"Sur les caractéristiques des systèmes de trois dimensions," Bull. Sci. math. et Astron. **4** (1873), 172-192.

On the characteristics of three-dimensional systems ([1])

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(Analysis made by the author)

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In my paper "Sur les caractéristiques des systèmes de trois dimensions," I assumed (following the example of Riemann) that the line element of a system is equal to the square root of a second-degree homogeneous function of the differentials dX_1, dX_2, dX_3 whose coefficients are functions of the variables X_1, X_2, X_3, in such a way that:

$$ds^2 = A_{11}\, dX_1^2 + 2A_{12}\, dX_1\, dX_2 + A_{22}\, dX_2^2 + 2A_{13}\, dX_1\, dX_3 + 2A_{23}\, dX_2\, dX_3 + A_{33}\, dX_3^2 . \tag{1}$$

The form of the functions A_{11}, A_{12}, … will depend upon the choice of coordinates, as well as the properties of the given three-dimensional system. Indeed, if:

$$ds^2 = a_{11}\, dx_1^2 + 2a_{12}\, dx_1\, dx_2 + a_{22}\, dx_2^2 + 2a_{13}\, dx_1\, dx_3 + 2a_{23}\, dx_2\, dx_3 + a_{33}\, dx_3^2 \tag{2}$$

is another form of the line element then in order for the latter to express the line element of the same system, it would be necessary that a transformation of the first form into the second one should be possible. However, upon supposing that x_1, x_2, x_3 are arbitrary functions of the X_1, X_2, X_3, one will see the theoretical possibility of reducing just three of the coefficients A_{11}, A_{12}, … to the form a_{11}, a_{12}, …, while the other three cannot take the other given forms, in general. Consequently, in order for one to be able to transform the form (1) into (2), the coefficients of the latter form must necessarily satisfy certain conditions that are three in number. In order to find those conditions, suppose that x_1, x_2, x_3 are arbitrary functions of X_1, X_2, X_3. Substitute the values of the differentials dx_1, dx_2, dx_3 in (2) and equate the coefficients of the various products of the differentials dX in the form (2) to the corresponding coefficients of the form (1). One will then obtain six equations of the form:

$$\sum_r \sum_s a_{rs} \frac{\partial x_r}{\partial X_m} \frac{\partial x_s}{\partial X_n} = A_{mn} , \tag{3}$$

([1]) F. SUVOROFF, "Sur les caractéristiques des systèmes de trois dimensions," Kazan (1871) (114 pages gr. (?) in octavo).

in which the summation signs $\sum_r$ and $\sum_s$ are extended over the values 1, 2, 3 of the indices r and s. One can eliminate the derivatives of the x with respect to the X from these six equations and their derivatives, and those derivatives will depend upon only some arbitrary relations that couple the coordinates of the two expressions for the differential element with each other. The result of that elimination will provide us with the necessary conditions that the transformation of the form must satisfy, and the verification of those conditions will depend upon the possibility of transforming one of the forms into the other one. If those conditions are expressed by means of the values of certain invariant functions of the coefficients A_{11} , A_{12} , …, and their derivatives then those invariants will serve as characteristics that relate to the three-dimensional system.

The number of resultants of the elimination can be determined with no difficulty. Upon raising equations (3) to second derivatives, one will get:

$$6\,(1 + 3 + 6) = 60$$

equations that contain:

$$3\,(3 + 6 + 10) = 57$$

quantities to be eliminated. As a result, we will have three resulting equations that contain the second derivatives of the coefficients of the forms of the line element. Other than those three necessary conditions for the transformation, there exist some other conditions that refer to the derivatives of order three and higher of the coefficient the form; however, the first three are the simplest ones.

Upon setting:

$$A_{hk}^{\alpha} = \frac{\partial A_{hk}}{\partial X_\alpha}, \qquad A_{hk}^{\alpha\beta} = \frac{\partial^2 A_{hk}}{\partial X_\alpha\,\partial X_\beta}, \qquad \begin{bmatrix} m\,n \\ i \end{bmatrix} = A_{mi}^{n} - A_{mn}^{i} + A_{ni}^{m},$$

$$\Omega = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}, \qquad \Omega_{hk} = \frac{\partial \Omega}{\partial A_{hk}},$$

