"Über die Wirkung rotierender ferner Massen in der Einsteinschen Gravitationstheorie," Phys. Zeit. **19** (1918), 33-39.

On the effect of rotating distant masses in Einstein's theory of gravitation

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The arguments that have the present paper to thank for their existence can be best clarified by a quote from **Einstein**'s groundbreaking work in the year 1914 ([1]). In its introduction, he said the following:

"At first, it would generally seem that such an extension of the theory of relativity should be rejected on physical grounds. Namely, let K be a coordinate system, in the **Galilei-Newton** sense of the term, and let K' be a coordinate system that rotates uniformly relative to K. Centrifugal forces will act upon masses that are at rest relative to K', while no such forces will act upon masses that are at rest relative to K. **Newton** already saw a proof in that of the fact that one had to regard the rotation of K' as an 'absolute,' so one could not treat K' as being 'at rest' with the same justification that one had with K. However, that argument is not cogent, as **E. Mach** showed, in particular. Namely, we do not necessarily need to attribute the existence of those centrifugal forces to a motion of K'. If **Newton**'s laws of mechanics do not admit such a concept then that would quite probably be rooted in some flaw in the theory…"

Now, since **Einstein**'s theory seems to have been completed in the publications of 1915, one must ask: Do the equations of the new theory (to the extent that it is free of the flaws in **Newton**'s theory) actually say that the rotation of distant masses will generate a gravitational field that is equivalent to a "centrifugal force?" One might attempt to table a discussion of that question by saying that the required equivalence is guaranteed by the general covariance of the field equations. However, things are not quite so simple, since the boundary conditions for the $g_{\mu\nu}$ at spatial infinity also play a role. The questions that are related to that were treated mainly in the papers of **De Sitter** ([2]) and **Einstein** ([3]). Now, we shall not go further into these general question in what follows; rather, we would like to carry out the calculations for a special concrete example of the field of

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([1]) **A. Einstein**, Berl. Ber. (1914), pp. 1030; cf., also Ann. Phys. (Leipzig) **49** (1916), pp. 769.
([2]) **De Sitter**, Amsterdam Proc. **19** (1917), pp. 527.
([3]) **A. Einstein**, Berl. Ber. (1917), pp. 142.

rotating distant masses and study them. The method that **Einstein** ([1]) gave for the approximate integration of the field equations is eminently suited to that end, and it shall serve as the basis for the following calculations. The example that we shall choose is that of the field in the interior of a uniformly-rotating, infinitely-thin, hollow sphere that is loaded with a constant mass density.

In the first section of this paper (which can be skipped without impairing one's understanding of what follows), the approximate calculation of the $g_{\mu\nu}$ will be performed for the interior of the spherical shell, and in the second section, the motion of a mass-point in that field will be discussed.

A. Computational part: The calculation of the $g_{\mu\nu}$ in the vicinity of the center of the rotating sphere. –

Notations:

a	radius of the hollow sphere
M	its mass
ω	its angular velocity
x, y, z	the rectangular coordinates of a point of the outer surface of the ball
x_0, y_0, z_0	coordinates of the reference point
κ	gravitational constant
ρ_0	naturally-measured spatial density of the matter

As far as the viewpoint of the approximation that will be used in the field calculations is concerned, let us say this in advance: The field in the vicinity of the center of the sphere will be considered to be weak enough that only those terms in the field equations that have order one relative to the quantities $\gamma_{\mu\nu}$ will be used in the calculations ($\gamma_{\mu\nu}$ is defined by $g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu}$). That approach will make it possible for us to employ **Einstein**'s method for the approximate integration of the field equations. With the second, oft-employed, approach to the approximation, we regard the components of the velocity of ponderable matter as small in comparison to unity (i.e., the speed of light), such that their first powers can already be neglected in that coarsest approximation that leads to **Newton**'s theory. We would like to apply this approximation (which is completely independent of the first one) only to the extent that we shall drop the terms of order three and higher in the velocities in comparison to 1. Finally, our calculations shall relate to the vicinity of the center of the sphere. Let r be the distance from the reference point to the center of the sphere, and let R be the distance from the reference point to the integration element, so we shall develop $1 / R$ into a power series in r / a that we shall truncate after the second power.

