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Forces of interaction in electrodynamics and in the field theory of nuclear forces. (Part I)

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Abstract. It will be shown that the quantum theory of wave fields leads to the same expressions for the interaction of charges as the classical treatment of retarded potentials.

The interaction operator has the following form: Retarded (or advanced) potential of one charge at the position of the second one, times the second charge. If one of those two charges can emit radiation in the first approximation then one must choose the retarded potential of that charge or the advanced potential of the other.

The present Part I contains the complete discussion for a scalar field. The generalization to a four-vector field is touched upon only briefly, and will be treated in a second part.

Introduction

Whether or not the quantum theory of wave fields contains great internal contradictions at the present time, it is still the single practical means of describing the corpuscular nature of radiation and matter.

On the one hand, it follows from quantum electrodynamics that there exist discrete *light quanta* – i.e., the result that radiation of frequency $k_0 c$ can be emitted or absorbed only in amounts of $h k_0 c$.

On the other hand, one can derive the *classical result* of the *retarded interaction* between two charges.

In many cases, we are interested in only the second property of the field. Since that retarded interaction does not contain Planck's constant, it must be possible to show that all of the laws of interaction that follow from a quantum theory of wave fields are identical with the corresponding classical results.

We would like to prove this for a scalar field A whose field equation has the form:

$$(\Box - l^2)\, A = -4\pi J. \tag{0.1}$$

On the basis of electrodynamics, we call $A(x)$ the *potential* and $J(x)$ the *charge*. x is the position four-vector, whose components are $x_0 = ct$, and x_1, x_2, x_3, the last three of

which we denote by **x**. The wave equation (0.1), whose static solution for a point charge at rest at the coordinate origin reads:

$$A = \frac{e^{-lr}}{r}, \tag{0.2}$$

defines the starting point of YUKAWA's field theory [**1**], which explains nuclear forces by means of the existence of a new field whose particle (which is associated with it by the quantum theory of wave fields) has the mass hl / c. l is then the reciprocal Compton wave length of that new particle. From the range of nuclear forces, that mass will have the order of magnitude of 100 electron masses. In what follows, we will show that the existence of charged and uncharged particles must be required. Charged particles were observed by various authors in cosmic rays. One will find more details on this in the notices of YUKAWA [**1**], KEMMER, BHABHA, and the author [**2**].

1. The retarded and advanced potential.

We look for a solution of (0.1). In order to do that, we develop $J(x)$ in a Fourier integral with the integrand $J(k)$, where k means the four-vector with the (real) temporal component k_0 and spatial components $\mathbf{k} = (k_1, k_2, k_3)$. dx^4, dk^4 and $d\mathbf{x}^3$, $d\mathbf{k}^3$ mean the four-dimensional and three-dimensional volume elements in space (the space of wave vectors, resp.). Therefore, let:

$$J(x) = \int dk^4\, e^{i(k,\, x)}\, J(k). \tag{1.1}$$

If one develops $A(x)$ in an analogous way then it will follow by comparing the coefficients that one has the relation:

$$A(k) = \frac{4\pi\, J(k)}{(k,k) + l^2} \tag{1.2}$$

for the coefficients $A(k)$, in which (k, x) and (k, k) are scalar products of four-vectors.

We say *eigenvectors of the field* when we mean vectors whose temporal components obey the relation:

$$k_0 = \pm \bar{k}_0(\mathbf{k}) = \pm \sqrt{(\mathbf{k},\mathbf{k}) + l^2}\ . \tag{1.3}$$

In this, $(\mathbf{k}, \mathbf{k})$ is the scalar product of the spatial part of k with itself.

We assume that $J(k)$ has no singularities for real k-values. The integrand $A(k)$ in the expression for $A(x)$ will then possess singularities on the real k_0-axis for both $k_0 = \pm \bar{k}_0(\mathbf{k})$. Since, by assumption, $J(k)$ possesses no singularities in the immediate neighborhood of the real k_0-axis, we can already deform the path of integration in (1.1) over k_0 from $-\infty$ to $+\infty$ before the comparison of coefficients. If we denote that deformed path by (..) then can write the solution of (0.1) as:

$$A^{(..)}(y) = \int d\mathbf{k}^3 \int_{(..)} dk_0\, A(k)\, e^{i(k,y)}$$
$$= \int dx^4\, J(x)\, D^{(..)}(y-x)\,. \qquad (1.4)$$

In this, $D^{(..)}(z)$ is a function that is defined by the Fourier integral:

$$D^{(..)}(z) = \frac{2}{(2\pi)^3} \int_{(..)} e^{i(k,x)} \frac{1}{(k,k)+l^2}\, dk^4\,. \qquad (1.5)$$

In order to derive the second identity in (1.4), one makes use of the four-dimensional δ-function:

$$\delta(k) = (2\pi)^{-4} \int dx^4 e^{i(k,x)}\,, \qquad (1.6)$$

with the property that:

$$\int_K dk^4\, f(k)\, \delta(k) = f(0) \quad \text{or} \quad = 0, \qquad (1.7)$$

according to whether the point $k_i = 0$ lies inside or outside of the path of integration K in (2.7), respectively.