$$2\,V_{lm} = (A_{lm}^{nn} - A_{mn}^{ln} + A_{nn}^{lm} - A_{ln}^{mn}) - \sum_\sigma \sum_\rho \frac{\Omega_{\rho\sigma}}{2\Omega} \left\{ \begin{bmatrix} l\,n \\ \sigma \end{bmatrix} \begin{bmatrix} m\,n \\ \rho \end{bmatrix} - \begin{bmatrix} n\,n \\ \sigma \end{bmatrix} \begin{bmatrix} m\,l \\ \rho \end{bmatrix} \right\},$$

$$2\,V_{ll} = (2A_{mn}^{mn} - A_{mm}^{nn} - A_{nn}^{mm}) - \sum_\sigma \sum_\rho \frac{\Omega_{\rho\sigma}}{2\Omega} \left\{ \begin{bmatrix} m\,m \\ \sigma \end{bmatrix} \begin{bmatrix} n\,n \\ \rho \end{bmatrix} - \begin{bmatrix} m\,n \\ \sigma \end{bmatrix} \begin{bmatrix} m\,n \\ \rho \end{bmatrix} \right\},$$

in which l, m, n are three of the indices 1, 2, 3 that are all different from each other, and the summations signs extend over the values 1, 2, 3 of the indices ρ and σ, and upon letting the lowercase letters ω and v denote the similar expressions that are composed from the coefficients a_{11}, a_{12}, … of the second form, the three resultants of the elimination in question will then be presented as:

(I) $$\frac{1}{\Omega}\sum_{p}\sum_{p'} A_{pp'} V_{pp'} = \frac{1}{\omega}\sum_{p}\sum_{p'} a_{pp'} v_{pp'} ,$$

in which the summation signs extend over the values 1, 2, 3 of the indices p, p'.

(II) $$\frac{1}{\Omega}\sum_{p}\sum_{p'} A_{pp'} V_{pp'} = \frac{1}{\omega}\sum_{p}\sum_{p'} a_{pp'} v_{pp'} ,$$

in which q, q', r, r' are the remainders when one divides $p + 1$, $p' + 1$, $p + 2$, $p' + 2$, resp., by 3, and the summation signs extend over the same values of p, p':

(III) $$\frac{1}{\Omega^2}\sum_{p,q,r} \pm V_{1p} V_{2q} V_{3r} = \frac{1}{\omega^2}\sum_{p,q,r} \pm v_{1p} v_{2q} v_{3r} ,$$

in which the sign $\sum_{p,q,r}\pm$ denotes the determinant of the elements under the summation sign. Those three functions, which Christoffel had obtained already before me in a general form (see Borchardt's Journal, v. **70**), do not contain the derivatives of the x with respect to the X. As a result, they will not depend upon the choice of the coordinates, and as a consequence, they will express the necessary conditions for the transformation. Moreover, the form of those conditions is remarkable insofar as they express the identical equality of functions that are similar to the coefficients of the line element. It results from this that in order for a transformation to take one form of the line element to another one, it is necessary that the three functions in question must preserve their values under the passage from one of the systems of variables to the other one; i.e., that is must be impossible to transform the given form into another one for which any of the three functions has a value that is different from the one that it had in the first form.

Those functions, which relate to the three-dimensional system, must imply the properties that characterize the system essentially, due to the independence that exists between their values and the choice of coordinates; i.e., the ones that distinguish that system from other three-dimensional systems for which the functions have another value. Hence, for example, each of those three functions must reduce to zero for the space whose line element is expressed by the equation:

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 .$$

Hence, the space is a system that is different from all of the other systems for which any of those functions is not annulled, but it is equal to an arbitrary constant magnitude or to a function of the coordinates of the point. The form of the line element of such a system cannot be reduced to the sum of the squares of coordinate differentials for any arbitrary choice of coordinates; i.e., in geometric language, for surfaces, there exists only one function of this nature ([1]) that does not change in value under the coordinate

([1]) CASORATI: "Ricerca fondamentale per lo studio di una certa classe di proprietà delle superficie curva," Annali di Matematica **3** (1860).

