"Zur Theorie der Stabilität des elastischen Gleichgewichts," Zeit. angew. Math. Mech. **13** (1933), 160-165.

On the stability of elastic equilibrium

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Translated D. H. Delphenich

1. Problem statement ([1]). – In the present article, I will develop a general theory of the stability of elastic equilibrium. In order to exhibit such a theory, one can either start from the fact that an equilibrium state is at the limit of stability when one has not only the equilibrium state under scrutiny, but also an "infinitely-close" equilibrium state, or one can appeal to the energetic formulation, in which an equilibrium state ceases to be stable when the potential energy is no longer a true minimum. I prefer the energetic formulation, because it seems simpler to me, and since it is simplest to adapt to non-rectangular coordinates. For the sake of brevity, I shall confine myself here to rectangular coordinates and external forces that do not change with the displacement. However, a generalization of that presents no complications.

2. Notations. Coordinates. – We shall refer to the equilibrium state whose stability is being examined as the "initial state" and a state that emerges from the initial state when we impart any "allowable" displacements to the points of the elastic body as a "neighboring state." "Allowable" means displacements that are compatible with the geometric constraints (e.g., support conditions). We introduce the following coordinates:

a) Spatially-fixed, rectangular normal coordinates ξ_1, ξ_2, ξ_3, which are referred to the three rectangular unit vectors $\mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3$. The vector from the origin to the point ξ_1, ξ_2, ξ_3 is then $\sum_\nu \xi_\nu \mathfrak{E}_\nu$.

([1]) Confer:
1. **R. v. Mises**, "Über die Stabilitätsprobleme der Elastizitätstheorie," this journal **3** (1923), pp. 406.
2. **H. Reissner**, "Energiekriterium der Knicksicherheit," this journal **5** (1925), pp. 475.
3. **C. B. Beizeno** and **H. Hencky**, "On the general theory of elastic stability I and II," Kon. Akademie van Wetenschappen te Amsterdam, Proceedings **31** (1928), pp. 569.
4. **E. Trefftz**, "Über die Ableitung der Stabilitätskriterien des elastischen Gleichgewichts, etc." *Verhandlungen des III intern. Kongresses für Techsniche Mechanik, Stockholm* 1930, v. 3, pp. 44.

Furthermore, confer the literature that is cited in those works.

b) Substantial coordinates x_1, x_2, x_3. They are attached to the mass-particles of the elastic body. In the initial state, they shall agree with the coordinates ξ_1, ξ_2, ξ_3, and in the neighboring state, they will be curvilinear coordinates.

Let the displacement that takes the initial state to the neighboring state be $\mathfrak{u} = u_1\,\mathfrak{E}_1 + u_2\,\mathfrak{E}_2 + u_3\mathfrak{E}_3$, where u_ν are functions of x_1, x_2, x_3. One then has that:

$$\xi_\nu = x_\nu \text{ and the radius vector from the origin is } \mathfrak{R} = \sum_\nu x_\nu\,\mathfrak{E}_\nu \tag{1}$$

in the initial state and:

$$\xi_\nu = x_\nu + u_\nu \text{ and the radius vector from the origin is } \mathfrak{r} = \sum_\nu (x_\nu + u_\nu)\,\mathfrak{E}_\nu \tag{2}$$

in the neighboring state.

We consider external forces that take the form of:

a) Volume forces with the components X_1, X_2, X_3 per unit initial volume.

They shall be constant in the following sense: The force $(X_1\,\mathfrak{E}_1 + X_2\,\mathfrak{E}_2 + X_3\,\mathfrak{E}_3)\,dx_1\,dx_2\,dx_3$ shall act upon the mass-particles that lie within a parallelepiped with edges dx_1, dx_2, dx_3 that are parallel to the axes in the initial state, and also when the parallelepiped has experienced a displacement and deformation into the neighboring state (example: gravity).

b) Surface tractions with the components Ξ_1, Ξ_2, Ξ_3 per unit area in the initial state, i.e., the force $(\Xi_1\,\mathfrak{E}_1 + \Xi_2\,\mathfrak{E}_2 + \Xi_3\,\mathfrak{E}_3)\,dO$ on the part of the surface that meets the element dO, and also when the element in the neighboring state has experienced a displacement, rotation, or deformation (example: pressure on the end surface of a buckled rod).

