b\frac{\partial^2 y}{\partial v^2} + c\frac{\partial^2 z}{\partial v^2} = \sum a\frac{\partial^2 x}{\partial v^2}.$$

Differentiating the equations that follow from (18) and (22) of § **1**, namely:

$$\text{(2)} \qquad \sum a\frac{\partial x}{\partial u} = 0, \quad \sum a\frac{\partial x}{\partial v} = 0,$$

will imply:

$$\text{(3)} \qquad \sum \frac{\partial a}{\partial u}\frac{\partial x}{\partial u} = -d, \quad \sum \frac{\partial a}{\partial v}\frac{\partial x}{\partial v} = -d'', \qquad \sum \frac{\partial a}{\partial u}\frac{\partial x}{\partial v} = \sum \frac{\partial a}{\partial v}\frac{\partial x}{\partial u} = -d',$$

and from that:

$$\sum \frac{\partial a}{\partial u}dx = \sum \frac{\partial x}{\partial u}da = -(d\,du + d'\,dv),$$

$$\text{(4)} \qquad \sum \frac{\partial a}{\partial v}dx = \sum \frac{\partial x}{\partial v}da = -(d'\,du + d''\,dv),$$

$$\sum a\,d^2x = -\sum da\,dx = d\,du^2 + 2\,d'\,du\,dv + d''\,dv^2.$$

Upon multiplying $\begin{vmatrix} a & \frac{\partial a}{\partial u} & \frac{\partial a}{\partial v} \end{vmatrix}$ by $\begin{vmatrix} a & \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix}$, one will get:

$$\text{(5)} \qquad d\,d'' - d'^2 = \delta \begin{vmatrix} a & \frac{\partial a}{\partial u} & \frac{\partial a}{\partial v} \end{vmatrix}.$$

Next, the derivatives $\frac{\partial a}{\partial u}$, $\frac{\partial a}{\partial v}$ can be represented linearly in terms of $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$, and conversely, with the help of the quantities e, f, g, d, d', d''. Namely, if one solves the equations that appear in (3):

$$a\frac{\partial a}{\partial u} + b\frac{\partial b}{\partial u} + c\frac{\partial c}{\partial u} = 0,$$

$$\frac{\partial x}{\partial u}\frac{\partial a}{\partial u} + \frac{\partial y}{\partial u}\frac{\partial b}{\partial u} + \frac{\partial z}{\partial u}\frac{\partial c}{\partial u} = -d,$$

$$\frac{\partial x}{\partial v}\frac{\partial a}{\partial u} + \frac{\partial y}{\partial v}\frac{\partial b}{\partial u} + \frac{\partial z}{\partial v}\frac{\partial c}{\partial u} = -d',$$

for the $\frac{\partial a}{\partial u}, \frac{\partial b}{\partial u}, \frac{\partial c}{\partial u}$, it will follow that ([1]):

$$\text{(6)}\qquad \frac{\partial a}{\partial u} = \rho'\frac{\partial x}{\partial u} + \sigma'\frac{\partial x}{\partial v}, \qquad \frac{\partial a}{\partial v} = \rho''\frac{\partial x}{\partial u} + \sigma''\frac{\partial x}{\partial v},$$

when one sets:

$$\text{(7)}\qquad \begin{aligned} f\,d' - g\,d &= \rho'\,\delta^2, & f\,d - e\,d' &= \sigma'\,\delta^2, \\ f\,d'' - g\,d' &= \rho''\,\delta^2, & f\,d' - e\,d'' &= \sigma''\,\delta^2, \end{aligned}$$

to abbreviate.

It follows from this that:

$$\text{(8)}\qquad \begin{aligned} e\,\rho'' + d'(\sigma'' - \rho') - d''\sigma' &= 0, \\ d\,\rho'' + d'(\sigma'' - \rho') - d''\sigma' &= 0. \end{aligned}$$

There exists another system of equations that shows that the second and higher partial derivatives of x, y, z or a, b, c with respect to u, v can all be represented linearly in terms of the first derivatives of x, y, z with respect to u, v, and in terms of the quantities a, b, c or also the former quantities alone. Namely, one has ([2]):

$$\text{(9)}\qquad \begin{aligned} \frac{\partial^2 x}{\partial u^2} - p\frac{\partial x}{\partial u} - q\frac{\partial x}{\partial v} &= d\cdot a = d\cdot\sqrt{1-\delta'(x)}, \\ \frac{\partial^2 x}{\partial u\,\partial v} - p'\frac{\partial x}{\partial u} - q'\frac{\partial x}{\partial v} &= d'\cdot a = d'\cdot\sqrt{1-\delta'(x)}, \\ \frac{\partial^2 x}{\partial v^2} - p''\frac{\partial x}{\partial u} - q''\frac{\partial x}{\partial v} &= d''\cdot a = d''\cdot\sqrt{1-\delta'(x)}, \end{aligned}$$

along with the corresponding equations in y and b, z and c.

In fact, from [§ **1**, (23)] and (1), when one employs the abbreviations in [§ **1**, (2)]:

$$d\cdot a = \frac{1}{\delta^2}\begin{vmatrix} 1 & x_1 & x_2 \\ 0 & y_1 & y_2 \\ 0 & z_1 & z_2 \end{vmatrix}\begin{vmatrix} x_{11} & x_1 & x_2 \\ y_{11} & y_1 & y_2 \\ z_{11} & z_1 & z_2 \end{vmatrix} = \frac{1}{\delta^2}\begin{vmatrix} x_{11} & x_1 & x_2 \\ m & e & f \\ n & f & g \end{vmatrix},$$

which will prove the first equation in (9) when one consults [§ **1**, (27)]; one gets the other two equations similarly.

We add two further remarks that will be used later. First, we convert equations (6) and (9). If we substitute the values [§ **1**, (22)] and consider equations [§ **1**, (21)] then (6)

([1]) **Rodrigues**, Corr. sur l'Ec. poly. **3** (1815), pp. 162 for parameter curves that are lines of curvature [cf., §**9**, (5)]; **Weingarten**, J. für Math. **59** (1861), pp. 382 for general parameter curves.

([2]) **Gauss**, *Disq. gen.*, art. 11.

and (9) will provide a representation of the differentials da, $d\alpha_1$, $d\alpha_2$ in terms of a, α_1, α_2, namely:

$$da = \alpha_1 \sqrt{c}\,(\rho' du + \rho'' dv) + \alpha_2 \sqrt{g}\,(\sigma' du + \sigma'' dv)\,, \tag{10}$$

$$\begin{aligned} d\alpha_1 &= \frac{a}{\sqrt{e}}(d\,du + d'\,dv) - (\alpha_1 f - \alpha_1\sqrt{eg})\frac{q\,du + q'dv}{e}\,, \\ d\alpha_2 &= \frac{a}{\sqrt{g}}(d'\,du + d''\,dv) - (\alpha_2 f - \alpha_1\sqrt{eg})\frac{p'\,du + p''\,dv}{g}\,. \end{aligned} \tag{11}$$

One then has the following: If one multiplies the three equations (9) by g, $-2f$, e, resp., adds them, and sets (cf., § **17**) :

$$\delta^2\,\delta''(x) = g\,(x_{11} - p\,x_1 - q\,x_2) - 2\,f\,(x_{12} - p'\,x_1 - q'\,x_2) + e\,(x_{11} - p''\,x_1 - q''\,x_2)\,, \tag{12}$$

to abbreviate, or also, as a simple calculation with the use of equations [§ **1**, (6), (7), (7a)] will show:

$$\delta''(x) = \frac{1}{\delta}\left[\frac{\partial}{\partial u}\left(\frac{g\,x_1 - f\,x_2}{\delta}\right) + \frac{\partial}{\partial v}\left(\frac{e\,x_2 - f\,x_1}{\delta}\right)\right], \tag{13}$$

then it will follow that:

$$a\,(e\,d'' - 2f\,d' + g\,d) = \delta^2 \cdot \delta''(x). \tag{14}$$

When briefly denotes the left-hand side of equations (9) by $(x)_{11}$, $(x)_{12}$, $(x)_{22}$, one will further get:

$$a^2\,(d\,d'' - d'^2) = (x)_{11}\,(x)_{22} - (x)_{12}^2\,. \tag{15}$$

Due to [§ **1**, (27)], one will get the following representations from (14) and (15):

$$h = \frac{e\,d'' - 2\,f\,d' + g\,d}{e\,g - f^2} = \frac{\delta''(x)}{\sqrt{1 - \delta'(x)}}\,, \tag{16}$$

$$k = \frac{d\,d'' - d'^2}{e\,g - f^2} = \frac{(x)_{11}\,(x)_{22} - (x)_{12}^2}{\delta^2(1 - \delta'(x))}\,. \tag{17}$$

The abbreviated values h and k represent the mean curvature and the curvature, resp., of the surface at the point (u, v) (cf., § **5**). Equations (16) and (17) give a curious representation of those quantities in terms of x (or y, z) and its derivatives with respect to u and v.

Remark. – In the formulas that were developed before, as well as the ones to follow, upon permuting u and v, the quantities:

$$e, f, g, d, d', d'', m, m', m'', p, p', p'', \rho', \sigma'$$

will go to:

$$g, f, e, d'', d', d, n'', n', n, q'', q', q, \sigma'', \rho'',$$

respectively.

§ 3. – Minimal lines. Isometric lines. Conformal mapping of a surface.

Any equation between (u, v) represents a curve on the surface. The ratio $du : dv$ that it implies determines the direction of advance or the direction of the line element of the curve at the point (u, v). In the following sections, we shall consider the *differential equations for the most important special surface curves.* Just like the character and properties of the surface itself, those curves are also determined in terms of the six fundamental quantities e, f, g ; d, d', d''.

One calls the system of curves on the surface that are defined by the equation:

$$ds^2 = dx^2 + dy^2 + dz^2 = 0 \tag{1}$$

minimal lines ([1]), so their *differential equation* reads:

$$ds^2 = e\,du^2 + 2\,f\,du\,dv + g\,dv^2 = 0. \tag{2}$$

From (1), the minimal lines are geometrically defined to be those curves on the surface whose tangents intersect the spherical circle at infinity.

From (2), the condition for the parameter curves u = const. and v = const. themselves to be minimal lines is that:

$$e = g = 0. \tag{2a}$$

Equation (2) has degree two in $du : dv$, so two minimal lines will go through each point. Indeed, those lines will be imaginary for real surfaces with real e, f, g, since the discriminant of (2), namely, $\delta^2 = eg - f^{\,2}$, is positive, which would follow from [§ **1**, (23)]. However, they will lead to important systems of real lines. In order to do that, one imagines that equation (2) has been decomposed into its two conjugate imaginary factors:

$$P = \frac{1}{\sqrt{e}}(e\,du + f\,dv + i\delta\,dv), \qquad Q = \frac{1}{\sqrt{e}}(e\,du + f\,dv - i\delta\,dv). \tag{3}$$

It is known that there are infinitely many integrating factors that will convert such differential expressions into complete differentials. Let one such factor for P be equal to

([1]) From the **Lie** process, Math. Ann. **14** (1878), pp. 337.

$\mu + i\nu$, and let the associated differential be $d\alpha + i\,d\beta$, where μ, ν and α, β are certain real functions of (u, v). One will then have:

$$P(\mu + i\nu) = d\alpha + i\,d\beta, \qquad Q(\mu - i\nu) = d\alpha - i\,d\beta,$$

so

$$ds^2 = PQ = \lambda^2 (d\alpha^2 + d\beta^2), \qquad \lambda^2 = \frac{1}{\mu^2 + \nu^2}. \tag{4}$$

Establishing that form (4) for the line element will then require the integrating the differential equation (2). The curves $\alpha(u, v)$ = const. and $\beta(u, v)$ = const. (or more briefly, α and β) possess a characteristic geometric property. Namely, if one introduces α, β as parameters, instead of (u, v), then ds^2 will assume the form (4), which is distinguished from (2) by the facts that the coefficient of $d\alpha\,d\beta$ is zero, and that the coefficients of $d\alpha^2$ and $d\beta^2$ are equal to each other. The first condition says that the system of curves α, β intersect orthogonally, while the second one says that the surface can be divided into *infinitely-many squares* by the system of curves α, β. If one then considers α, β simultaneously to be rectangular coordinates in a plane then it will follow from (4) that the surface can be mapped *conformally to the plane* by the equations $\alpha(u, v) = \alpha$, $\beta(u,v) = \beta$ ([1]) with the linear expansion λ, under which, the system of lines in the plane that are parallel to the axes corresponds to the system of curves α, β on the surface. Now, since the plane can be divided into an infinitude of small squares by the lines α, β, the same thing will be true for the surface. Due to that property, the system α, β is called an *isometric (isothermal) system of curves* on the surface, and the quantities α, β are its *thermal parameters.* That will give the **theorem**:

Any solution to the differential equation (2) *will imply an isometric system of curves on the surface whose parameters* α, β *will give the line element the form* (4), *and thus map the surface conformally to the plane.*

Corresponding to the infinitude of integrating factors to (3), there are infinitely many isometric systems of lines on any surface. In order to make the transition from one such system to another, one again lets $\mu + i\nu$ be an integrating factor of P and lets $d\alpha + i\,d\beta$ be its associated differential. It is then known that the most general integrating factor of P will have the form:

$$(\mu + i\nu)\,F(\alpha + i\beta),$$

where F denotes an arbitrary function. If one now sets:

$$P \cdot (\mu + i\nu)\,F(\alpha + i\beta) = F(\alpha + i\beta)\,(d\alpha + i\,d\beta) = d\,\Pi(\alpha + i\beta) = dA + i\,dB$$

then ds^2 will assume the form:

$$ds^2 = L^2 (dA^2 + dB^2)\,; \tag{5}$$

([1]) **Lagrange** (1779) for the ellipsoid of rotation (*Oeuvres*, t. IV, pp. 637). **Gauss** (1822) for the general surface (*Werke*, Bd. IV, pp. 193).

i.e., the two families of curves A, B will likewise define an isometric system in the plane. Since Π is arbitrary, just like F, one will have the **theorem:**

If (α, β) *is an isometric system then one will get infinitely many other isometric systems* (A, B) *from it when one understands* Π *to mean an arbitrary function and sets:*

$$A + i\,B = \Pi\,(\alpha + i\,\beta) \tag{6}$$

and splits that equation into its real and imaginary parts.

That is consistent with the known theorem in the theory of functions:

If one splits a function of the complex variables $\alpha + i\beta$ *into its real and imaginary parts then one will get two functions A, B that effect a conformal map of the plane* (α, β) *onto the plane* (A, B).

If (α, β) and (A, B) are two such families then they must be coupled by two equations $A = A\,(\alpha, \beta)$ and $B = B\,(\alpha, \beta)$ that will make:

$$ds^2 = \lambda^2\,(d\alpha + i\,d\beta)\,(d\alpha - i\,d\beta) = L^2\,(dA + i\,dB)\,(dA - i\,dB)\,.$$

Now, since dA and dB are linear and homogeneous in $d\alpha$ and $d\beta$, it will follow that:

$$dA + i\,dB = \rho\,(d\alpha \pm i\,d\beta),$$

and since the left-hand side of this is a complete differential, the same thing must be true for the right-hand side, so ρ will be a function of $(\alpha \pm i\,\beta)$, which will lead back to equation (6).

These considerations also imply the criterion for the *parameter curves u*, *v themselves to define an isometric system* on the surface. In that case, ds^2 must have the form:

$$ds^2 = l\,(U\,du^2 + V\,dv^2), \tag{7}$$

in which U is a function of just u, V is a function of just v, and l is an arbitrary function of (u, v). A comparison with (1) will give the necessary and sufficient conditions for the isometry as:

$$f = 0, \qquad e = l\,U, \qquad g = l\,U \qquad\qquad \text{or} \qquad\qquad e : g = U : V = V_1 : U_1\,. \tag{8}$$

One can give them another form. Since:

$$\frac{\partial}{\partial v}\left(\frac{e}{l}\right) = 0, \qquad \frac{\partial}{\partial u}\left(\frac{g}{l}\right) = 0,$$

upon eliminating l, one will get the equations:

(9) $$f = 0, \qquad \frac{\partial^2 \log e}{\partial u\,\partial v} = \frac{\partial^2 \log g}{\partial u\,\partial v},$$

in place of (8).

If those conditions are fulfilled then one can get l from:

(10) $$\log l = \int \left(\frac{\partial \log g}{\partial u} du + \frac{\partial \log e}{\partial v} dv \right).$$

In particular, if $U = V = 1$, so $f = 0$, $e = g$ then (u, v) will be thermal parameters.

As an *application* of this, we shall address the **problem**:

Map the sphere onto the plane conformally,

which will be useful later.

One first expresses the coordinates X, Y, Z of a point P on the sphere in terms of the parameters u, v of the minimal lines on the sphere. The sphere has a radius of 1 and its center at the origin; here, as well as later, we shall denote that sphere by K. Let $Z = 0$ be the equatorial plane, let ρ be the radius of the parallel circle through P, and let ψ be the angle that the meridian plane of P makes with the plane $Y = 0$. X, Y, Z will then have the values:

(11) $$X = \rho \cos \psi, \quad Y = \rho \sin \psi, \quad Z = \sqrt{1-\rho^2}.$$

If (u, v) are the parameters of the minimal lines and one expresses the line element dS of the sphere on the one hand by (ρ, ψ) and on the other hand by (u, v) then one will get:

(12) $$dS^2 = \frac{d\rho^2}{1-\rho^2} + \rho^2\, d\psi^2 = F(u, v)\, \frac{du}{u}\frac{dv}{v},$$

in which $F(u, v)$ is an as-yet-undetermined function. It will follow from decomposition that:

$$\frac{d\rho}{\rho\sqrt{1-\rho^2}} + i\, d\psi = \frac{du}{u}, \qquad \frac{d\rho}{\rho\sqrt{1-\rho^2}} - i\, d\psi = \frac{dv}{v}, \qquad \rho^2 = F(u, v),$$

or

$$\frac{d\rho}{\rho\sqrt{1-\rho^2}} = \frac{1}{2}\left(\frac{du}{u} + \frac{dv}{v}\right), \quad i\, d\psi = \frac{1}{2}\left(\frac{du}{u} - \frac{dv}{v}\right),$$

or

(13) $$\rho = \frac{2\sqrt{uv}}{1+uv}, \qquad i\,\psi = \frac{1}{2}\log\left(\frac{u}{v}\right),$$

$$\cos\psi = \frac{u+v}{2\sqrt{uv}}, \qquad \sin\psi = \frac{u-v}{2i\sqrt{uv}}. \tag{14}$$

Thus, $u \cdot v$ = const. are the parallel lines of the sphere K, while $u : v$ = const. are the meridian curves. It will follow from (11) that:

$$X = \frac{u+v}{1+uv}, \qquad Y = i\,\frac{v-u}{1+uv}, \qquad Z = \frac{uv-1}{1+uv}, \tag{15}$$

and upon solving these:

$$u = \frac{X+iY}{1-Z}, \qquad v = \frac{X-iY}{1-Z}. \tag{16}$$

We shall now couple the sphere K with the plane by means of stereographic projection ([1]). If the pole ($X = Y = 0$, $Z = 1$) is the center of projection, the equatorial plane $Z = 0$ is the projection plane, and X, Y, Z is a point on the sphere, while ξ, η is its image point, then those coordinates (when the ξ and η axes coincide with the X and Y axes, resp.) will be coupled by the equations:

$$\xi = \frac{X}{1-Z}, \qquad \eta = \frac{Y}{1-Z}, \qquad \xi^2 + \eta^2 = \frac{1+Z}{1-Z}, \tag{17}$$

or

$$X = \frac{2\xi}{\xi^2+\eta^2+1}, \qquad Y = \frac{2\eta}{\xi^2+\eta^2+1}, \qquad Z = \frac{\xi^2+\eta^2-1}{\xi^2+\eta^2+1}. \tag{18}$$

It follows from (16) and (17) that:

$$u = \xi + i\,\eta, \qquad v = \xi - i\,\eta. \tag{19}$$

Therefore, the parameters u, v of the minimal lines of the sphere are identical with the conjugate values $\xi + i\,\eta$, $\xi - i\,\eta$, that are defined by the coordinates of the projection point. From (12), (13), and (19), one will have:

$$dS^2 = \frac{4\,du\,dv}{(1+uv)^2} = \frac{4(d\xi^2+d\eta^2)}{(\xi^2+\eta^2+1)^2}. \tag{20}$$

One then has the theorem, which is easy to prove geometrically:

The sphere is mapped conformally to the plane by stereographic projection.

([1]) **Riemann**, *Ges. Werke*, 1867, pp. 286.

From (20), the curves $\xi = \alpha$, $\eta = \beta$ define a special isometric system on the sphere. One will get the most general isometric system when one sets the real and imaginary components of an arbitrary function of the complex variable $\xi + i\,\eta$ equal to constants. If L and M are conjugate functions, and:

$$L(u) = L(\xi + i\,\eta) = \Xi + i\,\mathrm{H}, \qquad M(v) = M(\xi - i\,\eta) = \Xi - i\,\mathrm{H} \tag{21}$$

then $\Xi(\xi, \eta) = A$, $\mathrm{H}(\xi, \eta) = B$ will represent the most general isometric system on the sphere. If one sets:

$$L(u) = \int \sqrt{P(u)}\,du\,, \quad M(v) = \int \sqrt{Q(v)}\,dv$$

then it will follow from (21) that:

$$2\,d\,\Xi = \sqrt{P}\,du + \sqrt{Q}\,dv\,, \qquad 2i\,d\,\mathrm{H} = \sqrt{P}\,du - \sqrt{Q}\,dv\,.$$

Hence, the differential equation of the most general isometric system on the sphere in terms of the parameters (u, v) will have the form:

$$P\,du^2 - Q\,dv^2 = 0, \tag{22}$$

in which P and Q are arbitrary conjugate functions relative to u and v. Conversely, any such solution to the differential equation will be an isometric system on the sphere.

§ 4. – Geodetic lines. Geodetic coordinates.

One calls the curves on a surface whose principal normal at each point coincides with the surface normal or whose osculating plane goes through the surface normal *geodetic lines on the surface*.

If ξ, η, ζ are the coordinates of a variable point on the surface normal at the point x, y, z, and r is the distance between those two points then the equations of that surface normal will be:

$$\xi - x = r\,a, \qquad \eta - y = r\,b, \qquad \zeta - z = r\,c\,. \tag{1}$$

Furthermore, the equation of the osculating plane at a point (x, y, z) on a space curve, for which x, y, z are functions of a parameter, and when ξ, η, ζ are the running coordinates, will be:

$$\left|\, \xi - x \;\vdots\; dx \;\vdots\; d^2x \,\right| = 0. \tag{2}$$

If one expresses the fact that the osculating plane (2) contains the normal (1) then one will get the *differential equation for geodetic lines* in the form:

$$(3) \qquad \left| a \;\vdots\; dx \;\vdots\; d^2x \right| = 0.$$

In order to obtain it in terms of the parameters (u, v) and its differentials, one multiplies the left-hand side by the determinant [§ **1**, (24)]. (3) will then go to:

$$(4) \qquad \begin{vmatrix} e\,du + f\,dv & m\,du^2 + 2m'\,du\,dv + m''dv^2 + e\,d^2u + f\,d^2v \\ f\,du + g\,dv & n\,du^2 + 2n'\,du\,dv + n''dv^2 + f\,d^2u + g\,d^2v \end{vmatrix} = 0.$$

That differential equation depends upon only the coefficients e, f, g of the line element. Moreover, it has order two, so its integral will include two arbitrary constants. One can prove that a geodetic line is determined completely by one of its points and the direction through it, while in general it is not determined by two of its points, even when they are very close to each other. In § **16**, we shall give the integration of equation (4) for a special type of surface that includes second-order surfaces and surfaces of revolution, among others.

By a comparison with (3), one can further show that the geodetic lines can also be defined by:

1. The shortest line between two points (within certain limits).

2. The locus of points moving on a surface in the absence of forces.

3. Finally, the curve that a tensed string will assume on the surface when no forces act upon it ([1]).

Geodetic lines can be employed to introduce certain parameters ([2]) that are also called geodetic coordinates. If one draws an arbitrary curve p on the surface and starts from each of its points and measures out equal lengths along the geodetic lines γ, γ_1, … that are normal to those points then the endpoints will define a curve p_1 that one calls a *geodetic parallel* to p. One easily convinces oneself that the geodetic parallels p, p_1, …, and the geodetic lines γ, γ_1, … intersect each other orthogonally everywhere. One then calls the system of those two families a *geodetic orthogonal system*. In fact, if one introduces the two families as parameter curves and denotes the first one by u = const. and the second one by v = const., and understands u to be the length of the geodetic lines, measured from the line p (viz., $u = 0$), in particular, then one will first have $e = 1$ in the expression for the line element (since one must have $ds = du$ for $dv = 0$), but one must also have $f = 0$. The differential equation (4) of the geodetic lines must be satisfied by $e = 1$ and $dv = 0$. However, it will reduce to $du^3 \cdot \dfrac{\partial f}{\partial u} = 0$ for those values. One must then have $f = V$; i.e., f must be a function of just v, or f must be likewise constant along any individual geodetic line γ, γ_1, … (v = const.). Now, since f must have the value 0 for the intersection point of

([1]) **Monge-Liouville**, *Applications*, pp. 401, *et seq.*
([2]) **Gauss**, *Disq. gen.*, art. 15, 16, 19.

the curve p with each of its orthogonal curves γ, γ_1, …, one must have $f = 0$ over the whole surface. (Q. E. D.)

The parameters u, v that are defined in this way are called *geodetic coordinates.* The line element will have the form:

$$ds^2 = du^2 + g\,dv^2 \tag{5}$$

in terms of them, in which g is a function of (u, v). Conversely, if the line element of a surface has the form $ds^2 = e\,du^2 + g\,dv^2$, with the condition that e is a constant or a function of only u, then one will get the form (5) by the substitution $e\,du^2 = du_1^2$, so u_1 will then represent a system of geodetic parallels, and u_1 itself will be the length of the geodetic line v = const., when it is measured from the first geodetic parallel $u_1 = 0$.

If one contracts the first geodetic parallel p ($u = 0$) to a point P in the discussion above, so the geodetic lines v = const. will emanate from that point then one will call the geodetic parallels u = const. *geodetic circles*, and (u, v) will be a *geodetic polar system* with its center at P. In that case, one must add the condition for the function g in (5) that one must also have $g = 0$ for $u = 0$, since one must have $ds = du$ for all directions at the point P ($u = 0$). Under the assumption that in a geodetic polar system v means the angle that the line v makes with the line $v = 0$ at the point P, one must add the further condition that for $u = 0$, one has $\frac{\partial\sqrt{g}}{\partial u} = 1$. The arc length element of a geodetic circle of infinitely small radius will then have the value $ds = u\,dv$ now; one must then have $\sqrt{g} = u$, or $\frac{\partial\sqrt{g}}{\partial u} = 1$, for $u = 0$.

Let it be further remarked that the minimal lines can be regarded as geodetic lines, since the osculating plane to a minimal line contacts the imaginary spherical circle (§ **3**). As a result, that plane will be normal to the tangent to the curve, so it will go through the surface normal. Analytically, that means the same thing, since equation (4) will be satisfied by $du = 0$, as well as $dv = 0$, for $e = g = 0$ (cf., also § **17**).

§ 5. – Conjugate directions. Lines of curvature. Asymptotic lines.

Two line elements ds_1 and ds_2 or directions $(du_1 : dv_1)$ and $(du_2 : dv_2)$ on the surface are called *conjugate* ([1]) when the tangential planes at the endpoints of the one line element intersect in the direction of the other element.

The fact that this relationship is reciprocal in the two directions follows from the proof of equation (1) below.

Conjugate systems of lines on the surface are ones that intersect along conjugate directions everywhere.

The condition for two directions $du_1 : dv_1$ and $du_2 : dv_2$ to be conjugate is that:

([1]) **Dupin**, *Développements*, pp. 44 and 91.

(1) $$d \cdot du_1\, du_2 + d' \cdot (du_1\, dv_2 + dv_1\, du_2) + d'' \cdot dv_1\, dv_2 = 0\,.$$

If X, Y, Z are the running coordinates, then the equations for the tangent plane at the origin (x, y, z) of ds_1 will be:

$$(X - x)\, a + (Y - y)\, b + (Z - z)\, c = 0,$$

$$(X - x - dx_1)\,(a + da_1) + (Y - y - dy_1)\,(b + db_1) + (Z - z - dz_1)\,(c + dc_1) = 0,$$

where:

$$dx_1 = \frac{\partial x}{\partial u} du_1 + \frac{\partial x}{\partial v} dv_1, \qquad da_1 = \frac{\partial a}{\partial u} du_1 + \frac{\partial a}{\partial v} dv_1\,.$$

Since $a\, dx_1 + b\, dy_1 + c\, dz_1 = 0$, one will then have:

$$(X - x)\, da_1 + (Y - z)\, db_1 + (Z - z)\, dc_1 = 0$$

for the point (X, Y, Z) on the line of intersection of the two tangent planes.

The condition for the endpoint $(x + dx_1\,, y + dy_1\,, z + dz_1)$ of ds_2 to lie on that line is that:

$$da_1\, dx_2 + db_1\, dy_2 + dc_1\, dz_2 = 0$$

However, from [§ **2**, (3)], that equation is identical to (1).

The necessary and sufficient condition for the *parameter curves* u = const *and* v = const. *to be themselves conjugate lines* is that:

(2) $$d' = 0\,.$$

If (1) is to be satisfied by $(du_1 = du\,;\, dv_1 = 0)$ and $(du_2 = 0\,;\, dv_2 = dv)$ then one must have $d' = 0$, and conversely, if $d' = 0$ then that differential must satisfy equation (1).

From [§ **1**, (9)], the angle V between a direction $du_1 : dv_1$ and the direction $du_2 : dv_2$ that is conjugate to it is determined by:

$$ds_1\, ds_2 \cos V = e\, du_1\, du_2 + f\,(du_1\, dv_2 + dv_1\, du_2) + g\, dv_1\, dv_2\,,$$

or when one substitutes the values:

$$du_2 = \lambda\,(d' du_1 + d'' dv_1), \qquad dv_2 = -\,\lambda\,(d\, du_1 + d' dv_1),$$

(where λ is a proportionality factor), from (1), by:

(3) $$ds_1\, ds_2 \cos V = \lambda \cdot [\, du_1^2\,(e\, d' - f\, d) + du_1\, dv_2\,(e\, d'' - g\, d) + dv_1^2\,(f\, d'' - g\, d')]\,.$$

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The systems of curves on a surface that are both conjugate and orthogonal are called *lines of curvature* ([1]).

From (3), the differential equation of the lines of curvature is cos $V = 0$, or when one writes (u, v) for (u_1, v_1):

$$(e\, d' - f\, d)\, du^2 + (e\, d'' - g\, d)\, du\, dv + (f\, d'' - g\, d')\, dv^2 = 0$$

or

$$\text{(4)} \qquad \begin{vmatrix} e\,du + f\,dv & d\,du + d'\,dv \\ f\,du + g\,dv & d'\,du + d''\,dv \end{vmatrix} = 0.$$

There are then two families of lines of curvature on the surface.

From (2) and [§ **1**, (10)] or from (4), the necessary and sufficient conditions for the *parameter curves* (u, v) *to be themselves lines of curvature* are:

$$\text{(5)} \qquad f = 0, \quad d' = 0.$$

Another definition of the lines of curvature will give rise to further considerations.

Lines of curvature are also the curves along which the consecutive surface normals intersect.

Should the normals at the points (u, v) and $(u + du, v + dv)$ intersect each other, then if ξ, η, ζ denote the coordinates of the intersection point and r denotes its distance from the point (u, v) or (x, y, z), one would have the condition:

$$x = x + ra = (x + dx) + (r + dr)(a + da)$$

or

$$\text{(6)} \qquad dx + a\, dr + r\, da = 0,$$

along with the corresponding equations in (y, b) and (z, c). Upon eliminating r and dr, it will follow that:

$$\text{(7)} \qquad |\, a\ da\ dx \,| = 0.$$

That equation can be easily brought into the form (4), either by multiplying the left-hand side by the determinant [§ **1**, (24)] or in the following way: If one multiplies (6) by a and adds the corresponding equations in (y, b) and (z, c) then, since $\sum a\, dx = 0$ and $\sum a^2 = 1$, it will follow that the directions of the lines in question have $dr = 0$, so:

$$\text{(8)} \qquad dx + r\, da = 0, \qquad dy + r\, db = 0, \qquad dz + r\, dc = 0.$$

If one multiplies those equations by $\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}$, resp., and then by $\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}$, and adds them each time then, from [§ **1**, (4)] and [§ **2**, (3)]:

([1]) **Monge-Liouville**, *Applications*, pp. 124, *et seq.*; **Dupin**, *Développements*, pp. 47 and 94.

(9) $$e\,du + f\,dv = r\,(d\,du + d'\,dv), \qquad f\,du + g\,dv = r\,(d'\,du + d''\,dv).$$

If one eliminates r from this then that will yield equation (4), with which, the identity of the aforementioned lines with the lines of curvature will be proved. Equations (8) give a third definition of the lines of curvature, namely, (cf., § **11**).

Lines of curvature are also the curves whose direction at each point is parallel to the direction of its spherical image.

On the other hand, if one eliminates $du : dv$ from (9) then one will get a quadratic equation in r, namely:

(10) $$\begin{vmatrix} r\,d - e & r\,d' - f \\ r\,d' - f & r\,d'' - g \end{vmatrix} = 0$$

or [1]:

$$r^2\,(d\,d'' - d'^2) - r\,(e\,d'' - 2f\,d' + g\,d) + (e\,g - f^2) = 0.$$

The two roots r_1 and r_2 of that equation are called the *radii of principal curvature* of the surface at the point (u, v). They are the radii of curvature of those planar normal sections that contact one line of curvature and the other at the point (u, v). The directions of advance that belong to r_1 and r_2 are implied uniquely from equations (9).

One obtains the values of the *curvature k* and the *mean curvature h* of the surface at the point (u, v), which were used briefly before, from (10):

(11) $$k = \frac{1}{r_1\,r_2} = \frac{d\,d'' - d'^2}{e\,g - f^2}, \qquad h = \frac{1}{r_1} + \frac{1}{r_2} = \frac{e\,d'' - 2\,f\,d' + g\,d}{e\,g - f^2}.$$

The lines of curvature, as well as the radii of principal curvature, are always real for real surfaces, since the discriminant of (4) in terms of $du : dv$ is identical to the discriminant of (10) in terms of r, and is therefore positive. In fact, that discriminant can be easily brought into the form:

(12) $$\begin{aligned} \delta^4 \left(\frac{1}{r_2} - \frac{1}{r_1}\right)^2 &= (e\,d'' - g\,d)^2 - 4\,(e\,d' - f\,d)\,(f\,d'' - g\,d') \\ &= [(e\,d'' - g\,d) - \frac{2f}{e}\,(e\,d' - f\,d)]^2 + \frac{4}{e^2}\,(e\,g - f^2)\,(e\,d' - f\,d)^2 . \end{aligned}$$

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[1] **Monge-Liouville**, *Applications*, pp. 129.

A direction on the surface that coincides with the associated conjugate direction is called an *asymptotic direction*. The curves that are defined by the asymptotic directions are called *asymptotic lines* ([1]).

One will get the differential equation of the asymptotic lines from (1) when one sets $du_1 : dv_1 = du_2 : dv_2 = du : dv$:

$$d\,du^2 + 2d'du\,dv + d''dv^2 = 0. \tag{13}$$

Thus, two asymptotic lines go through each point of the surface. Conversely, from (1) and (13), conjugate directions are ones that lie harmonically to the asymptotic lines.

If one considers the left-hand side of (13) and the expression for ds^2 [§ **1**, (3)] as a binary form in du, dv then the numerator and denominator of h and k in (11) will be simultaneous invariants of it, while the left-hand side of (4) will be the simultaneous covariant of it.

From (13), the necessary and sufficient conditions for the *parameter curves to be themselves asymptotic lines* are:

$$d = 0, \qquad d'' = 0. \tag{14}$$

In order to determine the angle W between the two asymptotic directions, let $du' : dv'$ and $du'' : dv''$ be the roots of the equation (13), and furthermore, let ds' and ds'' be the elements of the asymptotic lines, while μ is a proportionality factor.

$$du'\,du'' = \mu\,d'', \qquad dv'dv'' = \mu\,d, \qquad du'\,dv'' + dv'\,du'' = -\mu\cdot 2\,d'.$$

Due to (11) and (12), one will then have:

$$e\,du'du'' + f\,(du'\,dv'' + dv'\,du'') + g\,dv'dv'' = \mu\,(e\,d'' - 2f\,d' + g\,d) = \mu\,\delta^2\left(\frac{1}{r_1}+\frac{1}{r_2}\right),$$

$$ds'\,ds'' = \mu\,[(e\,d'' - 2f\,d' + g\,d)^2 - 4\,(e\,g - f^2)\,(d\,d'' - d'^2)]1/2 = \mu\,\delta^2\left(\frac{1}{r_1}-\frac{1}{r_2}\right),$$

and as a result, from [§ **1**, (8)]:

$$\cos W = \frac{r_1+r_2}{r_1-r_2}. \tag{15}$$

The angle between the asymptotic lines then depends upon only the ratio of the two radii of principal curvature ([2]).

The curve on the surface along which the asymptotic lines intersect rectangularly is:

([1]) **Dupin**, *Développements*, pp. 51.
([2]) **Dupin**, *Développements*, pp. 189.

(16) $$r_1 + r_2 = 0 \qquad \text{or} \qquad e\, d'' - 2f\, d' + g\, d = 0.$$

The asymptotic lines can be defined in yet another way. From [§ **2**, (4)], equation (13) is identical to:

(17) $$da\, dx + db\, dy + dc\, dz = 0\ ;$$

i.e., (cf., § **11**): The asymptotic lines are also those curves on the surface whose direction at each point is normal to the direction of the spherical image.