transformations, and from the geometric viewpoint, it represents the curvature of the surface according to Gauss. As far as the three functions that were found above is concerned, the theory of three-dimensional systems other than the present space is much too recent for me to decide how to give a special name to the properties of the systems that are expressed by those functions. However, just as several other geometers (Riemann, Kronecker, Beltrami) have applied geometric terminology to these properties, and in various ways, in Chapter III of my paper, I have sought to untangle the geometric laws of the analytical expressions that are found. To that end, in order to better understand the analogy between formulas (I), (II), and (III) and the formulas of ordinary geometry (which are analogies that one discovers by considering a three-dimensional system to be a locus in a four-dimensional system), I have sacrificed generality for the advantage of having a more intuitive picture of things, and instead of a four-dimensional system of the most general form, I took a system whose line element is expressed by the sum of the squares of the differentials of the four coordinates, namely:

$$ds^2 = dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2. \tag{4}$$

I shall denote the coordinates of a four-dimensional system by the letter y and I shall reserve the letter x for the coordinates of a three-dimensional system. Upon considering the latter to be a locus in the former, the coordinates x will play the role of curvilinear coordinates on a surface that is placed in space. As a result, similar to the coordinates of space, the coordinates y of the points of a three-dimensional system will be functions of the coordinates x, and the line element of a three-dimensional system that is a locus in a four-dimensional space, will be expressed by formula (4).

Upon replacing the differentials dy in (4) with their values as functions of the differentials dx, and then equating the coefficients of the same differentials dx in the forms (1) and (4), one will get the coefficients A, and therefore, the functions V and Ω, as well, expressed in terms of the derivatives of y with respect to x. Since a three-dimensional system, when it is considered to be a locus in a four-dimensional space, can be represented by an arbitrary function of the coordinates y_1, y_2, y_3, y_4 :

$$W = F(y_1, y_2, y_3, y_4) = 0,$$

we can regard one of the coordinates – y_4, for example – as a function of the other three. Upon expressing A, V, Ω in terms of the derivatives of y_4 with respect to the y_1, y_2, y_3, and setting:

$$Q = \sqrt{1 + \left(\frac{\partial y_4}{\partial y_1}\right)^2 + \left(\frac{\partial y_4}{\partial y_2}\right)^2 + \left(\frac{\partial y_4}{\partial y_3}\right)^2},$$

$$Y = \begin{vmatrix} \frac{\partial^2 y_4}{\partial y_1^2} & \frac{\partial^2 y_4}{\partial y_1\,\partial y_2} & \frac{\partial^2 y_4}{\partial y_1\,\partial y_3} \\ \frac{\partial^2 y_4}{\partial y_2\,\partial y_1} & \frac{\partial^2 y_4}{\partial y_2^2} & \frac{\partial^2 y_4}{\partial y_2\,\partial y_3} \\ \frac{\partial^2 y_4}{\partial y_3\,\partial y_1} & \frac{\partial^2 y_4}{\partial y_3\,\partial y_2} & \frac{\partial^2 y_4}{\partial y_3^2} \end{vmatrix},$$

to abbreviate, and letting Y_{pq} denote the second-order determinant that is obtained by suppressing the p^{th} row and the q^{th} column from Y, one will find expressions for the function (I), (II), (III) in terms of the derivatives of y_4 :

(I*a*) $$\frac{1}{\Omega}\sum_p\sum_{p'} A_{pp'}V_{pp'} = \frac{1}{Q^4}\left(Y_{11}+Y_{22}+Y_{33}+\sum_p\sum_{p'}\frac{\partial y_4}{\partial y_p}\cdot\frac{\partial y_4}{\partial y_{p'}}Y_{pp'}\right),$$

(II*a*) $$\begin{cases} \dfrac{1}{\Omega^2}\sum_p\sum_{p'}\Omega_{pp'}(V_{qq'}V_{rr'}-V_{qr'}V_{rq'}) \\ = \dfrac{Y}{Q^2}\left\{\dfrac{\partial^2 y_4}{\partial y_1^2}\left[1+\left(\dfrac{\partial y_4}{\partial y_2}\right)^2+\left(\dfrac{\partial y_4}{\partial y_3}\right)^2\right]+\dfrac{\partial^2 y_4}{\partial y_2^2}\left[1+\left(\dfrac{\partial y_4}{\partial y_1}\right)^2+\left(\dfrac{\partial y_4}{\partial y_3}\right)^2\right]\right. \\ +\dfrac{\partial^2 y_4}{\partial y_3^2}\left[1+\left(\dfrac{\partial y_4}{\partial y_1}\right)^2+\left(\dfrac{\partial y_4}{\partial y_2}\right)^2\right]-2\dfrac{\partial^2 y_4}{\partial y_1\,\partial y_2}\dfrac{\partial y_4}{\partial y_1}\dfrac{\partial y_4}{\partial y_2} \\ \left.-2\dfrac{\partial^2 y_4}{\partial y_1\,\partial y_3}\dfrac{\partial y_4}{\partial y_1}\dfrac{\partial y_4}{\partial y_3}-2\dfrac{\partial^2 y_4}{\partial y_2\,\partial y_3}\dfrac{\partial y_4}{\partial y_2}\dfrac{\partial y_4}{\partial y_3}\right\}, \end{cases}$$