Let E be the total internal energy of the elastically-deformed body, and let e be the same thing per unit initial volume. Let A be the work that the external forces do when going from the initial state to the neighboring state.

3. Elementary parallelepiped. Lattice vectors. Line element. – We consider an elementary parallelepiped that is included between the pair of surfaces x_ν = const. and $x_\nu + dx_\nu$ = const. In the initial state, it has the three edges dx_1, dx_2, dx_3 that are parallel to the axes. In a neighboring state, its edges will be defined by the three vectors $\frac{\partial \mathfrak{r}}{\partial x_1}dx_1$, $\frac{\partial \mathfrak{r}}{\partial x_2}dx_2$, $\frac{\partial \mathfrak{r}}{\partial x_3}dx_3$ (see. Fig. 1). We call those vectors $\mathfrak{e}_h = \frac{\partial \mathfrak{r}}{\partial x_1}$ the "lattice vectors." From (2), we calculate them to be:

$$\mathfrak{e}_h = \frac{\partial \mathfrak{r}}{\partial x_1} = \mathfrak{E}_h + \sum_\nu \frac{\partial u_\nu}{\partial x_h} \mathfrak{E}_\nu \,. \tag{3}$$

The lattice vectors produce the coefficients of the line element. Two points whose coordinates differ by dx_1, dx_2, dx_3 are connected by the vector:

$$d\,\mathfrak{r} = \sum_\nu \frac{\partial \mathfrak{r}}{\partial x_\nu} dx_\nu = \sum_\nu \mathfrak{e}_\nu \, dx_\nu \,. \tag{4}$$

The square of its length is:

$$d\,\mathfrak{r}^2 = [\sum_\nu \mathfrak{e}_\nu \, dx_\nu]^2 = \sum_\mu \sum_\nu \mathfrak{e}_\mu \cdot \mathfrak{e}_\nu \, dx_\mu \, dx_\nu \,. \tag{5}$$

Comparing the coefficients of that with the usual form $\sum_\mu \sum_\nu g_{\mu\nu} \, dx_\mu \, dx_\nu$ gives:

$$g_{\mu\nu} = \mathfrak{e}_\mu \cdot \mathfrak{e}_\nu \,. \tag{6}$$

From formula (3), that will give us:

$$g_{\mu\nu} = \mathfrak{e}_\mu \cdot \mathfrak{e}_\nu = \mathfrak{E}_\mu \cdot \mathfrak{E}_\nu + \left(\frac{\partial u_\mu}{\partial x_\nu} + \frac{\partial u_\nu}{\partial x_\mu} \right) + \sum_h \frac{\partial u_h}{\partial x_\mu} \frac{\partial u_h}{\partial x_\nu} \,, \tag{7}$$

in which one considers the fact that one will have $\mathfrak{E}_\mu \cdot \mathfrak{E}_\nu = 0$ for $\mu \neq \nu$ for rectangular unit vectors.

Set $u_\nu = 0$ for the initial state. If we denote the coefficients of the line element by upper-case $G_{\mu\nu}$ here then:

$$G_{\mu\nu} = \mathfrak{E}_\mu \cdot \mathfrak{E}_\nu \,. \tag{8}$$

In particular, we will later need the changes $\Delta g_{\mu\nu}$ that the $g_{\mu\nu}$ experience under the transition from the initial state to the neighboring one:

$$\Delta g_{\mu\nu} = g_{\mu\nu} - G_{\mu\nu} = \frac{\partial u_\mu}{\partial x_\nu} + \frac{\partial u_\nu}{\partial x_\mu} + \sum_h \frac{\partial u_h}{\partial x_\mu} \frac{\partial u_h}{\partial x_\nu} \,. \tag{9}$$