Moreover, from [§ **2**, (4)], equation (13) is identical to:

(18) $$a\, d^2 x + b\, d^2 y + c\, d^2 z = 0.$$

However, that is the condition for the tangent plane of the surface at the point (x, y, z) to contain the point $x + 2\, dx + d^2 x,\ y + 2\, dy + d^2 y,\ z + 2\, dz + d^2 z$, so:

The asymptotic lines are also the curves whose osculating planes coincide with the tangent plane at every point.

That result, in a different form, will give the **theorem**:

The curve of intersection of a tangent plane to the surface with that surface has a double point at the contact point. The two tangents to the double point osculate the surface. They give the directions of the asymptotic lines.

If (x, y, z) is a well-defined point on the surface with the fixed parameters (u, v) then the curve of intersection of the tangent plane of that point with the surface will have the equation:

(19) $$a\,(X - x) + b\,(Y - y) + c\,(Z - z) = 0,$$

when one expresses the quantities $x, y, z\ ;\ a, b, c$ in terms of the fixed parameters (u, v), and the quantities X, Y, Z in terms of the variable parameters (U, V). Equation (19), and likewise the equations that emerge from them by differentiating with respect to U and V :

$$a\frac{\partial X}{\partial U} + b\frac{\partial Y}{\partial U} + c\frac{\partial Z}{\partial U} = 0, \qquad a\frac{\partial X}{\partial V} + b\frac{\partial Y}{\partial V} + c\frac{\partial Z}{\partial V} = 0,$$

will be satisfied when one lets U, V coincide with u, v, resp., and X, Y, Z with x, y, z, resp.; i.e., the curve of intersection (19) has a double point at the point (u, v). In order to find the directions $du : dv$ of the tangents at (u, v), one must substitute the development:

(20) $$X = x + \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial^2 x}{\partial u^2}du^2 + 2\frac{\partial^2 x}{\partial u\,\partial v}du\,dv + \frac{\partial^2 x}{\partial v^2}dv^2 + \dots,$$

in (19), along with the corresponding developments for Y and Z. One will then get precisely equation (13) for the directions $du : dv$ of the tangents at the double point, which will imply the directions of the asymptotic lines. (Q. E. D.)

In § **15**, it will be shown that the asymptotic lines are also the curves for which every planar curve of intersection that contacts them (but is different from the intersection curve with the tangent plane) has an inflection point at the contact point.

One can distinguish between surfaces ([1]) according to whether:

$$
(21)\qquad
\begin{array}{llll}
k \quad \text{or} \quad d\,d'' - d'^2 > 0 & \text{in the neighborhood of an } \textit{elliptic} \text{ point}, \\
\;'' \quad\;\, '' \qquad\quad '' \qquad\; < 0 & \qquad '' \qquad\quad '' \qquad\quad \textit{hyperbolic} \text{ point}, \\
\;'' \quad\;\, '' \qquad\quad '' \qquad\; = 0 & \text{along the curve through a } \textit{parabolic} \text{ point}.
\end{array}
$$

The latter curve separates the neighborhoods of the elliptic and hyperbolic points. At the elliptic points, the radii of principal curvature have the same signs and the asymptotic lines are imaginary; viz., the surface is convex-convex. At the hyperbolic points, the radii of principal curvature have the opposite signs and the asymptotic points are real; viz., the surface is convex-concave. At a parabolic point, one radius of principal curvature is infinite, so the asymptotic lines will coincide. In general, the curve through the parabolic point will be a locus of cusps, while in singular cases, it can also envelope the asymptotic lines totally or partially. One can easily examine those relationships in the example of the surfaces of revolution.

The relations (21) have their origin in a line of reasoning that is connected with the remark above about the asymptotic lines.

In order to investigate the surface in the neighborhood of a point, one imagines the intersection curve of the surface with a plane that is parallel to the tangent plane at the point and infinitely close to it and replaces that intersection curve with the so-called *indicatrix* ([2]); i.e., the simplest curve that agrees with the intersection curve as closely as possible in the vicinity of the point.

In order to obtain the equation of the indicatrix in the simplest form, let u, v be the parameters of the lines of curvature, so from (5) and (11):

$$
(22)\qquad f = 0, \qquad d' = 0, \qquad \frac{1}{r_1} = \frac{d}{e}, \qquad \frac{1}{r_2} = \frac{d''}{g}.
$$

A surface that is parallel to the tangent plane (19) at a distance of ε from it has the equation:

$$
a\,(X - x - \varepsilon\,a) + b\,(Y - y - \varepsilon\,b) + c\,(Z - z - \varepsilon\,c) = 0,
$$

([1]) **Dupin**, *Développements*, pp. 49 and 154.
([2]) **Dupin**, *Développements*, pp. 48 and 147.

or

$$a\,(X-x) + b\,(Y-y) + c\,(Z-z) = \varepsilon\,.$$

If one substitutes the values (20) in this then one will get:

$$d\,du^2 + 2\,d'du\,dv + d''dv^2 = \varepsilon,$$

or when one sets the arc length of the parameter curves equal to $\sqrt{e}\,du = \xi$ and $\sqrt{g}\,dv = \eta$, and uses equations (22), one will get:

$$\frac{\xi^2}{r_1} + \frac{\eta^2}{r_2} = \varepsilon \tag{23}$$

as the *equation of the indicatrix,* in which ε is assumed to be infinitely small. Thus, the indicatrix at an elliptical point ($k > 0$) will consist of an ellipse, a hyperbola at a hyperbolic point ($k < 0$), and at a parabolic point ($r_2 = \infty$), it will consist of a pair of infinitely-close parallel lines.

Furthermore, conjugate directions at a point of the surface will correspond to conjugate diameters, the asymptotic lines through the point, to the asymptotes, and the directions of the lines of curvature, to the principal axes of the indicatrix, such that the directions of the lines of curvature will also be the directions for which the normal curvatures of the point are a maximum or minimum (cf., § **15**).

In particular, if $r_1 = r_2$ at a point, so the indicatrix is a circle then the point will be called a *circular point* ([1]) (or *umbilic point*); from (12), one will have:

$$\frac{1}{r_1} = \frac{1}{r_2} = \frac{d}{e} = \frac{d'}{f} = \frac{d''}{g} \tag{24}$$

for it.

One easily sees that one does not have infinitely many lines of curvature that go through a circular point, in general, but only three of them ([2]). Let A = const. be the finite equation of the two families of lines of curvature, when solved for the constants. If one sets $u + du$ and $v + dv$ for u and v, resp., in it and develops in powers of du and dv then when one goes up to terms of second order, one will get the differential equation for the lines of curvature in the form:

$$P\,du^2 + Q\,du\,dv + R\,dv^2 = 0,$$

in which the left-hand side coincides with (4), up to a factor.

At a circular point, where $P = Q = R = 0$, from (24), one will need to go on to third-order terms. When one sets $\frac{\partial P}{\partial u} = P_1$, $\frac{\partial P}{\partial v} = P_2$, etc., to abbreviate, one will then get the equation:

$$(P_1\,du + P_2\,dv)\,du^2 + (Q_1\,du + Q_2\,dv)\,du\,dv + (R_1\,du + R_2\,dv)\,dv^2 = 0,$$

([1]) **Dupin**, *Développements*, pp. 126.
([2]) **Dupin**, *loc. cit.*, pp. 164.

which implies three values for the ratio $du : dv$.

If $d = d' = d'' = 0$ then one will have a circular point with infinite radius. Upon carrying out the development (20), one will now find a third-order curvature for the indicatrix, and the intersection curve of the tangent plane will contain a triple point whose three tangents are parallel to the three asymptotes of the indicatrix and will give the asymptotic direction of the surface at that point.

The investigations in §§ **3-5** can be employed in order to simplify the formulas in § **1** and § **2** that were developed for general parameters; one must set:

	$f = 0$,	when the parameter curves are	orthogonal,
	$d' = 0$,	″ ″ ″	conjugate,
	$e = 0,\ g = 0$,	″ ″ ″	minimal lines,
(25)	$e = g,\ f = 0$,	″ ″ ″	isometric lines,
	$d = 0,\ d'' = 0$,	″ ″ ″	asymptotic lines,
	$f = 0,\ d' = 0$,	″ ″ ″	lines of curvature,
	$e = 1,\ f = 0$,	″ ″ ″	a geodetically-orthogonal system,

and u is the arc length of the geodetic lines, measured from one of the geodetic parallels v = const.

We add the following remark here ([1]) about when a surface can be considered to be flexible, but not extensible. A surface is called a *bending surface* of another one or *developable* from it when it can be obtained by bending. The points of two different surfaces might relate to each other in such a way that their points (x, y, z) and (x', y', z') are given as functions of the same parameter pairs (u, v) in the form:

$$(26)\qquad \begin{aligned} &x = x\,(u, v), \quad y = y\,(u, v), \quad z = z\,(u, v), \\ &x' = x'\,(u, v), \quad y' = y'\,(u, v), \quad z' = z'\,(u, v)\,. \end{aligned}$$

An obvious necessary and sufficient condition for two surfaces to be developable to each other is that the line elements ds and ds', or the expressions:

$$ds^2 = e\,du^2 + 2\,f\,du\,dv + g\,dv^2 \qquad \text{and} \qquad ds'^2 = e'\,du^2 + 2\,f'\,du\,dv + g'\,dv^2,$$

must be equal to each other for every direction, or that:

$$(27)\qquad e = e', \qquad f = f', \qquad g = g'.$$

It follows from this that all equations or quantities that depend upon only e, f, g will remain unchanged under bending.

([1]) **Gauss**, *Disq. gen.*, art. 12 and 13.

As is also geometrically clear, that will be true of the angle between two surface curves [§ **1**, (8)] and the differential equation of the shortest lines [§ **4**, (4)], and therefore those lines themselves. Later on (§ **7**), it will be shown that the curvature k and (§ **15**) the geodetic curvature of corresponding surface curves will have the same values at corresponding points of two surfaces that can be developed to each other.

We shall return to speak of the criterion for the developability of two given surfaces later in § **18**.

§ 6. – Application to central surfaces.

The developments up to now shall now be employed for a brief examination of central surfaces. Every point (u, v) of a given surface C corresponds to two centers of principal curvature that lie along the normal. The *central surface* of C – i.e., the locus of all centers of principal curvature of C – then has two sheets, which correspond to the radii r_1 and r_2 and are then denoted by C_1 and C_2, resp. A normal to C contacts C_1 and C_2 at the associated centers of curvature. The two sheets C_1 and C_2 are then also the loci of the edges of regression of the developable surface A that is associated with the two systems of lines of curvature (1).

For the sake of analytical studies, one chooses the lines of curvature on C to be the parameter curves. From § **5** and [§ **2**, (6)], one will then have the formulas:

$$f = 0, \qquad d' = 0, \qquad ds^2 = e\,du^2 + g\,dv^2, \tag{1}$$

$$\frac{1}{r_1} = \frac{d}{e}, \qquad \frac{1}{r_2} = \frac{d''}{g}, \tag{2}$$

$$\frac{\partial a}{\partial u} = -\frac{1}{r_1}\frac{\partial x}{\partial u}, \qquad \frac{\partial a}{\partial v} = -\frac{1}{r_2}\frac{\partial x}{\partial v}. \tag{3}$$

With their help, when the quantities that relate to the sheet C_1 are distinguished by the subscript 1 and one sets (cf., § **11**):

$$\frac{\sqrt{e}}{r_1} = \sqrt{E}, \qquad \frac{\sqrt{g}}{r_2} = \sqrt{G}, \tag{4}$$

one will get the following formulas by an easy calculation:

$$x_1 = x + r_1\,a, \qquad y_1 = y + r_1\,b, \qquad z_1 = z + r_1\,c, \tag{5}$$

$$a_1 = -\frac{1}{\sqrt{e}}\frac{\partial x}{\partial u}, \qquad b_1 = -\frac{1}{\sqrt{e}}\frac{\partial y}{\partial u}, \qquad c_1 = -\frac{1}{\sqrt{e}}\frac{\partial z}{\partial u}, \tag{6}$$

(1) **Monge-Liouville**, *Applications*, pp. 135, *et seq.*

$$(7) \qquad e_1 = \left(\frac{\partial r_1}{\partial u}\right)^2, \quad f_1 = \frac{\partial r_1}{\partial u}\frac{\partial r_1}{\partial v}, \quad g_1 = \left(\frac{\partial r_1}{\partial v}\right)^2 + G\,(r_2 - r_1)^2,$$

$$(8) \qquad d_1 = \sqrt{E}\,\frac{\partial r_1}{\partial u}, \qquad d_1' = 0, \qquad d_1'' = -\frac{G}{\sqrt{E}}\frac{\partial r_2}{\partial u},$$

$$(9) \qquad ds_1^2 = dr_1^2 + G\,(r_2 - r_1)^2\,dv^2, \qquad \delta_1^2 = G\,(r_2 - r_1)^2\left(\frac{\partial r_1}{\partial u}\right)^2,$$

$$(10) \qquad d_1\,d_1'' - d_1'^2 = -\frac{\partial r_1}{\partial u}\frac{\partial r_1}{\partial v}, \qquad k_1 = -\frac{1}{(r_2 - r_1)^2}\frac{\partial r_2/\partial u}{\partial r_1/\partial u}.$$

That will imply the corresponding formulas for the sheet C_2 when one switches u and v and sets:

$$(11) \qquad x_1,\; y_1,\, z_1, \qquad a_1,\, b_1,\, c_1, \qquad d_1, d_1',\; d_1', \qquad r_1,\, r_2, \quad s_1,\, s_2, \quad e,\, g,\, E,\, G$$

equal to:

$$(12) \qquad x_2,\; y_2,\, z_2, \qquad a,\, b_2,\, c_2, \qquad d_2'',\; d_2',\, d_2, \qquad r_2,\, r_1, \quad s_2,\, s_1, \quad g,\, e,\, G,\, E,$$

resp.

Those equations imply the following **theorems**:

From (6): *The normal to the surface C_1 is parallel to the associated line of curvature at the corresponding point on C. The normals to C_1 and C_2 at corresponding points will then be normal to each other.*

From (8): *The system of curves on C_1* (*or on C_2*) *that corresponds to the two families of lines of curvature on C are conjugates.*

From (9): *The curves* r_1 = const. *on C_1 (which correspond to the lines of constant radius r_1 on C) are geodetic parallels, and the curves* v = const. *(which correspond to one family of lines of curvature on C) are the associated orthogonal geodetics. At the same time, r_1 is the arc length along those geodetics when it is measured from a geodetic parallel.*

If $r_1 = U$, in particular, – i.e., it is a function of only u – then $f_1 = 0$, $d_1' = 0$; i.e., the lines of curvature of C_1 will correspond to the lines of curvature of C itself.

If $r_1 = V$, in particular, – i.e., it is a function of only v – then:

$$e_1 = f_1 = d_1 = d_1' = 0; \qquad k_1 = \infty, \qquad ds_1^2 = g_1\,dv^2;$$

i.e., the surface C_1 will become a curve. The surface C will then be a general channel surface; i.e., the envelope of a sphere with variable radius whose center describes an arbitrary space curve ([1]).

Of especial interest is the central surface of a surface C *whose radii of curvature* r_1 and r_2 are coupled *by an equation* $F(r_1, r_2) = 0$ ([2]), so for them:

$$\frac{\partial r_1}{\partial u}\frac{\partial r_2}{\partial v} - \frac{\partial r_1}{\partial v}\frac{\partial r_2}{\partial u} = 0 \qquad \text{or} \qquad \frac{\partial h}{\partial u}\frac{\partial k}{\partial v} - \frac{\partial h}{\partial v}\frac{\partial k}{\partial u} = 0. \tag{13}$$

For those surfaces, from (10), and due to (13), one has the following relation between the radii of curvature r_1, r_2 of C and the curvatures k_1 and k_2 of C_1 and C_2, resp. ([3]):

$$k_1 k_2 (r_1 - r_2)^4 = 1. \tag{14}$$

The values k_1 and k_2 then have the same signs, or the asymptotic lines on C_1 and C_2 are simultaneously real or imaginary.

Furthermore, from (8), the equations of the asymptotic lines on C_1 and C_2 are:

$$E\frac{\partial r_1}{\partial u}du^2 - G\frac{\partial r_2}{\partial u}dv^2 = 0, \qquad E\frac{\partial r_1}{\partial v}du^2 - G\frac{\partial r_2}{\partial v}dv^2 = 0,$$

resp.

However, those equations are identical, under the assumption in (13). Therefore ([4]):

The asymptotic lines on C_1 and C_2 are corresponding lines for the surfaces that are characterized by (13).

In order to derive a third property of such surfaces ([5]), we infer the formulas:

$$\frac{\partial \log\sqrt{E}}{\partial v} = \frac{1}{r_2 - r_1}\frac{\partial r_1}{\partial v}, \qquad \frac{\partial \log\sqrt{G}}{\partial u} = \frac{1}{r_1 - r_2}\frac{\partial r_2}{\partial u}, \tag{15}$$

from § **9**.

If one expresses r_2 as a function of r_1, and conversely, then upon integrating, one will get:

$$E = \varepsilon \exp\left(2\int\frac{dr_1}{r_2 - r_1}\right), \qquad G = \gamma \exp\left(2\int\frac{dr_2}{r_1 - r_2}\right),$$

(1) **Monge-Liouville**, *Applications*, pp. 238.
(2) **Weingarten**, J. für Math. **59** (1861), pp. 382.
(3) **Halphen**, Bull. Soc. math. Paris **4** (1876), pp. 94.
(4) **Ribaucour**, C. R. Acad. Sc. **74** (1872), pp. 1402.
(5) **Weingarten**, *loc. cit.*, pp. 384.

in which ε is a function of only u, and γ is a function of only v. For a suitable choice of parameters u and v, ε and γ can be made constant, and both of them can be set to 1 ([1]). Under that assumption, the quantities E and G, as well as the quantities e, g, from (4), will then be functions of one of the two radii r_1 or r_2 alone. Equation (9) will then go to:

$$ds_1^2 = dr_1^2 + \Phi(r_1)\, dv^2, \tag{16}$$

in which $\Phi(r_1)$ is a well-defined function of r_1 that depends upon the basic equation $F(r_1, r_2) = 0$. For each such equation, there will exist infinitely many surfaces C; however, from (16), the line element of the central sheet C_1 will be the same for all of those surfaces. One then has the **theorem**:

All of the central sheets C_1 that are associated with the various surfaces that are characterized by the same relation $F(r_1, r_2) = 0$ *can be developed to each other. The same thing will be true of the sheet C_2.*

However, the form (16) of the line element on C_1 is, at the same time, a surface of revolution, when it is referred to its meridian and parallel circles as the parameter curves. That implies the further **theorem**:

All surfaces for which the relation in question $F(r_1, r_2) = 0$ *is true have the property that the associated central sheet C_1 can be developed into one and the same surface of revolution, under which the geodetic lines* $v = \text{const.}$ *of C_1 will go to the meridians of the surface of revolution, and the curves* $r_1 = \text{const.}$ *on C_1 will go to the parallel circles.*

The same theorem will be true for C_2. If $F(r_1, r_2) = 0$ is symmetric in r_1 and r_2 then C_1 and C_2 can be developed to the same surface of revolution, and thus to each other, as well.

In order to ascertain the surface of revolution onto which C_1 can be developed in a given case – i.e., when $F(r_1, r_2) = 0$ is given – let the z-axis of the coordinate system be the rotational axis of the surface, let ρ be the radius of the parallel circle at a distance z from the xy-plane, and let v be the angle between an arbitrary meridian plane and the zero meridian. When the meridian curve is $z = P$ – i.e., when it is equal to a function of ρ – and P' is the derivative of P with respect to ρ, the square of the line element of the surface of revolution will then have the form:

$$ds^2 = (1 + P'^2)\, d\rho^2 + \rho^2\, dv^2. \tag{17}$$

A comparison with (16) will give:

([1]) Namely, one can replace u with a function u_0 of u and v with a function v_0 of v, which will make $E = E_0 \left(\frac{\partial u_0}{\partial u}\right)^2$, $G = G_0 \left(\frac{\partial v_0}{\partial v}\right)^2$. (Cf., § **18**).

(18) $$dr_1 = \sqrt{1+P'^2}\, d\rho\,, \qquad \sqrt{\Phi(r_1)} = \rho,$$

from which, one will get a differential equation for the determination of the function P when one eliminates r_1 .

The theorem above on the developability of C_1 to a surface of revolution has a converse; namely, one can prove the following **theorem** ([1]):

Any surface that can be developed to a surface of revolution can be considered to be one sheet of the central surface of a surface C, between whose radii of principal curvature an equation F (r_1 , r_2) = 0 *exists. (The only exception is defined by ruled surfaces that can be developed to the catenoid.)*

Hence, the problem of finding all surfaces for which a relation $F(r_1, r_2) = 0$ exists will become identical to the problem of finding all bending surfaces of a surface of revolution.

To conclude this discussion, we prove the following **theorem** ([2]):

The lines of curvature of a surface C that is characterized by an equation F (r_1 , r_2) = 0 *can be determined by mere quadratures.*

In order to prove this ([3]), we appeal to a lemma. If one chooses the lines of curvature on C to be the parameter curves (u, v), as usual, then, from (1) and (2), the expression:

(19) $$M = \frac{1}{\delta}\begin{vmatrix} e\,du + f\,dv & d\,du + d'\,dv \\ f\,du + g\,dv & d'\,du + d''\,dv \end{vmatrix},$$

which will give the differential equation for the lines of curvature when it is set equal to 0, will assume the form:

$$M = \sqrt{e\,g}\left(\frac{1}{r_2}-\frac{1}{r_1}\right)du\,dv = \sqrt{E\,G}\;(r_1 - r_2)\,du\,dv = 2\varphi\,du\,dv\,.$$

The expression:

(20) $$\lambda = -\frac{1}{\varphi}\frac{\partial^2 \log\varphi}{\partial u\,\partial v}$$

shall be constructed from the value of φ that is defined in this. If one uses the values:

$$\frac{\partial^2 \log\sqrt{E}}{\partial u\,\partial v} = \frac{1}{r_2 - r_1}\frac{\partial^2 r_1}{\partial u\,\partial v} - \frac{1}{(r_2-r_1)^2}\frac{\partial r_1}{\partial v}\left(\frac{\partial r_2}{\partial u}-\frac{\partial r_1}{\partial u}\right),$$

([1]) **Weingarten**, *loc. cit.*, pp. 384.
([2]) **Lie, Darboux**, Bull. (2) **4** (1880), pp. 300-304.
([3]) **Weingarten**, J. für Math. **103** (1888), pp. 184.

$$\frac{\partial^2 \log \sqrt{G}}{\partial u\,\partial v} = \frac{1}{r_1 - r_2}\frac{\partial^2 r_2}{\partial u\,\partial v} - \frac{1}{(r_1 - r_2)^2}\frac{\partial r_2}{\partial u}\left(\frac{\partial r_1}{\partial v} - \frac{\partial r_2}{\partial v}\right),$$

which are implied by (15), in that equation then one will get the equation for λ:

(21) $$\lambda \sqrt{EG} = 2\,(r_2 - r_1)^{-3}\left(\frac{\partial r_1}{\partial u}\frac{\partial r_2}{\partial v} - \frac{\partial r_1}{\partial v}\frac{\partial r_2}{\partial u}\right).$$

It will then follow from this that for a surface C of the type that was given above, and only for one for which the value λ vanishes, from (20), one will have $\varphi = UV$, where U is a function of only u and V is a function of only v. For a suitable choice of the parameters u and v (cf., the remark on page 28), one can make $\varphi = 1 / 2$, such that one will have:

(22) $$M = du\,dv.$$

Now, let a surface C for which one has $F(r_1, r_2) = 0$ be given in terms of arbitrary parameters (u_0, v_0), and let:

(23) $$M = (A_1\,du_0 + B_1\,dv_0)\,(A_2\,du_0 + B_2\,dv_0)$$

in terms of those parameters, in which A_1, B_1, A_2, B_2 are given functions of (u_0, v_0), then since the expression M in (19) is invariant under a transformation of the parameters (§ **18**), a comparison of (22) and (23), when ρ is an undetermined factor, will give:

(24) $$du = \rho\,(A_1\,du_0 + B_1\,dv_0), \qquad dv = \rho^{-1}\,(A_2\,du_0 + B_2\,dv_0).$$

The integrability conditions for these expressions, namely:

$$\frac{\partial(\rho A_1)}{\partial v_0} = \frac{\partial(\rho B_1)}{\partial u_0}, \qquad \frac{\partial(\rho^{-1} A_2)}{\partial v_0} = \frac{\partial(\rho^{-1} B_2)}{\partial u_0},$$

give the derivatives of ρ with respect to u_0 and v_0. One will then get ρ as a function of u_0, v_0 by a mere quadrature, and then get u, v as functions of u_0, v_0 from (24) by another quadrature. (Q. E. D.)

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CHAPTER II

DERIVING A SURFACE WITH GIVEN PROPERTIES.

§ 7. – The fundamental equations of Gauss and Mainardi.

The six functions e, f, g ; d, d', d'' of (u, v) that were defined in § **1** and **2** define the basis for not only examing the characteristic properties of a given surface, but at the same time, for also deriving a surface from given characteristic properties. Since a surface is already determined by three functions x, y, z of u, v, the six quantities e, f, g ; d, d', d'' cannot be independent of each other. That shows that they are coupled by a system of *three partial differential equations*. One obtains those equations when one exhibits the integrability conditions for the systems of equations (6) and (9) in § **2**.

When one introduces the notations:

$$\text{(1)} \qquad \begin{aligned} &\frac{\partial x}{\partial u} = a_1, \qquad \frac{\partial y}{\partial u} = b_1, \qquad \frac{\partial z}{\partial u} = c_1, \\ &\frac{\partial x}{\partial v} = a_2, \qquad \frac{\partial y}{\partial v} = b_2, \qquad \frac{\partial z}{\partial v} = c_2, \end{aligned}$$

to abbreviate, that system will read:

$$\text{(2)} \qquad \begin{aligned} &\frac{\partial a}{\partial u} - (\rho' a_1 + \sigma' a_2) = 0, &&\frac{\partial a}{\partial v} - (\rho'' a_1 + \sigma'' a_2) = 0, \\ &\frac{\partial a_1}{\partial u} - (d\,a + p\,a_1 + q\,a_2) = 0, &&\frac{\partial a_1}{\partial v} - (d'\,a + p'\,a_1 + q'\,a_2) = 0, \\ &\frac{\partial a_2}{\partial u} - (d'\,a + p'\,a_1 + q'\,a_2) = 0, &&\frac{\partial a_2}{\partial v} - (d''\,a + p''\,a_1 + q''\,a_2) = 0. \end{aligned}$$

If one next sets the two values of $\dfrac{\partial^2 a}{\partial u\,\partial v}$ that one gets from (2) equal to each other and once more expresses the values of $\dfrac{\partial a_1}{\partial u}, \dfrac{\partial a_2}{\partial u}, \dfrac{\partial a_1}{\partial v}, \dfrac{\partial a_2}{\partial v}$ in terms of a, a_1, a_2 then one will get a homogeneous linear equation in a, a_1, a_2 . If one appends the corresponding equations in b, b_1, b_2 and c, c_1, c_2 then one will have three equations of the form:

$$C\,a + C_1\,a_1 + C_2\,a_2 = 0, \qquad C\,b + C_1\,b_1 + C_2\,b_2 = 0, \qquad C\,c + C_1\,c_1 + C_2\,c_2 = 0,$$

with the same coefficients C, C_1, C_2 . Now, since the determinant $|\,a\;a_1\;a_2\,|$ does not vanish, from [§ **1**, (24)], it will follow that:

(3) $C = 0, \qquad C_1 = 0, \qquad C_2 = 0.$

If one treats the following equations (2) likewise – i.e., one sets the two values of $\dfrac{\partial^2 a_1}{\partial u\,\partial v}$ equal to each other – then one will get three equations in the same way:

(4) $A = 0, \qquad A_1 = 0, \qquad A_2 = 0,$

and similarly setting the two values of $\dfrac{\partial^2 a_2}{\partial u\,\partial v}$ equal to each other will give three equations:

(5) $B = 0, \qquad B_1 = 0, \qquad B_2 = 0 .$

As an easy calculation will show, the nine equations (3), (4), (5) reduce to three equations. Namely, $C = 0$ identically. Furthermore, A_1 , A_2 , B_1 , B_2 are equal to each other, up to a factor. Finally, C_1 and C_2 vanish along with A and B, and conversely. *The integrability conditions of the system* (2) *then consist of three equations.* The *first two* read:

(6)
$$\frac{\partial d}{\partial v} - \frac{\partial d'}{\partial u} = p'\,d + (q' - p)\,d' - q\,d'' \qquad \text{(from } A = 0\text{)},$$
$$\frac{\partial d''}{\partial u} - \frac{\partial d'}{\partial v} = q'\,d'' + (p' - q'')\,d' - p''\,d \qquad \text{(from } B = 0\text{)}.$$

By introducing the quantities:

(7) $$\frac{d}{\delta} = t, \qquad \frac{d'}{\delta} = t', \qquad \frac{d''}{\delta} = t'',$$

they will take on the form:

(8)
$$\frac{\partial t}{\partial v} - \frac{\partial t'}{\partial u} = 2\,q'\,t'' - q\,t'' - q''\,t,$$
$$\frac{\partial t''}{\partial u} - \frac{\partial t'}{\partial v} = 2\,p'\,t' - p''\,t - p\,t''.$$

One can also get equations (6) when one differentiates equations [§ **2**, (1)] with respect to u and v and employs equations [§ **2**, (6)], along with [§ **1**, (7)]. One will then get:

$$A_{30} = \sum a\,\frac{\partial^3 x}{\partial u^3} \qquad = \frac{\partial d}{\partial u} + p\,d + q\,d',$$

$$
(9)\qquad
\begin{aligned}
A_{21} &= \sum a \frac{\partial^3 x}{\partial u^2\,\partial v} = \frac{\partial d'}{\partial u} + p'\,d + q'\,d' = \frac{\partial d}{\partial v} + p\,d' + q\,d'',\\
A_{12} &= \sum a \frac{\partial^3 x}{\partial u\,\partial v^2} = \frac{\partial d''}{\partial u} + p''\,d + q''\,d' = \frac{\partial d'}{\partial v} + p'\,d' + q'\,d'',\\
A_{03} &= \sum a \frac{\partial^3 x}{\partial v^3} = \frac{\partial d''}{\partial v} + p''\,d' + q''\,d''.
\end{aligned}
$$

Hence, the derivatives of d, d', d'' with respect to u and v can be expressed in terms of the four sums A_{30}, A_{21}, A_{12}, A_{03}, as well as the six quantities e, f, g ; d, d', d'', and their derivatives with respect to u, v.

As the *third integrability condition*, one gets:

$$
(10)\qquad
\begin{aligned}
\frac{d\,d'' - d'^2}{eg - f^2} &= \frac{1}{f}\left(\frac{\partial p'}{\partial u} - \frac{\partial p}{\partial v} + p'q' - p''q\right) && \text{(from } A_1 = 0\text{)},\\
&= \frac{1}{f}\left(\frac{\partial q'}{\partial v} - \frac{\partial q''}{\partial u} + p'q' - p''q\right) && \text{(from } B_2 = 0\text{)},\\
&= \frac{1}{f}\left(\frac{\partial q}{\partial v} - \frac{\partial q'}{\partial u} + pq' - p'q + q\,q'' - q'^2\right) && \text{(from } A_2 = 0\text{)},\\
&= \frac{1}{f}\left(\frac{\partial p''}{\partial u} - \frac{\partial p'}{\partial v} + q''p' - q'p'' + p\,p'' - p'^2\right) && \text{(from } B_1 = 0\text{)}.
\end{aligned}
$$

Gauss ([1]) already presented this equation by a different derivation; he obtained the **theorem**:

The ***curvature*** *k of the surface* [cf., § **5**, (11)] *can be represented in terms of the coefficients e, f, g of the line element alone, so it will remain unchanged under a bending of the surface.*

Equations (6) define an essential extension of **Gauss**'s system of formulas. They were probably first given by **Mainardi** ([2]), but in a different form, although they are often referred to as the **Codazzi** equations ([3]). They were developed several times later ([4]) in the simpler form (6). Moreover, they appeared before **Mainardi** in **Lamé**'s study of triply-orthogonal systems of surfaces (cf., § **10**) in terms of special parameters.

([1]) **Gauss**, *Disq. gen.*, art. 11 and 12.

([2]) **Mainardi**, Giornale dell' Istituto Lombardo **9** (1856), pp. 395. Cf., **Knoblauch**'s remark in Jour. für Math., Bd. **103**, pp. 31.

([3]) **Codazzi**, Ann. di mat. **2** (1868), pp. 273.

([4]) **Knoblauch** gave equations (9) in *loc. cit.*, pp. 32.

The expression (10) for the *curvature* $k = 1 : r_1\, r_2$ can be put into yet another form. First, it will imply the simple representation ([1]):

$$(11)\qquad \frac{\delta}{r_1 r_2} = \frac{\partial}{\partial v}\left(\frac{\delta q}{e}\right) - \frac{\partial}{\partial u}\left(\frac{\delta q'}{e}\right) = \frac{\partial}{\partial u}\left(\frac{\delta p''}{g}\right) - \frac{\partial}{\partial v}\left(\frac{\delta p'}{g}\right).$$

In fact, if one recalls [§ **1**, (7a)] then one will get:

$$\frac{1}{e}\left(q'\frac{\partial e}{\partial u} - q\frac{\partial e}{\partial v}\right) = \frac{1}{2\delta^2}\left[\left(\frac{\partial e}{\partial v}\right)^2 + \frac{\partial e}{\partial u}\frac{\partial g}{\partial u} - 2\frac{\partial e}{\partial v}\frac{\partial f}{\partial u}\right] = 2\,(q'p - q\,p'),$$

which proves equation (11). If one considers [§ **1**, (14)] then it will further follow from (11) that:

$$(12)\qquad \frac{\delta}{r_1 r_2} = -\frac{\partial^2\omega}{\partial u\,\partial v} - \frac{\partial}{\partial u}\left(\frac{\delta q'}{e}\right) - \frac{\partial}{\partial v}\left(\frac{\delta p'}{g}\right) = +\frac{\partial^2\omega}{\partial u\,\partial v} + \frac{\partial}{\partial u}\left(\frac{\delta p''}{g}\right) + \frac{\partial}{\partial v}\left(\frac{\delta q'}{e}\right).$$

Since one further has:

$$\frac{\delta q'}{e} = \frac{1}{\delta e}\left[\frac{e}{2}\frac{\partial g}{\partial u} - \frac{f}{2}\frac{\partial e}{\partial v}\right] = \frac{1}{\sqrt{e}\sin\omega}\left[\frac{\partial\sqrt{g}}{\partial u} - \cos\omega\frac{\partial\sqrt{e}}{\partial v}\right],$$

$$\frac{\delta p'}{g} = \frac{1}{\delta g}\left[\frac{g}{2}\frac{\partial e}{\partial v} - \frac{f}{2}\frac{\partial g}{\partial u}\right] = \frac{1}{\sqrt{g}\sin\omega}\left[\frac{\partial\sqrt{e}}{\partial v} - \cos\omega\frac{\partial\sqrt{g}}{\partial u}\right],$$

one will then have ([2]):

$$(13)\qquad \frac{\delta}{r_1 r_2} = -\frac{\partial^2\omega}{\partial u\,\partial v} - \frac{\partial}{\partial u}\left[\frac{\dfrac{\partial\sqrt{g}}{\partial u} - \cos\omega\dfrac{\partial\sqrt{e}}{\partial v}}{\sqrt{e}\,\sin\omega}\right] - \frac{\partial}{\partial v}\left[\frac{\dfrac{\partial\sqrt{e}}{\partial v} - \cos\omega\dfrac{\partial\sqrt{g}}{\partial u}}{\sqrt{g}\,\sin\omega}\right],$$

instead of (12).

The formulas for the curvature that one obtains by introducing the geodetic curvature of the parameter curves have a more limited meaning. If one denotes the geodetic curvatures of v = const. and u = const. by $1 / \zeta_1$ and $1 / \zeta_2$, resp., then [cf., § **14**, (13)]:

$$\frac{\delta q}{e} = \frac{\sqrt{e}}{\zeta_1}, \qquad \frac{\delta p''}{g} = -\frac{\sqrt{g}}{\zeta_2},$$

([1]) When this calculation is performed, that will give **Liouville**'s formula, Jour. de Math. **16** (1851), pp. 131.

([2]) **Liouville**, *loc. cit.*

It will then follow from (12) that ([1]):

$$\frac{\delta}{r_1 r_2} = + \frac{\partial^2 \omega}{\partial u\,\partial v} + \frac{\partial}{\partial v}\left(\frac{\sqrt{e}}{\zeta_1}\right) - \frac{\partial}{\partial u}\left(\frac{\sqrt{g}}{\zeta_2}\right). \tag{14}$$

Finally, one has:

$$f = 0, \qquad \omega = 90^{\circ}, \qquad \delta = \sqrt{eg}\,, \qquad \frac{1}{\zeta_1} = -\frac{1}{\sqrt{eg}}\frac{\partial\sqrt{e}}{\partial v}, \qquad \frac{1}{\zeta_2} = +\frac{1}{\sqrt{eg}}\frac{\partial\sqrt{g}}{\partial u}$$

for the *rectangular* parameter curves. It will then follow from (14) that ([2]):

$$\frac{\delta}{r_1 r_2} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial v}\left(\frac{1}{\zeta_1}\right) - \frac{1}{\sqrt{e}}\frac{\partial}{\partial u}\left(\frac{1}{\zeta_2}\right) - \frac{1}{\zeta_1^2} - \frac{1}{\zeta_2^2}. \tag{15}$$

§ 8. – Bonnet's theorem.

From § **7**, the six fundamental functions e, f, g ; d, d', d'' of (u, v) for any surface satisfy three partial differential equations. Now, one has the very important converse **theorem** ([3]):

A surface is determined uniquely (up to its position in space and a reflection in a plane) when six functions e, f, g ; d, d', d'' exist that satisfy the three partial differential equations (6) *and* (10) *of* § **7**.

