"Über die formale Analogie zwischen den elektromagnetischen Grundgleichungen und den Einsteinschen Gravitationsgleichungen erster Näherung," Phys. Zeit. **19** (1918), 204-205.

On the formal analogy between the basic electromagnetic equations and Einstein's gravitational equations in the first approximation

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In what follows, some formal developments shall be pursued somewhat further that found their place in a prior paper ([1]) merely in a footnote. One is dealing with the analogy between the **Maxwell-Lorentz** equations, on the one hand, and the equations that determine the motion of a point in a weak gravitational field in the first approximation, on the other. **Einstein** himself had already spoken of that analogy occasionally in the Wiener Naturforschertag in 1913 ([2]). However, since his field equations have experienced an essential modification since then, it would not seem inappropriate to develop the formulas in question for the ultimate conception of the theory.

Let us remark from the outset that in what follows we will always employ a system of measurement in which the speed of light is equal to 1 and that we shall choose:

$$x_1 = x, \qquad x_2 = y, \qquad x_3 = z, \qquad x_4 = it$$

as coordinates.

We now consider an entirely special case of point motion in a quasi-stationary gravitational field. A mass-point moves so slowly in the field that the squares and products of its velocity components can be neglected in comparison to 1. Let the gravitational field itself be weak [such that the deviations of the $g_{\mu\nu}$ from the classical values – 1 (0, resp.) can be considered to be small of first order], and let it be generated by incoherent (i.e., stress-free) masses whose velocities are somewhat larger than those of the mass-point being examined such that we must consider the squares and binary products. If we denote the velocity of the mass-point by $\mathfrak{v}$ and that of the mass that generates the field by $\mathfrak{v}'$ then the calculations will include expressions with the orders of magnitude of: $\mathfrak{v}$, $\mathfrak{v}'$, $\mathfrak{v}'^2$, and $\mathfrak{v} \cdot \mathfrak{v}'$.

As is known, the equations of motion read:

$$\frac{d^2 x_\tau}{ds^2} = \Gamma_{\mu\nu}^{\tau} \frac{dx_\mu}{ds} \frac{dx_\nu}{ds}, \quad \tau = 1, \ldots, 4. \tag{1}$$

([1]) **H. Thirring**, this Zeit. **19** (1918), 33.

([2]) **A. Einstein**, this Zeit. **14** (1913), 1261.

The components of the velocity of the mass-point are on the right-hand side; if we neglect their squares and products, from our assumptions, then the equations will go to:

$$\frac{d^2 x_\tau}{dt^2} = 2i\left(\Gamma^\tau_{14}\frac{dx_1}{dt} + \Gamma^\tau_{24}\frac{dx_2}{dt} + \Gamma^\tau_{34}\frac{dx_3}{dt}\right) - \Gamma^\tau_{44} \qquad \tau = 1, \ldots, 4. \tag{2}$$

In what follows, we shall consider the spatial components of the equations of motion; then let $\tau = 1, 2, 3$. For weak fields, the three-index symbols are:

$$\Gamma^\tau_{\sigma 4} = -\begin{Bmatrix}\sigma\,4\\ \tau\end{Bmatrix} = \begin{bmatrix}\sigma\,4\\ \tau\end{bmatrix} = \frac{1}{2}\left(\frac{\partial g_{\sigma\tau}}{\partial x_4} + \frac{\partial g_{\tau 4}}{\partial x_\sigma} - \frac{\partial g_{\sigma 4}}{\partial x_\tau}\right). \tag{3}$$

The derivatives $\partial g_{\sigma\tau} / \partial x_4$ ($\sigma \neq 4$, $\tau \neq 4$) have order of magnitude $\mathfrak{v}'^2$, as we shall show shortly; they are multiplied by dx_σ / dt (order of magnitude $\mathfrak{v}$) in (2) and thus drop out in our approximation. Only the derivatives of $g_{\tau 4}$ and g_{44} enter into Γ^τ_{44}; for what follows, we will then need only those coefficients $g_{\mu\nu}$ that include the index 4 at least once. In order to calculate them, we employ **Einstein**'s approximate solution ([1]):

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu}, \qquad \delta_{\mu\nu} = \begin{cases} 1 & \mu = \nu, \\ 0 & \mu \neq \nu, \end{cases}$$

$$\gamma_{\mu\nu} = \gamma'_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu}\sum_\alpha \gamma'_{\alpha\alpha}, \qquad \gamma'_{\mu\nu} = -\frac{\kappa}{2\pi}\int \frac{T_{\mu\nu}(x', y', z', t' - R)}{R}\,dV_0, \tag{4}$$

in which $T_{\mu\nu}$ refers to the energy tensor, x', y', z', to the coordinates of the integration space, R, to the distance from the integration element to the reference point, and dV_0 is the naturally-measured volume element. The energy tensor for incoherent matter is given by:

$$T_{\mu\nu} = T^{\mu\nu} = \rho_0 \frac{dx_\mu}{ds}\frac{dx_\nu}{ds}. \tag{5}$$

The four coefficients $g_{\mu 4}$ that come under consideration will then read:

([1]) **A. Einstein**, Berl. Ber. (1916), pp. 688.