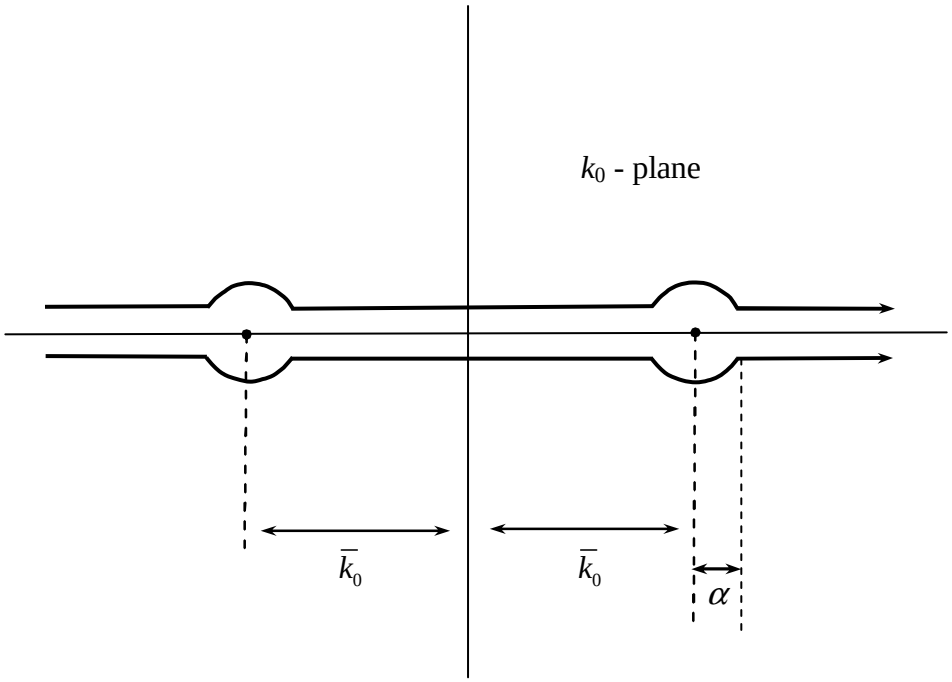

Figure 1.

If one chooses the integration paths to be the two paths that are denoted by (+) and (–) and run completely in the positive (negative, resp.) half-plane (Fig. 1), and considers that:

$$\int\limits_{(\pm)} dk_0 \frac{e^{-ik_0 x_0}}{k_0} = \lim_{\alpha=0}\left(-2i\int\limits_{\alpha}^{\infty} dk_0 \frac{\sin k_0 x_0}{k_0} + \int\limits_{-\alpha_{(\pm)}} \frac{dk_0}{k_0} \right)$$

$$= -i\pi\left(\frac{x_0}{|x_0|}\pm 1\right); \qquad \left(\frac{x_0}{|x_0|}=0 \quad \text{for} \quad x_0 = 0\right)$$

(α is a small, positive quantity in this)

then one will obtain the following expression for $D^{(..)}$:

$$D^{(..)}(x) = g^{(..)}(x_0)\,\frac{1}{|\mathbf{x}|}\frac{1}{\pi}\int\limits_0^{\infty} d\,|\mathbf{k}|\frac{|\mathbf{k}|}{k_0}[\cos(|\mathbf{k}||\mathbf{x}|-\bar{k}_0 x_0)-\cos(|\mathbf{k}||\mathbf{x}|+\bar{k}_0 x_0)] \tag{1.8}$$

with:

$$g^{(+)} = \begin{cases} 1 \\ 1/2\,, \\ 0 \end{cases} \qquad g^{(-)} = \begin{cases} 1 \\ -1/2 \\ -1 \end{cases} \qquad \text{for} \qquad \begin{matrix} > \\ x_0 = 0, \text{ resp.} \\ < \end{matrix}$$

Therefore, in formula (1.4), $A^{(+)}$ is determined by only the charge distribution in the past, and $A^{(-)}$ is determined by only the one in the future. As a generalization from electrodynamics, they will be referred to as the *retarded* and *advanced* potential, resp. $D^{(+)}$ and $D^{(-)}$ themselves are the retarded (advanced, resp.) potentials of a point charge at the origin that is non-zero only at the time point $x_0 = 0$, and whose spatial integral has the value unity. From their derivation, $D^{(+)}$ and $D^{(-)}$ are invariant functions.

For $l = 0$ (viz., the electrodynamic case), one will have:

$$D^{(\pm)}(x) = \frac{1}{|\mathbf{x}|}\delta(|\mathbf{x}| \mp x_0) \qquad (\delta = \text{one-dimensional } \delta\text{-function}).$$

We then define the likewise-invariant function:

$$D(x) = D^{(-)}(x) - D^{(+)}(x)$$

$$= \frac{1}{|\mathbf{x}|}\frac{1}{\pi}\int\limits_0^{\infty} d\,|\mathbf{k}|\frac{|\mathbf{k}|}{\bar{k}_0}[\cos(|\mathbf{k}||\mathbf{x}|+\bar{k}_0 x_0)-\cos(|\mathbf{k}||\mathbf{x}|-\bar{k}_0 x_0)]\,, \tag{1.9}$$

which goes to the Heisenberg-Pauli invariant δ-function for $l = 0$. With its help, one can write the retarded and advanced potentials in the form:

$$A^{(\pm)}(y) = \int d\mathbf{x}^3 \int\limits_{y_0}^{\mp\infty} dx_0\, D(y-x)J(x)\,. \tag{1.10}$$

In general, retarded and advanced potentials are two different expressions then. However, there are special charge distributions whose Fourier coefficients $J(k)$ vanish for those k that are eigenvectors of the field. It is not necessary to deviate from the integration path (1.4) in this case; i.e., both integration paths will give the same value. The advanced potential of that special charge distribution is equal to the retarded one.

MØLLER [**3**] showed that the electrodynamical interaction of two electrons in quantum theory can be regarded by using the following interaction Ansatz: One measures the retarded or advanced potential of the first electron at the position of the second one and multiplies it with the charge of the second electron. This argument will, in fact, lead to a symmetric expression for free electrons, since the Fourier-analyzed matrix element of the current will contain no eigenvectors of the field.

The Breit interaction [**4**] can then be obtained from the MØLLER Ansatz by developing it in $1/c$, and its reduction to the large components for Dirac functions will, in fact, yield the correct spin-spin and spin-orbit couplings of Pauli's theory of spin.

If one recalls the prescription that this interaction Ansatz can be employed for the calculation of eigenvalues in the first approximation then it will follow that only static charge distributions will come under consideration, and therefore the retarded and advanced potentials will, in fact, be the equal.

Later on, HULME [**5**] showed that MØLLER's prescription can be replaced with a stronger one whenever the one particle is capable of radiating: Namely, the retarded potential of that radiating particle must then be chosen at the location of the second particle, and multiplied by the charge of the second particle, or else the advanced potential of the non-radiating particle at the location of the radiating one, and multiplied by its charge.

In what follows, we will once more find Hulme's sharpening of Møller's prescription as the result of a more general quantum-theoretic argument. Its classical analogue is just this appearance of eigenvectors of the field in the "mixed charge density" of the one particle.