The proof rests upon a detailed examination of the integral of the system [§ **7**, (2)], whose integrability conditions are contained in equations (6) and (10) of § **7**.

To abbreviate, we denote the left-hand sides of the equations in the first column in [§**7**, (2)] by X_{11}, X_{12}, X_{13}, and those of the second column by X_{21}, X_{22}, X_{23}.

It is known that there are infinitely many systems of values a, a_1, a_2 that satisfy the three equations $X_{11} = 0$, $X_{12} = 0$, $X_{13} = 0$, which is also true for the quantities e, f, g ; d, d', d'' or the $\rho', \sigma', p, q, p', q'$ that are derived from them. Let (x_0, x_1, x_2), (y_0, y_1, y_2), (z_0, z_1, z_2) be three linearly-independent systems of values a, a_1, a_2 of that kind whose determinant does not vanish then. The most general system of values that satisfies equations [§ **7**, (2)] will then have the form:

$$a = \xi_0\, x_0 + \eta_0\, y_0 + \zeta_0\, z_0\,, \qquad a_1 = \xi_0\, x_1 + \eta_0\, y_1 + \zeta_0\, z_1\,, \qquad a_2 = \xi_0\, x_2 + \eta_0\, y_2 + \zeta_0\, z_2\,, \tag{1}$$

([1]) **Codazzi**, Ann. di mat. **2** (1868), pp. 271, eq. (54).

([2]) **Bonnet**, Journ. Ec. Poly., Cah. **32** (1848), pp. 53.

([3]) **Bonnet**, Journ. Ec. Poly., Cah. **42** (1867), pp. 33, *et seq.*, in which the proof is carried out under the assumption of rectangular parameter curves. Lipschitz gave another proof of **Bonnet**'s theorem that was not as simple in Sitzungsberichte d. Berl. Acad. (1883), pp. 541, *et seq.*

in which ξ_0 , η_0 , ζ_0 are more *arbitrary functions of v*. Should those values of a, a_1, a_2 also satisfy the three equations $X_{21} = 0$, $X_{22} = 0$, $X_{23} = 0$, then one would have the following equations for the determination of ξ_0 , η_0 , ζ_0 :

$$(2) \qquad \begin{aligned} x_0 \frac{\partial \xi_0}{\partial v} + y_0 \frac{\partial \eta_0}{\partial v} + z_0 \frac{\partial \zeta_0}{\partial v} &= -(\xi_0 X_{21} + \eta_0 Y_{21} + \zeta_0 Z_{21}) , \\ x_1 \frac{\partial \xi_0}{\partial v} + y_1 \frac{\partial \eta_0}{\partial v} + z_1 \frac{\partial \zeta_0}{\partial v} &= -(\xi_0 X_{22} + \eta_0 Y_{22} + \zeta_0 Z_{22}) , \\ x_2 \frac{\partial \xi_0}{\partial v} + y_2 \frac{\partial \eta_0}{\partial v} + z_2 \frac{\partial \zeta_0}{\partial v} &= -(\xi_0 X_{23} + \eta_0 Y_{23} + \zeta_0 Z_{23}) . \end{aligned}$$

In order for the solutions ξ_0 , η_0 , ζ_0 of this system to be functions of v alone, the system must be equivalent to the system that is obtained from the latter by differentiating with respect to u, in which ξ_0 , η_0 , ζ_0 and their derivatives with respect to v are considered to be constants; i.e., it will then be equivalent to the system:

$$(3) \qquad \begin{aligned} \frac{\partial x_0}{\partial u}\frac{\partial \xi_0}{\partial v} + \frac{\partial y_0}{\partial u}\frac{\partial \eta_0}{\partial v} + \frac{\partial z_0}{\partial u}\frac{\partial \zeta_0}{\partial v} &= -\left(\xi_0 \frac{\partial X_{21}}{\partial u} + \eta_0 \frac{\partial Y_{21}}{\partial u} + \zeta_0 \frac{\partial Z_{21}}{\partial u}\right), \\ \frac{\partial x_1}{\partial u}\frac{\partial \xi_0}{\partial v} + \frac{\partial y_1}{\partial u}\frac{\partial \eta_0}{\partial v} + \frac{\partial z_1}{\partial u}\frac{\partial \zeta_0}{\partial v} &= -\left(\xi_0 \frac{\partial X_{22}}{\partial u} + \eta_0 \frac{\partial Y_{22}}{\partial u} + \zeta_0 \frac{\partial Z_{22}}{\partial u}\right), \\ \frac{\partial x_2}{\partial u}\frac{\partial \xi_0}{\partial v} + \frac{\partial y_2}{\partial u}\frac{\partial \eta_0}{\partial v} + \frac{\partial z_2}{\partial u}\frac{\partial \zeta_0}{\partial v} &= -\left(\xi_0 \frac{\partial X_{23}}{\partial u} + \eta_0 \frac{\partial Y_{23}}{\partial u} + \zeta_0 \frac{\partial Z_{23}}{\partial u}\right). \end{aligned}$$

That is, in fact, the case. If one then multiplies the second and third of equations (2) by ρ' and σ', resp., and adds them then one will get precisely the first of equations (3) with the help of [§ **7**, (2)] and under the assumption of equations (6) and (10) in § **7**. In the same way, and under the same assumptions, one will get the second and third of equations (3) when one multiplies the three equations (2) by d, p, q or d', p', q', resp., and adds them.

It follows from this that under the assumption that equations (6) and (10) of § **7** are valid, infinitely many systems of three quantities a, a_1, a_2 can be determined that satisfy the *entire simultaneous system* [§ **7**, (2)]. If one then understands (x_0 , x_1 , x_2), (y_0 , y_1 , y_2), (z_0 , z_1 , z_2) to mean three linearly-independent systems of values *of that kind*, moreover, then the most general system of values that satisfies equations [§ **7**, (2)] and the corresponding equations that one forms in (b, b_1, b_2) and (c, c_1, c_2) will have the form:

$$(4) \qquad \begin{aligned} a &= \xi_0 x_0 + \eta_0 y_0 + \zeta_0 z_0 , \quad & a_1 &= \xi_0 x_1 + \eta_0 y_1 + \zeta_0 z_1 , \quad & a_2 &= \xi_0 x_2 + \eta_0 y_2 + \zeta_0 z_2 , \\ b &= \xi_1 x_0 + \eta_1 y_0 + \zeta_1 z_0 , \quad & b_1 &= \xi_1 x_1 + \eta_1 y_1 + \zeta_1 z_1 , \quad & b_2 &= \xi_1 x_2 + \eta_1 y_2 + \zeta_1 z_2 , \end{aligned}$$

$$c = \xi_2\, x_0 + \eta_2\, y_0 + \zeta_2\, z_0\,, \qquad c_1 = \xi_2\, x_1 + \eta_2\, y_1 + \zeta_2\, z_1\,, \qquad c_2 = \xi_2\, x_2 + \eta_2\, y_2 + \zeta_2\, z_2\,,$$

in which the nine coefficients ξ_0, η_0, ζ_0 ; ξ_1, η_1, ζ_1 ; ξ_2, η_2, ζ_2 are now *constants;* i.e., independent of u, v.

Six equations exist between these nine coefficients, as the following consideration will show: From equations (19), in conjunction with (9) and (22) of § **1**, one will have the following relations for the nine quantities a, a_1, a_2 ; b, b_1, b_2 ; c, c_1, c_2 :

$$(5)\qquad \begin{aligned} a^2 + \frac{1}{\delta^2}(g\,a_1^2 + e\,a_2^2 - 2f\,a_1\,a_2) &= 1, \\ ab + \frac{1}{\delta^2}[g\,a_1\,b_1 + e\,a_2\,b_2 - f\,(a_1\,b_2 + b_1\,a_2)] &= 0, \end{aligned}$$

along with two other pairs that emerge from them by cyclic permutation of a, b, c.

If one correspondingly defines the following quantities, which correspond to the left-hand sides in (5):

$$(6)\qquad \begin{aligned} M_{11} &= x_0^2 + \frac{1}{\delta^2}(g\,x_1^2 + e\,x_2^2 - 2f\,x_1\,x_2)\,, \\ M_{12} &= x_0\,y_0 + \frac{1}{\delta^2}[g\,x_1\,y_1 + e\,x_2\,y_2 - f\,(x_1\,y_2 + y_1\,x_2)], \end{aligned}$$

along with the quantities M_{22}, M_{33} and M_{23}, M_{31} that emerge from them by cyclic permutation of x, y, z, then those six constants M_{ik} will also be constant – i.e., independent of u and v – since their partial derivatives with respect to u and v will vanish, due to [§ **7**, (2)]. If one now substitutes the values (4) into equations (5) then, from (6), one will get:

$$(7)\qquad \begin{aligned} &M_{11}\,\xi_i^2 + M_{22}\,\eta_i^2 + M_{33}\,\zeta_i^2 + 2M_{23}\,\eta_i\,\zeta_i + 2M_{31}\,\zeta_i\,\xi_i + 2M_{12}\,\xi_i\,\eta_i = 1, \qquad (i = 0, 1, 2), \\ &M_{11}\,\xi_i\,\xi_k + M_{22}\,\eta_i\,\eta_k + M_{33}\,\zeta_i\,\zeta_i \\ &\quad + M_{23}\,(\eta_i\,\zeta_k + \zeta_i\,\eta_k) + M_{31}\,(\zeta_i\,\xi_k + \xi_i\,\zeta_k) + M_{12}\,(\xi_i\,\eta_k + \eta_i\,\xi_k) = 0 \quad (i, k = 0, 1; 1, 2; 2, 3). \end{aligned}$$

Those equations express the fact that the nine coefficients in (4) – namely, ξ_0, η_0, ζ_0 ; ξ_1, η_1, ζ_1 ; ξ_2, η_2, ζ_2 – are the coordinates of the endpoints of three conjugate diameters of a fixed second-order surface:

$$(8)\qquad M_{11}\,\xi^2 + M_{22}\,\eta^2 + M_{33}\,\zeta^2 + 2M_{23}\,\eta\,\zeta + 2M_{31}\,\zeta\,\xi + 2M_{12}\,\xi\,\eta = 1.$$

That surface is a midpoint surface, so its conjugate diameters are all finite, since the determinant:

$$\begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix} = \frac{1}{\delta^4} \begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{vmatrix} \cdot \begin{vmatrix} x_0 & x_1\,g - x_2\,f & x_2\,e - x_1\,f \\ y_0 & y_1\,g - y_2\,f & y_2\,e - y_1\,f \\ z_0 & z_1 g - z_2\,f & z_2\,e - z_1\,f \end{vmatrix} = \frac{1}{\delta^2} \begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{vmatrix}^2$$

is positive, from (6).

Therefore, every triple of conjugate diameters will correspond to a well-defined system of nine constants in (4). All that remains to be shown is how the various solutions (4) that are obtained in that way behave geometrically with respect to each other for a different choice of conjugate diameter.

If one inverts the direction of one of the three conjugate diameters then the sign of the coordinates of one of its endpoints will change (thus, e.g., the constants ξ_0, η_0, ζ_0, and from (4), a, a_1, a_2, as well, or from [§ **1**, (22)], α, α_1, α_2); i.e., the surface experiences a reflection in the *xy*-plane.

By contrast, if one chooses a second triple of conjugate diameters ξ', η', ζ', instead of the first one ξ, η, ζ, which is coupled to the first triple by the linear equations:

$$\begin{aligned} &\xi'_0 = \xi_0\,\lambda_0 + \xi_1\,\mu_0 + \xi_2\,\nu_0, \quad \xi'_1 = \xi_0\,\lambda_1 + \xi_1\,\mu_1 + \xi_2\,\nu_1, \quad \xi'_2 = \xi_0\,\lambda_2 + \xi_1\,\mu_2 + \xi_2\,\nu_2, \\ (9)\quad &\eta'_0 = \eta_0\,\lambda_0 + \eta_1\,\mu_0 + \eta_2\,\nu_0, \quad \eta'_1 = \xi_0\,\lambda_1 + \xi_1\,\mu_1 + \xi_2\,\nu_1, \quad \eta'_2 = \xi_0\,\lambda_2 + \xi_1\,\mu_2 + \xi_2\,\nu_2, \\ &\zeta'_0 = \zeta_0\,\lambda_0 + \zeta_1\,\mu_0 + \zeta_2\,\nu_0, \quad \zeta'_1 = \zeta_0\,\lambda_1 + \zeta_1\,\mu_1 + \zeta_2\,\nu_1, \quad \zeta'_2 = \zeta_0\,\lambda_2 + \zeta_1\,\mu_2 + \zeta_2\,\nu_2, \end{aligned}$$

then equations (7), when defined in terms of the triple ξ', η', ζ', will imply that the transformation coefficients λ_0, μ_0, ν_0; λ_1, μ_1, ν_1; λ_2, μ_2, ν_2 in (9) must satisfy the same conditions as the coefficients of an orthogonal substitution.

When one introduces ξ', η', ζ', instead of ξ, η, ζ, the values a, b, c in (4) will go to the new values a', b', c', which are coupled to the original ones by the equations:

$$\begin{aligned} &a' = a\,\lambda_0 + b\,\mu_0 + c\,\nu_0, \qquad b' = a\,\lambda_1 + b\,\mu_1 + c\,\nu_1, \qquad c' = a\,\lambda_2 + b\,\mu_2 + c\,\nu_2, \\ (10)\qquad &a'_1 = a_1\lambda_0 + b_1\mu_0 + c_1\nu_0, \qquad b'_1 = a_1\lambda_1 + b_1\mu_1 + c_1\nu_1, \qquad c'_1 = a_1\lambda_2 + b_1\mu_2 + c_1\nu_2, \\ &a'_2 = a_2\lambda_0 + b_2\mu_0 + c_2\nu_0, \qquad b'_2 = a_2\lambda_1 + b_2\mu_1 + c_2\nu_1, \qquad c'_2 = a_2\lambda_2 + b_2\mu_2 + c_2\nu_2. \end{aligned}$$

However, one will get precisely equations (10) when one subjects the coordinates of a point (x, y, z) of the surface to an orthogonal transformation with the same coefficients λ, μ, ν. Thus, the choice of a new triple will correspond to a rotation of the coordinate system or the surface around the origin.

With that, the theorem that was posed is proved. That theorem contains *a general and systematic method for deriving the equation of a surface that is defined by its characteristic properties.* The problem splits into two parts ([1]):

([1]) **Bour**, Journ. Ec. Poly., Cah. **39** (1862), pp. 23.

First of all, the six fundamental quantities e, f, g ; d, d', d'' must be ascertained as functions of u, v. In order to do that, one chooses a well-defined system of parameters u, v that are suited to the problem, from which, two of the six quantities can be determined or two relations between them can be established. A third relation will give the property that characterizes the surface. One then comes to the three differential equations (6) and (10) in § **7**. One will then have six equations that suffice to determine the six fundamental quantities.

Secondly, the point coordinates x, y, z must be represented as functions of u, v. That will come about when one integrates the system [§ **7**, (2)] in the way that was given above using equations (4). If one then gives the constants ξ, η, ζ arbitrary values that are compatible with (7) then one will get:

$$x = \int (a_1\,du + a_2\,dv), \quad y = \int (b_1\,du + b_2\,dv), \quad z = \int (c_1\,du + c_2\,dv) \tag{11}$$

from equations [§ **7**, (1)], so one will get x, y, z by performing quadratures, which will introduce three arbitrary additive constants. With that, one will have the equation of the surface for a well-defined position in space.

In § **9**, we shall give a derivation of the equation of a minimal surface by that method.

§ 9. – Applications. Differential equations of certain surfaces.

For some applications, we shall briefly summarize the most important formulas of § **7** for special parameter curves.

1. *Let the parameter curves* (u, v) *be minimal lines.*

From (11) and (25) of § **5**, and from (6) and (10) of § **7**, one will then have:

$$e = g = 0, \qquad ds^2 = 2f\,du\,dv,$$

$$h = \frac{2d'}{f}, \qquad k = -\frac{d\,d'' - d'^2}{f^2} = -\frac{1}{f}\frac{\partial^2 \log f}{\partial u\,\partial v}, \tag{1}$$

$$\frac{1}{d'}\frac{\partial d}{\partial v} = \frac{\partial \log}{\partial u}\left(\frac{d'}{f}\right), \qquad \frac{1}{d'}\frac{\partial d''}{\partial u} = \frac{\partial \log}{\partial v}\left(\frac{d'}{f}\right).$$

2. *Let the parameter curves* (u, v) *be isometric lines.*

$$e = g = \lambda, \qquad f = 0, \quad ds^2 = \lambda\,(du^2 + dv^2),$$

$$h = \frac{d + d''}{\lambda}, \quad k = -\frac{d\,d'' - d'^2}{\lambda^2} = -\frac{1}{2\lambda}\left(\frac{\partial^2 \log f}{\partial u^2} + \frac{\partial^2 \log f}{\partial v^2}\right), \tag{2}$$

$$\frac{\partial d''}{\partial u} - \frac{\partial d'}{\partial v} = \tfrac{1}{2}(d + d'')\frac{\partial \log \lambda}{\partial u}, \qquad \frac{\partial d}{\partial v} - \frac{\partial d'}{\partial u} = \tfrac{1}{2}(d + d'')\frac{\partial \log \lambda}{\partial v}.$$

3. *Let the parameter curves* (u, v) *be asymptotic lines.*

(3)
$$h = -\frac{2 f t'}{\delta}, \qquad -\delta k = +\delta t'^2 = \frac{\partial^2 \omega}{\partial u\, \partial v} + \frac{\partial}{\partial u}\left(\frac{\delta q'}{e}\right) + \frac{\partial}{\partial v}\left(\frac{\delta p'}{g}\right),$$
$$\frac{\partial \log t'}{\partial u} = -2q', \qquad \frac{\partial \log t'}{\partial v} = -2p'.$$

4. *Let the parameters* (u, v) *be orthogonal geodesic coordinates.*

One will then have:

$$e = 1, \quad f = 0, \quad ds^2 = du^2 + g\, dv^2,$$

(4)
$$h = d + \frac{d''}{g}, \qquad k = \frac{d\, d'' - d'^2}{\lambda^2} = -\frac{1}{\sqrt{g}}\frac{\partial^2 \sqrt{g}}{\partial u^2}.$$

5. *Let the parameter curves* (u, v) *be the lines of curvature.*

One will then have (cf., § **6**) ([1]):

$$f = 0, \quad d' = 0, \quad ds^2 = e\, du^2 + g\, dv^2,$$

(5)
$$\sqrt{E} = \frac{d}{\sqrt{e}} = \frac{\sqrt{e}}{r_1}, \qquad \sqrt{G} = \frac{d''}{\sqrt{g}} = \frac{\sqrt{g}}{r_2}, \quad F = 0,$$
$$\frac{\partial a}{\partial u} = -\frac{1}{r_1}\frac{\partial x}{\partial u}, \qquad \frac{\partial a}{\partial v} = -\frac{1}{r_2}\frac{\partial x}{\partial v}.$$

Furthermore, the **Mainardi** equations [§ **7**, (6)] read:

(6)
$$\frac{\partial d}{\partial v} = \frac{1}{2}\left(\frac{d}{e} + \frac{d''}{g}\right)\frac{\partial e}{\partial v}, \qquad \frac{\partial d''}{\partial u} = \frac{1}{2}\left(\frac{d}{e} + \frac{d''}{g}\right)\frac{\partial g}{\partial u}.$$

When one introduces r_1, r_2 in place of d, d'', they can be put into the form ([2]):

([1]) The quantities E, F, G will be defined more precisely in § **11**.
([2]) **Enneper**, Zeit. Math. Phys. **7** (1862), pp. 89.

$$\frac{\partial}{\partial v}\left(\frac{e}{r_1}\right)=\frac{1}{2}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\frac{\partial e}{\partial v}, \qquad \frac{\partial}{\partial v}\left(\frac{g}{r_2}\right)=\frac{1}{2}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\frac{\partial g}{\partial u},$$

or

$$\frac{\partial \log\sqrt{e}}{\partial v}=\frac{r_1 r_2}{r_1-r_2}\frac{\partial}{\partial v}\left(\frac{1}{r_1}\right), \qquad \frac{\partial \log\sqrt{g}}{\partial u}=\frac{r_1 r_2}{r_2-r_1}\frac{\partial}{\partial v}\left(\frac{1}{r_2}\right).$$

When one introduces E, G, and recalls (5), it will then follow that:

$$\frac{\partial \log\sqrt{E}}{\partial v}=\frac{1}{r_2-r_1}\frac{\partial r_1}{\partial v}, \qquad \frac{\partial \log\sqrt{G}}{\partial u}=\frac{1}{r_1-r_2}\frac{\partial r_2}{\partial u} \tag{7}$$

and

$$\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}=\frac{1}{\sqrt{g}}\frac{\partial\sqrt{e}}{\partial v}, \qquad \frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}=\frac{1}{\sqrt{e}}\frac{\partial\sqrt{g}}{\partial u}. \tag{8}$$

Finally, **Gauss**'s equation [§ **7**, (10)] will take on the form:

$$\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{e}}\frac{\partial\sqrt{g}}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{g}}\frac{\partial\sqrt{e}}{\partial v}\right)=-\frac{d\,d''}{\sqrt{eg}}=-\frac{\sqrt{eg}}{r_1 r_2} \tag{9}$$

or, from (8):

$$\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)+\sqrt{EG}=0, \tag{10}$$

which one also infers when one forms equation (9) for the sphere, for which $r_1 r_2 = 1$.

The *first* application is defined by the derivation of the differential equations for surfaces of constant curvature k and constant mean curvature h in various parameter systems.

When a surface possesses *constant negative curvature* $k = -1 : \mu^2$, the asymptotic lines will be real. If one chooses them to be the parameter curves and denotes the angle between them by ω then, from (3), one will have $t = t'' = 0$; $t'^2 = 1 : \mu^2$. As a result, $p' = q' = 0$, and from [§ **1**, (7)], $m' = n' = 0$. Hence, for a suitable choice ([1]) of parameters (u, v), from [§ **1**, (6)], one will have:

$$e = g = 1, \qquad f = \cos\omega, \qquad \delta = \sin\omega, \qquad d = d'' = 0, \qquad d' = \frac{\sin\omega}{\mu}. \tag{11}$$

Finally, from (3), one can determine ω or represent the *differential equation of the surface in the parameters* (u, v) *of the asymptotic lines* with ([2]):

([1]) Cf., the remark on page 28.
([2]) **Hazzidakis**, Jour. f. Math. **88** (1878), pp. 68.

$$\frac{\partial^2 \omega}{\partial u\,\partial v} = \frac{\sin\omega}{\mu}. \tag{12}$$

When that equation is solved, one will know the fundamental quantities e, f, g ; d, d', d''. However, from the theorem in § **8**, the surface will be determined uniquely with that.

From (11), the differential equation of the lines of curvature of the surface will be: $du^2 - dv^2 = 0$. If one then sets $u + v = u_1$; $u - v = v_1$, one will get the *differential equation for the surface in terms of the parameters* (u_1, v_1) *of the lines of curvature* from (12):

$$\frac{\partial^2 \omega}{\partial u_1^2} - \frac{\partial^2 \omega}{\partial v_1^2} = \frac{\sin\omega}{\mu^2}. \tag{13}$$

This equation can also be derived easily from equations (5) to (10).

Should a surface possess *constant mean curvature* $h = 2 / \mu$, and should one next choose the *minimal lines to be the parameter curves* then, from (1), one would get: $e = g = 0$; $d' = f : \mu$. Moreover, $\frac{\partial d}{\partial v} = 0$, $\frac{\partial d''}{\partial u} = 0$, so for a suitable choice of the parameters (u, v): $d = d'' = 1$, and one will get the *equation for f* or the *differential equation of the surface:*

$$\frac{\partial^2 \log f}{\partial u\,\partial v} = \frac{1}{f} - \frac{f}{\mu^2}. \tag{14}$$

By contrast, if one chooses the *lines of curvature to be the parameter curves* (u, v) then one will have $f = d' = 0$ and $\frac{d}{e} + \frac{d''}{g} = \frac{2}{\mu}$, so as a result, from (6), $\frac{\partial d}{\partial v} = \frac{1}{\mu}\frac{\partial e}{\partial v}$; $\frac{\partial d''}{\partial u} = \frac{1}{\mu}\frac{\partial g}{\partial u}$. For a suitable choice of u, v, one will then have: $d = e / \mu + 1$; $d'' = g / \mu - 1$, so $e = g$. That will give the theorem:

The surfaces of constant mean curvature (especially the minimal surfaces $\mu = \infty$*) possess isometric lines of curvature* ([1]).

Moreover, the fundamental quantities are:

$$e = g = \lambda, \qquad f = d' = 0, \qquad d = \frac{1}{\mu} + 1, \qquad d'' = \frac{1}{\mu} - 1, \tag{15}$$

and it will follow from (9) that the equation that determines λ, or *the differential equation of the surface* will be:

([1]) **Bonnet**, C. R. Acad. Sc. **37** (1853), pp. 529 and Jour. d. Math. **5** (1860), pp. 221.

(16)
$$\frac{\partial^2 \log\lambda}{\partial u^2}+\frac{\partial^2 \log\lambda}{\partial v^2}= 2\left(\frac{1}{\lambda}-\frac{1}{\mu^2}\right).$$

When one makes the substitutions $\lambda = \mu\, e^{i\theta}$, $u = u_1$, $v = i\, v_1$, that will go to an equation for θ as a function of u_1, v_1, namely:

(17)
$$\frac{\partial^2\theta}{\partial u_1^2}-\frac{\partial^2\theta}{\partial v_1^2}=-\frac{4\sin\theta}{\mu}.$$

The formal agreement between equations (13) and (17) shows that the derivation of surfaces of constant curvature and the derivation of surfaces of constant mean curvature are essentially the same problem. In fact, there exists a very simple geometric connection between both types of surface that is expressed by the **theorem** ([1]):

The parallel surface to a surface of constant curvature $1 : \mu^2$ *that is at a distance of* μ *is a surface of constant mean curvature* $1 : \mu$ *.*

If x, y, z are the coordinates of a surface in the parameters u, v then the coordinates x', y', z' of the parallel surface at a distance μ will be:

(18)
$$x' = x - \mu\, a, \quad y' = y - \mu\, b, \quad z' = z - \mu\, c\,.$$

If one defines the fundamental quantities for this surface then one will get the values $r_1' = r_1 + \mu$, $r_2' = r_2 + \mu$ for the radii of principal curvature, which would also follow geometrically from the fact that both surfaces possess the same normals, so they will have corresponding lines of curvature. Therefore, the mean curvature h' of the parallel surface will be:

(19)
$$h' = \frac{h+2\mu k}{1+\mu\, h+\mu^2 k}.$$

Now, $k = 1 : \mu^2$, so it will follow that $h' = 1 : \mu$. (Q. E. D.)

The differential equations (12), (13) or (14), (16), (17) can be integrated only under restricting assumptions.

As a *second application*, we give the *differential equation for surfaces with isometric lines of curvature* ([2]). If one chooses the lines of curvature to be the parameter curves then one will have:

([1]) **Bonnet**, Nouv. Ann. de Math. **12** (1853), pp. 437.

([2]) **H. Stahl**, J. für. Math., v. 111. If the equation of the surface is given in the form $F(x, y, z) = 0$ then F will be determined by a fourth-order partial differential equation. (This note no longer appears in the J. für. Math., since publication of this pamphlet has made printing it in the Journal superfluous.) Cf., **Weingarten**, Sitzber. d. Berl. Acad. (1883), pp. 1163.

(20) $$f = 0, \qquad d' = 0, \qquad e = g = \lambda.$$

Equations (2) then come down to these:

(21) $$d\,d'' = -\frac{\lambda}{2}L, \qquad \frac{\partial d}{\partial v} = \frac{d+d''}{2}\frac{\partial \log \lambda}{\partial v}, \qquad \frac{\partial d''}{\partial u} = \frac{d+d''}{2}\frac{\partial \log \lambda}{\partial u},$$

in which one has set:

(22) $$\frac{\partial^2 \log \lambda}{\partial u^2} + \frac{\partial^2 \log \lambda}{\partial v^2} = L,$$

to abbreviate.

If one eliminates the quantity d'' from the first and second equation (21) then one will get a linear differential equation in d^2:

$$\frac{\partial d^2}{\partial v} - \frac{\partial \log \lambda}{\partial u} d^2 + \frac{\lambda}{2}\frac{\partial \log \lambda}{\partial v} L = 0.$$

The integration of this is easy; when one sets:

(23) $$\frac{1}{2}\int L \frac{\partial \log \lambda}{\partial v} dv = L_1, \qquad \frac{1}{2}\int L \frac{\partial \log \lambda}{\partial u} du = L_2,$$

to abbreviate, and appends the corresponding equation in d'' that follows from (21), one will get:

(24) $$d^2 = \lambda (U - L_1), \qquad d''^2 = \lambda (V - L_2),$$

in which U is a function of u, and V is a function of v. In that way, the first equation in (21) will go to:

(25) $$4\,(U - L_1)\,(V - L_2) = L^2.$$

That is the *desired differential equation* that determines λ as a function of u, v. It already contains two arbitrary functions U, V; its solution will bring two more arbitrary functions of u, v with it. If (25) is solved or λ is determined then one will get L_1 and L_2 or d and d'' by quadratures. The six fundamental quantities e, f, g ; d, d', d'' are ascertained with that. The quadratures that determine d and d'' can be avoided when one defines $\frac{\partial d}{\partial u}$ and $\frac{\partial d''}{\partial v}$ from equations (21). When one sets the two values for $\frac{\partial^2 d}{\partial u\,\partial v}$ $\left(\text{or } \frac{\partial^2 d''}{\partial u\,\partial v}\right)$ equal to each other, one will then get an equation of the form $A\,d^2 + B\,d''^2 + C = 0$ whose coefficients are known and which will yield the first equation in (21) that couples d and d''.

One easily convinces oneself that the differential equation (16) is only a special case of (25). The latter equation will then be satisfied when one sets $U = V = 0$ and:

$$L = \frac{1}{2}\left(\frac{1}{\lambda} - \frac{\lambda}{\mu^2}\right).$$

A *further application* refers to the problem of ascertaining the *bending surface of a given surface* (1) or that of determining the surface when e, f, g are given as functions of (u, v). Here, one seeks the quantities d, d', d'', or from [§ **7**, (7)], the quantities t, t', t''. One must solve equations (8) and (10) of § **7** in order to determine them. The latter has the form $t\,t'' - t'^2 = k$, in which the curvature k is a given function of e, f, g or of (u, v). In order to arrive at a single differential equation for t' in a symmetric way, one must eliminate t and t''. One could, say, introduce $t = \mu(t' + i\sqrt{k})$, $t'' = \frac{1}{\mu}(t' - i\sqrt{k})$ into equations [§ **7**, (8)] and eliminate the function μ by repeated differentiation. In regard to that, one must remark (2) that the resulting differential equation in t' would possess a number of integrals that would exceed that of equations (8) and (10) in § **7**, which one started from. However, the isolation of the foreign elements from the solution to the problem would encounter grave difficulties.

One will arrive at a similar outcome when one introduces the equation of the asymptotic lines in the form $\varphi = a$, $\psi = b$ (with parameters a, b), instead of introducing the quantities d, d', d'', which represent the coefficients in the differential equation of those lines. If λ is a proportionality factor then one will have:

$$t = \lambda\,\varphi_1\,\psi_1\,, \quad t' = \lambda\,(\varphi_1\,\psi_2 + \psi_1\,\varphi_2), \qquad t'' = \lambda\,\varphi_2\,\psi_2\,,$$

and one will get two partial differential equations for the determination of the functions φ and ψ, in place of equations [§ **7**, (8)] (3). Solving them would yield the factor λ from the equation $t\,t'' - t'^2 = k$, which would determine d, d', d''.

The best solution of the problem seems to be the one that is directly connected with **Gauss**'s equations [§ **2**, (9)] (4). The quantities to be determined d, d', d'' are expressed by a single quantity x in them. However, in order to determine x, from [§ **2**, (17)], one has the equation:

$$(26)\qquad (x_{11} - p\,x_1 - q\,x_2)(x_{22} - p''x_1 - q''x_2) - (x_{12} - p'x_1 - q'x_2)^2 = \delta^2 \cdot k\,[1 - \delta'(x)],$$

in which the coefficients depend upon only e, f, g, so they are known. That equation can be regarded as the differential equation for the surface; in fact, any solution of it will give a surface of that kind. If e, f, g are real, and should the surface be real, then d, d', d''

(1) That problem, in particular, was treated in detail as a special case by **Bour**, J. Ec. Poly. Cah., **39** (1862) and **Bonnet**, *ibid.*, Cah. **41** and **42** (1865).

(2) **Weingarten**, Festschrift der techn. Hochschule zu Berlin (1884), pp. 32.

(3) **Darboux**, *Leçons*, III, pp. 285.

(4) **Bour**, *loc. cit.*, pp. 15; **Dini**, Giorn. di Mat. **2** (1864), pp. 287; **Bonnet**, *loc. cit.*, Cah. **42**, pp. 3; **Weingarten**, Festschrift, pp. 31, *et seq.*

would also have to be real, and from [§ **2**, (9)] one would then have to add the condition that $1 - \delta'(x) > 0$. Any real-valued solution of (26) that satisfies that condition would correspond to a real solution of the problem.

Equation (26) for x is linear in $x_{11}\, x_{22} - x_{12}^2$, x_{11}, x_{12}, x_{22}. From [§ **2**, (9)], the differential equation of its characteristics is:

$$d\, du^2 + 2d\, du\, dv + d\, dv^2 = 0\ ;$$

i.e., the characteristic of the differential equation (26) are the asymptotic lines of the surface ([1]). That remark is important for certain bending problems, such as, *inter alia*, the problem: Deform a given surface in such a way that a curve that lies on it will go to a prescribed space curve. [Cf., **Darboux**, *Leçons*, III, pp. 277; in a different treatment, **Weingarten**, J. für Math. **100** (1886), pp. 296.]

In conclusion, we shall mention the surfaces *with one or two systems of planar or spherical lines of curvature.* If one chooses the lines of curvature to be the parameter curves (u, v) then $f = d' = 0$, and the three differential equations (6) and (9) will exist between e, g, d, d''. If one now lets (cf., § **14** and § **15**) ρ_u and ρ_v denote radii of absolute torsion for the lines of curvature v = const. and u = const., resp., and lets R_u and R_v, resp., denote their radii of osculation then the condition for a family of lines of curvature (v = const.) to be planar will be that $\rho_u = \infty$. The condition for the two families to be planar is that $\rho_u = \infty$ and $\rho_v = \infty$. Furthermore, the condition for a family (v = const.) to be spherical is that $R_u = V$, and the condition for both families to be spherical is $R_u = V$ and $R_v = U$, where U depends upon u and V depends upon v. When one appends one or the other of those conditions to equations (6) and (9), one will have the partial differential equations of the corresponding problem that the quantities e, g, d, d'' must satisfy.

Under some further simplifying assumptions, those partial differential equations can be converted into ordinary ones, or also solved completely. However, a direct evaluation of the simplified assumptions will often reach the objective more rapidly ([2]). Another treatment of surfaces with planar lines of curvature will be indicated in § **11**.

§ 10. – Application to a triply-orthogonal system of surfaces ([3]).

The equations that were developed in §§ **7** and **8** also prove to be useful in the *study of a triply-orthogonal systems of surfaces.* The equations:

(1) $$x = x(u, v, w), \qquad y = y(u, v, w), \qquad z = z(u, v, w)$$

([1]) **Darboux**, *Leçons*, III, pp. 252.

([2]) From the literature that refers to that problem, we cite:

Monge-Liouville, Appl., pp. 161. **Joachimsthal**, Programm des franz. Gymnasium in Berlin (1848) and Jour. f. Math, **54** (1857). **Bonnet**, C. R. Acad. Sc. **36** (1853). **Serret**, C. R. Acad. Sc. **36** (1853). **Enneper**, Abh. d. Kgl. Ges. d. Wiss. zu Göttingen **23** (1878). **Pirondini**, Giornale di Mat. **22** (1884).

([3]) **Lamé**, Jour. de Math. **5** (1840), pp. 313; *ibid.*, **8** (1843), pp. 397 and *Leçons sur les coordonées curvilignes* (1859).

represent a system of infinitely-many surfaces in a three-fold way, since, e.g., each value of w corresponds to a surface with the parameter curves (u, v). In order to be able to apply the previous formulas, nothing more will be necessary besides exhibiting the six fundamental quantities e, f, g ; d, d', d'' for the surface w = const. The corresponding quantities for the surfaces u = const. and v = const. will then be obtained by cyclic permutation of u, v, w.

The fact that the surfaces of the system (1) intersect orthogonally along the parameter curves is expressed by the equations:

$$\sum \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} = 0, \qquad \sum \frac{\partial x}{\partial v}\frac{\partial x}{\partial w} = 0, \qquad \sum \frac{\partial x}{\partial w}\frac{\partial x}{\partial u} = 0, \tag{2}$$

in which the summation extends over three expressions in x, y, z that take the same form. The line element ds of space between two points (u, v, w) and $(u + du, v + dv, w + dw)$ is then determined by:

$$ds^2 = H_1^2\,du^2 + H_2^2\,dv^2 + H_3^2\,dw^2, \tag{3}$$

in which:

$$H_1^2 = \sum\left(\frac{\partial x}{\partial u}\right)^2, \qquad H_2^2 = \sum\left(\frac{\partial x}{\partial v}\right)^2, \qquad H_3^2 = \sum\left(\frac{\partial x}{\partial w}\right)^2. \tag{4}$$

It can be shown that the three quantities H_1, H_2, H_3 are fundamental and definitive of the character of triply-orthogonal systems. First of all, from (2) and (4), the *first three fundamental quantities e, f, g of the surface w* = const. will be equal to H_1^2, 0, H_2^2, resp.