Einstein's approximation method of integration yields the following prescription for the calculation of the $g_{\mu\nu}$:

([1]) **A. Einstein**, Berl. Ber. (1916), pp. 688.

$$g_{\mu\nu} = -\,\delta_{\mu\nu} + \gamma_{\mu\nu}\,, \qquad \delta_{\mu\nu} = \begin{cases} 1 & \mu = \nu, \\ 0 & \mu \neq \nu, \end{cases} \tag{1}$$

$$\gamma_{\mu\nu} = \gamma'_{\mu\nu} - \tfrac{1}{2}\sum_{\alpha} \gamma'_{\alpha\alpha}\,, \tag{2}$$

$$\gamma'_{\mu\nu} = -\frac{\kappa}{2\pi}\int \frac{T_{\mu\nu}(x, y, z, t - r)}{R}\, dV_0\,. \tag{3}$$

In this, $T_{\mu\nu}$ is the covariant energy tensor of matter, dV_0 is the spatial volume element of the integration space, and:

$$R^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\,.$$

The coefficients $g_{\mu\nu}$ of the line element refer to the coordinates $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = i\,t$.

With the first approach to the approximation, we can replace the covariant energy tensor with the contravariant one; it is given by neglecting the stresses:

$$T_{\mu\nu} = T^{\mu\nu} = \rho_0 \frac{dx_\mu}{ds}\frac{dx_\nu}{ds} = \rho_0 \frac{dx_\mu}{dx_4}\frac{dx_\nu}{dx_4}\left(\frac{dx_4}{ds}\right)^2. \tag{4}$$

The sphere rotates around the z-axis with an angular velocity of ω, so one will have:

$$\left.\begin{aligned} \frac{dx_1}{dx_4} &= -i\frac{dx}{dt} = i\,a\,\omega \sin\vartheta \sin\varphi, \\ \frac{dx_2}{dx_4} &= -i\frac{dy}{dt} = -i\,a\,\omega \sin\vartheta \cos\varphi, \\ \frac{dx_3}{dx_4} &= 0 \end{aligned}\right\} \tag{5}$$

for one of its points that has the polar coordinates a, ϑ, φ. When these are substituted in (4), that will imply the following matrix:

$$T_{\mu\nu} = \rho_0 \left(\frac{dx_4}{ds}\right)^2 \begin{Bmatrix} -a^2\omega^2 \sin^2\vartheta \sin^2\varphi & +a^2\omega^2 \sin^2\vartheta \sin\varphi \cos\varphi & 0 & i\,a\,\omega \sin\vartheta \sin\varphi \\ +a^2\omega^2 \sin^2\vartheta \sin\varphi \cos\varphi & -a^2\omega^2 \sin^2\vartheta \cos^2\varphi & 0 & -i\,a\,\omega \sin\vartheta \cos\varphi \\ 0 & 0 & 0 & 0 \\ i\,a\,\omega \sin\vartheta \sin\varphi & -i\,a\,\omega \sin\vartheta \cos\varphi & 0 & 1 \end{Bmatrix}. \tag{6}$$

Since we understand ρ_0 to mean the naturally-measured density of matter, we must likewise set dV_0 equal to the naturally-measured spatial volume element in order to ensure the tensor character of the integral (3). The formula ([1]):

$$dV_0 = \sqrt{g}\; i \frac{dx_4}{ds} dV \tag{7}$$

is true for it. We introduce polar coordinates for the integration, so:

$$\sqrt{g}\; dV = a^2\, da \sin\vartheta\, d\vartheta\, d\varphi . \tag{8}$$

Finally, we still have to express $1 / R$ in terms of the integration variable. We choose the coordinate system in such a way that the reference point falls in the ZX-plane, so its coordinate will then be:

$$x_0 = r \sin \vartheta_0 , \qquad y_0 = 0, \qquad z_0 = r \cos \vartheta_0 .$$

One will then have:

$$R^2 = (a \sin\vartheta \cos\vartheta - r \sin\vartheta_0)^2 + a^2 \sin^2\vartheta \sin^2\vartheta + (a \cos\vartheta - r \cos\vartheta_0)^2$$

$$= a^2 \left[1 - \frac{2r}{a} (\sin\vartheta \cos\varphi \sin\vartheta_0 + \cos\vartheta \cos\vartheta_0) + \frac{r^2}{a^2} \right].$$