2. Lagrange and Hamilton function of the field.

For the sake of generality, the field and the charge shall be regarded as complex. Formula (0.1) will then follow from the Lagrange function:

$$L = \int d\mathbf{x}\ \mathcal{L},$$

with the Lagrange density function:

$$\mathcal{L} = -\frac{1}{8\pi}\left[\left(\frac{\partial A^*}{\partial x},\frac{\partial A}{\partial x}\right)+l^2 A^* A\right]+\frac{1}{2}\left[A^* J+\left(\frac{\partial A^*}{\partial x},S\right)+\text{conj.}\right] \tag{2.1}$$

(conj. means the complex conjugate of the expression in brackets), in the usual way (when one considers A and A^* to be two independent functions). In place of J, one will find the expression:

$$J_{\text{eff}} = J - \left(\frac{\partial}{\partial x}, S\right). \tag{2.2}$$

If one lets J denote *charge* then (on dimensional grounds) the four-vector S can be regarded as the *polarization.* (2.2) is then the *effective charge* in eq. (0.1).

The conjugate momenta follow in the usual way:

$$P = \frac{\delta L}{\delta \dot{A}} = \frac{\partial \mathcal{L}}{\partial A} - \left(\frac{\partial}{\partial \mathbf{x}}, \frac{\partial \mathcal{L}}{\partial \frac{\partial \dot{A}}{\partial \mathbf{x}}}\right) = \frac{1}{8\pi c}\frac{\partial A^*}{\partial x_0} + \frac{1}{2c} S_0^*, \tag{2.3}$$

and there is a corresponding expression for P^*. The Hamilton function is calculated by using the relation:

$$H = -L + \int d\mathbf{x}^3 (\dot{A}P + \dot{A}^* P^*) = W + V. \tag{2.4}$$

In this, W and V are the volume integrals of the energy densities $\mathcal{W}$ and $\mathcal{V}$, which have the forms (*):

$$\mathcal{W} = \frac{1}{8\pi}\left[\left(\frac{\partial A^*}{\partial x}, \frac{\partial A}{\partial x}\right) + l^2 A^* A\right] + 8\pi c^2 P^* P \tag{2.5}$$

and

$$\mathcal{V} = -\frac{1}{2}\left[A^* J + \left(\frac{\partial A^*}{\partial \mathbf{x}}, \mathbf{S}\right) + \text{conj},\right] + 4\pi c\,(PS_0 + \text{conj.})\,. \tag{2.6}$$

The canonical equations follow from the functional differentiation of the Hamiltonian functional with respect to A^* (P, resp.), which is defined by the second and third term of eq. (2.3):

$$\dot{A} = \frac{\delta H}{\delta P} = 8\pi c^2 P^* - 4\pi c\, S_0 \tag{2.7}$$

and

$$\dot{P}^* = -\frac{\delta H}{\delta A^*} = \frac{1}{8\pi}(\Delta - l^2)A + \frac{1}{2}\left(J - \left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{S}\right)\right). \tag{2.8}$$

(*) Formula (2.4) will yield only a supplementary term of the form:

$$+ 2\pi \int d\mathbf{x}^3 S_0^* S_0 \tag{2.6a}$$

that depends upon only the matter (and is therefore superfluous to the field equations), in addition to the W and V that were written down. However, as will be shown in Part Two, that term will be necessary if the equations of motion for matter are to make sense.

Differentiating (2.7) with respect to time and eliminating $\dot{P}^*$ from (2.8) then gives, in fact, eq. (0.1) with the effective charge that was defined in (2.2) in place of *J*.

3. Quantization of the field theory.

The quantization of the scalar field theory was carried out by PAULI and WEISSKOPF [**6**]. The operator identities:

$$\begin{aligned} \dot{P} &= (i/h)[H,P] = -\,\delta H/\delta A, \\ \dot{A} &= (i/h)[H,A] = \delta H/\delta P \end{aligned} \tag{3.1}$$

follow for each analytic functional *H* of the operators *P* and *A* when one has:

$$[P(\mathbf{x}),\, A(\mathbf{x}')] = (h/i)\;\delta(\mathbf{x}-\mathbf{x}'), \tag{3.2}$$

and when the starred operators A^* and P^* commute with the un-starred ones *A* and *P*. In this, a square bracket means:

$$[a,\, b] = ab - ba.$$

Therefore, the equations of motion (2.7) and (2.8) also follow for the quantum-theoretic operators when we employ the operator for *H* that follows from the classical equations (2.4), (2.5), and (2.6) when one replaces the P^*, A^*, *P*, and *A* with operators that fulfill (3.2).

The same thing will be true when we employ an operator *K* for which one has:

$$\begin{aligned} (i/h)[K,P^*] &= -\delta K/\delta A^* = \frac{1}{2}\left(J - \left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{S}\right)\right), \\ (i/h)[K,A] &= \quad \delta K/\delta P = -4\pi c\, S_0, \end{aligned} \tag{3.3}$$

instead of *V*. This equation can also be regarded as the definition of the charge quantities *J* and *S*.

J and *S* are functions of the canonical variables that describe the state of matter. If we denote them by *p* and *q* then the (classical) equations of motion for matter shall follow from:

$$\begin{aligned} \dot{p} &= (i/h)[K,\, p], \\ \dot{q} &= (i/h)[K,q]. \end{aligned} \tag{3.4}$$

The quantum-theoretic problem consists of solving the Schrödinger equation:

$$\left(H + \frac{h}{i}\frac{\partial}{\partial t}\right)\Psi(t) = 0. \tag{3.5}$$

We shall follow a method that was proposed by DIRAC, FOCK, and PODOLSKI [**7**]:

A function (functional, resp.) $\Psi'(T, t)$ for which one has:

$$\Psi'(T, t) = \Psi(t) \tag{3.6}$$

is introduced, along with an operator $K'(T, t)$ that likewise satisfies the relation:

$$K'(T, t) = K. \tag{3.7}$$

In (3.7), it was assumed that the operator K does not contain time t explicitly. K' will then depend upon T and t explicitly, in general. Now, if the function Ψ simultaneously satisfies the two equations:

$$C_T\Psi' = \left(W + \frac{h}{i}\frac{\partial}{\partial T} \right)\Psi' (T, t) = 0, \tag{3.8}$$

$$C_t\Psi' = \left(K'(T,t) + \frac{h}{i}\frac{\partial}{\partial t} \right)\Psi' (T, t) = 0 \tag{3.9}$$

then the Schrödinger equation will follow from this for the one-sided Ψ with $H = W + K$ that was defined by (3.6).