One further finds the expression:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = H_1\,H_2\,H_3$$

for the functional determinant of x, y, z with respect to u, v, w from equations (2) and (4), which is implied immediately by squaring the left-hand side. If one differentiates the three equations (2) with respect to w, u, v then one will get:

$$\sum\left(\frac{\partial x}{\partial u}\frac{\partial^2 x}{\partial v\,\partial w} + \frac{\partial x}{\partial v}\frac{\partial^2 x}{\partial u\,\partial w}\right) = 0, \qquad \sum\left(\frac{\partial x}{\partial v}\frac{\partial^2 x}{\partial u\,\partial w} + \frac{\partial x}{\partial w}\frac{\partial^2 x}{\partial u\,\partial v}\right) = 0,$$

$$\sum\left(\frac{\partial x}{\partial w}\frac{\partial^2 x}{\partial u\,\partial v} + \frac{\partial x}{\partial u}\frac{\partial^2 x}{\partial v\,\partial w}\right) = 0,$$

resp. If one adds the first and third of these equations and subtracts the second one then it will follow that:

$$(6)\qquad \sum\frac{\partial x}{\partial u}\frac{\partial^2 x}{\partial v\,\partial w}=0,\qquad \sum\frac{\partial x}{\partial v}\frac{\partial^2 x}{\partial w\,\partial u}=0,\qquad \sum\frac{\partial x}{\partial w}\frac{\partial^2 x}{\partial u\,\partial v}=0,$$

in which the last two equations are obtained from the first one by cyclically permuting *u*, *v*, *w*. Moreover, when one differentiates the first equation in (4) with respect to *u*, *v*, *w* and recalls (2), one will get the equations:

$$(7)\qquad \begin{aligned}
&\sum\frac{\partial x}{\partial u}\frac{\partial^2 x}{\partial u^2} && = H_1\frac{\partial H_1}{\partial u},\\
&\sum\frac{\partial x}{\partial u}\frac{\partial^2 x}{\partial u\,\partial v}=-\sum\frac{\partial x}{\partial v}\frac{\partial^2 x}{\partial u^2} && = H_1\frac{\partial H_1}{\partial v},\\
&\sum\frac{\partial x}{\partial u}\frac{\partial^2 x}{\partial u\,\partial w}=-\sum\frac{\partial x}{\partial w}\frac{\partial^2 x}{\partial u^2} && = H_1\frac{\partial H_1}{\partial w},
\end{aligned}$$

which correspond to equations [§ **1**, (6)].

If one lets α_3, β_3, γ_3 denote the cosines of the inclination angles that the normal to the surface *w* = const. at the point (*u*, *v*) makes with the coordinate axes then, from [§ **1**, (22)], one will have:

$$(8)\qquad \alpha_3=\frac{1}{H_3}\frac{\partial x}{\partial w},\qquad \beta_3=\frac{1}{H_3}\frac{\partial y}{\partial w},\qquad \gamma_3=\frac{1}{H_3}\frac{\partial z}{\partial w}.$$

One will now get the *last three fundamental quantities d, d′, d″ for the surface w* = const. from this when one starts with the definition [§ **2**, (1)] and employs equations (6) and (7):

$$(9)\qquad \begin{aligned}
&\alpha_3\frac{\partial^2 x}{\partial u^2}+\beta_3\frac{\partial^2 y}{\partial u^2}+\gamma_3\frac{\partial^2 z}{\partial u^2} = \frac{1}{H_3}\sum\frac{\partial x}{\partial w}\frac{\partial^2 x}{\partial u^2} = -\frac{H_1}{H_3}\frac{\partial H_1}{\partial w},\\
&\alpha_3\frac{\partial^2 x}{\partial u\,\partial v}+\beta_3\frac{\partial^2 y}{\partial u\,\partial v}+\gamma_3\frac{\partial^2 z}{\partial u\,\partial v} = \frac{1}{H_3}\sum\frac{\partial x}{\partial w}\frac{\partial^2 x}{\partial u\,\partial v}=0,\\
&\alpha_3\frac{\partial^2 x}{\partial v^2}+\beta_3\frac{\partial^2 y}{\partial v^2}+\gamma_3\frac{\partial^2 z}{\partial v^2} = \frac{1}{H_3}\sum\frac{\partial x}{\partial w}\frac{\partial^2 x}{\partial v^2} = -\frac{H_2}{H_3}\frac{\partial H_2}{\partial w}.
\end{aligned}$$

One will then have the following *summary of the values of the six fundamental quantities e, f, g*; *d, d′, d″, when they are referred to the three orthogonal systems of surfaces* (1) *w* = *c*, *u* = *a*, *v* = *b* (*a*, *b*, *c* constants) with the corresponding parameters (*u*, *v*), (*v*, *w*), (*w*, *u*):

(10)

	u	v	w	f	g	d	d'	d''
$w = c$	u	v	H_1^2	0	H_2^2	$-\dfrac{H_1}{H_3}\dfrac{\partial H_1}{\partial w}$	0	$-\dfrac{H_2}{H_3}\dfrac{\partial H_2}{\partial w}$
$u = a$	v	w	H_2^2	0	H_3^2	$-\dfrac{H_2}{H_1}\dfrac{\partial H_2}{\partial u}$	0	$-\dfrac{H_2}{H_1}\dfrac{\partial H_3}{\partial u}$
$v = b$	w	u	H_3^2	0	H_1^2	$-\dfrac{H_3}{H_2}\dfrac{\partial H_3}{\partial v}$	0	$-\dfrac{H_1}{H_2}\dfrac{\partial H_1}{\partial v}$

One can then express all of those fundamental quantities in terms of the functions H_1, H_2, H_3, and their derivatives with respect to u, v, w.

One can infer a series of consequences from the table (10) and the previous developments, of which only the most important shall be mentioned, for the sake of brevity.

Dupin's Theorem (1) follows from the vanishing of f and d' for the three systems of surfaces:

When three families of surfaces intersect orthogonally, the intersection curves will always be lines of curvature of the surfaces.

If one denotes the radii of principle curvature for the lines of curvature $v = b$ and $u = a$ on the surface w = const. by r_{31} and r_{32}, resp., then equations [§ **9**, (5)] will give them the expressions:

$$\frac{1}{r_{31}} = -\frac{1}{H_1 H_3}\frac{\partial H_1}{\partial w}, \qquad \frac{1}{r_{32}} = -\frac{1}{H_2 H_3}\frac{\partial H_2}{\partial w}, \tag{11}$$

along with the corresponding ones for r_{12}, r_{13}, and r_{23}, r_{21}, and a series of relations between the radii r_{ik} that we shall pass over (2).

We shall further define the *fundamental equations* of § **7** for a triply-orthogonal system of surfaces.

If one denotes the direction cosines of the normals to the surfaces $u = a$, $v = b$, $w = c$ (a, b, c constants) by (α_1, β_1, γ_1), (α_2, β_2, γ_2), (α_3, β_3, γ_3), resp., then from [§ **1**, (22), cf., also (8)], one will have:

$$\alpha_1 = \frac{1}{H_1}\frac{\partial x}{\partial u}, \qquad \alpha_2 = \frac{1}{H_2}\frac{\partial x}{\partial v}, \qquad \alpha_3 = \frac{1}{H_3}\frac{\partial x}{\partial w}, \tag{12}$$

along with the corresponding equations in (β, y) and (γ, z). The system [§ **7**, (2)] leads to (3):

(1) **Dupin**, *Développements*, pp. 239.
(2) **Lamé**, *Leçons*, pp. 80, *et seq.*
(3) **Lamé**, *Leçons*, pp. 89, eq. (28) and pp. 91, eq. (30).

$$
\begin{aligned}
&\frac{\partial\alpha_1}{\partial u} = -\frac{1}{H_2}\frac{\partial H_1}{\partial v}\alpha_2 - \frac{1}{H_3}\frac{\partial H_1}{\partial w}\alpha_3, &&\frac{\partial\alpha_1}{\partial v} = \frac{1}{H_1}\frac{\partial H_2}{\partial u}\alpha_2,\\
&\frac{\partial\alpha_2}{\partial u} = \frac{1}{H_2}\frac{\partial H_1}{\partial v}\alpha_1, &&\frac{\partial\alpha_2}{\partial v} = -\frac{1}{H_3}\frac{\partial H_2}{\partial w}\alpha_3 - \frac{1}{H_1}\frac{\partial H_2}{\partial u}\alpha_1,\\
&\frac{\partial\alpha_3}{\partial u} = \frac{1}{H_2}\frac{\partial H_1}{\partial w}\alpha_1, &&\frac{\partial\alpha_3}{\partial v} = \frac{1}{H_3}\frac{\partial H_2}{\partial w}\alpha_2,
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\frac{\partial\alpha_1}{\partial w} &= \frac{1}{H_1}\frac{\partial H_3}{\partial u}\alpha_3,\\
\frac{\partial\alpha_2}{\partial w} &= \frac{1}{H_2}\frac{\partial H_3}{\partial v}\alpha_3,\\
\frac{\partial\alpha_3}{\partial w} &= -\frac{1}{H_1}\frac{\partial H_3}{\partial u}\alpha_1 - \frac{1}{H_2}\frac{\partial H_3}{\partial v}\alpha_2,
\end{aligned}
$$

along with the corresponding equations in $(\beta_1, \beta_2, \beta_3)$ and $(\gamma_1, \gamma_2, \gamma_3)$. It follows from (12) that:

$$x = \int(\alpha_1 H_1\,du + \alpha_2 H_2\,dv + \alpha_3 H_3\,dw)\,. \tag{14}$$

For an orthogonal system of surfaces, **Mainardi**'s equations [§ **7**, (6)] or [§ **9**, (6)] reduce to three equations, namely ([1]):

$$
\begin{aligned}
\frac{\partial^2 H_1}{\partial v\,\partial w} &= \frac{1}{H_2}\frac{\partial H_2}{\partial w}\frac{\partial H_1}{\partial v} + \frac{1}{H_3}\frac{\partial H_2}{\partial v}\frac{\partial H_1}{\partial w},\\
\frac{\partial^2 H_2}{\partial w\,\partial u} &= \frac{1}{H_3}\frac{\partial H_3}{\partial u}\frac{\partial H_2}{\partial w} + \frac{1}{H_1}\frac{\partial H_1}{\partial w}\frac{\partial H_2}{\partial u},\\
\frac{\partial^2 H_3}{\partial u\,\partial v} &= \frac{1}{H_1}\frac{\partial H_1}{\partial v}\frac{\partial H_2}{\partial u} + \frac{1}{H_2}\frac{\partial H_2}{\partial u}\frac{\partial H_3}{\partial v}.
\end{aligned}
\tag{15}
$$

Gauss's equations [§ **7**, (10)] or [§ **9**, (9)] take on the form ([2]):

([1]) **Lamé**, *Leçons*, pp. 76, eq. (8).
([2]) **Lamé**, *Leçons*, pp. 78, eq. (9).

$$\frac{\partial}{\partial v}\left(\frac{1}{H_2}\frac{\partial H_2}{\partial v}\right)+\frac{\partial}{\partial w}\left(\frac{1}{H_3}\frac{\partial H_2}{\partial w}\right)+\frac{1}{H_1^2}\frac{\partial H_2}{\partial u}\frac{\partial H_3}{\partial u}=0,$$

(16)
$$\frac{\partial}{\partial w}\left(\frac{1}{H_3}\frac{\partial H_1}{\partial w}\right)+\frac{\partial}{\partial u}\left(\frac{1}{H_1}\frac{\partial H_3}{\partial u}\right)+\frac{1}{H_2^2}\frac{\partial H_3}{\partial v}\frac{\partial H_1}{\partial v}=0,$$

$$\frac{\partial}{\partial u}\left(\frac{1}{H_1}\frac{\partial H_3}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{1}{H_2}\frac{\partial H_1}{\partial v}\right)+\frac{1}{H_3^2}\frac{\partial H_1}{\partial w}\frac{\partial H_2}{\partial w}=0\,.$$

If one sets:

(17)
$$\frac{1}{H_1}\frac{\partial H_2}{\partial u}=H_{12}\,,\qquad \frac{1}{H_1}\frac{\partial H_3}{\partial u}=H_{13}\,,\qquad \text{etc.}$$

then one can replace the six equations (15) and (16) in the three quantities H_1, H_2, H_3 with the following nine equations in the six functions H_{12}, H_{21}, H_{23}, H_{32}, H_{31}, H_{13} ([1]):

(18)
$$\frac{\partial H_{23}}{\partial u}=H_{21}\,H_{13}\,,\qquad \frac{\partial H_{31}}{\partial v}=H_{32}\,H_{21}\,,\qquad \frac{\partial H_{12}}{\partial w}=H_{13}\,H_{32}\,,$$

$$\frac{\partial H_{32}}{\partial u}=H_{31}\,H_{12}\,,\qquad \frac{\partial H_{13}}{\partial v}=H_{12}\,H_{23}\,,\qquad \frac{\partial H_{21}}{\partial w}=H_{23}\,H_{31}\,,$$

and

(19)
$$\frac{\partial H_{12}}{\partial u}+\frac{\partial H_{21}}{\partial v}+H_{31}\,H_{32}=0,\qquad \frac{\partial H_{23}}{\partial v}+\frac{\partial H_{32}}{\partial w}+H_{12}\,H_{13}=0,$$

$$\frac{\partial H_{31}}{\partial w}+\frac{\partial H_{13}}{\partial u}+H_{23}\,H_{21}=0.$$

One gets the following equation from (19) by an easy combination:

(20)
$$\frac{\partial(H_{23}\,H_{32})}{\partial u}=\frac{\partial(H_{31}\,H_{13})}{\partial v}=\frac{\partial(H_{12}\,H_{21})}{\partial w}=H_{12}\,H_{23}\,H_{31}+H_{21}\,H_{32}\,H_{13}\,.$$

Equations (13) to (16) solve the problem of finding the *most general triply-orthogonal system of surfaces.* The solution divides into three parts: The first part consists of determining the three quantities H_1, H_2, H_3 by integrating the systems (15) and (16), the second part involves determining the quantities α_i, β_i, γ_i, and with that, the coordinates *x*, *y*, *z* will be obtained by integrating the system (13). Just as in § **8**, one can show that an orthogonal system of surfaces is defined uniquely (up to its position space) by each system of values H_1, H_2, H_3 that satisfy equations (15) and (16). The determination of the three quantities H_1, H_2, H_3 from the six equations (15) and (16) is possible in infinitely-many ways. In fact, it will be shown below that the problem of

([1]) **Lamé**, *Leçons*, pp. 76 and 79.

finding the most general orthogonal system of surfaces can also be reduced to the solution of a single third-order partial differential equation.

The problem of determining an orthogonal system of surfaces or of integrating equations (15) and (16) can be reduced, and in that way simplified when one adds the condition that the *three-fold system is at the same time isometric*. One can then give the form of the functions H_1, H_2, H_3 more precisely from the outset. Namely, if U, U_1 are functions that are *independent* of u, V, V_1 are independent of v, and finally U, U_1 are independent of u, then from [§ **3**, (8)], one will have the conditions:

$$H_1 : H_2 = U : V_1\,, \qquad H_2 : H_3 = V : W_1\,, \qquad H_3 : H_1 = W : U_1\,.$$

It follows from this that:

$$U\,V\,W = U_1\,V_1\,W_1\,, \quad \text{so} \quad U_1 = U, \qquad V_1 = V, \qquad W_1 = W,$$

so:

$$ds^2 = K\,(U\,du^2 + V\,dv^2 + W\,dw^2) = L\,(V'W'du^2 + W'U'dv^2 + U'V'dw^2)\,, \tag{21}$$

where U', V', W' are functions of the same kind as U, V, W, and K or L are functions of (u, v, w).

Lamé ([1]) has shown that the further assumption that L is a constant in (21) will lead to the system of second-order confocal surfaces. **Darboux** ([2]) has treated the question for general L, determined the values of H_1, H_2, H_3, or the form of the line element for all orthogonal and isometric systems of surfaces, and has exhibited the surface equations for one part of those cases.

One can further employ **Lamé**'s formulas to prove the **theorem** ([3]):

The only conformal map of space onto itself is a similarity and inversion by reciprocal radii.

Namely, if u, v, w are the rectangular coordinates of a spatial point, and x, y, z are those of the image point then one will have the following condition for the two spaces to be conformal to each other:

$$ds^2 = dx^2 + dy^2 + dz^2 = \lambda^{-2}\,(du^2 + dv^2 + dw^2), \tag{22}$$

in which λ is a function of (u, v, w). A series of equations will serve to determine them that are implied immediately by **Lamé**'s equations when one sets $H_1 = H_2 = H_3 = \lambda^{-1}$. Equations (16) will then lead to the conditions:

([1]) **Lamé**, *Leçons*, pp. 93, *et seq.*

([2]) **Darboux**, Ann. Ec. Norm., v. III, 1866, pp. 130 and C. R. Acad. Sc., v. **84**, pp. 298; cf., **Maschke**, Diss. Göttingen, 1880.

([3]) **Monge-Liouville**, *Applications*, Note VI, pp. 609, *et seq.*

(23) $$\frac{\partial^2\lambda}{\partial u^2}+\frac{\partial^2\lambda}{\partial v^2}=\frac{\partial^2\lambda}{\partial v^2}+\frac{\partial^2\lambda}{\partial w^2}=\frac{\partial^2\lambda}{\partial w^2}+\frac{\partial^2\lambda}{\partial u^2}=\frac{1}{\lambda}\left\{\left(\frac{\partial\lambda}{\partial u}\right)^2+\left(\frac{\partial\lambda}{\partial v}\right)^2+\left(\frac{\partial\lambda}{\partial w}\right)^2\right\},$$

and equations (15) will lead to:

(24) $$\frac{\partial^2\lambda}{\partial u\,\partial v}=\frac{\partial^2\lambda}{\partial v\,\partial w}=\frac{\partial^2\lambda}{\partial w\,\partial u}=0.$$

The latter yield:

(25) $$\lambda = U + V + W,$$

in which U, V, W are functions of u, v, w, resp. When U' and U'' are the first and second derivatives of U with respect to u, resp., it will further follow from (23) that:

(26) $$U' + V'' = V' + W'' = W' + U'' = \frac{U'^2+V'^2+W'^2}{U+V+W}.$$

Hence, each of those expressions is independent of u, v, w and one will have:

(27) $$U'' = V'' = W'' = \frac{2}{c},$$

in which c is constant that is independent of u, v, w. With that, one will have:

$$U = \frac{1}{c}[(u-a_1)^2+a^2],\qquad V = \frac{1}{c}[(v-b_1)^2+b^2],\qquad W = \frac{1}{c}[(w-c_1)^2+c^2],$$

and one will get from (26) that:

$$(u-a_1)^2+(v-b_1)^2+(w-c_1)^2 = (u-a_1)^2+(v-b_1)^2+(w-c_1)^2+a_2+b_2+c_2,$$

or

$$a_2 + b_2 + c_2 = 0.$$

If one locates the coordinate origin at (a_2, b_2, c_2) then one will finally have:

$$\lambda = \frac{1}{c}(u^2+v^2+w^2),$$

or

(28) $$dx^2+dy^2+dz^2 = c^2\cdot\frac{du^2+dv^2+dw^2}{(u^2+v^2+w^2)^2}.$$

If one sets $u^2 + v^2 + w^2 = \rho$, to abbreviate, then one will have:

$$H_1 = H_2 = H_3 = c\,\rho^{-1}.$$

If one then exhibits the system (13) and integrates it then it will follow that:

$$\alpha_1 = 1 - 2\,\rho^{-1}\,u^2, \qquad \alpha_2 = 1 - 2\,\rho^{-1}\,u\,v, \qquad \alpha_3 = 1 - 2\,\rho^{-1}\,u\,w,$$

and one will get from (14) that:

$$(29) \qquad x = \frac{c\,u}{u^2+v^2+w^2}, \qquad y = \frac{c\,v}{u^2+v^2+w^2}, \qquad z = \frac{c\,w}{u^2+v^2+w^2}.$$

However, that is an inversion through reciprocal radii relative to the sphere $u^2 + v^2 + w^2 = c^2$. There will be an exception when $c = \infty$. From (27) and (26), U, V, W, and λ constant are constant then; i.e., the map consists of a similarity. (Q. E. D.)

The study of triple-orthogonal systems of surfaces can also be carried out in such a way that one does not start with equations (1), but with their solution in terms of u, v, w ([1]). One will then have:

$$(30) \qquad u\,(x, y, z) = u, \qquad v\,(x, y, z) = v, \qquad w\,(x, y, z) = w\,.$$

The orthogonality of those three systems of equations is expressed by the equations:

$$(31) \qquad \sum \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} = 0, \qquad \sum \frac{\partial v}{\partial x}\frac{\partial w}{\partial x} = 0, \qquad \sum \frac{\partial w}{\partial x}\frac{\partial u}{\partial x} = 0,$$

in which the summation is once more extended over x, y, z. Furthermore, one has:

$$(32) \qquad \alpha_3 = \frac{1}{h_3}\frac{\partial w}{\partial x}, \quad \beta_3 = \frac{1}{h_3}\frac{\partial w}{\partial y}, \quad \gamma_3 = \frac{1}{h_3}\frac{\partial w}{\partial z},$$

when:

$$(33) \qquad h_3^2 = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2,$$

along with the corresponding equations for the index 1 and u or the index 2 and v.

Since, from (30) and (1):

$$\frac{\partial w}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial w} = 1,$$

it will follow from (8) and (32) that:

$$(34) \qquad H_i\,h_i = 1 \qquad (i = 1, 2, 3).$$

([1]) **Lamé**, *Leçons*, pp. 7, *et seq.*

The problem of determining a triply-orthogonal system of surfaces will now lead to the **theorem** ([1]):

The necessary and sufficient condition for a system of equations w (x, y, z) = *w, with the parameter w, to belong to a triply-orthogonal system is a third-order partial differential equation for the function w* (x, y, z).

When one differentiates the three equations (31) with respect to x, y, z one and two times, that will yield 3 + 9 + 27 = 39 equations, in all, into which 2 (3 + 6 + 10) = 38 different derivatives of u and v with respect to x, y, z will enter. One will then obtain a single third-order partial differential equation for the function w (x, y, z) by eliminating the latter. We refer to the literature ([2]) for the exhibition of that equation and the proof that it is also the sufficient condition for the existence of a triply-orthogonal system or that every system of surfaces $w(x, y, z) = w$ that satisfies that equation is associated with two other ones $u(x, y, z) = u$ and $v(x, y, z) = v$ that define an orthogonal system with the first one.

That is connected with the following remark: From [§ **2**, (9)], one also has the equation:

$$\frac{\partial^2 x}{\partial u\,\partial v} = \frac{1}{H_1}\frac{\partial H_1}{\partial v}\frac{\partial x}{\partial u} + \frac{1}{H_2}\frac{\partial H_2}{\partial u}\frac{\partial x}{\partial v},$$

which is also included in the system (13), along with the corresponding equations in y and z. A comparison with the last equation in (15) will show that the quantities H_3 satisfy the same differential equation as x, y, z, and more generally, so will the function $\alpha(x^2 + y^2 + z^2) + \beta x + \gamma y + \delta z + \varepsilon$, in which α, β, γ, δ, ε are independent of u, v. That includes the theorem that a family of surfaces $w(x, y, z) = w$ that satisfies the differential equation:

$$\frac{1}{h_3} = \alpha(x^2 + y^2 + z^2) + \beta x + \gamma y + \delta z + \varepsilon, \tag{35}$$

in which h_3 is determined by (33), and α, β, γ, δ, ε are arbitrary functions of w, will always belong to an orthogonal system ([3]).

§ 11 – Spherical map. Plane coordinates. Applications.

If one interprets the quantities a, b, c (§ **1**), which are coupled by the equation $a^2 + b^2 + c^2 = 1$, as the coordinates of a point on an auxiliary sphere K of radius 1 around the origin then one will have the *spherical map of the surface* that **Gauss** introduced ([4]). Every pair of values (u, v) simultaneously corresponds to a point on the surface and an image point on the sphere in such a way that both points have normals that point in the

([1]) **Bouquet**, Jour. de Math. **11** (1846), pp. 446. **Bonnet**, C. R. Acad. Sc. (1862).
([2]) **Darboux**, Ann. Ec. Norm. **3** (1866), pp. 110. **Weingarten**, Jour. f. Math. **83** (1877), pp. 1.
([3]) **Darboux**, Ann. Ec. Norm. (2) **7** (1878), pp. 110. **Weingarten**, *loc. cit.*, pp. 11.
([4]) **Gauss**, *Disq. gen.*, art. 6.

same direction. In order to study the spherical map, we shall denote all quantities that refer to the sphere in the same way as the corresponding ones that refer to the surface, except with upper-case symbols, instead of lower-case ones. Hence, the image point of (x, y, z) – i.e., (a, b, c) – will be denoted by (X, Y, Z), the cosines of the inclination angles of the normal to the sphere, by (A, B, C), etc. We will then have:

$$X = A = a, \qquad Y = B = b, \qquad Z = C = c \tag{1}$$

$$X^2 + Y^2 + Z^2 = 1. \tag{2}$$

From [§ **2**, (3)], when $d' = 0$, that will imply the following **theorems**:

1. *The spherical image of one of two conjugate directions is perpendicular to the other one*.

As a result, since the tangent plane to the surface and the tangent plane to the sphere are parallel at corresponding points:

2. *The spherical image of the direction of an asymptotic line is perpendicular to that direction.*

3. *The spherical image of the direction of a line of curvature is parallel to that direction.*

As was remarked before (§ **5**), those theorems can also be used as definitions of conjugate lines, asymptotic lines, and lines of curvature.

Most important of all are the six *fundamental quantities* E, F, G ; D, D', D'' *that are defined for the sphere.* Since **Gauss**'s auxiliary sphere represents a surface whose points are given by equations (1), one can set both x, y, z and a, b, c equal to X, Y, Z, resp., in the previous equations. Moreover, the quantities e, f, g ; d, d', d'' can be set to E, F, G ; D, D', D'', resp., and the quantities p, p', p''; q, q', q'', which are constructed from e, f, g, can be set to the corresponding quantities P, P', P'' ; Q, Q', Q'', resp., that are constructed from E, F, G. It will then follow from equations [§ **2**, (3)] that:

$$D = -E, \qquad D' = -F, \qquad D'' = -G, \tag{3}$$

and furthermore, equations [§ **2**, (6), (7), (16), (17)] will imply that:

$$E = h\,d - ke, \quad F = h\,d' - kf, \quad G = h\,d'' - kg, \tag{4}$$

and from this:

$$H = -2, \qquad K = 1, \qquad R_1 = R_2 = -1.$$

Finally, one will have:

(5) $$\Delta = \sqrt{EG - F^2} = k\delta = \frac{d\,d'' - d'^2}{\delta},$$

and the line element on the sphere will assume the form:

(6) $$\begin{aligned} dS^2 &= dX^2 + dY^2 + dZ^2 \\ &= h\,(d\,du^2 + 2\,d'du\,dv + d''dv^2) - k\,(e\,du^2 + 2\,f\,du\,dv + g\,dv^2). \end{aligned}$$

(4) implies the **theorems**:

The lines of curvature on the surface will go to an orthogonal system on the sphere under the spherical map.

That is because $f = 0$, $d' = 0$ implies that $F = 0$.

Conversely: *Of the orthogonal systems on the surface (when it is not a minimal surface), only the lines of curvature will go to an orthogonal system on the sphere.*

That is because $f = 0$, $F = 0$ will imply that $d' = 0$ when one does not have $h = 0$; i.e., when the surface is not a minimal surface.

It follows from (4) that:

(7) $$E\;g - 2\,F\,f\;+ G\,e\;= \delta^2\,(h^2 - 2k),$$

(8) $$E\,d'' - 2\,F\,d' + G\,d\;= \delta^2\,hk\,.$$

If one combines the last equation with (5) then one will get the following expressions for the mean curvature h and the curvature k of the surface ([1]):

(9) $$h = \frac{E\,d'' - 2Fd' + Gd}{d\,d'' - d'^2}, \qquad k = \frac{E\,G - F^2}{d\,d'' - d'^2}.$$

If one substitutes the values of h and k from [§ **2**, (16) and (17)] in (4) then one will get:

(10) $$\begin{aligned} \delta^2\,E &= e\,d'^2 - 2\,f\,d\,d' + g\,d^2, \\ \delta^2\,F &= e\,d'd'' - f\,(d\,d' + d'^2) + g\,d\,d', \\ \delta^2\,G &= e\,d''^2 - 2\,f\,d'd'' + g\,d'^2. \end{aligned}$$

Solving these will yield equations for e, f, g that have a similar form, namely:

([1]) **Weingarten**, Festschrift der tech. Hochschule zu Berlin, 1884, pp. 41.

$$\begin{aligned}\Delta^2 e &= E\,d'^2 - 2\,F\,d\,d' + G\,d^2,\\ \Delta^2 f &= E\,d'd'' - F(d\,d' + d'^2) + G\,d\,d',\\ \Delta^2 g &= E\,d''^2 - 2\,F\,d'd'' + G\,d'^2.\end{aligned} \tag{11}$$

The plane coordinates of the surface shall be given now as functions of (u, v), instead of the point coordinates (x, y, z), and the formulas that will serve to investigate the surface shall be summarized briefly ([1]). Let the equation of the tangent plane at the point (x, y, z) be:

$$X\,\xi + Y\,\eta + Z\,\zeta - T = 0, \tag{12}$$

if ξ, η, ζ are the running coordinates.

The coefficients X, Y, Z, T are the *plane coordinates of the surface.* We add the condition that:

$$X^2 + Y^2 + Z^2 = 1, \tag{13}$$

which is no loss of generality, since one needs only to divide the coefficients X, Y, Z, T by $\sqrt{X^2 + Y^2 + Z^2}$ when that condition is not fulfilled. Under the assumption (13), X, Y, Z will be identical with the values (1) – i.e., with the cosines of the inclination angles of the surface normal with respect to the coordinate axes – while T means the distance from the tangent plane to the origin.

We shall next show how one can get the *point coordinates* (x, y, z) of the surface when the plane coordinates X, Y, Z, T are given as functions of (u, v). Since (x, y, z) is the contact point of the tangent plane (12), one will have:

$$x\,X + y\,Y + z\,Z - T = 0 \tag{14}$$

identically.

If one differentiates with respect u and v then from [§ **2**, (2)], when one uses the abbreviations in [§ **1**, (2)], one will get:

$$\begin{aligned}x\,X_1 + y\,Y_1 + z\,Z_1 - T_1 &= 0,\\ x\,X_2 + y\,Y_2 + z\,Z_2 - T_2 &= 0.\end{aligned} \tag{15}$$

([1]) The equations of the point and plane coordinates will first become completely dualistic when one introduces a general second-order surface at infinity in place of the imaginary spherical circle as what is definitive, using the process of **A. Cayley**, Phil. Trans. **149** (1859), pp. 61.

The three equations (14) and (15) are solved for x, y, z. From [§ **1**, (24)], the common denominator will be equal to Δ. When one applies [§ **1**, (23) and (25)] (defined in terms of X, Y, Z), the numerator of x will become:

$$\begin{vmatrix} T & Y & Z \\ T_1 & Y_1 & Z_1 \\ T_2 & Y_2 & Z_2 \end{vmatrix} = \Delta\, T\, X + [T_1\,(G\,X_1 - F\,X_2) + T_2\,(E\,X_2 - F\,X_1)].$$

One will then have:

$$(16)\qquad x = X\,T + \Delta\,(X, T), \qquad y = Y\,T + \Delta'\,(Y, T), \qquad z = Z\,T + \Delta''\,(Z, T),$$

when one sets:

$$(17)\qquad \Delta\,(X, T) = [\,E\,X_2\,T_2 - F\,(X_1\,T_2 + T_1\,X_2) + G\,X_1\,T_1],$$

to abbreviate (cf., [§ **17**, (8)]).

We shall further develop some *partial differential equations for the quantities* X, Y, Z, T that are analogous to equations [§ **2**, (9)] for x, y, z. If one recalls (3) then [§ **2**, (9)] will immediately imply the following equations for X :

$$(18)\qquad \begin{aligned} &X_{11} - P\,X_1 - Q\,X_2 + E\,X = 0, \\ &X_{12} - P'X_1 - Q'X_2 + F\,X = 0, \\ &X_{22} - P''X_1 - Q''X_2 + G\,X = 0. \end{aligned}$$

Corresponding equations with the same coefficients are true for Y, Z. The equations for T read somewhat differently. When one differentiates (15) with respect to u, v and recalls [§ **2**, (3)], one will get:

$$(19)\qquad \begin{aligned} &x\,X_{11} + y\,Y_{11} + z\,Z_{11} - T_{11} = d, \\ &x\,X_{12} + y\,Y_{12} + z\,Z_{12} - T_{12} = d', \\ &x\,X_{22} + y\,Y_{22} + z\,Z_{22} - T_{22} = d''. \end{aligned}$$

If one multiplies equation (14) by E and equations (15) by $-P$ and $-Q$, resp., and adds them to the first equation in (19) then, due to (18), one will get [1]:

$$(20)\qquad \begin{aligned} &T_{11} - P\,T_1 - Q\,T_2 + E\,T = -d, \\ &T_{12} - P'T_1 - Q'T_2 + F\,T = -d', \end{aligned}$$

[1] **Weingarten**, Festschrift, (1884), pp. 41.

$$T_{22} - P''T_1 - Q''T_2 + G\,T = -\,d''.$$

With that, the *six fundamental quantities of the surface e, f, g ; d, d′, d″, and their derivatives with respect to u, v* are represented in terms of X, Y, Z, T. Namely, one gets the quantities d, d', d'' from (20) and then the quantities e, f, g from (11).

In order to now return to the plane coordinates, we shall also present the quantities here that are important for the *study of a surface curve* – viz., ds^2, L, M, N (§ **14**) – in terms of X, Y, Z, T, and their derivatives with respect to u, v. From (11), one will have:

(21) $$\Delta^2\, ds^2 = E\,(d\,du + d\,dv)^2 - F\,(d\,du + d\,dv)\,(d\,du + d\,dv) + G\,(d\,du + d\,dv)^2$$

for ds^2.

The expression for L is obtained immediately from the values of d, d', d'' in (20), and upon consulting equations (18), it can be easily brought into the form:

(22) $$L = |\,X \quad X_1 \quad X_2 \quad X_{11}\,du^2 + 2\,X_{12}\,du\,dv + X_{22}\,dv^2\,|,$$

in which the determinant on the right-hand side includes four rows that are formed in the same way, of which, only the first one is written out.

In order to represent M, we use (4) to form:

$$E\,du + F\,dv = h\,(d\,du + d'dv) - k\,(e\,du + f\,dv),$$

$$F\,du + G\,dv = h\,(d'du + d''dv) - k\,(f\,du + g\,dv),$$

from which, we will obtain the expression for M :

(23) $$M = \frac{1}{\Delta}\begin{vmatrix} d\,du + d'\,dv & E\,du + F\,dv \\ d'\,du + d''\,dv & F\,du + G\,dv \end{vmatrix}.$$

If one substitutes the values of d, d', d'' in (20) and arranges them in terms of the derivatives of T then one will get the differential equation for the lines of curvature (M = 0):

(23a) $$A_1\frac{\partial T}{\partial u} + A_2\frac{\partial T}{\partial v} + B_1\,d\left(\frac{\partial T}{\partial u}\right) + B_2\,d\left(\frac{\partial T}{\partial v}\right) = 0,$$

in which the coefficients A_1, A_2, B_1, B_2 have values that are easy to give. Since, from (18), equation (23a) is also true for X, Y, Z with the same coefficients, if one eliminates those coefficients then one will also get the equation for the lines of curvature in the form ([1]):

(24) $$\left|\, \frac{\partial X}{\partial u} \;\vdots\; \frac{\partial X}{\partial v} \;\vdots\; d\left(\frac{\partial X}{\partial u}\right) \;\vdots\; d\left(\frac{\partial X}{\partial v}\right) \right| = 0,$$

([1]) **Darboux**, *Leçons*, I, pp. 240.

with the same abbreviation as in (22).

Finally, the expression for N, which will yield the differential equation for the geodetic lines on the surface when it is set equal to zero, is more complicated, and therefore also less interesting. One constructs it most simply from [§ **14**, (1) and (2)]:

$$\Delta N = \begin{vmatrix} X & dx & d^2x \end{vmatrix} \cdot \begin{vmatrix} X & \dfrac{\partial X}{\partial u} & \dfrac{\partial X}{\partial v} \end{vmatrix} \tag{25}$$

by multiplying those two determinants, whose second third rows are defined the same way in terms of Y, y and Z, z, as the first one was in terms of X, x.

We shall also add a pair of *applications*. From (11), the condition for the parameter curves u, v to be orthogonal ($f = 0$) is:

$$E\, d'd'' - F\,(d\,d'' + d'^2) + G\,d\,d' = 0. \tag{26}$$

Moreover, from (20), the condition for the parameter curves u, v to be conjugate ($d' = 0$):

$$T_{12} - P'T_1 - Q'T_2 + F\,T = 0. \tag{27}$$

If one couples equation (27) with the corresponding equations (18) in X, Y, Z then the condition for the parameter curves to be conjugate will go to ([1]):

$$|\, X_{12}\ \ X_1\ \ X_2\ \ X \,| = 0, \tag{28}$$

in which the determinant of the left-hand side is written with the same abbreviation as in (22).

Finally, from (4) and (20), and when one substitutes the values for P' and Q', the two conditions for the parameter curves u, v to be the lines of curvature of the surface ($f = 0$, $d' = 0$) will be:

$$F = 0, \qquad \frac{\partial^2 T}{\partial u\,\partial v} - \frac{\partial \log\sqrt{E}}{\partial v}\frac{\partial T}{\partial u} - \frac{\partial \log\sqrt{G}}{\partial u}\frac{\partial T}{\partial v} = 0, \tag{29}$$

from which, in conjunction with the corresponding equations (18) in X, Y, Z, one can once more derive the vanishing of certain determinants with three terms in them.