(III*a*) $$\frac{1}{\Omega^2}\sum_{p,q,r}\pm V_{1p}V_{2q}V_{3r} = \frac{Y^2}{Q^{10}}.$$

Upon taking the square root of the last expression, so:

(III*b*) $$\frac{1}{\Omega}\sqrt{\sum_{p,q,r}\pm V_{1p}V_{2q}V_{3r}} = \frac{Y}{Q^5},$$

one will easily see that the right-hand side is somewhat analogous to Gauss's formula for the curvature of a surface, and that it will coincide with that formula if we extend to three dimensions, just as it will coincide with the formula that Kronecker ([1]) called the *curvature* of a system of an arbitrary number of dimensions. Indeed, if one expresses the

([1]) Monatsberichte d. König. Akad. zu Berlin, August 1869.

derivatives of the y_4 in terms of partial derivatives of W with respect to the coordinates y_1, y_2, y_3, y_4 then when one substitutes those values into formula (III*b*), one will get:

$$\frac{1}{(W_1^2+W_2^2+W_3^2+W_4^2)^{1/2}}\begin{vmatrix} 0 & W_1 & W_2 & W_3 & W_4 \\ W_1 & W_{11} & W_{12} & W_{13} & W_{14} \\ W_2 & W_{21} & W_{22} & W_{23} & W_{24} \\ W_3 & W_{31} & W_{32} & W_{33} & W_{34} \\ W_4 & W_{41} & W_{42} & W_{43} & W_{44} \end{vmatrix},$$

in which one sets:

$$W_p = \frac{\partial W}{\partial y_p}, \qquad W_{rs} = \frac{\partial^2 W}{\partial y_r\, \partial y_s},$$

to abbreviate.

However, I do not think that the term *curvature of the system* can be properly applied to that function. Indeed, following Gauss, we are accustomed to use the term *curvature of a system* to describe the ratio of the area of an infinitely-small triangle in the given system to the corresponding area in a system of positive constant curvature (viz., a spherical system). That is, at least, what Riemann and Beltrami meant by the curvature of the system, and it is only in that sense that the non-Euclidian or pseudo-spherical systems of the latter geometry will have a constant negative curvature, and the spherical systems will have positive curvature. Effectively, the line element of a spherical system can be put into the form:

$$(p) \qquad ds^2 = R^2\left(\frac{dx_1^2+dx_2^2+dx_3^2+dx^2}{x^2}\right),$$

in which $x^2 = a^2 + x_1^2 + x_2^2 + x_3^2$, x_1, x_2, x_3 are the coordinates of the system, while R and a are parameters. The value of the functions (III*b*) for that form of the line element will be $\pm\, 1 / R^3$, and not $+ 1 / R^2$, which is the value that the curvature of the system must have, according to Riemann. For systems with constant negative curvature, the function (III*b*) will give an imaginary value $\sqrt{-1/R^6}$. It seems to me that the second term of the *condensation of the system* that Kronecker gave to that function at the end of the cited paper upon comparing it to the Kummer functions that represent the condensation of light rays had a simpler geometric explanation. Indeed, compare the two three-dimensional systems: viz., the given system:

$$W = F(y_1, y_2, y_3, y_4) = 0$$

and a spherical system; i.e., one with constant positive curvature. We will find the equation of the latter system when we know the expression (p) for its line element. Attribute any values to y_1, y_2, y_3, y_4 and set:

$$R\frac{a}{x} = y_4, \qquad R\frac{x_1}{x} = y_1, \qquad R\frac{x_2}{x} = y_2, \qquad R\frac{x_3}{x} = y_3.$$