Developing this into a binomial series and making the omissions that were cited in the introduction will yield:

$$\frac{1}{R} = \frac{1}{a} \left\{ 1 + \frac{r}{a} (\sin\vartheta \cos\varphi \sin\vartheta_0 + \cos\vartheta \cos\vartheta_0) - \frac{1}{2}\frac{r^2}{a^2} + \frac{3}{2}\frac{r^2}{a^2} (\sin\vartheta \cos\varphi \sin\vartheta_0 + \cos\vartheta \cos\vartheta_0)^2 \right\} . \tag{9}$$

We denote the expression in the curly brackets by K and write:

$$\frac{1}{R} = \frac{K}{a} . \tag{9a}$$

Substituting (6), (7), (8), and (9a) in (3) gives:

([1]) Cf., **Einstein**, Berl. Ber. (1914), pp. 1058, eq. (47a).

$$\left.\begin{aligned}
\gamma'_{11} &= \frac{i\kappa}{2\pi}\rho_0\, a^3\omega^2 da \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \left(\frac{dx_4}{ds}\right)^3 \sin^3\vartheta \sin^2\varphi K, \\
\gamma'_{22} &= \frac{i\kappa}{2\pi}\rho_0\, a^3\omega^2 da \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \left(\frac{dx_4}{ds}\right)^3 \sin^3\vartheta \cos^2\varphi K, \\
\gamma'_{44} &= -\frac{i\kappa}{2\pi}\rho_0\, a\, da \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \left(\frac{dx_4}{ds}\right)^3 \sin\vartheta\, K, \\
\gamma'_{12} &= -\frac{i\kappa}{2\pi}\rho_0\, a^3\omega^2 da \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \left(\frac{dx_4}{ds}\right)^3 \sin^3\vartheta \sin\varphi\cos\varphi K, \\
\gamma'_{14} &= \frac{i\kappa}{2\pi}\rho_0\, a^2\omega da \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \left(\frac{dx_4}{ds}\right)^3 \sin^2\vartheta \sin\varphi K, \\
\gamma'_{24} &= -\frac{\kappa}{2\pi}\rho_0\, a^2\omega da \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \left(\frac{dx_4}{ds}\right)^3 \sin^2\vartheta \cos\varphi K, \\
\gamma'_{13} &= \gamma'_{23} = \gamma'_{33} = \gamma'_{43} = 0.
\end{aligned}\right\} \qquad (10)$$

The absolute value of the quantity dx_4 / ds differs from unity only by terms of order $\omega^2\ a^2$; they appear as factors in the small first-order quantities $\gamma'_{\mu\nu}$. Therefore, it is sufficient for one to calculate them from the expression for the "zeroth" approximation to the line element:

$$\left.\begin{aligned}
ds^2 &= -dx_1^2 - dx_2^2 - dx_3^2 - dx_4^2, \\
\frac{ds^2}{dx_4^2} &= -1 - \frac{dx_1^2 + dx_2^2 + dx_3^2}{dx_4^2} = -1 + \omega^2 a^2 \sin^2\vartheta, \\
\frac{ds}{dx_4} &= i\left(1 - \frac{\omega^2 a^2}{2}\sin^2\vartheta\right), \\
\left(\frac{dx_4}{ds}\right)^2 &= i\left(1 + \frac{3}{2}\omega^2 a^2 \sin^2\vartheta\right).
\end{aligned}\right\} \qquad (11)$$

Since our computational precision extends to only terms of order $\omega^2\ a^2$, we can set $\left(\frac{dx_4}{ds}\right)^3 = i$ in all $\gamma'_{\mu\nu}$ that already contain the factor $\omega\, a$, except that we employ the expression (11) for γ'_{44}. If we set:

$$\rho_0\, da = \sigma$$

then (10) will go to:

$$\left.\begin{aligned}
\gamma'_{11} &= -\frac{\kappa}{2\pi}\sigma a^3\omega^2 \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin^3\vartheta \sin^2\varphi K,\\
\gamma'_{22} &= -\frac{\kappa}{2\pi}\sigma a^3\omega^2 \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin^3\vartheta \cos^2\varphi K,\\
\gamma'_{44} &= -\frac{\kappa}{2\pi}\sigma a \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin\vartheta\, K\left(1+\frac{3}{2}\omega^2 a^2 \sin^2\vartheta\right),\\
\gamma'_{12} &= \frac{i\kappa}{2\pi}\sigma a^3\omega^2 \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin^3\vartheta \sin\varphi \cos\varphi K,\\
\gamma'_{14} &= \frac{i\kappa}{2\pi}\sigma a^2\omega \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin^2\vartheta \sin\varphi K,\\
\gamma'_{24} &= -\frac{i\kappa}{2\pi}\sigma a^2\omega \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin^2\vartheta \cos\varphi K,\\
\gamma'_{13} &= \gamma'_{23} = \gamma'_{33} = \gamma'_{34} = 0.
\end{aligned}\right\} \tag{12}$$