4. The stresses and the internal energy of the elastic body. – We again consider an elementary parallelepiped (Fig. 1). If the force $\mathfrak{k}_1 \, dx_2 \, dx_3$ acts on the vectors $\mathfrak{e}_2 \, dx_2$ and $\mathfrak{e}_3 \, dx_3$ of the stressed parallelogram that bounds the parallelepiped on the side of increasing x_1 then we will call $\mathfrak{k}_1$ the "stress vector" that belongs to the surface x_1 = const. The stress vectors $\mathfrak{k}_2$ and $\mathfrak{k}_3$ will be defined correspondingly on the surfaces x_2 = const. and x_3 = const. If we decompose the stress vectors along the lattice vectors:

$$\mathfrak{k}_h = \sum_\nu k_{h\nu} \, \mathfrak{e}_\nu \,, \tag{10}$$

then we will get the stress components $k_{\mu\nu}$. If we considers rotational equilibrium then we will establish the symmetry:

$$k_{\mu\nu} = k_{\nu\mu} . \tag{11}$$

Now let $\mathfrak{e}\, dx_1\, dx_2\, dx_3$ be the energy that is stored in the parallelepiped. If we then impart the infinitesimal displacement $\delta\,\mathfrak{r}$ on the points of the elastic body then that will raise the internal energy by the amount of work that is performed by the stresses in that way. If $\delta\,\mathfrak{r}$ is the displacement at the center of the parallelepiped then the displacement on the left boundary face will be $\delta\,\mathfrak{r} - \frac{1}{2}\frac{\partial\,\delta\mathfrak{r}}{\partial x_1}\,dx_1$ and $\delta\,\mathfrak{r} + \frac{1}{2}\frac{\partial\,\delta\mathfrak{r}}{\partial x_1}\,dx_1$ on the right-hand one, and the associated stress vectors $-\,\mathfrak{k}_1\,dx_2\,dx_3$ ($+\,\mathfrak{k}_1\,dx_2\,dx_3$, resp.) will perform an amount of work equal to:

$$-\,\mathfrak{k}_1\,dx_2\,dx_3\cdot\left(\delta\,\mathfrak{r} - \frac{1}{2}\frac{\partial\,\delta\mathfrak{r}}{\partial x_1}\,dx_1\right) + \mathfrak{k}_1\,dx_2\,dx_3\cdot\left(\delta\,\mathfrak{r} - \frac{1}{2}\frac{\partial\,\delta\mathfrak{r}}{\partial x_1}\,dx_1\right) = \mathfrak{k}_1\cdot\frac{\partial\delta\mathfrak{r}}{\partial x_1}\,dx_1\,dx_2\,dx_3 . \tag{12}$$

When the same considerations are applied to the vectors $\mathfrak{k}_2$ and $\mathfrak{k}_3$, that will give the total work, i.e., the increase in the internal energy, as:

$$\delta\,\mathfrak{e}\,dx_1\,dx_2\,dx_3 = \sum_{\nu}\mathfrak{k}_\nu\cdot\frac{\partial\delta\mathfrak{r}}{\partial x_\nu}\,dx_1\,dx_2\,dx_3 . \tag{13}$$

If we substitute $\mathfrak{k}_h = \sum_{\nu} k_{h\nu}\,\mathfrak{e}_\nu$ here and observe that $\mathfrak{e}_h = \frac{\partial\,\mathfrak{r}}{\partial x_h}$, so $\delta\,\mathfrak{e}_h = \frac{\partial\,\delta\mathfrak{r}}{\partial x_h}$, then we will get:

$$\delta\,\mathfrak{e} = k_{11}\,\mathfrak{e}_1\cdot\delta\,\mathfrak{e}_1 + k_{11}\,(\mathfrak{e}_1\cdot\delta\,\mathfrak{e}_1 + \mathfrak{e}_2\cdot\delta\,\mathfrak{e}_2) + \text{etc.} = \sum_{\mu}\sum_{\nu} k_{h\nu}\,\delta\mathfrak{e}_\mu\cdot\mathfrak{e}_\nu ,$$