Upon substituting these values in (p), one will get:

$$ds^2 = dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2,$$

$$R^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2.$$

As a result, the equation for that locus will be that of a three-dimensional spherical system. I will use the symbols y_1', y_2', y_3', y_4' in order to distinguish the coordinates of the points of the system $W = 0$ from the coordinates of the system:

$$V = y_1^2 + y_2^2 + y_3^2 + y_4^2 - 1 = 0,$$

in which I have set $R = 1$, for more simplicity. For each point (y_1, y_2, y_3, y_4) of the system $W = 0$, one can find a point (y_1', y_2', y_3', y_4') of the system $V = 0$ for which one will have:

$$\frac{\partial y_4'}{\partial y_1'} = \frac{\partial y_4}{\partial y_1}, \qquad \frac{\partial y_4'}{\partial y_2'} = \frac{\partial y_4}{\partial y_2}, \qquad \frac{\partial y_4'}{\partial y_3'} = \frac{\partial y_4}{\partial y_2}.$$

For this to be true, it is sufficient to suppose that the coordinates y_1', y_2', y_3', y_4' are equal to the derivatives:

$$\frac{\partial W}{\partial y_1}, \quad \frac{\partial W}{\partial y_2}, \quad \frac{\partial W}{\partial y_3}, \quad \frac{\partial W}{\partial y_4},$$

respectively, divided by:

$$\sqrt{\left(\frac{\partial W}{\partial y_1}\right)^2 + \left(\frac{\partial W}{\partial y_2}\right)^2 + \left(\frac{\partial W}{\partial y_3}\right)^2 + \left(\frac{\partial W}{\partial y_4}\right)^2}.$$

I will call such points *corresponding points.* Take four points in the system $W = 0$ that are determined by the coordinates:

$$\begin{array}{llllll} y_1, & y_2, & y_3; & y_1 + dy_1, & y_2 + dy_2, & y_3 + dy_3; \\ y_1 + \partial y_1, & y_2 + \partial y_2, & y_3 + \partial y_3; & y_1 + \delta y_1, & y_2 + \delta y_2, & y_3 + \delta y_3, \end{array}$$

in which the fourth coordinate is determined as a function of the first three. One determines the coordinates of the four corresponding points in the system $V = 0$ from the agreed-upon rule, and they will be functions of the coordinates of the point of the system $W = 0$. Denote them by:

$$\begin{array}{llllll} y_1', & y_2', & y_3'; & y_1' + dy_1', & y_2' + dy_2', & y_3' + dy_3'; \\ y_1' + \partial y_1', & y_2' + \partial y_2', & y_3' + \partial y_3'; & y_1' + \delta y_1', & y_2' + \delta y_2', & y_3' + \delta y_3'. \end{array}$$

Upon considering d, ∂, d to be symbols for infinitely-small increments and confining the approximation to the first-order terms, the ratio:

$$\begin{vmatrix} dy_1 & dy_2 & dy_3 \\ \partial y_1 & \partial y_2 & \partial y_3 \\ \delta y_1 & \delta y_2 & \delta y_3 \end{vmatrix} : \begin{vmatrix} dy'_1 & dy'_2 & dy'_3 \\ \partial y'_1 & \partial y'_2 & \partial y'_3 \\ \delta y'_1 & \delta y'_2 & \delta y'_3 \end{vmatrix} = c$$

will represent the ratio of the volumes of the loci, taken in the systems $W = 0$, $V = 0$ and corresponding to the tetrahedra in space. However, since each point of the first system corresponds to a well-defined point of the second one, each volume of the first system will correspond to a volume in the second system, and the latter volume will be like a transformation of the first one, since each well-defined point inside the volume in the first system will correspond to a well-defined point inside the volume of the second system. The ratio c that was found above is the ratio of two of those corresponding infinitely-small volumes. The value of that ratio will give a measure of the difference between the volume in the first system and the volume in the second system from the viewpoint of a well-defined property (the curvature, according to Kronecker), or if the volumes are infinitely small, a measure of the difference between each point of the first system relative to the corresponding point of the second system from the viewpoint of that property. Since the function c is a function of the derivatives of $W = 0$, it will be different for different systems, and as a result, it can serve as a characteristic for a system. However, upon expressing the differentials of the coordinates y' as functions of the differentials of the coordinates y and substituting their values in the function c, after reduction, one will find the expression for that ratio as a function of the second derivatives of y_4, namely:

$$c = \frac{Y}{Q^5};$$

i.e., that ratio is expressed by the value of the function (III*b*). It will then follow from this that the function (III*b*) does not represent the ratio of the areas of infinitely-small triangles (which we call the *curvature* of the system, in accord with Gauss and Riemann), but the ratio of the volumes of infinitely-small tetrahedra. For that reason, I believe that the second term that Kronecker gave to the property that is expressed by that function – namely, referring to it as a measure of the *condensation* of the system – is the most convenient one.