If one substitutes the value of K from (9) in these expressions and evaluates the integrals then one will get:

$$\left.\begin{aligned}
\gamma'_{11} &= -\frac{\kappa}{2\pi}\frac{M}{3a}a^3\omega^2\left(1-\frac{r^2}{5a^2}\right),\\
\gamma'_{22} &= -\frac{\kappa}{2\pi}\frac{M}{3a}a^3\omega^2\left\{1-\frac{r^2}{5a^2}(1-3\sin^2\vartheta_0)\right\},\\
\gamma'_{44} &= \frac{\kappa}{2\pi}\frac{M}{a}\left\{1+a^2\omega^2\left[1-\frac{r^2}{5a^2}\left(1-\frac{3}{2}\sin^2\vartheta_0\right)\right]\right\},\\
\gamma'_{24} &= -\frac{i\kappa}{2\pi}\frac{M}{3a}\omega r\sin\vartheta_0,\\
\gamma'_{12} &= \gamma'_{14} = \gamma'_{13} = \gamma'_{23} = \gamma'_{33} = \gamma'_{34} = 0.
\end{aligned}\right\} \tag{13}$$

One then gets the $\gamma_{\mu\nu}$, and thus, the $g_{\mu\nu}$, from these values with the use of equations (1) and (2). One now replaces the polar coordinates r and ϑ_0 of the reference point with its rectangular coordinates and replaces **Einstein**'s gravitational constant κ with the usual one: $k = \kappa / 8\pi$ (speed of light = 1). That will then yield:

$$\left.\begin{aligned}
g_{11} &= -1 - \frac{2kM}{a}\left\{1 + a^2\omega^2 - \frac{\omega^2}{10}(2z_0^2 + x_0^2)\right\}, \\
g_{22} &= -1 - \frac{2kM}{a}\left\{1 + a^2\omega^2 - \frac{\omega^2}{10}(2z_0^2 - 3x_0^2)\right\}, \\
g_{33} &= -1, \\
g_{44} &= -1 + \frac{2kM}{a}\left\{1 + \frac{5a^2\omega^2}{3} - \frac{\omega^2}{6}(2z_0^2 - x_0^2)\right\}, \\
g_{24} &= -i\frac{4kM}{3a}\omega x_0,
\end{aligned}\right\} \tag{14}$$

while all of the remaining $g_{\mu\nu}$ vanish.

We would now like to free ourselves from the special choice of coordinate system. (Indeed, we put the reference point in the *zx*-plane.) To that end, we make the transformation:

$$\left.\begin{aligned}
x_1' &= x_1 \cos a + x_2 \sin a, \\
x_2' &= -x_1 \sin a + x_2 \cos a, \\
x_3' &= x_3, \\
x_4' &= x_4.
\end{aligned}\right\} \tag{15}$$

By means of the transformation formula for a covariant tensor of rank two:

$$g'_{\sigma\tau} = \frac{\partial x_\mu}{\partial x'_\sigma}\frac{\partial x_\nu}{\partial x'_\tau} g_{\mu\nu},$$

the coefficient matrix will go to:

$$g_{\mu\nu} = \left\{\begin{matrix}
-1 - \frac{2kM}{a}\left[1 + a^2\omega^2 - \frac{\omega^2}{10}(2z^2 + x^2 - 3y^2)\right] & +\frac{2kM}{a}\frac{\omega^2}{5}xy & 0 & +i\frac{4kM}{3a}\omega y \\
+\frac{2kM}{a}\frac{\omega^2}{5}xy & -1 - \frac{2kM}{a}\left[1 + a^2\omega^2 - \frac{\omega^2}{10}(2z^2 + x^2 - 3y^2)\right] & 0 & -i\frac{4kM}{3a}\omega y \\
0 & 0 & -1 & 0 \\
i\frac{4kM}{3a}\omega y & -i\frac{4kM}{3a}\omega x & 0 & -1 + \frac{2kM}{a}\left[1 + \frac{5a^2\omega^2}{3} + \frac{\omega^2}{6}(2z^2 - x^2 - y^2)\right]
\end{matrix}\right\}. \tag{16}$$

The index 0 for the coordinates has been dropped here; from now on, *x*, *y*, *z* will mean the coordinates of the reference point.