A second application might refer to the derivation of surfaces of a certain kind. If the *spherical image of a surface is given* – i.e., if either X, Y, Z are given as functions of two parameters u, v with the condition that $X^2 + Y^2 + Z^2 = 1$ or if E, F, G are given as functions of u, v with the condition that $K = 1$, where K is the **Gauss**ian expression for the curvature in terms of E, F, G (i.e., the right-hand side of equation [§ **7**, (10)], when it

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([1]) **Darboux**, *Leçons*, I, pp. 121.

is formed in terms of E, F, G) then the surface will still not be defined by that, since T is still an arbitrary function of the parameters u, v. In order to determine the surface, a further condition will be necessary; we shall consider the following cases:

1. Let the surface be arranged such that the *parameter curves are orthogonal* ($f = 0$). That suggests equation (26), or when one denotes the left-hand sides in (20) by $(T)_{11}$, $(T)_{12}$, $(T)_{22}$, resp., the equation for T :

$$E\,(T)_{12}\,(T)_{22} - F\,[(T)_{11}\,(T)_{22} + (T)_{12}^2] + G\,(T)_{11}\,(T)_{12} = 0, \tag{30}$$

which can be regarded as the differential equation of the surface. If T is ascertained from (30) then one will find x, y, z directly from (16) when X, Y, Z are given. By contrast, when E, F, G are given, one can find d, d', d'' from (20) and e, f, g from (11), from which, the surface will be determined uniquely.

2. *Let the parameter curves of the surface be conjugate* ($d' = 0$). One will then have the differential equation of the surface in (27).

3. The given quantities X, Y, Z be arranged such that $F = 0$, and the desired surface, such that the *parameter curves are the lines of curvature of the surface* ($f = 0$, $d' = 0$). The differential equation of the surface will then be equation (29) in T ([1]).

4. Let the *mean curvature h be constant* for the surface [or also given as a function of (u, v)]. From (9) and (20), the differential equation of the surface in terms of T will then be ([2]):

$$E\,(T)_{22} - 2F\,(T)_{12} + G\,(T)_{11} + h\,[(T)_{11}\,(T)_{22} - (T)_{12}^2] = 0. \tag{31}$$

5.

$$k\,[(T)_{11}\,(T)_{22} + (T)_{12}^2] = E\,G - F^2. \tag{32}$$

One can evaluate those conditions in order to derive *surfaces with a system of planar lines of curvature* (cf., § **9**, as well as the literature).

From **Joachimsthal**'s theorem (§ **15**), the spherical image of a planar line of curvature is a minor circle whose plane is parallel to the plane of the line of curvature. One now gets the most general simple infinitude of circles on the sphere (2) when one intersects it with a plane whose equation has the form:

$$U_1\,X + U_2\,Y + U_3\,Z = 1, \tag{33}$$

([1]) **Enneper**, Nachr. d. Kgl. Ges. d. Wiss. Göttingen (1870), pp. 78, with a different derivation.
([2]) **Weingarten**, Festschrift, pp. 42.

in which U_1, U_2, U_3 are arbitrary functions of one parameter u. One can perhaps regard that plane as the normal plane to an arbitrary, but given, space curve.

We focus on the circular sections of the planes (33) with the sphere (2) as a first system of curves C_u with the parameter u, and the orthogonal trajectories to them on the sphere as a second system of curves C_v with the parameter v. In order to find the curves C_v, one can use stereographic projection. The system of circles C_u on the sphere will once more go to a system of circles E_u in the plane under that projection. If one seeks the orthogonal trajectories E_v to E_u and carries them back to the sphere then one will have the system C_v. One now expresses the coordinates (X, Y, Z) of a point on the sphere (2) in terms of the coordinates (ξ, η) of its stereographic projection by way of [§ **3**, (18)] using the equations:

$$X = \frac{2\xi}{\xi^2+\eta^2+1}, \qquad Y = \frac{2\eta}{\xi^2+\eta^2+1}, \qquad Z = \frac{\xi^2+\eta^2-1}{\xi^2+\eta^2+1}. \tag{34}$$

If one substitutes those values in (33) then one will have the system E_u in the plane (ξ, η) in the form $f(\xi, \eta, u) = 0$. If the orthogonal system is $E_v : \varphi(\xi, \eta, u) = 0$ then upon substituting the values of (ξ, η) in (34), one will get the system C_v in the form:

$$\Phi(X, Y, Z, v) = 0. \tag{35}$$

The coupling of equations (2), (33), and (35) gives the coordinates X, Y, Z of the spherical map of the desired surface as functions of the parameters u, v.

One knows the quantities E and G (while $F = 0$), moreover, and only has to solve the differential equation (29). That is entirely possible here. From § **15**, the condition that the system of lines of curvature u = const. is planar says that the associated radius of torsion $\rho_v = \infty$, or from [§ **14**, (26)], that H_v is a function of u, or finally, from [§ **14**, (25)], that:

$$\frac{1}{\sqrt{EG}}\frac{\partial\sqrt{G}}{\partial u} = \frac{1}{U}, \qquad \text{so} \qquad \sqrt{E} = U\frac{\partial \log\sqrt{G}}{\partial u},$$

in which U is a function of u. The differential equation (29) for T will then assume the form:

$$\frac{1}{U}\frac{\partial T}{\partial v} = \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{T}}{\partial u}\right).$$

One will get:

$$\frac{\partial T}{\partial u} = \frac{\sqrt{E}}{U}(T - U_0)$$

as a first integral, and when one sets:

$$\exp\left(\int\frac{\sqrt{E}}{U}du\right) = P, \quad \frac{\sqrt{E}\,U_0}{U} = Q, \tag{36}$$

to abbreviate, one will get:

$$T = P\,(V - \int P^{-1} Q\, du\,) \tag{37}$$

as a second integral, in which V is a function of v. The problem is solved with that. Apart from the quadratures in (36) and (37), the solution will require only the integration of a first-order ordinary differential equation, which will yield the orthogonal trajectories (35) of the system of curves u = const. on the sphere. Five arbitrary functions enter into the solution, namely, a function V of v and four functions U_0 , U_1 , U_2 , U_3 of u, since one can set $U = u$.

Should the surface possess *two systems of planar lines of curvature* then the spherical image would be given from the outset. Namely, the images of two systems of lines of curvature will be two systems of minor circles that intersect at right angles. However, as is known, or as would follow from stereographic projection, two such systems of circles will only consist of the intersection curves of the sphere with two pencils of planes with the parameters u, v whose axes are reciprocal polars of the sphere. When one chooses such axes arbitrarily, one will get X, Y, Z immediately, and one will then get T as a function of (u, v) from (29). From (36) and (37), the solution of the problem will then require only two quadratures and will lead to only two arbitrary functions U_0 and V. A geometric consideration gives the **theorem**:

For every surface with two systems of planar lines of curvature, the planes of the one system are parallel to one line g, while those of the other system are parallel to a line g_1 *. Those two lines are perpendicular to each other.*

§ 12 – Application to minimal surfaces.

Minimal surfaces ([1]) – i.e., those simple surfaces that have the smallest area for a given boundary – are characterized geometrically by the condition that the radii of principal curvature r_1 and r_2 are equal and opposite at each point. They are then defined by $r_1 + r_2 = 0$ or $h = 0$, or from [§ **2**, (16)], by the equation:

$$e\,d'' - 2\,f\,d' + g\,d = 0. \tag{1}$$

That will imply the following *properties of minimal surfaces:*

Their minimal lines are conjugate.

That is because it follows from $e = g = 0$ that $d' = 0$.

([1]) Exhibiting the differential equation of minimal surfaces by means of the calculus of variations was presented by **Lagrange**, Misc. Taur. (1760-61); *Oeuvres*, I, pp. 335. **Meusnier** gave its geometric interpretation (viz., h = 0), Sav. Etr. **10** (1785), pp. 477. The integration was performed by **Monge-Liouville**, *Applications*, pp. 211, *et seq.*

Their asymptotic lines are orthogonal.

It follows from $d = d'' = 0$ that $f = 0$.

The converses of those theorems are obviously true, as well; namely:

If the minimal lines on a surface are conjugate or the asymptotic lines are orthogonal then the surface will be a minimal surface.

The fact that the lines of curvature of minimal surfaces are isometric was proved in § **9**. It likewise follows from equations [§ **7**, (6)] that the asymptotic lines are isometric. In § **17**, it will be shown that a system of isometric lines can also be obtained from each system of parallel planar intersections and its orthogonal trajectories, so, e.g., from the level curves of the surface and the lines of greatest inclination.

Furthermore, from [§ **11**, (4)], one will have for the spherical map on the sphere K that:

$$E = -D = h\,d - k\,e, \quad F = -D' = h\,d' - k\,f, \quad G = -D'' = h\,d'' - k\,g\,. \tag{2}$$

When one sets $h = 0$, these equations will yield the following **theorems**:

1. *The spherical images of the minimal lines are once more minimal lines, and likewise asymptotic lines (viz., the rectilinear generators of the sphere).*

That is because it will follow from $e = g = 0$ that $E = -D = 0$, $G = -D'' = 0$.

2. *The spherical images of isometric lines (in particular, the lines of curvature again) are again isometric lines.*

That is because it follows from $e = g$, $f = 0$ that $E = G$, $F = 0$.

3. *The spherical image of a minimal surface is conformal to the original one (**Urbild**)* ([1]).

That is because (2) implies that $dS^2 = -k\,ds^2$ or $ds = r\,dS$, when r is the radius of principal curvature of the surface.

Those three theorems are, in turn, characteristic of minimal surfaces; i.e., they can be inverted in the following way:

4. *If the spherical images of the minimal lines on a surface are again minimal lines then the surface is either a minimal surface or a sphere.*

That is because when $e = g = 0$ and $E = G = 0$, it will follow from (2) that $h\,d = 0$, $h\,d'' = 0$, so:

([1]) **Bonnet**, Jour. d. Math. **5** (1860), pp. 227.

either $h = 0$; i.e., the surface is a minimal surface

or $d = d'' = 0$, so from [§ **5**, (10)], $r_1 = r_2$; i.e., the surface is a sphere.

5. *If the spherical images of isometric lines on a surface are again isometric lines then the surface is either a minimal surface or a sphere.*

That is because, from (2), if $e = g$, $f = 0$ and $E = G$, $F = 0$ then $h\,d = h\,d''$, $h\,d' = 0$, so:

either $h = 0$ or $d = d''$, $d' = 0$, so, from [§ **5**, (10)], $r_1 = r_2$.

6. *If the spherical map of a surface is conformal to the original then the surface is either a minimal surface or a sphere.*

That is because (2) implies the following conditions for a surface to be conformal to its spherical image:

$$h\,d - k\,e = \mu\,e, \qquad h\,d' - k\,f = \mu\,f, \qquad h\,d'' - k\,g = \mu\,g,$$

in which μ is a factor that is yet-to-be-determined. If one multiplies those equations, once by d'', $-2d'$, d, resp., and then by g, $-2f$, e and adds them each time then, from [§ **2**, (16) and (17)], one will get:

$$2hk = h\,(\mu + k), \qquad h^2 = 2\,(\mu + k).$$

It then follows that:

Either $h = 0$, $\mu = -k$; i.e., the surface is a minimal surface

Or $\mu = k$, $h^2 = 4k$, so $r_1 = r_2$; i.e., the surface is a sphere.

For the further study of minimal surfaces, we *derive their equation using the methods of* § **8** ([1]). Next, the fundamental quantities e, f, g ; d, d', d'' will be presented. If one chooses the minimal lines to be the parameter curves (u, v) then:

(3) $$e = g = 0, \qquad d' = 0.$$

When one sets $d = d'' = 1$, one will then have to integrate the differential equation [§ **9**, (14)], under the assumption that $\mu = \infty$. According to **Liouville** ([2]), the general integral includes two arbitrary functions U and V, the first of which depends upon only u, while

([1]) **Enneper**, Zeit. Math. **9** (1864). **Enneper**'s derivations has been altered here in such a way that one can immediately get **Weierstrass**'s [Sitz. Berl. Acad., (1866)] fundamental form for them from his well-known investigations on minimal surfaces.

([2]) **Monge-Liouville**, *Applications*, Note IV, pp. 597.

the second one depends upon only v. We shall change the treatment somewhat and likewise introduce two such functions U, V when we set:

$$d = -U, \qquad d'' = -V. \tag{4}$$

The equation for f will then go to:

$$f \cdot \frac{\partial^2 \log f}{\partial u\, \partial v} = U\, V.$$

It is satisfied by a particular integral; one such is:

$$f = \tfrac{1}{2}\, UV\, (1 + uv)^2. \tag{5}$$

The six fundamental quantities of minimal surfaces are determined by the equations (3), (4), (5). One gets the *line element* $ds^2 = 2f\, du\, dv$ from them:

$$ds^2 = UV\, (1 + uv)^2\, du\, dv, \tag{6}$$

as well as the *radius of principal curvature* $r = f : \sqrt{d\, d''}$:

$$r = \tfrac{1}{2}\sqrt{U\, V}\, (1 + uv)^2\,. \tag{7}$$

Moreover, one gets the *differential equation for the lines of curvature:*

$$U\, du^2 - V\, dv^2 = 0 \tag{8}$$

and the *differential equation for the asymptotic lines:*

$$U\, du^2 + V\, dv^2 = 0\,. \tag{9}$$

The lines of curvature and the asymptotic lines are then determined by quadratures ([1]).

From the form of (8) and (9), and the differential equations $du = 0$, $dv = 0$ for the minimal lines, one will also find that the differential equations of the minimal lines, the lines of curvature, and the asymptotic curves for minimal surface that is given in terms of arbitrary parameters are also integrable by quadratures ([2]).

([1]) **Roberts**, J. d. Math. **11** (1846), pp. 300.

([2]) From the equations of § **9**, the theorem is true for surfaces of constant mean curvature, while for the surfaces of constant curvature, only the differential equations of the lines of curvature and asymptotic lines can be determined by quadratures, and for surfaces with isometric lines of curvature, that will only be true of the latter lines and the minimal lines. The common source of those, and similar, theorems is the following theorem that **Lie** proved in the theory of multipliers: If one knows that a linear relation with constant coefficients $Z \equiv \lambda\, X + \mu\, Y$ exists between the integrals $X = \alpha$, $Y = \beta$, $Z = \gamma$ (where α, β, γ are integration constants) of three first-order ordinary differential equations then those three differential

One must now *integrate the simultaneous system* [§ **2**, (2)]. In our case, it reads:

$$\frac{\partial a}{\partial u} = \frac{2a_2}{(1+uv)^2 V}, \qquad \frac{\partial a_1}{\partial u} = -Ua + \left(\frac{2v}{1+uv} + \frac{U'}{U}\right)a_1,$$

$$\frac{\partial a}{\partial v} = \frac{2a_1}{(1+uv)^2 U}, \qquad \frac{\partial a_2}{\partial v} = -Va + \left(\frac{2u}{1+uv} + \frac{V'}{V}\right)a_2,$$

while $\frac{\partial a_2}{\partial u} = \frac{\partial a_1}{\partial v} = 0$. One easily obtains three systems of particular a, a_1, a_2 whose determinant does not vanish, namely, with the notations of § **8** :

$$x_0 = \frac{2u}{1+uv}, \qquad y_0 = \frac{2v}{1+uv}, \qquad z_0 = \frac{uv-1}{1+uv},$$

$$x_1 = -u^2\,U, \qquad y_1 = U, \qquad z_1 = u\,V,$$

$$x_2 = V, \qquad y_2 = -v^2\,V, \qquad z_2 = v\,V.$$

It follows from this that:

$$M_{11} = M_{22} = M_{23} = M_{31} = 0, \qquad M_{33} = 1, \qquad M_{12} = 2,$$

and as the second-order equation of the surface [§ **8**, (8)]:

$$4\xi\eta + \zeta^2 = 1.$$

The coefficients (ξ_i , η_i , ζ_i) are determined from the associated equations:

$$4\,\xi_i\,\eta_i + \zeta_1^2 = 1, \qquad 2\,(\xi_i\,\eta_k + \eta_i\,\xi_k) + \zeta_i\,\zeta_k = 0.$$

The system of values (or the position in space) that corresponds to the **Weierstrass** form is:

$$\xi_0 = \eta_0 = \tfrac{1}{2}, \quad \xi_1 = -\frac{i}{2}, \quad \eta_1 = \frac{i}{2}, \quad \zeta_2 = 1, \quad \zeta_0 = \zeta_1 = \xi_2 = \eta_2 = 0.$$

With that, equations [**8**, (4)] will become:

$$a = \frac{u+v}{1+uv}, \qquad b = i\,\frac{v-u}{1+uv}, \qquad c = \frac{uv-1}{1+uv},$$

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equations can be integrated by quadratures. Cf., **Lie-Scheffers**, *Vorlesungen über Differentialgleichungen*, Leipzig, Teubner, pp. 163, as well as pp. 169, *et seq.* (1891).

$$a_1 = \frac{1}{2}(1 - u^2)\,U, \qquad b_1 = \frac{i}{2}(1 + u^2)\,U, \qquad c_1 = u\,U, \tag{10}$$

$$a_2 = \frac{1}{2}(1 - v^2)\,V, \qquad b_2 = -\frac{i}{2}(1 + v^2)\,V, \qquad c_2 = v\,V.$$

If one then sets:

$$U = F(u), \qquad V = \Phi(v) \tag{11}$$

then, from [§ **8**, (11)], one will ultimately get the equations for the minimal surface in the form ([1]):

$$x = \frac{1}{2}\int(1-u^2)\,F(u)\,du + \frac{1}{2}\int(1-v^2)\,\Phi(v)\,dv,$$

$$y = \frac{i}{2}\int(1+u^2)\,F(u)\,du - \frac{i}{2}\int(1+v^2)\,\Phi(v)\,dv, \tag{12}$$

$$z = \int u\,F(u)\,du + \int v\,\Phi(v)\,dv.$$

For:

$$U = f''(u), \qquad V = \varphi''(v), \tag{13}$$

one will get:

$$x = \frac{1}{2}(1 - u^2)\,f''(u) + u\,f'(u) - f(u) + \frac{1}{2}(1 - v^2)\,\varphi''(v) + v\,\varphi'(v) - \varphi(v),$$

$$y = \frac{i}{2}(1 + u^2)\,f''(u) - i\,u\,f'(u) + i\,f(u) - \frac{i}{2}(1 + v^2)\,\varphi''(v) + i\,v\,\varphi'(v) - i\,\varphi(v), \tag{14}$$

$$z = u\,f''(u) - f'(u) + v\,\varphi''(v) - \varphi'(v).$$

This derivation of the equation of a minimal surface is purely analytical. One can abbreviate the solution when one uses the geometric theorem that was proved above that the spherical map of the minimal lines on a minimal surface will again be minimal lines on the sphere. Now, represent the coordinates (a, b, c) of a point on the sphere of radius 1 in terms of the parameters (u, v) that belong to the minimal lines as in [§ **3**, (15)] using the expressions in (10). If one substitutes those values, as well as the values (3), (4), (5), in equations [§ **2**, (6)] then one will get equations (12) by quadrature.

([1]) **Weierstrass**, *loc. cit.*, pp. 619. If one sets:

$$\sqrt{2U}\,du = d\alpha, \qquad \sqrt{2V}\,dv = d\beta, \qquad u = A(\alpha), \qquad v = B(\beta), \qquad \text{so} \qquad U = \frac{1}{2A'^2}, \qquad V = \frac{1}{2B'^2}$$

then one will get **Enneper**'s equations, *loc. cit.*, pp. 107.

Monge ([1]) gave the shortest path to the equation of the minimal surface. One must satisfy equation (1), or when one once more chooses the minimal lines to be the parameter curves, the equations $e = g = 0$; $d' = 0$. Upon applying them, the middle equation in [§ **2**, (9)] will go to:

$$\frac{\partial^2 x}{\partial u\,\partial v} = 0, \qquad \frac{\partial^2 y}{\partial u\,\partial v} = 0, \qquad \frac{\partial^2 z}{\partial u\,\partial v} = 0. \tag{15}$$

Hence:

$$x = U_1 + V_1\,, \quad y = U_2 + V_2\,, \quad z = U_3 + V_3\,. \tag{16}$$

The equation $d' = 0$ will be satisfied with that. The equations $e = 0$, $g = 0$ will then give the following conditions for the functions U and V in (16):

$$U_1'^2 + U_2'^2 + U_3'^2 = 0, \qquad V_1'^2 + V_2'^2 + V_3'^2 = 0. \tag{17}$$

One can now set x and y equal to the values that were given (12), and one will then also get the value of z that was given in (12) from (17). It follows from (16) and (17) that the line element ds of the surface will be:

$$ds^2 = 2\,(dU_1\,dV_1 + dU_2\,dV_2 + dU_3\,dV_3). \tag{18}$$

Equations (16) and (17) include the **theorem**:

The general minimal surface can be generated by translating a minimal curve of one system along a minimal curve of the other system.

That theorem defines the foundation for further important studies of the degree and class of *algebraic minimal surfaces* and the determination of such surfaces from given elements. We refer to the literature ([2]) for that theory.

The following considerations are connected with **Weierstrass**'s form (12) or (14) for a minimal surface. When one chooses F and Φ or f and φ to be arbitrary functions, those equations will represent all possible minimal surfaces. All of those surfaces are related to each other by the parameters (u, v) in such a way that they will have parallel normals at corresponding points (u, v) [eq. (10)]. The minimal surface (12) or (14) is *real* when and only when u and v are complex-conjugate variables, and when F and Φ or f and φ are conjugate functions such that each function $F(u)$ or $f(u)$ of the complex variable u will belong to a well-defined real minimal surface ([3]).

Under that assumption, one can write the equations (12) conveniently as ([4]):

([1]) **Monge-Liouville**, *Applications*, pp. 211, *et seq.*

([2]) **Lie**, Arch. før Math. **2** (1877), pp. 295. Math. Ann. **14** (1878), pp. 331 and **15** (1870), pp. 465. **Darboux**, *Leçons*, I, Book III, Chap. VI-IX.

([3]) **Bonnet**, C. R. Acad. Sc. **37** (1853), pp. 529.

([4]) **Weierstrass**, *loc. cit.*, pp. 619.

$$(19) \qquad x = \Re\int (1-u^2)\,F(u)\,du\,, \qquad y = \Re\int i\,(1+u^2)\,F(u)\,du\,, \qquad z = \Re\int 2u\,F(u)\,du\,,$$

in which the leading symbol $\Re$ means that one should take the real part of the following complex quantity.

One can further show that the minimal surface is *algebraic* if and only if the function f in (14) is an algebraic function ([1]). From (8) and (9), an algebraic minimal surface will have algebraic lines of curvature and asymptotic lines when $\int f'''(u)\,du$ and $\int \varphi'''(u)\,du$ are algebraic functions of u and v.

Equations (12) or (14) give the general minimal surface in terms of point coordinates (x, y, z). One gets the equation of the surface in planar coordinates (X, Y, Z, T) from them by using the formulas of § **11**. From (10), one has:

$$(20) \qquad X = \frac{u+v}{1+uv}, \qquad Y = i\,\frac{v-u}{1+uv}, \qquad Z = \frac{uv-1}{1+uv}.$$

In order to form T in terms of (u, v), one must substitute the values (20) and (12) or (14) in:

$$T = x\,X + y\,Y + z\,Z\,.$$

One will then get ([2]):

$$(1 + uv)\,T = \int^{u} (u-u_1)(1+u_1 v)\,F(u_1)\,du_1 + \int^{v} (v-v_1)(1+u v_1)\,\Phi(v_1)\,dv_1$$

or

$$(21) \qquad T = f'(u) + \varphi'(v) - \frac{2uv}{1+uv}\left[\frac{f(u)}{u} + \frac{\varphi(v)}{v}\right].$$

This expression is, at the same time, the general integral of the partial differential equation:

$$T_{12} + \frac{2}{(1+uv)^2}\,T = 0,$$

which one gets from the middle equation in [§ **11**, (20)] for the values (20) of X, Y, Z, and the values $E = G = 0$, $P' = Q' = 0$ that one gets from $e = g = 0$, $h = 0$, $d' = 0$.

One will get the equation for the minimal surface in terms of only the planar coordinates (X, Y, Z, T) by eliminating (u, v) from (20) and (21). One will have:

$$(22) \qquad u = \frac{X+iY}{1-Z}, \qquad v = \frac{X-iY}{1-Z},$$

and from that:

([1]) **Weierstrass**, *loc. cit.*, pp. 621.
([2]) **Weierstrass**, *loc. cit.*, pp. 623.

$$T = f'\left(\frac{X+iY}{1-Z}\right) + \varphi'\left(\frac{X-iY}{1-Z}\right) - (X-iY)\,f\left(\frac{X+iY}{1-Z}\right) - (X-iY)\,\varphi\left(\frac{X+iY}{1-Z}\right).$$

If the surface is real, so f and φ are conjugate functions, then one can give a real form to this equation. Namely, if one sets:

$$f(u) = f\left(\frac{X+iY}{1-Z}\right) = P + i\,Q, \qquad \varphi(v) = \varphi\left(\frac{X-iY}{1-Z}\right) = P - i\,Q$$

then it will follow that:

$$P = \tfrac{1}{2}[f(u) + \varphi(v)], \qquad \frac{\partial P}{\partial X} = \frac{f'(u)+\varphi'(v)}{2(1-Z)},$$

and one will have:

(23) $$T = 2\,(1-Z)\,\frac{\partial P}{\partial X} - 2\,(X\,P + Y\,Q)\,.$$

One will then have, e.g.:

$$T = \frac{2(2-Z)(X^2-Y^2)}{(1-Z)^2}$$

for $f(u) = u^3$.

Some special minimal surfaces that are of interest shall now be mentioned. If one sets:

(24) $$u = r\,e^{i\varphi}, \qquad v = r\,e^{-i\varphi}, \qquad F(u) = e^{i\alpha}\,u^{-2}, \qquad \Phi(v) = e^{-i\alpha}\,v^{-2},$$

in which r, φ, α mean real quantities, then an easy calculation will give one the surface:

(25) $$\begin{aligned} -x &= r\cos(\varphi+\alpha) + r^{-1}\cos(\varphi-\alpha), \\ -y &= r\cos(\varphi+\alpha) + r^{-1}\sin(\varphi-\alpha), \\ z &= 2\log r\cos\alpha - 2\varphi\sin\alpha \end{aligned}$$

for a suitable determination of the constants.

If one sets $\alpha = 0$ and inverts the signs of x and y then one will have the surface:

(26) $$x = (r + r^{-1})\cos\varphi, \qquad y = (r + r^{-1})\sin\varphi, \qquad z = 2\log r$$

or

$$\sqrt{x^2+y^2} = r + r^{-1} = e^{z/2} + e^{-z/2}.$$

By contrast, if one sets $\alpha = \pi / 2$ and inverts the signs of y and z then one will get the surface:

$$x = (r - r^{-1}) \sin \varphi, \qquad y = (r - r^{-1}) \cos \varphi, \qquad z = 2\varphi \tag{27}$$

or

$$\frac{z}{2} = \arctan \frac{x}{y}.$$

Equations (25) represent a helicoid ([1]), equations (26), a surface of revolution (viz., the catenoid)([2]), and equations (27) represent a ruled surface [namely, the helical conoid (*die flächgängige Schraubenfläche*)] ([3]). Those surfaces can be developed to each other, since for the values (24), the expression (6) will go to:

$$ds^2 = r^{-4} (dr^2 + r^2 \, d\varphi^2),$$

so the line element will be independent of α, or it will be the same for the three surfaces (25), (26), (27).

One easily convinces oneself that equations (25), (26), (27) represent the only helicoids, surfaces of revolution, and ruled surfaces that are, at the same time, minimal surfaces ([4]). We shall show this briefly for the surfaces of revolution. The general surface of revolution has the equation:

$$x = \rho \cos \psi, \quad y = \rho \sin \psi, \quad z = P,$$

in which P is an arbitrary function of ρ. If one denotes the first and second derivatives of P by P' and P'', resp., then one will get the values:

$$h = \frac{\rho P'' + P'(1 + P'^2)}{\rho (1 + P'^2)^{3/2}}, \qquad k = \frac{P' P''}{\rho (1 + P'^2)^2}$$

for the mean curvature h and the curvature k . The minimal surfaces that are found among the surfaces of revolution are then determined by the differential equation ($h = 0$):

$$\rho P'' + P'(1 + P'^2) = 0.$$

When α is the integration constant, the first integral is:

$$P' = \frac{-\alpha}{\sqrt{\rho^2 - \alpha^2}},$$

([1]) **Scherk**, J. für Math. **13** (1835), pp. 185.
([2]) **Meusnier**, Sav. Etr. **10** (1785), pp. 477.
([3]) **Meusnier**, *ibid.*
([4]) **Catalan**, J. de Math. **7** (1842), pp. 203.

and a second integration will give the equation of the surface as:

$$z = P = -\alpha \log \frac{\rho + \sqrt{\rho^2 - \alpha^2}}{\alpha} \qquad \text{or} \qquad \rho = \frac{\alpha}{2}(e^{z/\alpha} + e^{-z/\alpha}) .$$

Furthermore, we pose the **problem**:

Determine those minimal surfaces for which the spherical representation of the lines of curvature is a given system of isometric lines on the sphere.

From [§ **3**, (22)], the differential equation of the most general system of isometric lines on the sphere has the form:

$$P\,(u)\,du^2 - Q\,(v)\,dv^2 = 0.$$

On the other hand, from (8), the differential equation of the lines of curvature of the minimal surfaces is:

$$U\,du^2 - V\,dv^2 = 0.$$

Therefore, if P and Q are given then U and V can be determined from the equations:

$$U = P\,(u), \qquad V = Q\,(v). \tag{28}$$

An example of that is defined by the determination of the minimal surfaces that have *two families of planar lines of curvature* ([1]).

The two families of planar lines of curvature correspond to two mutually-orthogonal systems of minor circles in the sphere that are intersected by two pencils of planes whose axes are reciprocal polars of the sphere (cf., § **11**). Under stereographic projection, the two families of circles in the plane (ξ, η) map to a pencil of circles that go through two fixed points and the associated orthogonal system of circles. One puts the two fixed points at the points $+a$ and $-a$ on the ξ-axis. The equation of the pencil of circles with the parameter λ is then:

$$\xi^2 + \eta^2 - a^2 = 2\lambda\,\xi . \tag{29}$$

The coordinates (ξ, η) are coupled with the parameters (u, v) of the minimal lines on the sphere by equations [§ **3**, (19)]:

$$u = \xi + i\,\eta\,, \qquad v = \xi + i\,\eta\,. \tag{30}$$

When one solves (29) for λ, it will then follow that:

([1]) **Bonnet**, C. R. Acad. Sc. **41** (1855), pp. 41.

$$\frac{uv - a^2}{u + v} = \lambda,$$

and the differential equation of the pencil of circles or the first system of minor circles on the sphere will become:

$$\frac{du}{u^2 + a^2} + \frac{dv}{v^2 + a^2} = 0.$$

One gets the differential equation of the second orthogonal system of minor circles from this when one replaces $d\xi$, $d\eta$ with – $d\eta$, $d\xi$, resp., or from (30), du, dv with $i\,du$ and $-i\,dv$, resp.:

$$\frac{du}{u^2 + a^2} - \frac{dv}{v^2 + a^2} = 0.$$

The product of the last two equations is the differential equation of the given system of isometric lines on the sphere. From (28), the associated minimal surface with two systems of planar lines of curvature is then determined by the functions:

$$(31)\qquad U = F(u) = \frac{1}{(u^2 + a^2)^2}, \qquad V = \Phi(v) = \frac{1}{(v^2 + a^2)^2}.$$

The cases $a = 0$ and $a = \infty$ are interesting; both assumptions lead to the same surface ([1]), as is easy to see. For $a = \infty$, U and V are the same real constants. From (29), the system of circles in the plane that correspond to the lines of curvature on the minimal surface will become lines parallel to the axes ξ, η; therefore, ξ, η are the parameters of the lines of curvature of the minimal surface. The surface itself is algebraic and has order nine, while the lines of curvature are third-order plane curves. If one sets $F(u) = \Phi(v) = 3$, defines x, y, z using (12), and replaces u, v with the values ξ, η from (20) then one will get:

$$(32)\qquad x = 3\xi + 3\xi\,\eta^2 - \xi^3, \quad -y = 3\eta + 3\eta\,\xi^2 - \eta^3, \qquad z = 3\xi^2 - 3\eta^2.$$

Eliminating η or ξ will give the equations of the planes in which the lines of curvature $\xi = a$ and $\eta = b$ lie:

$$x + \xi z - 3\,\xi - 2\xi^2 = 0, \qquad\qquad y + \eta z + 3\eta + 2\eta^2 = 0,$$

resp.

The *line element of the minimal surface*, which is determined by (6), is connected with some further considerations in regard to the conformal map and the development of minimal surfaces onto other surfaces. The conformal map onto the sphere K was mentioned before on pp. 66. For the *conformal map onto the plane*, one must only

([1]) **Enneper**, Zeit. Math. **9** (1864), pp. 108 and Gött. Nachr. (1882), pp. 40.

introduce two complex-conjugate quantities $\xi_1 + i\,\eta_1$ and $\xi_1 - i\,\eta_1$ in place of u and v, resp., and consider ξ_1 and η_1 to be the rectangular coordinates of a point in the plane.

If one starts from, e.g., the differential equation of the lines of curvature (8) and sets:

$$\sqrt{U}\,du + \sqrt{V}\,dv = 2\,d\xi_1\,, \qquad \sqrt{U}\,du - \sqrt{V}\,dv = 2i\,d\eta_1$$

or

$$(33) \qquad \sqrt{U}\;du = d\xi_1 + i\,d\eta_1\,, \qquad \sqrt{V}\;du = d\xi_1 - i\,d\eta_1$$

then it will follow from (6) and (7) that:

$$(33a) \qquad ds^2 = \sqrt{UV}\;(1 + uv)^2\;(d\xi_1^2 + d\eta_1^2) = 2r\,(d\xi_1^2 + d\eta_1^2)\,,$$

with which, one will arrive at a conformal map of the minimal surface for which the lines of curvature correspond to the parallels to the axes $\xi_1 = \alpha_1$, $\eta_1 = \beta_1$, and the asymptotic lines correspond to the parallels to the bisectors of the axis angles $\xi_1 + \eta_1 = \gamma_1$, $\xi_1 - \eta_1 = \delta_1$.

By contrast, if one sets:

$$(34) \qquad u = \xi + i\eta\,, \qquad v = \xi - i\eta$$

then it will follow from (6) that:

$$(34a) \qquad ds^2 = (1 + \xi^2 + \eta^2)\,F\,(\xi + i\eta)\,\Phi\,(\xi - i\eta)\,(d\xi^2 + d\eta^2),$$

and one will have a conformal map of the minimal surface onto the plane under which the point (ξ, η) in the plane that corresponds to the point (u, v) on the minimal surface will be obtained when one projects the spherical image point (u, v) stereographically onto the plane (cf., § **3**).

That suggests certain *development problems.* If one introduces polar coordinates (r, φ) in the plane, in place of the rectangular coordinates (ξ, η), and one sets:

$$(35) \qquad u = \xi + i\eta = r\,e^{i\varphi}, \quad v = \xi - i\eta = r\,e^{-i\varphi}, \quad uv = r^2, \qquad u : v = e^{2i\varphi}$$

then one will get:

$$(36) \qquad ds^2 = (1 + r^2)\,F\,(r\,e^{i\varphi})\,\Phi\,(r\,e^{-i\varphi})\,(dr^2 + r^2\,d\varphi^2),$$

in place of (34a). For:

$$(37) \qquad F\,(u) = A\,u^m, \qquad \Phi\,(v) = B\,v^m,$$

in which A, B are arbitrary and m is a real constant, the product $F\,(u)\,\Phi\,(v)$ will be independent of φ, and one will get:

(38) $$ds^2 = AB\,(1 + r^2)\,r^{2m}\,(dr^2 + r^2\,d\varphi^2)\,.$$

A comparison of that expression with [§ **6**, (17)] will show that the minimal surface that is defined by the functions (37) can be developed onto a surface of revolution that is determined by the equations:

$$\frac{dr}{r} = \frac{\sqrt{1+P'^2}}{\rho}\,d\rho, \qquad \sqrt{AB}\,(1 + r^2)\,r^{m+1} = \rho.$$

In that way, the parallel circles of the surface of revolution correspond to the curves uv = const. on the minimal surface. That is, the curves that go to parallel circles on the sphere under the spherical map and the meridian curves on the surface revolution correspond to the curves $u : v$ = const. on the minimal surface; i.e., to curves that go to the meridians on the sphere under the spherical map.

It can also be shown that the minimal surfaces that are determined by (37) are the only ones that can be developed onto a surface of revolution ([1]).

We further treat the question of finding *all minimal surfaces that can be developed onto a given minimal surface* ([2]). For a given surface (12), one can construct a second surface by replacing $x, y, z, u, z, F(u), \Phi(v)$ with $x_1, y_1, z_1, u_1, z_1, F_1(u_1), \Phi_1(v_1)$, resp. Should both surfaces be developable to each other then it would have to be possible to determine u_1, v_1 as functions of u, v in such a way that one would have $ds = ds_1$, or from (6):

(39) $$(1 + uv)^2\,F(u)\,\Phi(v)\,du\,dv = (1 + u_1v_1)^2\,F_1(u_1)\,\Phi_1(v_1)\,du_1\,dv_1\,.$$

However, if one sets:

$$du_1 = \frac{\partial u_1}{\partial u}du + \frac{\partial u_1}{\partial v}dv\,, \qquad dv_1 = \frac{\partial v_1}{\partial u}du + \frac{\partial v_1}{\partial v}dv$$

then equation (39) will give the conditions $\frac{\partial u_1}{\partial u}\frac{\partial v_1}{\partial u} = \frac{\partial u_1}{\partial v}\frac{\partial v_1}{\partial v} = 0$, from which, it will follow that u_1 is a function of u and v_1 is a function of v, or conversely, that v_1 is a function of u and u_1 is a function of v. We set $u_1 = \lambda(u)$, $v_1 = \mu(v)$. The functions λ and μ are then determined from:

$$(1 + uv)^2\,F(u)\,\Phi(v) = (1 + \lambda\mu)^2\,F_1(\lambda)\,\Phi_1(\mu)\,\lambda\,\mu\,.$$

If one takes the logarithm and forms the second derivatives with respect to u and v then it will follow that:

(40) $$\frac{1}{(1+uv)^2} = \frac{\lambda'\mu'}{(1+\lambda\mu)^2} \qquad \text{or} \qquad \frac{4\,du\,dv}{(1+uv)^2} = \frac{4\,du_1\,dv_1}{(1+u_1v_1)^2}\,.$$

([1]) **Schwarz**, Jour. für Math., Bd. 80, pp. 295.
([2]) **Bonnet**, C. R. Acad. Sc. **37** (1853), pp. 532. **Darboux**, *Leçons*, I, pp. 334.