but $g_{\mu\nu} = \mathfrak{e}_\mu\cdot\mathfrak{e}_\nu$, so $\delta\,g_{\mu\nu} = \mathfrak{e}_\mu\cdot\delta\,\mathfrak{e}_\nu + \mathfrak{e}_\nu\cdot\delta\,\mathfrak{e}_\mu$. With that, we will have:

$$\delta\,\mathfrak{e} = \sum_{\mu,\nu} k_{h\nu}\,\delta g_{\mu\nu} . \tag{14}$$

We would now like to calculate the change in the elastic energy under the transition from the initial state to the neighboring state. In order to do that, we decompose the stresses $k_{\mu\nu}$ into the stresses $\sigma_{\mu\nu}$ in the initial state and the additional stresses $\tau_{\mu\nu}$ that are produced by the deformation:

$$k_{\mu\nu} = \sigma_{\mu\nu} + \tau_{\mu\nu} . \tag{15}$$

The change in internal energy per unit volume will then be:

$$\Delta e = \int \delta e = \tfrac{1}{2}\int \sum_{\mu,\nu} \sigma_{\mu\nu}\, \delta g_{\mu\nu} + \tfrac{1}{2}\int \sum_{\mu,\nu} \tau_{\mu\nu}\, \delta g_{\mu\nu} \,. \tag{16}$$

The first integral on the right can be evaluated because the initial stresses $\sigma_{\mu\nu}$ are constant during the transition, i.e., for the integration. That will give [see formula (9)]:

$$\tfrac{1}{2}\int \sum_{\mu,\nu} \sigma_{\mu\nu}\, \delta g_{\mu\nu} = \tfrac{1}{2} \sum_{\mu,\nu} \sigma_{\mu\nu}\, \Delta g_{\mu\nu} = \tfrac{1}{2} \sum_{\mu,\nu} \sigma_{\mu\nu} \left[\frac{\partial u_\mu}{\partial x_\nu} + \frac{\partial u_\nu}{\partial x_\mu} + \sum_h \frac{\partial u_h}{\partial x_\mu} \frac{\partial u_h}{\partial x_\nu} \right]. \tag{17}$$

The second integral $\tfrac{1}{2}\int \sum_{\mu,\nu} \tau_{\mu\nu}\, \delta g_{\mu\nu}$ is the work done by the additional stresses $\tau_{\mu\nu}$ under the deformation that produced them. We make the assumption for it that is at the basis of all of the cases that were calculated up to now, namely, that we find ourselves in the domain of validity of the law of superposition, such that we can formulate the connection between the displacements u_ν and the stresses $\tau_{\mu\nu}$ precisely as we do in the classical theory of elasticity. If we then denote the shear modulus by G and the lateral contraction number of the material m, and set:

$$\varepsilon_\nu = \frac{\partial u_\mu}{\partial x_\nu}, \qquad \gamma_{\mu\nu} = \frac{\partial u_\mu}{\partial x_\nu} + \frac{\partial u_\nu}{\partial x_\mu}, \qquad \Theta = \sum_\nu \frac{\partial u_\nu}{\partial x_\nu}, \tag{18}$$

to abbreviate, then the deformation energy will become:

$$\tfrac{1}{2}\int \sum_{\mu,\nu} \tau_{\mu\nu}\, \delta g_{\mu\nu} = a = G \left\{ \frac{m-1}{m-2}\Theta^2 - 2(\varepsilon_1 \varepsilon_2 + \varepsilon_2 \varepsilon_3 + \varepsilon_3 \varepsilon_1) + \tfrac{1}{2}(\gamma_{12}^2 + \gamma_{23}^2 + \gamma_{31}^2) \right\}, \tag{19}$$

and the known stress-extension equations will be valid:

$$\tau_{11} = \frac{\partial a}{\partial \varepsilon_1} = 2G\left(\varepsilon_1 + \frac{\Theta}{m-2}\right), \qquad \tau_{12} = \frac{\partial a}{\partial \gamma_{12}} = G\, \gamma_{12}, \quad \text{and cycl. perm.} \tag{20}$$