Now, in order for the two eq. (3.8) and (3.9) to be simultaneously soluble, the operators C_T and C_t must commute. This condition:

$$[C_T, C_t] = [W, K'(T, t)] + \frac{h}{i}\frac{\partial K'(T,t)}{\partial T} = 0 \tag{3.10}$$

represents a differential equation for the dependency of the operator K' on T. When one considers the initial condition (3.7), its solution will be:

$$K'(T, t) = e^{iW(t-T)/h}\, K\, e^{-iW(t-T)/h}. \tag{3.11}$$

Eq. (3.8) can be solved formally by:

$$\Psi'(T, t) = e^{-iWT/h}\ \psi(t). \tag{3.12}$$

The function $\psi(t)$, which depends upon only time, then satisfies the following Schrödinger equation, which follows from (3.9), (3.11), and (3.12):

$$\left(K'' + \frac{h}{i}\frac{\partial}{\partial t} \right)\psi(t) = 0 \tag{3.13}$$

in which $K''(t)$ means the operator:

$$K''(t) = e^{iWt/h}\, K\, e^{-iWt/h}. \tag{3.14}$$

The Hamilton operator K'' in eq. (3.13) is then the original operator K, in which the operator $A(\mathbf{x})$ has been replaced with the operator $A''(\mathbf{x}, t)$, which can be calculated from $A(\mathbf{x})$ by the same unitary transformation (3.14) that produced $K''(t)$ from K. If one ignores the operator character of A then (3.13) will be the Schrödinger equation of matter under the influence of a given potential field $A''(\mathbf{x}, t)$ that depends upon time t explicitly.

The same thing is true for P. In order to have an explicit form for the new operators A'' and P'', we develop A'' in a series:

$$A''(\mathbf{x}, t) = A(\mathbf{x}, 0) + t\frac{\partial A''}{\partial t}(\mathbf{x}, 0) + \frac{t^2}{2}\frac{\partial^2 A''}{\partial t^2}(\mathbf{x}, 0) + \ldots$$

In this, one has the following relations that follow from (3.14):

$$A''(\mathbf{x}, 0) = A(\mathbf{x}),$$

$$\frac{\partial A''}{\partial t}(\mathbf{x}, 0) = (i/h)\,[W, A(\mathbf{x})] = \frac{\delta W}{\delta P} = 8\pi c^2 P^*(\mathbf{x}),$$

$$\frac{\partial^2 A''}{\partial t^2}(\mathbf{x}, 0) = (i/h)^2\,[W, [W, A(\mathbf{x})]] = 8\pi c^2 (i/h)\,[W, P^*],$$

$$= -8\pi c^2 \frac{\delta W}{\delta A^*} = c^2(\Delta^2 - l^2) A(\mathbf{x}),$$

etc. If one then introduces the symbolic operator:

$$b(\mathbf{x}) = \sqrt{-\Delta + l^2}\,, \tag{3.15}$$

which acts upon only the parameter $\mathbf{x}$, then one will generally get:

$$\begin{aligned} &\frac{\partial^{2n} A''}{\partial t^{2n}}(\mathbf{x}, 0) = (ic\,b(\mathbf{x}))^{2n} A(\mathbf{x}), \\ &\frac{\partial^{2n+1} A''}{\partial t^{2n+1}}(\mathbf{x}, 0) = 8\pi c^2 (icb)^{-1} (icb)^{2n} P^*(\mathbf{x}). \end{aligned} \tag{3.16}$$

To abbreviate the series development of the symbolic operator b, we introduce the expressions $\cos(bct) = \cos(b\,x_0)$ and $\sin(bct) = \sin(b\,x_0)$. One can then write:

$$A(\mathbf{x}, t) = A(x) = \cos(b(\mathbf{x})\,x_0)\,A(\mathbf{x}) + 8\pi c^2\, b(\mathbf{x})^{-1} \sin(b(\mathbf{x})\,x_0)\,P^*(\mathbf{x}). \tag{3.17}$$

In what follows, we will always employ the symbol $A(x)$ for $A''(\mathbf{x}, t)$, in which x again means the world vector.

This potential that depends upon the world vector x obeys well-defined commutation relations. Let x and y be two events, so:

$$[A(x), A(y)] = [A(x)^*, A(y)^*] = 0, \tag{3.18}$$

since $A(\mathbf{x})$ and $P(\mathbf{x})^*$ commute in (3.17).

If one employs the commutation relation (3.2) and its conjugate, and one writes the δ-function in the form of its Fourier development [the analogue of (1.6)]:

$$\delta(\mathbf{x}) = (2\pi)^{-3} \int d\mathbf{k}^3 e^{i(\mathbf{k},\mathbf{x})}$$

then, upon considering that:

$$b(\mathbf{x})\, e^{i(\mathbf{k},\,\mathbf{x})} = \bar{k}_0(\mathbf{k})\, e^{i(\mathbf{k},\mathbf{x})} \tag{3.19}$$

[in which $k_0(\mathbf{k})$ again means the (positive) square root of (1.3)], one will have the commutation relation:

$$[A(x)^*, A(y)] = -\,2\frac{hc}{i} D(x-y). \tag{3.20}$$

In this, D is once more the invariant function that is defined by its Fourier representation (1.9).

The $P''(\mathbf{x}, t)$ that can possibly enter into K can be expressed by a series that is analogous to (3.17). One then finds that:

$$P''(\mathbf{x}, t) = P(x) = \frac{1}{8\pi c^2}\frac{\partial A''(\mathbf{x},t)^*}{\partial t} = (8\pi c)^{-1}\,\frac{\partial A(x)^*}{\partial x_0}. \tag{3.21}$$

In particular, if the part of K that represents the interaction of field and matter has the form V, whose density is given by (2.6), then the corresponding density $\mathfrak{D}''(\mathbf{x}, t)$ will be given by the invariant expression:

$$\mathfrak{D}''(\mathbf{x}, t) = -\,\frac{1}{2}\left[A(x)^*\, J(\mathbf{x}) + \left(\frac{\partial A(x)}{\partial x}, S(\mathbf{x})\right) + \text{conj.}\right]. \tag{3.22}$$

$J(\mathbf{x})$ and $S(\mathbf{x})$ are the (scalar) charge and the four-vector of its polarization at the location $\mathbf{x}$. In the event that it does not depend upon the potential A, as in Dirac's theory, t will not appear explicitly in it, as before.