That equation says that the spherical images dS and dS_1 that correspond to the line elements ds and ds_1 on the two minimal surfaces are equal to each other or that the spherical images of two corresponding figures on the two minimal surfaces are either equal or symmetric. Here, they must be equal, since we have taken u_1 to be a function of u and v_1 as a function of v, and since u_1 must go to u and v_1 into v under a continuous change in the coefficients of λ and μ. By a suitable orientation or placement of the second surface, one can make $u_1 = u$, $v_1 = v$. One will then get the following condition from (39):

$$F(u)\,\Phi(v) = F_1(u)\,\Phi_1(v)\,,$$

which can be satisfied only when one sets:

$$F_1(u) = e^{i\alpha} F(u), \qquad \Phi_1(v) = e^{-i\alpha}\Phi(v), \tag{41}$$

in which α is an arbitrary constant. The second surface (x_1, y_1, z_1) that is determined in that way is called an *associated* surface to the first one, and when $\alpha = \pi / 2$, its *adjunct* surface. One then has the **theorem:**

The minimal surfaces that can be developed from a given minimal surface are its associated surfaces.

The constant quantity α in (41) has a simple geometric meaning: It is equal to the angle ϑ that two corresponding line elements ds and ds_1 on both minimal surfaces define with each other. That angle is then the same for any two corresponding elements ds and ds_1 . In fact, if one defines the values of dx, dy, dz from (12) and recalls (41) in order to define the associated values of dx_1 , dy_1 , dz_1 then one will get:

$$\cos\vartheta = \frac{dx\,dx_1 + dy\,dy_1 + dz\,dz_1}{ds\,ds_1} = \frac{e^{i\alpha}+e^{-i\alpha}}{2} = \cos\alpha, \tag{41a}$$

from which $\vartheta = \alpha$. (Q. E. D.)

If one sets:

$$\begin{aligned} U_1 &= \frac{1}{2}\int(1-u^2)\,F(u)\,du\,, & U_2 &= \frac{i}{2}\int(1+u^2)\,F(u)\,du\,, & U_3 &= \int u\,F(u)\,du\,, \\ V_1 &= \frac{1}{2}\int(1-v^2)\,\Phi(v)\,dv\,, & V_2 &= -\frac{i}{2}\int(1+v^2)\,\Phi(v)\,dv\,, & V_3 &= \int v\,\Phi(v)\,dv\,, \end{aligned} \tag{42}$$

to abbreviate [cf., (12) and (16)], then one will have:

$$x_1 = U_1 + V_1\,, \qquad y = U_2 + V_2\,, \qquad z = U_3 + V_3 \tag{43}$$

for a point (x, y, z) on the original surface ($\alpha = 0$):

$$(44) \qquad x_0 = i\,(U_1 - V_1), \qquad y_0 = i\,(U_2 - V_2) \qquad z_0 = i\,(U_3 - V_3)$$

for a point (x_0, y_0, z_0) on the adjoint surface ($\alpha = \pi / 2$), and:

$$(45) \qquad x_\alpha = e^{i\alpha} U_1 + e^{-i\alpha} V_1 , \qquad y_\alpha = e^{i\alpha} U_2 + e^{-i\alpha} V_2 , \qquad z_\alpha = e^{i\alpha} U_3 + e^{-i\alpha} V_3$$

for a point $(x_\alpha, y_\alpha, z_\alpha)$ on the general associated surface, or:

$$x_\alpha = x \cos \alpha + x_0 \sin \alpha, \qquad y_\alpha = y \cos \alpha + y_0 \sin \alpha, \qquad z_\alpha = z \cos \alpha + z_0 \sin \alpha .$$

Examples of these three mutually-developable minimal surfaces are defined by the ruled surface, surface of rotation, and helicoid that were mentioned on pp. 73.

The quantity α in (41) can be regarded as a variable parameter. It is therefore possible to bend parts of a minimal surface continually in such a way that it will remain a minimal surface during the bending. Since the minimal surfaces that are defined by the functions in (37) can be developed onto a surface of revolution, they can also be bent into themselves continuously; **Enneper**'s surface (32) will serve as an example of that.

With that, we conclude with some further remarks that will lead to the *determination of a minimal surface from a curve that lies on it.* If u and v are complex-conjugate quantities, and F and Φ are conjugate functions then the surfaces (43) and (44) will be real. It will follow from (43) and (44) that:

$$(46) \qquad 2U_1 = x - i\,x_0 , \qquad 2U_2 = y - i\,y_0 , \qquad 2U_3 = z - i\,z_0 .$$

Furthermore, one has:

$$X\,dx_0 + Y\,dy_0 + Z\,dz_0 = 0, \qquad dx\,dx_0 + dy\,dy_0 + dz\,dz_0 = 0$$

[the latter is true from (41a)], and as a result, when ρ is a proportionality factor:

$$(47) \qquad \rho\,dx_0 = Y\,dz - Z\,dy, \quad \rho\,dy_0 = Z\,dx - X\,dz, \quad \rho\,dz_0 = X\,dy - Y\,dx .$$

The factor ρ is equal to + 1, as a direct construction of the two sides of (47) using (43), (44), and (20) will easily show. One will then have the equations:

$$(48) \quad 2U_1 = x - i\int (Y\,dz - Z\,dy) , \; 2U_2 = y - i\int (Z\,dx - X\,dx) , \; 2U_3 = z - i\int (X\,dy - Y\,dx) .$$

The expressions will serve as the solution to the following **problem** ([1]):

([1]) Posed by **Bjørling**, Archiv før Math. **4** (1864), pp. 290. Solved by **Bonnet**, C. R. Acad. Sc. **40** (1855), pp. 1107, and **42** (1856), pp. 532. **Schwarz** gave the solution above and equations (48) in Jour. für Math. **80** (1875), pp. 291.

Determine a minimal surface that goes through a given curve and has the same given normals (or tangent planes) along it.

Let the coordinates (x, y, z) and direction cosines (X, Y, Z) of the normal to the desired surface be given as analytic functions ([1]) of a real parameter t at each point to the curve, such that the equations:

$$X^2 + Y^2 + Z^2 = 1, \qquad X\,dx + Y\,dy + Z\,dz = 0 \tag{49}$$

will then be satisfied identically. The functions U_1, U_2, U_3 can then be determined for real values of t using equations (48), up to additive constants that influence only the position of the surface in space. From (49), those functions satisfy the equations:

$$X\,dU_1 + Y\,dU_2 + Z\,dU_3 = 0, \quad dU_1^2 + dU_2^2 + dU_3^2 = 0 \tag{50}$$

identically.

Now, U_1, U_2, U_3 also have a well-defined meaning as analytic functions of t for all complex values of t that t can assume as the argument of x, y, z and X, Y, Z. If one then allows the variable t to also take on complex values then the equations:

$$x' = 2\,\mathfrak{R}\,(U_1), \qquad y' = 2\,\mathfrak{R}\,(U_2), \qquad z' = 2\,\mathfrak{R}\,(U_3) \tag{51}$$

will represent a minimal surface with the desired behavior. Namely, it will go through the given curve and have the same normals along it, since the mean of the complex values of t that is defined by the values of x', y', z' and X', Y', Z' for real t will go to the given values of x, y, z and X, Y, Z.

The determination of the surface is unique. At the same, one has the **corollary** ([2]):

Any straight line on a minimal surface is a symmetry axis of the surface.

That is because the parts of the surface that are found on the two sides of the line will have the normals along that line. If one then rotates the one part through 180° around the line then its normals will coincide with those of the other part. Therefore, the first part will cover the second one. Similarly, one has:

Any plane that cuts a minimal surface orthogonally along a curve is a symmetry plane of the surface.

The **Bjørling** problem includes the following special problem ([3]):

Determine a minimal surface on which one is given either a geodetic line, an asymptotic line, or a line of curvature.

([1]) I. e., functions that can be developed in power series and extended to complex t.
([2]) **Weierstrass**, cf., **Schwarz**, Jour. für Math., Bd. 80, pp. 292.
([3]) **Bonnet**, C. R. Acad. Sc. **42** (1856), pp. 532.

That is because in all three cases, one knows the normals to the surface along the given curve. In the first case, they are the principal normals, in the second case, the binormals, and in the third case, they are a normal system of the curve that defines a developable surface.

Some examples that belong with those are:

The minimal surfaces for which a parabola is a geodetic line ([1])

and:

The minimal surfaces for which a prescribed algebraic curve is a geodetic line, when applied to conic sections and their evolutes, a cycloid, etc. ([2])

For the case of the ellipse, e.g., one will get a transcendental minimal surface on which a simply-infinite family of fourth-degree space curves lies, each of which possesses an isolated double point and has confocal spherical conic sections for its spherical images.

We shall briefly discuss the **more general problem**:

Lay a minimal surface through a closed (or open) line that is simple and continuous inside of it.

This problem was posed by **Gergonne** ([3]), and its various solutions were made intuitive experimentally by **Plateau** ([4]) using films of soapy water with glycerin.

Analytically, the problem has been solved up to now only in the case where the surface is bounded by rectilinear polygons ([5]) or also be rectilinear line segments and planes that intersect the surface orthogonally ([6]). Since every line on a surface is an asymptotic line, because the tangent planes to the surface along the line are osculating planes of that line and since every plane curve on a surface whose plane cuts the surface orthogonally is a line of curvature of the surface, because the successive surface normals along the curve intersect each other, one can also express the latter problem as:

Determine a minimal surface M that is bounded partly by planar lines of curvature and partly by straight asymptotic lines.

([1]) **Catalan**, C. R. Acad. Sc. **41** (1855), pp. 1019. Jour. Ec. poly. Cah. **37** (1858), pp. 160.

([2]) **Schwarz**, Jour. für Math. **80** (1675), pp. 293. **Herzog**, Vierteljahrschrift v. Wolf Zurich **20** (1875), pp. 217. **Henneberg**, Diss. Zurich 1875 and Zeit. v. Wolf **21** (1875), pp. 17.

([3]) **Gergonne**, Ann. Math. pure et appl. **7** (1816), pp. 68, 143-147.

([4]) **Plateau,** *Statique expér. et théor. des liquides*, Ghent and Leipzig, 1873.

([5]) **Riemann**, *Ges. Werke*, (1867), pp. 283 and 417. **Weierstrass**, Montsber. d. Berl. Acad., 1866, pp. 855.

([6]) **Schwarz**, Monatsber. d. Berl. Acad. 1865, pp. 149 and *Bestimmung einer speciellen Minimalfläche*, Berlin, 1871.

We shall briefly suggest the solution to that problem ([1]). Let the minimal surface M be referred to the parameters (u, v) of the minimal lines, and let it be assumed to have the form (19).

The surface M is first mapped conformally to a plane E_1 with the axes ξ_1, η_1 using equations (33), so by:

$$\sqrt{F(u)}\,du = d\xi_1 + i\,d\eta_1\,, \tag{52}$$

such that the images of the lines of curvature are then parallel to the axes ξ_1, η_1, and the images of the asymptotic lines are parallel to the bisector of the angle between those axes.

Secondly, the surface M is mapped conformally onto a sphere K of radius 1 around the origin, which will make the image of a planar line of curvature parallel to a great circle in the plane of the line of curvature, and the image of a straight asymptotic line parallel to a great circle in the normal plane to the asymptotic line. Moreover, the sphere K will then be mapped stereographically, and thus conformally, onto a plane E with the axes ξ, η using equations (34), so by:

$$u = \xi + i\,\eta, \tag{53}$$

which will make the great circles on K correspond to certain circles on E.

With that, the piece of the minimal surface M that is bounded by the asymptotic lines and lines of curvature will be mapped conformally to two different planes E_1 and E. The image in the plane E_1 is bounded by straight lines with well-defined positions, and the image in the plane E is bounded by circles with well-defined positions. From (52) and (53), the desired function $F(u)$ depends upon the points (ξ_1, η_1) and (ξ, η) in the planes E_1 and E by way of the equation:

$$\sqrt{F(u)} = \frac{d\xi_1 + i\,d\eta_1}{d\xi + i\,d\eta}. \tag{54}$$

One still has to represent $\xi_1 + i\,\eta_1$ as a function of $\xi + i\,\eta = u$. That emerges from the fact that the circular polygon in the plane E will be mapped conformally onto the rectilinear polygon on the plane E_1. That problem will be solved most easily when one maps both polygons conformally onto the upper half of a third plane E_0 with the axes ξ_0, η_0, and thus represents $\xi + i\,\eta$ and $\xi_1 + i\,\eta_1$ in terms of that third variable $\xi_0 + i\,\eta_0$, which can always be done, from considerations that belong to the theory of functions.

As a simplest example of a solution to the problem, we shall determine the minimal surface M that is bounded by two arbitrary lines in space.

The images of two lines on M on the sphere K are two great circles that one can regard as meridians (the Z-axis is the polar axis), one of which, viz., the zero meridian (the XZ-plane), makes an angle of $\alpha\kappa$ with the other one. Hence, the images of the two lines in the plane E will again be two lines that go through the origin, subtend the angle $\alpha\kappa$, and the first of which falls along the ξ-axis. By contrast, two lines on the surface M

([1]) One will find a beautiful and thorough treatment of that problem in **Darboux**, *Leçons*, I, Book III, Chap. X-XIII.

map to two parallel lines in the plane E_1 that subtend an angle of $\pi / 4$ with the ξ_1-axis. One will then have to map, on the one hand, a sector of the E-plane, and on the other hand, the parallel strips in the E_1-plane conformally into the E_0-plane. It is known that this will happen as a result of the equations:

$$\xi + i\,\eta = (\xi_0 + i\,\eta_0)^{\alpha}, \qquad e^{i\pi/4}(\xi_1 + i\,\eta_1) = C \cdot \log(\xi_0 + i\,\eta_0), \tag{55}$$

as one also easily verifies by introducing polar coordinate into the planes E and E_0. From (54), when $\xi + i\,\eta$ is again replaced with u, one will then have:

$$\sqrt{F(u)} = \frac{C}{\alpha} e^{-i\pi/4} u^{-1}, \qquad F(u) = i\,K\,u^{-2}, \tag{56}$$

in which K is a real constant. One will get the desired surface upon introducing that function into (19). As a comparison of (56) with (24) will show, it is identical with the helical conoid, which could also have been predicted geometrically. The constant K in (56) can be determined when one knows the shortest distances from the two lines to the minimal surface.

CHAPTER III

STUDY OF GENERAL SURFACE CURVES

§ 13. – The general space curve.

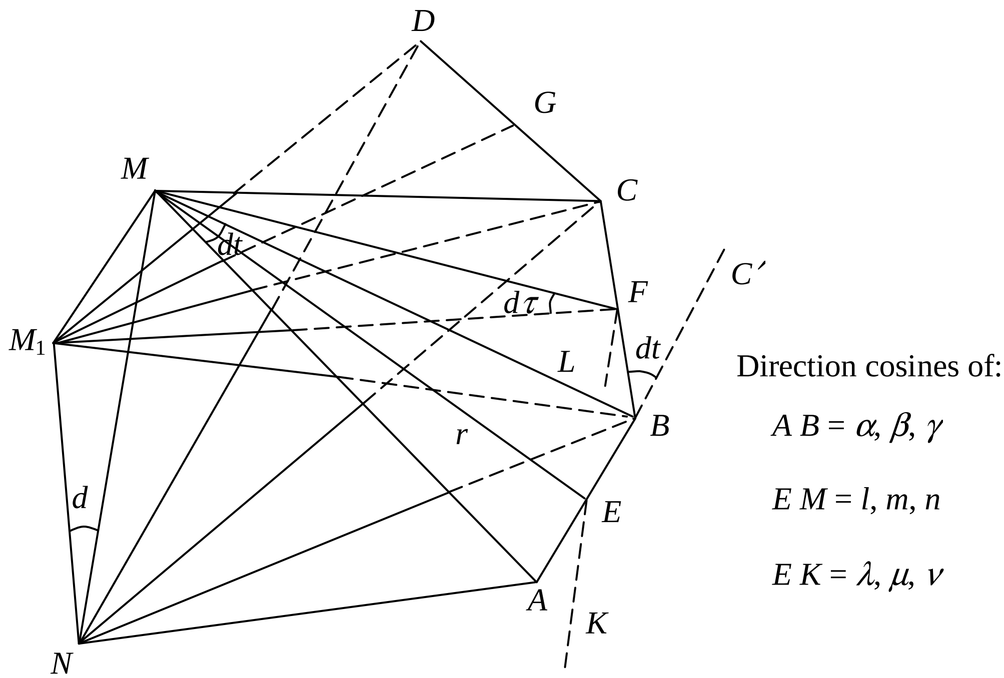

Figure 1.

For our study of surface curves, we shall briefly present the definitions and equations that will be true for a *general space curve* ([1]). (Fig. 1):

1. Let *A*, *B*, *C*, *D*, … be successive points on the curve, and let *E*, *F*, *G*, … be the midpoints of the arc elements *AB*, *BC*, *CD*, …

2. Let *ABCM* be the *osculating plane of the point E*, let *MEK* be the *normal plane*, and let *AEK* be the *rectifying plane*. Let $BCDM_1$, M_1FL, and *BFL*, resp., be the corresponding planes at the point *F*.

3. Let *ABC*′ be the *tangent* to the point *E*, let *EM* be the *principal normal* (viz., the normal in the osculating plane), and let *EK* be the *binormal* (viz., the normal

([1]) **Clairaut**, *Recherches sur les courbes à courbures*, 1731.
Lancret, "Mémoire sur les courbes à double courbures," Sav. etr. **1** (1805).
Monge-Liouville, *Applications*, pp. 392, 407, 418, *et seq.*, and pp. 547 (Note I).
Saint-Venant, J. Ec. poly., Cah. **30** (1845), in which the older literature is also summarized.

perpendicular to the osculating plane). Let *BC*, FM_1, and *FL*, resp., be the corresponding lines at the point *F*.

4, Let *M* be the *center* and let $AM = BM = CM$ be the *radius* of the *curvature circle* at *E*; i.e., the circle that goes through the three points *A*, *B*, *C*. Let M_1 and $BM_1 = CM_1 = DM_1$, resp., be the same quantities for the point *F*.

5. Let *N* be the *center* and let $NA = NB = NC = ND$ be the *radius of the osculating sphere of the point E;* i.e., the sphere that goes through the four points *A*, *B*, *C*, *D*. The center *N* is the point at which three consecutive normal planes at *E*, *F*, *G* intersect.

6. Let *MN* be the *curvature axis* (polar line) of the point *E* ; i.e., the line of intersection of two consecutive normal planes at *E* and *F*. The curvature axis *MN* is parallel to the binormal *EK*, and the point *M* is the point at which two consecutive curvature axes at *E* and *F* intersect.

7. Let $CBC' = AMB = EMF$ be the *contingency angle* of *E* ; i.e., the angle between two consecutive tangents or normal planes at *E* and *F*.

8. Let MNM_1 be the *torsion angle* of *E* ; i.e., the angle between the two consecutive binormals or curvature axes or osculating planes at the points *E* and *F*.

The envelope of the normal planes or the locus of curvature axes is a developable surface, namely, the *polar surface of the space curve.* The centers of the osculating circles define the edge of regression of that surface.

Let the following *notations be defined for those quantities* at a point (x, y, z) on a space curve:

The cosines of the inclination angles between the three principal direction and the coordinate axes:

α, β, γ	for the tangent,
l, m, n	for the principal normal,
λ, μ, ν	for binormal .

Furthermore, let:

ds	be the arc length of the curve,
dt	be the contingency angle,
$d\tau$	be the torsion angle,
$d\sigma$	be the angle between two consecutive principal normals,
$r = \dfrac{ds}{dt}$	be the radius of curvature,
$\rho = \dfrac{ds}{d\tau}$	be the radius of torsion,

R	be the radius of the osculating sphere,
x', y', z'	be the coordinates of the center of curvature,
X', Y', Z'	be the coordinates of the center of the osculating sphere.

The definition of the radius of torsion ρ is only an analogue to the radius of curvature r. A torsion circle that would correspond to the curvature circle does not exist.

In order to *represent those quantities analytically* ([1]), let the coordinates (x, y, z) of a point of the curve be given as functions of one parameter. One will then have:

(1) $$ds^2 = dx^2 + dy^2 + dz^2, \qquad ds\, d^2s = dx\, d^2x + dy\, d^2y + dz\, d^2z$$

for the arc length element ds at the point (x, y, z), and:

(2) $$|\,X - x \quad dx \quad d^2x\,| = 0$$

for the equation of the osculating plane when X, Y, Z are the running coordinates.

Moreover, one has:

$$\alpha^2 + \beta^2 + \gamma^2 = 1, \qquad l\lambda + m\mu + n\nu = 0,$$

(3) $$l^2 + m^2 + n^2 = 1, \qquad \lambda\alpha + \mu\beta + \nu\gamma = 0,$$

$$\lambda^2 + \mu^2 + \nu^2 = 1, \qquad \alpha l + \beta m + \gamma n = 0,$$

so as a result:

(4) $$\begin{vmatrix} \alpha & \beta & \gamma \\ l & m & n \\ \lambda & \mu & \nu \end{vmatrix} = \pm 1.$$

Under the assumption that the + sign has been chosen, so the positive directions of the tangent, principal normal, and binormal have the same relationships to each other as the positive x, y, and z-axes, one will have:

(5) $$\alpha = m\nu - n\mu, \qquad l = \mu\gamma - \nu\beta, \qquad \lambda = \beta n - \gamma m,$$

along with the corresponding equations for β, γ; m, n, and μ, ν.

First, we *calculate the quantities* α, l, λ; one obviously has:

(6) $$\alpha = \frac{dx}{ds}, \qquad \beta = \frac{dy}{ds}, \qquad \gamma = \frac{dz}{ds}.$$

Furthermore, if r temporarily represents a proportionality factor then one will obtain the osculating plane from equation (2):

([1]) **Frenet**, *Thesis*, Toulouse, 1847, and J. de Math. **17** (1852), pp. 437.

$$(7)\quad \lambda = \frac{r}{ds^2}(dy\, d^2z - dz\, d^2y), \quad \mu = \frac{r}{ds^2}(dz\, d^2x - dx\, d^2z), \quad \nu = \frac{r}{ds^2}(dx\, d^2y - dy\, d^2x),$$

and from (6) and (7), using (5):

$$(8)\qquad l = \frac{r}{ds^2}(ds\, d^2x - dx\, d^2s) = r\frac{d\alpha}{ds},$$

along with the corresponding values for m, n. Since $l^2 + m^2 + n^2 = 1$, the factor r is determined from (8) to be:

$$(9)\qquad \frac{1}{r^2} = \frac{d\alpha^2 + d\beta^2 + d\gamma^2}{ds^2}.$$

Secondly, we express $d\alpha$, dl, $d\lambda$ from a, l, λ; from (8), one has:

$$(10)\qquad \frac{d\alpha}{ds} = \frac{l}{r}, \qquad \frac{d\beta}{ds} = \frac{m}{r}, \qquad \frac{d\gamma}{ds} = \frac{n}{r}.$$

Furthermore, from (3), one has $\lambda\, d\lambda + \mu\, d\mu + \nu\, d\nu = 0$, and from (3) and (10), $\alpha\, d\lambda + \beta\, d\mu + \gamma\, d\nu = 0$; therefore, when one recalls (5):

$$d\lambda : d\mu : d\nu = l : m : n,$$

or, when ρ temporarily represents a proportionality factor:

$$(11)\qquad \frac{d\lambda}{ds} = \frac{l}{\rho}, \qquad \frac{d\mu}{ds} = \frac{m}{\rho}, \qquad \frac{d\nu}{ds} = \frac{n}{\rho}.$$

Finally, it follows from the middle equation (5), when one takes the differential and uses (10) and (11), that:

$$\frac{dl}{ds} = \frac{1}{r}(n\,\mu - m\,\nu) + \frac{1}{\rho}(m\,\gamma - n\,\beta),$$

or, from (5):

$$(12)\qquad \frac{dl}{ds} = -\left(\frac{\alpha}{r} + \frac{\lambda}{\rho}\right), \qquad \frac{dm}{ds} = -\left(\frac{\beta}{r} + \frac{\mu}{\rho}\right), \qquad \frac{dn}{ds} = -\left(\frac{\gamma}{r} + \frac{\nu}{\rho}\right).$$

From (11), the factor ρ is determined by the equation:

$$(13)\qquad \frac{1}{\rho^2} = \frac{d\lambda^2 + d\mu^2 + d\nu^2}{ds^2}.$$

Finally, one easily convinces oneself that the quantities r and ρ, which were introduced temporarily as proportionality factors, are nothing but the *radius of curvature* and the *radius of torsion*. In order to do that, we determine the values of dt and $d\tau$.

As is known, one has:

$$\cos v = \cos a \cos a' + \cos b \cos b' + \cos c \cos c'$$

for the angle v between two lines whose angles with the axes are (a, b, c) and (a', b', c'). It will follow from this that:

$$4\sin^2\frac{v}{2} = (\cos a' - \cos a)^2 + (\cos b' - \cos b)^2 + (\cos c' - \cos c)^2.$$

If one now sets $a' = a + da$, so $\cos a' - \cos a = d \cos a$, and $4\sin^2\frac{v}{2} = dv^2$, then one will get the angle dv between two lines whose directions differ infinitely little:

$$dv^2 = da^2 + db^2 + dc^2.$$

When that is applied to tangent and the binormal, that will give:

$$(14) \qquad dt^2 = d\alpha^2 + d\beta^2 + d\gamma^2, \qquad d\tau^2 = d\lambda^2 + d\mu^2 + d\nu^2.$$

A comparison with (9) and (13) shows that, in fact, r is identical to the radius of curvature, and ρ is identical to the radius of torsion. It will then follow from (12) that:

$$(15) \qquad ds^2 = dl^2 + dm^2 + dn^2 = dt^2 + d\tau^2.$$

Finally, one has:

$$(16) \qquad x' = x + r\,l, \qquad y' = y + r\,m, \qquad z' = z + r\,n$$

for the coordinates (x', y', z') of the center of curvature.

Equations (6), (10), (11), (12) give the **theorems**:

The higher differentials of the coordinates x, y, z of a point on the curve can be represented in terms of the nine cosines (α, β, γ), (l, m, n), (λ, μ, ν), *along with two quantities r and ρ, and their differentials.*

If (α, β, γ) *are given as functions of the arc length then one can derive the values of r and l, m, n from* (9) *and* (10), *and the values of ρ, and λ, μ, ν from* (13) *and* (12).

One can further prove the **theorem**:

A space curve is determined uniquely (except for its position in space) when its radius of curvature r and radius of torsion ρ are given as functions of the arc length s.

The *radius R* and *center* (X', Y', Z') of the *osculating* sphere at the point (x, y, z) still remain to be determined. As the point of intersection of three consecutive normal planes, the center is determined from the equations:

$$(17)\qquad \begin{aligned} (X'-x)\,\alpha + (Y'-y)\,\beta + (Z'-z)\,\gamma &= 0, \\ (X'-x)\,l + (Y'-y)\,m + (Z'-z)\,n &= r, \\ (X'-x)\,\lambda + (Y'-y)\,\mu + (Z'-z)\,\nu &= -\rho\frac{dr}{ds}, \end{aligned}$$

of which, the last two are formed by applying (10) to (12) to a first and second differentiation of the first equation.

Upon solving that, one will get:

$$(18)\qquad X'-x = r\,l - \rho\frac{dr}{ds}\lambda, \qquad Y'-y = r\,m - \rho\frac{dr}{ds}\mu, \qquad Z'-z = r\,n - \rho\frac{dr}{ds}\nu,$$

and upon squaring and adding:

$$(19)\qquad R^2 = r^2 + \rho^2\left(\frac{dr}{ds}\right)^2.$$

For variable parameters, equations (18) represent the edge of regression C' of the developable polar surface of the original curve C. There is nothing difficult about presenting the corresponding values (α, β, γ), (l, m, n), (λ, μ, ν), r, ρ, R for the curve C', and deriving a series of theorems on the connection between the curves C and C' by analogy, such as, e.g., the **theorems**:

The principal normals of the curves C and C′ are parallel at corresponding points.

The binormal of each of the two curves C and C′ is parallel to the tangent of the other.

We refer to some other presentations for the evolutes of a space curve ([1]).

([1]) This was first investigated by **Monge-Liouville**, *Applications*, pp. 396.

§ 14. – The general surface curve. Determining the radius.

For the study of the general surface curve, the radii r, ρ, R, and the angles (α, β, γ), (l, m, n), (λ, μ, ν), that appear in § **13** must be represented in terms of the parameters u, v. If one then lets H denote the angle between the surface normal and the principal normal of the curve then one can show that the *radii r, ρ, R and the angle H can also be expressed very simply in terms of the four quantities ds^2, L, M, N*, which will represent the most important surface curves when they are set to zero ([1]). Namely, we let:

$ds^2 = 0$	define the differential equation of the minimal lines,
$L = 0$	define the differential equation of the asymptotic lines,
$M = 0$	define the differential equation of the lines of curvature,
$N = 0$	define the differential equation of the geodetic lines.

From §§ **4** and **5**, one has:

$$L = a\,d^2x + b\,d^2y + c\,d^2z,$$

(1)
$$M = |\,a \quad da \quad dx\,|,$$

$$N = |\,a \quad dx \quad d^2x\,|.$$

The last two forms can be represented directly in terms of the parameters u, v, and their differentials when one multiplies them by the expression [§ **1**, (24)]:

(2)
$$\delta = \left| a \quad \frac{\partial x}{\partial u} \quad \frac{\partial x}{\partial v} \right|.$$

One then gets:

$$ds^2 = e\,du^2 + 2f\,du\,dv + g\,dv^2,$$

$$L = d\,du^2 + 2\,d'\,du\,dv + d''\,dv^2,$$

(3)
$$M = \frac{1}{\delta}\begin{vmatrix} e\,du + f\,dv & d\,du + d'\,dv \\ f\,du + g\,dv & d'\,du + d''\,dv \end{vmatrix},$$

$$N = \frac{1}{\delta}\begin{vmatrix} e\,du + f\,dv & m\,du^2 + 2m'\,du\,dv + m''dv^2 + e\,d^2u + f\,d^2v \\ f\,du + g\,dv & n\,du^2 + 2n'\,du\,dv + n''dv^2 + f\,d^2u + g\,d^2v \end{vmatrix}.$$

([1]) The form and the proof that we shall give to the following equations also makes it possible to extend to non-Euclidian geometry and higher-dimensional space, which has been done up to now for only one part of the theory of curvature.

If one sets:

$$(4)\qquad \begin{aligned} N_1 &= p\,du^2 + 2p'du\,dv + p''\,dv^2 + d^2 u, \\ N_2 &= q\,du^2 + 2q'du\,dv + q''\,dv^2 + d^2 v \end{aligned}$$

then N will assume the form:

$$(5)\qquad N = \delta(N_2\,du - N_1\,dv),$$

and from [§ **2**, (9)], the second differential of x will become:

$$(6)\qquad d^2 x = L\,a + N_1 \frac{\partial x}{\partial u} + N_2 \frac{\partial x}{\partial v}.$$

If one squares the middle equation (1) and uses the values (16) and (17) of § **2** then it will follow that:

$$(7)\qquad M^2 = h\,L\,ds^2 - L^2 - k\,ds^4.$$

If one multiplies the determinant $|\,da\ \ dx\ \ d^2x\,|$ times (2) then one will get:

$$(8)\qquad |\,da\ \ dx\ \ d^2x\,| = L\,M.$$

Finally, from (1), and due to [§ **2**, (4)], one will have:

$$(9)\qquad M\,N = |\,a\ \ da\ \ dx\,|\ |\,a\ \ dx\ \ d^2x\,| = -\,ds\,(L\,d^2s + ds \sum da\,d^2x).$$

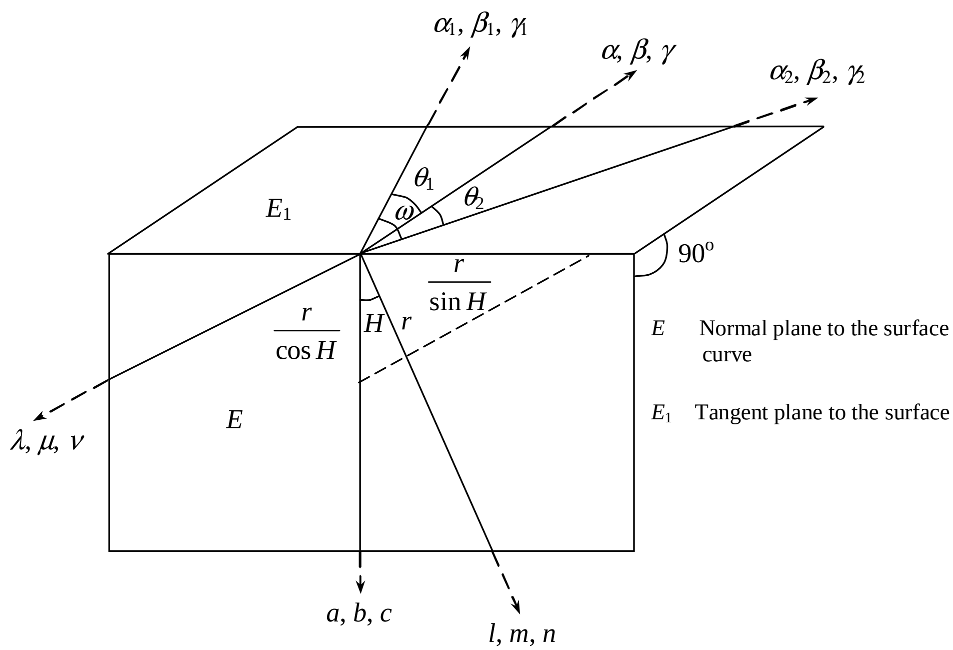

Figure 2.

With these preparations, we shall go on to the determination of the radii r, ρ, R, and the angle H. From (7) and (8) in § **13**, one will have (cf., Fig. 2):

$$\text{(10)}\qquad \begin{aligned} \cos H &= a\,l + b\,m + c\,n = \frac{r}{ds^2}\sum a\,d^2x, \\ \sin H &= a\,\lambda + b\,\mu + c\,\nu = \frac{r}{ds^2}\begin{vmatrix} a & dx & d^2x \end{vmatrix}. \end{aligned}$$

Hence [1]:

$$\text{(11)}\qquad \frac{\cos H}{r} = \frac{L}{ds^2}, \qquad \frac{\sin H}{r} = \frac{N}{ds^2}.$$

That will then imply the *angle H* and the *radius of curvature r* of the curve from:

$$\text{(12)}\qquad \tan H = \frac{N}{L\,ds}, \qquad \frac{1}{r^2} = \frac{N^2 + L^2 ds^2}{ds^2}.$$

If one differentiates the second equation in (10) and uses the formulas [§ **13**, (11)] then one will get:

$$\cos H\,dH = \sum a\,d\lambda + \sum \lambda\,da = \frac{ds}{\rho}\cos H + \frac{r}{ds^2}\begin{vmatrix} da & dx & d^2x \end{vmatrix}.$$

When one recalls (8) and (11), that will imply the following formula for the *radius of torsion* [2]:

$$\text{(13)}\qquad \frac{1}{\rho} = \frac{dH}{ds} - \frac{M}{ds^2}.$$

The expression for dH that one obtains from (12) contains the third differentials of u and v, in addition to the first and second ones.

Finally, in order to represent the *radius of osculation R*, one must get dr from [§ **13**, (19)]. If one differentiates the first equation in (11) then one can determine dr from:

$$\text{(14)}\qquad -\frac{\cos H}{r^2}\frac{dr}{ds} - \frac{\sin H}{r}\frac{dH}{ds} = \frac{1}{ds}\,d\left(\frac{L}{ds^2}\right) = \frac{dL}{ds^2} - 2L\frac{d^2r}{ds^2}.$$

This equation can be converted [3]. Due to (9), from (11) and (13), one will have:

$$\frac{2\sin H}{r}\left(\frac{1}{\rho} - \frac{dH}{ds}\right) = -\frac{2M\,N}{ds^2} = \frac{2L\,ds^2}{ds^4} + \frac{2\sum da\,d^2x}{ds^3}.$$

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[1] **Minding**, J. für Math. **6** (1830), pp. 159.

[2] **Bonnet**, J. Ec. poly., Cah. **32** (1848), pp. 14, in which the equation is first given when the parameter curves are lines of curvature. **Darboux**, *Leçons*, II, pp. 357 and 386, in a different form.

[3] **Laguerre**, Bull. soc. philomathique **7** (1870), pp. 49. **Darboux**, *Leçons*, II, pp. 395.

If one adds that to (14) then it will follow that:

$$-\frac{\cos H}{r^2}\frac{dr}{ds}+\frac{\sin H}{r}\left(\frac{2}{\rho}-3\frac{dH}{ds}\right)$$

(15)

$$=\frac{dL}{ds^2}+\frac{2\sum da\,d^2x}{ds^3}=\frac{1}{ds^2}\left(\sum da\,d^2x-\sum dx\,d^2a\right).$$

This expression, which will be denoted by P, will be completely free of d^2u and d^2v when it is represented in terms of u and v. A simple calculation will yield the expression for $P\,ds^3$:

$$P\,ds^3 = \sum da\,d^2x-\sum dx\,d^2a$$

(16)

$$= P_{20}\,du^3 + 3\,P_{21}\,du^2\,dv + 3\,P_{12}\,du\,dv^2 + P_{03}\,dv^3,$$

in which:

$$P_{30} = \frac{\partial d}{\partial u} - 2\,(p\,d + q\,d''), \qquad P_{12} = \frac{\partial d''}{\partial u} - 2\,(p'd' + q'd''),$$

(17)

$$P_{31} = \frac{\partial d}{\partial v} - 2\,(p'd + q'd'), \qquad P_{03} = \frac{\partial d''}{\partial v} - 2\,(p''d' + q''d'')\,.$$

If one applies equation (15) to a normal section of the surface, so one sets $H = 0$, then the left-hand side of (15) will reduce to $\frac{d}{ds}\left(\frac{1}{r}\right)$; i.e.:

The third-order differential equation $P = 0$ (16) defines the curves on the surface for which the contacting normal section at each point is contacted by its curvature circle at four points ([1]).