After integrating over the entire body, we will get the total change in internal elastic energy in the form:

$$\left.\begin{aligned} \Delta E &= \iiint \Delta e\, d\omega \\ &= \tfrac{1}{2}\iiint \sum_{\mu,\nu} \sigma_{\mu\nu} \left(\frac{\partial u_\mu}{\partial x_\nu} + \frac{\partial u_\mu}{\partial x_\nu} \right) d\omega + \tfrac{1}{2}\iiint \sum_{\mu,\nu,h} \sigma_{\mu\nu} \frac{\partial u_h}{\partial x_\mu} \frac{\partial u_h}{\partial x_\nu}\, d\omega + \iiint a\, d\omega \quad (d\omega = dx_1\, dx_2\, dx_3). \end{aligned}\right\} \tag{21}$$

5. The energy criterion for the stability limit. – An elastic body is then found to be in stable equilibrium when the increase ΔE in the internal elastic energy under an allowable displacement to a neighboring state is greater than the work ΔA that is done by the external forces. If the stability limit is attained then there will be at least one system of displacements for which $\Delta E = \Delta A$, but no system for which $\Delta E < \Delta A$. As long as $\Delta E < \Delta A$, instability will prevail. Therefore, at the limit of stability, one will have:

$$\left.\begin{aligned}\Delta E &= \tfrac{1}{2}\iiint\sum_{\mu,\nu}\sigma_{\mu\nu}\left(\frac{\partial u_\mu}{\partial x_\nu}+\frac{\partial u_\mu}{\partial x_\nu}\right)d\omega+\tfrac{1}{2}\iiint\sum_{\mu,\nu,h}\sigma_{\mu\nu}\frac{\partial u_h}{\partial x_\mu}\frac{\partial u_h}{\partial x_\nu}d\omega+\iiint a\,d\omega \\ &\geq \Delta A = \iiint\sum_{\nu}X_\nu\,u_\nu\,d\omega+\iint\sum_{\nu}\Xi_\nu\,u_\nu\,dO.\end{aligned}\right\} \tag{22}$$

If the left-hand side is to be greater than the left-hand side for all allowable displacements u_ν then the predominantly linear terms must drop out for sufficiently-small displacements, i.e., one must have:

$$\tfrac{1}{2}\iiint\sum_{\mu,\nu}\sigma_{\mu\nu}\left(\frac{\partial u_\mu}{\partial x_\nu}+\frac{\partial u_\mu}{\partial x_\nu}\right)d\omega = \iiint\sum_{\nu}X_\nu\,u_\nu\,d\omega+\iint\sum_{\nu}\Xi_\nu\,u_\nu\,dO \tag{23}$$

for all allowable u_ν. After partial integration, that will yield the known equilibrium conditions of the initial state:

in the interior:
$$\sum_{\nu}\frac{\partial\sigma_{h\nu}}{\partial x_\nu}+X_h = 0\,, \tag{24}$$

$(h = 1, 2, 3)$,

on the surface:
$$\sum_{\nu}\sigma_{h\nu}\cos(\nu, N) = \Xi_h\,. \tag{25}$$

In the last equation, $\cos(\nu, N)$ is the direction cosine of the exterior normal to the elastic body with respect to the ν^{th} axis.

After dropping the linear terms, what will remain in (22) are the quadratic terms, which we would like to denote by Q :

$$Q = \tfrac{1}{2}\iiint\sum_{\mu,\nu,h}\sigma_{\mu\nu}\frac{\partial u_h}{\partial x_\mu}\frac{\partial u_h}{\partial x_\nu}d\omega+\iiint a\,d\omega \geq 0\,. \tag{26}$$