B. Physical part: The motion of a mass-point inside of a rotating hollow sphere. – We would like to present the equations of motion for a mass-point that is found near the center of our rotating spherical shell. The field in that neighborhood is characterized by the coefficient matrix of the $g_{\mu\nu}$ [eq. (16) of the first section].

The law of motion for a mass-point in **Einstein**'s theory is known to be given by the condition:

$$\delta \int ds = 0,$$

or, when one performs the variation ([1]):

$$\frac{d^2 x_\tau}{ds^2} = \Gamma^\tau_{\mu\nu} \frac{dx_\mu}{ds} \frac{dx_\nu}{ds} \qquad (\tau = 1, \ldots, 4). \tag{17}$$

With the first approach to approximation, one will have:

$$\Gamma^\tau_{\mu\nu} = -\begin{Bmatrix} \mu\,\nu \\ \tau \end{Bmatrix} = \begin{bmatrix} \mu\,\nu \\ \tau \end{bmatrix} = \frac{1}{2}\left(\frac{\partial g_{\tau\nu}}{\partial x_\mu} + \frac{\partial g_{\mu\tau}}{\partial x_\nu} - \frac{\partial g_{\mu\nu}}{\partial x_\tau} \right) \tag{18}$$

for the "field components" $\Gamma^\tau_{\mu\nu}$. We would like to consider only those motions of mass-points that are small in comparison to the speed of light, such that we can neglect the squares and products of the velocity components. We can then drop all terms in which the index 4 does not occur from the right-hand side of equations (17), and replace the derivatives with respect to *s* with ones with respect to *t*, as well. If we give consideration to $dx_4 / dt = i$ then equations (17) will go to:

$$\frac{d^2 x_\tau}{ds^2} = 2i \left(\Gamma^\tau_{14} \frac{dx_1}{dt} + \Gamma^\tau_{24} \frac{dx_2}{dt} + \Gamma^\tau_{34} \frac{dx_3}{dt} \right) - \Gamma^\tau_{44}. \tag{19}$$

Only those field components $\Gamma^\tau_{\mu\nu}$ that contain the index 4 at least once will come under consideration then. They amount to sixteen quantities that can be represented in our case as a matrix of an antisymmetric tensor of rank two (although they are not actually tensor components!). Since the partial derivatives with respect to x_4 all vanish for stationary fields, the quantities can then be written as follows ([2]):

([1]) Indices that occur twice will be summed from 1 to 4.

([2]) One should observe that this matrix corresponds completely to the six-vector $\mathfrak{M}$ of the electromagnetic field. The analogy between electrodynamics and the (approximate) theory of gravitation goes even further than that when one ponders the fact that in the approximate integration, the quantities g_{14}, g_{24}, g_{34}, g_{44} can be computed from the density and velocity of matter in precisely the same way that the potentials $\mathfrak{A}_x$, $\mathfrak{A}_y$, $\mathfrak{A}_z$, Φ can be computed from the electric four-current, and furthermore, the fact that in our case, the right-hand sides of eq. (19) correspond completely with the components of the ponderomotor force $\mathfrak{E} + [\mathfrak{v}\,\mathfrak{H}]$, up to a numerical factor!