$$\left.\begin{aligned}
g_{14} &= -i\frac{\kappa}{2\pi}\int\frac{\rho_0\,\mathfrak{v}'_x}{R}\left(\frac{dt'}{ds}\right)^2 dV_0,\\
g_{24} &= -i\frac{\kappa}{2\pi}\int\frac{\rho_0\,\mathfrak{v}'_y}{R}\left(\frac{dt'}{ds}\right)^2 dV_0,\\
g_{34} &= -i\frac{\kappa}{2\pi}\int\frac{\rho_0\,\mathfrak{v}'_z}{R}\left(\frac{dt'}{ds}\right)^2 dV_0,\\
g_{44} &= -1+\frac{\kappa}{4\pi}\int\frac{\rho_0}{R}\left(\frac{dt'}{ds}\right)^2 dV_0.
\end{aligned}\right\}\tag{6}$$

The field components that appear in eq. (2) can now be calculated from the $g_{\mu\nu}$ as follows, while neglecting the terms that were discussed above:

$$\left.\begin{array}{llll}
\Gamma^1_{14}=0 & \Gamma^1_{24}=\frac{1}{2}\left(\frac{\partial g_{14}}{\partial x_2}-\frac{\partial g_{24}}{\partial x_1}\right) & \Gamma^1_{34}=\frac{1}{2}\left(\frac{\partial g_{14}}{\partial x_3}-\frac{\partial g_{34}}{\partial x_1}\right) & \Gamma^1_{44}=-\frac{1}{2}\frac{\partial g_{44}}{\partial x_1}\\
\Gamma^2_{14}=\frac{1}{2}\left(\frac{\partial g_{24}}{\partial x_1}-\frac{\partial g_{14}}{\partial x_2}\right) & \Gamma^2_{24}=0 & \Gamma^2_{34}=\frac{1}{2}\left(\frac{\partial g_{24}}{\partial x_3}-\frac{\partial g_{34}}{\partial x_2}\right) & \Gamma^2_{44}=-\frac{1}{2}\frac{\partial g_{44}}{\partial x_2}\\
\Gamma^3_{14}=\frac{1}{2}\left(\frac{\partial g_{34}}{\partial x_1}-\frac{\partial g_{14}}{\partial x_3}\right) & \Gamma^3_{24}=\frac{1}{2}\left(\frac{\partial g_{34}}{\partial x_2}-\frac{\partial g_{24}}{\partial x_3}\right) & \Gamma^3_{34}=0 & \Gamma^3_{44}=-\frac{1}{2}\frac{\partial g_{44}}{\partial x_3}\\
\Gamma^4_{14}=\frac{1}{2}\frac{\partial g_{44}}{\partial x_1} & \Gamma^4_{34}=\frac{1}{2}\frac{\partial g_{44}}{\partial x_2} & \Gamma^4_{34}=\frac{1}{2}\frac{\partial g_{44}}{\partial x_3} & \Gamma^4_{44}=0
\end{array}\right\}\tag{7}$$

Equations (2), (6), and (7) correspond completely to the basic electrodynamical equations, up to a numerical factor. In order to let this similarity emerge better, we set:

$$\left.\begin{array}{c}
\mathfrak{A}_x = i\,g_{14},\quad \mathfrak{A}_y = i\,g_{24},\quad \mathfrak{A}_z = i\,g_{34},\quad \Phi=\frac{g_{44}+1}{2},\\
\mathfrak{H}_x = 2i\,\Gamma^3_{24} = -2i\,\Gamma^2_{34},\quad \mathfrak{H}_y = 2i\,\Gamma^1_{34} = -2i\,\Gamma^3_{14},\quad \mathfrak{H}_z = 2i\,\Gamma^2_{14} = -2i\,\Gamma^1_{24},\\
\mathfrak{E}_x = \Gamma^1_{41} = -\Gamma^4_{14},\qquad \mathfrak{E}_y = \Gamma^2_{44} = -\Gamma^4_{24},\qquad \mathfrak{E}_z = \Gamma^3_{44} = -\Gamma^4_{34},\\
k=\frac{\kappa}{8\pi}.
\end{array}\right\}\tag{8}$$

With these notations, equations (6), (7), and (2) go to:

$$\mathfrak{A} = 4k\int\frac{\rho_0\,\mathfrak{v}'}{R}\left(\frac{dt'}{ds}\right)^2 dV_0\,,\qquad \Phi = k\int\frac{\rho_0}{R}\left(\frac{dt'}{ds}\right)^2 dV_0\,,\tag{6a}$$

$$\mathfrak{H} = \mathrm{rot}\,\mathfrak{A},\qquad \mathfrak{E} = -\,\mathrm{grad}\,\Phi - \frac{\partial\mathfrak{A}}{\partial t},\tag{7a}$$

$$\ddot{\mathfrak{s}} = -\mathfrak{E} - [\mathfrak{v}\,\mathfrak{H}]. \tag{2a}$$

Except for the factor $\left(\frac{dt'}{ds}\right)^2$, which deviates from unity only by quantities of order $\mathfrak{v}'^2$, equations (6a), (7a), and (2a) differ from the corresponding electrodynamical ones only by the inverted sign on the right-hand side of (2a) and the appearance of the factor 4 in (6a). The analogue of the magnetic field in the theory of gravitation is therefore four times as large as it is in electrodynamics.

Let us add a remark of a fundamental nature to this formal analogy: It seems very unlikely from the outset that the same mathematical laws that represent approximate formulas for certain special cases in one domain of phenomena should describe a different domain of phenomena *exactly*. For that reason, one suspects that (apart from physical necessity, on formal grounds) the **Maxwell-Lorentz** equations are likewise only approximate formulas that are indeed sufficiently precise for the fields that can be produced in engineering practice, but for much stronger fields, such as the ones that would appear at atomic and electronic dimensions, one would require a corresponding generalization, which **Hilbert** and **Mie** have already made of first draft of (but while starting from a much different viewpoint).

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