Some further representations are connected with that ([2]). From (17) and (6) in § **7**, the partial derivatives of d, d', d'' with respect to u, v are expressed in terms of e, f, g ; d, d', d'', and the four quantities P_{30}, P_{21}, P_{12}, P_{03} . That will lead to a simple representation of the derivatives of h and k with respect to u and v. Namely, if one differentiates the equation $\delta^2 k = d\,d'' - d'^2$ [§ **2**, (12)] with respect to u then one will get:

$$\delta^2\frac{\partial k}{\partial u}+2\delta k\frac{\partial \delta}{\partial u} = d\frac{\partial d''}{\partial u}-2d'\frac{\partial d'}{\partial u},$$

or, from [§ 1, (7a)], [§ **7**, (6)], and (17):

([1]) **De la Gournerie**, J. de Math. **20** (1855), pp. 145, in which that differential equation was first presented for the form $z = f(x, y)$ of the surface. Cf., **Laguerre**, *loc. cit.*, **Knoblauch**, J. für Math. **103**, pp. 32, *et seq.* **Darboux**, *Leçons*, II, pp. 396.

([2]) **Knoblauch**, *loc. cit.*

$$\delta^2 \frac{\partial k}{\partial u} = -2\,(p + q')\,(d\,d - d'^2) - 2d'(P_{21} + dp' + d'p + d'q' + d''q)$$
$$+ d''(P_{30} + 2\,dp + 2\,d'q) + d\,(P_{12} + 2\,d'p' + 2\,d''q'),$$

or

$$\text{(18)} \qquad \delta^2 \frac{\partial k}{\partial u} = d''P_{30} - 2\,d'P_{21} + d\,P_{12}\,, \qquad \delta^2 \frac{\partial k}{\partial v} = d\,P_{03} - 2\,d'P_{12} + d''P_{21}\;.$$

Similarly, one finds that:

$$\text{(19)} \qquad \delta^2 \frac{\partial h}{\partial u} = g\,P_{30} - 2\,f\,P_{21} + e\,P_{12}\,, \qquad \delta^2 \frac{\partial h}{\partial v} = eP_{03} - 2\,f\,P_{12} + g\,P_{21}\;.$$

Equations (18) and (19) can be used in order to represent the quadratic covariant of the cubic form (16), namely:

$$\text{(20)} \qquad \frac{1}{\delta^2}\begin{vmatrix} P_{30}\,du + P_{21}\,dv & P_{21}\,du + P_{12}\,dv \\ P_{21}\,du + P_{11}\,dv & P_{12}\,du + P_{03}\,dv \end{vmatrix},$$

in terms of only the radius of curvature r_0 of the normal section that belongs to the direction $du : dv$ and the radii of principal curvature r_1 and r_2 .

In conclusion, we shall now give the values of ds^2, L, M, N, or of H, r, ρ for the *parameter curves* that are used most often in applications. If one affixes subscripts u and v to the elements that refer to the curves v = const. and u = const., resp., then:

$$\text{(21)} \qquad \begin{aligned} ds_u^2 &= e\,du^2, & L_u &= d\,du^2, \\ ds_v^2 &= e\,dv^2, & L_v &= d''dv^2, \end{aligned}$$

$$\text{(22)} \qquad \begin{aligned} \delta M_u &= (e\,d' - f\,d)\,du^2, & \delta N_u &= (e\,n - f\,m)\,du^3 = \delta^2\,q\,du^3, \\ \delta M_v &= (f\,d' - g\,d)\,dv^2, & \delta N_v &= (f\,n'' - f\,m'')\,dv^3 = -\,\delta^2\,p''dv^3, \end{aligned}$$

and as a result:

$$\text{(23)} \qquad \frac{\sin H_u}{r_u} = \frac{N_u}{ds_u^3} = \frac{\delta q}{e^{3/2}}\,, \qquad \frac{\sin H_v}{r_v} = \frac{N_v}{ds_v^3} = -\frac{\delta p''}{g^{3/2}}\,.$$

If the parameter curves are *orthogonal*, so $f = 0$, then one will have:

$$\text{(23a)} \qquad \frac{\sin H_u}{r_u} = -\frac{1}{\sqrt{eg}}\frac{\partial\sqrt{e}}{\partial v}\,, \qquad \frac{\sin H_v}{r_v} = +\,\frac{1}{\sqrt{eg}}\frac{\partial\sqrt{g}}{\partial u}\,.$$

If the parameter curves are *lines of curvature*, in particular, so $f = d = 0$, then one will have:

$$\frac{\cos H_u}{r_u} = \frac{1}{r_1}, \quad \frac{\cos H_v}{r_v} = \frac{1}{r_2}, \tag{24}$$

$$\tan H_u = -\frac{r_1}{\sqrt{eg}}\frac{\partial\sqrt{e}}{\partial v} = -\frac{1}{\sqrt{EG}}\frac{\partial\sqrt{E}}{\partial v},$$
(25)
$$\tan H_v = +\frac{r_2}{\sqrt{eg}}\frac{\partial\sqrt{g}}{\partial u} = +\frac{1}{\sqrt{EG}}\frac{\partial\sqrt{G}}{\partial u},$$

and

$$\frac{1}{\rho_u} = \frac{1}{\sqrt{e}}\frac{\partial H_u}{\partial u}, \qquad \frac{1}{\rho_v} = \frac{1}{\sqrt{g}}\frac{\partial H_v}{\partial v}, \tag{26}$$

in addition to (23a).

From (23a), one easily proves the **theorem**:

If two systems of curves of constant geodetic curvature are orthogonal to each other then they will define an isometric system,

and other similar theorems.

§ 15. – Theorems on surface curves.

The formulas that were developed in § **14** imply a series of *general theorems on surface curves* ([1]). We appeal to the usual notation ([2]) and call:

$$\frac{1}{r} = \frac{\sqrt{N^2 + L^2 ds^2}}{ds^2} \qquad \text{the (absolute) curvature}$$

$$\frac{\cos H}{r} = \frac{L}{ds^2} \qquad \text{the normal curvature}$$

$$\frac{\sin H}{r} = \frac{N}{ds^2} \qquad \text{the tangential or geodetic curvature} \tag{1}$$

$$\frac{1}{\rho} = \frac{dH}{ds} - \frac{M}{ds^2} \qquad \text{the (absolute) torsion}$$

([1]) Cf., **Darboux**, *Leçons*, t. II, Book V, Chap. III.

([2]) The reason for the term "geodetic torsion" will follow below, and the reason for the term "geodetic curvature" will follow in § **16**.

$$\frac{1}{\bar{\rho}} = -\frac{M}{ds^2} \qquad \text{the geodetic torsion.}$$

For a curve whose osculating plane always cuts the surface at *the same angle*, one will have H = const., while one must set $\rho = \infty$ for a *planar surface curve* and set R = const. for *spherical curve*. The following statements are also true:

1. If a *surface curve is an asymptotic line* and one denotes its r, ρ, $\bar{\rho}$ by r_a, ρ_a, $\bar{\rho}_a$, resp., then $L = 0$, $H = 90^{\circ}$, so when $N \neq 0$ ([1]):

$$(2) \qquad \frac{1}{r_a} = \frac{N}{ds^3}, \qquad \frac{1}{\rho_a} = \frac{1}{\bar{\rho}_a} = -\frac{M}{ds^2}, \qquad \frac{1}{\rho_a^2} = -\frac{1}{r_1 r_2}.$$

The last equation is obtained as follows: Since $L = 0$, it will follow from [§ **14**, (7)] that $M^2 = -k\, ds^4$, so:

$$\frac{1}{\rho_a^2} = \frac{M^2}{ds^4} = -k = -\frac{1}{r_1 r_2}.$$

Equations (2) give the **theorems**:

The absolute curvature of an asymptotic line is the geodetic curvature; both are determined by the first equation (2).

The absolute torsion of an asymptotic line is equal to the geodetic torsion; its square is equal to minus the curvature of the surface, so:

The asymptotic lines on a surface of constant curvature all have the same constant torsion.

One likewise has $L = 0$ *for the curve that contacts that asymptotic line, so:*

Either $H \neq 90^{\circ}$ *and* $r = \infty$, *such as, e.g., for all planar sections that contact the asymptotic lines without falling in the tangent plane…*

One can then also regard asymptotic lines as the curves for which every planar intersection curve that contacts them (but not the intersection curve of the tangent plane) will have an inflection point at the contact point.

…or one has $H = 90^{\circ}$ *and* r *is finite.*

In the latter case, since M and P [§ **4**, (16)] include only the first differentials du and dv, they will have the same values for the asymptotic lines and the curves that contact them.

([1]) **Enneper**, Nachr. Kön Ges. Wiss. zu Göttingen (1870), pp. 499.

Now, $H = 90^{\circ}$ for both curves, but $dH : ds$ vanishes only for asymptotic lines, so it will follow from (1) and (15) of § **14** that:

$$\frac{1}{\rho} - \frac{dH}{ds} = \frac{1}{\rho_a}, \qquad \frac{1}{r}\left(\frac{2}{\rho} - 3\frac{dH}{ds}\right) = \frac{2}{r_a\,\rho_a}.$$

The elimination of $dH : ds$ will yield the relation:

$$\frac{1}{\rho} - \frac{3}{\rho_a} = -\frac{2r}{r_a\,\rho_a}. \tag{3}$$

One has, e.g., $r = \infty$ for the intersection curve of the tangent plane with the surface, so: $2r = 3r_a$ ([1]).

2. *If the surface curve is a geodetic line* then $N = 0$, $H = 0$, hence, when $L \neq 0$:

$$\frac{1}{r} = \frac{L}{ds^2}, \qquad \frac{1}{\rho} = \frac{1}{\bar{\rho}} = -\frac{M}{ds^2}. \tag{4}$$

I.e.: for a geodetic line, the absolute curvature is equal to the normal curvature; both of them are determined by the first equation in (4).

Furthermore, the geodetic torsion of an arbitrary curve is equal to the absolute torsion of the geodetic line that contacts it; that was the motivation for the term “geodetic torsion” for $1 : \bar{\rho}$ in (1).

3. *If the surface curve is a line of curvature then* $M = 0$.

Hence, $1 : \bar{\rho} = 0$;

i.e., the geodetic torsion of a line of curvature is equal to zero.

and $dH = ds : d\tau$;

i.e., the angle of torsion of a line of curvature is equal to the differential of the angle H between the surface normal and the principal normal of the curve ([2]).

That further implies **Joachimsthal’s theorem**, and one then has its generalization ([3]):

If the intersection curve between two surfaces F_1 and F_2 is a line of curvature for each of them then the angle between both surfaces will be constant.

([1]) **Beltrami**, Nouv. Ann. de Math. (2) **4** (1865), pp. 258.
([2]) **Lancret**, *Mém. sur les lignes à double courbure*, 1806.
([3]) **Bonnet**, J. Ec. poly., Cah. 35 (1853), pp. 119.

In fact, since $M_1 = M_2 = 0$, one has:

$$\frac{dH_1}{ds} - \frac{1}{\rho} = 0, \qquad \frac{dH_2}{ds} - \frac{1}{\rho} = 0,$$

and since the value $1 : \rho$ – i.e., the torsion of the intersection curve – is the same for both sides, it will follow that $H_1 – H_2$ = const. (Q. E. D.)

Conversely, when two surfaces F_1 and F_2 intersect each other at a constant angle, if the intersection curve is a line of curvature of one surface then it will also be one for the other surface.

It will then follow from $dH_1 = dH_2$ that $M_1 = M_2$; i.e., if $M_1 = 0$ then one will also have $M_2 = 0$.

One gets the special **theorems** for planar and spherical lines of curvature ([1]):

When a line of curvature is planar or spherical, its plane or sphere will cut the surface at a constant angle, and conversely.

When a plane or a sphere cuts a surface at a constant angle, the intersection curve will be a line of curvature.

Furthermore:

The spherical image of a planar line of curvature is a minor circle whose plane is parallel to the plane of the line of curvature.

Finally, one gets some theorems on radii of curvature. If one considers an arbitrary curve and the normal section that contacts it at the point (u, v), while the normal curvature of that curve is $1 : r_0$ then since L and ds are equal for both curves and $H = 0$ for the normal section, one will have:

$$\frac{1}{r_0} = \frac{\cos H}{r}. \tag{5}$$

That is **Meusnier's theorem** ([2]):

The radius of curvature of an arbitrary curve is equal to the projection of the radius of curvature of the associated normal section onto the osculating plane of the curve.

From (1), the normal curvature of a curve at the point (u, v) is:

$$\frac{1}{r_0} = \frac{d\,du^2 + 2d'\,du\,dv + d''\,dv^2}{e\,du^2 + 2f\,du\,dv + g\,dv^2}. \tag{6}$$

([1]) **Joachimsthal**, J. für Math. **30** (1846), pp. 347.
([2]) **Meusnier**, Mém. des Sav. étrangers **10** (1785), pp. 477.

The directions $du : dv$ for which r_0 is a maximum or a minimum (1) are determined from this to be:

$$(7)\qquad \begin{aligned} &r_0\,(d\,du + d'dv) - (e\,du + f\,dv) = 0, \\ &r_0\,(d'du + d''dv) - (f\,du + g\,dv) = 0. \end{aligned}$$

Those equations are identical with [§ **5**, (9)]. Eliminating r_0 will lead to the *differential equation for the lines of curvature*, and eliminating $du : dv$ will lead to the *equation of the radii of principal curvature*. Therefore, the lines of curvature are also the curves on the surface for which the normal curvature is a maximum or minimum.

Euler’s *expression* for the radius of curvature r_0 of an arbitrary normal section in terms of the radii of principal curvature r_1 and r_2 also follows from (6). If one introduces the lines of curvature as parameter curves, such that $f = 0$ and $d' = 0$, and denotes the elements of the lines of curvature by ds_1 and ds_2 , the element of the normal section by ds, and the angle between the normal section and the curve v = const., between ds and ds_1, by Θ_1 then one will have:

$$ds_1 = \sqrt{e}\,du = ds\cos\Theta_1\,, \qquad ds_2 = \sqrt{g}\,dv = ds\sin\Theta_1\,.$$

Hence, from (6):

$$\frac{1}{r_0} = \frac{d\,du^2 + d''dv^2}{ds^2} = \frac{d}{c}\left(\frac{ds_1}{ds}\right)^2 + \frac{d''}{g}\left(\frac{ds_1}{ds}\right)^2,$$

or, from [§ **9**, (5)] (2):

$$(8)\qquad \frac{1}{r_0} = \frac{\cos^2\Theta_1}{r_1} + \frac{\sin^2\Theta_1}{r_2}\,.$$

4. *If the surface curve is simultaneously a line of curvature and a geodetic line then* $M = 0$, $N = 0$, $H = 0$, $L \neq 0$; as a result:

$$\rho = \infty\,;$$

i.e., the curve is planar and lies in a normal plane of the surface.

If the curve is simultaneously an asymptotic line and a geodetic line, so $L = 0$, $N = 0$, *then:*

$$r = \infty\,;$$

i.e., the surface is a ruled surface, and the curve is one of the rectilinear generators.

(1) **Dupin**, *Développements*, pp. 106.
(2) **Euler**, Abh. d. Berl. Acad., 1760.

If the curve is simultaneously a line of curvature and an asymptotic line then $L = 0$, $M = 0$, $H = 90^{\mathrm{o}}$, $\rho = \bar{\rho} = \infty$, and:

$$r_1\, r_2 = \infty\,;$$

i.e., the surface is a developable surface, and the curve is one of the generating lines.

The same thing will be true when the curve is simultaneously an asymptotic line, a line of curvature, and a geodetic line.

§ 16. – Calculating the angle. Applications.

In § **14**, the angle H and the radii r, ρ, R for a surface curve are represented in terms of the parameters u, v ; the cosines (α, β, γ), (l, m, n), (λ, μ, ν) shall now be likewise represented in terms of u, v. One first has:

$$\alpha = \frac{dx}{ds} = \frac{\partial x}{\partial u}\frac{du}{ds} + \frac{\partial x}{\partial v}\frac{dv}{ds}\,. \tag{1}$$

However, at the same time, the *three cosines* α, l, λ *can be represented in terms of* a, α_1 , α_2 , *in conjunction with* ω, H, *and two auxiliary angles* Θ_1 , Θ_2 , and all that will remain is to express Θ_1 , Θ_2 in terms of u, v. Hence, we let:

Θ_1 and Θ_2 denote *the angles that the curves* v = const. *and* u = const., resp., *define with the surface curve,* and are measured in such a way that:

$$\Theta_1 + \Theta_2 = \omega. \tag{2}$$

If one chooses the upper signs in equations [§ **13**, (4)] and [§ **1**, (20)], as before, so in:

$$\begin{vmatrix} \alpha & \beta & \gamma \\ l & m & n \\ \lambda & \mu & \nu \end{vmatrix} = \pm 1, \qquad \begin{vmatrix} a & b & c \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} = \pm \sin \omega, \tag{3}$$

then the positive senses of the six directions will be established in a certain way, such as is given in Fig. 2. One will now get the following equations, either analytically or from known theorems in spherical trigonometry:

$$\sum \alpha\, a = 0, \qquad \sum l\, a = \cos H, \qquad \sum \lambda\, a = \sin H,$$

$$\sum \alpha\, \alpha_1 = \cos \Theta_1\,, \qquad \sum l\, \alpha_1 = -\sin H \sin \Theta_1\,, \qquad \sum \lambda\, \alpha_1 = \cos H \sin \Theta_1, \tag{4}$$

$$\sum \alpha\,\alpha_2 = \cos\Theta_2\,, \qquad \sum l\,\alpha_2 = \quad \sin H \sin\Theta_2\,, \qquad \sum \lambda\,\alpha_2 = -\cos H \sin\Theta_2\,,$$

when one sets $\alpha\,a + \beta\,b + \gamma\,c = \sum \alpha\,a$, etc., to abbreviate.

If one solves these equations for (α, β, γ), (l, m, n), (λ, μ, ν), and recalls (2) and (21) in § **1** then it will follow that α, l, λ have the values:

$$(5)\qquad \begin{aligned} \alpha \sin\omega &= \alpha_1 \sin\Theta_1 + \alpha_2 \sin\Theta_2\,, \\ l \sin\omega &= a\cos H \sin\omega - \alpha_1 \sin H \cos\Theta_2 + \alpha_2 \sin H \cos\Theta_1\,, \\ \lambda \sin\omega &= a \sin H \sin\omega + \alpha_1 \cos H \cos\Theta_2 - \alpha_2 \cos H \cos\Theta_1\,. \end{aligned}$$

Along with α, l, λ, the differentials $d\alpha$, dl, $d\lambda$ are also represented in terms of u, v using (10), (11), and (12) in § **13**. The angles Θ_1, Θ_2 must then be expressed. It follows from (1) and (22) in § **1** that ([1]):

$$(6)\qquad \begin{aligned} \cos\Theta_1 &= \sum \alpha\,\alpha_1 = \frac{e\,du + f\,dv}{ds\sqrt{e}}, \qquad \sin\Theta_1 = \frac{\delta\,dv}{ds\sqrt{e}}, \\ \cos\Theta_2 &= \sum \alpha\,\alpha_2 = \frac{f\,du + g\,dv}{ds\sqrt{g}}, \qquad \sin\Theta_2 = \frac{\delta\,du}{ds\sqrt{g}}\,. \end{aligned}$$

The differentials $d\,\Theta_1$ and $d\,\Theta_2$ are also important. If one differentiates the first equation in (6) then it will follow that:

$$-\sin\Theta_1\,d\,\Theta_1 = \sum \alpha\,d\alpha_1 + \sum \alpha_1\,d\alpha\,.$$

From [§ **13**, (10)], (4), and [§ **15**, (1)], one has:

$$\sum \alpha_1\,d\alpha = \frac{ds}{r}\sum \alpha_1\,l = -\frac{\sin H \sin\Theta_1 ds}{r} = -\frac{N}{ds^2}\sin\Theta_1\,,$$

and furthermore, from [§ **2**, (11)] and (6):

$$\sum \alpha\,d\alpha_1 = -\frac{q\,du + q'dc}{e}(f\cos\Theta_1 - \sqrt{e\,g}\cos\Theta_1) = \frac{\delta}{e}(q\,du + q'\,dv)\sin\Theta_1\,.$$

Thus, one has:

$$(7)\qquad d\,\Theta_1 = -\frac{\delta}{e}(q\,du + q'\,dv) + \frac{N}{ds^2}\,,$$

([1]) **Gauss**, *Disq. gen.*, art. **12**.

and similarly:

$$d\,\Theta_2 = -\frac{\delta}{g}(p'\,du + q''\,dv) - \frac{N}{ds^2}\,.$$

Equation [§ **1**, (13)] will again follow upon adding these two values.

From [§ **14**, (5)], (7) will imply that:

(8)
$$\frac{\partial\Theta_1}{\partial u} = -\frac{\delta q}{e} + \frac{\delta N_2}{ds^2}\,, \qquad \frac{\partial\Theta_1}{\partial v} = -\frac{\delta q'}{e} + \frac{\delta N_1}{ds^2}\,,$$
$$\frac{\partial\Theta_2}{\partial u} = -\frac{\delta p'}{g} - \frac{\delta N_2}{ds^2}\,, \qquad \frac{\partial\Theta_2}{\partial v} = -\frac{\delta p''}{g} + \frac{\delta N_1}{ds^2}\,,$$

which are equations that will be used later.

We shall now give two *applications* of the formulas that were developed.

First, let a second definition of the *geodetic curvature* ([1]) of a surface curve be mentioned. (Eq. [§ **15**, (1)]) If one refers to the angle $d\varepsilon$ between the two geodetic lines on the surface that contact the line element ds of the curve at the endpoints as the *geodetic contingency angle* of the curve then one will have the **theorem**:

The geodetic curvature of a surface curve is equal to the geodetic contingency angle $d\varepsilon$ divided by the line element ds.

That property gives rise to the terminology "geodetic curvature" and "geodetic contingency angle."

In fact: Θ_1 is the angle that the curve v = const. defines with the line element ds of the surface curve at the point (u, v), so $\Theta_1 + d\,\Theta_1$ will be the corresponding angle for the next line element. Therefore, if $\Theta_1 + d\Theta_1'$ denotes the corresponding angle for the geodetic line that contacts the surface curve then, since $N = 0$ for a geodetic line, it will follow from (7) that:

$$d\Theta_1' = -\frac{\delta}{e}(q\,du + q'\,dv)\,.$$

If one subtracts this from (7) then one will get:

(9)
$$d\varepsilon = d\,\Theta_1 - d\Theta_1' = \frac{N}{ds^2} \qquad \text{or} \qquad \frac{\sin H}{r} = \frac{d\varepsilon}{ds}$$

for the geodetic contingency angle. (Q. E. D.)

([1]) **Bonnet**, J. Ec. poly., Cah. **32** (1848), pp. 42, eq. (a). **Monge-Liouville**, *Applications*, Note II, pp. 574.

A *second application* relates to the *geodetic lines* on a surface. Their differential equation $N = 0$ has order two, but, as **Gauss** showed ([1]), it can be decomposed in the following way: For $N = 0$, one will get from (7) and (6) that:

$$(10)\qquad \begin{aligned} d\,\Theta_1 &= -\frac{\delta}{e}(q\,du + q'\,dv), & \cos\Theta_1 &= \frac{e\,du + f\,dv}{ds\sqrt{e}}, \\ d\,\Theta_2 &= -\frac{\delta}{g}(p'\,du + p''\,dv), & \cos\Theta_2 &= \frac{f\,du + g\,dv}{ds\sqrt{g}}. \end{aligned}$$

The upper, like the lower, pair of these equations is equivalent to the differential equation of the geodetic line $N = 0$. If one again combines the equations of each pair then one will get one or the other of the following two equations, instead of $N = 0$ ([2]):

$$(11)\qquad \begin{aligned} 2\,ds\cdot d\left(\frac{e\,du + f\,dv}{ds}\right) &= \frac{\partial e}{\partial u}du^2 + 2\frac{\partial f}{\partial u}du\,dv + \frac{\partial g}{\partial u}dv^2, \\ 2\,ds\cdot d\left(\frac{f\,du + g\,dv}{ds}\right) &= \frac{\partial e}{\partial v}du^2 + 2\frac{\partial f}{\partial v}du\,dv + \frac{\partial g}{\partial v}dv^2. \end{aligned}$$

From equations (10), one proves the **theorem** ([3]):

For surfaces whose line element has the form (viz., the ***Liouville*** *form):*

$$(12)\qquad ds^2 = (U + V)(du^2 + dv^2),$$

in which U depends upon only u and V depends upon only v, one can get the equation of the geodetic line, as well as the arc length, from mere quadratures.

Namely, if one chooses isometric coordinates u, v then one will have:

$$(13)\qquad e = g = \lambda, \qquad f = 0, \qquad q = -\frac{1}{2\lambda}\frac{\partial\lambda}{\partial v}, \qquad q' = \frac{1}{2\lambda}\frac{\partial\lambda}{\partial u}, \qquad \Theta_1 + \Theta_2 = \frac{\pi}{2}.$$

It will then follow from (10) that:

$$(14)\qquad ds\cos\Theta_1 = \sqrt{\lambda}\,du, \quad ds\sin\Theta_1 = \sqrt{\lambda}\,dv, \quad \cos\Theta_1\,dv = \sin\Theta_1\,du,$$

and

([1]) **Gauss**, *Disq. gen.*, art. 18.
([2]) **Darboux**, *Leçons*, II, pp. 405.
([3]) **Monge-Liouville**, *Applications*, Note III, pp. 579.

(15) $$2\,\lambda\,d\,\Theta_1 = \frac{\partial\lambda}{\partial v}du - \frac{\partial\lambda}{\partial u}dv.$$

If one multiplies the last equation by $\sin\Theta_1 \cos\Theta_1$ and uses (14) then one will get:

(16) $$\lambda\,d\,(\sin^2\Theta_1) = \cos^2\Theta_1\,\frac{\partial\lambda}{\partial v}dv - \sin^2\Theta_1\frac{\partial\lambda}{\partial u}du.$$

Now, if $\lambda = U + V$, as in (12), then if U', V' are the derivatives of U, V with respect to u, v, resp., it will follow from (16) that:

$$U\,d\,(\sin^2\Theta_1) + U'\sin^2\Theta_1\,du = V\,d\,(\cos^2\Theta_1) + V'\cos^2\Theta_1\,dv,$$

or upon integration:

(17) $$U\sin^2\Theta_1 - V\cos^2\Theta_1 = a,$$

in which a is a first arbitrary constant. One then has:

(18) $$\cos^2\Theta_1 = \frac{U-a}{U+V}, \qquad \sin^2\Theta_1 = \frac{V+a}{U+V},$$

or, from (14):

$$\tan\Theta_1 = \frac{dv}{du} = \sqrt{\frac{V+a}{U-a}}.$$

Hence, a second quadrature will give the differential equation for the geodetic lines on the surface with the character of (12) in the form:

(19) $$\int\frac{du}{\sqrt{U-a}} - \int\frac{dv}{\sqrt{V+a}} = b,$$

in which b is a second arbitrary constant. However, from (14) and (18), the arc length of the geodetic lines is:

(20) $$ds = \sqrt{\lambda}\,(\Theta_1\cos du + \Theta_1\sin dv) = \sqrt{U-a}\,du + \sqrt{V+a}\,dv,$$

so it is likewise determined by quadrature.

That method of integration finds applications, *inter alia*, to second-order surfaces and surfaces of revolution. As for the latter, one will have:

(21) $$ds^2 = (1 + P'^2)\,d\rho^2 + \rho^2\,dv^2 = U\,(du^2 + dv^2)$$

for the line element (cf., [**6**, (17)]), when one sets:

$$U = \rho^2 \qquad \text{and} \qquad du = \sqrt{1+P'^2}\,\frac{d\rho}{\rho}.$$

Hence, $V = 0$ here, and, from (17), the first integral of the differential equation of the geodetic line will give **Clairaut**'s equation ([1]):

$$\rho \sin \Theta_1 = \text{const.} \tag{22}$$

I.e.: For all points of a geodetic line on a surface of revolution, the product of the radius vector ρ with the sine of azimuth Θ_1 will be constant.

A *third application* leads to **Gauss**'s theorems on the *total curvature of a surface patch* ([2]). If one denotes the spherical map of the surface element *do* by *dO* then, from [§ **1**, (11)], one will have: $do = \delta\,du\,dv$, $dO = \Delta\,du\,dv$, so from [§ **11**, (5)], one will have:

$$\frac{dO}{do} = \frac{\Delta}{\delta} = k = \frac{1}{r_1 r_2}. \tag{23}$$

Gauss called the spherical image O of a finite surface patch o its total curvature; from (23) and [§ **7**, (11)], one will have:

$$O = \iint \frac{\delta\,du\,dv}{r_1 r_2} = \iint \left[\frac{\partial}{\partial v}\left(\frac{\delta q}{e}\right) - \frac{\partial}{\partial u}\left(\frac{\delta q'}{e}\right)\right] du\,dv \tag{24}$$

$$= \iint \left[\frac{\partial}{\partial u}\left(\frac{\delta p''}{g}\right) - \frac{\partial}{\partial v}\left(\frac{\delta p'}{g}\right)\right] du\,dv.$$

Now, when two functions R and S of (u, v) on O are single-valued and continuous, and possess derivatives, **Green**'s theorem will read:

$$\iint \left(\frac{\partial S}{\partial u} - \frac{\partial R}{\partial v}\right) du\,dv = \int (R\,du + S\,dv),$$

in which the double integral on the left-hand side extends over the surface, while the simple integral on the right-hand side runs around its boundary, such that the surface lies to its left.

Hence, it will follow from (7) that ([3]):

$$0 = -\int \frac{\delta}{e}(q\,du + q'\,dv) = \int d\Theta_1 - \int \frac{N}{ds^2} = \int d\Theta_1 - \int \frac{\sin H}{r}\,ds, \tag{25}$$

([1]) **Clairaut**, *Recherches sur les courbes à double courbure*, 1731.
([2]) **Gauss**, *Disq. gen.*, Art. 6.
([3]) **Bonnet**, J. Ec. poly., Cah. 32 (1848), pp. 124, *et seq.* **Darboux**, *Leçons*, III, pp. 126.

$$0 = +\int\frac{\delta}{q}(p'\,du + p''\,dv) = -\int d\Theta_2 - \int\frac{N}{ds^2} = -\int d\Theta_2 - \int\frac{\sin H}{r}\,ds\,.$$

One gets **Gauss**' theorem ([1]) for the geodetic triangle from this. For its sides, one has $N = 0$; hence, if A, B, C are the angles of the triangle then one will have, from (25), that:

$$0 = \int d\Theta_1 = 2\pi - (p - A + \pi - B + \pi - C) = A + B + C - \pi\,. \tag{26}$$

§ 17. – Conversions. Differential parameters. Applications.

In the derivation of the fundamental formulas for a surface curve (§ **14**), the behavior of the curve in the neighborhood of its points (u, v) was thought to be determined by the first and second differentials of u, v. It shall now be assumed that *the surface curve has the form* $\varphi(u, v) = a$ (where a is a constant), and the equations of § **14** shall be converted accordingly. One will then arrive at the known expressions for the *differential parameters*, *inter alia*, in the most natural way.

If one sets:

$$\frac{\partial\varphi}{\partial u} = \varphi_1\,, \qquad \frac{\partial\varphi}{\partial v} = \varphi_2\,, \tag{1}$$

to abbreviate, then the equation $d\varphi(u, v) = 0$, which is true for the first differentials of u and v, and when one applies a proportionality factor λ, can be replaced with:

$$du = \lambda\,\varphi_2\,, \qquad dv = -\lambda\,\varphi_1\,. \tag{2}$$

From § **14**, our problem consists of merely forming the expressions for L, M, N, ds^2. On initially has for L and M that:

$$\begin{aligned} L &= \lambda^2(d\,\varphi_2^2 - 2d'\varphi_1\varphi_2 + d''\varphi_1^2)\,, \\ M &= \frac{\lambda^2}{\delta}[(e\,d' - f\,d)\,\varphi_2^2 - (e\,d'' - g\,d)\,\varphi_1\varphi_2 + (f\,d'' - g\,d')\,\varphi_1^2]\,. \end{aligned} \tag{3}$$

The vanishing of these expressions is the condition for the system of curves $\varphi = a$, with the parameter a, to represent one family of asymptotic lines or lines of curvature of the surface.

If one further sets ([2]):

([1]) **Gauss**, *Disq. gen.*, Art. 20.

([2]) For the following, cf., **Beltrami**, Giorn. di Mat. **2** (1864) and **3** (1865). Math. Ann. **1** (1869), pp. 575, *et seq.* **Darboux**, *Leçons*, III, Book VII, Chap. 1.

(4) $$\delta'(\varphi) = \frac{1}{\delta^2}[e\,\varphi_2^2 - 2\,f\,\varphi_1\,\varphi_2 + g\,\varphi_1^2],$$

(5) $$\delta'''(\varphi) = \frac{1}{\delta^2}\left[\frac{\partial}{\partial u}\left(\frac{g\,\varphi_1 - f\,\varphi_2}{\delta\sqrt{\delta'(\varphi)}}\right) + \frac{\partial}{\partial v}\left(\frac{e\,\varphi_2 - f\,\varphi_1}{\delta\sqrt{\delta'(\varphi)}}\right)\right],$$

to abbreviate, then one will get ds^2 and N as:

(6) $$ds^2 = \lambda^2\,\delta^2\,\delta'(\varphi), \qquad N = -\lambda^3\,\delta^3\,(\delta'(\varphi))^{3/2}\cdot\delta'''(\varphi)\,.$$

The expression for N requires proof. From [§ **16**, (6)], one has:

$$\delta'''(\varphi) = \frac{1}{\delta}\left[\frac{\partial}{\partial v}\left(\frac{e\,du + f\,dv}{ds}\right) - \frac{\partial}{\partial u}\left(\frac{f\,du + g\,dv}{ds}\right)\right]$$

$$= \frac{1}{\delta}\left[\frac{\partial}{\partial v}\left(\sqrt{e}\cos\Theta_1\right) - \frac{\partial}{\partial u}\left(\sqrt{g}\cos\Theta_2\right)\right].$$

If one performs the differentiation and uses formulas (6) and (8) of § **16** and [§ **15**, (1)] then one will get ([1]):

(6a) $$\delta'''(\varphi) = \frac{\delta(N_1\,dv - N_2\,du)}{ds^2} = -\frac{N}{ds^2} = -\frac{\sin H}{r}.$$

(Q. E. D.)

The function $\delta'''(\varphi)$ can be replaced with another similar function, namely:

(7) $$\delta'''(\varphi) = \frac{1}{\delta}\left[\frac{\partial}{\partial u}\left(\frac{g\,\varphi_1 - f\,\varphi_2}{\delta}\right) + \frac{\partial}{\partial v}\left(\frac{e\,\varphi_2 - f\,\varphi_1}{\delta}\right)\right].$$

In fact, if performs the differentiation with respect to $\delta'(\varphi)$ in (5) and sets:

(8) $$\delta'(\varphi, \psi) = [e\,\varphi_2\,\psi_2 - f\,(\varphi_1\,\psi_2 + \psi_1\,\varphi_2) + g\,\varphi_1\,\psi_1]$$

then:

(9) $$\delta'''(\varphi) = \frac{\delta''(\varphi)}{\sqrt{\delta'(\varphi)}} - \frac{1}{2}\frac{\delta'(\varphi, \delta'(\varphi))}{(\delta'(\varphi))^{3/2}}.$$

([1]) **Bonnet**, J. de Math. (2) **5** (1860), pp. 166.

Following **Lamé**'s process for the special case of the plane (1), **Beltrami** called the expressions $\delta'(\varphi)$ and $\delta''(\varphi)$ the *first and second-order differential parameters* (2) *of the function* φ, and he called $\delta'(\varphi, \psi)$ the *mixed parameter of the functions* φ *and* ψ. $\delta'''(\varphi)$ can be expressed in terms of them using (9). It is convenient to introduce yet another abbreviating expression, namely:

$$\vartheta(\varphi, \psi) = \frac{1}{\delta}(\varphi_1\, \psi_2 - \psi_1\, \varphi_2). \tag{10}$$

It can be expressed in terms of the first-order differential parameter, since, as is easy to see:

$$\vartheta^2(\varphi, \psi) + \delta'^2(\varphi, \psi) = \delta'(\varphi)\cdot\delta'(\psi)\,. \tag{11}$$

If Φ and Ψ are functions of (φ, ψ) then one can express $\delta'(\Phi), \vartheta(\Phi, \Psi)$, $\delta'(\Phi, \Psi)$, $\delta''(\Phi)$ easily in terms of $\delta'(\varphi)$, $\delta'(\psi)$, $\vartheta(\varphi, \psi)$, $\delta'(\varphi, \psi)$, $\delta''(\varphi)$, $\delta''(\psi)$, as well as in terms of the partial derivatives of Φ and Ψ with respect to φ, ψ, which is required for certain transformations (3).

Besides the functions φ and ψ, the differential parameters contain only the coefficients e, f, g of the line element and will remain unchanged when the surface is bent. They then play an important role as bending invariants (cf., § **18**) in all problems of surface theory that depend upon only the form of the line element, and in particular, the problem of geometry on a surface.