4. Derivation of the interaction operator in configuration space.

If one describes the matter, not by a matter field, but by the coordinates q^s (s = 1, 2, …, n) of n particles (viz., configuration space) then the Schrödinger functions Ψ, Ψ', and ψ that we employed in paragraph 3, will become functionals of the functions $A(\mathbf{x})$ and functions of the q^s. The Hamilton operator of matter K will then be decomposed into a

sum $\sum K^s$, in which K^s (along with the field operators) contains only operators that act upon q^s. Each K^s will itself decompose into a term R^s (q^s) that contains only operators that act upon q^s and a term:

$$V^s\,(q^s, A''(\mathbf{x}, t)) = V^s\,(t)$$

that contains field operators that contain time t explicitly, according to the foregoing paragraph, in addition to the operators that act upon q^s.

We solve the Schrödinger equation (3.13):

$$\left(K'' + \frac{h}{i}\frac{\partial}{\partial t}\right)\psi = \left(R + V + \frac{h}{i}\frac{\partial}{\partial t}\right)\psi = 0 \tag{4.1}$$

[in which, K means the sum of the K^s, and R (V, resp.) means the sum of the R^s (V^s, resp.)] by an approximation procedure. In it, we imagine that the *perturbing term* V is proportional to a small number, in which we develop.

Therefore, let:

$$\psi = \psi^0 + \psi^1 + \psi^2 + \ldots, \tag{4.2}$$

in which ψ^m is proportional to the m^{th} power of that small number. If one sets the factors of the individual powers equal to zero then one will obtain the following system of equations from (4.1):

$$\begin{aligned}
&\left(R + \frac{h}{i}\frac{\partial}{\partial t}\right)\psi^0 = 0,\\
&\left(R + \frac{h}{i}\frac{\partial}{\partial t}\right)\psi^1 + V\psi^0 = 0,\\
&\left(R + \frac{h}{i}\frac{\partial}{\partial t}\right)\psi^2 + V\psi^0 = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..
\end{aligned} \tag{4.3}$$

The solution $\psi^0 + \psi^1 + \psi^2$ of the three equations that were written down in (4.3) then gives the *second approximation* to the solution of (4.1). Along with the effect of the matter on the field, it already contains the reaction of the field that is generated by the matter on that matter. Therefore, the interaction of two particles with each other is contained in *its first approximation*. If we succeed in finding a relation:

$$V^s(t)\ \psi^1 = \sum_r U^{sr}\psi^0 \tag{4.4}$$

then we can combine the last two equations in the system that was written down in (4.3) into the equation:

$$R + \frac{h}{i}\frac{\partial}{\partial t}(\psi^1 + \psi^2) + \left(V + \sum_s\sum_r U^{rs}\right)\psi^0 = 0, \tag{4.5}$$

by addition.

If one develops the solution to the equation:

$$\left(R + V + \sum_r \sum_s U^{rs} + \frac{h}{i} \frac{\partial}{\partial t} \right) \psi = 0, \tag{4.6}$$

in which one thinks of V and $\sum\sum U^{rs}$ as being multiplied by *the same* small parameter, then one will get the first equation in (4.3) for the first equation (for ψ^0), while one will get eq, (4.5), with ψ^{I}, in place of $\psi^1 + \psi^2$, as the *first approximation* ψ^{I} [in which the Roman index means the development in the perturbing term $V + U^{rs}$ in (4.6)]. Thus, the *first approximation* of (4.6) will be identical with the *second approximation* of the correct eq. (4.1). However, it is clear that the higher approximations of (4.6) will lead to incorrect expressions for ψ; i.e., ones that do not satisfy the Schrödinger equation of the problem.

Obviously, the term:

$$U^{rs} + U^{sr}, \qquad r \neq s \tag{4.7}$$

will then represent the *interaction energy in the first approximation* between the particles r and s.

In order to discover a relation (4.4), we consider the system of equations of the n-equations:

$$\left(R^s + V^s(t^s) + \frac{h}{i} \frac{\partial}{\partial t^s} \right) \psi(t^1, \ldots, t^s, \ldots, t^n) = 0. \tag{4.8}$$

If we find the solution to this many-timed system then the function:

$$\psi(t, t, \ldots, t) = \psi(t) \tag{4.9}$$

will represent a solution to the problem (4.1). t^s is the "particle time" of the s^{th} particle. However, equations (4.8) are not, in fact, simultaneously soluble. One will find more details on this in BLOCH [**8**]. Nonetheless, we will see that it is soluble in the first approximation, and we will, in fact, ascertain a relation (4.4).

In order to do that, one must demand the existence of a Lorentz system in which R^s contains no explicitly time-dependent forces. Naturally, that is the case for free particles, and likewise for a bound electron in the rest system of the atomic nucleus. There then exist eigensolutions for which one has:

$$f(R^s)\, u_{\nu_1 \cdots \nu_n} = u_{\nu_1 \cdots \nu_n} \cdot f(h\, \nu_s) \tag{4.10}$$

[if $f(z)$ is an arbitrary function of z that is representable by a series development]. The u_{ν_s} are functions that are independent of the t^s. The time-dependent functions:

$$v_{\nu_1 \cdots \nu_n} = u_{\nu_1 \cdots \nu_n} e^{-i \sum \nu_s t^s} \tag{4.11}$$

then fulfill (4.8) in the first approximation.