As far as the interpretation of formulas (I) and (II) are concerned, upon replacing the function (II*a*) by the function:

$$\text{(II}b\text{)} \qquad \frac{\sum_p \sum_{p'} \Omega_{pp'} (V_{qq'} V_{rr'} - V_{qr'} V_{rq'})}{\Omega \sqrt{\sum_{p,q,r} \pm V_{1p} V_{2q} V_{2r}}},$$

it is easy to see that the expressions for the functions (I*a*), (II*b*), and (III*b*) in terms of the derivatives of y_4 represent the coefficients of the cubic equation $1 / R$ that are obtained by eliminating $y_1 - y_1'$, $y_2 - y_2'$, $y_3 - y_3'$, dy_1, dy_2, dy_3 from the equation:

$$(q) \qquad (y_1 - y_1')^2 + (y_2 - y_2')^2 + (y_3 - y_3')^2 + (y_4 - y_4')^2 = R^2,$$

its three partial derivatives with respect to y_1, y_2, y_3 (upon considering y_4 to be a function of the other coordinates), and the three total differentials of those partial derivatives. Consequently, upon letting $1 / R_1$, $1 / R_2$, $1 / R_3$, be the roots of that equation, we will have:

$$(\text{III}c) \qquad \frac{1}{\Omega}\sqrt{\sum_{p,q,r} \pm V_{1p}\,V_{2q}\,V_{3r}} = -\frac{1}{R_1\,R_2\,R_3},$$

$$(\text{I}c) \qquad \frac{1}{\Omega}\sum_{p}\sum_{p'} A_{pp'}\,V_{pp'} = \frac{1}{R_2\,R_3} + \frac{1}{R_3\,R_1} + \frac{1}{R_1\,R_2},$$

$$(\text{II}c) \qquad \frac{\sum_{p}\sum_{p'} \Omega_{pp'}(V_{qq'}\,V_{rr'} - V_{qr'}\,V_{rq'})}{\Omega\sqrt{\sum_{p,q,r} \pm V_{1p}\,V_{2q}\,V_{2r}}} = -\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right).$$

Since equation (q) represents a system of constant positive curvature, the three roots $1/R_1$, $1 / R_2$, $1 / R_3$ will be inversely proportional to the radii of curvature of the three directions that are determined by the two equations between the differential dy_1, dy_2, dy_3 that one can obtain by eliminating the difference $y_4 - y_4'$ between the differentials of the partial derivatives of equation (p). Since the square root of the function (III) can have an imaginary value, which will be true for a system of constant negative curvature, in order to avoid that inconvenience, I shall multiply the functions (II*c*) and (III*c*) by (III*c*); upon then setting:

$$\alpha = \frac{1}{R_2\,R_3}, \qquad \beta = \frac{1}{R_3\,R_1}, \qquad \gamma = \frac{1}{R_1\,R_3},$$

one can write:

$$\frac{1}{\Omega}\sum_{p}\sum_{p'} A_{pp'}\,V_{pp'} = \alpha + \beta + \gamma,$$

$$\frac{1}{\Omega^2}\sum_{p}\sum_{p'} \Omega_{pp'}(V_{qq'}\,V_{rr'} - V_{qr'}\,V_{rq'}) = \beta\gamma + \gamma\alpha + \alpha\beta,$$

$$\frac{1}{\Omega^3}\sum_{p,q,r} \pm V_{1p}\,V_{2q}\,V_{3r} = \alpha\beta\gamma.$$

The parameters α, β, γ represent the (Gaussian) surface curvatures along the three principal sections of the three-dimensional system along which the curvature presents the property of being a maximum or minimum.

Indeed, Lipschitz has proved ([1]) that the general form of the line element of a three-dimensional system can be converted into the form that Riemann gave for a system of constant curvature when the necessary and sufficient condition:

$$(r) \qquad \frac{\sum_p \sum_{p'} V_{pp'}\,(\delta^1 x_q\, d^1 x_r - \delta^1 x_r\, d^1 x_q)(\delta x_{q'}\, dx_{r'} - \delta x_{r'}\, dx_{q'})}{\sum_p \sum_{p'} \Omega_{pp'}\,(\delta^1 x_q\, d^1 x_r - \delta^1 x_r\, d^1 x_q)(\delta x_{q'}\, dx_{r'} - \delta x_{r'}\, dx_{q'})} = \alpha$$

is satisfied, in which α is the measure of the constant curvature.