$$\left.\begin{array}{llll}
\Gamma^1_{14}=0 & \Gamma^1_{24}=\frac{1}{2}\left(\frac{\partial g_{14}}{\partial x_2}-\frac{\partial g_{24}}{\partial x_1}\right) & \Gamma^1_{34}=\frac{1}{2}\left(\frac{\partial g_{14}}{\partial x_3}-\frac{\partial g_{34}}{\partial x_1}\right) & \Gamma^1_{44}=-\frac{1}{2}\frac{\partial g_{44}}{\partial x_1} \\
\Gamma^2_{14}=\frac{1}{2}\left(\frac{\partial g_{24}}{\partial x_1}-\frac{\partial g_{14}}{\partial x_2}\right) & \Gamma^2_{24}=0 & \Gamma^2_{34}=\frac{1}{2}\left(\frac{\partial g_{24}}{\partial x_3}-\frac{\partial g_{34}}{\partial x_2}\right) & \Gamma^2_{44}=-\frac{1}{2}\frac{\partial g_{44}}{\partial x_2} \\
\Gamma^3_{14}=\frac{1}{2}\left(\frac{\partial g_{34}}{\partial x_1}-\frac{\partial g_{14}}{\partial x_3}\right) & \Gamma^3_{24}=\frac{1}{2}\left(\frac{\partial g_{34}}{\partial x_2}-\frac{\partial g_{24}}{\partial x_3}\right) & \Gamma^3_{34}=0 & \Gamma^3_{44}=-\frac{1}{2}\frac{\partial g_{44}}{\partial x_3} \\
\Gamma^4_{14}=\frac{1}{2}\frac{\partial g_{44}}{\partial x_1} & \Gamma^4_{34}=\frac{1}{2}\frac{\partial g_{44}}{\partial x_2} & \Gamma^4_{34}=\frac{1}{2}\frac{\partial g_{44}}{\partial x_3} & \Gamma^4_{44}=0
\end{array}\right\} \quad (20)$$

If one substitutes the special values for the $g_{\mu\nu}$ from (16) in this then one will get the following matrix:

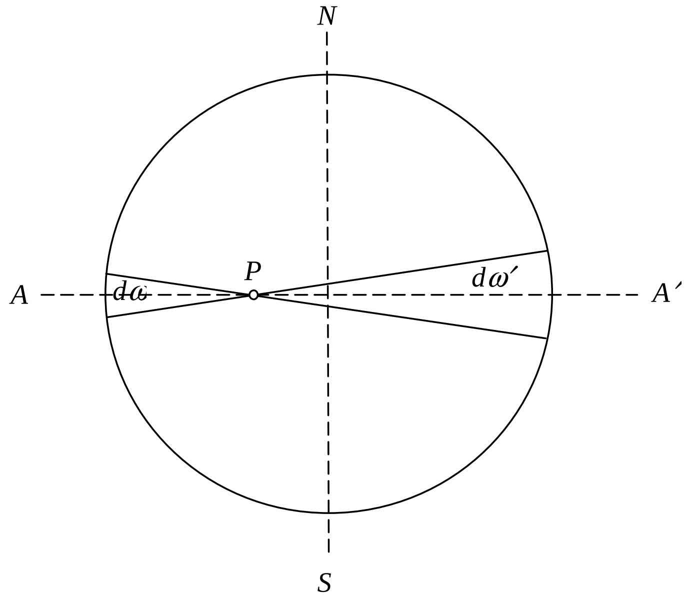
$$\left.\begin{array}{cccc}
0 & i\frac{4kM}{3a}\omega & 0 & -\frac{kM}{3a}\omega^2 x \\
-i\frac{4kM}{3a}\omega & 0 & 0 & -\frac{kM}{3a}\omega^2 y \\
0 & 0 & 0 & \frac{2kM}{3a}\omega^2 z \\
\frac{kM}{3a}\omega^2 x & \frac{kM}{3a}\omega^2 y & -\frac{2kM}{3a}\omega^2 z & 0
\end{array}\right\} \quad (21)$$

We then obtain the equations of motion for our special problem from (19) and (21):

$$\left.\begin{array}{l}
\ddot{x}=-\frac{8kM}{3a}\omega\dot{y}+\frac{kM}{3a}\omega^2 x, \\
\ddot{y}=\frac{8kM}{3a}\omega\dot{x}+\frac{kM}{3a}\omega^2 y, \\
\ddot{z}=-\frac{2kM}{3a}\omega^2 z.
\end{array}\right\} \quad (22)$$

The right-hand sides of the equations represent the components of the force that our field exerts on the mass-point with mass 1. As one sees, the first terms correspond completely to the X and Y components of the Coriolis force, and the second term, to the centrifugal force. The third equation implies the at-first-surprising result that this "centrifugal force" possesses an axial component. Its appearance in the field of the rotating ball can be explained as follows: From the standpoint of the observer at rest, the surface elements of the hollow sphere that are found near the equator will have larger velocities, and as a result, also larger apparent (inertial and gravitating) masses, when compared to the ones in the neighborhood of the poles. The field of a rotating hollow sphere that is loaded with constant surface density will then correspond to that of a spherical shell at rest whose surface density increases with increasing distance ϑ from the pole. The fact that points in the latter case that lie outside of the equatorial plane will be drawn towards it is self-explanatory.