We now consider the expression:

$$f\left(R^s + \frac{h}{i}\frac{\partial}{\partial t^s}\right) w\,(t^1, \ldots, t^n), \tag{4.12}$$

in which w is an arbitrary function of the q^s and t^s, and a functional of the field variables whose time-dependency is representable in the form:

$$w = \sum_{\omega_1} \cdots \sum_{\omega_n} e^{-i\sum \omega_s t^s} w_{\omega_1 \cdots \omega_n} \ . \tag{4.12b}$$

(4.12) will then become the sum:

$$\sum_{\omega_1} \cdots \sum_{\omega_n} e^{-\sum \omega_s t^s} f\,(R^s - h\omega_s) w_{\omega_1 \cdots \omega_n} \ .$$

The integral then has the following meaning:

$$\begin{aligned} \int dq^s \ \ldots &= \int dq^s v^*_{\nu_1 \cdots \nu_s} f\left(R^s + \frac{h}{i}\frac{\partial}{\partial t^s}\right) w \\ &= \int dq^s \sum_{\omega_1} \cdots \sum_{\omega_n} e^{-i\sum_r (\nu_r - \omega_r) t^r} \ (f\,(R^s - h\omega_s) * u_{\nu_1 \cdots \nu_n}) * w_{\omega_1 \cdots \omega_n} \ . \end{aligned}$$

In this, f^* is the Hermitian conjugate of the operator f. If $f(z)$ is a real function then $f^* = f(R^s - h\ \omega_s^*)$, since R^s is a Hermitian operator. Thus, the integral can be written:

$$\int dq^s \ \ldots = \int dq^s \sum_{\omega_1} \cdots \sum_{\omega_n} \ f\,(h(\nu_s - \omega_s))\, e^{i\sum (\nu_r - \omega_r) t^r} u^*_{\nu_1 \cdots \nu_n} w_{\omega_1 \cdots \omega_n} \ . \tag{4.13}$$

Since every function of the operator $R^s + h\ \partial\ /\ i\ \partial t^s$ that is representable as a series development is therefore well-defined, the equations of the system that is analogous to (4.3), which will be contained in (4.8) by developing ψ, can be solved in the first approximation by:

$$\psi^1 = -\sum \left(R^r + \frac{h}{i}\frac{\partial}{\partial t^r}\right)^{-1} V^r\,(t^r)\ \psi^0 \ . \tag{4.14}$$

On the other hand, we have the relation:

$$\psi^0\,(t^1, \ldots, t^n) = \ e^{-i\sum R^r (t^r - t)}\ \psi^0\,(t) \ . \tag{4.15}$$

This is easy to verify when one pays attention to the fact that ψ^0 must be a linear combination of eigenfunctions $u_{\nu_1 \cdots \nu_n}$ whose coefficients have the time-dependency of $\exp(-\sum i\ \nu_r\ t)$.

The operator:

$$U^{rs} = -\left\{\left(R^r + \frac{h}{i}\frac{\partial}{\partial t^r}\right)^{-1} V^s(t^s)V^r(t^r)\,e^{-i\sum R^m(t^m - t)}\right\}_{t^1=t^2=\cdots=t} \tag{4.16}$$

will actually have the property (4.4) then.

According to the derivation of the operator U^{rs}, we can apply this only in the first approximation. Thus, we are interested in only matrices that are time-independent. If one calculates the matrix element (*):

$$U^{sr}_{\nu',\nu} = \int dq^1 \cdots \int dq^n \int dA\, v^*_{\nu'_1\cdots\nu'_n}\left(R^r + \frac{h}{i}\frac{\partial}{\partial t^r}\right)^{-1} V^s(t^s)V^t(t^r)\,v_{\nu_1\cdots\nu_n}, \tag{4.17}$$

which is many-timed, for the time being, and employs the relation (4.13) then it will contain the factor:

$$\frac{i}{h}\frac{e^{i(\nu'_r-\omega_r)t^r}\cdot e^{i(\nu'_s-\omega_s)t^s}}{i(\nu'_r-\omega_r)}. \tag{4.18}$$

The ω_s, t^s come from the development of:

$$w = V^s(t^s)\,V^r(t^r)\,v_{\nu_1\cdots\nu_n}(t^1, \ldots, t^n),$$

according to (4.12a).

With the help of the expression:

$$\lim_{\gamma=0}\frac{e^{i(\nu\pm i\gamma)t}}{i(\nu\pm i\gamma)} = -\lim_{\gamma=0}\int_t^{\pm\infty} dt\, e^{i(\nu\pm i\gamma)t},$$

which has been made convergent, this factor can be converted into an integral, and one will obtain the form:

$$U^{sr}_{\nu',\nu}(t^1,\cdots,t^n) = \int_{t^r}^{\pm\infty} dt^r \int dq^1 \cdots \int dq^n \int dA\, v^*_{\nu'_1\cdots\nu'_n}\frac{i}{h}V^s(t^s)V^t(t^r)\,v_{\nu_1\cdots\nu_n}. \tag{4.19}$$

If we consider the condition that only time-independent matrix elements play a role then that will mean that only terms whose exponents obey the relation:

$$(\nu'_s - \omega_s) + (\nu'_r - \omega_r) + \sum_{m\mp s,r}(\nu'_m - \nu_m) = 0 \tag{4.19a}$$

will be employed. If one chooses the $v_{\nu_1\cdots\nu_n}$ to be an orthogonal system of functions in the usual way then one must have $\nu'_m = \nu_m$, since the operator will contain no operators

(*) $\int dA$ means integration over the function space $A(\mathbf{x})$ (e.g., the space of light quanta).

that act upon q^m for $m \neq r, s$. Therefore, $(v'_r - \omega_r)$ can be replaced with $-(v'_s - \omega_s)$ everywhere. In eq. (4.18), that means that the integral over dt^r in (4.19) can be replaced with:

$$-\int_{t^s}^{\pm\infty} dt^s .$$

Therefore, if one adds the matrix element U^{rs} to that of U^{sr} in order to obtain that of the interaction operator (4.7), and makes use of the second of these alternatives then one will find that the time-free matrix elements of the interaction operator will be equal to the operator:

$$\int_{t}^{\pm\infty} dt^r e^{iR^r(t^r-t)} \frac{i}{h}[V^s(t), V^r(t^r)] e^{-iR^r(t^r-t)} . \tag{4.20}$$

Now, if V^s has the form:

$$V^s = -\int d\mathbf{x}^3 [(A(x)^* J^s(\mathbf{x})] + \left(\frac{\partial A(x)^*}{\partial x}, S^s(\mathbf{x}) \right) + \text{conj.} + \text{terms in } A^2,$$

in which J^s and S^s mean the charge and polarization operators *in the absence of fields* (their variation in the absence of a field is included in the "+ terms in A^2"), then it will suffice to consider only the terms that are linear in A in (4.20). It would be incorrect to include the higher-order terms, since we would like to consider the small quantities that we are developing only up to the second power in our approximation process. The expressions J^s and S^s will then no longer contain field operators, and will then commute with the field operators.