If one supposes that α is variable then formula (r) will give the expression for the variable curvature as a function of the differentials of the surface directions. The cubic equation in α that is provided by the discriminant of equation (r) will have the three values of α for its roots as functions of the coefficients of the form of the line element that represent the maximum and minimum values of α.

It will then follow that in order to specify a three-dimensional system, one must necessarily know the three roots of the equation in α, or the three coefficients of that equation, and none of those coefficients can remain arbitrary. That shows that among the three-dimensional systems, there cannot exist two of them that can be mapped to each other.

For a system with constant curvature, all three of the roots of the equation in α must be constant and equal. That condition will be fulfilled when one supposes that the functions V_{lm} are proportional to the Ω_{lm}, and that α is the factor of proportionality, which is, at the same time, the measure of the curvature of the system. One will then obtain six conditions for the determination of the coefficients of the form of the line element. Those conditions will take a very simple form when one chooses x_1, x_2, x_3 to be orthogonal coordinates. The line element of a three-dimensional system in the case of rectangular coordinates axes is presented in the form:

$$ds^2 = B_1^2\, dx_1^2 + B_2^2\, dx_2^2 + B_3^2\, dx_3^2 ,$$

and the condition for the system to have constant curvature will become:

$$\frac{\partial}{\partial x_3}\left(\frac{1}{B_3}\cdot\frac{\partial B_2}{\partial x_3}\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{B_2}\cdot\frac{\partial B_3}{\partial x_2}\right) + \frac{1}{B_1^2}\cdot\frac{\partial B_2}{\partial x_1}\cdot\frac{\partial B_3}{\partial x_1} = -\,\alpha\, B_2\, B_3 ,$$

$$\frac{\partial}{\partial x_1}\left(\frac{1}{B_1}\cdot\frac{\partial B_2}{\partial x_1}\right) + \frac{\partial}{\partial x_3}\left(\frac{1}{B_3}\cdot\frac{\partial B_1}{\partial x_3}\right) + \frac{1}{B_2^2}\cdot\frac{\partial B_3}{\partial x_2}\cdot\frac{\partial B_1}{\partial x_2} = -\,\alpha\, B_3\, B_1 ,$$

([1]) Borchardt's Journal, v. **72**, pp. 52.

$$\frac{\partial}{\partial x_2}\left(\frac{1}{B_2}\cdot\frac{\partial B_1}{\partial x_1}\right)+\frac{\partial}{\partial x_1}\left(\frac{1}{B_1}\cdot\frac{\partial B_2}{\partial x_1}\right)+\frac{1}{B_3^2}\cdot\frac{\partial B_1}{\partial x_3}\cdot\frac{\partial B_2}{\partial x_3} = -\,\alpha\,B_1\,B_2\,,$$

$$\frac{\partial^2 B_1}{\partial x_2\,\partial x_3}-\frac{1}{B_2}\cdot\frac{\partial B_2}{\partial x_3}\cdot\frac{\partial B_1}{\partial x_2}-\frac{1}{B_3}\cdot\frac{\partial B_3}{\partial x_2}\cdot\frac{\partial B_1}{\partial x_3}=0,$$

$$\frac{\partial^2 B_2}{\partial x_3\,\partial x_1}-\frac{1}{B_3}\cdot\frac{\partial B_3}{\partial x_1}\cdot\frac{\partial B_2}{\partial x_3}-\frac{1}{B_1}\cdot\frac{\partial B_1}{\partial x_3}\cdot\frac{\partial B_2}{\partial x_1}=0,$$

$$\frac{\partial^2 B_3}{\partial x_1\,\partial x_2}-\frac{1}{B_1}\cdot\frac{\partial B_1}{\partial x_2}\cdot\frac{\partial B_3}{\partial x_1}-\frac{1}{B_2}\cdot\frac{\partial B_2}{\partial x_1}\cdot\frac{\partial B_2}{\partial x_2}=0.$$

If one takes $\alpha = 0$, i.e., if one suppose that the curvature is zero then one will get the conditions for a given form of the line element to represent the line element of ordinary space. The latter conditions have been given already in the case of $\alpha = 0$ by Lamé ([1]), like the conditions for the transformation of a given form of the line element into the sum of the squares of three differentials.

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([1]) *Leçons sur les coordonnées curvilignes*, pp. 76.