(By the way, it is not difficult to also comprehend the fact that forces that are analogous to centrifugal forces will appear inside of such a hollow sphere that is non-uniformly loaded with mass density. It is known in potential theory that one can show the vanishing of the force field inside of a hollow sphere that is charged with a constant surface density as follows: The force of attraction between surface elements that will be seen inside the viewing angle $d\omega$ when they are viewed from P outward (cf., the figure) is equal and opposite to the force that is exerted by the surface elements that lie in the solid angle $d\omega'$ that belongs to $d\omega$. Naturally, that will no longer be the case for non-uniformly loaded densities. Let AA' be the equatorial plane, so the surface elements that lie inside of the viewing angle $d\omega$ at the position of the point P that is drawn in the figure will have midpoints that lie closer to the equator, so they will be more massive than those of $d\omega'$, in particular. A force in the direction $A'A$ will then result – i.e., a pull perpendicular to the axis of rotation from the outside that will get smaller as the reference point P moves closer to the center.)



The fact that we have come to know of merely a radial component to centrifugal force in nature, but never an axial one, can now be brought into agreement with the results that were found here somewhat by saying: The approximation of the fixed stars by an infinitely-thin hollow sphere is simply incorrect. However, even if we would like to improve our approximation (perhaps by a spatial mass distribution), we would never get a field that is completely equivalent to a centrifugal force by the method of integration that was employed here. We would obtain such a field then if we thought of all of the masses that are found in outer space (Milky Way system, etc) as rotating and calculated their gravitational effects. However, the solution by retarded potentials [eq. (3)] assumes the boundary condition lim $\gamma_{\mu\nu} = 0$ for spatial infinity. Now, as **Einstein** said in his cosmological paper ([1]), those boundary conditions will be fulfilled approximately for a coordinate system that is at rest with respect to the center of the fixed stars. Our solution (16) does not represent the field of a rotating hollow sphere that is "alone in the world"

([1]) Berl. Ber. (1917), pp. 142.

then, but the field inside such a hollow sphere, and outside of which masses are found at much larger distances from the origin and are at rest in the mean with respect to the chosen reference system. The field that is represented by eq. (16) is then, for example, the one that exists at the position of the center of the Sun, if, instead of the Sun and all of the planets, a large hollow sphere (say, with a radius that equals the orbit of Neptune) were present that rotates with an angular velocity ω relative to the fixed stars. If an observer were found at the center of this ball on a world-body whose own gravity field could be neglected and that rotated around the same axis with the angular velocity ω' then he would perceive centrifugal and Coriolis forces that would combine the effects of his own rotation and that of the rotating hollow sphere. The influence of the field of the hollow sphere on the centrifugal field that arises from the proper rotation shall be studied in what follows.

To that end, we introduce a coordinate system that is rigidly coupled with the reference body that rotates with angular velocity ω'. That happens by means of the transformation:

$$\left.\begin{aligned} x' &= \quad x\cos\omega'\frac{x_4}{i}+\sin\omega'\frac{x_4}{i}, \qquad z' = z, \\ y' &= -x\sin\omega'\frac{x_4}{i}+y\cos\omega'\frac{x_4}{i}, \quad x_4' = x_4. \end{aligned}\right\} \tag{23}$$

The quantities $g_{\mu 4}$ that are of interest to us will go to:

$$\begin{aligned} g'_{14} &= -\,i\,y'\left[\omega'\left(1+\frac{2kM}{3a}\right)-\omega\frac{4kM}{3a}\right] \\ g'_{24} &= \quad i\,x'\left[\omega'\left(1+\frac{2kM}{3a}\right)-\omega\frac{4kM}{3a}\right], \\ g'_{44} &= -\,1+\frac{2kM}{3a}\left[1+\frac{5a^2\omega^2}{3}-\frac{\omega^2}{3}z^2\right]+(x'^2+y'^2)\left\{\omega'^2\left(1+\frac{2kM}{a}\right)-\omega\omega'\frac{4kM}{3a}+\omega^2\frac{kM}{3a}\right\} \end{aligned} \tag{24}$$