We introduce the operator:

$$J^s(y) = e^{iR^r(y_0-ct)/ch} J^s(\mathbf{y}) e^{-iR^r(y_0-ct)/ch} , \tag{4.21}$$

whose expectation values yield the expectation values of the charge of the particle s at the position $\mathbf{y}$ at the time y_0 / c (which is generally different from the time t that enters into the Schrödinger equation) (viz., the *operator of the retarded charge*). Due to the commutation relations (3.18) and (3.20), the interaction operator can be written in the form:

$$U^{rs} + U^{sr} = -\tfrac{1}{2}\int d\mathbf{x}^3 \left\{ J^s(\mathbf{x}) A^r(x)^* + \left(S^s(\mathbf{x}), \frac{\partial A^r(x)^*}{\partial x} \right) + \text{conj.} \right\}_{x_0 = ct} , \tag{4.22}$$

with (*):

(*) In the temporal part of the scalar product, one must actually have:

$$-\tfrac{1}{2}\int d\mathbf{x}^3 S_0^s(\mathbf{x}) \int_x^{\pm\infty} dy_0 \frac{\partial}{\partial x_0} \int d\mathbf{y}^3 (J^r(y)^* D \cdots) ,$$

$$A^r(x) = \int_{x_0}^{\pm\infty} dy_0 \int d\mathbf{y}^3 \left(J^r(y) D(x-y) + \left(S^r(y), \frac{\partial D(x-y)}{\partial y} \right) \right).$$

One can convert the second term in the integrand by partial derivatives, since the *D*-function will vanish on the outer surface of the four-dimensional half-space, and one will obtain:

$$A^r(x) = \int_{x_0}^{\pm\infty} dy_0 \int d\mathbf{y}^3 J^r_{\text{eff}}(y) D(x-y), \tag{4.23}$$

where:

$$J^r_{\text{eff}}(y) = J^r(y) - \left(\frac{\partial}{\partial y}, S^r(y) \right) \tag{4.24}$$

means the effective charge of the particle *r* at the position **y** at the time y_0 / c that was defined in (2.2).

However, (4.23) is nothing but the classical formula (1.10) for the retarded or advanced potential in operator form.

The operator (4.22) no longer contains any field operators, and will lead to Møller's prescription with no further assumptions. Obviously, the retarded and advanced potentials are equal with that.

However, if the particle *r* can radiate then that will mean that the function ψ^1 in (4.14) will be infinite; i.e., the term with $\nu'_r = \omega_+$ that enters into the denominator of (4.18). That will always be the case when the Fourier analysis of the matrix element of the operator $J^r(x)$ contains the eigenvectors of the field that were defined in paragraph 1 for certain transitions. The first approximation gives an essential contribution to the perturbation of the wave function ψ^0. The initial state will be decay exponentially. If one carries out the calculation that one already has in the zeroth approximation, which one can write:

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i.e., $\partial / \partial x_0$ appears after dy_0 *under the integral sign.* If one puts it *in front of the integral sign*, as in formula (4.22), then due to the relations:

$$D(x)_{x_0=0} = 0; \quad \left\{ \frac{\partial D(x)}{\partial x_0} \right\}_{x_0=0} = -4\pi\delta(\mathbf{x}); \quad \left\{ \frac{\partial D(x)}{\partial x_i} \right\}_{x_0=0, i\neq 0} = 0,$$

one will obtain a supplementary term of the form:

$$-2\pi \int dx^3 (S^s_0(\mathbf{x}) S^r_0(\mathbf{x})^* + \text{conj.}),$$

which will drop out, in contrast to the term that was mentioned in the rem. on pp. 6 (2.6a).

$$\left.\begin{aligned} v_{\nu_1\cdots\nu_n} &= u_{\nu_1\cdots\nu_n} e^{-i\sum_{m\neq r}\nu_m t^m}\cdot F(-\nu_r + i\gamma, t^r) \\ \text{with}\quad F(\nu,t) &= \begin{cases} 0 & \text{for } t<0, \\ e^{i\nu t} & \text{for } t<0, \end{cases} \end{aligned}\right\} \tag{4.25}$$

in place of (4.11), then, as one easily convinces oneself by Fourier representation (or integration over time in the usual way), (4.18) will read:

$$-\frac{i}{h}\int_{t^r}^{-\infty} dt\, F(\nu'_r - \omega_r + i\gamma, t)\cdot e^{i(\nu'_s-\omega_s)t^s}\,. \tag{4.18a}$$

Formula (4.19) will then remain true when one keeps only the sign $-\infty$ for the upper integration limit.

As before, the formula (4.17) for U^{rs} contains the factor (4.18) with the r and s switched. If one now adds the term:

$$\frac{i}{h}\cdot\frac{-e^{i(\nu'_s-\omega_s)t^s}}{i(\nu'_s-\omega_s)}$$

to that factor then one will convince oneself that this addition will make *no contribution* to the time-independent matrix element when the s^{th} particle cannot radiate. Since γ means a quantity that is vanishingly small in comparison to $\nu'_s - \omega_s$, the energy relation (4.19a) will be true, as before, which will once more allow one to express (4.18), when provided with a supplementary term (with s and r switched), as an integral over dt^r from t^r to $-\infty$.

The operator (4.23) that appears in (4.22) will then contain only the limit $-\infty$, and according to the definition (1.10), it will be the retarded potential operator of the r^{th} (radiating) particle [**5**].

5. Derivation of the interaction operator in ordinary space.

The representation of matter by the coordinates of n spatial points q^s is possible only when the number of particles of matter quanta remains preserved. However, as both experiment and theory show, that is not the case (e.g., pair production). One then appeals to the field representation of matter, whose simplest form is the representation by a Schrödinger scalar field $q(\mathbf{x})$. Its quantization was given by PAULI and WEISSKOPF [**6**]. Another possibility (which is the only one that the half-integer spin of matter quanta permits) is the Dirac representation in terms of a four-component field spinor $q^\alpha(\mathbf{x})$ (α = 1, 2, 3, 4). The quantization of the Dirac field brings certain difficulties with it (e.g., the theory of holes). That hole representation does not appear explicitly in method of quantization that was proposed by MAJORANA [**9**].