At the limit of stability, there will now be at least one “dangerous” (*gefährlich*) system of displacements u_1, u_2, u_2 for which the equal sign will be valid in (26). Since Q cannot become negative, the value 0 that is attained by the “dangerous” system must be a minimum then. It follows from that minimal property that we must have:

$$\delta Q = 0 \tag{27}$$

for all allowable variations δu_ν . We will get the differential equations and boundary conditions for the "dangerous" displacements from that variational demand. If we form δQ then the first integral will yield:

$$\delta \tfrac{1}{2} \iiint \sum_{\mu,\nu,h} \sigma_{\mu\nu} \frac{\partial u_h}{\partial x_\mu} \frac{\partial u_h}{\partial x_\nu} d\omega = \iiint \sum_{\mu,\nu,h} \sigma_{\mu\nu} \frac{\partial u_h}{\partial x_\mu} \frac{\partial \delta u_h}{\partial x_\nu} d\omega .$$

(Two summands actually appear, but the differ by only the notation of the summation sign, so they are equal.) A partial integration gives:

$$- \iiint \sum_{\mu,\nu,h} \delta u_h \frac{\partial}{\partial x_\nu} \left(\sigma_{\mu\nu} \frac{\partial u_h}{\partial x_\mu} \right) d\omega + \iint \sum_{\mu,\nu,h} \delta u_h \, \sigma_{\mu\nu} \frac{\partial u_h}{\partial x_\mu} \cos(\nu, N) \, dO ,$$

in which the first integral is taken over the interior of the body and the second one, over its surface, and if we use eq. (25) to set $\sum_\nu \sigma_{\mu\nu} \cos(\nu, N) = \Xi_\mu$ then that will give:

$$- \iiint \sum_{\mu,\nu,h} \delta u_h \frac{\partial}{\partial x_\nu} \left(\sigma_{\mu\nu} \frac{\partial u_h}{\partial x_\mu} \right) d\omega + \iint \sum_{\mu,h} \delta u_h \, \Xi_\mu \frac{\partial u_h}{\partial x_\mu} \, dO .$$

The variation of the second integral is known from the classical theory of elasticity:

$$\delta \iiint a(\varepsilon, \gamma) \, d\omega = - \iiint \sum_h \delta u_h \sum_\nu \frac{\partial \tau_{\mu\nu}}{\partial x_\nu} d\omega + \iint \sum_h \delta u_h \sum_\mu \tau_{h\mu} \cos(\mu, N) \, dO .$$

In total, we will then get:

$$\left. \begin{aligned} \delta Q = \; & - \iiint \sum_h \delta u_h \left\{ \sum_\nu \frac{\partial \tau_{h\nu}}{\partial x_\nu} + \sum_{\mu,\nu} \frac{\partial}{\partial x_\nu} \left(\sigma_{\mu\nu} \frac{\partial u_h}{\partial x_\nu} \right) \right\} d\omega \\ & + \iint \sum_h \delta u_h \left\{ \sum_\mu \tau_{\mu\nu} \cos(\mu, N) + \sum_\mu \Xi_\mu \frac{\partial u_h}{\partial x_\nu} \right\} dO . \end{aligned} \right\} \tag{28}$$

Should that vanish for all allowable δu_h, then the spatial and surface integrals would have to vanish by themselves, and that would only be possible if the differential equations:

$$\sum_\nu \frac{\partial \tau_{h\nu}}{\partial x_\nu} + \sum_{\mu,\nu} \frac{\partial}{\partial x_\nu} \left(\sigma_{\mu\nu} \frac{\partial u_h}{\partial x_\nu} \right) = 0 \tag{29}$$

were fulfilled in the interior, while the boundary conditions:

$$\sum_{\mu} \tau_{\mu\nu} \cos(\mu, N) + \sum_{\mu} \Xi_{\mu} \frac{\partial u_h}{\partial x_\nu} = 0 \tag{30}$$

were fulfilled on the surface. Mechanically, those equations mean that, along with the initial state, there exists an infinitely-close equilibrium state. Mathematically, one is dealing with an eigenvalue problem: Equations (29) and (30), along with the stress-extension equations (20), are homogeneous. If the loading remains outside the stability limit then they will have no non-zero solution, since such a thing will first appear as soon as the stability limit is attained.