We shall now give some *applications of the differential parameters.*

First, the *line element ds* of the surface can be expressed in terms of two systems of curves $\varphi = a$, $\psi = v$. Namely, if one sets $\varphi = u$, $\psi = v$ then:

$$\delta'(u) = \frac{g}{\delta^2}, \quad \delta'(v) = \frac{e}{\delta^2}, \quad \delta'(u, v) = -\frac{f}{\delta^2}, \qquad \vartheta(u, v) = \frac{1}{\delta},$$

or

$$e = \frac{\delta'(v)}{\vartheta^2(u,v)}, \qquad f = -\frac{\delta'(u,v)}{\vartheta^2(u,v)}, \qquad g = \frac{\delta'(u)}{\vartheta^2(u,v)}. \tag{12}$$

Therefore:

$$ds^2 = \frac{\delta'(v)\,du^2 - 2\,\delta'(u,v)\,du\,dv + \delta'(u)\,dv^2}{\vartheta^2(u,v)},$$

---

(1) **Lamé**, *Leçons sur les coord. curvil.*, 1859, pp. 6. If the surface goes to a plane, and if (u, v) are the rectilinear parallel coordinates in it then one will have $e = g = 1$, $f = 0$, so:

$$\delta'(\varphi) = \left(\frac{\partial\varphi}{\partial u}\right)^2 + \left(\frac{\partial\varphi}{\partial v}\right)^2, \quad \delta''(\varphi) = \frac{\partial^2\varphi}{\partial u^2} + \frac{\partial^2\varphi}{\partial v^2}.$$

(2) The differential parameter $\delta'(\varphi)$ had appeared already in **Gauss**, *Disq. gen.*

(3) **Beltrami**, Giorn. II, pp. 367.

or due to the invariant property of the differential parameter (cf., § **18**), when one introduces the general parameters φ, ψ, instead of u, v :

(13) $$ds^2 = \frac{\delta'(\psi)\,d\varphi^2 - 2\delta'(\varphi,\psi)\,d\varphi\,d\psi + \delta'(\varphi)\,d\psi^2}{\vartheta^2(\varphi,\psi)}.$$

Furthermore, from [§ **1**, (18)], the angle $(\varphi,\ \psi)$ that the two surface curves $\varphi = a$ and $\psi = b$ make with each other is determined by:

(14) $$\cos(\varphi,\ \psi) = \frac{\delta'(\varphi,\psi)}{\sqrt{\delta'(\varphi)\,\delta'(\psi)}}, \qquad \sin(\varphi,\ \psi) = \frac{\vartheta(\varphi,\psi)}{\sqrt{\delta'(\varphi)\,\delta'(\psi)}}.$$

Hence, $\delta'(\varphi,\ \psi) = 0$ is the condition for two curves $\varphi = a$ and $\psi = b$ to intersect orthogonally. We shall use that in order to express the differential parameters of the orthogonal system, which will be denoted by $\psi = b$, to the system of curves $\varphi = a$ (with the parameter a), in terms of the differential parameter of φ. By assumption, one has:

$$\psi_2\,(e\,\varphi_2 - f\,\varphi_1) - \psi_1\,(f\,\varphi_2 - g\,\varphi_1) = 0,$$

or when μ is a proportionality factor:

$$\psi_1 = \mu\,(e\,\varphi_2 - f\,\varphi_1), \qquad -\,\psi_2 = \mu\,(g\,\varphi_1 - f\,\varphi_2),$$

or

$$e\,\psi_2 - f\,\psi_1 = -\,\delta^2\,\mu\,\varphi_1, \qquad g\,\psi_1 - f\,\psi_2 = \delta^2\,\mu\,\varphi_2.$$

Hence, from (4) and (5):

(15) $$\delta'(\psi) = -\,\mu\,(\varphi_1\,\psi_2 - \psi_1\,\varphi_2) = \delta^2\,\mu^2\,\delta'(\varphi),$$

(16) $$\delta'''(\psi) = \frac{1}{\delta}\left[\frac{\partial}{\partial u}\left(\frac{\varphi_2}{\sqrt{\delta'(\varphi)}}\right) - \frac{\partial}{\partial v}\left(\frac{\varphi_1}{\sqrt{\delta'(\varphi)}}\right)\right]$$

$$= -\,\frac{1}{2}(\delta'(\varphi))^{-3/2}\left[\varphi_2\,\frac{\partial\delta'(\varphi)}{\partial u} - \varphi_1\,\frac{\partial\delta'(\varphi)}{\partial v}\right],$$

which we shall make use of later.

We further consider *systems of curves* $\varphi = a$ *of a well-defined character;* one immediately has the theorems:

I. $\delta'(\varphi) = 0$ *is the necessary and sufficient condition for* $\varphi = a$ *to be a system of* ***minimal lines***.

2. $\delta'''(\varphi) = 0$ *is the condition for* $\varphi = a$ *to be a system of* ***geodetic lines***, *and*

$\delta'''(\varphi) = F(\varphi)$ is the condition for $\varphi = a$ to be a system of ***lines of constant geodetic curvature****, and finally:*

3. *$\delta'(\varphi) = F(\varphi)$ is the condition for $\varphi = a$ to be a system of* ***geodetic parallels****, and*

   $\delta'(\varphi) = 1$ is the condition for $\varphi = a$ to be a system of geodetic parallels and, at the same time, for φ to be the arc length of the associated geodetic lines ([1]).

The last one requires proof. If $\varphi = a$ is a system of geodetic parallels, and $\psi = b$ is the system of associated orthogonal geodetic lines, and should φ be the arc length of the geodetic lines, when measured from those geodetic parallels, then the problem of determining the system $\varphi = a$ will lead to that of ascertaining three functions φ, ψ, Q of u, v in such a way that square of the line element will take on the form:

$$ds^2 = e\,du^2 + 2f\,du\,dv + g\,dv^2 = d\varphi^2 + Q\,d\psi^2 . \tag{17}$$

One will then have:

$$e = \varphi_1^2 + Q\psi_1^2, \qquad f = \varphi_1\,\varphi_2 + Q\,\psi_1\,\psi_2, \qquad g = \varphi_2^2 + Q\psi_2^2 .$$

Eliminating the two quantities $\sqrt{Q}\,\psi_1$ and $\sqrt{Q}\,\psi_2$ will lead to a partial differential equation in φ, namely:

$$e\varphi_2^2 - 2f\varphi_1\varphi_2 + g\,\psi_2^2 = \delta^2, \quad \text{so} \qquad \delta'(\varphi) = 1. \tag{18}$$

If one drops the condition that φ is the arc length of the orthogonal geodetic lines and introduces an arbitrary function φ' in place of φ then one will have:

$$\varphi = \int \frac{d\varphi'}{\sqrt{F(\varphi')}}, \qquad \text{so} \qquad \varphi_1 = \frac{\varphi_1'}{\sqrt{F(\varphi')}}, \qquad \varphi_2 = \frac{\varphi_2'}{\sqrt{F(\varphi')}} .$$

Hence, (18) will go to $\delta'(\varphi') : F(\varphi') = 1$.
(Q. E. D.)

One treats the problem of determining a *system of isometric lines (isotherms)* on the surface similarly. If $\varphi = a$ and $\psi = b$ are the two families of an isometric system then the solution will emerge from the problem (cf., § **3**) of bringing the line element into the form:

$$ds^2 = e\,du^2 + 2f\,du\,dv + g\,dv^2 = e_0\,d\varphi^2 + g_0\,d\psi^2, \tag{19}$$

in which e_0 and g_0 are functions of φ and ψ with the condition:

([1]) **Gauss**, *Disq. gen.*, art. **22**. **Beltrami**, Giorn. II, pp. 277.

$$\frac{\partial^2 \log}{\partial\varphi\partial\psi}\left(\frac{g_0}{e_0}\right) = 0 \qquad \text{or} \qquad \frac{\partial \log}{\partial\varphi}\left(\frac{g_0}{e_0}\right) = 2\,F(\varphi),$$

and $F(\varphi)$ is a function of only φ. Now, when the differential parameters that are constructed from the quantities e_0, g_0 are denoted with the subscript 0, from (4) and (7), one will have:

$$\delta_0'(\varphi) = \frac{1}{e_0}, \quad \delta_0''(\varphi) = \frac{1}{\sqrt{e_0\,g_0}}\frac{\partial}{\partial\varphi}\sqrt{\frac{g_0}{e_0}} = \frac{1}{2e_0}\frac{\partial \log}{\partial\varphi}\left(\frac{g_0}{e_0}\right).$$

It follows from this that:

$$(20) \qquad \frac{\delta_0''(\varphi)}{\delta_0'(\varphi)} = \frac{1}{2}\frac{\partial \log}{\partial\varphi}\left(\frac{g_0}{e_0}\right) = F(\varphi) \qquad \text{or} \qquad \frac{\delta_0''(\varphi)}{\delta_0'(\varphi)} = F(\varphi),$$

when one considers the invariant nature of the differential parameter. (cf., § **18**).

If $\varphi(u, v)$ is any solution to the partial differential equation (20) then $\varphi = a$ will represent one family of an isometric system with the parameter a. The associated orthogonal family $\psi = b$ with the parameter b is obtained by integrating the differential equation:

$$(21) \qquad \frac{1}{\delta}[(e\,\varphi_2 - f\,\varphi_1)\,du - (g\,\varphi_1 - f\,\varphi_2)\,dv] = 0,$$

which can be solved by quadrature, since the integrating factor is a function of only φ, as one proves easily with the help of (20). If φ and ψ are ascertained then e_0 and g_0 will follow from $1 : e_0 = \delta'(\varphi)$, $1 : g_0 = \delta'(\psi)$.

Should φ and ψ be simultaneously thermal parameters, so $e_0 = g_0 = \lambda_0$, then the condition (20) for φ would go to $\delta_0''(\varphi) = 0$ or $\delta''(\varphi) = 0$. Therefore, the left-hand side in (21) will be a complete differential, or ψ will be determined from:

$$(22) \qquad d\psi = \frac{1}{\delta}[(e\,\varphi_2 - f\,\varphi_1)\,du - (g\,\varphi_1 - f\,\varphi_2)\,dv],$$

and λ_0 will be determined from $1 : \lambda_0 = \delta'(\varphi) = \delta'(\psi)$. We then find the **theorem** ([1]):

IV. $\delta''(\varphi) : \delta'(\varphi) = F(\varphi)$ is the necessary and sufficient condition for $\varphi = a$ to be *one family of an isometric system* with the parameter a, and:

$\delta''(\varphi) = 0$ *is the condition for* $\varphi = a$ *to be one family of an isometric system and, at the same time, for* φ *to be its thermal parameter.*

Therefore, this will imply, e.g., from [§ **2**, (14)], the **theorem**:

([1]) **Beltrami**, Giorn. II, pp. 369.

The intersection curves of a minimal surface ($h = 0$) *with parallel planes will define one family of an isometric system.*

II and IV will also imply the **theorems** ([1]):

1. *If a system of curve $\varphi = a$ (with the parameter a) exists on a surface, and its curves are both geodetically parallel and have constant geodetic curvature, then the system will also be isothermal.*

That is because from (9), $\delta'(\varphi) = F_1(\varphi)$, and $\delta'''(\varphi) = F_3(\varphi)$ will imply that $\delta''(\varphi) : \delta'(\varphi) = F(\varphi)$, so from (8), $F_1'(\varphi) \cdot \delta'(\varphi)$.

2. *Conversely: If a system of curves $\varphi = a$ that is both isometric and geodetic exists on a surface then the curves of the system will also have constant geodetic curvature.*

That is because, from (9), $\delta'(\varphi) = F_1(\varphi)$ and $\delta''(\varphi) : \delta'(\varphi) = F(\varphi)$ will imply that $\delta'''(\varphi) = F_3(\varphi)$.

3. *In both cases, the surface can be developed to a surface of revolution, under which the geodetic parallels will go to parallel circles on the surface revolution.*

That is because in both cases, one will then have $\delta'(\varphi) = F_1(\varphi)$ and $\delta''(\varphi) = F_2(\varphi)$. Now, if (u, v) are geodetic coordinates (i.e., $u = a$ are geodetic parallels, $v = b$ are geodetically orthogonal, and if, in addition, u is the arc length of the geodetic lines, when measured from one of the geodetic parallels) then one will have $ds^2 = du^2 + g\,dv^2$; as a result:

$$\delta'(u) = 1, \qquad \delta''(u) = \frac{1}{2}\frac{\partial \log g}{\partial u} = F_2(u) .$$

Hence, $g = UV$, in which U depends upon u, and V depends upon only v ; therefore:

$$ds^2 = du^2 + UV\,dv^2 = du^2 + U\,dv_1^2 ;$$

i.e., the line element of the surface is, at the same time, the line element of a surface of revolution. (Q. E. D.)

Finally, we shall give an application of the differential parameters to the *line of striction of a system of curves* ([2]) $\varphi = a$, with the parameter a. One understands that to mean the locus of point on the curves of the system for which the distance to the next system of curves is a minimum. One must first calculate the normal distance between two curves φ and $\varphi + d\varphi$. Let u, v and $u + du, v + dv$ be two consecutive points on φ,

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([1]) **Beltrami**, Giorn. III, pp. 89 and 90.
([2]) **Beltrami**, Giorn. II, pp. 276 and III, pp. 230.

and let ds be the associated line element, and furthermore let $u + \delta u$, $v + \delta v$ be a point at a distance δs from (u, v) through which the curve $\varphi + \delta\varphi$ goes, such that:

$$\delta\varphi = \varphi_1\,\delta u + \varphi_2\,\delta v\,.$$

Finally, let ϑ be the angle between the line elements ds and δs, and let δn be the distance between the curves φ and $\varphi + \delta\varphi$ at the point (u, v). From [§ **1**, (8)] and (6), one will then have:

(23) $$\delta n = \delta s \sin\vartheta = \delta\left[\frac{du}{ds}\,\delta v - \frac{dv}{ds}\,\delta u\right] = \frac{\varphi_1\,\delta u + \varphi_2\,\delta v}{\sqrt{\delta'(\varphi)}} = \frac{\delta\varphi}{\sqrt{\delta'(\varphi)}}\,.$$

The line of striction is the locus of points (u, v) for which δn is a minimum. Hence, when one considers $\delta\varphi$ to be constant, it will be determined from the condition $d\left(1:\sqrt{\delta'(\varphi)}\right) = 0$, or:

(24) $$\frac{\partial\delta'(\varphi)}{\partial u}du + \frac{\partial\delta'(\varphi)}{\partial v}dv = 0 \qquad \text{or} \qquad \varphi_2\frac{\partial\delta'(\varphi)}{\partial u} - \varphi_1\frac{\partial\delta'(\varphi)}{\partial v} = 0.$$

However, from (16), that equation will be identical to $\delta'''(\psi) = 0$ when $\psi = b$ is the orthogonal to $\varphi = a$ with the parameter b. Hence, one has the **theorem**:

The line of striction of a system of curves $\varphi = a$ is the locus of points at which each curve of the orthogonal system $\psi = b$ has geodetic curvature zero.

§ 18. – Transforming the parameters. Application.

At the conclusion of this general study of surfaces and surface curves, we shall briefly consider *transformations of the parameters* and the expressions that are *invariant* under such transformations. It is geometrically clear that a series of metric quantities, such as the radii of principal curvature r_1 and r_2, and furthermore, the line element ds, the angle (C, C_1) between two surface curves, and finally the quantities H, r, ρ, R that determine the character of a surface curve, are independent of the fundamental parameters u, v or that the values of those quantities do not change when one introduces new parameters u_0, v_0, instead of u, v ([1]). However, one can also easily convince oneself of that analytically when one looks for the invariants of the transformation of parameters; i.e., the expressions that will be constructed in the same way and have the same values in terms of the old and new parameters.

Let:

(1) $$u_0 = \rho(u, v), \quad v_0 = \sigma(u, v)$$

([1]) **Gauss**, *Disq. gen.*, art. 21. **Weingarten**, Festschrift der tech. Hochschule zu Berlin, 1884.

be the equations that couple the old and new parameters, which effect the following transformation:

(2) $x = x(u, v) = x_0(u_0, v_0), \quad y = y(u, v) = y_0(u_0, v_0), \quad z = z(u, v) = z_0(u_0, v_0),$

and which implies the following equations for arbitrary surface curves $\varphi = a$, $\psi = b$, …:

(3) $$\varphi(u, v) = \varphi_0(u_0, v_0), \qquad \psi(u, v) = \psi_0(u_0, v_0) .$$

We denote all of the quantities that are defined in terms of the new parameters u_0, v_0 with a subscript 0 and set the determinant of the transformation to:

(4) $$\frac{\partial u_0}{\partial u}\frac{\partial v_0}{\partial v} - \frac{\partial u_0}{\partial v}\frac{\partial v_0}{\partial u} = \Omega.$$

One next has:

$$e = e_0\left(\frac{\partial u_0}{\partial u}\right)^2 + 2 f_0 \frac{\partial u_0}{\partial u}\frac{\partial v_0}{\partial u} + g_0\left(\frac{\partial v_0}{\partial u}\right)^2,$$

(5) $$f = e_0 \frac{\partial u_0}{\partial u}\frac{\partial u_0}{\partial v} + f_0\left(\frac{\partial u_0}{\partial u}\frac{\partial v_0}{\partial v} + \frac{\partial v_0}{\partial u}\frac{\partial u_0}{\partial v}\right) + g_0 \frac{\partial v_0}{\partial u}\frac{\partial v_0}{\partial v},$$

$$g = e_0\left(\frac{\partial u_0}{\partial v}\right)^2 + 2 f_0 \frac{\partial u_0}{\partial v}\frac{\partial v_0}{\partial v} + g_0\left(\frac{\partial v_0}{\partial v}\right)^2.$$

The same equations will be true when e, f, g and e_0, f_0, g_0 are replaced with d, d', d'' and d_0, d_0', d_0'', resp., or also E, F, G and E_0, F_0, G_0, resp. It follows immediately from these equations that:

(6) $$\frac{\delta^2}{\delta_0^2} = \frac{e\,g - f^2}{e_0\,g_0 - f_0^2} = \frac{d\,d'' - d'^2}{d_0\,d_0'' - d_0'^2} = \frac{e\,d'' - 2\,f\,d' + g\,d}{e_0\,d_0'' - 2\,f_0\,d_0' + g_0\,d_0} = \Omega^2 ;$$

i.e., the mean curvature h and the curvature k are invariant under transformations of the parameters. The same thing is true of the cosines a, b, c that the direction of the surface normal determines. They are the most important invariants that appear in the investigation of the surface itself. The partial differential equations (6) and (10) in § **7** between the six fundamental quantities e, f, g ; d, d', d'' will likewise remain the same under the transformation ([1]).

It further follows from (5) that:

$$e\,du + f\,dv = (e\,du_0 + f\,dv_0)\frac{\partial u_0}{\partial u} + (f\,du_0 + g\,dv_0)\frac{\partial v_0}{\partial u},$$

(7)

([1]) For more details: **Weingarten**, *loc. cit.*, pp. 19.

$$f\,du + g\,dv = (e\,du_0 + f\,dv_0)\,\frac{\partial u_0}{\partial v} + (f\,du_0 + g\,dv_0)\,\frac{\partial v_0}{\partial v},$$

along with the corresponding equations when one defines them in terms of d, d', d'' or E, F, G. It will then follow that (¹):

$$ds = ds_0\,, \qquad L = L_0\,, \qquad M = M_0\,, \qquad N = N_0\,;$$

i.e., the metric quantities H, r, ρ, R are invariant under transformations of the parameters.

The expression for dS, the expression for P [§ **14**, (16)], and the expression that appears in [§ **1**, (8)]:

$$e\,du\,d_1u + f\,(du\,d_1v + dv\,d_1u) + g\,dv\,d_1v\,, \tag{8}$$

are likewise invariant. Those are the most important invariants that appear in the study of a surface curve. In addition, the expressions that one gets from N when one replaces e, f, g with d, d', d'', resp., or E, F, G, resp., are also invariant.

The differential parameters and converted forms will appear in place of these invariants when one defines the surface curve by an equation $\varphi(u, v) = a$. One has:

$$\frac{\partial\varphi}{\partial u} = \frac{\partial\varphi_0}{\partial u_0}\frac{\partial u_0}{\partial u} + \frac{\partial\varphi_0}{\partial v_0}\frac{\partial v_0}{\partial u}, \qquad \frac{\partial\varphi}{\partial v} = \frac{\partial\varphi_0}{\partial u_0}\frac{\partial u_0}{\partial v} + \frac{\partial\varphi_0}{\partial v_0}\frac{\partial v_0}{\partial v}. \tag{9}$$

It will then follow from (5) and (6) that:

$$\begin{aligned}
\frac{1}{\delta}\left(e\frac{\partial\varphi}{\partial v} - f\frac{\partial\varphi}{\partial u}\right) &= \frac{1}{\delta}\left[\left(e_0\frac{\partial\varphi_0}{\partial v_0} - f\frac{\partial\varphi_0}{\partial u_0}\right)\frac{\partial u_0}{\partial u} + \left(f_0\frac{\partial\varphi_0}{\partial v_0} - g\frac{\partial\varphi_0}{\partial u_0}\right)\frac{\partial v_0}{\partial u}\right],\\
\frac{1}{\delta}\left(f\frac{\partial\varphi}{\partial v} - g\frac{\partial\varphi}{\partial u}\right) &= \frac{1}{\delta}\left[\left(e_0\frac{\partial\varphi_0}{\partial v_0} - f\frac{\partial\varphi_0}{\partial u_0}\right)\frac{\partial u_0}{\partial v} + \left(f_0\frac{\partial\varphi_0}{\partial v_0} - g\frac{\partial\varphi_0}{\partial u_0}\right)\frac{\partial v_0}{\partial v}\right].
\end{aligned} \tag{10}$$

One next has $\delta'(\varphi) = \delta'_0(\varphi_0)$ then, where δ'_0 is the first-order differential parameter that is constructed from the quantities e_0, f_0, g_0. Furthermore, when one differentiates the first of equations (10) with respect to v and the second one with respect to u and subtracts them, one will get: $\delta''(\varphi) = \delta''_0(\varphi_0)$. One will then have:

$$\delta'(\varphi) = \delta'_0(\varphi_0)\,, \qquad \delta'(\varphi, \psi) = \delta'_0(\varphi_0, \psi_0)\,, \qquad \delta''(\varphi) = \delta''_0(\varphi_0)\,, \tag{11}$$

and then also:

$$\vartheta(\varphi, \psi) = \vartheta_0(\varphi_0, \psi_0), \qquad \delta'''(\varphi) = \delta'''_0(\varphi_0)\,.$$

(¹) The proof of the last equation $N = N_0$ is easy to carry out by means of some computation.

The first and second-order differential parameters are then invariant. However, the expressions that one obtains when one replaces the quantities e, f, g with d, d', d'', resp., or E, F, G, resp., are also invariant.

Among the invariants that have been enumerated, one finds, in particular, the so-called *bending invariants* ([1]); i.e., the invariants that also remain unchanged under a bending of the surface.

We shall not consider one and the same surface (x, y, z), which is referred to two different parameter systems u, v and u_0 and v_0, as we did up to now, but two different surfaces (x, y, z) and (x', y', z') that are referred to the same parameter system (u, v). The equations and definitions at the end of § **5** will then be valid.

Bending invariants are those of the invariants that were considered above that include only the three fundamental quantities e, f, g, and their derivatives, but not the quantities d, d', d''. They include, first of all, the expression for the curvature k ([2]), and then the expressions ds, N, and the expression (8), which still includes the differentials of u and v, and finally, the corresponding differential parameters $\delta'(\varphi), \delta'(\varphi, \psi), \delta''(\varphi)$, which still include the derivatives of φ and ψ with respect to u and v. Of the metric quantities, the bending invariants include not only the curvature of the surface, but also the geodetic curvature of any surface curve ([3]), and the angle between two surface curves, which is also geometrically clear.

It is clear that one will get further bending invariants when one repeats the processes ϑ, δ', and δ'' on one or more functions φ, ψ, … Conversely, every bending invariant can be represented in that way ([4]). In fact, if:

$$J = F\left(e, \frac{\partial e}{\partial u}, \cdots, \varphi, \frac{\partial \varphi}{\partial u}, \cdots, \psi, \frac{\partial \psi}{\partial u}, \cdots\right) \tag{12}$$

is a general bending invariant that includes e, f, g, φ, ψ, …, along with the derivatives with respect to u, v, then one can represent all of its elements by repeated application of the operations ϑ and δ' to the functions u, v, φ, ψ, … That is because, from [§ **17**, (12)], e, f, g, δ can be expressed in terms of the $\delta'(u)$, $\delta'(v)$, $\vartheta(u, v)$, and from [§ **17**, (10)], one has:

$$\frac{\partial \varphi}{\partial u} = \delta\vartheta(\varphi, v), \qquad \frac{\partial \varphi}{\partial v} = \delta\vartheta(u, \varphi), \tag{13}$$

so

$$\frac{\partial^2 \varphi}{\partial u^2} = \delta\vartheta\left(\frac{\partial \varphi}{\partial u}, v\right) = \delta\vartheta[\delta\vartheta(\varphi, v), v], \qquad \text{etc.}$$

If one now introduces two of the functions φ and ψ that appear in J in place of u and v then one will get the representation of J that was given. If only one function φ enters into J then one can replace u and v with φ and $\delta'\varphi$ or $\delta''\varphi$ and then get all invariants of a

([1]) **Beltrami**, Giorn., II, pp. 356.
([2]) **Gauss**, *Disq. gen.*, art. 12.
([3]) **Minding**, J. für Math., **6** (1830), pp. 159.
([4]) **Darboux**, *Leçons*, III, pp. 204.

single function φ by repeated application of the process δ' and δ'' when they are applied to that function.

As an application of the formulas for the bending invariants, we shall develop the *criteria for two surfaces with arbitrarily given parameters:*

$$(14) \qquad x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v),$$

and

$$(15) \qquad x_0 = x_0(u_0, v_0), \qquad y_0 = y_0(u_0, v_0), \qquad z_0 = z_0(u_0, v_0)$$

to be developable onto each other ([1]). The two surfaces are related to each other point-by-point by means of two equations between the parameters (u, v) and (u_0, v_0):

$$(16) \qquad \varphi(u, v) = \varphi_0(u_0, v_0), \qquad \psi(u, v) = \psi_0(u_0, v_0),$$

with the condition that φ and ψ are mutually-independent functions, so the determinant $\varphi_1\,\psi_2 - \psi_1\,\varphi_2$ does not vanish identically. The necessary and sufficient conditions for the developability of the surface (14) and (15) to each other will then be contained in the three equations (5). It is preferable to give those conditions a different form. Namely, if one uses φ and ψ as parameters on the surface (14) and φ_0 and ψ_0 as parameters on (15) then the form [§ **17**, (13)] of the line element will imply the necessary and sufficient conditions for developability in the form:

$$(17) \qquad \delta'(\varphi) = \delta'_0(\varphi_0), \qquad \delta'(\psi) = \delta'_0(\psi_0), \qquad \delta'(\varphi, \psi) = \delta'_0(\varphi_0, \psi_0),$$

in which differential parameters that are provided with the index 0 refer to the surface (15).

If one assumes that the surfaces (14) and (15) that relate to each other by means of equations (16) are mutually-developable then it will be necessary that the curvature k and the first and second-order differential parameters must have the same value at corresponding points, or that one must have:

$$(18) \qquad k = k_0, \qquad \delta'(k) = \delta'_0(k_0), \qquad \delta''(k) = \delta''_0(k_0), \qquad \delta'''(k) = \delta'''_0(k_0),$$

in which the third or fourth equation is a consequence of the remaining three.

The first of these equations, $k = k_0$, says that the curves on both surfaces with equal constant curvature k = const. must correspond to each other. The second one says that along the curves with k = const., points for which the normal distance dn between the curve k and $k + dk$ is the same must correspond to each other (cf., [§ **17**, (23)]). The third

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([1]) **Minding**, J. für Math. **19** (1839), pp. 180. **Monge-Liouville**, *Applications*, Notes IV and V. **Bonnet**, J. Ec. poly., Cah. **41** (1863), pp. 211, *et seq.* The presentation above is connected with the discussion in **Darboux**, *Leçons,* III, pp. 223, *et seq.*

or fourth one says that the geodetic curvature of the curve k = const. must also be the same at corresponding points (cf., [§ **17**, (6a)]).

One now asks, conversely, whether and under what condition on the given equations (14) and (15) one can find two equations of the form (16) such that φ and ψ (and as a result φ_0 and ψ_0, as well) will be independent of each other, and at the same time, that the three equations (17) will be satisfied, or to examine whether, and to what extent, the *necessary* conditions (18) for the developability are also *sufficient.* There will then be *several cases* to distinguish:

1. Under the assumption that k and k_0 are not constant and that $\delta'(k)$ is not a function of k, and $\delta'_0(k_0)$ is not a function of k_0, one sets:

$$k = k_0, \qquad \delta'(k) = \delta'_0(k_0). \tag{19}$$

One will then have two equations of the form (16) between (u, v) and (u_0, v_0), from which two well-defined families of curves on the two surfaces with the aforementioned geometric meaning will be related to each other. The sufficient conditions for developability are then that the two equations:

$$\delta'(\delta' k) = \delta'_0(\delta'_0 k_0), \qquad \delta'(k, \delta' k) = \delta'_0(k_0, \delta'_0 k_0), \tag{20}$$

will also be fulfilled by means of equations (19).

2. By contrast, if $\delta'(k)$ is a function of k then if developability is to even be possible, one must also have that $\delta'_0(k_0)$ is a function of k_0, and in fact, the same function.

Therefore let:

$$\delta'(k) = f(k), \qquad \delta'_0(k_0) = f(k_0). \tag{21}$$

The two equations (19) will then give only the first of equations (16) or only one relations between (u, v) and (u_0, v_0). Under the assumption that $\delta'''(k)$ is independent of k and $\delta'''_0(k_0)$ is independent of k_0, one appends the equation:

$$\delta'''(k) = \delta'''_0(k_0) \tag{22}$$

to (19).

From (21), the surfaces (14) and (15) will have the special form that makes the system of curves k = const. at the same time a system of geodetic parallels. From (21), those curves on the two surfaces will relate to each other for which the curves k = const. will intersect at point of equal geodetic curvature.

Moreover, from (17), the sufficient condition for the developability is the one that follows from equations (19), (21), and (22):

$$\delta'(\delta'''(k)) = \delta'_0(\delta'''_0(k_0)). \tag{23}$$

The further condition that:

(24) $$\delta'(k,\, \delta'''k) = \delta_0'(k_0, \delta_0'''k_0),$$

which is included in (17), is fulfilled by itself by means of (19), (21), and (22). That follows from an identity for k into which equation (21), which is characteristic of the surface in question, can be converted, and which includes only k and its differential parameters. Namely, if one introduces a system of geodetic coordinates (u_1, v_1) on an arbitrary surface (§ **4**) then the square of the line element will assume the form:

(25) $$ds^2 = du_1^2 + g_1\, dv_1^2,$$

in which u_1 , v_1 , g_1 are well-defined functions of (u, v).

Moreover, when one applies the differential parameters for the index 1 that are constructed for the form (25), one will have:

(26) $$\delta_1'(u_1) = 1, \qquad \delta_1''(u_1) = \frac{\partial \log \sqrt{g_1}}{\partial u_1}, \quad \text{and} \qquad k = -\frac{1}{\sqrt{g_1}}\frac{\partial^2 \sqrt{g_1}}{\partial u_1^2},$$

in which the last equation is inferred from [§ **9**, (4)].

Eliminating g_1 will give the relation:

(27) $$\delta_1'(u_1, \delta_1'' u_1) = -k - [\delta_1'' u_1]^2,$$

which will be true for any surface. For the surfaces that are characterized by (21), k = const. will now be a system of geodetic parallels. If one introduces that system into (25) in place of u_1 = const. then a comparison of (25) with [§ **17**, (13)], in conjunction with [§ **17**, (15)], and when one recalls the invariant nature of the differential parameter, will yield:

(28) $$u_1 = \int \frac{dk}{\sqrt{\delta'(k)}} = \int \frac{dk}{\sqrt{f(k)}}.$$

If one substitutes that value in (27) then, since the differential parameters are invariant, and since:

$$\delta''\left(\int \frac{dk}{\sqrt{\delta' k}}\right) = \delta'''(k),$$

so

$$\delta'\left(\int \frac{dk}{\sqrt{\delta' k}}, \delta''\int \frac{dk}{\sqrt{\delta' k}}\right) = \frac{1}{\sqrt{\delta' k}}\, \delta'(k,\, \delta''' k),$$

one will get the desired identity for k :

(29) $$\frac{1}{\sqrt{\delta' k}}\,\delta'(k,\ \delta''' k) = -k - (\delta''' k)^2.$$

However, due to (19) and (22), the equation (24) will follow from that. (Q. E. D.)

3. If not only is $\delta'(k) = f(k)$, but also $\delta'''(k) = f_1(k)$, then developability will first require that in addition to $\delta_0'(k_0) = f(k_0)$, one also has $\delta_0'''(k_0) = f_1(k_0)$. Those assumptions are already sufficient for developability, and the latter is also possible in an infinitude of ways. One gets that equation and the second equation between (u, v) and (u_0, v_0) that is appended to $k = k_0$ as follows:

The condition $\delta'''(k) = f_1(k)$ further characterizes the surface (14) by the fact that the geodetic parallels k = const., which already relate to each other, are at the same time curves of constant geodetic curvature. If one again bases the form of the line element on (25) then that condition will also lead to a determination of the value of the quantity g_1 by a quadrature. Namely, it follows form $\delta'''(k) = f_1(k)$ that:

(30) $$\delta_1''(u_1) = \frac{\partial \log \sqrt{g_1}}{\partial u_1} = \varphi_1(u_1), \qquad \text{from which,} \qquad \sqrt{g_1} = U_1\,;$$

i.e., $\sqrt{g_1}$ will be equal to a well-defined function of u_1 when v_1 is chosen suitably.

Therefore, we now have:

(31) $$ds^2 = du_1^2 + U_1^2\,dv_1^2$$

for (14), in which v_1 is a well-defined function of (u, v). However, since k or u_1 and $\delta'''(k)$ or U_1 is the same for both surfaces, the line element of the surface (15) will have the form:

$$ds^2 = du_1^2 + U_1^2\,dv_2^2,$$

in which v_2 is a well-defined function of (u_0, v_0). When one sets $dv_1^2 = dv_2^2$ or:

(32) $$v_1 = \pm v_2 + c,$$

in which c is an arbitrary constant, one will get the second relation between (u, v) and (u_0, v_0). Due to the arbitrary constant c, the development will be possible in a simple infinitude of ways. In fact, it will be determined when one associates an arbitrary point p_1 on the first surface that goes through a curve $k = a$ with an arbitrary point p_2 on the corresponding curve $k = a$ on the second surface. The double sign on v_2 in (32) says that one can carry out the bending of the second surface onto the first one on either one side of the curve $k = a$ or the other.

From (31), the mutually-developable surfaces that are treated here are, at the same time, developable onto a surface of rotation. In that way, the meridian curves will correspond to the curves v_1 = const., and the parallel circles will correspond to the curves u_1 = const. or k = const.

The derivation of equation (31) or the determination of the quantities u_1, U_1, and v_1 as functions of (u, v) required only quadratures. That is because u_1 is known from (28), U_1 is known from (30), and v_1 follows from (31):

$$dv_1 = \frac{1}{U_1}\sqrt{ds^2 - du_1^2}\ .$$

The same thing can be inferred from the fact that the system k = const. is both geodetically parallel and has constant geodetic curvature. Hence, from Theorem 1 on pp 112, it will also be isometric, and the associated orthogonal family will be obtained from [§ **17**, (21)] by quadrature.

4. Finally, the case can arise that the *curvature k* of the surface (14) is not a function of (u, v), but only a *constant*. It will then be necessary for developability that the curvature on the surface (15) must also be constant and have the same value. However, that condition is also sufficient for developability, which is also true here in infinitely many ways, but in a different sense.

Namely, if one introduces geodetic polar coordinates (§ **4**) on the surfaces (14) and (15) then the square of the line element will have the form:

$$ds^2 = du_1^2 + g_1\,dv_1^2\,, \qquad ds^2 = du_2^2 + g_2\,dv_2^2\,, \tag{33}$$

and from [§ **9**, (4)], g_1 and g_2 will be determined from the equations:

$$\frac{\partial^2\sqrt{g_1}}{\partial u_1^2} + k\sqrt{g_1} = 0, \qquad \frac{\partial^2\sqrt{g_2}}{\partial u_2^2} + k\sqrt{g_2} = 0\ .$$

One will then have that (1):

$$\begin{array}{llll} \sqrt{g_1} \text{ will be equal to} & \sin\left(u_1\sqrt{k}\right), & \sinh\left(u_1\sqrt{k}\right), & u_1\,, \\ \sqrt{g_2} \quad ” \qquad ” & \sin\left(u_2\sqrt{k}\right), & \sinh\left(u_2\sqrt{k}\right), & u_2\,, \end{array} \tag{34}$$

according to whether:

$$k \qquad > 0, \qquad < 0, \qquad = 0,$$

respectively.

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(1) In which $\sinh(u) = \frac{1}{2}(e^u - e^{-u})$.

If the poles of the polar systems on both surfaces are to correspond then one would need to set $u_1 = u_2$, since those values represent equal geodetic lengths. It will then follow from (33) that $dv_1^2 = dv_2^2$ or that $v_1 = \pm v_2 + c$.

That will give the **theorem**:

If two surfaces have the same constant curvatures then they will be developable to each other in a triple infinitude of different ways.

Namely, one can associate two points p_1 and q_1 on the first surface with two points p_2 and q_2 on the second surface in an entirely arbitrary way, while assuming only that the geodetic distance between those point-pairs is the same on both surfaces.

From (33) and (34), the surfaces of constant curvature k can be developed onto surfaces of revolution of the same constant curvature k. For $k = 0$, that surface will be the cylinder of revolution or the plane, for $k > 0$, it will be the sphere, and for $k < 0$, it will be the pseudo-sphere, whose simplest manifestation has the tractrix (viz., the evolvent of the catenary) for its meridian curve.

The derivation of the form (33) or the determination of the geodetic lines on a surface of constant curvature requires one to integrate a **Ricatti** differential equation ([1]).

([1]) **Darboux**, *Leçons*, III, pp. 223.