under this transformation. If one forms the equations of motion from these quantities according to (19) and (20) then that will yield:

$$\left.\begin{aligned} \ddot{x} &= \quad 2\left[\omega'\left(1+\frac{2kM}{a}\right)-\omega\frac{4kM}{3a}\right]\dot{y}+\left\{\omega'^2\left(1+\frac{2kM}{a}\right)-\omega\omega'\frac{4kM}{3a}+\omega^2\frac{kM}{3a}\right\}x, \\ \ddot{y} &= -2\left[\omega'\left(1+\frac{2kM}{a}\right)-\omega\frac{4kM}{3a}\right]\dot{y}+\left\{\omega'^2\left(1+\frac{2kM}{a}\right)-\omega\omega'\frac{4kM}{3a}+\omega^2\frac{kM}{3a}\right\}y, \\ \ddot{z} &= \qquad -\frac{2kM}{3a}\omega^2 z. \end{aligned}\right\} \tag{25}$$

If one sets $M = 0$ in this then one will get the usual centrifugal-Coriolis field:

$$\left.\begin{aligned} \ddot{x} &= 2\omega'\dot{y} + \omega'^2 x, \\ \ddot{y} &= -2\omega'\dot{x} + \omega'^2 y, \\ \ddot{z} &= 0. \end{aligned}\right\} \tag{26}$$

If one sets $M \neq 0$, $\omega = 0$ then one will get:

$$\left.\begin{aligned} \ddot{x} &= 2\omega'\left(1+\frac{2kM}{a}\right)\dot{y} + \omega'^2\left(1+\frac{2kM}{a}\right)x, \\ \ddot{y} &= -2\omega'\left(1+\frac{2kM}{a}\right)\dot{x} + \omega'^2\left(1+\frac{2kM}{a}\right)y, \\ \ddot{z} &= 0, \end{aligned}\right\} \tag{27}$$

from which one can see how the inertial effects can be influenced by the presence of surrounding masses M: Centrifugal and Coriolis forces will be multiplied by the factor $\left(1+\frac{2kM}{a}\right)$.

Finally, one sees from eq. (25) itself that the effect of a rotation of the hollow sphere with the same sense consists of a reduction of the centrifugal and Coriolis force. If one sets:

$$\omega' = \omega\,\frac{4kM}{3(2kM+a)} \tag{28}$$

then the Coriolis force will vanish. One can refer to the quantity $\frac{4kM}{3(2kM+a)}$ as the "dragging coefficient" of the hollow sphere relative to the Coriolis force. The centrifugal force cannot be made to vanish, since the expressions in the curly brackets in eq. (25) will yield no real roots for ω when they are set to zero. The value of the centrifugal force in the "rest" system ($\omega' = 0$) was:

$$\frac{kM}{3a}\,\omega^2\sqrt{x^2+y^2}\,.$$

If one could now rotate the reference system in the same sense as the hollow sphere then the centrifugal force would initially go down with small values of ω' and attain a minimum when ω'/ω was equal to one-half the value of dragging coefficient ([1]). From there on, it would once more increase and again attain the original value that it had for ω' = 0, as long as ω'/ω was equal to the dragging coefficient. With increasing ω', it would then rise again and attain a magnitude for large ω' that differed only slightly from the

([1]) One can convince oneself of that fact directly by differentiating the quantity in brackets.

value that it had in the absence of the hollow sphere (namely, $\omega'^2 \sqrt{x^2 + y^2}$), since $2kM$ / a is indeed small compared to 1, from our assumptions.

On first glance, the fact that the right-hand sides of the equations of motion (25) do not depend upon merely the difference $\omega - \omega'$ seems to contradict the very essence of a theory of relativity. However, it should not be forgotten that we are not dealing with just two bodies (namely, the mass-point and hollow sphere) in the problem that we treat here, but that even-more-distant masses that are at rest in the initially-chosen reference system must be brought into play as a third element that is determined by the field by way of the boundary condition lim $\gamma_{\mu\nu} = 0$.

Summary.

It will be shown in a concrete example that further forces that are due to rotating masses will appear in the (Einsteinian) gravitational field that are analogous to the centrifugal (Coriolis, resp.) force. The peculiarities that the calculated special case exhibits will be discussed.

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