In order to show that the solubility of equations (29) with the boundary conditions (30) is not only necessary, but sufficient, for the attainment of the stability limit, one must further show that one also has $Q(u_\nu) = 0$ for the solutions u_ν. Up to now, it has only been shown that $\delta Q = 0$ for all allowable δu_ν. However, if one makes $\delta u_\nu = \frac{1}{2} u_\nu$, in particular, then one will have $\delta Q = Q(u_\nu)$. Since δQ vanishes, it will follow that $Q(u_\nu) = 0$. Q. E. D.

6. Application. – If one would like to the use the general three-dimensional theory of the stability of elastic equilibrium that was developed here in order to arrive at the usual approximation methods in engineering then one would do best to start with the variational problem that is formulated in equations (26) and (27). One must only replace the expression for the elastic energy of the additional deformation $\iiint a(\varepsilon, \gamma)\, d\omega$ with the approximate expressions in the elementary methods. In order to show that in an example, I would like to derive the equations for the tipping of a rectangular cantilever (Fig. 2).

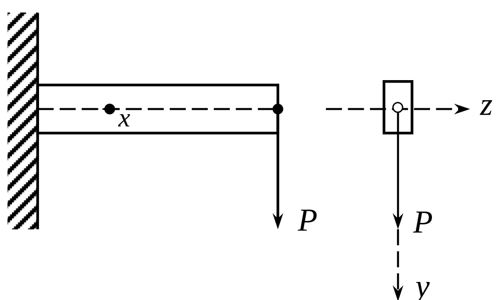

Figure 2.

Let a beam of narrow rectangular cross-section be anchored at one end ($x = 0$), and at the other end ($x = l$), let it be loaded with an isolated force P in the direction of the longer side of the rectangle (viz., the y-direction). The notations are the usual ones: The x-axis is along the centerline of the beam, J_y and J_z are the moments of inertia of the cross-section, $E\,J$ is the bending stiffness, and $G\Theta$ is the torsional stiffness. From the elementary theory, the force P produces the stresses σ_{xx} and σ_{xy}, where one has:

$$\sigma_{xx} = -\frac{P(l-x)\,y}{J_z}, \qquad \iint \sigma_{xy}\, dy\, dz = P\,. \tag{31}$$

(N. B. Double integrals are taken over the cross-section.)

For a sufficiently-large load P, the rod will buckle laterally, i.e., in the z-direction, and simultaneously experience a torsion. We make the following simplifying assumption for that displacement from the equilibrium configuration:

$$v = -\vartheta(x)\, z\,, \qquad w = W(x) + \varphi(x)\, y\,. \tag{32}$$

We must now calculate the two integrals for Q in equation (26). From the elementary theory, the deformation energy of the bent and twisted rod will be:

$$\iiint a\,d\omega = \tfrac{1}{2}\int_0^l E\,J_y\,W''(x)^2\,dx + \tfrac{1}{2}\int_0^l G\,\Theta\,\vartheta'(x)^2\,dx\ . \tag{33}$$

When one uses the stress formulas (31) and the law of displacement (32), the integral that represents the work done by the initial stresses will be, with the altered notation:

$$\left.\begin{aligned}\tfrac{1}{2}\iiint \sum_{\mu,\nu,h}\sigma_{\mu\nu}\frac{\partial u_h}{\partial x_\mu}\frac{\partial u_h}{\partial x_\nu}\,d\omega = \quad & -\tfrac{1}{2}\iiint \frac{P(l-x)\,y}{J_z}\{[W'(x)+\vartheta'(x)\,y]^2+[\vartheta'(x)\,z]^2\}\,dx\,dy\,dz \\ & +\tfrac{1}{2}\iiint 2\,\sigma_{xy}\,[W'(x)+\vartheta'(x)\,y]\,\vartheta(x)\,dx\,dy\,dz.\end{aligned}\right\} \tag{34}$$