The Schrödinger equation (3.13) can also be decomposed into a field representation of the form (4.1), except that R and V will no longer be sums of n terms R^s and V^s, but spatial integrals of certain densities:

$$R = \int d\mathbf{x}^3\, \mathfrak{R}\,(\mathbf{x}), \qquad V = \int d\mathbf{x}^3\, \mathfrak{D}(\mathbf{x}, t)\,. \tag{5.1}$$

The argument that led up to formula (4.4) remains valid (*). By contrast, (4.4) must be replaced with the demand that there exists an operator $\mathfrak{A}(\mathbf{x}, \mathbf{y})$ for which one has:

$$\mathfrak{D}(\mathbf{x}, t)\ \psi^1 = \int d\mathbf{y}^3\, \mathfrak{A}\,(\mathbf{x}, \mathbf{y})\ \psi^0. \tag{5.2}$$

Eq. (4.6) then includes a double integral over the operator $\mathfrak{A}(\mathbf{x}, \mathbf{y})$, instead of the double sum over U^{rs}. Corresponding to the function $\psi\,(t^1, \ldots, t^n)$, one introduces a functional $\psi(t(\mathbf{x}))$ that satisfies the equation:

$$\left(\int d\mathbf{x}^3 \delta(\mathbf{x}-\mathbf{y})\, \mathfrak{R}(\mathbf{x}) + \int d\mathbf{x}^3 \delta(\mathbf{x}-\mathbf{y})\, \mathfrak{D}(\mathbf{x}, t(\mathbf{y})) + \frac{h}{i}\frac{\partial}{\partial t(\mathbf{y})} \right) \psi = 0. \tag{5.3}$$

In this, $\delta\,(\mathbf{x})$ shall fill up a finite domain, for now. The time-free eigenfunctions of the zeroth approximation u of equation (5.3) are then the eigenfunctions of the operators that correspond to the energy of the matter that is contained in a volume around the position $\mathbf{y}$ that is defined by the δ-function. $t(\mathbf{y})$ is referred to as the "local time," in analogy to the "particle time." One can now easily perform the rest of the calculations in complete analogy to the ones in the foregoing paragraphs and obtain:

$$U = -\frac{1}{4}\int d\mathbf{x}^3 \left\{ J(\mathbf{x})A(x) + \left(S(\mathbf{x}), \frac{\partial A(x)}{\partial x} \right) + \text{conj.} \right\}_{x_0 = ct}, \tag{5.4}$$

with

$$A(x) = \int_{x_0}^{\pm\infty} dy_0 \int d\mathbf{y}^3\, J_{\text{eff}}\,(y)\, D(x-y), \tag{5.5}$$

instead of (4.22), after the transition to the δ-function. In this, $J(y)$ and $S(y)$ are once more the retarded charge operators of the zeroth approximation (A = 0), which are obtained from the operators $J(\mathbf{y})$ and $S(\mathbf{y})$ by:

$$J(y) = e^{iR(y_0 - ct)/hc}\, J(\mathbf{y})\, e^{-iR(y_0 - ct)/hc}\,. \tag{5.6}$$

The factor 1 / 4 in (5.4) comes from the fact that $\mathbf{x}$ and $\mathbf{y}$ are integrated over all of space, and therefore, each volume element will be counted twice. Naturally, it will be impossible to isolate the self-energy term that will come about in (4.22) by omitting the

(*) Naturally, the Schrödinger function ψ is now a functional of A ($\mathbf{x}$) and q ($\mathbf{x}$).

term $r = s$. Due to the non-commutation of the charge operators with each other, in addition to (5.4), one will obtain terms that are proportional to the squares of the field operators, and which contribute to the self-energy, double absorption and emission, and the Compton effect.

6. Generalization to multi-component fields.

The generalization to multi-component tensor fields can be achieved formally quite simply by giving further tensor indices to A, J, and S_i ; e.g., the four-vector fields A_i , J_i , and S_{ik} .

However, the energy W will then lose its positive-definite character. That can be remedied only by introducing auxiliary conditions.

For the case $l = 0$, the auxiliary condition in matter-free space is known to be [**7**]:

$$\left(\frac{\partial}{\partial x}, A\right)\psi = 0. \tag{6.1}$$

It is compatible with itself and its complex conjugate (i.e., for two events x and y).

For the case $l \neq 0$, it is not possible to find an auxiliary condition. However, if one introduces a scalar field B that satisfies a field equation of type (0.1) with the same l, along with the field A_i, then:

$$\left(\left(\frac{\partial}{\partial x}, A\right) + lB\right)\psi = 0 \tag{6.2}$$

will yield a positive-definite energy, and it will be compatible with itself and its complex conjugate.

In the absence of matter, the auxiliary conditions must be changed in a certain way in order for them to remain preserved in the course of time t.

Maxwell's equations follow in a known way from (6.1). Among other things, they say that a photon possesses a further degree of freedom beyond its impulse: viz., its polarization. That is possible in only two ways (e.g., left and right-circular), so the photon has only two possible orientations for its spin. Certain Maxwell-like equations follow from (6.2) that were discussed by PROCA [**10**]: The field knows three linearly-independent forms of polarization, which corresponds to a particle with three possible orientations to its spin. Naturally, the eigenvalues of the spin are the whole-number multiples h, 0, and $-h$ of h.

In Part Two, the Proca field shall be treated as an application of the method that was proposed here.

As Herr Dr. N. KEMMER has kindly informed me, various papers by him and other authors [**11**] will appear that treat the Proca field and its application to the interaction between neutrons and protons.

The advantage of the present method lies in the fact that it allows one to calculate the retarded interaction terms (and naturally, the non-retarded ones, as well) without going

into the quantum structure of the fields explicitly and without any special assumptions on the representation of matter.

In agreement with Kemmer, it seems possible to me that a complex Proca field (whose particles are "heavy electrons" and "heavy anti-electrons") and a Proca field (whose particles are uncharged) might describe the nuclear forces completely.

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