If one integrates over y and z, which will make the integrals $\iint y\,dy\,dz$, $\iint y^3\,dy\,dz$, $\iint y\,z^3\,dy\,dz$, and $\iint \sigma_{xy}\,y\,dy\,dz$ drop out, due to the double symmetry in the cross-section, then one will get:

$$-\int_0^l P(l-x)W'(x)\,\vartheta'(x)\,dx + \int_0^l P\,W'(x)\,\vartheta(x)\,dx = -\int_0^l P\,W'(x)\,\frac{d}{dx}[(l-x)\,\vartheta(x)]\,dx\ . \tag{35}$$

In that way, one will have:

$$Q = -\int_0^l P\,W'(x)\,\frac{d}{dx}[(l-x)\,\vartheta(x)]\,dx + \tfrac{1}{2}\int_0^l E\,J_y\,W''(x)^2\,dx + \tfrac{1}{2}\int_0^l G\,\Theta\,\vartheta'(x)^2\,dx\,. \tag{36}$$

We now form:

$$\left.\begin{aligned}\delta Q = \quad & -\int_0^l P\,\delta W'(x)\,\frac{d}{dx}[(l-x)\,\vartheta(x)]\,dx - \int_0^l P\,W'(x)\,\frac{d}{dx}[(l-x)\,\delta\vartheta(x)]\,dx \\ & +\int_0^l E\,J_y\,W''(x)\,\delta W''(x)\,dx + \int_0^l G\,\Theta\,\vartheta'(x)\,\delta\vartheta'(x)\,dx.\end{aligned}\right\} \tag{37}$$

In the partial integration by which we eliminate the derivatives $\delta W'$ and $\delta\vartheta'$ from the integrals, we must observe that due to the initial stress conditions for $x = 0$:

$$W(0) = 0\,, \qquad W'(0) = 0\,, \qquad \vartheta(0) = 0 \tag{38}$$

we must also have:

$$\delta W(0) = 0\,, \qquad \delta W'(0) = 0\,, \quad \delta\vartheta(0) = 0\,.$$

δQ can vanish for all allowable variations δW and $\delta\,\vartheta$ only when the factors of $\delta W(l)$, $\delta W'(l)$, and $\delta\,\vartheta(l)$, and bracketed expressions in the integrals vanish in their own right. With that, we will get the differential equations:

$$E\,J\,\frac{d^4 W}{dx^4} + \frac{d^2}{dx^2}\,[P\,(l - x)\,\vartheta] = 0\,, \qquad G\,\Theta\,\frac{d^2\,\vartheta}{dx^2} - P\,(l - x)\,\frac{d^2 W}{dx^2} = 0\,, \tag{40}$$

and the boundary conditions:

$$W''(l) = 0\,, \qquad E\,J_y\,W'''(l) = P\;\vartheta(l)\,, \qquad \vartheta'(l) = 0 \tag{41}$$

Those differential equations and boundary conditions are identical to the equations that **Prandtl** gave.

One can arrive at the differential equations and boundary conditions for all known problems of the buckling and tipping of plates and rods from the general Ansatz in the same way.

7. Concluding remark. – In conclusion, I would like to go into the connection between what I did and the theory of **Biezeno** and **Hencky**, as well as my own previous investigations. **Biezeno** and **Hencky** ("On the general theory of elastic stability," *loc. cit.*, footnote on pp. 1) based their theory on a consideration of the infinitely-close equilibrium states. The essential difference between their method and my own consists of the way that the "neighboring" stress state is described. **B** and **H** always referred to a spatially-fixed coordinate system, while I employed the substantial coordinates that move with the mass-particles. I believe that I have achieved a simplification in the representation with the use of that natural coordinate system. Naturally, the results are the same in both cases, but the work of **B** and **H** goes further than my own.

My own previous investigations ("Über die Ableitung der Stabilitätskrieterien des elastischen Gleichgewichts, etc.," *loc. cit.*, footnote on pp. 1) are more general than the theory that was developed here, insofar as they did not include the assumption that the stress-extension equations for the additional stresses and the displacements had the simple form of the classical theory of elasticity. The advantage of the greater generality of my previous investigations is to be contrasted with the greater practical utility of the theory that is developed here.

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